Some invariants of nonsingular projective varieties and complete local rings

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If you can't fix it you've got to stand it.

Annie Proulx, *Brokeback Mountain*

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Abstract

In this thesis, we establish some results concerning invariants of nonsingular projective varieties and complete local rings (in characteristic zero) which are defined using local cohomology and de Rham cohomology.

We first study Lyubeznik numbers, invariants of local rings with coefficient fields defined using iterated local cohomology. If V is a *nonsingular* projective variety defined over a field of characteristic zero, we prove that the Lyubeznik numbers of the local ring at the vertex of the affine cone over V (viewing V as a closed subvariety of some projective space $\mathbb{P}^n_k$) are independent of the chosen embedding into projective space, by expressing these numbers in terms of the dimensions of the algebraic de Rham cohomology spaces of V.

We next consider Matlis duality. We give an equivalent definition of the Matlis dual over a local ring with coefficient field k in terms of certain k-linear maps, which we call Σ-*continuous* maps. We use this definition to develop a theory of Matlis duality for $\mathscr{D}$-modules over formal power series rings in characteristic zero. If $R = k[[x_1, \ldots, x_n]]$ is such a ring, and $\mathscr{D} = \mathrm{Diff}(R,k)$ is the ring of k-linear differential operators on R, we show that the Matlis dual $D(M)$ of any left $\mathscr{D}$-module M can again be given a structure of left $\mathscr{D}$-module; and if M is holonomic, the de Rham cohomology spaces of $D(M)$ are k-dual to those of M.

Finally, we examine the Hodge-de Rham spectral sequences associated with Hartshorne's algebraic de Rham homology and cohomology theories for a complete local ring A with a coefficient field k of characteristic zero. *A priori*, these objects depend on a choice of k-algebra surjection $k[[x_1, \ldots, x_n]] \to A$. We prove that, beginning with their E_2-terms, these spectral sequences depend only on A (and possibly the choice of coefficient field) and consist of finite-dimensional k-spaces, thus producing another set of numerical invariants of A. What is more, using our results on Matlis duality, we conclude that the E_2-objects in the homology and cohomology spectral sequences are k-dual to each other; whether this duality holds (as we conjecture) for the rest of the spectral sequences remains open.

Contents

Chapter 1

Introduction

This thesis assumes background knowledge of commutative algebra, homological algebra, and algebraic geometry, for which our basic references are, respectively, [1], [2], and [3]. In this introductory chapter, we collect some preliminary material on local cohomology, $\mathscr{D}$-modules, spectral sequences, and Matlis duality.

1.1 Preliminaries on local cohomology

We begin with the definition of *local cohomology modules*. Our basic reference for facts about local cohomology modules is [4].

Throughout this section, we assume that R is a commutative, Noetherian ring with 1, and that $I \subset R$ is an ideal.

Definition 1.1.1. Let M be an R-module. The *I-power torsion submodule* of M is the R-submodule $\Gamma_I(M)$ consisting of those $m \in M$ annihilated by some power of I.

The operation Γ_I defines an additive functor from the category of R-modules to itself, and it is easy to see that this functor is left-exact. We may therefore consider its right derived functors.

Definition 1.1.2. Let M be an R-module. The *local cohomology modules* of M supported at I, which we denote $H^i_I(M)$, are the right derived functors $R^i\Gamma_I(M)$ evaluated at M.

Given any R-module M, we compute its local cohomology modules in the following way: choose an injective resolution $0 \to M \to \mathscr{I}^\bullet$ in the category of R-modules, and calculate the cohomology objects of the complex $\Gamma_I(\mathscr{I}^\bullet)$ (the resulting objects, as is well-known, are independent of the chosen resolution). In particular, we have $\Gamma_I(M) = H_I^0(M)$ for all M, and given any short exact sequence of R-modules, we obtain the usual long exact sequence for local cohomology supported at I.

Example 1.1.3. Let $n \neq 0$ be an integer, and consider the $\mathbb{Z}$-module $\mathbb{Z}$. The short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

is an injective resolution of $\mathbb{Z}$, since the injective $\mathbb{Z}$-modules are precisely the *divisible* Abelian groups. Therefore, the local cohomology modules $H^i_{(n)}(\mathbb{Z})$ are the cohomology objects of the complex

$$0 \to \Gamma_{(n)}(\mathbb{Q}) \to \Gamma_{(n)}(\mathbb{Q}/\mathbb{Z}) \to 0.$$

As $\mathbb{Q}$ has no torsion, $\Gamma_{(n)}(\mathbb{Q})$, and hence $H^0_{(n)}(\mathbb{Z})$, is zero. On the other hand, given any rational number whose denominator is a power of n, we see that its class in $\mathbb{Q}/\mathbb{Z}$ is annihilated by that power of n, and so $\Gamma_{(n)}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_n/\mathbb{Z}$, where $\mathbb{Z}_n$ is the localization of $\mathbb{Z}$ at the multiplicative system consisting of all powers of n. It follows that $H^1_{(n)}(\mathbb{Z}) \simeq \mathbb{Z}_n/\mathbb{Z}$.

Here are some basic facts about local cohomology modules, all of which will be used later in this thesis:

Proposition 1.1.4. *(a) Local cohomology supported at I depends only on the radical $\sqrt{I}$: if I and J are ideals of R such that $\sqrt{I} = \sqrt{J}$, then $H^i_I(M) \simeq H^i_J(M)$ for all M and i. [4, Remark 1.2.3]*

(b) Local cohomology commutes with direct sums: if $\{M_j\}$ is a family of R-modules, we have $H^i_I(\oplus_j M_j) \simeq \oplus_j H^i_I(M_j)$ for all i. [4, Thm. 3.4.10]

(c) If $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module, then $H^i_{\mathfrak{m}}(M)$ is an Artinian R-module for all i. [4, Thm. 7.1.3]

(d) *If $(R,\mathfrak{m})$ is a* Gorenstein *local ring, $H^i_{\mathfrak{m}}(R)$ vanishes unless $i=\dim(R)$, and $H^{\dim(R)}_{\mathfrak{m}}(R)$ is an injective hull of $R/\mathfrak{m}$ as an R-module. [4, Cor. 6.2.9, Lemma 11.2.3]*

(e) *The change of ring principle: If $R\to R'$ is a ring homomorphism, I is an ideal of R, and N is an R'-module, then $H^i_I(N)\simeq H^i_{IR'}(N)$ as R-modules for all i. [4, Thm. 4.2.1]*

(f) *Local cohomology commutes with flat base change: if R' is a flat R-algebra, I is an ideal of R, and M is an R-module, then $H^i_I(M)\otimes_R R'\simeq H^i_{IR'}(M\otimes_R R')$ as R'-modules for all i. [4, Thm. 4.3.2]*

In addition to the standard long exact cohomology sequence for derived functors, there is another useful long exact sequence for local cohomology: the *Mayer-Vietoris sequence* for two ideals.

Proposition 1.1.5. *[4, Thm. 3.2.3] Let I and J be ideals of R. There is a long exact sequence of R-modules*

$$\cdots\to H^{i-1}_{I\cap J}(M)\to H^i_{I+J}(M)\to H^i_I(M)\oplus H^i_J(M)\to H^i_{I\cap J}(M)\to H^{i+1}_{I+J}(M)\to\cdots$$

for any R-module M, functorial in M.

The definition of local cohomology modules as derived functors given above is sometimes inconvenient to work with. There are many equivalent definitions, three of which we will use in this thesis and which we give now. In each case, we single out a particular feature of the definition that is useful to us.

Proposition 1.1.6. *[4, Thm. 1.3.8] Let M be an R-module. For all i, we have isomorphisms*

$$H^i_I(M)\simeq\varinjlim_l \operatorname{Ext}^i_R(R/I^l,M)$$

of R-modules, functorial in M.

This definition is useful because it expresses local cohomology modules, which are almost never finitely generated, as countable direct limits of finitely generated R-modules. The next definition uses sheaf theory:

Definition 1.1.7. Let X be a topological space and let $Y \subset X$ be a closed (or even locally closed) subset. For any sheaf $\mathscr{F}$ of Abelian groups on X, let $\Gamma_Y(X, \mathscr{F})$ be the Abelian group of global sections of $\mathscr{F}$ whose support is contained in Y. This defines a left-exact additive functor Γ_Y from sheaves of Abelian groups on X to Abelian groups, and we let $H^i_Y(X, \mathscr{F})$, the local cohomology groups of $\mathscr{F}$ supported at Y, be its right derived functors evaluated at $\mathscr{F}$.

Proposition 1.1.8. *[5, Exp. II, Prop. 5] If $X = \operatorname{Spec}(R)$ is an affine scheme, $Y = V(I)$ is a closed subscheme, and $\mathscr{F} = \widetilde{M}$ is a quasi-coherent sheaf, the local cohomology groups $H^i_Y(X, \mathscr{F})$ have the structure of R-modules, and we have isomorphisms*

$$H^i_Y(X, \mathscr{F}) \simeq H^i_I(M)$$

of R-modules for all i.

This definition (the original definition of local cohomology from [5]) is useful because it makes clear the fact that any Abelian group homomorphism between R-modules induces an Abelian group homomorphism on local cohomology, which does not follow from its definition above as a functor on the category of R-modules.

Finally, we have a definition in terms of the *Čech complex*, which is useful because it is often the easiest to compute.

Definition 1.1.9. Let f be an element of R. The *Čech complex* of R with respect to f, which we denote $C^\bullet(f;R)$, is the complex

$$0 \to R \to R_f \to 0$$

where the sole differential is the natural localization map $r \mapsto \frac{r}{1}$. If $f_1, \dots, f_s$ is a sequence of elements of R, the *Čech complex* of R with respect to $f_1, \dots, f_s$, which we denote $C^\bullet(f_1, \dots, f_s;R)$, is the tensor product

$$C^\bullet(f_1, \dots, f_s;R) = C^\bullet(f_1;R) \otimes_R \cdots \otimes_R C^\bullet(f_s;R)$$

If M is any R-module, the *Čech complex* of M with respect to $f_1, \dots, f_s$, which we denote $C^\bullet(f_1, \dots, f_s;M)$, is the tensor product $C^\bullet(f_1, \dots, f_s;R) \otimes_R M$.

Writing out the definitions, we see that $C^\bullet(f_1,\ldots,f_s;M)$ takes the form

$$0 \to M \to \oplus_i M_{f_i} \to \oplus_{i<j} M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_s} \to 0$$

where the components of the differentials are either zero or the natural localization maps, up to appropriate signs.

Proposition 1.1.10. *[4, Thm. 5.1.20] Let M be an R-module, and let I be an ideal. Fix generators $f_1,\ldots,f_s$ for I. Then we have isomorphisms*

$$H^i_I(M) \simeq h^i(C^\bullet(f_1,\ldots,f_s;M))$$

of R-modules for all i.

Proposition 1.1.10 has two immediate consequences. The first is that the cohomology objects of the Čech complex do not depend on the chosen generators for I. The second, apparent from the length of the complex, is that $H^i_I(M) = 0$ for $i > s$. Recalling Proposition 1.1.4(a), we see that local cohomology supported at an ideal I always vanishes in degrees higher than the *arithmetic rank* of I, that is, the minimal number of generators of I *up to radical*.

Example 1.1.11. We revisit Example 1.1.3, this time using the Čech complex $C^\bullet(n;\mathbb{Z})$. This complex takes the form

$$0 \to \mathbb{Z} \to \mathbb{Z}_n \to 0$$

where the sole differential is the map $m \mapsto \frac{m}{1}$. Since $\mathbb{Z}$ is an integral domain, this localization map is injective, and so the cohomology objects of this complex are 0 in degree 0 and $\mathbb{Z}_n/\mathbb{Z}$ in degree 1.

1.2 Preliminaries on $\mathscr{D}$-modules

We give a general definition of differential operators, following EGA, and then consider the special case of a formal power series ring.

Definition 1.2.1. [6, §16] Let R be a commutative ring and $k \subset R$ a commutative subring. The ring $\mathrm{Diff}(R,k)$ of k-linear differential operators on R, a subring of $\mathrm{End}_k(R)$, is defined recursively as follows. A differential operator $R \to R$ of order zero is multiplication by an element of R. Supposing that differential operators of order $\leq j-1$ have been defined, $d \in \mathrm{End}_k(R)$ is said to be a differential operator of order $\leq j$ if, for all $r \in R$, the commutator $[d,r] \in \mathrm{End}_k(R)$ is a differential operator of order $\leq j-1$, where $[d,r] = dr - rd$ (the products being taken in $\mathrm{End}_k(R)$). We write $\mathrm{Diff}^j(R)$ for the set of differential operators on R of order $\leq j$ and set $\mathrm{Diff}(R,k) = \cup_j \mathrm{Diff}^j(R)$. Every $\mathrm{Diff}^j(R)$ is naturally a left R-module. If $d \in \mathrm{Diff}^j(R)$ and $d' \in \mathrm{Diff}^l(R)$, it is easy to prove by induction on $j+l$ that $d' \circ d \in \mathrm{Diff}^{j+l}(R)$, so $\mathrm{Diff}(R,k)$ is a ring.

We consider now the special case in which k is a field of characteristic zero and $R = k[[x_1,\dots,x_n]]$ is a formal power series ring over k. A standard reference for facts about the ring $\mathscr{D} = \mathrm{Diff}(R,k)$ and left modules over $\mathscr{D}$ in this case is [7, Ch. 3]; we summarize some of these facts now. The ring $\mathscr{D}$, viewed as a left R-module, is freely generated by monomials in the partial differentiation operators $\partial_1 = \frac{\partial}{\partial x_1}, \dots, \partial_n = \frac{\partial}{\partial x_n}$ ([6, Thm. 16.11.2]: here the characteristic-zero assumption is necessary). This ring has an increasing filtration $\{\mathscr{D}(\nu)\}$, called the *order filtration*, where $\mathscr{D}(\nu)$ consists of those differential operators of order $\leq \nu$ (the order of an element of $\mathscr{D}$ is the maximum of the orders of its summands, and the order of a single summand $\rho \partial_1^{a_1} \cdots \partial_n^{a_n}$ with $\rho \in R$ is $\sum a_i$). The associated graded object $\mathrm{gr}(\mathscr{D}) = \oplus \mathscr{D}(\nu)/\mathscr{D}(\nu-1)$ with respect to this filtration is isomorphic to $R[\zeta_1,\dots,\zeta_n]$ (a commutative ring), where ζ_i is the image of ∂_i in $\mathscr{D}(1)/\mathscr{D}(0) \subset \mathrm{gr}(\mathscr{D})$.

If M is a finitely generated left $\mathscr{D}$-module, there exists a *good filtration* $\{M(\nu)\}$ on M, meaning that M becomes a filtered left $\mathscr{D}$-module with respect to the order filtration on $\mathscr{D}$ *and* $\mathrm{gr}(M) = \oplus M(\nu)/M(\nu-1)$ is a finitely generated $\mathrm{gr}(\mathscr{D})$-module. We let J be the radical of $\mathrm{Ann}_{\mathrm{gr}(\mathscr{D})} \mathrm{gr}(M) \subset \mathrm{gr}(\mathscr{D})$ and set $d(M) = \dim \mathrm{gr}(\mathscr{D})/J$ (Krull dimension). The ideal J, and hence the number $d(M)$, is independent of the choice of good filtration on M. By *Bernstein's theorem*, if $M \neq 0$ is a finitely generated left $\mathscr{D}$-module, we have $n \leq d(M) \leq 2n$. In the case $d(M) = n$ we say that M is *holonomic*. It is known that submodules and quotients of holonomic

$\mathscr{D}$-modules are holonomic, an extension of a holonomic $\mathscr{D}$-module by another holonomic $\mathscr{D}$-module is holonomic, holonomic $\mathscr{D}$-modules are of finite length over $\mathscr{D}$, and holonomic $\mathscr{D}$-modules are cyclic (generated over $\mathscr{D}$ by a single element).

Remark 1.2.2. For any ideal $I \subset R = k[[x_1, \dots, x_n]]$, the local cohomology modules $H^i_I(R)$ have a natural structure of left $\mathscr{D}$-modules [8]; indeed, they are *holonomic* $\mathscr{D}$-modules [8, 2.2(d)], a fact which will repeatedly prove crucial for us.

To any left $\mathrm{Diff}(R,k)$-module M, we can associate its *de Rham complex*. This is a complex of length n, denoted $M \otimes \Omega^\bullet_R$ (or simply $\Omega^\bullet_R$ in the case $M = R$), whose objects are R-modules but whose differentials are merely k-linear. It is defined as follows [7, §1.6]: for $0 \le i \le n$, $M \otimes \Omega^i_R$ is a direct sum of $\binom{n}{i}$ copies of M, indexed by i-tuples $1 \le j_1 < \cdots < j_i \le n$. The summand corresponding to such an i-tuple will be written $M\, dx_{j_1} \wedge \cdots \wedge dx_{j_i}$.

Convention 1.2.3. *The subscript R in $\Omega^\bullet_R$ indicates over which ring the tensor products of objects are being taken. To simplify notation, we will follow this convention when de Rham complexes over different rings are being simultaneously considered.*

The k-linear differentials $d^i : M \otimes \Omega^i_R \to M \otimes \Omega^{i+1}_R$ are defined by

$$d^i(m\, dx_{j_1} \wedge \cdots \wedge dx_{j_i}) = \sum_{s=1}^{n} \partial_s(m)\, dx_s \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_i},$$

with the usual exterior algebra conventions for rearranging the wedge terms, and extended by linearity to the direct sum. The cohomology objects $h^i(M \otimes \Omega^\bullet_R)$, which are k-spaces, are called the *de Rham cohomology spaces* of the left $\mathscr{D}$-module M, and are denoted $H^i_{dR}(M)$. In the case of a holonomic module, van den Essen proved that these spaces are finite-dimensional:

Theorem 1.2.4. *[9, Prop. 2.2] If M is a holonomic left $\mathscr{D}$-module, its de Rham cohomology $H^i_{dR}(M)$ is a finite-dimensional k-space for all i.*

Finally, we recall the well-known fact that the de Rham complex of a $\mathscr{D}$-module is independent of the coordinates $x_1, \dots, x_n$ for R.

Proposition 1.2.5. *If $R = k[[x_1, \dots, x_n]]$ and $\mathscr{D} = \mathrm{Diff}(R,k)$, the de Rham complex of any left $\mathscr{D}$-module M is independent of the choice of a regular system of parameters $x_1, \dots, x_n$ for R.*

Proof. Replacing $x_1,\dots,x_n$ by another regular system of parameters induces a k-algebra automorphism of R and hence a ring automorphism of $\mathscr{D}$. The $\mathscr{D}$-module M is viewed as a $\mathscr{D}$-module with respect to the new coordinates "by transport of structure", that is, via the ring automorphism $\mathscr{D} \to \mathscr{D}$. It therefore suffices to prove the claim in the case where M is the $\mathscr{D}$-module R. The module Ω^1_R and the map $d^0 : R \to \Omega^1_R$, which we have defined above using the chosen regular system of parameters $\{x_i\}$, are in fact intrinsic to R, as is clear from the following characterization [10, Example 12.7]: Ω^1_R is the *universally finite module of differentials* of R [10, Def. 11.1] and d^0 is the corresponding *universally finite derivation*, that is, the universal object among finitely generated R-modules M endowed with a k-derivation $R \to M$. For all i, d^i is simply the map induced by d^0 via the formula $d^i(r\omega) = d^0(r) \wedge \omega$ for $\omega \in \Omega^i_R = \bigwedge^i \Omega^1_R$. It follows that all the exterior powers Ω^i_R and all the differentials d^i are intrinsic to R, that is, the entire de Rham complex $\Omega^\bullet_R$ is independent of the chosen parameters. $\square$

1.3 Preliminaries on spectral sequences

As we will be working with morphisms of spectral sequences, we collect some basic facts and definitions in this section concerning them. References for this material include Weibel [2, Ch. 5] and EGA [11, §11]. We will not need to consider convergence issues for unbounded spectral sequences and hence make no mention of such issues here.

Definition 1.3.1. Let $\mathscr{C}$ be an Abelian category. A (cohomological) *spectral sequence* consists of the following data: a family $\{E^{p,q}_r\}$ of objects of $\mathscr{C}$ (where $p,q \in \mathbb{Z}$ and $r \geq 1$ or ≥ 2; with r fixed and p,q varying, we obtain the *E_r-term* of the spectral sequence), and morphisms (the *differentials*) $d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ for all p,q,r such that $d^{p,q}_r \circ d^{p-r,q+r-1}_r = 0$ and $\ker(d^{p,q}_r)/\operatorname{im}(d^{p-r,q+r-1}_r) \xrightarrow{\sim} E^{p,q}_{r+1}$: a family of such isomorphisms, denoted $\alpha^{p,q}_r$, is part of the data of the spectral sequence.

Let E be a spectral sequence in an Abelian category $\mathscr{C}$, and suppose that for all l and for all r, there are only finitely many nonzero objects $E^{p,q}_r$ with $p+q=l$. Such a spectral sequence is called *bounded*. (For example, this occurs if $E^{p,q}_r = 0$ whenever p or q is negative, in which case E is called a *first-quadrant* spectral sequence.) If E is a bounded spectral sequence, for every pair (p,q), there exists r_0 such that for all $r \geq r_0$, $d_r^{p,q}$ has zero target, $d_r^{p-r,q+r-1}$ has zero source, and so $E_{r+1}^{p,q} \simeq E_r^{p,q}$. We denote this stable object by $E_\infty^{p,q}$. We can now define the *abutment* of such a spectral sequence.

Definition 1.3.2. Let E be a bounded spectral sequence in an Abelian category $\mathscr{C}$. Suppose we are given a family E^m of objects of $\mathscr{C}$, all endowed with a finite decreasing filtration $E^m = E_s^m \supset E_{s+1}^m \supset \cdots \supset E_t^m = 0$, and for all p, an isomorphism $\beta^{p,m-p} : E_\infty^{p,m-p} \xrightarrow{\sim} E_p^m / E_{p+1}^m$. Then we say that the spectral sequence *abuts* or *converges* to $\{E^m\}$ (the *abutment*), and write $E_1^{p,q} \Rightarrow E^m$ or $E_2^{p,q} \Rightarrow E^m$.

For example, if E is a first-quadrant spectral sequence with abutment $\{E^m\}$, every E^m has a filtration of length $m+1$ (we take $s=0$ and $t=m+1$ in the definition above), with $E^m/E_1^m \simeq E_\infty^{0,m}$ and $E_m^m \simeq E_\infty^{m,0}$.

Given two spectral sequences, there is a natural notion of a morphism between them, which consists of morphisms between the objects in the E_r-terms for all r, each of which induces its successor on cohomology. There is also a natural notion of morphisms between bounded spectral sequences with given abutments.

Definition 1.3.3. Let E and E' be two spectral sequences in $\mathscr{C}$ with respective differentials d and d'. A *morphism* $u : E \to E'$ is a family of morphisms $u_r^{p,q} : E_r^{p,q} \to E_r^{'p,q}$ such that the $u_r^{p,q}$ are compatible with the differentials ($d_r^{'p,q} \circ u_r^{p,q} = u_r^{p+r,q-r+1} \circ d_r^{p,q}$ for all p,q,r) and the morphisms $\overline{u}_r^{p,q} : \ker(d_r^{p,q})/\operatorname{im}(d_r^{p-r,q+r-1}) \to \ker(d_r^{'p,q})/\operatorname{im}(d_r^{'p-r,q+r-1})$ induced by $u_r^{p,q}$ commute with the given isomorphisms $\alpha_r^{p,q}$ (that is, $\alpha_r^{'p,q} \circ \overline{u}_r^{p,q} = \overline{u}_{r+1}^{p,q} \circ \alpha_r^{p,q}$) so that, in the appropriate sense, $u_{r+1}^{p,q}$ is the morphism induced by $u_r^{p,q}$. If E and E' are bounded spectral sequences with abutments $\{E^m\}$ and $\{E'^m\}$, a *morphism with abutments* between the spectral sequences is a morphism $u : E \to E'$ as just defined together with a family of morphisms $u^m : E^m \to E'^m$ compatible with the filtrations on E^m and E'^m such that, if we denote by $u_\infty^{p,q}$ the map induced by $u_1^{p,q}$ (or $u_2^{p,q}$) between the stable objects $E_\infty^{p,q}$ and $E_\infty^{'p,q}$, this map must commute with the isomorphisms $\beta^{p,q}$: if we denote by u_p^m the morphism $E_p^m/E_{p+1}^m \to E_p^{'m}/E_{p+1}^{'m}$ induced by u^m, which is required to be filtration-compatible, we must have $\beta'^{p,q} \circ u_\infty^{p,q} = u_p^{p+q} \circ \beta^{p,q}$.

Convention 1.3.4. *For the remainder of this thesis, every spectral sequence will be a bounded spectral sequence with abutment, and every morphism of spectral sequences will be a morphism with abutments. Consequently, we suppress the phrase "with abutment".*

To show that two spectral sequences are *isomorphic*, it suffices to construct a morphism between them which is an isomorphism on the objects of the initial ($r = 1$ or $r = 2$) terms. This result is crucial to our work in both this section and in section 4.2, so we record a version here:

Proposition 1.3.5. *[2, Thm. 5.2.12] Let $\mathscr{C}$ be an Abelian category, and let $u = (u_r^{p,q}, u^n)$ be a morphism between two spectral sequences E, E' in $\mathscr{C}$. If there exists r such that $u_r^{p,q}$ is an isomorphism for all p and q, then $u_s^{p,q}$ is an isomorphism for all p and q and all $s \geq r$, and u^m is an isomorphism for all m. It follows that the abutments of E and E' are isomorphic as filtered objects.*

There is also a notion of a degree-shifted morphism of spectral sequences (see, for example, [12, §4.1]), and a degree-shifted analogue of Proposition 1.3.5, which we will make use of in this thesis. Again, for us, all spectral sequences will be bounded and all morphisms will be morphisms with abutments.

Definition 1.3.6. Let E and E' be two spectral sequences in $\mathscr{C}$ with respective differentials d and d'. If $a, b \in \mathbb{Z}$, a *morphism $u : E \to E'$ with bidegree (a,b)* is a family of morphisms $u_r^{p,q} : E_r^{p,q} \to E_r'^{p+a,q+b}$ such that the $u_r^{p,q}$ are compatible with the differentials ($d_r'^{p+a,q+b} \circ u_r^{p,q} = u_r^{p+r,q-r+1} \circ d_r^{p,q}$ for all p,q,r) and $u_{r+1}^{p,q}$ is induced on cohomology by $u_r^{p,q}$. If E and E' are bounded spectral sequences with abutments $\{E^m\}$ and $\{E'^m\}$, a *morphism with abutments* between the spectral sequences *with bidegree* (a,b) is a morphism $u : E \to E'$ with bidegree (a,b) as just defined together with a family of morphisms $u^m : E^m \to E'^{m+a+b}$ such that $u^m(E_p^m) \subset E_{p+a}'^{m+a+b}$ for all p and satisfying the obvious compatibility conditions analogous to those in the non-degree-shifted definition.

The degree-shifted analogue of Proposition 1.3.5 is proved in exactly the same way, but in the conclusion (that the abutments are isomorphic as filtered objects), it is worth recording precisely which filtrations are being compared and what the corresponding degree shifts are:

Proposition 1.3.7. *Let $\mathscr{C}$ be an Abelian category, and let $u = (u_r^{p,q}, u^n)$ be a morphism of bidegree (a,b) between two spectral sequences E, E' in $\mathscr{C}$. If there exists r such that $u_r^{p,q}$ is an isomorphism for all p and q, then $u_s^{p,q}$ is an isomorphism for all p and q and all $s \geq r$, and u^m is an isomorphism for all m. This has the following consequence for the abutments: for all m, E^m, endowed with the filtration $\{E_p^m\}_{p=0}^{m+1}$ where $E_p^m/E_{p+1}^m \simeq E_\infty^{p,m-p}$, is isomorphic (as a filtered object) to E'^{m+a+b}, endowed with the filtration $\{E'^{m+a+b}_{p+a}\}_{p=0}^{m+1}$, where $E'^{m+a+b}_{p+a}/E'^{m+a+b}_{p+a+1} \simeq E'^{p+a,m+b-p}_\infty$.*

Double complexes are a common source of spectral sequences: the cohomology of the totalization of a double complex can be approximated, and in some cases even computed, by the objects in the early terms of either of two spectral sequences associated to the double complex. To be precise, let $K^{\bullet,\bullet}$ be a double complex in an Abelian category $\mathscr{C}$, which we think of abusively as the "E_0-term" of a spectral sequence, and let $T^\bullet$ be its totalization. Our conventions for double complexes are those of EGA: the horizontal ($d_h^{\bullet,\bullet}$) and vertical ($d_v^{\bullet,\bullet}$) differentials of $K^{\bullet,\bullet}$ commute, we define $T^i = \oplus_{p+q=i} K^{p,q}$, and the differentials of $T^\bullet$ require signs, namely $d(x) = d_h(x) + (-1)^p d_v(x)$ for $x \in K^{p,q}$. The two spectral sequences associated to $K^{\bullet,\bullet}$ [11, §11.3] are the *column-filtered* ("vertical differentials first") spectral sequence, for which $E_1^{p,q} = h_v^q(K^{p,\bullet})$ (and the differentials are those induced on vertical cohomology by the maps $d_h^{p,q}$), and the *row-filtered* ("horizontal differentials first") spectral sequence, for which $E_1^{p,q} = h_h^p(K^{\bullet,q})$ (and the differentials are those induced on horizontal cohomology by the maps $d_v^{p,q}$). Both have $h^{p+q}(T^\bullet)$, the cohomology of the totalization, for their abutment. A morphism $K^{\bullet,\bullet} \to K'^{\bullet,\bullet}$ of double complexes induces morphisms between their column-filtered spectral sequences as well as between their row-filtered spectral sequences [11, p.30].

The spectral sequences of a double complex are useful for computing hyperderived functors of left-exact functors between Abelian categories. Suppose $\mathscr{A}, \mathscr{B}$ are Abelian categories, $\mathscr{A}$ has enough injective objects, and $F : \mathscr{A} \to \mathscr{B}$ is a left-exact additive functor. If $K^\bullet$ is a complex with differential d in $\mathscr{A}$, the (right) hyperderived functors of F evaluated at $K^\bullet$ are defined as follows [11, §11.4]: if $K^\bullet \to I^\bullet$ is a quasi-isomorphism and $I^\bullet$ is a complex of injective objects in $\mathscr{A}$, then $\mathbb{R}^i F(K^\bullet) = h^i(F(I^\bullet))$, and the objects of $\mathscr{B}$ thus obtained are independent of the choice of $I^\bullet$. Such a complex $I^\bullet$ can be produced as the totalization of a *Cartan-Eilenberg resolution* of $K^\bullet$, which is a double complex $J^{\bullet,\bullet}$ with differentials d_h, d_v such that every $J^{p,q}$ is an injective object of $\mathscr{A}$ and, for all p, $J^{p,\bullet}$ (resp. $\ker(d_v^{p,\bullet})$, $\mathrm{im}(d_v^{p,\bullet})$, $h_v^\bullet(J^{p,\bullet})$) is an injective resolution of K^p (resp. $\ker(d^p)$, $\mathrm{im}(d^p)$, $h^p(K^\bullet)$). It follows that $\mathbb{R}^i F(K^\bullet)$ is the cohomology of the totalization of the double complex $F(J^{\bullet,\bullet})$, and so, by the preceding paragraph, we have two spectral sequences whose abutment is this cohomology. For example, the column-filtered spectral sequence begins $E_1^{p,q} = h_v^q(F(J^{p,\bullet}))$ and has abutment $\mathbb{R}^{p+q}F(K^\bullet)$. But since $J^{p,\bullet}$ is an injective resolution of K^p, we see that $h_v^q(F(J^{p,\bullet})) = R^q F(K^p)$, the ordinary qth right derived functor of F applied to K^p; this is the form in which the "first" hyperderived functor spectral sequence is usually given [11, 11.4.3.1].

Now suppose $K^\bullet$, $K'^\bullet$ are complexes in $\mathscr{A}$ with respective Cartan-Eilenberg resolutions $J^{\bullet,\bullet}$, $J'^{\bullet,\bullet}$. A morphism of complexes $f : K^\bullet \to K'^\bullet$ induces a morphism of double complexes $J^{\bullet,\bullet} \to J'^{\bullet,\bullet}$ which is unique up to homotopy [11, p. 33]. This implies that f induces a well-defined morphism between the spectral sequences for the hyperderived functors of F evaluated at $K^\bullet$ and at $K'^\bullet$ [11, p. 30], since two double complex morphisms that are chain homotopic induce the same morphisms on horizontal and vertical cohomology, hence the same spectral sequence morphisms. By taking $K'^\bullet = K^\bullet$ and f to be the identity, we see that the isomorphism class of the spectral sequence for F evaluated at $K^\bullet$ is independent of the Cartan-Eilenberg resolution.

Later in this section, we will need to build a spectral sequence for hyperderived functors using a double complex that is not a Cartan-Eilenberg resolution of the original complex, and for this purpose, the following comparison lemma will be useful.

Lemma 1.3.8. *Let $\mathscr{A}$ be an Abelian category with enough injective objects, $\mathscr{B}$ another Abelian category, and $F : \mathscr{A} \to \mathscr{B}$ a left-exact additive functor. Suppose that $K^\bullet$ is a complex in $\mathscr{A}$,* *concentrated in degrees $p \geq 0$, and that $L^{\bullet,\bullet}$ is a double complex in $\mathscr{A}$ whose objects $L^{p,q}$ are all F-acyclic and such that, for all $p \geq 0$, $L^{p,\bullet}$ is a resolution of K^p. Then the first spectral sequence for the hyperderived functors of F applied to $K^\bullet$, which begins $E_1^{p,q} = R^qF(K^p)$ and has $\mathbb{R}^{p+q}F(K^\bullet)$ for its abutment, is isomorphic to the column-filtered spectral sequence of the double complex $F(L^{\bullet,\bullet})$.*

Proof. By definition, the first spectral sequence is the column-filtered spectral sequence of the double complex $F(J^{\bullet,\bullet})$, where $J^{\bullet,\bullet}$ is a choice of Cartan-Eilenberg resolution of $K^\bullet$ in $\mathscr{A}$. The assertion of the lemma is that we can replace $J^{\bullet,\bullet}$ with the resolution $L^{\bullet,\bullet}$, which is generally not a Cartan-Eilenberg resolution and whose objects may not even be injective.

Our strategy will be to compare both of these double complexes to a third one. Let $\mathscr{C}^+$ denote the category of complexes in $\mathscr{A}$ that are concentrated in degrees $p \geq 0$. Then $\mathscr{C}^+$ is an Abelian category with enough injective objects, and if $I^\bullet \in \mathscr{C}^+$ is injective, then I^p is an injective object of $\mathscr{A}$ for all p [13, Thms. 10.42, 10.43; Remark, p. 652].

We return now to the complex $K^\bullet$. Choose an injective resolution $0 \to K^\bullet \to I^{\bullet,\bullet}$ of $K^\bullet$ in $\mathscr{C}^+$. In particular, $I^{\bullet,\bullet}$ is a double complex of injective objects in $\mathscr{A}$. Now note that the two double complex resolutions $J^{\bullet,\bullet}$ and $L^{\bullet,\bullet}$ can also be regarded as resolutions of $K^\bullet$ in the category $\mathscr{C}^+$. Any resolution in $\mathscr{C}^+$ can be compared with an injective one by [14, Lemma XX.5.2]: there exist morphisms $J^{\bullet,\bullet} \to I^{\bullet,\bullet}$ and $L^{\bullet,\bullet} \to I^{\bullet,\bullet}$ extending the identity on $K^\bullet$ and unique up to homotopy as maps in $\mathscr{C}^+$. These morphisms of double complexes induce morphisms between the column-filtered spectral sequences corresponding to the double complexes after applying the functor F. To finish the proof, by Proposition 1.3.5, it is enough to check that these morphisms of spectral sequences are isomorphisms at the E_1-level. We first consider the morphism $F(J^{\bullet,\bullet}) \to F(I^{\bullet,\bullet})$. For all p, $J^{p,\bullet} \to I^{p,\bullet}$ is a morphism between two injective resolutions of K^p extending the identity on K^p, which induces an isomorphism $h^q(F(J^{p,\bullet})) \xrightarrow{\sim} h^q(F(I^{p,\bullet}))$, both sides being equal to $R^qF(K^p)$ by definition and being the $E_1^{p,q}$-terms of the respective spectral sequences. In the case of the morphism $F(L^{\bullet,\bullet}) \to F(I^{\bullet,\bullet})$, we do not have injective resolutions of K^p (only F-acyclic ones) on the left-hand side, but by [14, Thm. XX.6.2], this is enough: the $L^{p,\bullet} \to I^{p,\bullet}$ also give rise to isomorphisms after applying F and taking cohomology. We conclude that the three column-filtered spectral sequences corresponding to the double complexes $F(J^{\bullet,\bullet})$, $F(I^{\bullet,\bullet})$, and $F(L^{\bullet,\bullet})$ are isomorphic beginning with their E_1-terms, completing the proof. □

We will need one more type of spectral sequence, the *Grothendieck composite-functor spectral sequence*:

Proposition 1.3.9. *[2, Thm. 5.8.3] Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be Abelian categories, and suppose $\mathscr{A}$ and $\mathscr{B}$ have enough injective objects. Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be left-exact additive functors. Suppose that for every injective object I of $\mathscr{A}$, the object $F(I)$ of $\mathscr{B}$ is acyclic for G. Then for every object A of $\mathscr{A}$, there is a spectral sequence which begins $E_2^{p,q} = (R^pG)((R^qF)(A))$ and abuts to $R^{p+q}(G \circ F)(A)$.*

Example 1.3.10. For our purposes, the most important example of a composite-functor spectral sequence is the spectral sequence for *iterated local cohomology*. Let R be a Noetherian ring, and let I and J be ideals of R. If $\mathscr{I}$ is an injective R-module, then $\Gamma_J(\mathscr{I})$ is again injective [3, Lemma III.3.2], hence acyclic for the functor Γ_I. It follows that the left-exact functors $F = \Gamma_J$ and $G = \Gamma_I$ satisfy the conditions of Proposition 1.3.9. Since R is Noetherian, $\Gamma_I \circ \Gamma_J = \Gamma_{I+J}$. For any R-module M, the corresponding spectral sequence for the derived (local cohomology) functors begins $E_2^{p,q} = H_I^p(H_J^q(M))$ and abuts to $H_{I+J}^{p+q}(M)$.

1.4 Preliminaries on Matlis duality

Finally, we will need some facts about injective hulls and Matlis duals. Later in this thesis (chapter 3), we will consider Matlis duality at greater length; for now, we content ourselves with recalling the basic definitions. A standard reference for this theory is [1, §18], and its original statement by Matlis appears in [15]; as an additional source, we recommend [16, App. A]. (All definitions and results in this section can be found in this appendix.)

Definition 1.4.1. Let R be a commutative ring, M an R-module, and $N \subset M$ a submodule. We say that M is an *essential extension* of N if every nonzero R-submodule of M has nonzero intersection with N. (More generally, if $N \hookrightarrow M$ is an injective homomorphism of R-modules, we say that M is an essential extension of N if it is an essential extension of the image of N.)

It is well-known that given any R-module M, there exists an essential extension $M \hookrightarrow E$ where E is an injective module. It is not hard to see that any injective module I containing M must have a copy of E as a direct summand; we may therefore think of such an E as a "smallest injective module containing M", and we call E an *injective hull* of M. The indefinite article is used because, although any two injective hulls of M are isomorphic, the isomorphism between them need not be unique, and so the assignment $M \mapsto E$ is not a functor. We write $E(M)$, or sometimes $E_R(M)$, for an injective hull of M.

Over Noetherian rings, there is a classification theorem for injective modules which makes use of injective hulls:

Proposition 1.4.2. *[16, Thm. A.21] Let R be a Noetherian ring. The* indecomposable *injective R-modules (that is, those which cannot be written as a direct sum of two nonzero submodules) are precisely the injective hulls $E_R(R/\mathfrak{p})$ where $\mathfrak{p} \in \operatorname{Spec}(R)$. If I is an injective R-module, there exists a decomposition*

$$I \simeq \oplus_{\mathfrak{p}\in \operatorname{Spec}(R)} E_R(R/\mathfrak{p})^{\mu_\mathfrak{p}}$$

where the $\mu_\mathfrak{p}$ (possibly infinite cardinals) depend only on I.

If $(R,\mathfrak{m})$ is a *local* Noetherian ring, the indecomposable injective R-module $E = E_R(R/\mathfrak{m})$ (an injective hull of the residue field $k = R/\mathfrak{m}$) plays a special role.

Definition 1.4.3. Let $(R,\mathfrak{m})$ and E be as above. If M is an R-module, its *Matlis dual* is the R-module $D(M) = \operatorname{Hom}_R(M,E)$.

Note that D is a contravariant functor on the category of R-modules, exact since E is injective.

Remark 1.4.4. We have $D(R) = E$ (obviously) and $D(E) = \widehat{R}$, the $\mathfrak{m}$-adic completion of R [16, Thm. A.31]. This last identification is a crucial step in the proof of the Matlis duality theorem (below). In particular, $D(E) = R$ if R is a complete local ring.

The main result of the Matlis duality theory is the following theorem:

Theorem 1.4.5 (Matlis duality)**.** *Let* $(R,\mathfrak{m})$ *be a* complete *local ring, and let* $E = E(R/\mathfrak{m})$ *be an injective hull of* $k = R/\mathfrak{m}$ *as an R-module.*

(a) If M is a finitely generated R-module, $D(M) = \mathrm{Hom}_R(M,E)$ *is an Artinian R-module.*

(b) Conversely, if M is Artinian, $D(M)$ *is finitely generated.*

(c) If M is either finitely generated or Artinian, there is a canonical isomorphism $M \simeq D(D(M))$ *of R-modules.*

Finally, we have a useful lemma concerning the *socle* of E:

Lemma 1.4.6. *[16, Thm. A.20(2)] If* $(R,\mathfrak{m})$ *is a local ring and E is an injective hull of its residue field k, the socle* $\mathrm{Soc}(E) = (0 :_E \mathfrak{m})$ *is a one-dimensional k-space.*

1.5 Thesis preview

The remainder of this thesis is structured as follows.

In chapter 2, we define the Lyubeznik numbers of a local ring containing a field and give some of their basic properties, then review Hartshorne's (local and global) algebraic de Rham homology and cohomology theories in characteristic zero. The main result of this chapter, which we prove using some results from Hartshorne's theory, is that in characteristic zero, the Lyubeznik numbers of the local ring at the vertex of the affine cone over a nonsingular projective variety (under some embedding) are independent of the embedding. This result is known [17] for *all* projective varieties over a field of positive characteristic, and remains open for *singular* varieties in characteristic zero.

In chapter 3, we develop a theory of Matlis duality for $\mathscr{D}$-modules; besides being necessary for our work in chapter 4, this theory may be of independent interest. We begin by giving an alternate definition of the Matlis dual over a local ring R with a coefficient field k. If R is complete, this definition enables us to give a structure of right $\mathscr{D}$-module to the Matlis dual of any left $\mathscr{D}$-module (where $\mathscr{D} = \text{Diff}(R,k)$); if, moreover, R is regular, we may transpose the action to get another left $\mathscr{D}$-module. If we begin with a holonomic left $\mathscr{D}$-module, its de Rham cohomology and that of its Matlis dual are k-dual to each other. After giving the rather involved proof of this last fact, we work out in detail what happens in the case of a local cohomology module and its Matlis dual.

Finally, in chapter 4, we apply the work of the previous chapter to Hartshorne's de Rham homology and cohomology for a complete local ring A with a coefficient field k (of characteristic zero). The definition of these objects in [18] involves a choice of k-algebra surjection $R \to A$ where R is a complete regular local k-algebra, but the objects themselves depend only on A (and are finite-dimensional). We consider the corresponding Hodge-de Rham spectral sequences with these objects as abutments, and show that, beginning with their E_2-terms, the spectral sequences depend only on A and have finite-dimensional objects. We are able to give a proof for the de Rham homology spectral sequence that does not depend on the theory developed earlier in this thesis, but we use the results of chapter 3 to "export" our results for homology to the spectral sequence for cohomology. We conclude by conjecturing that the homology and cohomology spectral sequences are k-dual to each other, and present a preliminary result giving some evidence for this conjecture: the E_2-objects of the two spectral sequences are indeed k-dual to each other.

Chapter 2

Local cohomology invariants of nonsingular projective varieties

2.1 Introduction to Lyubeznik numbers

In 1993, Lyubeznik [8] introduced invariants of local rings with coefficient fields, defined in terms of local cohomology. These invariants have since come to be known as *Lyubeznik numbers* (or sometimes *local cohomology multiplicities*). In this section, we give the definition of these invariants and some of their basic properties, quoting liberally from [8]. Following the appearance of this paper, some more properties of these invariants were worked out in [19] and [20], and a topological interpretation (in the case of ground field $\mathbb{C}$) was given in [21] (see section 2.3 below). Much more information can be found in the survey article [22].

Definition 2.1.1. Let A be a local ring, let R be a regular local ring of dimension n containing a field, and suppose that $\pi : R \to A$ is a surjective ring homomorphism. Write I for $\ker \pi \subset R$, $\mathfrak{m}$ for the maximal ideal of R, and $k = R/\mathfrak{m}$ for the residue field. For all i and j, the (i,j)th *Lyubeznik number* of A is

$$\lambda_{i,j}(A) = \dim_k \operatorname{Ext}^i_R(k, H^{n-j}_I(R)).$$

A priori, it is not even clear that these numbers are finite, much less that they depend only on A. Both of these facts are established in [8]. The numbers $\lambda_{i,j}(A)$ can be thought of as Bass numbers of local cohomology modules. Recall that if M is an R-module and $0 \to M \to \mathscr{I}^\bullet$ is its minimal injective resolution in the category of R-modules, the ith *Bass number* $\mu_i(\mathfrak{m}, M)$ of M with respect to $\mathfrak{m}$ is the number of copies of the injective hull $E_R(R/\mathfrak{m})$ appearing as direct summands of $\mathscr{I}^i$ (by Proposition 1.4.2, this module, being injective, can be decomposed as a direct sum of indecomposable injective modules, which take the form $E_R(R/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec}(R)$). On the one hand, it is well-known that

$$\mu_i(\mathfrak{m}, M) = \dim_k \operatorname{Ext}^i_R(k, M)$$

for all i and any M [1, Thm. 18.7]. On the other hand, since R is regular, the Bass numbers of the local cohomology modules $H^{n-j}_I(R)$ with respect to any prime ideal $\mathfrak{p} \subset R$ are finite ([8, Cor. 3.6(d)]; the case $\operatorname{char}(k) > 0$ appeared first in [23]). This gives the first part of the following proposition:

Proposition 2.1.2. *[8, Thm.-Def. 4.1] Let A, R, and π be as in Definition 2.1.1.*

(a) $\lambda_{i,j}(A)$ is finite for all i and j.

(b) $\lambda_{i,j}(A)$ depends only on A, i, and j; neither on R nor on π.

By [8, Lemma 4.2], the numbers $\lambda_{i,j}(A)$ do not change if we replace A with its completion $\widehat{A}$. Given any local ring A containing a field, its completion is a quotient of a complete regular local ring containing a field by Cohen's structure theorem [1, Thm. 29.4], and so upon replacing A with $\widehat{A}$, we can always find a surjection π as in Definition 2.1.1. It follows that we can define $\lambda_{i,j}(A)$ for any local ring A containing a field.

For any i and j, the iterated local cohomology module $H^i_{\mathfrak{m}}(H^{n-j}_I(R))$ is injective [8, Cor. 3.6(a)] and supported only at $\mathfrak{m}$. Therefore, this module is isomorphic to a direct sum of copies of the injective hull $E = E(R/\mathfrak{m})$ (finitely many, by [8, Cor. 3.6(d)]). From this fact, we can obtain an equivalent definition of $\lambda_{i,j}(A)$:

Proposition 2.1.3. *[24, Lemma 2.2] Let A, R, and π be as in Definition 2.1.1. For all i and j, we have*

$$H^i_{\mathfrak{m}}(H^{n-j}_I(R)) \simeq E^{\lambda_{i,j}(A)}$$

as R-modules.

By Lemma 1.4.6, the dimension of the socle of the right-hand side (and hence of the left-hand side) is precisely $\lambda_{i,j}(A)$.

We record some properties of the numbers $\lambda_{i,j}(A)$:

Lemma 2.1.4. *[8, p. 54] Let A, R, and π be as in Definition 2.1.1, and let $d = \dim(A)$.*

(a) $\lambda_{i,j}(A) = 0$ *if* $j > d$ *or* $i > j$.

(b) $\lambda_{d,d}(A) \neq 0$.

(c) If A is a complete intersection, $\lambda_{d,d}(A) = 1$ *and all other* $\lambda_{i,j}(A) = 0$.

The number $\lambda_{d,d}(A)$ is called the *highest* or *top* Lyubeznik number of A. Zhang [25] has given a characterization of the top Lyubeznik number in terms of the *Hochster-Huneke graph* of A.

Definition 2.1.5. [26, Def. 3.4] Let B be any local ring. The *Hochster-Huneke graph* Γ_B has one vertex for each top-dimensional minimal prime ideal $\mathfrak{p} \subset B$, and two distinct vertices $\mathfrak{p}$ and $\mathfrak{q}$ are joined by an edge if and only if the ideal $\mathfrak{p} + \mathfrak{q}$ has height 1.

Zhang's description of the top Lyubeznik number is the following:

Proposition 2.1.6. *[25, Main Thm.] Let A be a local ring containing a field, and let* $d = \dim(A)$. *The top Lyubeznik number $\lambda_{d,d}(A)$ is equal to the number of connected components of the Hochster-Huneke graph Γ_B, where $B = \widehat{A^{sh}}$ is the completion of the strict Henselization of A.*

2.2 Hartshorne's algebraic de Rham cohomology

In [27], Grothendieck introduced a purely algebraic de Rham cohomology theory for smooth schemes over a field k of characteristic 0. Let X be such a scheme. Since X is smooth, the sheaf Ω_X of Kähler differentials on X is a locally free $\mathscr{O}_X$-module. The *de Rham complex* $\Omega_X^\bullet$ on X is the complex

$$0 \to \mathscr{O}_X \to \Omega_X \to \Omega_X^2 \to \cdots \to \Omega_X^n \to 0$$

where Ω_X^i is the ith exterior power of Ω_X, n is the dimension of X, and the differentials are locally given by the usual exterior derivative formulas. Though the objects of this complex are coherent $\mathscr{O}_X$-modules, the differentials are merely k-linear.

Definition 2.2.1. [27] Let X be a smooth scheme over a field k of characteristic zero, and let $\Omega_X^\bullet$ be its de Rham complex. The (algebraic) *de Rham cohomology* $H_{dR}^*(X)$ of X is the hypercohomology $\mathbb{H}^*(X, \Omega_X^\bullet)$.

Recall that this hypercohomology is computed by choosing an injective resolution $\Omega_X^\bullet \to \mathscr{I}^\bullet$ in the category of sheaves of Abelian groups (or k-spaces) on X (a quasi-isomorphism where each $\mathscr{I}^i$ is an injective object in the category of sheaves) and setting $\mathbb{H}^i(X, \Omega_X^\bullet) = h^i(\Gamma(X, \mathscr{I}^\bullet))$ for all i. That is, we compute the right hyperderived functors of the global section functor Γ evaluated at $\Omega_X^\bullet$. If $Y \subset X$ is a closed subset, by replacing the functor Γ with Γ_Y, we obtain the *local* hypercohomology $\mathbb{H}_Y^*(X, \Omega_X^\bullet)$.

By the general theory of section 1.3, there is an associated *Hodge-de Rham* spectral sequence, which begins $E_1^{p,q} = H^q(X, \Omega_X^p)$ and has abutment $H_{dR}^{p+q}(X)$. Note that if X is affine, the coherent sheaves Ω_X^p on X have no higher cohomology, and so the Hodge-de Rham spectral sequence degenerates at E_1. It is easy to see from this that $H_{dR}^i(X) = h^i(\Gamma(X, \Omega_X^\bullet))$ for all i.

Remarkably, this purely algebraic theory recovers topological information: in the case $k = \mathbb{C}$, algebraic de Rham cohomology coincides with Betti (singular) cohomology. This is *Grothendieck's comparison theorem*:

Theorem 2.2.2. *[27, Thm. 1] Let X be a smooth scheme over $\mathbb{C}$, and let X^{an} be its associated* *complex-analytic space. Then*

$$H^i_{dR}(X) \simeq H^i(X;\mathbb{C})$$

as $\mathbb{C}$-spaces, for all i, where the right-hand side is singular cohomology with $\mathbb{C}$-coefficients.

This comparison theorem fails if X is not smooth, essentially because in this case the sheaves Ω^p_X are not locally free. In [18], Hartshorne constructed a more general theory which coincides with Grothendieck's in the smooth case and computes the Betti cohomology in the case where the ground field is $\mathbb{C}$. Hartshorne defines both de Rham homology and de Rham cohomology theories, both global (for Y a finite-type scheme over a field of characteristic zero) and local (for Y the spectrum of a complete local ring containing a field of characteristic zero).

Before giving the definition of Hartshorne's de Rham homology and cohomology, we recall the notion of the *formal completion* of a scheme along a closed subscheme:

Definition 2.2.3. If X is any Noetherian scheme and $Y \subset X$ is a closed subscheme defined by an ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$, the *formal completion* $\widehat{X}$ of X along Y is the locally ringed space $\widehat{X} = Y$ with the sheaf of rings $\mathscr{O}_{\widehat{X}} = \varprojlim \mathscr{O}_X/\mathscr{I}^l$ (note that each $\mathscr{O}_X/\mathscr{I}^l$ is supported on Y). If $\mathscr{F}$ is any coherent $\mathscr{O}_X$-module, its *formal completion* (an $\mathscr{O}_{\widehat{X}}$-module) is $\varprojlim \mathscr{F}/\mathscr{I}^l\mathscr{F}$.

By the Leibniz rule, the differentials in the complex $\Omega^\bullet_X$ are $\mathscr{I}$-adically continuous (for any sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$), and hence pass to $\mathscr{I}$-adic completions. We can define in this way the *completed* de Rham complex $\widehat{\Omega}^\bullet_X$.

Definition 2.2.4. [18, Ch. II] Let Y be a scheme of finite type over a field k of characteristic zero, and let $Y \hookrightarrow X$ be a closed immersion where X is a smooth scheme over k. Let $\Omega^\bullet_X$ be the de Rham complex on X, and let $\widehat{\Omega}^\bullet_X$ be its formal completion along Y. Let $n = \dim(X)$.

(a) The *de Rham homology* of Y is defined by $H_i^{dR}(Y) = \mathbb{H}^{2n-i}_Y(X, \Omega^\bullet_X)$ for all i.

(b) The *de Rham cohomology* of Y is defined by $H^i_{dR}(Y) = \mathbb{H}^i(\widehat{X}, \widehat{\Omega}^\bullet_X)$ for all i.

A priori, the definitions of $H^i_{dR}(Y)$ and $H_i^{dR}(Y)$ depend on the choice of closed immersion into a smooth scheme X. In fact, these k-spaces depend only on Y, and they satisfy many other desirable properties, some of which we list below:

Proposition 2.2.5. *(a) The k-spaces $H^i_{dR}(Y)$ and $H^{dR}_i(Y)$ depend only on Y, not on the choice of X nor on the embedding $Y \hookrightarrow X$. [18, Thm. II.1.4, Thm. II.3.2]*

(b) Both $H^i_{dR}(Y)$ and $H^{dR}_i(Y)$ are finite-dimensional k-spaces for all i. [18, Thm. II.6.1]

(c) If Y is smooth over k and has dimension n, then $H^{dR}_i(Y) \simeq H^{2n-i}_{dR}(Y)$ for all i. [18, Prop. II.3.4]

(d) If Y is proper over k, then $H^i_{dR}(Y) \simeq (H^{dR}_i(Y))^*$ for all i, where the asterisk denotes k-dual. [18, Thm. II.5.1]

Since both $H^i_{dR}(Y)$ and $H^{dR}_i(Y)$ are defined in terms of hypercohomology (or local hypercohomology), the general theory of section 1.3 furnishes us with associated Hodge-de Rham spectral sequences. The Hodge-de Rham spectral sequence for homology begins $E_1^{n-p,n-q} = H_Y^{n-q}(X, \Omega_X^{n-p})$ and abuts to $H^{dR}_{p+q}(Y)$, and the Hodge-de Rham spectral sequence for cohomology begins $\tilde{E}_1^{p,q} = H^q(\widehat{X}, \widehat{\Omega}^p_X)$ and abuts to $H^{p+q}_{dR}(Y)$.

The statement of Grothendieck's comparison theorem holds for finite-type schemes over $\mathbb{C}$ with this definition of de Rham cohomology:

Theorem 2.2.6. *[18, Thm. IV.1.1] Let Y be a scheme of finite type over $\mathbb{C}$, and let Y^{an} be its associated complex-analytic space. Then*

$$H^i_{dR}(Y) \simeq H^i(Y; \mathbb{C})$$

as $\mathbb{C}$-spaces, for all i, where the right-hand side is singular cohomology with $\mathbb{C}$-coefficients.

There is an analogous theory for complete local rings. Let A be a complete local ring with coefficient field k of characteristic zero. By Cohen's structure theorem, there exists a surjection of k-algebras $\pi : R \to A$ where R is a complete regular local ring containing k, which must take the form $R = k[[x_1, \dots, x_n]]$ for some n. Let $I \subset R$ be the kernel of this surjection. We have a corresponding closed immersion $Y \hookrightarrow X$ where $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(R)$. Let $\Omega^\bullet_X$ be the *continuous* de Rham complex on X. Here, the sheaf Ω^1_X is free of rank n with basis $dx_1, \dots, dx_n$, and the other sheaves in the complex are its corresponding exterior powers. (In fact, the complex $\Omega^\bullet_X$ is the sheafified version of the de Rham complex $\Omega^\bullet_R$ of the left $\mathrm{Diff}(R,k)$-module R as defined in section 1.2.)

Definition 2.2.7. [18, §III.1] Let A be a complete local ring with coefficient field k of characteristic zero. Let $\pi : R \to A$ be a surjection of k-algebras where $R = k[[x_1,\dots,x_n]]$ for some n, and let $Y \hookrightarrow X$ (where $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(R)$) be the corresponding closed immersion. Let $\Omega^\bullet_X$ be the *continuous* de Rham complex on X, and $\widehat{\Omega}^\bullet_X$ its formal completion along Y. Finally, let $P \in Y$ be the closed point.

(a) The *de Rham homology* of Y is defined by $H_i^{dR}(Y) = \mathbb{H}_Y^{2n-i}(X, \Omega^\bullet_X)$ for all i.

(b) The *(local) de Rham cohomology* of Y is defined by $H^i_{P,dR}(Y) = \mathbb{H}^i_P(\widehat{X}, \widehat{\Omega}^\bullet_X)$ for all i.

As in the global case, we have Hodge-de Rham spectral sequences for homology and cohomology. We also have the following analogues of the various facts stated in Proposition 2.2.5:

Proposition 2.2.8. *(a) The k-spaces $H^i_{P,dR}(Y)$ and $H_i^{dR}(Y)$ depend only on Y, not on the choice of X nor on the embedding $Y \hookrightarrow X$. [18, Prop. III.1.1]*

(b) Both $H^i_{P,dR}(Y)$ and $H_i^{dR}(Y)$ are finite-dimensional k-spaces for all i. [18, Thm. III.2.1]

(c) We have $H^i_{P,dR}(Y) \simeq (H_i^{dR}(Y))^*$ for all i, where the asterisk denotes k-dual. [18, Thm. III.2.3]

Parts (a) and (b) of the above proposition remain true if P is replaced with any closed subset of Y, including Y itself. It is part (c), which requires supports in P, that shows why we are particularly interested in the case of de Rham cohomology supported at P. Note also that part (c) holds unconditionally in the local case, in contrast to its global analogue.

If Y is a scheme of finite type over k and $P \in Y$ is a closed point, there are two possible definitions for the de Rham cohomology $H^i_{P,dR}(Y)$ of Y supported at P. The first possible definition is to replace Γ with Γ_P in Definition 2.2.4, that is, to take $H^i_{P,dR}(Y) = \mathbb{H}^i_P(\widehat{X}, \widehat{\Omega}^\bullet_X)$. Alternatively, we can consider the complete local ring $\widehat{\mathscr{O}_{X,P}}$ and compute the de Rham cohomology of its spectrum as in Definition 2.2.7. In fact, there is no ambiguity, as these two definitions coincide:

Theorem 2.2.9 (The strong excision theorem)**.** *[18, Prop. III.3.1] Let Y be a scheme of finite type over k, and let P be a closed point of Y. Then we have*

$$\mathbb{H}^i_P(\widehat{X}, \widehat{\Omega}^\bullet_X) \simeq H^i_{P,dR}(\operatorname{Spec} \widehat{\mathscr{O}_{X,P}})$$

as k-spaces for all i.

One important case of cohomology supported at a closed point P is the case where P is the vertex of the affine cone $C = C(V)$ over a projective variety V. Recall that if V is a closed subvariety of some projective space $\mathbb{P}^n_k$ over k, the *affine cone* over V is the closed affine subvariety of $\mathbb{A}^{n+1}_k$ cut out by the same (homogeneous) polynomial equations that define V. We have the following exact sequences relating the (global) de Rham cohomology of V and the (local) de Rham cohomology of C (which, by the strong excision theorem, we can think of as the de Rham cohomology of Spec $\widehat{\mathscr{O}_{C,P}}$):

Proposition 2.2.10. *[18, Prop. III.3.2] Let $V \subset \mathbb{P}^n_k$ be a projective variety, $C \subset \mathbb{A}^{n+1}_k$ the affine cone over V, and $P \in C$ the vertex. Then $H^0_P(C) = 0$ and we have the following two exact sequences, where $H^i_{P,dR}$ and H^i_{dR} denote local algebraic de Rham cohomology and global algebraic de Rham cohomology, respectively:*

$$0 \to k \to H^0_{dR}(V) \to H^1_{P,dR}(C) \to 0$$

and

$$0 \to H^1_{dR}(V) \to H^2_{P,dR}(C) \to H^0_{dR}(V) \to H^2_{dR}(V) \to H^3_{P,dR}(C) \to H^1_{dR}(V) \to \cdots$$

Here the maps $H^i_{dR}(V) \to H^{i+2}_{dR}(V)$ for $i \geq 0$ are given by cup product with the class of a hyperplane section $\zeta \in H^2_{dR}(V)$.

If V is nonsingular and k is algebraically closed, we can obtain more information about the cup product maps of Proposition 2.2.10:

Theorem 2.2.11 (Hard Lefschetz theorem for algebraic de Rham cohomology)**.** *If V is a nonsingular projective variety of dimension r over an algebraically closed field k of characteristic* *zero, then for each i with $0 \leq i \leq r$, the map $H^{r-i}_{dR}(V) \to H^{r+i}_{dR}(V)$ defined by i-fold cup product with the hyperplane section ζ is an isomorphism.*

Proof. The classical hard Lefschetz theorem [28, p. 122] is the analogue (with $k = \mathbb{C}$) of the preceding assertion with the de Rham cohomology of V replaced by the singular cohomology $H^j(V^{an};\mathbb{C})$ of the associated complex-analytic space. By Theorem 2.2.6, we have, for all j, $H^j_{dR}(V) \simeq H^j(V^{an};\mathbb{C})$ (this only requires that V be a scheme of finite type over $\mathbb{C}$), and this isomorphism carries the de Rham hyperplane class to the Betti hyperplane class, so that the result is true when the base field is $\mathbb{C}$. The statement for an arbitrary algebraically closed base field k with $\mathrm{char}(k) = 0$ follows from the statement for $k = \mathbb{C}$ by the "Lefschetz principle" as follows. Take a finite set of generators for the homogeneous defining ideal of V, in which only finitely many coefficients from k appear, and consider the field k' obtained by adjoining this finite set of coefficients to $\mathbb{Q}$. V may thus be defined over the field k': if V' is the variety defined over k' by the same homogeneous polynomials as V, then $V = V' \times_{k'} k$, and V' is also nonsingular. Since $\mathbb{Q} \subset k' \subset k$ and k is algebraically closed, we may embed k' in $\mathbb{C}$. As V is nonsingular, $H^j_{dR}(V)$ is simply the hypercohomology of the de Rham complex of V, which is seen immediately to commute with extensions of the scalar field because formation of the module of Kähler differentials commutes with such extensions; this reduces the hard Lefschetz theorem for V over k' to the hard Lefschetz theorem for V over $\mathbb{C}$, which we have already verified. $\square$

Remark 2.2.12. An immediate consequence of Theorem 2.2.11, which we will use later in this thesis, is that for any $j \geq 0$, the map $H^j_{dR}(V) \to H^{j+2}_{dR}(V)$ defined by cup product with ζ is injective, since there is an isomorphism which factors through this map.

Finally, we state a result of Ogus relating (local) de Rham cohomology and local cohomology on a formal scheme:

Theorem 2.2.13. *[29, Thm. 2.3] Let k be a field of characteristic zero, let $R = k[[x_0,\dots,x_n]]$, and let Y be a closed subset of $X = \mathrm{Spec}(R)$, defined by an ideal I. Let $\hat{X}$ be the formal completion of X along Y, and let P be the closed point. Assume s is an integer such that* $\mathrm{Supp}(H^i_I(R)) \subset$

$\{P\}$ for all $i > n+1-s$. Then there are natural maps $R \otimes_k H^j_{P,dR}(Y) \to H^j_P(\hat{X}, \mathscr{O}_{\hat{X}})$ which are isomorphisms for $j < s$ and injective for $j = s$.

2.3 Lyubeznik numbers for nonsingular projective varieties

In section 2.1, Lyubeznik numbers are defined for any local ring A containing a field. We will be particularly interested in the case where A is the local ring at the vertex of the affine cone over a projective variety (see the discussion preceding Proposition 2.2.10). Let V be a projective variety of dimension r over a field k of characteristic zero. Under an embedding $V \hookrightarrow \mathbb{P}^n_k$, we can write $V = \mathrm{Proj}(k[x_0, \ldots, x_n]/I)$ where I is a homogeneous defining ideal for V. Let $\mathfrak{m} = (x_0, \ldots, x_n)$ be the homogeneous maximal ideal of $k[x_0, \ldots, x_n]$, so that $I \subset \mathfrak{m}$. Then $A = (k[x_0, \ldots, x_n]/I)_{\mathfrak{m}}$ is the local ring at the vertex of the affine cone over V.

In [30, p. 133], Lyubeznik asked whether $\lambda_{i,j}(A)$ depend only on V, i, and j, and not on the embedding $V \hookrightarrow \mathbb{P}^n_k$ (or, for that matter, on n). Zhang settled this question in the affirmative in the case of the "top" Lyubeznik number $\lambda_{r+1,r+1}(A)$, in any characteristic, in [25]; he went on to give an affirmative answer for all $\lambda_{i,j}(A)$ in the $\mathrm{char}(k) = p > 0$ case in [17]. In [17], several preliminary results are established in a characteristic-free setting, but the main line of argument makes crucial use of the Frobenius morphism. This left the characteristic-zero case open for all but the "top" Lyubeznik number.

In this section, we determine completely the Lyubeznik numbers $\lambda_{i,j}(A)$, in the case in which V is a *nonsingular* variety, in terms of quantities known already to be embedding-independent. In this case, the vertex is an isolated singularity of the affine cone. If the ground field is $\mathbb{C}$, García López and Sabbah have given a topological interpretation of the Lyubeznik numbers of an isolated singularity:

Theorem 2.3.1. *[21, Thm.] Let V be a scheme of finite type over $\mathbb{C}$ with an isolated singularity at the point $P \in V$. Let $A = \mathscr{O}_{V,P}$ be the local ring at P, and let $d = \dim(A)$.*

(a) If $d = 1$, then $\lambda_{1,1}(A) = 1$ and all other $\lambda_{i,j}(A)$ vanish.

(b) If $d \geq 2$, then $\lambda_{0,j}(A) = \dim_{\mathbb{C}} H_P^j(V;\mathbb{C})$ if $1 \leq j \leq d-1$, $\lambda_{i,d}(A) = \dim_{\mathbb{C}} H_P^{i+d}(V;\mathbb{C})$ if $2 \leq i \leq d$, and all other $\lambda_{i,j}(A)$ vanish.

Here $H_P^j(V;\mathbb{C})$ denotes singular cohomology supported at P and with $\mathbb{C}$-coefficients.

We will use algebraic de Rham cohomology instead of singular cohomology, thereby removing the need to take $\mathbb{C}$ for the ground field.

We fix some further notation. V, I, and A remain as above, except that now and for the remainder of this section we will assume V is nonsingular. Write $R = (k[x_0,\dots,x_n])_{\mathfrak{m}}$, a regular local ring, so that $A = R/I$; it is this R which will intervene in the definition of the $\lambda_{i,j}(A)$. The following quantities will appear repeatedly: $\dim(R) = \mathrm{ht}(\mathfrak{m}) = n+1$, $\mathrm{ht}(I) = \mathrm{codim}(V,\mathbb{P}_k^n) = n-r$, and $\dim(A) = r+1$. In particular, since $\dim(R) = n+1$, our definition of the Lyubeznik numbers of A reads

$$\lambda_{i,j}(A) = \dim_k(\mathrm{Ext}_R^i(k, H_I^{n+1-j}(R))).$$

We remark that, as explained in [17, §8], it is harmless to assume that k is algebraically closed, since Lyubeznik numbers are unaltered under extension of the base field. In the algebraically closed case, we have the following embedding-independent description of $\lambda_{r+1,r+1}$, which is simply a translation into algebro-geometric language of Zhang's result (Proposition 2.1.6):

Proposition 2.3.2. *[25, Theorem 2.7] Let $V_1,\dots,V_s$ be the r-dimensional irreducible components of V, and let Γ_V be the graph on the vertices $V_1,\dots,V_s$ in which V_i and V_j are joined by an edge if and only if $\dim(V_i \cap V_j) = r-1$. Then $\lambda_{r+1,r+1}(A)$ equals the number of connected components of Γ_V.*

We know by Lemma 2.1.4 that $\lambda_{i,j}(A) = 0$ if either i or j is greater than $r+1$, Proposition 2.3.2 deals with the case $i = j = r+1$, and Theorem 2.3.3 below deals with the remaining cases. This furnishes a complete description of all $\lambda_{i,j}(A)$.

Theorem 2.3.3. *Let V be a nonsingular projective variety over a field k of characteristic zero (we may assume k is algebraically closed) and define the Lyubeznik numbers $\lambda_{i,j}$ of the local* *ring at the vertex of the affine cone over V with respect to some choice of embedding as above. Write $\beta_j = \dim_k(H^j_{dR}(V))$, where H_{dR} denotes algebraic de Rham cohomology.*

1. $\lambda_{i,j} = 0$ *if* $i > 0$ *and* $j < r+1$*;*

2. $\lambda_{0,0} = 0, \lambda_{0,1} = \beta_0 - 1$*;*

3. $\lambda_{0,2} = \beta_1, \lambda_{0,j} = \beta_{j-1} - \beta_{j-3}$ *for* $j = 3, \ldots, r$*;*

4. $\lambda_{0,r+1} = \lambda_{1,r+1} = 0$ *[21, Thm. (c)];*

5. $\lambda_{l,r+1} = \lambda_{0,r+2-l}$ *(determined above) for* $l = 2, \ldots, r$ *[21, Remark 1].*

Since algebraic de Rham cohomology is intrinsic to V (Proposition 2.2.5(b)), the above list, together with Proposition 2.3.2, immediately implies the following:

Corollary 2.3.4. *If V is a* nonsingular *projective variety over a field k of characteristic zero and $\lambda_{i,j}(A)$ is calculated as above, then for all i and j, $\lambda_{i,j}(A)$ depends only on V, i, and j, not on n nor on the embedding $V \hookrightarrow \mathbb{P}^n_k$.*

The rest of this section consists in a proof of the first three parts of Theorem 2.3.3. Part (1) will follow from known results on the local cohomology $H^{n+1-j}_I(R)$ and from the definition of $\lambda_{i,j}$. To establish parts (2) and (3), we will first relate local cohomology supported at I to local cohomology of the formal spectrum of the I-adic completion of R, then we will use a result of Ogus (Theorem 2.2.13) to relate this formal local cohomology to local de Rham cohomology. Finally, we will appeal to the exact sequence of Proposition 2.2.10 connecting local de Rham cohomology at the vertex of the affine cone over V to the de Rham cohomology of V itself. Parts (4) and (5) appear already in [21] and thus are not proven here; see also [31] for an alternative proof of parts (4) and (5) using the Grothendieck composite-functor spectral sequence.

We begin with a lemma on the support of certain local cohomology modules:

Lemma 2.3.5. *Let V, R, A, and I be as above (V is nonsingular). Then* $\mathrm{Supp}(H^i_I(R)) \subset \{\mathfrak{m}\}$ *whenever* $i \neq \mathrm{ht}(I) = n - r$*.*

Proof. Since V is nonsingular, the affine cone over V has an isolated singularity at its vertex, and so all its other local rings are regular local rings: for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$ of the coordinate ring $k[x_0,\dots,x_n]/I$, the localization $(k[x_0,\dots,x_n]/I)_{\mathfrak{p}} = (k[x_0,\dots,x_n])_{\mathfrak{p}}/(I \cdot (k[x_0,\dots,x_n])_{\mathfrak{p}}) \simeq R_{\mathfrak{p}}/I_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$ is a regular local ring, where we write $I_{\mathfrak{p}} = IR_{\mathfrak{p}}$. We will show that for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$ of R, $\mathfrak{p}$ does not belong to the support of $H^i_I(R)$ for any $i \neq \operatorname{ht}(I)$, that is, the localization $(H^i_I(R))_{\mathfrak{p}}$ is zero. This is clear if $I \not\subset \mathfrak{p}$, so we assume that $I \subset \mathfrak{p}$, in which case $\operatorname{ht}(I) = \operatorname{ht}(I_{\mathfrak{p}})$. By the flat base change principle for local cohomology (Proposition 1.1.4(f)), we have $(H^i_I(R))_{\mathfrak{p}} \simeq H^i_{I_{\mathfrak{p}}}(R_{\mathfrak{p}})$. Since both $R_{\mathfrak{p}}$ and its quotient $R_{\mathfrak{p}}/I_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$ are regular local rings, we conclude [32, Proposition 2.2.4] that $I_{\mathfrak{p}}$ is generated by part of a regular system of parameters of $R_{\mathfrak{p}}$, which must have $\operatorname{ht}(I_{\mathfrak{p}}) = \operatorname{ht}(I) = n-r$ elements. But if an ideal is generated by a regular sequence of length $n-r$, local cohomology supported at this ideal cannot be nonzero in any degree other than $n-r$. $\square$

Corollary 2.3.6. *If $i > \operatorname{ht}(I)$, $H^i_I(R)$ is an injective R-module.*

Proof. By [8, Theorem 3.4(b)], $\operatorname{injdim}(H^i_I(R)) \leq \dim \operatorname{Supp}(H^i_I(R))$, and the right-hand side is zero if $\operatorname{Supp}(H^i_I(R)) \subset \{\mathfrak{m}\}$. $\square$

Proof of Theorem 2.3.3(1). Suppose $j < r+1$. Then $n+1-j > n-r = \operatorname{ht}(I)$, so by the preceding corollary, $H^{n+1-j}_I(R)$ is an injective R-module. This implies that $\operatorname{Ext}^i_R(k, H^{n+1-j}_I(R))$ vanishes for all $i > 0$, so that if $i > 0$ and $j < r+1$, the dimension of this Ext module (which, by definition, is $\lambda_{i,j}(A)$) is zero, proving part (1). $\square$

Proof of Theorem 2.3.3(2,3). We compute $\lambda_{0,j}(A)$ for $0 \leq j \leq r$. By definition, this is

$$\dim_k \operatorname{Ext}^0_R(k, H^{n+1-j}_I(R)) = \dim_k \operatorname{Hom}_R(k, H^{n+1-j}_I(R)),$$

the dimension of the socle of $H^{n+1-j}_I(R)$. As discussed in the previous section, for $j < r+1$, $H^{n+1-j}_I(R)$ is an injective R-module supported only at $\mathfrak{m}$. By Proposition 1.4.2, $H^{n+1-j}_I(R)$ is thus isomorphic to a direct sum of copies of $E = E(R/\mathfrak{m})$, a chosen injective hull of the residue field $k = R/\mathfrak{m}$; by [8, Theorem 3.4(d)], the number of copies is finite. Therefore we can write

$H_I^{n+1-j}(R) \simeq E^{t_j}$ for some non-negative integer t_j. Since the socle of E is one-dimensional by Lemma 1.4.6, the socle of $H_I^{n+1-j}(R) \simeq E^{t_j}$ is t_j-dimensional, from which it follows that $\lambda_{0,j}(A) = t_j$ (for $0 \leq j \leq r$). We therefore turn our attention to the determination of t_j.

We may complete R at the maximal ideal $\mathfrak{m}$ without changing the $\lambda_{i,j}(A)$ [8, Lemma 4.2], so we can, and will, assume that R is the complete regular local ring $k[[x_0,\dots,x_n]]$. Let D denote the Matlis dual functor (Definition 1.4.3) from the category of R-modules to itself, so that $D(M) = \mathrm{Hom}_R(M,E)$ for an R-module M, where $E = E(R/\mathfrak{m})$ is the injective hull mentioned above. We will compute $D(H_I^{n+1-j}(R))$ in two different ways and equate the two answers. On the one hand, D is an exact functor (since E is injective) and $D(E) \simeq R$ by Remark 1.4.4, so that

$$D(H_I^{n+1-j}(R)) \simeq D(E^{t_j}) \simeq (D(E))^{t_j} \simeq R^{t_j}$$

for each j with $0 \leq j \leq r$. On the other hand, by [29, Proposition 2.2.3], we have isomorphisms

$$D(H_I^{n+1-j}(R)) \simeq H_P^j(\hat{X}, \mathscr{O}_{\hat{X}})$$

for all j. Here $X = \mathrm{Spec}(R)$, $Y \subset X$ is the closed subscheme defined by I (that is, the spectrum of $A = R/I$), $P \in Y$ is the closed point, $\hat{X}$ is the formal completion of X along Y and $\mathscr{O}_{\hat{X}}$ is the structure sheaf of $\hat{X}$. (See section 3.4 below for more discussion of this isomorphism.)

Equating the results of our two calculations of $D(H_I^{n+1-j}(R))$, we see that $H_P^j(\hat{X}, \mathscr{O}_{\hat{X}}) \simeq R^{t_j}$. It therefore suffices to calculate the R-rank of $H_P^j(\hat{X}, \mathscr{O}_{\hat{X}})$, for which we use Theorem 2.2.13. By Lemma 2.3.5, $H_I^i(R)$ is supported only at P for $i > n-r$, so we can take $s = r+1$ in the statement of Theorem 2.2.13. Consequently, for $j < r+1$, we have isomorphisms $R \otimes_k H_{P,dR}^j(Y) \simeq H_P^j(\hat{X}, \mathscr{O}_{\hat{X}})$, where the right-hand side is isomorphic to R^{t_j}. Furthermore, by Proposition 2.2.8(b), the local de Rham cohomology $H_{P,dR}^j(Y)$ is a finite-dimensional k-vector space for each j. This, together with the isomorphism $R \otimes_k H_{P,dR}^j(Y) \simeq R^{t_j}$, implies that $\lambda_{0,j}(A) = t_j = \dim_k H_{P,dR}^j(Y)$, which means we have reduced ourselves further to the calculation of the dimension of this local de Rham cohomology space. Ogus indicates a way to compute this dimension in [29, Remark, p. 354]. For the convenience of the reader, we give the full details, showing how the explicit formulas of Theorem 2.3.3(2,3) are obtained.

We use the exact sequences of Proposition 2.2.10, which take the form

$$0 \to k \to H^0_{dR}(V) \to H^1_{P,dR}(Y) \to 0$$

and

$$0 \to H^1_{dR}(V) \to H^2_{P,dR}(Y) \to H^0_{dR}(V) \to H^2_{dR}(V) \to H^3_{P,dR}(Y) \to H^1_{dR}(V) \to \cdots;$$

recall that by Theorem 2.2.9, the de Rham cohomology of the affine cone C supported at the vertex P and the local de Rham cohomology $H^j_{P,dR}(Y)$ of the complete local ring A coincide.

Let β_i denote the dimension of the k-vector space $H^i_{dR}(V)$ (finite by Proposition 2.2.5(2)). We have $\lambda_{0,0}(A) = 0$ by Proposition 2.2.10 and $\lambda_{0,1}(A) = \beta_0 - 1$ from the short exact sequence above, proving Theorem 2.3.3(2).

Since V is nonsingular and k may be assumed algebraically closed, the hypotheses of Theorem 2.2.11 are satisfied. It follows by Remark 2.2.12 that if $j < r$, the map $H^j_{dR}(V) \to H^{j+2}_{dR}(V)$ defined by cup product with ζ, which occurs in the long exact sequence above, is injective. That long exact sequence hence splits into short exact sequences:

$$0 \to H^1_{dR}(V) \to H^2_{P,dR}(Y) \to 0$$

and, for all $j \geq 3$,

$$0 \to H^{j-3}_{dR}(V) \to H^{j-1}_{dR}(V) \to H^j_{P,dR}(Y) \to 0.$$

We see at once from these exact sequences of finite-dimensional k-vector spaces that $\lambda_{0,2}(A) = \dim_k H^2_{P,dR}(Y) = \beta_1$ and, for $j \geq 3$, $\lambda_{0,j}(A) = \dim_k H^j_{P,dR}(Y) = \beta_{j-1} - \beta_{j-3}$, proving part (3). $\square$

Chapter 3

Matlis duality for $\mathscr{D}$-modules

The main goal of this chapter is the proof of the following theorem (its two assertions are proved separately below as Proposition 3.2.13 and Theorem 3.3.1):

Theorem 3.0.7. *Let k be a field of characteristic zero, let $R = k[[x_1, \dots x_n]]$ be a formal power series ring over k, and let $\mathscr{D} = \mathrm{Diff}(R,k)$ be the ring of k-linear differential operators on R. If M is a left $\mathscr{D}$-module, the Matlis dual $D(M)$ of M with respect to R can also be given a natural structure of left $\mathscr{D}$-module. We write $H^*_{dR}(M)$ for the de Rham cohomology of a left $\mathscr{D}$-module. If M is a* holonomic *left $\mathscr{D}$-module, then for every i, we have an isomorphism of k-spaces*

$$(H^i_{dR}(M))^* \simeq H^{n-i}_{dR}(D(M))$$

where the asterisk denotes k-linear dual.

If M is holonomic, its de Rham cohomology spaces are known to be finite-dimensional (see Theorem 1.2.4), and so it follows from Theorem 3.0.7 that $D(M)$ also has finite-dimensional de Rham cohomology. However, $D(M)$ is not, in general, a holonomic $\mathscr{D}$-module, so this is not clear *a priori*. Indeed, Hellus has shown [33, Cor. 2.6] that Matlis duals of local cohomology modules (which are holonomic $\mathscr{D}$-modules) have, in general, infinitely many associated primes, implying that they need not be holonomic $\mathscr{D}$-modules (which always have finitely many associated primes [8, Cor. 3.6(c)]).

Much of our theory of Matlis duality for $\mathscr{D}$-modules can be developed in more generality. We first give an equivalent definition in section 3.1 of the Matlis dual (over any local ring with a coefficient field k) in terms of k-linear maps to k. This definition allows us to give a structure of right $\mathrm{Diff}(R,k)$-module to the Matlis dual of a left $\mathrm{Diff}(R,k)$-module whenever R is a *complete* local ring with coefficient field k, which we do in section 3.2. Let $\mathscr{D} = \mathrm{Diff}(R,k)$. In the regular case, there is a transposition operation, by means of which we can regard Matlis duals of left $\mathscr{D}$-modules as left $\mathscr{D}$-modules. The question of how, in this case, the de Rham complexes of a *holonomic* $\mathscr{D}$-module and of its Matlis dual are related occupies us in section 3.3, where we finish the proof of Theorem 3.0.7. Finally, in section 3.4, we work out an example in detail, removing any potential ambiguity in the definition of the $\mathscr{D}$-module structure on the Matlis duals of local cohomology modules over formal power series rings.

3.1 Matlis duality and Σ-continuous maps

In this section, we describe formulations of Matlis duality for local rings containing a field k in terms of continuous k-linear maps to k. Many of these results are not new; some of them are stated without proof in SGA2 [5, Exp. IV, Exemple 5.2] as well as in [34, p. 63, Ex. 1]. For lack of adequate references for the proofs, we provide their full details here. We also define the class of k-linear maps (the "Σ-continuous" maps) between arbitrary modules over such rings that admit Matlis duals.

We refer to section 1.4 for the definition of (classical) Matlis duality.

If $f : M \to N$ is an R-linear homomorphism of R-modules, its Matlis dual is the R-linear homomorphism $f^* : D(N) \to D(M)$ defined by pre-composition with f: $f^*(\phi) = \phi \circ f$. Using this definition, it does not make sense *a priori* to speak of the Matlis dual of a map $\delta : M \to N$ that is not R-linear. However, we will show that a more general class of maps can be dualized. We will make use of functorial identifications of the Matlis dual of a finite-length (resp. finitely generated) R-module with the set of k-linear (resp. k-linear and $\mathfrak{m}$-adically continuous) maps from the module to k; from these identifications, we will see that any k-linear map between finite-length R-modules, and any $\mathfrak{m}$-adically continuous k-linear map between finitely generated R-modules, has a Matlis dual. We will also explain how this theory can be extended to the case of arbitrary modules, which are not treated in [5].

Remark 3.1.1. In [5] the following results are stated in a slightly more general setting: $(R,\mathfrak{m})$ is a (Noetherian) local ring containing a field k_0, such that the residue field $k = R/\mathfrak{m}$ is a finite extension of k_0. Since we need only the case where $k = k_0$, we make this assumption throughout to simplify the discussion. However, with only minor modifications to the arguments, all of what follows in this section is true at the level of generality of [5].

Let R be as above, and let N be any R-module, hence *a fortiori* a k-space. We can define an R-module structure on the k-space $\mathrm{Hom}_k(N,k)$ as follows. Given a k-linear homomorphism $\lambda : N \to k$, we define $r\cdot\lambda : N \to k$ by $(r\cdot\lambda)(n) = \lambda(rn)$, which is again k-linear since, if $\alpha \in k$, we have $(r\cdot\lambda)(\alpha n) = \lambda(r(\alpha n)) = \alpha\lambda(rn) = \alpha(r\cdot\lambda)(n)$ by the k-linearity of λ. (We will use the dot $\cdot$ throughout this section to denote an R-action on maps which is defined by multiplication on the input of the map when multiplying the output by $r \in R$ may not make sense.) Now recall that the socle $\mathrm{Soc}(E) = (0 : \mathfrak{m})$ of E is a one-dimensional k-space (Lemma 1.4.6). Fix, once and for all, a k-linear splitting $\sigma : E \to k$ of the inclusion, and define, for any R-module N, a map $\Phi_N : \mathrm{Hom}_R(N,E) \to \mathrm{Hom}_k(N,k)$ by $\Phi_N(g) = \sigma\circ g$. Clearly, if g is R-linear (and hence k-linear), the composition $\sigma\circ g$ is k-linear.

Lemma 3.1.2. *The map Φ_N defined above is an R-linear homomorphism.*

Proof. Let $g \in \mathrm{Hom}_R(N,E)$ and $r \in R$ be given. Then for any $n \in N$, we have $\Phi_N(rg)(n) = \sigma((rg)(n)) = \sigma(rg(n)) = \sigma(g(rn)) = (\sigma\circ g)(rn) = (r\cdot\Phi_N(g))(n)$, so that $\Phi_N(rg) = r\cdot\Phi_N(g)$. □

We list some more elementary properties of the maps Φ_N. Suppose we have an R-linear homomorphism $f : M \to N$ of R-modules. The map $f^* : \mathrm{Hom}_R(N,E) \to \mathrm{Hom}_R(M,E)$ (that is, the Matlis dual of f) is clearly R-linear.

Lemma 3.1.3. *The map $f^\vee : \mathrm{Hom}_k(N,k) \to \mathrm{Hom}_k(M,k)$ defined by pre-composition with f (*i.e.*, $f^\vee(\lambda) = \lambda\circ f$ for a k-linear $\lambda : N \to k$) is R-linear.*

Proof. Let $r \in R$ and $\lambda \in \mathrm{Hom}_k(N,k)$ be given. Then

$$f^\vee(r\cdot\lambda)(m) = (r\cdot\lambda)(f(m)) = \lambda(rf(m)) = \lambda(f(rm)) = f^\vee(\lambda)(rm) = (r\cdot f^\vee(\lambda))(m)$$

for any $m \in M$, as desired. □

Moreover, we note that given any $g \in \mathrm{Hom}_R(N,E)$, both $\Phi_M(f^*(g))$ and $f^\vee(\Phi_N(g))$ are equal to the composite $\sigma \circ g \circ f : M \to k$. Therefore, the diagram below is commutative and all its arrows are R-linear maps:

$$\begin{array}{ccc} \mathrm{Hom}_R(N,E) & \xrightarrow{f^*} & \mathrm{Hom}_R(M,E) \\ \downarrow{\Phi_N} & & \downarrow{\Phi_M} \\ \mathrm{Hom}_k(N,k) & \xrightarrow{f^\vee} & \mathrm{Hom}_k(M,k). \end{array}$$

We have now established enough preliminaries to prove the following:

Proposition 3.1.4. *The map* $\Phi_N : \mathrm{Hom}_R(N,E) \to \mathrm{Hom}_k(N,k)$ *defined by* $\Phi_N(\phi) = \sigma \circ \phi$ *is an isomorphism of* R*-modules whenever* N *is of finite length.*

Proof. We proceed by induction on the length $l(N)$ in the category of R-modules, remarking that any finite-length N is a k-space of dimension $l(N)$. The base case, $l(N) = 1$, is the case $N \simeq k$; here Φ_N is simply the chosen isomorphism identifying $\mathrm{Hom}_k(k,k) \simeq k$ with the socle $\mathrm{Soc}(E) \simeq \mathrm{Hom}_R(k,E) = \mathrm{Hom}_R(R/\mathfrak{m},E)$ of E. Now suppose $l(N) \geq 2$, in which case there is a short exact sequence $0 \to k \to N \to N' \to 0$ of R-modules where $l(N') = l(N) - 1$. As E is an injective R-module, the functor $\mathrm{Hom}_R(-,E)$ is exact. Moreover, $\mathrm{Hom}_k(-,k)$ is also an exact functor on the category of k-spaces (all k-spaces are injective objects). We therefore obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(N',E) & \longrightarrow & \mathrm{Hom}_R(N,E) & \longrightarrow & \mathrm{Hom}_R(k,E) & \longrightarrow & 0 \\ & & \downarrow{\Phi'_N} & & \downarrow{\Phi_N} & & \downarrow{\Phi_k} & & \\ 0 & \longrightarrow & \mathrm{Hom}_k(N',k) & \longrightarrow & \mathrm{Hom}_k(N,k) & \longrightarrow & \mathrm{Hom}_k(k,k) & \longrightarrow & 0 \end{array}$$

All maps in this diagram are R-linear, and the bottom row is exact as a sequence of R-modules, since it is exact as a sequence of k-spaces. The map Φ_k is an isomorphism by our base case, and

Φ'_N is an isomorphism by the induction hypothesis, so Φ_N is an isomorphism by the five-lemma and the proof is complete. $\square$

The discussion so far has involved multiple arbitrary choices, meaning that the maps Φ_N are not canonical. The Matlis module E is a choice of R-injective hull of k, which is known to be defined only up to non-unique isomorphism [4, Ex. 10.1.2]; what is more, we have introduced an arbitrary splitting $\sigma : E \to k$. At the level of generality in which we are working in this section (a local ring R containing its residue field), it seems that some non-canonicity must be accepted. However, our intended application of this theory is to the case where R is a formal power series ring over a field, and here there are canonical choices for E and σ, which we now describe.

Let $R = k[[x_1, \dots, x_n]]$ for k a field, and let $\mathfrak{m}$ be its maximal ideal. Since R is Gorenstein, the local cohomology module $H^n_{\mathfrak{m}}(R)$ is isomorphic to E by Proposition 1.1.4(d). Thus we can take $H^n_{\mathfrak{m}}(R)$ as a canonical choice for the Matlis module. A canonical choice for the map $H^n_{\mathfrak{m}}(R) \to k$ comes from *residue theory*. The original reference for this theory is [35]: see also [5, Exp. IV, Remarque 5.5] and [36, Ch. 5] for concrete descriptions in our special case. The residue map is a canonical map $H^n_{\mathfrak{m}}(\Omega^n_R) \to k$. In our case, $R \simeq \Omega^n_R$ and so $H^n_{\mathfrak{m}}(R) \simeq H^n_{\mathfrak{m}}(\Omega^n_R)$. By [5, Exp. IV, Remarque 5.5], the resulting composite $\sigma : H^n_{\mathfrak{m}}(R) \to k$ is independent of the choice of basis of Ω^n_R, hence is canonically defined.

Moreover, we can give an explicit formula for σ. If we compute $H^n_{\mathfrak{m}}(R)$ using the Čech complex of R with respect to $x_1, \dots, x_n$ (Definition 1.1.9), the resulting R-module consists of all k-linear combinations of “inverse monomials” $x_1^{s_1} \cdots x_n^{s_n}$ where $s_1, \dots, s_n < 0$. The R-action is defined by the usual exponential rules with the caveat that non-negative powers of the variables are set equal to zero, so the product of a formal power series in R with such an “inverse polynomial” has only finitely many nonzero terms. The residue map σ is defined by setting

$$\sigma\left(\sum \alpha_{s_1,\dots,s_n} x_1^{s_1} \cdots x_n^{s_n}\right) = \alpha_{-1,\dots,-1} \in k,$$

that is, by taking the $(-1, \dots, -1)$-coefficient of such an “inverse polynomial” [36, Ch. 5].

Convention 3.1.5. *Whenever we work with the Matlis dual of a k-linear map between modules over the ring $R = k[[x_1,\dots,x_n]]$, we assume we are using the canonical choices of $H^n_{\mathfrak{m}}(R)$ for E and the residue map, as defined above, for $\sigma : E \to k$.*

We next consider the case of R-modules that are not of finite length, for which we need to restrict attention to $\mathfrak{m}$-adically continuous homomorphisms. We recall the general definition here:

Definition 3.1.6. Let R be a commutative ring, $I \subset R$ an ideal, and M,N two finitely generated R-modules. The *I-adic topology* on M (resp. N) is defined by stipulating that $\{I^nM\}$ (resp. $\{I^nN\}$) be a fundamental system of neighborhoods of 0. An Abelian group homomorphism $f : M \to N$ is *I-adically continuous* if it is continuous with respect to these topologies on M and N: that is, if for all t there exists an s such that $f(I^sM) \subset I^tN$.

Remark 3.1.7. Note that any R-linear map is automatically I-adically continuous. Also note that this definition makes sense for arbitrary R-modules, not necessarily finitely generated. We insist on finite generation here because we will make use of a different notion of continuity later in the case of arbitrary modules.

In the case of the local ring $(R,\mathfrak{m})$, the only fundamental neighborhood of $0 \in k$ in the $\mathfrak{m}$-adic topology on k is $\{0\}$ itself. Therefore, if M is a finitely generated R-module, a $\mathfrak{m}$-adically continuous map $M \to k$ is one that annihilates $\mathfrak{m}^tM$ for some $t \geq 0$. Let $\mathrm{Hom}_{cont,k}(M,k)$ be the k-space of k-linear maps $M \to k$ that are $\mathfrak{m}$-adically continuous.

Now note that if M is a finitely generated R-module and $\phi : M \to E$ is an R-linear map, ϕ annihilates $\mathfrak{m}^tM$ for some t. Indeed, let $m_1,\dots,m_n$ be generators for M over R. Since $E = E(R/\mathfrak{m})$ is $\mathfrak{m}$-power torsion (every element of E is annihilated by some power of $\mathfrak{m}$ [1, Thm. 18.4(v)]), there exist t_i for $i = 1,\dots,n$ such that $\phi(m_i)$ is annihilated by $\mathfrak{m}^{t_i}$; but then, setting $t = \max\{t_1,\dots,t_n\}$, we have $\phi(\mathfrak{m}^tM) = 0$. Therefore every such ϕ factors through $M/\mathfrak{m}^tM$ for some t, that is, $\mathrm{Hom}_R(M,E) = \varinjlim \mathrm{Hom}_R(M/\mathfrak{m}^tM,E)$, and all $M/\mathfrak{m}^tM$ are of finite length. Since $\mathrm{Hom}_R(M/\mathfrak{m}^tM,E)$ is isomorphic via $\Phi_{M/\mathfrak{m}^tM}$ to $\mathrm{Hom}_k(M/\mathfrak{m}^tM,k)$, and these isomorphisms form a compatible system as t varies, we deduce the existence of an isomorphism

$$\Phi_M : \mathrm{Hom}_R(M,E) \xrightarrow{\sim} \varinjlim \mathrm{Hom}_k(M/\mathfrak{m}^t M, k) = \mathrm{Hom}_{cont,k}(M,k),$$

again defined by $\Phi_M(\phi) = \sigma \circ \phi$. Our definition of the action of an element $r \in R$ on a k-linear map $\lambda : M \to k$ by pre-composition by multiplication with r preserves the property of $\mathfrak{m}$-adic continuity, so $\mathrm{Hom}_{cont,k}(M,k)$ is indeed an R-module. As in the finite-length case, we see that Φ_M is functorial in the R-module M. Note that if M is of finite length, every k-linear map $M \to k$ is $\mathfrak{m}$-adically continuous, so that the isomorphism Φ_M just defined coincides with the Φ_M defined earlier. We summarize the above discussion in the following

Theorem 3.1.8. *[5, Exp. IV, Exemple 5.2] Let $(R,\mathfrak{m})$ be a local ring containing its residue field k. Let E be an R-injective hull of k, and fix a k-linear splitting σ of the inclusion $k \hookrightarrow E$. For every finitely generated R-module M, post-composition with σ defines an isomorphism*

$$\Phi_M : \mathrm{Hom}_R(M,E) \xrightarrow{\sim} \mathrm{Hom}_{cont,k}(M,k)$$

of R-modules, functorial in the R-module M. (As a special case, if M is of finite length, $\mathrm{Hom}_R(M,E) \simeq \mathrm{Hom}_k(M,k)$.)

For the rest of this section, the assumptions on R and k are as in Theorem 3.1.8. A consequence of this theorem is that we can define the Matlis dual of a map between finitely generated (resp. finite-length) R-modules as long as the map is k-linear and $\mathfrak{m}$-adically continuous (resp. k-linear). The definition is simply pre-composition:

Definition 3.1.9. Let $\delta : M \to N$ be a k-linear map between finitely generated R-modules that is continuous with respect to the $\mathfrak{m}$-adic topologies on M and N. Then pre-composition with δ is k-linear and carries $\mathfrak{m}$-adically continuous k-linear maps $\lambda : N \to k$ to $\mathfrak{m}$-adically continuous k-linear maps $\lambda \circ \delta : M \to k$, so we can define the *Matlis dual* $\delta^* = \Phi_M^{-1} \circ \delta^\vee \circ \Phi_N : D(N) \to D(M)$ to be the composite

$$\mathrm{Hom}_R(N,E) \xrightarrow[\Phi_N]{} \mathrm{Hom}_{cont,k}(N,k) \xrightarrow[\delta^\vee]{} \mathrm{Hom}_{cont,k}(M,k) \xrightarrow[\Phi_M^{-1}]{} \mathrm{Hom}_R(M,E)$$

Remark 3.1.10. If M and N are of finite length, δ^* can be defined in the same way as above for any k-linear δ. (More generally, we will see below in Corollary 3.1.16 that any k-linear map between *Artinian* modules can be dualized.)

Our dual construction behaves well with respect to composition:

Proposition 3.1.11. *If M,N,P are finitely generated R-modules and $\delta : M \to N$, $\delta' : N \to P$ are $\mathfrak{m}$-adically continuous k-linear maps, then $(\delta' \circ \delta)^* = \delta^* \circ \delta'^*$ as maps $D(P) \to D(M)$.*

Proof. We calculate using the definition: $(\delta' \circ \delta)^* = \Phi_M^{-1} \circ (\delta' \circ \delta)^\vee \circ \Phi_P = \Phi_M^{-1} \circ \delta^\vee \circ \delta'^\vee \circ \Phi_P = (\Phi_M^{-1} \circ \delta^\vee \circ \Phi_N) \circ (\Phi_N^{-1} \circ \delta'^\vee \circ \Phi_P) = \delta^* \circ \delta'^*$, as desired. □

In the case of an R-linear map (which is automatically k-linear and $\mathfrak{m}$-adically continuous) between finitely generated R-modules, our definition of the Matlis dual of this map agrees with the usual one, so our notation is unambiguous and our definition is in fact a generalization of the usual one. We make this precise in the following nearly tautological lemma:

Lemma 3.1.12. *Let M and N be finitely generated R-modules. If $f : M \to N$ is an R-linear homomorphism, then $f^* = \Phi_M^{-1} \circ f^\vee \circ \Phi_N$, so the usual definition of the Matlis dual of f coincides with ours.*

Proof. Suppose $\phi : N \to E$ is R-linear. Then $(f^\vee \circ \Phi_N)(\phi) = \sigma \circ \phi \circ f$. As Φ_M is an isomorphism, $(\Phi_M^{-1} \circ f^\vee \circ \Phi_N)(\phi)$ is the unique R-linear map $M \to E$ which gives $\sigma \circ \phi \circ f$ upon post-composition with σ; by the unicity, it cannot be anything but $\phi \circ f = f^*(\phi)$. Therefore the left- and right-hand sides of the asserted equality agree upon evaluation at every $\phi \in D(N)$. □

What can be said in the case of arbitrary (not necessarily finitely generated) modules? An arbitrary R-module M can be regarded as the filtered direct limit of its finitely generated R-submodules M_λ. The Matlis dual functor converts direct limits into inverse limits, so $D(M) = \varprojlim D(M_\lambda)$. (This is true for any contravariant Hom functor. Note that its converse is not true.) If $\phi : M \to E$ is R-linear (an element of $D(M)$), the restriction of ϕ to a fixed M_λ corresponds by Theorem 3.1.8 to an $\mathfrak{m}$-adically continuous map $M_\lambda \to k$. Therefore, we will be able to repeat our earlier constructions in the case of arbitrary modules if we impose some conditions on how the maps we are studying behave under restriction to finitely generated submodules. We make this precise with the following definition.

Definition 3.1.13. Let M and N be R-modules, and let $\delta : M \to N$ be a k-linear map. We say that δ is Σ-*continuous* if for every finitely generated R-submodule $M_\lambda \subset M$, the R-submodule $\langle \delta(M_\lambda) \rangle \subset N$ generated by the image of M_λ under δ is finitely generated and the restriction $\delta|_{M_\lambda} : M_\lambda \to \langle \delta(M_\lambda) \rangle$ is $\mathfrak{m}$-adically continuous in the sense of Definition 3.1.6. We write $\mathrm{Hom}_k^\Sigma(M,N)$ for the set (indeed, R-module) of Σ-continuous k-linear maps $M \to N$, and refer to such maps simply as "Σ-continuous", the k-linearity being understood.

By restricting attention to Σ-continuous maps to k, we obtain a generalization of Theorem 3.1.8 to the case of an arbitrary module:

Theorem 3.1.14. *Let M be an R-module, and suppose a choice of $\sigma : E \to k$ as in Theorem 3.1.8 has been made. There is an isomorphism of R-modules*

$$\Phi_M : D(M) = \mathrm{Hom}_R(M,E) \xrightarrow{\sim} \mathrm{Hom}_k^\Sigma(M,k)$$

defined by post-composition with σ and functorial in the R-module M.

Proof. Consider the family $\{M_\lambda\}$ of finitely generated R-submodules of M. We view M as the filtered direct limit of the M_λ. As the Matlis dual functor converts direct limits into inverse limits, we see that $D(M) = \varprojlim D(M_\lambda)$, the transition maps being pre-composition with inclusions $M_\lambda \hookrightarrow M_{\lambda'}$ of finitely generated submodules. For all M_λ, post-composition with σ defines an isomorphism $\Phi_{M_\lambda} : D(M_\lambda) \to \mathrm{Hom}_{cont,k}(M_\lambda, k)$ by Theorem 3.1.8. Since this isomorphism is functorial in M_λ, the $\{\Phi_{M_\lambda}\}$ form a compatible system of R-linear isomorphisms, inducing an R-linear isomorphism

$$\Phi'_M : D(M) = \varprojlim D(M_\lambda) \xrightarrow{\sim} \varprojlim \mathrm{Hom}_{cont,k}(M_\lambda, k);$$

but the right-hand side of this isomorphism can be identified with $\mathrm{Hom}_k^\Sigma(M,k)$, essentially by definition. The natural isomorphism

$$\theta : \varprojlim \mathrm{Hom}_{cont,k}(M_\lambda, k) \to \mathrm{Hom}_k^\Sigma(M,k)$$

is defined on a compatible system $\{f_\lambda \in \mathrm{Hom}_{cont,k}(M_\lambda, k)\}$ by taking $\theta(\{f_\lambda\})$ to be the unique k-linear map $M \to k$ whose restriction to every M_λ is f_λ (this map is Σ-continuous by definition), and the composition $\Phi_M = \theta \circ \Phi'_M$ is nothing but post-composition with σ, completing the proof. □

Remark 3.1.15. If M is a finitely generated R-module, then $\mathrm{Hom}_k^\Sigma(M,k) = \mathrm{Hom}_{cont,k}(M,k)$ clearly, so in this case the Φ_M above is the same as the Φ_M appearing in Theorem 3.1.8.

This theorem allows us to identify the Matlis dual with the *full* k-linear dual in the case of an Artinian module:

Corollary 3.1.16. *If M is an R-module such that* $\mathrm{Supp}(M) = \{\mathfrak{m}\}$ *(for instance if M is Artinian), then* $\mathrm{Hom}_R(M,E) \simeq \mathrm{Hom}_k(M,k)$ *as R-modules.*

Proof. Let M_λ be a finitely generated R-submodule of M. The hypothesis on M implies that M_λ is annihilated by a power of $\mathfrak{m}$ and consequently is of finite length. Given any k-linear map $M \to k$, its restriction to M_λ is therefore $\mathfrak{m}$-adically continuous. We conclude that $\mathrm{Hom}_k^\Sigma(M,k) = \mathrm{Hom}_k(M,k)$: the corollary now follows from Theorem 3.1.14. □

Remark 3.1.17. Suppose now that R is *complete*. If M is a finitely generated or Artinian R-module, the double Matlis dual $D(D(M))$ is canonically isomorphic to M by Theorem 3.1.9, and if $f: M \to N$ is an R-linear homomorphism between finitely generated R-modules, $f^{**}: M = D(D(M)) \to D(D(N)) = N$ can be canonically identified with f. In fact, the same is true for the more general maps we are dualizing in this section (by applying the R-linear result at the level of every finite-length quotient of M and N), but we need to make sure the double dual is defined. If M and N are finitely generated R-modules and $\delta: M \to N$ is k-linear and $\mathfrak{m}$-adically continuous, then $D(N)$ and $D(M)$ are Artinian R-modules and δ^* is a k-linear map. By Corollary 3.1.16, δ^{**} exists. On the other hand, if M and N are Artinian R-modules, $D(N)$ and $D(M)$ are finitely generated R-modules, and for any k-linear map $\delta: M \to N$, the Matlis dual $\delta^*: D(N) \to D(M)$ is not only k-linear but $\mathfrak{m}$-adically continuous as well (by Lemma 3.1.18 below), so that again δ^{**} exists. Therefore, the equality $\delta^{**} = \delta$ holds for any k-linear δ (in the

Artinian case) or any k-linear and $\mathfrak{m}$-adically continuous δ (in the finitely generated case) over complete local rings R.

Lemma 3.1.18. *Let M and N be Artinian R-modules, and let $\delta : M \to N$ be a k-linear map. The Matlis dual $\delta^* : D(N) \to D(M)$ is $\mathfrak{m}$-adically continuous.*

Proof. Let t be a natural number, and consider the annihilator M_t of $\mathfrak{m}^t$ in M, a finite-length R-submodule of M. As N is Artinian, every element of N is annihilated by some power of $\mathfrak{m}$: indeed, given any $y \in N$, $R \cdot y \simeq R/\operatorname{Ann}(y)$ is a submodule of N, hence Artinian itself, and so $\mathfrak{m}^l \subset \operatorname{Ann}(y)$ for some l. Since M_t is of finite length, it follows that there exists some s such that $\delta(M_t) \subset N_s$, where N_s is the annihilator of $\mathfrak{m}^s$ in N. Now apply Matlis duality. The containment $\delta(M_t) \subset N_s$ implies that the kernel of the map $D(N) \to D(N_s)$ is carried by δ^* into the kernel of $D(M) \to D(M_t)$. But since $M_t = (0 :_M \mathfrak{m}^t)$, this kernel (which we may identify with $D(M/M_t)$ by the exactness of D) is $\mathfrak{m}^t D(M)$; this means that $\delta^*(\mathfrak{m}^s D(M)) \subset \mathfrak{m}^t D(N)$, that is, that δ^* is continuous with respect to the $\mathfrak{m}$-adic topologies on $D(M)$ and $D(N)$. $\square$

We can now extend the definition of the Matlis dual of an $\mathfrak{m}$-adically continuous map between finitely generated R-modules to a definition of the Matlis dual of a Σ-continuous map between arbitrary modules as follows.

Proposition 3.1.19 (Proposition-Definition)**.** *Let M and N be R-modules, and let $\delta : M \to N$ be a Σ-continuous map. Define the **Matlis dual** $\delta^* = \Phi_M^{-1} \circ \delta^\vee \circ \Phi_N : D(N) \to D(M)$ to be the composite*

$$\operatorname{Hom}_R(N,E) \xrightarrow[\Phi_N]{} \operatorname{Hom}_k^\Sigma(N,k) \xrightarrow[\delta^\vee]{} \operatorname{Hom}_k^\Sigma(M,k) \xrightarrow[\Phi_M^{-1}]{} \operatorname{Hom}_R(M,E),$$

where again $\delta^\vee$ is pre-composition with δ. This construction satisfies the following properties: given another R-module P and a Σ-continuous map $\delta' : N \to P$, we have $(\delta' \circ \delta)^* = \delta^* \circ \delta'^*$ as k-linear maps $D(P) \to D(M)$, and δ^* is the usual Matlis dual in the case in which δ is R-linear.

Proof. We first check that δ^* is well-defined. Suppose that $\tau : N \to k$ is Σ-continuous. Then the same is true for $\delta^\vee(\tau) = \tau \circ \delta : M \to k$, since given any finitely generated R-submodule

$M_\lambda \subset M$, $\tau \circ \delta$ factors through a finitely generated R-submodule $N_\mu \subset N$ by assumption, and as both $\delta|_{M_\lambda} : M_\lambda \to N_\mu$ and $\tau|_{N_\mu} : N_\mu \to k$ are $\mathfrak{m}$-adically continuous, so is their composite, which is $(\tau \circ \delta)|_{M_\lambda}$. Thus $\delta^\vee$ is a well-defined k-linear map $\mathrm{Hom}_k^\Sigma(N,k) \to \mathrm{Hom}_k^\Sigma(M,k)$. The final two assertions follow immediately since both Proposition 3.1.11 and Lemma 3.1.12 hold for the restriction of δ to any finitely generated R-submodule of M. □

Remark 3.1.20. Finally, we note that it is not necessary in applications to consider the direct system of *all* finitely generated R-submodules of a given module, as is done in the proof of Theorem 3.1.14. Suppose M, N, and δ are as above, and suppose we are given *cofinal* families $\{M_\lambda\}$ (resp. $\{N_\mu\}$) of finitely generated R-submodules of M (resp. N), meaning that every finitely generated R-submodule of M (resp. N) is contained in some M_λ (resp. some N_μ), with the further condition that for all λ, there exists μ such that $\delta(M_\lambda) \subset N_\mu$. Then the above construction still defines $\delta^* : D(N) \to D(M)$.

3.2 Matlis duality for $\mathscr{D}$-modules over complete local rings

We now specialize to the case where $(R,\mathfrak{m})$ is a *complete* local ring with coefficient field k. This assumption is in force throughout the section unless otherwise indicated.

If $\mathscr{D} = \mathrm{Diff}(R,k)$ is the non-commutative ring of k-linear differential operators on R and M is a left $\mathscr{D}$-module, elements of $\mathscr{D}$ act on M via Σ-continuous maps. This fact allows us to immediately apply the formalism of the previous section to define the Matlis dual of the action of an element of $\mathscr{D}$; in this way, $D(M)$ becomes a *right* $\mathscr{D}$-module. In the case of a complete *regular* local ring with coefficient field *k of characteristic zero*, we describe a “transposition” operation allowing us to regard these Matlis duals as *left* $\mathscr{D}$-modules.

Let $\mathscr{D} = \mathrm{Diff}(R,k)$ be the ring of k-linear differential operators on R. Suppose M is a left $\mathscr{D}$-module, and let $\delta \in \mathrm{Diff}(R,k)$ be a differential operator. The map $\delta : M \to M$ defined by the action of δ (that is, $\delta(m) = \delta \cdot m$) is a k-linear map (we abusively use the same letter δ to simplify notation).

We need a lemma on the continuity of differential operators’ action:

Lemma 3.2.1. *Let M be a left* $\mathrm{Diff}(R,k)$*-module (here R is any commutative ring, and $k \subset R$ any commutative subring), and let $I \subset R$ be any ideal. For any $\delta \in \mathrm{Diff}^j(R)$ and $s \geq 0$, we have $\delta(I^{s+j}M) \subset I^s M$.*

Proof. We proceed by induction on $s+j$. If $s+j=0$, there is nothing to prove. Now suppose $s+j \geq 0$ and the containment established for smaller values of $s+j$. Since $\delta \in \mathrm{Diff}^j(R)$, it follows that $[\delta, r] \in \mathrm{Diff}^{j-1}(R)$ for any $r \in R$, and so $[\delta, r](I^{s+j-1}M) \subset I^s M$ by the induction hypothesis for s and $j-1$. That is, $[\delta, r](tm) = \delta(rtm) - r\delta(tm) \in I^s M$ for any $r \in R$, $t \in I^{s+j-1}$ and $m \in M$. If we further suppose that $r \in I$ and put $x = rt \in I^{s+j-1}I = I^{s+j}$, we see that $\delta(xm) = r\delta(tm) + [\delta, r](tm)$. By the induction hypothesis for $s-1$ and j, $\delta(I^{s+j-1}M) \subset I^{s-1}M$, so $\delta(tm) \in I^{s-1}M$ and $r\delta(tm) \in I^s M$. Thus both terms on the right-hand side (and their sum $\delta(xm)$) belong to $I^s M$. As any element of $I^{s+j}M$ can be expressed as a finite sum $\sum_\alpha r_\alpha t_\alpha m_\alpha$ with $r_\alpha \in I$ and $t_\alpha \in I^{s+j-1}$, the result follows. $\square$

If a left $\mathscr{D}$-module M is finitely generated as an R-module, then given any $\delta \in \mathscr{D}$, the lemma shows that the corresponding map $\delta : M \to M$ is $\mathfrak{m}$-adically continuous. Therefore the formalism of the previous section applies, and we can define the k-linear Matlis dual $\delta^* : D(M) \to D(M)$ in such a way that if $\delta' \in \mathscr{D}$ is another differential operator, $(\delta' \circ \delta)^* = \delta^* \circ (\delta')^*$ (note that the order of composition is reversed). This allows us to define a structure of right $\mathscr{D}$-module on $D(M)$ by $\phi \cdot \delta = \delta^*(\phi)$. Here is an example of this dual construction:

Example 3.2.2. Let $R = k[[x]]$, where k is a field, equipped with the k-linear derivation $\delta = \frac{d}{dx}$. Denote by $\mathfrak{m}$ its maximal ideal (x). Take $M = R$, a finitely generated R-module. Recall that by Convention 3.1.5, we take for E the local cohomology module $H^1_{(x)}(R)$, which is the module of finite sums $\sum_{s>0} \frac{\alpha_s}{x^s}$ where $\alpha_s \in k$, and we define Matlis duals of k-linear maps between R-modules using the residue map $\sigma : E \to k$ given by $\sigma(\sum_{s>0} \frac{\alpha_s}{x^s}) = \alpha_1$.

We determine explicitly the Matlis dual $\delta^* : D(R) \to D(R)$, which we will identify with a map $E \to E$ since canonically $\mathrm{Hom}_R(R,E) \simeq E$. Let $\mu = \sum_{s>0} \frac{\alpha_s}{x^s}$ be an element of E and suppose that t is the greatest integer such that $\alpha_t \neq 0$. Then μ can be identified with an R-linear map $R \to E$ in the following way: given an element $\sum_{i=0}^{\infty} \beta_i x^i$ of R, its image in E is the product

$(\sum_{i=0}^{\infty}\beta_i x^i)(\sum_{s>0}\frac{\alpha_s}{x^s})$, which is carried by σ to $\sum_{s>0}\alpha_s\beta_{s-1}$. Therefore the corresponding k-linear map $\sigma\circ\mu : R\to k$ is given by $\sum_{i=0}^{\infty}\beta_i x^i \mapsto \sum_{s>0}\alpha_s\beta_{s-1}$. Both μ and $\sigma\circ\mu$ send $\mathfrak{m}^t$ to zero; write $\overline{\mu}$ for the R-linear map $R/\mathfrak{m}^t \to E$ induced on the quotient.

The derivation δ induces a k-linear map $R/\mathfrak{m}^{t+1}\to R/\mathfrak{m}^t$ by the Leibniz rule, and the k-linear composite $R/\mathfrak{m}^{t+1}\to R/\mathfrak{m}^t \xrightarrow{\sigma\circ\overline{\mu}} k$ is defined by

$$\sum_{i=0}^{t}\lambda_i x^i \mapsto \sum_{i=0}^{t-1}(i+1)\lambda_{i+1}x^i \mapsto \sum_{s>0}s\alpha_s\lambda_s$$

where the scalars α_s are the numerators in the expansion of μ. Since $R/\mathfrak{m}^{t+1}$ is of finite length as an R-module, this composite corresponds to a unique R-linear map $R/\mathfrak{m}^{t+1}\to E$, that is, a map $R\to E$ defined by multiplication by an element of E that is annihilated by $\mathfrak{m}^{t+1}$. Examining the formula above, we see that this element of E cannot be anything but $\sum_{s>0}\frac{s\alpha_s}{x^{s+1}}$. Therefore, viewed at the level of a map $E\to E$, the map δ^* is defined by

$$\delta^*(\frac{1}{x^s}) = \frac{s}{x^{s+1}},$$

which differs from the standard ("quotient rule") differentiation map on E only by a minus sign. (Our discussion of "transposition" at the end of this section will indicate the reason for this sign.) This concludes Example 3.2.2.

In order to extend this definition of Matlis dual to arbitrary $\mathscr{D}$-modules, removing the finite generation hypothesis, we need to show that differential operators act on $\mathscr{D}$-modules via Σ-continuous maps. We prove this now using the alternate characterization of differential operators given in EGA [6].

Definition 3.2.3. Suppose that R is any commutative ring and $k\subset R$ a commutative subring. We denote by B the ring $R\otimes_k R$, by $\mu : B\to R$ the multiplication map $\mu(r\otimes s) = rs$, and by $J\subset B$ the kernel of μ. We identify the subring $R\otimes_k 1 = \{r\otimes 1 | r\in R\}$ of B with R, and view B as an R-algebra using this identification. In this way, B and all ideals of B can be regarded as R-modules. Finally, for any $j\geq 0$, we denote by $P^j_{R/k}$ (or P^j) the quotient B/J^{j+1}.

In [6], differential operators are described in terms of R-linear maps via the following correspondence:

Proposition 3.2.4. *[6, Prop. 16.8.4] For any commutative ring R and commutative subring $k \subset R$, there is an isomorphism*

$$\mathrm{Hom}_R(P^j, R) \simeq \mathrm{Diff}^j(R)$$

of R-modules, where the differential operators on the right-hand side are understood to be k-linear.

In our case, where R is a complete local ring and k a coefficient field, both sides of the isomorphism of Proposition 3.2.4 are finitely generated R-modules:

Proposition 3.2.5. *Let $(R, \mathfrak{m})$ be a (Noetherian) complete local ring with coefficient field k. For all j, the R-module $\mathrm{Diff}^j(R) \subset \mathrm{Diff}(R, k)$ is finitely generated.*

The following definition will be used in the proof of Proposition 3.2.5:

Definition 3.2.6. Let $(R, \mathfrak{m})$ be a local ring. For any R-module L, the *maximal $\mathfrak{m}$-adically separated quotient* of L is $L^{sep} = L/(\cap_s \mathfrak{m}^s L)$. Note that L^{sep} is $\mathfrak{m}$-adically separated, and $L/\mathfrak{m}L \simeq L^{sep}/\mathfrak{m}L^{sep}$.

Proof of Proposition 3.2.5. Let j be given. By Proposition 3.2.4, it suffices to show that the R-module $\mathrm{Hom}_R(P^j, R)$ is finitely generated. As R is $\mathfrak{m}$-adically separated, every $f \in \mathrm{Hom}_R(P^j, R)$ factors uniquely through $(P^j)^{sep}$. Therefore $\mathrm{Hom}_R((P^j)^{sep}, R) \simeq \mathrm{Hom}_R(P^j, R)$ as R-modules. We claim that $(P^j)^{sep}$ is itself a finitely generated R-module, from which it will follow at once that

$$\mathrm{Hom}_R((P^j)^{sep}, R) \simeq \mathrm{Hom}_R(P^j, R) \simeq \mathrm{Diff}^j(R)$$

is finitely generated as well.

The elements $r \otimes 1 - 1 \otimes r$ with $r \in R$ generate J as an R-module: given $b = \sum_i r_i' \otimes r_i \in B$, we have $b \in J$ if and only if $\sum_i r_i' r_i = 0$, and in this case we see that

$$b = \sum_i (r_i' \otimes r_i - r_i' r_i \otimes 1) = -\sum_i (r_i' \otimes 1)(r_i \otimes 1 - 1 \otimes r_i) = \sum_i (-r_i') \cdot (r_i \otimes 1 - 1 \otimes r_i)$$

is an R-linear combination of elements of the form described. The equation

$$r \otimes r' = rr' \otimes 1 - r \cdot (r \otimes 1 - 1 \otimes r')$$

for $r, r' \in R$ also shows that we have an R-module direct sum decomposition $B = (R \otimes_k 1) \oplus J$. Since R is Noetherian, we can fix a finite set of generators $x_1, \ldots, x_s$ for its maximal ideal $\mathfrak{m}$. Moreover, since R contains its residue field k, we have a direct sum decomposition (as k-spaces) $R = k \oplus \mathfrak{m}$, so given any $r \in R$, we can write $r = c + x_1 y_1 + \cdots + x_s y_s$ where $c \in k$ and $y_i \in R$. We then have $r \otimes 1 - 1 \otimes r = \sum_i (x_i y_i \otimes 1 - 1 \otimes x_i y_i)$, since $c \otimes 1 = 1 \otimes c$ for $c \in k$. For all i, we can express $x_i y_i \otimes 1 - 1 \otimes x_i y_i$ as

$$(y_i \otimes 1)(x_i \otimes 1 - 1 \otimes x_i) - (x_i \otimes 1 - 1 \otimes x_i)(y_i \otimes 1 - 1 \otimes y_i) + (x_i \otimes 1)(y_i \otimes 1 - 1 \otimes y_i)$$

where the second summand belongs to J^2 and the third summand belongs to $\mathfrak{m}J$. We conclude that $r \otimes 1 - 1 \otimes r - (\sum y_i \cdot (x_i \otimes 1 - 1 \otimes x_i)) \in J^2 + \mathfrak{m}J$. Therefore, if we write $b_i = x_i \otimes 1 - 1 \otimes x_i$, the classes of the b_i generate $J/(J^2 + \mathfrak{m}J)$ as an R-module. Since $\mathfrak{m}$ annihilates $J/(J^2 + \mathfrak{m}J)$, and $R = k \oplus \mathfrak{m}$ as k-spaces, we see that moreover the classes of the b_i span $J/(J^2 + \mathfrak{m}J)$ as a k-space. Let L_j be the k-span of the monomials of degree at most j in the b_i, and let L'_j be the k-span of such monomials of degree precisely j, so that $L_j = \cup_{l=0}^{j} L'_l$: clearly all L_j and L'_j are finite-dimensional k-spaces. With this notation, what we have just shown is that $J = L'_1 + J^2 + \mathfrak{m}J$.

Now let $b \in B$ be arbitrary. Using the R-module direct sum decomposition $B = (R \otimes_k 1) \oplus J$, we write $b = (r \otimes 1) + x$ where $x \in J$. Using the k-space direct sum decomposition $R = k \oplus \mathfrak{m}$, we write $r = c + y$ where $c \in k$ and $y \in \mathfrak{m}$, so that $y \otimes 1 \in \mathfrak{m}B$. Our work above shows that there exist $\beta \in J^2$ and $\gamma \in \mathfrak{m}J$ such that $x - \beta - \gamma$ lies in L'_1. We have $(y \otimes 1) + \gamma \in \mathfrak{m}B$. We conclude from the decomposition

$$b = (c \otimes 1) + (x - \beta - \gamma) + \beta + (\gamma + (y \otimes 1))$$

that $B \subset L_1 + J^2 + \mathfrak{m}B$ (since $c \otimes 1 = c \cdot (1 \otimes 1) \in L_0 \subset L_1$), and hence that $B/(\mathfrak{m}B + J^2)$ is spanned as a k-space by L_1. Moreover, it follows by induction that for all j, $B/(\mathfrak{m}B + J^{j+1})$ is spanned as a k-space by L_j. Assume the conclusion for $j - 1$, that is, that $B \subset L_{j-1} + \mathfrak{m}B +$

J^j. We have already shown $J = L'_1 + \mathfrak{m}J + J^2$. Taking the jth power of both sides, we find $J^j \subset \mathfrak{m}J + \sum_{l=0}^{j} L'_{j-l} J^{2l} \subset \mathfrak{m}B + L'_j + J^{j+1}$, since $b_i \in J$ and for any $l > 0$, $2l + (j-l) \geq j+1$. Therefore

$$B \subset (L_{j-1} + L'_j) + \mathfrak{m}B + J^{j+1}.$$

Since $L_{j-1} + L'_j = L_j$, we have shown $B/(\mathfrak{m}B + J^{j+1})$ is spanned over k by L_j, completing the induction. It follows that every $B/(\mathfrak{m}B + J^{j+1})$ is a finite-dimensional k-space. We have

$$B/(\mathfrak{m}B + J^{j+1}) = P^j/\mathfrak{m}P^j \simeq (P^j)^{sep}/\mathfrak{m}(P^j)^{sep}$$

as k-spaces. Since $(P^j)^{sep}$ is $\mathfrak{m}$-adically separated and R is $\mathfrak{m}$-adically complete, the fact that $(P^j)^{sep}/\mathfrak{m}(P^j)^{sep}$ is a finite-dimensional k-space implies that $(P^j)^{sep}$ is a finitely generated R-module by a form of Nakayama's lemma [37, Lemma 2.1.16], completing the proof. $\square$

Remark 3.2.7. Our proof of Proposition 3.2.5 above, together with the proof of [37, Lemma 2.1.16], has the following consequence which we record separately for reference: if $x_1, \ldots, x_s$ generate the maximal ideal $\mathfrak{m}$, then $(P^j)^{sep}$ is generated over R by the classes of monomials in $b_0, b_1, \ldots, b_n$ of degree at most j, where $b_0 = 1 \otimes 1$ and $b_i = x_i \otimes 1 - 1 \otimes x_i$ for $i = 1, \ldots, n$.

We have now assembled enough preliminaries to prove the Σ-continuity of differential operators over a complete local ring:

Proposition 3.2.8. *Let $(R, \mathfrak{m})$ be a (Noetherian) complete local ring with coefficient field k, and let $\mathscr{D} = \mathrm{Diff}(R,k)$. If M is a left $\mathscr{D}$-module and $\delta \in \mathrm{Diff}(R,k)$, then $\delta : M \to M$ is Σ-continuous.*

Proof. We verify the conditions of Definition 3.1.13. Let $\delta \in \mathrm{Diff}(R,k)$ and let M_λ be a finitely generated R-submodule of M. We assert that the R-submodule $\langle \delta(M_\lambda) \rangle$ of M generated by the image of M_λ under δ is finitely generated over R. (By Lemma 3.2.1, we already know the restriction of δ to M_λ will be $\mathfrak{m}$-adically continuous, so this is all that must be proved.) With λ fixed, we proceed by induction on the order j of δ. Fix a finite set of generators $m_1, \ldots, m_n$ for M_λ. If $j = 0$, then δ is R-linear, so $\delta(m_1), \ldots, \delta(m_n)$ generate $\langle \delta(M_\lambda) \rangle$. Now suppose $j > 0$ and the statement proved for smaller values of j. By Proposition 3.2.5, we can find a finite set of R-module generators $d_1, \dots, d_s$ for $\mathrm{Diff}^{j-1}(R)$. Then we claim

$$\{\delta(m_i)\}_i \cup \{d_l(m_i)\}_{l,i}$$

is a finite set of generators for $\langle \delta(M_\lambda) \rangle$. Indeed, given any element $m_\lambda \in M_\lambda$, we can write it as a linear combination $r_1 m_1 + \cdots + r_n m_n$, and $\delta(r_i m_i) = r_i \delta(m_i) + [\delta, r_i](m_i)$ for all i. Since $[\delta, r_i] \in \mathrm{Diff}^{j-1}(R)$, we can write $[\delta, r_i] = \rho_{1,i} d_1 + \cdots + \rho_{s,i} d_s$ for some $\rho_{1,i}, \dots, \rho_{s,i} \in R$. Then

$$\delta(r_i m_i) = r_i \delta(m_i) + \rho_{1,i} d_1(m_i) + \cdots + \rho_{s,i} d_s(m_i)$$

for all i. The sum $\sum \delta(r_i m_i) = \delta(m_\lambda)$ thus belongs to the R-submodule of M generated by the specified finite set, completing the proof. $\square$

Corollary 3.2.9. *Let* $(R, \mathfrak{m})$ *be a complete local ring with coefficient field k, and let* $\mathscr{D} = \mathrm{Diff}(R, k)$*. Let M be a* left $\mathscr{D}$*-module. Then the Matlis dual* $D(M) = \mathrm{Hom}_R(M, E)$ *has a natural structure of* right $\mathscr{D}$*-module.*

Proof. Given any $\delta \in \mathrm{Diff}(R, k)$, Proposition 3.2.8 and Proposition 3.1.19 imply that the Matlis dual $\delta^* : D(M) \to D(M)$ is defined. We define a right $\mathrm{Diff}(R, k)$-action on $D(M)$ by $\phi \cdot \delta = \delta^*(\phi)$. This definition satisfies the axioms for a right action since, given another differential operator $\delta' \in \mathrm{Diff}(R, k)$, we have $(\delta' \circ \delta)^* = \delta^* \circ (\delta')^*$, again by Proposition 3.1.19. $\square$

We give some examples of Matlis duals with right $\mathrm{Diff}(R, k)$-structures, mostly involving local cohomology:

Example 3.2.10. Let $(R, \mathfrak{m})$ be a complete local ring with coefficient field k.

(a) Since R is a left $\mathrm{Diff}(R, k)$-module and $E = D(R)$, E has a natural structure of right $\mathrm{Diff}(R, k)$-module.

(b) If M is a left $\mathrm{Diff}(R, k)$-module, so is M_S for any multiplicatively closed subset $S \subset R$ [8, Example 5.1(a)]. If $I \subset R$ is an ideal, the local cohomology modules $H^i_I(R)$ supported at I have the structure of left $\mathrm{Diff}(R, k)$-modules, because $H^i_I(R)$ is the ith cohomology object of a complex whose objects are localizations of the left $\mathrm{Diff}(R,k)$-module R and whose maps, sums of natural localization maps, are $\mathrm{Diff}(R,k)$-homomorphisms [8, Example 5.1(c)]. By the previous theorem, the Matlis duals $D(H^i_I(R))$ are right $\mathrm{Diff}(R,k)$-modules.

(c) Now suppose $I = \mathfrak{m}$. If R is Cohen-Macaulay, then $H^i_{\mathfrak{m}}(R)$ is zero unless $i = \dim(R)$. The Matlis dual $D(H^{\dim(R)}_{\mathfrak{m}}(R))$, which is the *canonical module* of R, therefore has a structure of right $\mathrm{Diff}(R,k)$-module. If R is not Cohen-Macaulay, there exists some $i < \dim(R)$ such that $H^i_{\mathfrak{m}}(R)$ is nonzero. For such an i, the Matlis dual $D(H^i_{\mathfrak{m}}(R))$ is a right $\mathrm{Diff}(R,k)$-module that is finitely generated (since $H^i_{\mathfrak{m}}(R)$ is Artinian) as an R-module and whose dimension is strictly less than the dimension of R (in fact, its dimension is bounded above by i: [5, Exp. V, Thm. 3.1(ii)]). We remark that such an inequality is impossible for R regular, by Bernstein's theorem [7, Thm. I.4.1].

We give some more details related to Example 3.2.10(a). Since R is a finitely generated R-module, we know by Theorem 3.1.8 that $E = D(R) \simeq \mathrm{Hom}_{cont,k}(R,k)$, and the right $\mathscr{D}$-action on $\mathrm{Hom}_{cont,k}(R,k)$ is defined by $\lambda \cdot \delta = \lambda \circ \delta$ for $\delta \in D$ and $\lambda \in \mathrm{Hom}_{cont,k}(R,k)$. If M is any R-module, we have an isomorphism

$$\mathrm{Hom}^{\Sigma}_k(M,k) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}_{cont,k}(R,k))$$

by combining Theorem 3.1.14 and Theorem 3.1.8. Concretely, this isomorphism carries a Σ-continuous map $\lambda : M \to k$ to the R-linear map $M \to \mathrm{Hom}_{cont,k}(R,k)$ defined by $m \mapsto (r \mapsto \lambda(rm))$, and its inverse carries an R-linear map $\psi : M \to \mathrm{Hom}_{cont,k}(R,k)$ to the Σ-continuous map $M \to k$ defined by $m \mapsto \psi(m)(1)$.

By identifying E with $\mathrm{Hom}_{cont,k}(R,k)$ endowed with the right $\mathscr{D}$-structure above, we can give an alternate description of the right action of *derivations* (differential operators of order precisely 1) on the Matlis dual of any left $\mathscr{D}$-module, using the formulas of [38, Prop. 1.2.9]: if $\delta \in \mathscr{D}$ is a derivation and M is a left $\mathscr{D}$-module, then for any R-linear map $\phi : M \to E$, we define an R-linear map $\phi \cdot \delta : M \to E$ by $(\phi \cdot \delta)(m) = \phi(\delta \cdot m) - \phi(m) \cdot \delta$. This formula extends to define a right action of the R-subalgebra of $\mathscr{D}$ generated by R together with the derivations (e.g., if R is regular, this subalgebra is all of $\mathscr{D}$). This right action coincides with the right action we have defined in Corollary 3.2.9:

Proposition 3.2.11. *With the hypotheses of Corollary 3.2.9, let M be a left $\mathscr{D}$-module, and let $D(M)$ be its Matlis dual. If $\delta \in \mathscr{D}$ is a derivation and $\phi : M \to E$ is R-linear, we have $\delta^*(\phi) = \phi \cdot \delta$, where the right-hand side is defined as in [38, Prop. 1.2.9].*

Proof. We use the various identifications between equivalent forms of $D(M)$. Under the isomorphism $D(M) \xrightarrow{\Phi_M} \operatorname{Hom}_k^{\Sigma}(M,k)$ of Theorem 3.1.14, the map ϕ corresponds to $\sigma \circ \phi$, and $\delta^*(\phi) = \sigma \circ \phi \circ \delta$. Under the identification $\operatorname{Hom}_k^{\Sigma}(M,k) \simeq \operatorname{Hom}_R(M, \operatorname{Hom}_{cont,k}(R,k))$ given earlier, $\sigma \circ \phi \circ \delta$ corresponds to the map

$$m \mapsto (r \mapsto \sigma(\phi(\delta(rm)))).$$

On the other hand, for any $m \in M$, we have $(\phi \cdot \delta)(m) = \phi(\delta \cdot m) + \phi(m) \cdot \delta \in E$, where we identify E with the right $\mathscr{D}$-module $\operatorname{Hom}_{cont,k}(R,k)$ in order to define $\phi(m) \cdot \delta$. Under this identification, $\phi(m) \cdot \delta$ is the map $r \mapsto \sigma(r\phi(\delta \cdot m))$, and $\phi(\delta \cdot m)$ is the map $r \mapsto \sigma(\delta(r)\phi(m))$. It is therefore enough to show that

$$\sigma(\phi(\delta(rm))) = \sigma(r\phi(\delta \cdot m)) + \sigma(\delta(r)\phi(m))$$

for all $r \in R$ and $m \in M$, which follows immediately from the relations $\delta(rm) = \delta(r)m + r\delta(m)$ holding in any left $\mathscr{D}$-module M. $\square$

We now specialize further to the case in which $R = k[[x_1, \ldots, x_n]]$ is a formal power series ring over a field of characteristic zero. In this case, there is a transposition operation that converts left modules over $\operatorname{Diff}(R,k)$ to right modules and conversely. We recall its definition. Since k is of characteristic zero, the ring $\operatorname{Diff}(R,k)$ is a free left R-module generated by monomials in $\partial_1 = \frac{\partial}{\partial x_1}, \ldots, \partial_n = \frac{\partial}{\partial x_n}$.

Definition 3.2.12. Let $R = k[[x_1, \ldots, x_n]]$ where k is a field of characteristic zero, and let M be a left $\operatorname{Diff}(R,k)$-module. Let $\rho \partial_1^{a_1} \cdots \partial_n^{a_n}$ be an element of $\operatorname{Diff}(R,k)$, where $\rho \in R$. If $\cdot$ denotes the given left action of $\mathrm{Diff}(R,k)$, then for any $m \in M$, the formula

$$m * (\rho \partial_1^{a_1} \cdots \partial_n^{a_n}) = ((-1)^{a_1+\cdots+a_n} \partial_n^{a_n} \cdots \partial_1^{a_1} \rho) \cdot m$$

defines the *transpose* action, a *right* $\mathrm{Diff}(R,k)$-action $*$ on M.

There is, of course, a symmetric notion of the transpose of a right $\mathscr{D}$-module, which is a left $\mathscr{D}$-module. To see that the right action given above is well-defined, we view it in the following way. Let $\mathrm{Diff}(R,k)^\circ$ be the opposite algebra of $\mathrm{Diff}(R,k)$. There exists a unique isomorphism $\phi : \mathrm{Diff}(R,k)^\circ \to \mathrm{Diff}(R,k)$ such that $\phi(\rho) = \rho$ for all $\rho \in R$ and $\phi(\partial_i) = -\partial_i$ for all i, which when viewed as a map $\phi : \mathrm{Diff}(R,k) \to \mathrm{Diff}(R,k)$ is called the *principal anti-automorphism* of $\mathrm{Diff}(R,k)$. To see that this is an isomorphism, note that since all elements of R commute with each other and all ∂_i commute with each other, the only non-trivial relations among elements of $\mathrm{Diff}(R,k)$ are the relations $\partial_i \rho = \rho \partial_i + \partial_i(\rho)$. The map ϕ, which is clearly bijective, carries $\partial_i \rho$ to $-\rho \partial_i$ and $\rho \partial_i + \partial_i(\rho)$ to $-\partial_i \rho + \partial_i(\rho)$; since $-\rho \partial_i = -\partial_i \rho + \partial_i(\rho)$, the relations are respected. The transposed action $*$ is then simply defined by $m * \delta = \phi(\delta) \cdot m$ for $\delta \in \mathrm{Diff}(R,k)$.

Proposition 3.2.13. *Let $R = k[[x_1, \ldots, x_n]]$ with k a field of characteristic zero, $\mathscr{D} = \mathrm{Diff}(R,k)$, and M any* left *$\mathscr{D}$-module. There is a natural structure of* left *$\mathscr{D}$-module on the Matlis dual $D(M) = \mathrm{Hom}_R(M,E)$.*

Proof. Apply the right-to-left version of the transposition operation described above to the right $\mathscr{D}$-module $D(M)$ with structure defined in Corollary 3.2.9. □

Therefore, in this case, Matlis duality provides a (contravariant) functor from left $\mathscr{D}$-modules to left $\mathscr{D}$-modules.

3.3 The de Rham complex of a Matlis dual

Let $R = k[[x_1, \ldots, x_n]]$ where k is a field of characteristic zero, and let $\mathscr{D} = \mathrm{Diff}(R,k)$. For any left $\mathscr{D}$-module M, we can define its de Rham complex $M \otimes \Omega_R^\bullet$. The Matlis dual $D(M)$ is also a left $\mathscr{D}$-module by Proposition 3.2.13, so we can also consider the de Rham complex $D(M) \otimes \Omega_R^\bullet$.

Our goal in this section is to compare the cohomology of these two complexes. Specifically, we will show the following:

Theorem 3.3.1. *Let R and $\mathscr{D}$ be as above. If M is a* holonomic *left $\mathscr{D}$-module, then for all i, we have isomorphisms*

$$(H^i_{dR}(M))^* \simeq H^{n-i}_{dR}(D(M))$$

where the asterisk denotes k-linear dual.

By Proposition 3.2.8, the differentials in the complex $M \otimes \Omega^\bullet_R$ are Σ-continuous, so the entire complex can be Matlis dualized. Since the functor D is contravariant, the ith object in this dualized complex $D(M \otimes \Omega^\bullet_R)$ is $D(M \otimes \Omega^{n-i}_R)$. Theorem 3.3.1 is a trivial consequence of the following pair of propositions.

Proposition 3.3.2. *Let R and $\mathscr{D}$ be as above. If M is a* holonomic *left $\mathscr{D}$-module, then for all i, we have isomorphisms*

$$(h^i(M \otimes \Omega^\bullet_R))^* \simeq h^{n-i}(D(M \otimes \Omega^\bullet_R)).$$

Proposition 3.3.3. *Let R and $\mathscr{D}$ be as above. If M is* any *left $\mathscr{D}$-module, then for all i, we have isomorphisms*

$$h^i(D(M \otimes \Omega^\bullet_R)) \simeq h^i(D(M) \otimes \Omega^\bullet_R).$$

Proposition 3.3.3 is relatively straightforward, and we prove it next. The proof of Proposition 3.3.2, which is the longest and most involved proof in this thesis, will take up the remainder of this section.

Proof of Proposition 3.3.3. We first compute the differentials in the complex $D(M \otimes \Omega^\bullet_R)$. Let i be given, and consider the differential $d^i : M \otimes \Omega^i_R \to M \otimes \Omega^{i+1}_R$. An element of $M \otimes \Omega^i_R$ is a sum of terms of the form $m_{j_1 \cdots j_i}\, dx_{j_1} \wedge \cdots \wedge dx_{j_i}$ where $1 \leq j_1 < \cdots < j_i \leq n$, and the formula for d^i is

$$d^i(m\, dx_{j_1} \wedge \cdots \wedge dx_{j_i}) = \sum_{s=1}^{n} \partial_s(m)\, dx_s \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_i}.$$

Now consider the Matlis dual of this differential. Since the Matlis dual functor commutes with finite direct sums, we can identify $D(M \otimes \Omega^i_R)$ with a direct sum of $\binom{n}{i}$ copies of $D(M)$, again indexed by the $dx_{j_1} \wedge \cdots \wedge dx_{j_i}$. If $\phi \in D(M)$, we have the formula

$$(d^i)^*(\phi\, dx_{j_1} \wedge \cdots \wedge dx_{j_{i+1}}) = \sum_{s=1}^{i+1} (-1)^{s-1} \partial_s^*(\phi)\, dx_{j_1} \wedge \cdots \wedge \widehat{dx_{j_s}} \wedge \cdots \wedge dx_{j_{i+1}},$$

where the $\widehat{dx_{j_s}}$ means that symbol is omitted ($(d^i)^*$ is obtained by extending this formula by linearity).

We now introduce Koszul complexes, for which a reference is [2, §4.5]. Consider the commutative subring $\Delta = k[\partial_1, \ldots, \partial_n] \subset \mathscr{D}$. The $\mathscr{D}$-module $D(M)$ is *a fortiori* a Δ-module, and the de Rham complex $D(M) \otimes \Omega^\bullet_R$ is the *cohomological* Koszul complex $K^\bullet(D(M); \partial)$ [2, Notation 4.5.1] of the Δ-module $D(M)$ with respect to $\partial = (\partial_1, \ldots, \partial_n)$, where the ∂_i act on $D(M)$ via the Matlis dual maps ∂_i^*. On the other hand, by the formula above, it is clear that $D(M \otimes \Omega^\bullet_R)$ is the *homological* Koszul complex $K_\bullet(D(M); \partial)$. We have

$$h^i(K^\bullet(D(M); \partial)) \simeq h_{n-i}(K_\bullet(D(M); \partial))$$

by [2, Ex. 4.5.2]; regarding the complex on the right as being *cohomologically* indexed (as we do when considering it as the Matlis dual of $M \otimes \Omega^\bullet_R$), we see that the right-hand side is its ith cohomology object, completing the proof. □

The remainder of this section is long and contains a great deal of preliminary material necessary for the proof of Proposition 3.3.2. Before giving this preliminary material, we first outline it for the reader's benefit, then introduce some notation that we will use repeatedly.

- First, we prove some lemmas concerning direct and inverse systems of modules. The key definition here, Definition 3.3.7 (*strong-sense stability*), is a dual version of the Mittag-Leffler condition.
- We then introduce some definitions and results due to van den Essen, who in a series of papers studied the kernels and cokernels of differential operators. Not only his results, but

also some of the ideas in his proofs, will be of necessary use to us. We discuss changes of variable and prove a technical lemma, Lemma 3.3.17, that relies on van den Essen's work.

- We next describe how to "stratify" the de Rham complex $M \otimes \Omega^\bullet_R$, writing it as a direct limit of "de Rham-like" complexes whose objects are finitely generated R-modules. The crucial result concerning this direct system is Proposition 3.3.19, which asserts that the cohomology objects of these complexes satisfy strong-sense stability with finite-dimensional stable images. We also give a more general version of this result, Corollary 3.3.20, which will be of no use to us in this section but to which we will need to appeal in section 4.2.

- Finally, we give the proof of Proposition 3.3.2, using our work in section 3.1 on Matlis duality.

We need to work not only with the rings R and $\mathscr{D}$, but also with subrings defined using proper subsets of $\{x_1, \ldots, x_n\}$:

Definition 3.3.4. Let $j \geq 0$ be given. We denote by R_j (resp. R^j) the subring $k[[x_1, \ldots, x_j]]$ (resp. $k[[x_{n-j+1}, \ldots, x_n]]$) of R. (Thus, $R = R_n = R^n$ and $k = R_0 = R^0$.) We denote by $\mathscr{D}_j = \mathrm{Diff}(R_j, k)$ and $\mathscr{D}^j = \mathrm{Diff}(R^j, k)$ the corresponding rings of differential operators, which are subrings of $\mathscr{D}$.

This notation will be in force throughout the section, and will not be repeated in the statements of results.

Remark 3.3.5. If M is any left $\mathscr{D}$-module, it is also a left module over $\mathscr{D}_j$ and $\mathscr{D}^j$ for all j, and we have short exact sequences relating its "partial" and "full" de Rham complexes of the form

$$0 \to M \otimes \Omega^\bullet_{R^j}[-1] \to M \otimes \Omega^\bullet_{R^{j+1}} \to M \otimes \Omega^\bullet_{R^j} \to 0$$

where the first map is given by $\wedge dx_{n-j}$. The cohomology objects of the complex $M \otimes \Omega^\bullet_{R^j}$ are $\mathscr{D}_{n-j}$-modules, and if we consider the associated long exact cohomology sequence, the connecting homomorphisms (up to a sign) are simply ∂_{n-j}. Of course, there is also a version of this sequence involving de Rham complexes over R_j and R_{j+1} instead of R^j and R^{j+1}. (See section 4.1, especially Definition 4.1.6 and Lemma 4.1.7, for more details.)

In the proof of Proposition 3.3.2, we will consider various direct and inverse systems of complexes of R-modules and k-spaces. (All our direct and inverse systems will be indexed by the natural numbers, but the following discussion applies to any filtered index set.) The interaction of cohomology with *direct* limits in these categories is not complicated: the direct limit is an exact functor, and so it commutes with cohomology. For *inverse* limits, more caution is required, as the inverse limit is, in general, only left exact. In order to commute cohomology with inverse limits, we will need to verify the Mittag-Leffler condition for many of the inverse systems involved.

Definition 3.3.6. [11, 13.1.2] Let $\{M_i\}$ be an inverse system of modules (indexed by $\mathbb{N}$) over a commutative ring R, with inverse limit $M = \varprojlim M_i$ and transition maps $f_{ji} : M_j \to M_i$ for $i \leq j$. We say that the system $\{M_i\}$ satisfies the *Mittag-Leffler condition* if for all l, the descending chain $\{f_{l+s,l}(M_{l+s})\}$ of submodules of M_l becomes stationary: there exists some s such that for all $t \geq s$, $f_{l+s,l}(M_{l+s}) = f_{l+t,l}(M_{l+t})$.

The following condition on *direct* systems can be thought of as a sort of dual to the Mittag-Leffler condition:

Definition 3.3.7. Let $\{M_i\}$ be a direct system of modules (indexed by $\mathbb{N}$) over a commutative ring R, with direct limit $M = \varinjlim M_i$, transition maps $f_{ij} : M_i \to M_j$ for $i \leq j$, and insertion maps $f_i : M_i \to M$. If for some j, $N_j \subset M_j$ is an R-submodule, we say that the images of N_j under the transition maps *stabilize in the strong sense* if there exists l such that f_{j+l} induces an isomorphism

$$f_{j,j+l}(N_j) \xrightarrow{\sim} f_j(N_j);$$

equivalently, $f_{j,j+l}(N_j) \cap \ker f_{j+l} = 0$. (If we say that the images of some object stabilize in the strong sense, it will be clear from context with respect to which transition maps and direct system we mean.)

Note that since $\ker f_{j+l,j+l'} \subset \ker f_{j+l}$ for any $l' \geq l$, it follows at once from Definition 3.3.7 that there are also isomorphisms $f_{j,j+l}(N_j) \xrightarrow{\sim} f_{j,j+l'}(N_j)$ induced by the transition maps.

This condition is automatic if R is Noetherian and N_j is finitely generated:

Lemma 3.3.8. *Let R be a commutative Noetherian ring. If $\{M_i\}$ is a direct system of R-modules as in Definition 3.3.7, and $N_j \subset M_j$ is a finitely generated R-submodule, then the images of N_j stabilize in the strong sense.*

Proof. Since R is Noetherian, $\ker f_j \cap N_j$ is also a finitely generated R-module. Fix R-generators $x_1, \ldots, x_s$ for $\ker f_j \cap N_j$. For all $i \in \{1, \ldots, s\}$, we have $f_j(x_i) = 0$, and so there exists $l_i \geq 0$ such that $f_{j,j+l_i}(x_i) = 0 \in M_{j+l_i}$. If we put $l = \max\{l_i\}$, then $f_{j,j+l}$ annihilates $\ker f_j \cap N_j$. We claim that

$$f_{j,j+l}(N_j) \cap \ker f_{j+l} = 0.$$

Suppose that $x \in M_{j+l}$ belongs to this intersection. Then we have $f_{j+l}(x) = 0$ and $x = f_{j,j+l}(y)$ for some $y \in N_j$. From $f_j(y) = f_{j+l}(f_{j,j+l}(y)) = f_{j+l}(x) = 0$, we conclude $y \in \ker f_j$; but $y \in N_j$ as well, and since $f_{j,j+l}$ annihilates $\ker f_j \cap N_j$, it follows that $x = f_{j,j+l}(y) = 0$, completing the proof. $\square$

In the case of vector spaces over a field, the connection between the Mittag-Leffler condition and strong-sense stability can be made more precise:

Lemma 3.3.9. *Let $\{U_i\}$ be an inverse system (indexed by $\mathbb{N}$) of vector spaces over a field k, with transition maps f_{ji}. For all i, let $V_i = U_i^*$ be the k-space dual of U_i, and regard $\{V_i\}$ as a direct system with transition maps $\lambda_{ij} = f_{ji}^*$ for $i \leq j$. Suppose that for all l, the images of V_l under the transition maps $\lambda_{l,l+s}$ stabilize in the strong sense. Then $\{U_i\}$ satisfies the Mittag-Leffler condition.*

Proof. We prove the contrapositive. Suppose that the system $\{U_i\}$ does not satisfy Mittag-Leffler. Then there is an l such that for all s, there exists $t \geq s$ such that $f_{l+t,l}(U_{l+t}) \subsetneq f_{l+s,l}(U_{l+s})$. Since k-space dual is an exact functor, proper injections dualize to surjections with nontrivial kernels, so the surjection $(f_{l+s,l}(U_{l+s}))^* \twoheadrightarrow (f_{l+t,l}(U_{l+t}))^*$ is not an isomorphism. Given any linear map $g : W \to W'$ of k-spaces, we can identify $\operatorname{im} g^*$ with $(\operatorname{im} g)^*$ as follows: factor g as $W \twoheadrightarrow \operatorname{im} g \subset W'$, and dualize this factorization to obtain $(W')^* \twoheadrightarrow (\operatorname{im} g)^* \hookrightarrow W^*$. The image of the last map is $\operatorname{im}(g^*)$, and the injectivity allows us to identify this image with $(\operatorname{im} g)^*$. After identifying $(f_{l+s,l}(U_{l+s}))^*$ (resp. $(f_{l+t,l}(U_{l+t}))^*$) with $\lambda_{l,l+s}(V_l)$ (resp. $\lambda_{l,l+t}(V_l)$) as a subspace of V_{l+s} (resp. V_{l+t}) in this way, we see that the surjection $(f_{l+s,l}(U_{l+s}))^* \twoheadrightarrow (f_{l+t,l}(U_{l+t}))^*$ is nothing but $\lambda_{l+s,l+t}$ restricted to $\lambda_{l,l+s}(V_l)$. We conclude that the images of V_l cannot stabilize in the strong sense. $\square$

By applying the previous proof's reasoning twice, we can draw a conclusion about double duals:

Lemma 3.3.10. *Let k be a field, and let $\{U_i\}_{i\in\mathbb{N}}$ be an inverse system of k-spaces, indexed by $\mathbb{N}$, with transition maps $f_{ji} : U_j \to U_i$ for $i \leq j$, and inverse limit $U = \varprojlim U_i$. Suppose that the inverse system $\{U_i\}$ satisfies the Mittag-Leffler condition*[1] *and that the inverse limit $\varprojlim U_i^{**}$ is a finite-dimensional k-space. Then $\varprojlim U_i \simeq \varprojlim U_i^{**}$.*

Proof. The canonical map from a k-space U_i to its double dual U_i^{**} is always injective, and inverse limit is left exact, so there is a natural injection $U = \varprojlim U_i \hookrightarrow \varprojlim U_i^{**}$. Therefore U is also finite-dimensional. For all i, let $U_i' = \cap_{i\leq j} f_{ji}(U_j)$ be the stable image of the transition maps inside U_i (this descending chain stabilizes by the Mittag-Leffler hypothesis). Then [3, p. 191] $\{U_i'\}$ is also an inverse system, now with surjective transition maps, such that $\varprojlim U_i' = U$ and U maps surjectively to all U_i'. In particular, all U_i' are finite-dimensional k-spaces and hence isomorphic to their double duals $(U_i')^{**}$. By applying the reasoning in the proof of Lemma 3.3.9 twice, we see that since the inverse system $\{U_i\}$ satisfies the Mittag-Leffler condition, so does the inverse system $\{U_i^{**}\}$. Moreover, we can identify the double dual of a stable image, $(U_i')^{**}$, with the stable image of the double dual, $(U_i^{**})' \subset U_i^{**}$, again by using the argument of Lemma 3.3.9 twice. Since for all i, U_i' is canonically isomorphic to its double dual, the corresponding

[1] *Paul Garrett (private communication) has pointed out that the Mittag-Leffler hypothesis is superfluous; however, the proof is more difficult.*

inverse limits are also isomorphic. Thus we have isomorphisms

$$\varprojlim U_i \simeq \varprojlim U_i' \simeq \varprojlim (U_i')^{**} \simeq \varprojlim (U_i^{**})' \simeq \varprojlim U_i^{**},$$

completing the proof. □

In a series of papers [39, 40, 41, 42, 9], van den Essen examined the effect of the operator ∂_n acting on a holonomic left $\mathscr{D}$-module. His first result was that the kernel of ∂_n behaves as well as we might hope:

Theorem 3.3.11. *[40, Thm.] If M is a holonomic left $\mathscr{D}$-module and M_* is the kernel of $\partial_n : M \to M$, then M_* is a holonomic $\mathscr{D}_{n-1}$-module.*

The key step in the proof of Theorem 3.3.11 is the following lemma, which we will use below in the proof of Lemma 3.3.17:

Lemma 3.3.12. *[40, Cor. 2] Let M be a left $\mathscr{D}$-module and let M_* be the kernel of $\partial_n : M \to M$. If $M = R \cdot M_*$, then $M = M_* \oplus x_n M$ as $\mathscr{D}_{n-1}$-modules.*

Without some conditions, there is no analogous result for cokernels; van den Essen gives a concrete example in [41, Thm.] in which M is a holonomic $\mathscr{D}$-module but $M/\partial_n(M)$ is not holonomic.

If M is holonomic, there exists [41, Prop. 1] an element $g \in R$ such that the localization M_g is a finitely generated R_g-module. If g can be chosen to be x_n*-regular* (that is, such that $g(0,0,\ldots,0,x_n) \neq 0$), then [39, Ch. II, prop. 1.16] for all $m \in M$, there exists an x_n-regular $f \in R$ (in fact, f can be taken to be a power of g) such that the R-submodule $E_{f\partial_n}(m)$ of M generated by the family $\{(f\partial_n)^i(m)\}_{i \geq 0}$ is in fact *finitely* generated over R. That is, m is x_n-*regular* in the sense of the following definition:

Definition 3.3.13. If M is a left $\mathscr{D}$-module, an element $m \in M$ is x_n*-regular* if there exists an x_n-regular element $f \in R$ such that $E_{f\partial_n}(m)$, as defined in the preceding paragraph, is a finitely generated R-module. A holonomic $\mathscr{D}$-module M is x_n*-regular* if some (hence, every) m such that $M = \mathscr{D} \cdot m$ is x_n-regular. (More generally, a $\mathscr{D}$-module M is x_n-regular if every $m \in M$ is x_n-regular.)

We make two observations about the notion of x_n-regularity. The first is that, given any holonomic $\mathscr{D}$-module M, we can always change variables (that is, replace $x_1,\dots,x_n$ by another regular system of parameters for R) to make M x_n-regular [9, Cor. 1.8]: indeed, given $g \in R$ such that M_g is finitely generated over R_g, simply choose coordinates in which g is x_n-regular. By Proposition 1.2.5, the de Rham cohomology $H^*_{dR}(M)$ is independent of this choice. The second observation is that this condition is precisely what is required for the holonomy of the cokernel of ∂_n:

Theorem 3.3.14. *[42, Thm.] If M is a holonomic left $\mathscr{D}$-module that is x_n-regular, then $M/\partial_n(M)$ is a holonomic $\mathscr{D}_{n-1}$-module.*

The key step in the proof of Theorem 3.3.14 is the following lemma, which we will also use below in the proof of Lemma 3.3.17:

Lemma 3.3.15. *[42, Cor. 2] Let M be an $\mathscr{D}$-module. Suppose that $m \in M$ is x_n-regular. Then there exists a finitely generated R_{n-1}-submodule L of $R\cdot m$ and a natural number p such that $R\cdot m \subset L + \sum_{i=1}^{p} \partial_n^i(R\cdot m)$. In particular, $R\cdot m \subset L + \partial_n(R\cdot m)$.*

Note that the kernel M_* and the cokernel $M/\partial_n(M)$ are the cohomology objects of the complex $M \otimes \Omega^\bullet_{R^1}$. Consider the short exact sequence

$$0 \to M \otimes \Omega^\bullet_{R^1}[-1] \to M \otimes \Omega^\bullet_{R^2} \to M \otimes \Omega^\bullet_{R^1} \to 0$$

of Remark 3.3.5. The connecting homomorphisms in the corresponding long exact sequence in cohomology, up to a sign, are defined by ∂_{n-1}. Thus we obtain short exact sequences

$$0 \to K_{i-1} \to h^i(M \otimes \Omega^\bullet_{R^2}) \to C_i \to 0$$

where K_{i-1} is the kernel of ∂_n acting on $h^{i-1}(M \otimes \Omega^\bullet_{R^1})$ and C_i is the cokernel of ∂_n acting on $h^i(M \otimes \Omega^\bullet_{R^1})$. If K_{i-1} and C_i are holonomic $\mathscr{D}_{n-1}$-modules, then $h^i(M \otimes \Omega^\bullet_{R^2})$ is a holonomic $\mathscr{D}_{n-1}$-module, in which case we can consider the kernel and cokernel of ∂_{n-2} acting on it and continue the process. For each $\mathscr{D}_{n-j}$-module N which appears in this process, there exists an element g of R_{n-j} such that N_g is a finitely generated $(R_{n-j})_g$-module, and we can make a change of variables after which g is x_{n-j}-regular (so the kernel and cokernel of ∂_{n-j} acting on N are holonomic $\mathscr{D}_{n-j-1}$-modules). As the process terminates, there are only finitely many such g, and the failure of a given g to be x_{n-j}-regular after a change of variables is governed by a family of polynomial equations in the coefficients of the change of variables. Therefore, choosing a single coordinate change after which all g are regular is a matter of avoiding finitely many proper Zariski-closed subsets of the affine coordinate space, which is always possible since k is of characteristic zero and thus infinite.

We record the conclusion just drawn for reference:

Proposition 3.3.16. *Let M be a holonomic $\mathscr{D}$-module. There exists a change of variables (which, by Proposition 1.2.5, does not alter the de Rham cohomology of M) after which, for all i and j, $h^i(M \otimes \Omega^{\bullet}_{R^j})$ is a holonomic $\mathscr{D}_{n-j}$-module which is x_{n-j}-regular.*

We now prove a technical lemma which makes crucial use of van den Essen's results. The situation we examine is that of a $\mathscr{D}$-module expressible as a direct limit of R-modules, and a family of R_{n-1}-linear maps between these modules whose direct limit is the R_{n-1}-linear map ∂_n. We now describe the situation more precisely. Let M be a left $\mathscr{D}$-module, and suppose that $M = \varinjlim M_i$ as R-modules, where $\{M_i\}$ is a direct system (indexed by $\mathbb{N}$) of R-submodules of M. Let f_{ij} (resp. f_i) be the transition (resp. insertion) maps for this direct system. Suppose furthermore that there exist R_{n-1}-linear maps $\delta_i : M_i \to M_{i+1}$ for all i, compatible with the transition maps, such that $\varinjlim \delta_i = \partial_n$. (We write ∂ for ∂_n.) Given an element $m \in M$, there exists some j and $m_j \in M_j$ such that $m = f_j(m_j)$, and we have $\partial(m) = f_{j+1}(\delta_j(m_j))$, independently of the choice of j and m_j.

The transition maps f_{ij} induce, by the compatibility, maps on the kernels and cokernels of the δ_i: for all l and s, we see that $f_{l,l+s}(\ker \delta_l) \subset \ker \delta_{l+s}$ and that $f_l(\ker \delta_l) \subset \ker \partial$. From this, it follows that $f_{l,l+s}$ (resp. f_l) induces R_{n-1}-linear maps $f'_{l,l+s} : \ker \delta_l \to \ker \delta_{l+s}$ and $f''_{l,l+s}$: $\operatorname{coker} \delta_{l-1} \to \operatorname{coker} \delta_{l+s-1}$ (resp. $f'_l : \ker \delta_l \to \ker \partial$ and $f''_l : \operatorname{coker} \delta_{l-1} \to \operatorname{coker} \partial$). We have, for all l and s, $f'_{l+s} \circ f'_{l,l+s} = f'_l$, and a similar compatibility for the f''.

If M is holonomic and x_n-regular, we have a stability property for the images of these induced maps on kernels and cokernels:

Lemma 3.3.17. *Let M be a holonomic, x_n-regular $\mathscr{D}$-module. Let $\partial = \partial_n \in \mathscr{D}$. Suppose that $\{M_i\}$ is a direct system of R-submodules of M with $M = \varinjlim M_i$, and that $\{\delta_i : M_i \to M_{i+1}\}$ is a family of R_{n-1}-linear maps, compatible with the transitions, such that $\partial = \varinjlim \delta_i$. Let f_{ij} (resp. f_i) be the transition (resp. insertion) maps for this direct system, and define the induced maps f' and f'' as in the preceding paragraph. Fix l and let N_l be a finitely generated R-submodule of M_l.*

(a) Let $N'_l = N_l \cap \ker \delta_l$. The images of N'_l under the $f'_{l,l+s}$ stabilize in the strong sense, and the stable image is a finitely generated R_{n-1}-submodule of $\ker \partial$.

(b) Let N''_l be the image $N_l/(N_l \cap \delta_{l-1}(M_{l-1}))$ of N_l in $\operatorname{coker} \delta_{l-1}$. *The images of N''_l under the $f''_{l,l+s}$ stabilize in the strong sense, and the stable image is a finitely generated R_{n-1}-submodule of* $\operatorname{coker} \partial$.

Proof. Let $M_* = \ker \partial$. The Leibniz rule implies that $R \cdot M_*$ is a $\mathscr{D}$-submodule of M, and it is clear that $M_* = (R \cdot M_*)_*$ as $\mathscr{D}_{n-1}$-submodules of M. Moreover, for all i, $\ker \delta_i \subset f_i^{-1}(M_*)$ (since $\partial = \varinjlim \delta_i$), so we lose no information about the kernels (either of ∂ or of the δ_i) if we replace M by $M' = R \cdot M_*$ and all M_i by $M'_i = f_i^{-1}(M')$. Since $M' = \varinjlim M'_i$ (the transition maps being the restrictions of f_{ij} to M'_i) and $\partial|_{M'} = \varinjlim \delta_i|_{M'_i}$, we reduce (in the proof of (a)) to the case $M = M'$.

We now prove (a). By Lemma 3.3.12, the hypothesis that $M = M'$ implies that there is an R_{n-1}-module direct sum decomposition $M = M_* \oplus x_n M$. Let π be the R_{n-1}-linear projection $M \to M_*$. Since $f_l(N'_l) \subset M_*$, π restricts to an isomorphism $f_l(N'_l) \xrightarrow{\sim} \pi(f_l(N'_l))$, and the composite $N'_l \hookrightarrow N_l \xrightarrow{f_l} M$ factors through the projection $N_l \to N_l/x_n N_l$ (since $\ker(\pi \circ f_l)$ contains $x_n N_l$). Since N_l is a finitely generated R-module, $N_l/x_n N_l$ is a finitely generated R_{n-1}-module. This implies that $\pi(f_l(N'_l))$ is the R_{n-1}-linear image of a finitely generated R_{n-1}-module, so it is a finitely generated R_{n-1}-module, and so is its isomorphic copy $f_l(N'_l) = f'_l(N'_l)$. It follows that *if* the images of N'_l stabilize in the strong sense, the stable image is finitely generated.

As N_l is a finitely generated R-module, its images under the R-linear maps $f_{l,l+s}$ stabilize in the strong sense by Lemma 3.3.8; since $N_l' \subset N_l$ and $f_{l,l+s}'$ is simply a restriction of $f_{l,l+s}$ for all s, the fact that the images of N_l' under the $f_{l,l+s}'$ stabilize in the strong sense as well is automatic. This proves (a).

In our proof of (b) we cannot assume, but do not need, that $M = M'$, so we drop this assumption now. We now fix a set $\{n_1, \ldots, n_{\alpha_l}\}$ of R-generators for N_l. Then by assumption all $f_l(n_i) \in M$ are x_n-regular (since M itself is x_n-regular), so we can apply Lemma 3.3.15 to every $f_l(n_i)$ in turn. Let $i \in \{1, \ldots, \alpha_l\}$ be fixed, and consider the R-submodule $R \cdot f_l(n_i) \subset M$ generated by $f_l(n_i)$. By Lemma 3.3.15, there exist a positive natural number p_i and a finitely generated R_{n-1}-submodule L_i of $R \cdot f_l(n_i)$ such that

$$R \cdot f_l(n_i) \subset L_i + \partial\Big(\sum_{j=0}^{p_i-1} R \cdot \partial^j(f_l(n_i))\Big).$$

Write n_{ijl} for $\partial^j(f_l(n_i))$. Then if we let L_l' be the finitely generated R_{n-1}-module $L_1 + \cdots + L_{\alpha_l}$ and N_l''' the R-module generated by

$$\{n_{ijl}\}_{1 \le i \le \alpha_l, 0 \le j \le p_i - 1},$$

we see that

$$f_l(N_l) = \sum_{i=1}^{\alpha_l} R \cdot f_l(n_i) \subset L_l' + \partial(N_l''').$$

Let $\{f_l(y_1), \ldots, f_l(y_{\eta_l})\}$ be a set of R_{n-1}-generators for $L_l' \subset f_l(N_l)$, and write L_l'' for the R_{n-1}-submodule of N_l generated by $y_1, \ldots, y_{\eta_l}$, so that $f_l(L_l'') = L_l'$. Then since f_l'' is induced by f_l, the containment $f_l(N_l) \subset f_l(L_l'') + \partial(N_l''')$ implies that the image of $N_l'' = N_l/(N_l \cap \delta_{l-1}(M_l))$ in $M/\partial(M)$ (that is, under f_l'') is contained in the image of $L_l''/(L_l'' \cap \delta_{l-1}(M_l))$ under f_l''. We know that f_l'' is R_{n-1}-linear and that L_l'', and hence its quotient $L_l''/(L_l'' \cap \delta_{l-1}(M_l))$, is a finitely generated R_{n-1}-module, so we can conclude that $f_l''(N_l'')$ is a finitely generated R_{n-1}-module. It follows that *if* the images of N_l'' stabilize in the strong sense, the stable image is finitely generated.

Finally, we choose t so large that every n_{ijl} has a representative in M_{l+t-1}, as follows: for all i and j, there exists a natural number t_{ij} and an element $m_{l+t_{ij}} \in M_{l+t_{ij}}$ such that $f_{l+t_{ij}}(m_{l+t_{ij}}) =$

n_{ijl}. Put $t = \max\{t_{ij}\}+1$. Then since the n_{ijl} are R-generators for N_l''', any element of the R_{n-1}-module $\partial(N_l''')$ can be expressed as $f_{l+t}(\delta_{l+t-1}(m))$ for some $m \in M_{l+t-1}$. Therefore, if $n_l \in N_l$ is such that $f_l(n_l) \in \partial(M)$, the containment $f_l(N_l) \subset L_l' + \partial(N_l''')$ implies (since $L_l' \subset N_l'''$ by definition) that $f_l(n_l) = f_{l+t}(\delta_{l+t-1}(m))$ for some $m \in M_{l+t-1}$. Since $f_l = f_{l+t} \circ f_{l,l+t}$, we see that

$$f_{l,l+t}(n_l) - \delta_{l+t-1}(m) \in \ker f_{l+t}.$$

The kernel of the restriction of f_{l+t} to $f_{l,l+t}(N_l)$ is a finitely generated R-module. Therefore, enlarging t if necessary, we may assume that the difference $f_{l,l+t}(n_l) - \delta_{l+t-1}(m)$ is zero; that is, that $f_{l,l+t}(n_l) \in \delta_{l+t-1}(M_{l+t-1})$. (This follows by the same argument given in the proof of Lemma 3.3.8.) Therefore the restriction of f''_{l+t} to $f''_{l,l+t}(N_l'')$ is injective, so the images of N_l'' under the f''_{l+t} stabilize in the strong sense, completing the proof of (b) and the lemma. $\square$

The strategy of the proof of Proposition 3.3.2 is to write the de Rham complex $M \otimes \Omega^\bullet_R$ of a holonomic $\mathscr{D}$-module M as a direct limit of complexes whose objects are finitely generated R-modules (using the degree, or order, filtration on the ring $\mathscr{D}$). Fix, once and for all, both a holonomic left $\mathscr{D}$-module M and an element $m \in M$ such that $M = \mathscr{D} \cdot m$. Let $M_0 = R \cdot m$ be the cyclic R-submodule of M generated by m; more generally, let $M_l = \mathscr{D}^l \cdot m$ for $l \geq 0$, where $\mathscr{D}^l$ denotes the R-submodule of $\mathscr{D}$ consisting of differential operators of order at most l. Note that $M = \cup_l M_l$; for any i, $\partial_i : M \to M$ induces k-linear maps $\partial_i : M_l \to M_{l+1}$ for all l; and every M_l is a finitely generated R-module (a set of R-generators is given by $\{\delta m\}$ where δ runs through the monomials in $\partial_1, \ldots, \partial_n$ of total degree at most l).

Definition 3.3.18. Let M and m be as above. For all $j \in \{0, \ldots, n\}$ and $l \in \mathbb{N}$, let $\mathscr{M}_l^{j,\bullet}$ be the complex

$$0 \to M_l \to \oplus_{1 \leq i \leq n} M_{l+1} \to \cdots \to M_{l+j} \to 0$$

whose differentials are the restrictions of those in the complex $M \otimes \Omega^\bullet_{R^j}$. We simply write $\mathscr{M}_l^\bullet$ for $\mathscr{M}_l^{n,\bullet}$.

Every $\mathscr{M}_l^{j,\bullet}$ is a complex whose objects are finitely generated R-modules ($\mathscr{M}_l^{j,i}$ is a direct sum of $\binom{j}{i}$ copies of the finitely generated R-module M_{l+i}) and whose differentials are k-linear.

We have short exact sequences of complexes

$$0 \to \mathscr{M}_{l+1}^{j,\bullet}[-1] \to \mathscr{M}_l^{j+1,\bullet} \to \mathscr{M}_l^{j,\bullet} \to 0$$

for all l, analogous to the sequence of Remark 3.3.5; the first nonzero morphism is given by $\wedge dx_{n-j}$ and, in the induced long exact cohomology sequence, the connecting homomorphisms are given by ∂_{n-j}, up to a sign.

We have $M \otimes \Omega^\bullet_{R^j} = \varinjlim \mathscr{M}_l^{j,\bullet}$, and as filtered direct limits commute with cohomology, this implies

$$h^i(M \otimes \Omega^\bullet_{R^j}) \simeq \varinjlim h^i(\mathscr{M}_l^{j,\bullet})$$

for all i. In particular, taking $j = n$, we have $H^i_{dR}(M) = \varinjlim h^i(\mathscr{M}_l^\bullet)$ as k-spaces. By Theorem 1.2.4, the left-hand side is a finite-dimensional k-space for all i. Thus, for all l, the image of $h^i(\mathscr{M}_l^\bullet)$ in $H^i_{dR}(M)$ is a finite-dimensional k-space. The key technical result in the proof of Proposition 3.3.2 is the following:

Proposition 3.3.19. *Let $M = \mathscr{D} \cdot m$ be a holonomic $\mathscr{D}$-module, and define the approximations $\mathscr{M}_l^\bullet$ to its de Rham complex as above. For all i and l, the images of $h^i(\mathscr{M}_l^\bullet)$ stabilize in the strong sense, with finite-dimensional stable image.*

Proof of Proposition 3.3.19. We may assume, after possibly making a change of variables as in Proposition 3.3.16, that for all i and j, $h^i(M \otimes \Omega^\bullet_{R^j})$ is a holonomic $\mathscr{D}_{n-j}$-module which is x_{n-j}-regular. Let $(*_j)$ be the following statement: for all $i \in \{0, \dots, j\}$ and all $l \geq 0$, the images of $h^i(\mathscr{M}_l^{j,\bullet})$ stabilize in the strong sense, and the stable image is a finitely generated R_{n-j}-submodule of $h^i(M \otimes \Omega^\bullet_{R^j})$. The statement of our proposition is $(*_n)$, and we will prove $(*_j)$ for $j = 1, \dots, n$ by induction on j.

In the base case, $j = 1$, the complex $M \otimes \Omega^\bullet_{R^1}$ takes the form $0 \to M \xrightarrow{\partial_n} M \to 0$, and its l-th approximation $\mathscr{M}_l^{1,\bullet}$ takes the form $0 \to M_l \xrightarrow{\partial_n} M_{l+1} \to 0$. By hypothesis, M is x_n-regular, so we may apply Lemma 3.3.17 to the direct system $\{M_l\}$: every M_l is already a finitely generated R-module, so we simply take $N_l = M_l$ in the statement of that lemma. The statement $(*_1)$ follows at once.

Now suppose $j \geq 1$ and $(*_j)$ established. By hypothesis, the $\mathscr{D}_{n-j}$-modules $h^i(M \otimes \Omega^{\bullet}_{R^j})$ are holonomic and x_{n-j}-regular. By the inductive hypothesis $(*_j)$, the images of $h^i(\mathscr{M}_l^{j,\bullet})$ stabilize in the strong sense, and the stable image is a finitely generated R_{n-j}-module; applying this reasoning to all i at once, we see that given any l, there exists s such that, for all i, $h^i(\mathscr{M}_{l+s}^{j,\bullet})$ contains the stable image of $h^i(\mathscr{M}_l^{j,\bullet})$. (To keep the notation as simple as possible, we will not always record the dependence on i of various objects and maps. At each stage, we assume that constructions are being carried out for all i at once, and that indices large enough to work for all i have been chosen.) Let $N_l \subset h^i(\mathscr{M}_{l+s}^{j,\bullet})$ be this stable image, which we can identify with an R_{n-j}-submodule of $h^i(M \otimes \Omega^{\bullet}_{R^j})$.

For all l, we have R_{n-j-1}-linear maps $\partial_{n-j}^l : h^i(\mathscr{M}_l^{j,\bullet}) \to h^i(\mathscr{M}_{l+1}^{j,\bullet})$, which are (up to a sign) the connecting homomorphisms in the long exact cohomology sequence associated with the short exact sequence

$$0 \to \mathscr{M}_{l+1}^{j,\bullet}[-1] \to \mathscr{M}_l^{j+1,\bullet} \to \mathscr{M}_l^{j,\bullet} \to 0$$

described earlier. The direct limit $\varinjlim \partial_{n-j}^l$ is $\partial_{n-j} : h^i(M \otimes \Omega^{\bullet}_{R^j}) \to h^i(M \otimes \Omega^{\bullet}_{R^j})$. By Lemma 3.3.17, the images of $N_l \cap \ker \partial_{n-j}^{l+s}$ stabilize in the strong sense, and the stable image is a finitely generated R_{n-j-1}-submodule of $\ker \partial_{n-j}$. The image of $\ker \partial_{n-j}^l$ in $\ker \partial_{n-j}^{l+s}$ is contained in $N_l \cap \ker \partial_{n-j}^{l+s}$; consequently, enlarging s if necessary, we may assume that the images of $\ker \partial_{n-j}^l$ stabilize in the strong sense, at the $(l+s)$th stage, and the stable image is a finitely generated R_{n-j-1}-module.

We now extract more information from the long exact cohomology sequences. By the exactness, we have surjections

$$h^i(\mathscr{M}_l^{j+1,\bullet}) \twoheadrightarrow \ker \partial_{n-j}^l$$

for all l. We also have commutative diagrams

$$\begin{array}{ccc} h^i(\mathscr{M}_l^{j+1,\bullet}) & \xrightarrow[\psi_l]{} & h^i(\mathscr{M}_l^{j,\bullet}) \\ \downarrow{\scriptstyle \iota_t^{j+1}} & & \downarrow{\scriptstyle \iota_t^{j}} \\ h^i(\mathscr{M}_{l+t}^{j+1,\bullet}) & \xrightarrow[\psi_{l+t}]{} & h^i(\mathscr{M}_{l+t}^{j,\bullet}) \end{array}$$

for all l and t (where the vertical arrows are transition maps and the horizontal arrows are maps in the long exact cohomology sequences), so the previous surjections induce surjections

$$\iota_t^{j+1}(h^i(\mathscr{M}_l^{j+1,\bullet})) \twoheadrightarrow \iota_t^j(\ker \partial_{n-j}^l)$$

whose kernels we denote K_{l+t}. Consider, for any $t \geq s$, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{l+s} & \longrightarrow & \iota_s^{j+1}(h^i(\mathscr{M}_l^{j+1,\bullet})) & \longrightarrow & \iota_s^j(\ker \partial_{n-j}^l) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_{l+t} & \longrightarrow & \iota_t^{j+1}(h^i(\mathscr{M}_l^{j+1,\bullet})) & \longrightarrow & \iota_t^j(\ker \partial_{n-j}^l) & \longrightarrow & 0 \end{array}$$

with exact rows. The right vertical arrow is an isomorphism, since the images of $\ker \partial_{n-j}^l$ have already stabilized at the sth stage, and the middle vertical arrow is a surjection, since ι_t^{j+1} factors through ι_s^{j+1} by the compatibility of transition maps in a direct system. It follows by the snake lemma that the left vertical arrow is also a surjection, so for $t \geq s$, the image of K_{l+s} in K_{l+t} is all of K_{l+t}.

For any t, we have $K_{l+t} \subset \ker \psi_{l+t}$, and the right-hand side can be identified (by the exactness of the long cohomology sequence) with the cokernel of the map

$$\partial_{n-j}^{l+t} : h^{i-1}(\mathscr{M}_{l+t}^{j,\bullet}) \to h^{i-1}(\mathscr{M}_{l+t+1}^{j,\bullet}).$$

We now repeat the reasoning we gave for the kernels of ∂_{n-j} at the beginning of this proof. First, by the inductive hypothesis $(*_j)$, the images of $h^{i-1}(\mathscr{M}_{l+s+1}^{j,\bullet})$ stabilize in the strong sense, and the stable image is a finitely generated R_{n-j-1}-submodule L_{l+s+1} of $h^{i-1}(M \otimes \Omega_{R^j}^\bullet)$, realized as a submodule of $h^{i-1}(\mathscr{M}_{l+s+u+1}^{j,\bullet})$ for some u. If we consider the image of L_{l+s+1} in $\operatorname{coker} \partial_{n-j}^{l+s+u}$, Lemma 3.3.17 implies that its images, and hence the images of $\operatorname{coker} \partial_{n-j}^{l+s}$, stabilize in the strong sense, and the stable image is a finitely generated R_{n-j-1}-submodule of $\operatorname{coker} \partial_{n-j}$. This stable image occurs at the $(l+s+u+1+v)$th stage for some v: for simplicity, let $w = u+1+v$.

We have just shown that the image of $\ker \psi_{l+s} \simeq \operatorname{coker} \partial_{n-j}^{l+s}$ in $\ker \psi_{l+s+w}$ is a finitely generated R_{n-j-1}-module. This module contains the image of K_{l+s} in K_{l+s+w}, and we have already seen that this image is all of K_{l+s+w}. It follows that K_{l+s+w} is itself a finitely generated R_{n-j-1}-module. We have an exact sequence

$$0 \to K_{l+s+w} \to \iota_{s+w}^{j+1}(h^i(\mathscr{M}_l^{j+1,\bullet})) \to \iota_{s+w}^j(\ker \partial_{n-j}^l) \to 0$$

where the left and right terms are finitely generated R_{n-j-1}-modules. It follows that the middle term is also a finitely generated R_{n-j-1}-module. Therefore, the images of $h^i(\mathscr{M}_l^{j+1,\bullet})$ are eventually finitely generated R_{n-j-1}-modules; by Lemma 3.3.8, it follows that they also stabilize in the strong sense at some potentially later stage. This completes the proof of $(*_{j+1})$. By induction we conclude $(*_n)$, that is, Proposition 3.3.19. $\square$

By reducing to the case of Proposition 3.3.19, it is possible to draw the same conclusion about more general direct systems of complexes with the same limit. We record this conclusion now; the reader is warned that the following statement will not be used until section 4.2.

Corollary 3.3.20. *Let M be a holonomic $\mathscr{D}$-module. Suppose that $\{N_l^\bullet\}$ is a direct system of complexes with the following properties: the objects of the complexes are finitely generated R-modules, the differentials are k-linear, the transition maps $\lambda_{l,l+s}^\bullet$ are R-linear, $\varinjlim N_l^\bullet \simeq M \otimes \Omega_R^\bullet$ in the category of complexes of k-spaces, and for all i, the isomorphism $\varinjlim N_l^i \simeq M \otimes \Omega_R^i$ is an isomorphism of R-modules. Then for all l and i, the images of $h^i(N_l^\bullet)$ stabilize in the strong sense, with finite-dimensional stable image.*

Proof. We note first that it is harmless to assume the isomorphism $\varinjlim N_l^\bullet \simeq M \otimes \Omega_R^\bullet$ is an equality, and we do so. Fix l. For all i, the images of N_l^i stabilize in the strong sense by Lemma 3.3.8, since by assumption the N_l^i are finitely generated and the transition maps are R-linear. Choose s large enough that the transition maps $\lambda_{l,l+s}^i$ realize the stable image for all i at once. The stable images form a subcomplex $\lambda_{l,l+s}(N_l^\bullet)$ of $N_{l+s}^\bullet$ which we can identify (via the insertion map, which we denote $\lambda_{l+s}^\bullet$) with the subcomplex $\lambda_l(N_l^\bullet)$ of $M \otimes \Omega_R^\bullet$. Since every N_l^i is a finitely generated R-module and the insertion maps are R-linear, this is a subcomplex of $M \otimes \Omega_R^\bullet$ whose objects are finitely generated R-modules.

Now define $M_0 = \lambda_l^0(N_l^0)$, a finitely generated R-submodule of the holonomic $\mathscr{D}$-module M, and for all $j \geq 0$, let $M_j = \mathscr{D}^j M_0$ where $\mathscr{D}^j$ is the R-submodule of $\mathscr{D}$ consisting of differential operators of order at most j. Define the complex $\mathscr{M}_j^\bullet$ in exactly the same way as in Definition 3.3.18. Since $\lambda_l(N_l^\bullet)$ is a subcomplex of $M \otimes \Omega_R^\bullet$, we see that in fact $\lambda_l(N_l^\bullet)$ is a subcomplex of $\mathscr{M}_0^\bullet$ (and hence of $\mathscr{M}_j^\bullet$ for all $j \geq 0$). The only hypotheses on M_0 that we used in the proof of Proposition 3.3.19 were that M_0 was a finitely generated R-submodule of the holonomic $\mathscr{D}$-module M and that $M = \cup_j M_j$ (we did not need M_0 to be cyclic). In our case, since $\varinjlim N_l^\bullet = M \otimes \Omega_R^\bullet$, we also have $\varinjlim \mathscr{M}_j^\bullet = M \otimes \Omega_R^\bullet$, where this last direct limit is an ascending union of complexes. (In particular, looking at the 0th term of this complex, we see that the ascending union $\cup_j M_j$ is indeed the whole of M.) Therefore we can invoke Proposition 3.3.19 to conclude that the images of $h^i(\mathscr{M}_j^\bullet)$ stabilize in the strong sense. In particular, there exists t large enough that for all i, the images of $h^i(\mathscr{M}_0^\bullet)$ stabilize for $j \geq t$ with a finite-dimensional stable image.

The complex $\mathscr{M}_t^\bullet$ is still a complex of finitely generated R-modules, since $\mathscr{D}^t$ is a finitely generated R-submodule of $\mathscr{D}$. We now return to the stable image of the complex $N_l^\bullet$, which is $\lambda_{l,l+s}^\bullet(N_l^\bullet)$. Write s_l for s to display its dependence on l. As this argument can be carried out for any l, we obtain a direct system $\lambda_{l,l+s_l}^\bullet(N_l^\bullet)$ of the stable images of $N_l^\bullet$ as l varies. We may assume the s_l have been chosen so that $\{l+s_l\}$ is strictly increasing; then the transition maps in this direct system are the restrictions, for all pair $l \leq l'$, of $\lambda_{l+s_l,l'+s_{l'}}^\bullet$. Because the source and target complexes are complexes of stable images under the λ, these restricted transition maps are injective, and the direct system so constructed also has $M \otimes \Omega_R^\bullet$ for its direct limit. Thus $M \otimes \Omega_R^\bullet$ can be regarded as the ascending union of the complexes $\lambda_{l',l'+s_{l'}}^\bullet(N_{l'}^\bullet)$, all of which have finitely generated R-modules as objects. Any subcomplex of $M \otimes \Omega_R^\bullet$ whose objects are all finitely generated R-modules is thus a subcomplex of some $\lambda_{l',l'+s_{l'}}^\bullet(N_{l'}^\bullet)$. Let l' be so large that $\mathscr{M}_t^\bullet$ is a subcomplex of $\lambda_{l',l'+s_{l'}}^\bullet(N_{l'}^\bullet)$, and consider the composite morphism of complexes

$$N_l^\bullet \xrightarrow{\lambda_{l,l+s_l}^\bullet} N_{l,l+s_l}^\bullet \xrightarrow{\lambda_{l+s_l}^\bullet} \mathscr{M}_0^\bullet \hookrightarrow \mathscr{M}_t^\bullet \hookrightarrow \lambda_{l',l'+s_{l'}}^\bullet(N_{l'}^\bullet) \subset N_{l'+s_{l'}}^\bullet,$$

in which all morphisms but possibly the first are injections. Every step in this composite is a morphism of complexes, hence induces a morphism on cohomology. If we choose $l'' \leq l'$ such that $\mathscr{M}_0^\bullet$ is a subcomplex of $\lambda_{l'',l''+s_{l''}}^\bullet(N_{l''}^\bullet)$, the diagram

$$\begin{array}{ccc} \mathscr{M}_0^\bullet & \longrightarrow & \lambda_{l'',l''+s_{l''}}^\bullet(N_{l''}^\bullet) \\ \downarrow & & \downarrow \\ \mathscr{M}_t^\bullet & \longrightarrow & \lambda_{l',l'+s_{l'}}^\bullet(N_{l'}^\bullet) \end{array}$$

is commutative and all arrows are injections, so we can regard the vertical inclusion $\mathscr{M}_0^\bullet \hookrightarrow \mathscr{M}_t^\bullet$

as a restriction of $\lambda_{l''+s_{l''},l'+s_{l'}}$. This compatibility implies that the composite morphism above induces morphisms on cohomology through which the morphisms induced by $\lambda^i_{l,l'+s_{l'}}$ factor for all i. But we have seen already that partway through this composite morphism (at the $\mathscr{M}_0^\bullet \hookrightarrow \mathscr{M}_t^\bullet$ stage) the cohomology has attained a finite-dimensional stable image. Therefore, for all i, the images of $h^i(N_l^\bullet)$ stabilize, with finite-dimensional stable image, for $\tau \geq l' + s_{l'}$. $\square$

We now return to our original case, and prove Proposition 3.3.2 (and hence Theorem 3.3.1).

Proof of Proposition 3.3.2. For all l, the differentials in the complex $\mathscr{M}_l^\bullet$ are Σ-continuous, and therefore we can consider the Matlis dual $D(\mathscr{M}_l^\bullet)$, a complex whose ith object is the R-module $D(\mathscr{M}_l^{n-i})$ (which is Artinian, since $\mathscr{M}_l^{n-i}$ is finitely generated) and whose differentials are k-linear. By Proposition 3.1.16, the Matlis dual of this complex coincides with its k-linear dual, since for all i, $D(D(\mathscr{M}_l^{n-i})) = \mathrm{Hom}_k(D(\mathscr{M}_l^{n-i}), k)$. Together with Remark 3.1.17, this implies that for all l, the complexes $\mathscr{M}_l^\bullet$ and $(D(\mathscr{M}_l^\bullet))^*$ are naturally isomorphic as complexes of k-spaces, and we identify these complexes. Note that $\varprojlim D(\mathscr{M}_l^\bullet) \simeq D(M \otimes \Omega_R^\bullet)$ as complexes of k-spaces, by Remark 3.1.20.

For all i, the inverse system $\{D(\mathscr{M}_l^i)\}$ of R-modules satisfies the Mittag-Leffler condition, since the transition maps are surjective (they are the Matlis duals of the R-linear inclusions $\mathscr{M}_l^i \hookrightarrow \mathscr{M}_{l+1}^i$). Furthermore, the inverse system $\{h^i(D(\mathscr{M}_l^\bullet))\}$ of k-spaces also satisfies the Mittag-Leffler condition. To see this, note that for all i, we have

$$h^{n-i}(\mathscr{M}_l^\bullet) = h^{n-i}(D(\mathscr{M}_l^\bullet)^*) \simeq (h^i(D(\mathscr{M}_l^\bullet)))^*.$$

By Proposition 3.3.19, the images of $h^{n-i}(\mathscr{M}_l^\bullet)$ stabilize in the strong sense, so by Lemma 3.3.9, the original system $\{h^i(D(\mathscr{M}_l^\bullet))\}$ satisfies Mittag-Leffler. These two Mittag-Leffler conditions imply that cohomology commutes with taking the inverse limit of the system $\{D(\mathscr{M}_l^i)\}$: by [11, Prop. 13.2.3], we have isomorphisms

$$h^i(\varprojlim D(\mathscr{M}_l^\bullet)) \xrightarrow{\sim} \varprojlim h^i(D(\mathscr{M}_l^\bullet)).$$

Therefore, we have a chain of isomorphisms

$$
\begin{aligned}
h^i(M\otimes\Omega_R^\bullet) &\simeq (h^i(M\otimes\Omega_R^\bullet))^{**} && (3.1)\\
&\simeq (h^i(\varinjlim \mathscr{M}_l^\bullet))^{**} && (3.2)\\
&\simeq (\varinjlim h^i(\mathscr{M}_l^\bullet))^{**} && (3.3)\\
&\simeq (\varprojlim (h^i(\mathscr{M}_l^\bullet))^*)^* && (3.4)\\
&\simeq (\varprojlim (h^i((D(\mathscr{M}_l^\bullet))^*))^*)^* && (3.5)\\
&\simeq (\varprojlim (h^{n-i}(D(\mathscr{M}_l^\bullet)))^{**})^* && (3.6)\\
&\simeq (\varprojlim h^{n-i}(D(\mathscr{M}_l^\bullet)))^* && (3.7)\\
&\simeq (h^{n-i}(\varprojlim D(\mathscr{M}_l^\bullet)))^* && (3.8)\\
&\simeq (h^{n-i}(D(M\otimes\Omega_R^\bullet)))^*, && (3.9)
\end{aligned}
$$

where (3.1) holds because $h^i(M\otimes\Omega_R^\bullet)$ is a finite-dimensional k-space, (3.3) since $\varinjlim$ is exact and thus commutes with cohomology, (3.4) because taking k-dual converts direct limits into inverse limits, (3.5) by our identification of $\mathscr{M}_l^\bullet$ with $(D(\mathscr{M}_l^\bullet))^*$, (3.6) since k-dual is contravariant and exact, (3.7) by Lemma 3.3.10 applied to the inverse system $\{h^i(D(\mathscr{M}_l^\bullet))\}$, and (3.8) by the Mittag-Leffler condition as worked out above. Since $h^i(M\otimes\Omega_R^\bullet)$ is a finite-dimensional k-space, all of the k-spaces appearing in the chain of isomorphisms are finite-dimensional as well. Therefore $h^{n-i}(D(M\otimes\Omega_R^\bullet))$ is canonically isomorphic to its own double dual, so we can dualize the isomorphism obtained above to find $h^{n-i}(D(M\otimes\Omega_R^\bullet)) \simeq (h^i(M\otimes\Omega_R^\bullet))^*$, as desired. $\square$

3.4 An example: local cohomology of formal schemes

In this section, we describe the Matlis duals of local cohomology modules (over any local Gorenstein ring) in terms of local cohomology on a formal scheme. (A reference for this description is Ogus's thesis [29].) Specializing to the case of a complete local ring with a coefficient field, we obtain a right $\mathscr{D}$-module structure on the local cohomology of a formal scheme by applying the results of section 3.2 to dualize a left $\mathscr{D}$-module structure on ordinary local cohomology. We then independently introduce a natural left $\mathscr{D}$-module structure on the local cohomology of the formal scheme: the structure used to define its de Rham complex in [18]. The main result of this section is that in the case where the complete local ring is regular and of characteristic zero, these left and right $\mathscr{D}$-module structures are *transposes* of each other, so our theory and that of [18] are compatible.

Let $(R,\mathfrak{m})$ be a Gorenstein local ring. The key ingredient in the identification of the Matlis duals of local cohomology modules over R with local cohomology of a formal scheme is Grothendieck's *local duality theorem*:

Theorem 3.4.1. *[34, Thm. 6.3] Let $(R,\mathfrak{m})$ be a Gorenstein local ring of dimension n, E an injective hull of its residue field, and $D(M) = \operatorname{Hom}_R(M,E)$ the Matlis dual functor for R-modules M. If M is a finitely generated R-module, there are isomorphisms $H^{n-i}_{\mathfrak{m}}(M) \simeq D(\operatorname{Ext}^i_R(M,R))$, for all i, that are functorial in M.*

Remark 3.4.2. If moreover R is complete, we also have $\operatorname{Ext}^{n-i}_R(M,R) \simeq D(H^i_{\mathfrak{m}}(M))$, since the double Matlis dual and the identity functor are naturally isomorphic when restricted to the full subcategories of finitely generated or Artinian R-modules (as we have seen in Remark 3.1.17).

Now recall the definition of local cohomology as a direct limit of Ext modules: by Proposition 1.1.6, for any module M over any commutative Noetherian ring R, and any ideal $I \subset R$, we have isomorphisms $H^i_I(M) \simeq \varinjlim \operatorname{Ext}^i_R(R/I^t, M)$. Now put $M = R$, a Gorenstein local ring of dimension n with maximal ideal $\mathfrak{m}$. Taking Matlis duals (and using the fact that any contravariant Hom functor converts direct limits into inverse limits) we have isomorphisms

$$D(H^i_I(R)) \simeq D(\varinjlim \operatorname{Ext}^i_R(R/I^t,R)) \simeq \varprojlim D(\operatorname{Ext}^i_R(R/I^t,R)) \simeq \varprojlim H^{n-i}_{\mathfrak{m}}(R/I^t),$$

where in the last step we have used local duality (Theorem 3.4.1) for every t and passed to the inverse limit. If we let $X = \operatorname{Spec}(R)$ and Y the closed subscheme $V(I) \subset X$ defined by I, and if we write $\widehat{X}$ for the formal completion of Y in X (defined in section 2.2), P for its closed point, and $\mathscr{O}_{\widehat{X}}$ for its structure sheaf, then the last object in the sequence of isomorphisms above is precisely the local cohomology $H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ supported at the closed point [29, Prop. 2.2].

Therefore, for all i, the R-modules $D(H^i_I(R))$ and $H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ are isomorphic. We record this conclusion separately for future reference:

Proposition 3.4.3. *[29, Prop. 2.2] Let R be a Gorenstein local ring of dimension n with maximal ideal $\mathfrak{m}$. For all i, we have isomorphisms*

$$D(H^i_I(R)) \simeq \varprojlim H^{n-i}_{\mathfrak{m}}(R/I^t) \simeq H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$$

of R-modules, where the rightmost object is defined in the preceding paragraph.

For the remainder of the section, we identify the R-modules $D(H^i_I(R))$ and $H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ using the proposition above, suppressing any explicit mention of this isomorphism.

Now suppose that $(R, \mathfrak{m})$ is a complete local ring with coefficient field k. By Example 3.2.10, we know that if $I \subset R$ is an ideal, every $H^i_I(R)$ is a left $\mathrm{Diff}(R,k)$-module, and since R is complete, $D(H^i_I(R))$ is a right $\mathrm{Diff}(R,k)$-module. If moreover R is Gorenstein, we have $D(H^i_I(R)) = H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$, obtaining by transport of structure a right $\mathrm{Diff}(R,k)$-module structure on $H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$.

There is also a natural left $\mathrm{Diff}(R,k)$-module structure on $H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$, defined without Matlis duality: see [18, §I.7 and §III.2] and [33, §7.2]. Namely, let $d \in \mathrm{Diff}(R,k)$ be a differential operator of order j. For all $t \geq 0$, $d(I^{t+j}) \subset I^t$, and so d induces a k-linear map $d_t : R/I^{t+j} \to R/I^t$. Taking local cohomology, we obtain k-linear maps $d_t : H^{n-i}_{\mathfrak{m}}(R/I^{t+j}) \to H^{n-i}_{\mathfrak{m}}(R/I^t)$. To see this, either look at the maps induced by d_t on the Čech complexes of R/I^{t+j} and R/I^t relative to $x_1, \ldots, x_n$, whose cohomology objects are the local cohomology modules supported at $\mathfrak{m}$ by Proposition 1.1.10, or use Grothendieck's definition (Definition 1.1.7) of local cohomology as a functor on sheaves of Abelian groups on a topological space.

These maps are compatible with the projection maps $R/I^{t+j} \to R/I^t$ and so, upon passing to inverse limits, define a k-linear map $d : \varprojlim H^{n-i}_{\mathfrak{m}}(R/I^t) \to \varprojlim H^{n-i}_{\mathfrak{m}}(R/I^t)$. If γ belongs to $\varprojlim H^{n-i}_{\mathfrak{m}}(R/I^t) = H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$, we set $d \cdot \gamma = d(\gamma)$, and in this way define a left action of $\mathrm{Diff}(R,k)$ on $H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$.

Definition 3.4.4. If $(R,\mathfrak{m})$ is a complete Gorenstein local ring of dimension n with coefficient field k and $I \subset R$ is an ideal, the *Matlis dual action* of $\mathscr{D} = \mathrm{Diff}(R,k)$ on $D(H^i_I(R))$ for any i is the *right* action defined by dualizing the natural structure of left $\mathscr{D}$-module on $H^i_I(R)$, and the *inverse limit action* of $\mathscr{D}$ on $D(H^i_I(R)) = H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ is the *left* action defined in the preceding paragraph.

In the case of a characteristic-zero complete *regular* local ring $(R,\mathfrak{m})$ containing its residue field k, we can state precisely how these two $\mathscr{D}$-module structures are related: they are *transposes* of each other, as in Definition 3.2.12. This is the main result of this section.

Theorem 3.4.5. *Let k be a field of characteristic zero, let $R = k[[x_1,\cdots,x_n]]$, and let $I \subset R$ be an ideal. Denote by $\mathfrak{m}$ the maximal ideal of R, and by $\mathscr{D} = \mathrm{Diff}(R,k)$ the ring of k-linear differential operators on R. Then for all i, the Matlis dual action on $D(H^i_I(R)) = H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ is the transpose of the inverse limit action.*

Remark 3.4.6. It follows from Theorem 3.4.5 that the identification of Proposition 3.4.3 can be extended to an identification of left $\mathscr{D}$-modules, regarding $D(H^i_I(R))$ as a *left* $\mathscr{D}$-module by transposing the Matlis dual action as in Proposition 3.2.13. We can therefore identify the de Rham complexes of both sides as well.

Every element of $\mathscr{D}$ is a finite sum of terms of the form $\rho\partial_1^{a_1}\cdots\partial_n^{a_n}$ where $\rho \in R$, and Matlis duality respects composition of operators (reversing the order), so we need only check the statement of the theorem in the case of the action of a single ρ or ∂_i. There is nothing to prove in the case of an element $\rho \in R$, since the Matlis dual of the R-linear multiplication by such an element is again multiplication by ρ. Therefore, to prove Theorem 3.4.5, we need only to show that if i is fixed ($1 \le i \le n$), the Matlis dual and inverse limit actions of the partial derivative $\partial_i \in \mathscr{D}$ on $D(H^i_I(R)) = H^{n-i}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ differ by a sign (as transposing the action of ∂_i introduces a sign by Definition 3.2.12). Without loss of generality, we may assume $i = 1$ and write x for x_1, so that $R = k[[x,x_2,\ldots,x_n]]$ and $\partial = \partial_1$.

The proof of Theorem 3.4.5 is long and involved, so we first break it up into five steps, then give the idea of the proof of each step, and only afterward address the details. The steps in the proof are as follows:

1. Identify the Matlis dual of the differential operator $\partial : R \to R$, and therefore prove the theorem in the case where I is the zero ideal.

2. Prove the theorem in the case where I is a principal ideal $(f) \subset R$ generated by a regular element.

3. Prove the theorem in the case where I is a complete intersection ideal.

4. Prove the theorem in the case where I is an arbitrary ideal and $i = \mathrm{ht}(I)$ (that is, the case of the *top* local cohomology module).

5. Prove the theorem in general.

Here are the ideas for each step of the proof given above:

1. This was already done, as Example 3.2.2.

2. The derivation ∂ defines k-linear maps $R/f^{t+1} \to R/f^t$ for all t. Applying the functor $H^{n-1}_{\mathfrak{m}}$ to these maps, we get a compatible system of maps which define the inverse limit action of ∂ on $\varprojlim H^{n-1}_{\mathfrak{m}}(R/f^t) = D(H^1_{(f)}(R))$. On the other hand, $H^1_{(f)}(R) = R_f/R$ is the direct limit of the quotients R/f^t, with multiplication by f as the transition maps. It is easy to see the formula for the action of ∂ on R_f/R, using the quotient rule. We pass this quotient rule action through the isomorphism $R_f/R \simeq \varinjlim R/f^t$ in order to realize it via k-linear maps $R/f^t \to R/f^{t+1}$. The Matlis duals of these k-linear maps patch together to give the Matlis dual action of ∂ on $D(H^1_{(f)}(R))$. By using long exact sequences in local cohomology to set up a diagram involving the inverse-limit-action maps as well as the maps from step (1), and then taking the Matlis dual of this diagram, we produce both the Matlis-dual-action maps *and* the required signs.

3. This is a straightforward induction on the length of the regular sequence generating I, with step (2) as the base case. The diagrams and arguments used are direct analogues of those in step (2).

4. Given an ideal $I \subset R$ of height h, we find a regular sequence of length h in I, let J be the complete intersection ideal generated by this sequence, and reduce the result for $D(H^h_I(R))$ to the result for $D(H^h_J(R))$ which we know by step (3). The reduction involves writing both local cohomology modules as direct limits of finitely generated Ext modules.

5. With I and J as in step (4), we use a primary decomposition of J to find ideals I' and I'' such that $\mathrm{ht}(I') = h, \mathrm{ht}(I' + I'') = h+1$, $J = I' \cap I''$, and $\sqrt{I} = \sqrt{I'}$. The Mayer-Vietoris sequence of local cohomology with respect to these two ideals allows us to conclude the result for $D(H^{h+1}_I(R))$ as well. Knowing the result for two consecutive degrees (h and $h+1$) is enough to set up an induction on $\mathrm{ht}(I)$ which finishes the proof.

As is clear from the description above, the most important step of the proof is (2). It is the calculations used in this step that actually explain why the actions agree. The later steps are necessary, but tedious, technical reductions of the general case to step (2).

We now give the details.

Proof of step (2). In this step $I = (f)$ is principal and generated by a regular element. Here we need only examine $H^1_{(f)}(R)$ (resp. $\varprojlim H^{n-1}_{\mathfrak{m}}(R/f^t)$), as all other local cohomology modules vanish. By the Čech complex definition of local cohomology, $H^1_{(f)}(R)$ is the cokernel of the (injective) localization map $R \to R_f$, which we write R_f/R and can express as the direct limit $\varinjlim R/f^t$ with transition maps given by multiplication with f. Given an element $\frac{r}{f^t} \in R_f/R = H^1_{(f)}(R)$, the action of ∂ on this element is given by the quotient rule:

$$\partial(\frac{r}{f^t}) = \frac{f^t\partial(r) - tf^{t-1}\partial(f)r}{f^{2t}} = \frac{f\partial(r) - t\partial(f)r}{f^{t+1}}.$$

Viewing $H^1_{(f)}(R)$ as the direct limit $\varinjlim R/f^t$, we can describe the action of ∂ in the following way: given an element of $H^1_{(f)}(R)$, let $\gamma = r + f^tR \in R/f^t$ be a representative. Define

$$\partial_t(\gamma) = (f\partial(r) - t\partial(f)r) + f^{t+1}R \in R/f^{t+1}.$$

The image of the original cohomology class under ∂ is defined to be the element of $H^1_{(f)}(R)$ represented by $(f\partial(r) - t\partial(f)r) + f^{t+1}R$. That is, ∂ is described by passing to the direct limit of the family of k-linear maps $\partial_t : R/f^t \to R/f^{t+1}$.

On the other hand, we consider the inverse limit action. Let $t \geq 1$ be given, and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow[f^{t+1}]{} & R & \longrightarrow & R/f^{t+1} & \longrightarrow & 0 \\ & & & & \big\downarrow \partial & & \big\downarrow \overline{\partial}^t & & \\ 0 & \longrightarrow & R & \xrightarrow[f^{t}]{} & R & \longrightarrow & R/f^{t} & \longrightarrow & 0 \end{array}$$

of R-modules with R-linear (and exact) rows but merely k-linear vertical arrows. Here $\overline{\partial}^t$ is simply the map induced by ∂ by passing to quotients, that is, $\overline{\partial}^t(r + f^{t+1}R) = \partial(r) + f^tR$. Since

$$\partial(f^{t+1}r) = (t+1)f^t\partial(f)r + f^{t+1}\partial(r) = f^t((t+1)\partial(f)r + f\partial(r)),$$

we see that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow[f^{t+1}]{} & R & \longrightarrow & R/f^{t+1} & \longrightarrow & 0 \\ & & \big\downarrow \delta^t & & \big\downarrow \partial & & \big\downarrow \overline{\partial}^t & & \\ 0 & \longrightarrow & R & \xrightarrow[f^{t}]{} & R & \longrightarrow & R/f^{t} & \longrightarrow & 0 \end{array}$$

commutes if we set $\delta^t = (t+1)\partial(f) + f\partial \in \mathscr{D}$.

Both rows of the above diagram are short exact sequences of R-modules and so induce long exact sequences of local cohomology supported at $\mathfrak{m}$. Since R is a Gorenstein local ring, $H^i_{\mathfrak{m}}(R) = 0$ if $i \neq n$ and $H^n_{\mathfrak{m}}(R) = E$, the Matlis module (Proposition 1.1.4(d)). Likewise, the only non-vanishing local cohomology of R/f^{t+1} (resp. R/f^t) is $H^{n-1}_{\mathfrak{m}}(R/f^{t+1})$ (resp. $H^{n-1}_{\mathfrak{m}}(R/f^t)$) because, as f is a regular element, these quotients of R are Cohen-Macaulay local rings of dimension $n-1$. Therefore, the only nonzero portion of the resulting diagram is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{n-1}_{\mathfrak{m}}(R/f^{t+1}) & \longrightarrow & H^n_{\mathfrak{m}}(R) & \longrightarrow & H^n_{\mathfrak{m}}(R) & \longrightarrow & 0 \\ & & \big\downarrow \overline{\partial}^t & & \big\downarrow \delta^t & & \big\downarrow \partial & & \\ 0 & \longrightarrow & H^{n-1}_{\mathfrak{m}}(R/f^{t}) & \longrightarrow & H^n_{\mathfrak{m}}(R) & \longrightarrow & H^n_{\mathfrak{m}}(R) & \longrightarrow & 0 \end{array}$$

where, abusively, we have used the same symbols for the maps induced on cohomology to simplify the notation. Since $H^n_{\mathfrak{m}}(R)$ is the Matlis module E, we can rewrite the diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & (0 :_E f^{t+1}) \arrow[r] \arrow[d, "\overline{\partial}^t"] & E \arrow[r, "f^{t+1}"'] \arrow[d, "\delta^t"] & E \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & (0 :_E f^t) \arrow[r] & E \arrow[r, "f^t"'] & E \arrow[r] & 0
\end{tikzcd}$$

Now we apply Matlis duality to the entire diagram. An asterisk denotes the Matlis dual of a map, given by Definition 3.1.19. We obtain

$$\begin{tikzcd}
0 & R/f^{t+1} \arrow[l] & R \arrow[l] & R \arrow[l, "f^{t+1}"] & 0 \arrow[l] \\
0 & R/f^t \arrow[l] \arrow[u, "(\overline{\partial}^t)^*"] & R \arrow[l] \arrow[u, "(\delta^t)^*"] & R \arrow[l, "f^t"] \arrow[u, "\partial^*"] & 0 \arrow[l]
\end{tikzcd}$$

Our goal is to identify the map $(\overline{\partial}^t)^*$. Consider the map $\delta^t = (t+1)\partial(f) + f\partial$, which we can rewrite as $t\partial(f) + (\partial(f) + f\partial) = t\partial(f) + \partial f$ using the Leibniz rule. As we saw in sections 3.1 and 3.2, Matlis duality for differential operators is k-linear and respects composition in the contravariant sense. Multiplication by an element of R (for example, the element $t\partial(f)$) dualizes to again give multiplication by that element of R, and Example 3.2.2 (our step (1)) implies that $\partial^* = -\partial$, so the Matlis dual of δ^t is

$$(\delta^t)^* = (t\partial(f) + \partial f)^* = t\partial(f) + f(\partial)^* = t\partial(f) - f\partial.$$

Since the previous diagram must commute by functoriality of the Matlis dual, we see from our explicit description of $(\delta^t)^*$ that the only k-linear map $(\overline{\partial}^t)^*$ making the left square commute is defined by

$$(\overline{\partial}^t)^*(r + f^tR) = (t\partial(f) - f\partial)(r) + f^{t+1}R,$$

which is precisely the negative of the map ∂_t used to define the natural action of ∂ on $H^1_{(f)}(R) = \varinjlim R/f^t$.

Taking the Matlis dual of the equality $(\overline{\partial}^t)^* = -\partial_t$, we find

$$-(\partial_t)^* = (-\partial_t)^* = (\overline{\partial}^t)^{**} = \overline{\partial}^t,$$

where the last equality holds by Remark 3.1.17 since R is complete and the source and target of $\overline{\partial}^t$ are finitely generated R-modules. The Matlis dual action on $D(H^1_{(f)}(R)) = \varprojlim D(R/f^t)$ is defined by patching together the maps ∂_t^*, and the inverse limit action on $\varprojlim H^{n-1}_{\mathfrak{m}}(R/f^t)$ is defined by patching together the maps $\overline{\partial}^t$, so the above equalities imply (by passing to the inverse limit) that these actions differ by a sign, as claimed. □

Proof of step (3). In this case $I = (f_1, \dots, f_s)$, where $f_1, \dots, f_s$ form a regular sequence, $s > 1$, and we have assumed the result for smaller values of s. We begin with some observations. First, it suffices to examine the local cohomology modules $H^s_I(R)$ (resp. $\varprojlim H^{n-s}_{\mathfrak{m}}(R/I^t)$) since all other local cohomology modules vanish. Second, $f_1^t, \dots, f_s^t$ for every $t \geq 1$ is again a regular sequence [1, Thm. 18.1]. Third, if we denote by $I^{[t]}$ the ideal generated by the regular sequence $f_1^t, \dots, f_s^t$, the families $\{I^t\}$ and $\{I^{[t]}\}$ are cofinal with each other, so $\varprojlim H^{n-s}_{\mathfrak{m}}(R/I^t) \simeq \varprojlim H^{n-s}_{\mathfrak{m}}(R/I^{[t]})$. Fourth, by the Leibniz rule, $\partial(I^{[t+1]}) \subset I^{[t]}$ for all t.

We first work out the action of ∂ on the local cohomology module $H^s_I(R)$. To see this action and how to dualize it, it will be useful to consider two of the equivalent definitions of local cohomology: the Čech complex definition (where it is easy to see the formula for the action) and the definition as a direct limit of Ext modules (Proposition 1.1.6, where we can dualize one finitely generated piece at a time). Consider the Čech complex on $f_1, \dots, f_s$:

$$0 \to R \to \oplus_i R_{f_i} \to \oplus_{i<j} R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_s} \to 0$$

where the components of the differentials are either the natural localization maps (modified by appropriate signs) or zero; its cohomology objects are the local cohomology modules $H^i_I(R)$. In our case, the only module to study is the sth one. Given an element $\frac{r}{(f_1 \cdots f_s)^t} \in R_{f_1 \cdots f_s}$, the derivation ∂ acts on it via the quotient rule:

$$\begin{aligned} \partial\left(\frac{r}{(f_1 \cdots f_s)^t}\right) &= \frac{(f_1 \cdots f_s)^t \partial(r) - t(f_1 \cdots f_s)^{t-1} \partial(f_1 \cdots f_s) r}{(f_1 \cdots f_s)^{2t}} \\ &= \frac{f_1 \cdots f_s \partial(r) - t \partial(f_1 \cdots f_s) r}{(f_1 \cdots f_s)^{t+1}}. \end{aligned}$$

Now we note that for all t, the Koszul complex on R relative to $f_1^t,\dots,f_s^t$ is a projective resolution of R [1, Thm. 16.5(i)], so $\mathrm{Ext}^s_R(R/I^{[t]},R)\simeq R/I^{[t]}$. By cofinality,

$$H^s_I(R)\simeq \varinjlim \mathrm{Ext}^s_R(R/I^{[t]},R)\simeq \varinjlim R/I^{[t]}$$

where the transition maps are multiplication with $f_1\cdots f_s$. Viewing $H^s_I(R)$ as this direct limit, we can describe the action of ∂ in the following way: given an element of $H^s_I(R)$, let $\gamma = r+I^{[t]}\in R/I^{[t]}$ be a representative. Define

$$\partial_t(\gamma) = (f_1\cdots f_s\partial(r) - t\partial(f_1\cdots f_s)r) + I^{[t+1]}\in R/I^{[t+1]}.$$

The image of the original cohomology class under ∂ is defined to be the element of $H^s_I(R)$ represented by $(f_1\cdots f_s\partial(r)-t\partial(f_1\cdots f_s)r)+I^{[t+1]}\in R/I^{[t+1]}$. That is, ∂ is described by passing to the direct limit of the family of k-linear maps $\partial_t : R/I^{[t]}\to R/I^{[t+1]}$.

As in step (2), we examine the effect of ∂ on the local cohomology modules supported at $\mathfrak{m}$, pass to the inverse limit, and then compare the resulting formula with the previous "quotient rule" formula.

We write J for the ideal generated by $(f_1,\dots,f_{s-1})$. Since by hypothesis f_s^t (resp. f_s^{t+1}) is $R/J^{[t]}$-regular (resp. $R/J^{[t+1]}$-regular), we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & R/J^{[t+1]} \arrow[r, "f_s^{t+1}"'] & R/J^{[t+1]} \arrow[r] \arrow[d, "\partial_J^t"] & R/I^{[t+1]} \arrow[r] \arrow[d, "\partial_I^t"] & 0 \\
0 \arrow[r] & R/J^{[t]} \arrow[r, "f_s^{t}"'] & R/J^{[t]} \arrow[r] & R/I^{[t]} \arrow[r] & 0
\end{tikzcd}$$

where both ∂_J^t and ∂_I^t are k-linear maps induced by ∂ on the corresponding quotients of R. As in step (2), we calculate (for $r\in R$)

$$\partial_J^t(f_s^{t+1}(r+J^{[t+1]})) = ((t+1)f_s^t\partial(f_s)r + f_s^{t+1}\partial(r)) + J^{[t]} = f_s^t(((t+1)\partial(f_s)+f_s\circ\partial)(r)) + J^{[t]}$$

so that if we set $\delta_J^t = (t+1)\partial(f_s)+f_s\partial = t\partial(f_s)+\partial f_s\in\mathscr{D}$, we have a completed commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R/J^{[t+1]} \arrow[r, "f_s^{t+1}"'] \arrow[d, "\delta_J^t"] & R/J^{[t+1]} \arrow[r] \arrow[d, "\partial_J^t"] & R/I^{[t+1]} \arrow[r] \arrow[d, "\partial_I^t"] & 0 \\ 0 \arrow[r] & R/J^{[t]} \arrow[r, "f_s^t"'] & R/J^{[t]} \arrow[r] & R/I^{[t]} \arrow[r] & 0 \end{tikzcd}$$

Both rows of the above diagram are short exact sequences of R-modules and so induces a long exact sequence of local cohomology supported at $\mathfrak{m}$. All the quotients of R that appear are quotients by regular sequences, hence Cohen-Macaulay local rings, and so have only one nonzero local cohomology module supported at $\mathfrak{m}$. The relevant portion of the diagram is

$$\begin{tikzcd} 0 \arrow[r] & H_{\mathfrak{m}}^{n-s}(R/I^{[t+1]}) \arrow[r] \arrow[d, "\partial_I^t"] & H_{\mathfrak{m}}^{n-s+1}(R/J^{[t+1]}) \arrow[r] \arrow[d, "\delta_J^t"] & H_{\mathfrak{m}}^{n-s+1}(R/J^{[t+1]}) \arrow[r] \arrow[d, "\partial_J^t"] & 0 \\ 0 \arrow[r] & H_{\mathfrak{m}}^{n-s}(R/I^{[t]}) \arrow[r] & H_{\mathfrak{m}}^{n-s+1}(R/J^{[t]}) \arrow[r] & H_{\mathfrak{m}}^{n-s+1}(R/J^{[t]}) \arrow[r] & 0 \end{tikzcd}$$

If E is the Matlis module for R, this diagram takes the form

$$\begin{tikzcd} 0 \arrow[r] & (0 :_E I^{[t+1]}) \arrow[r] \arrow[d, "\partial_I^t"] & (0 :_E J^{[t+1]}) \arrow[r, "f_s^{t+1}"'] \arrow[d, "\delta_J^t"] & (0 :_E J^{[t+1]}) \arrow[r] \arrow[d, "\partial_J^t"] & 0 \\ 0 \arrow[r] & (0 :_E I^{[t]}) \arrow[r] & (0 :_E J^{[t]}) \arrow[r, "f_s^t"'] & (0 :_E J^{[t]}) \arrow[r] & 0 \end{tikzcd}$$

Applying Matlis duality to the entire diagram, we obtain

$$\begin{tikzcd} 0 & R/I^{[t+1]} \arrow[l] & R/J^{[t+1]} \arrow[l] & R/J^{[t+1]} \arrow[l, "f_s^{t+1}"] & 0 \arrow[l] \\ 0 & R/I^{[t]} \arrow[l] \arrow[u, "(\partial_I^t)^*"] & R/J^{[t]} \arrow[l] \arrow[u, "(\delta_J^t)^*"] & R/J^{[t]} \arrow[l, "f_s^t"] \arrow[u, "(\partial_J^t)^*"] & 0 \arrow[l] \end{tikzcd}$$

Our goal is to identify the map $(\partial_I^t)^*$, which we will do by finding a formula for $(\delta_J^t)^*$ and using the fact that the diagram must commute. Recall that

$$\delta_J^t = (t+1)\partial(f_s) + f_s\partial = t\partial(f_s) + \partial f_s.$$

We consider the two summands separately. Although $t\partial(f_s)$ is simply multiplication by an element of R, we must note that its dual is a map $R/J^{[t]} \to R/J^{[t+1]}$. Since the natural map $(0 :_E$

$J^{[t+1]}) \to (0 :_E J^{[t]})$ induced on top local cohomology by the natural surjection $R/J^{[t+1]} \to R/J^{[t]}$ is multiplication by $f_1 \cdots f_{s-1}$, we see that the dual $(t\partial(f_s))^* : R/J^{[t]} \to R/J^{[t+1]}$ is defined by

$$(t\partial f_s)^*(r + J^{[t]}) = t f_1 \cdots f_{s-1} \partial(f_s) r + J^{[t+1]}.$$

Now we consider the second summand, $\partial_J^t f_s$, which we write as a composite

$$(0 :_E J^{[t+1]}) \xrightarrow{f_s} (0 :_E J^{[t+1]}) \xrightarrow{\partial_J^t} (0 :_E J^{[t]}).$$

The first map, induced on local cohomology by multiplication with f_s on the quotient $R/J^{[t+1]}$, dualizes directly to multiplication with f_s. By the induction hypothesis, since J is a complete intersection ideal with fewer than s generators, we know that the formula for $(\partial_J^t)^*$ is

$$(\partial_J^t)^*(r + J^{[t]}) = (t\partial(f_1 \cdots f_{s-1}) r - (f_1 \cdots f_{s-1})\partial(r)) + J^{[t+1]},$$

since this is the negative of the "quotient rule" formula for the action of ∂ on $H_J^{s-1}(R) \simeq \varinjlim R/J^{[t]}$, and so we see that

$$(\delta_J^t)^* = (t\partial(f_s) + \partial_J^t f_s)^* = (t\partial(f_s))^* + f_s(\partial_J^t)^*$$

has the formula

$$\begin{aligned}
(\delta_J^t)^*(r + J^{[t]}) &= t f_1 \cdots f_{s-1}\partial(f_s) r + f_s((t\partial(f_1 \cdots f_{s-1}) r - (f_1 \cdots f_{s-1})\partial(r)) + J^{[t+1]} \\
&= t(f_1 \cdots f_{s-1}\partial(f_s) + \partial(f_1 \cdots f_{s-1}) f_s) r - f_1 \cdots f_{s-1} f_s \partial(r) + J^{[t+1]} \\
&= (t\partial(f_1 \cdots f_s) - f_1 \cdots f_s \circ \partial)(r) + J^{[t+1]}.
\end{aligned}$$

From this formula and the fact that the diagram must commute by the functoriality of Matlis duality, we see that the map $(\partial_I^t)^*$ must have the formula

$$(\partial_I^t)^*(r + I^{[t]}) = (t\partial(f_1 \cdots f_s) - f_1 \cdots f_s \circ \partial)(r) + I^{[t+1]}.$$

As this is precisely the negative of the formula for the ∂_t used to define the action of ∂ on $H_I^s(R) \simeq \varinjlim R/I^{[t]}$, the result follows by the same argument used at the end of the proof of step (2). $\square$

Proof of step (4). Let $I \subset R$ be an ideal of height h. Choose a regular sequence $f_1, \dots, f_h \in I$ and denote by J the ideal generated by $f_1, \dots, f_h$; then $J \subset I$, J is a complete intersection ideal, and both R/J and R/I are local rings of dimension $n-h$. (Since R is Cohen-Macaulay, the depth of I is h and so such a sequence can be found.) Moreover, R/I^t is a local ring of dimension $n-h$ for any $t \geq 1$; therefore, whenever $t' \geq t$, the canonical projection $R/I^{t'} \to R/I^t$ induces a map

$$H^{n-h}_{\mathfrak{m}}(R/I^{t'}) \to H^{n-h}_{\mathfrak{m}}(R/I^t),$$

which is surjective because $H^{n-h}_{\mathfrak{m}}$ is right-exact on R-modules of dimension $n-h$. By the local duality theorem, the Matlis dual of this map is an *injective* map

$$\operatorname{Ext}^h_R(R/I^t, R) \to \operatorname{Ext}^h_R(R/I^{t'}, R).$$

Since this holds for every pair of indices $t' \geq t$, and since $H^h_I(R) = \varinjlim \operatorname{Ext}^h_R(R/I^t, R)$, we see that in this case the direct limit can be regarded as an ascending union: every $\operatorname{Ext}^h_R(R/I^t, R)$ injects into $H^h_I(R)$, and every finitely generated R-submodule of $H^h_I(R)$ is contained in one of the $\operatorname{Ext}^h_R(R/I^t, R)$ (we identify the $\operatorname{Ext}^h_R(R/I^t, R)$ with submodules of $H^h_I(R)$).

We now repeat this reasoning with the family $\{J^{[t]}\}$, where again $J^{[t]}$ is the ideal generated by $f_1^t, \dots, f_s^t$; since $H^h_J(R) \simeq \varinjlim \operatorname{Ext}^h_R(R/J^{[t]}, R)$, where we can again regard the $\operatorname{Ext}^h_R(R/J^{[t]}, R)$ as submodules of $H^h_J(R)$ and the direct limit as an ascending union, every finitely generated submodule of $H^h_J(R)$ is contained in one of the $\operatorname{Ext}^h_R(R/J^{[t]}, R)$. Moreover, by the Leibniz rule, the image under ∂ of a finitely generated R-submodule of $H^h_J(R)$ is contained in a finitely generated R-submodule of $H^h_J(R)$. Consequently, given any $t \geq 1$, there exists t' such that $\partial(\operatorname{Ext}^h_R(R/I^t, R)) \subset \operatorname{Ext}^h_R(R/I^{t'}, R)$ and $\partial(\operatorname{Ext}^h_R(R/J^{[t]}, R)) \subset \operatorname{Ext}^h_R(R/J^{[t']}, R)$, giving us a diagram (with injective R-linear horizontal maps and k-linear vertical maps)

$$\begin{array}{ccc}
\operatorname{Ext}^h_R(R/I^t, R) & \longrightarrow & \operatorname{Ext}^h_R(R/J^{[t]}, R) \\
\Big\downarrow \partial & & \Big\downarrow \partial \\
\operatorname{Ext}^h_R(R/I^{t'}, R) & \longrightarrow & \operatorname{Ext}^h_R(R/J^{[t']}, R).
\end{array}$$

Taking the Matlis dual of this entire diagram, we get a commutative diagram

$$\begin{array}{ccc} H^{n-h}_{\mathfrak{m}}(R/I^t) & \longleftarrow & H^{n-h}_{\mathfrak{m}}(R/J^{[t]}) \\ \uparrow{\scriptstyle \partial^*} & & \uparrow{\scriptstyle \partial^*} \\ H^{n-h}_{\mathfrak{m}}(R/I^{t'}) & \longleftarrow & H^{n-h}_{\mathfrak{m}}(R/J^{[t']}), \end{array}$$

where the horizontal arrows are surjections.

On the other hand, we can begin with the diagram

$$\begin{array}{ccc} R/J^{[t']} & \longrightarrow & R/I^{t'} \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ R/J^{[t]} & \longrightarrow & R/I^t \end{array}$$

where the vertical arrows are induced by $\partial : R \to R$, and take local cohomology, obtaining a commutative diagram

$$\begin{array}{ccc} H^{n-h}_{\mathfrak{m}}(R/J^{[t']}) & \longrightarrow & H^{n-h}_{\mathfrak{m}}(R/I^{t'}) \\ \downarrow & & \downarrow \\ H^{n-h}_{\mathfrak{m}}(R/J^{[t]}) & \longrightarrow & H^{n-h}_{\mathfrak{m}}(R/I^t) \end{array}$$

The (surjective) horizontal arrows in this diagram are the same as the horizontal arrows in the diagram above. By step (3), the left vertical arrow in this diagram is precisely the negative of the right vertical arrow in the previous diagram. Therefore, due to the surjectivity, the *right* vertical arrow in this diagram must be precisely the negative of the *left* vertical arrow in the previous diagram. Since this holds for arbitrary t, we may pass to the inverse limit. We conclude that the Matlis dual action and the inverse limit action of ∂ on $D(H^h_I(R)) = H^{n-h}_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ (which, by Remark 3.1.20, are defined by patching together the vertical arrows in the above diagrams as t varies) differ by a sign. □

Proof of step (5). We retain the notation I and J from step (4). Let $J = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a primary decomposition of $J = (f_1, \ldots, f_h)$. Reindexing if necessary, we may assume $I \subset \mathfrak{q}_1, \cdots, I \subset \mathfrak{q}_s, I \not\subset \mathfrak{q}_{s+1}, \cdots, I \not\subset \mathfrak{q}_r$. We may assume $s < r$, as otherwise $\sqrt{I} = \sqrt{J}$ and there is nothing left to prove. Put $I' = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ and $I'' = \mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_r$; then we have $\mathrm{ht}(I') = h, \mathrm{ht}(I' + I'') = h + 1$, $J = I' \cap I''$, and $\sqrt{I} = \sqrt{I'}$. This last equality implies that $H^i_I(R) \simeq H^i_{I'}(R)$ for all i by Proposition 1.1.4(a).

The ideals I' and I'' give rise to a Mayer-Vietoris sequence of local cohomology (Proposition 1.1.5):

$$\cdots \to H_J^{h-1}(R) \to H_{I'+I''}^h(R) \to H_{I'}^h(R) \oplus H_{I''}^h(R) \to H_J^h(R) \to H_{I'+I''}^{h+1}(R) \to \cdots$$

Since J is a complete intersection ideal of height h, $H_J^i(R) = 0$ unless $i = h$, so from the long exact sequence above we conclude that $H_{I'+I''}^{h+1}(R)$ maps surjectively to $H_{I'}^{h+1}(R) \oplus H_{I''}^{h+1}(R)$ and that $H_{I'+I''}^j(R) \simeq H_{I'}^j(R) \oplus H_{I''}^j(R)$ for $j \geq h+2$.

On the other hand, for any $t \geq 1$, there is a short exact sequence of R-modules

$$0 \to R/((I')^t \cap (I'')^t) \to R/(I')^t \oplus R/(I'')^t \to R/((I')^t + (I'')^t) \to 0$$

which gives rise to a long exact sequence of local cohomology supported at the maximal ideal $\mathfrak{m}$ (using the fact that local cohomology commutes with direct sums):

$$\begin{aligned} \cdots \to H_{\mathfrak{m}}^{n-h-1}(R/((I')^t \cap (I'')^t)) &\to H_{\mathfrak{m}}^{n-h-1}(R/(I')^t) \oplus H_{\mathfrak{m}}^{n-h-1}(R/(I'')^t) \\ \to H_{\mathfrak{m}}^{n-h-1}(R/((I')^t + (I'')^t)) &\to H_{\mathfrak{m}}^{n-h}(R/((I')^t \cap (I'')^t)) \to \cdots \end{aligned}$$

As t varies, the short exact sequences form an inverse system, and so by functoriality the corresponding long exact sequences in local cohomology form an inverse system as well. For all t, all the local cohomology modules appearing in the long exact sequence (and thus all their submodules and quotients) are Artinian by Proposition 1.1.4(c), and so the term-by-term inverse systems all satisfy the Mittag-Leffler condition. Therefore, upon passing to the inverse limit, we get a long exact sequence [11, Prop. 13.2.2]:

$$\begin{aligned} \cdots \to \varprojlim H_{\mathfrak{m}}^{n-h-1}(R/((I')^t \cap (I'')^t)) \to \varprojlim H_{\mathfrak{m}}^{n-h-1}(R/(I')^t) \oplus H_{\mathfrak{m}}^{n-h-1}(R/(I'')^t) \\ \to \varprojlim H_{\mathfrak{m}}^{n-h-1}(R/((I')^t + (I'')^t)) \to \cdots \end{aligned}$$

To make these cumbersome expressions slightly shorter, we express them in terms of formal completions. Let $X = \operatorname{Spec}(R)$ as before. Let Y (resp. Y', Y'', Y''', Z) be the closed subscheme of X defined by I (resp. $I', I'', I'+I'', J$). If W is any of the closed subschemes Y, Y', Y'', Y''', or Z, let $X_{/W}$ denote the formal completion of W in X and $\mathscr{O}_{X_{/W}}$ its structure sheaf. (See sections 1 and 4.2 for the definition of the formal completion.) In each case let P denote the closed point. We will now identify the terms of the previous long exact sequence with local cohomology on these formal completions. Since $(I'+I'')^{2t-1} \subset (I')^t + (I'')^t \subset (I'+I'')^t$ for all t, the families $\{(I')^t + (I'')^t\}$ and $\{(I'+I'')^t\}$ are cofinal with each other, so we have an isomorphism

$$\varprojlim H^j_{\mathfrak{m}}(R/((I')^t + (I'')^t)) \simeq \varprojlim H^j_{\mathfrak{m}}(R/(I'+I'')^t) = H^j_P(X_{/Y'''}, \mathscr{O}_{X_{/Y'''}}).$$

As filtered inverse limits and finite direct sums commute, the term

$$\varprojlim (H^j_{\mathfrak{m}}(R/(I')^t) \oplus H^j_{\mathfrak{m}}(R/(I'')^t))$$

is simply $H^j_P(X_{/Y'}, \mathscr{O}_{X_{/Y'}}) \oplus H^j_P(X_{/Y''}, \mathscr{O}_{X_{/Y''}})$. (In fact, we can replace Y' with Y in the first direct summand, since I and I' agree up to radical and thus define the same closed subscheme.) Finally, we consider $\varprojlim H^j_{\mathfrak{m}}(R/((I')^t \cap (I'')^t))$. Using the Artin-Rees lemma, it can be shown that for all t there exists an integer $q(t)$ such that

$$(I')^{q(t)} \cap (I'')^{q(t)} \subset (I' \cap I'')^t = J^t$$

(see the proof of [4, Corollary 3.1.5]). Since the $q(t)$ can be chosen to increase with t, and clearly $(I' \cap I'')^t \subset (I')^t \cap (I'')^t$ for all t, we see that the families $\{(I')^t \cap (I'')^t\}$ and $\{J^t\}$ are cofinal with each other, so we have an isomorphism

$$\varprojlim H^j_{\mathfrak{m}}(R/((I')^t \cap (I'')^t)) \simeq \varprojlim H^j_{\mathfrak{m}}(R/J^t) = H^j_P(X_{/Z}, \mathscr{O}_{X_{/Z}})$$

However, since J is a complete intersection ideal, R/J^t is Cohen-Macaulay for all t [1, Ex. 17.4]. Regardless of t, the dimension of R/J^t is $n-h$; the Cohen-Macaulay condition implies that $H^j_{\mathfrak{m}}(R/J^t) = 0$ unless $j = n-h$. Therefore $\varprojlim H^j_{\mathfrak{m}}(R/((I')^t \cap (I'')^t)) = 0$ as well for all $j \neq n-h$.

We can now draw some conclusions from the inverse limit long exact sequence. We see that since $H^{n-h-1}_P(X_{/Z}, \mathscr{O}_{X_{/Z}}) = 0$, there is an injection

$$H^{n-h-1}_P(X_{/Y}, \mathscr{O}_{X_{/Y}}) \oplus H^{n-h-1}_P(X_{/Y''}, \mathscr{O}_{X_{/Y''}}) \hookrightarrow H^{n-h-1}_P(X_{/Y'''}, \mathscr{O}_{X_{/Y'''}}).$$

Since all lower-degree local cohomology on $X_{/Z}$ also vanishes, the long exact sequence implies isomorphisms

$$H_P^{n-h-i}(X_{/Y}, \mathscr{O}_{X_{/Y}}) \oplus H_P^{n-h-i}(X_{/Y''}, \mathscr{O}_{X_{/Y''}}) \simeq H_P^{n-h-i}(X_{/Y'''}, \mathscr{O}_{X_{/Y'''}})$$

for all $i \geq 2$.

Consider now the local cohomology module $H_I^{h+1}(R)$ and its Matlis dual $H_P^{n-h-1}(X_{/Y}, \mathscr{O}_{X_{/Y}})$. The Matlis dual of the R-linear surjection

$$H_{I'+I''}^{h+1}(R) \to H_I^{h+1}(R) \oplus H_{I''}^{h+1}(R)$$

is an R-linear injection

$$D(H_I^{h+1}(R)) \oplus D(H_{I''}^{h+1}(R)) \hookrightarrow D(H_{I'+I''}^{h+1}(R))$$

since E is injective and so the contravariant functor D is exact.

Our construction of the Matlis dual of ∂ respects direct sums. Since the height of $I' + I''$ is $h+1$, step (4) implies that the Matlis dual action and the inverse limit action of ∂ on $D(H_{I'+I''}^{h+1}(R)) = H_P^{n-h-1}(X_{/Y'''}, \mathscr{O}_{X_{/Y'''}})$ differ by a sign; the same is therefore true of the actions upon restriction to the direct summand $D(H_I^{h+1}(R)) = H_P^{n-h-1}(X_{/Y}, \mathscr{O}_{X_{/Y}})$ of its submodule. Here we are using the fact that the inverse limit action of ∂ respects the injection $\iota : H_P^{n-h-1}(X_{/Y}, \mathscr{O}_{X_{/Y}}) \hookrightarrow H_P^{n-h-1}(X_{/Y'''}, \mathscr{O}_{X_{/Y'''}})$, which can be seen as follows: given any $t \geq 1$, we have a commutative diagram

$$\begin{array}{ccc} R/(I')^{t+1} \oplus R/(I'')^{t+1} & \longrightarrow & R/(I'+I'')^{t+1} \\ \big\downarrow \partial & & \big\downarrow \partial \\ R/(I')^{t} \oplus R/(I'')^{t} & \longrightarrow & R/(I'+I'')^{t} \end{array}$$

where the horizontal arrows are the surjective maps from the inverse system of short exact sequences (the bottom map, for example, carries $(r' + (I')^t, r'' + (I'')^t)$ to $r' - r'' + (I')^t + (I'')^t$) and the ($k$-linear) vertical maps are induced by $\partial : R \to R$. Since $H_{\mathfrak{m}}$ is a functor on the category of sheaves of Abelian groups, the diagram obtained by applying $H_{\mathfrak{m}}^{n-h-1}$ to this diagram remains commutative. The resulting vertical arrows patch together as t varies to define the inverse limit actions of ∂, and the inverse limit of the horizontal arrows is ι, so we conclude that the inverse limit action of ∂ on $H_P^{n-h-1}(X_{/Y}, \mathscr{O}_{X_{/Y}})$ is indeed the restriction of that on $H_P^{n-h-1}(X_{/Y'''}, \mathscr{O}_{X_{/Y'''}})$.

Since the result holds for arbitrary ideals in cohomological degrees equal to the height of the ideal *and* the height plus one, we have now proved enough to begin an induction on the height. Replace I by $I' + I''$. Since $\mathrm{ht}(I' + I'') + 1 = h + 2$, we can conclude that the two actions of ∂ on $D(H^{h+2}_{I'+I''}(R)) = H_P^{n-h-2}(X_{/Y'''}, \mathscr{O}_{X_{/Y'''}})$ differ by a sign, but since

$$H_P^{n-h-2}(X_{/Y}, \mathscr{O}_{X_{/Y}}) \oplus H_P^{n-h-2}(X_{/Y''}, \mathscr{O}_{X_{/Y''}}) \simeq H_P^{n-h-2}(X_{/Y'''}, \mathscr{O}_{X_{/Y'''}})$$

from the long exact sequence, the same is true for the direct summand

$$D(H_I^{h+2}(R)) = H_P^{n-h-2}(X_{/Y}, \mathscr{O}_{X_{/Y}}).$$

By induction we conclude the actions on $D(H_I^{h+i}(R))$ differ by a sign for all $i \geq 0$. We have now accounted for all potentially nonzero local cohomology modules, since R is Cohen-Macaulay and therefore the depth of I is equal to h. This completes the proof of step (5) and thus of Theorem 3.4.5. $\square$

Chapter 4

De Rham invariants of complete local rings

In this last chapter, we turn our attention to the local algebraic de Rham theory for complete local rings defined in section 2.2; in particular, we study the associated Hodge-de Rham spectral sequences. We will recall many of the definitions from section 2.2 for the reader's convenience.

The embedding-independence, finiteness, and duality results for de Rham homology and cohomology in the local setting (Proposition 2.2.8) are proved in [18] by reducing to the global case and using resolution of singularities. We are able to give a purely local proof for both the embedding-independence and the finiteness of local de Rham homology and cohomology, replacing the global methods of algebraic geometry (including resolution of singularities) by the theory of algebraic $\mathscr{D}$-modules over a formal power series ring in characteristic zero as developed in chapter 3. Along the way, we will define a new set of invariants for complete local rings, analogous to the Lyubeznik numbers. Our proof shows more than is contained in [18], namely that the entire Hodge-de Rham spectral sequences for homology and cohomology (with the exception of the first term) are embedding-independent (up to a degree shift in the homology case) and consist of finite-dimensional spaces.

There is a sketch in [18, pp. 70-71] of a failed attempt to prove that local algebraic de Rham homology and cohomology are dual using Grothendieck's local duality theorem. The ideas in this sketch can now be profitably pursued using Lyubeznik's work on the $\mathscr{D}$-module structure of local cohomology [8]. This structure allows us to speak of the de Rham complex of a local cohomology module, and knowledge of such complexes in turn enables us to better understand the early terms of the spectral sequences appearing in Hartshorne's work.

Let A be a complete local ring with coefficient field k of characteristic zero. (We view A as a k-algebra via this coefficient field.) By Cohen's structure theorem, there exists a surjection of k-algebras $\pi : R \to A$ where R is a complete regular local k-algebra, which must take the form $R = k[[x_1, \dots, x_n]]$ for some n. Let $I \subset R$ be the kernel of this surjection. We have a corresponding closed immersion $Y \hookrightarrow X$ where $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(R)$. Recall that the *de Rham homology* of the local scheme Y is defined as $H_i^{dR}(Y) = \mathbb{H}_Y^{2n-i}(X, \Omega_X^\bullet)$, the hypercohomology (supported at Y) of the complex of *continuous* differential forms on X. (The differentials in this complex are merely k-linear, so the $H_i^{dR}(Y)$ are k-spaces.) The differentials in the complex $\Omega_X^\bullet$ are $\mathscr{I}$-adically continuous (where $\mathscr{I} \subset \mathscr{O}_X$ is the sheaf of ideals defining Y) and thus pass to $\mathscr{I}$-adic completions. We obtain in this way a complex $\widehat{\Omega}_X^\bullet$ of sheaves on $\widehat{X}$, the formal completion of $\Omega_X^\bullet$, whose differentials are again merely k-linear. Recall that the *(local) de Rham cohomology* of the local scheme Y is defined as $H_{P,dR}^i(Y) = \mathbb{H}_P^i(\widehat{X}, \widehat{\Omega}_X^\bullet)$, where P is the closed point of Y.

We also recall the Hodge-de Rham spectral sequences with the de Rham homology and cohomology of Y for abutments. The *Hodge-de Rham homology spectral sequence* begins $E_1^{n-p,n-q} = H_Y^{n-q}(X, \Omega_X^{n-p})$ and has abutment $H_{p+q}^{dR}(Y)$, and the *Hodge-de Rham cohomology spectral sequence* begins $\tilde{E}_1^{p,q} = H_P^q(\widehat{X}, \widehat{\Omega}_X^p)$ and has abutment $H_{P,dR}^{p+q}(Y)$.

The remainder of this chapter will be dedicated to the proofs of the following two theorems:

Theorem 4.0.7. *Let A be a complete local ring with coefficient field k of characteristic zero. Viewing A as a k-algebra via this coefficient field, let $R \to A$ be a choice of k-algebra surjection from a complete regular local k-algebra. Associated to this surjection we have a Hodge-de Rham spectral sequence for homology as above.*

(a) Beginning with the E_2-term, the isomorphism class of the homology spectral sequence with its abutment is independent of the choice of regular k-algebra and surjection $R \to A$, up to a degree shift.

(b) The k-spaces $E_2^{p,q}$ appearing in the E_2-term of the homology spectral sequence are finite-dimensional.

The meaning of "up to a degree shift" in the statement of part (a) is the following: given two surjections $R \to A$ and $R' \to A$ from complete regular local k-algebras, where $\dim(R) = n$ and $\dim(R') = n'$, we obtain two Hodge-de Rham spectral sequences $E^{\bullet,\bullet}_{\bullet,R}$ and $\mathbf{E}^{\bullet,\bullet}_{\bullet,R'}$. Part (a) asserts that there is a morphism $E^{\bullet,\bullet}_{\bullet,R} \to \mathbf{E}^{\bullet,\bullet}_{\bullet,R'}$ of bidegree $(n'-n, n'-n)$ between these spectral sequences which is an isomorphism on the objects of the E_2- (and later) terms (see section 1.3 for the precise definitions of the terms used here).

We also have the analogue (without a degree shift) for de Rham cohomology:

Theorem 4.0.8. *Let A and R be as in Theorem 4.0.7. Associated to this surjection we also have a local Hodge-de Rham spectral sequence for cohomology.*

(a) Beginning with the E_2-term, the isomorphism class of the cohomology spectral sequence with its abutment is independent of the choice of regular k-algebra and surjection $R \to A$.

(b) The k-spaces $\tilde{E}_2^{pq}$ appearing in the E_2-term of the cohomology spectral sequence are finite-dimensional.

Our proofs of Theorems 4.0.7(a) and 4.0.8(a), while thematically similar, are independent of each other. We believe that Theorem 4.0.8(a) should follow from Theorem 4.0.7(a) via a more general relationship between the homology and cohomology spectral sequences, which we precisely state as follows:

Conjecture 4.0.9. *Let A, k, and R be as in the statement of Theorem 4.0.7. Beginning with the E_2-terms, the homology and cohomology spectral sequences are k-dual to each other: for all $r \geq 2$, and for all p and q, $E_r^{n-p,n-q} \simeq (\tilde{E}_r^{pq})^*$, and similarly for the differentials.*

At the moment, however, we are able only to prove the following weak version of this conjecture:

Theorem 4.0.10. *Let A, k, and R be as in the statement of Theorem 4.0.7. The objects in the E_2-terms of the homology and cohomology spectral sequences are k-dual to each other: for all p and q, $E_2^{n-p,n-q} \simeq (\tilde{E}_2^{pq})^*$.*

4.1 The de Rham homology of a complete local ring

We repeat our assumptions and notation. Let k be a field of characteristic zero and A be a complete local ring with coefficient field k. By Cohen's structure theorem, there exists a surjection of k-algebras $\pi : R \to A$ where $R = k[[x_1, \ldots, x_n]]$ for some n. Let $I \subset R$ be the kernel of this surjection. We have a corresponding closed immersion $Y \hookrightarrow X$ where $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(R)$. The *de Rham homology* of the local scheme Y is the local hypercohomology $H_i^{dR}(Y) = \mathbb{H}_Y^{2n-i}(X, \Omega_X^\bullet)$ where $\Omega_X^\bullet$ is the *continuous* de Rham complex of sheaves of k-spaces on X.

The de Rham homology spaces defined above are known to be independent of the choice of R and π (Proposition 2.2.8(a)) and to be finite-dimensional k-spaces (Proposition 2.2.8(b)). In this section we give arguments for the embedding-independence and the finiteness which are purely local and provide new information. The *Hodge-de Rham* spectral sequence for homology has E_1-term given by $E_1^{n-p,n-q} = H_Y^{n-q}(X, \Omega_X^{n-p})$ and abuts to $H_{p+q}^{dR}(Y)$. (When needed, we will write $\{E_{r,R}^{n-p,n-q}\}$ for this spectral sequence, recording the dependence on the base ring R.) The assertion of Theorem 4.0.7 is that, beginning with the E_2-term, this spectral sequence consists of finite-dimensional k-spaces and its isomorphism class is independent (up to a bidegree shift) of R and π; this immediately recovers the embedding-independence and finiteness for the abutment $H_*^{dR}(Y)$. To make the line of argument clearer, we give the proof first for the E_2-term only (Proposition 4.1.2), then explain the additional steps needed to make the basic strategy work for the rest of the spectral sequence.

Lemma 4.1.1. *Let the surjection $\pi : R = k[[x_1,\dots,x_n]] \to A$ (and the associated objects I,X,Y) be as above, and $\{E_r^{p,q}\}$ the corresponding Hodge-de Rham spectral sequence for homology.*

(a) For all q, $H_Y^q(X,\mathscr{O}_X) \simeq H_I^q(R)$ as R-modules; indeed, if M is any R-module and $\mathscr{F}$ the associated quasi-coherent sheaf on X, we have $H_Y^q(X,\mathscr{F}) \simeq H_I^q(M)$.

(b) For all p and q, we have

$$E_2^{p,q} \simeq H_{dR}^p(H_Y^q(X,\mathscr{O}_X)) \simeq H_{dR}^p(H_I^q(R))$$

as k-spaces, where the $\mathscr{D}$-module structure on $H_I^q(R) \simeq H_Y^q(X,\mathscr{O}_X)$ is defined as in Remark 1.2.2.

Proof. As X is affine, part (a) is Proposition 1.1.8. Now consider the E_1-term of the spectral sequence. Its differentials are horizontal and so its E_2-objects are the cohomology objects of its rows. Fix such a row, say the qth row, which takes the form $E_1^{\bullet,q} = H_Y^q(X,\Omega_X^\bullet)$. The Ω_X^p are finite free sheaves on X and local cohomology H_Y^q commutes with direct sums, so for all p, $E_1^{p,q} \simeq H_Y^q(X,\mathscr{O}_X) \otimes \Omega_R^p$. As p varies, we obtain the complex $H_Y^q(X,\mathscr{O}_X) \otimes \Omega_R^\bullet$, whose k-linear maps are the de Rham differentials of the $\mathscr{D}$-module $H_Y^q(X,\mathscr{O}_X)$. Therefore its cohomology objects, the E_2-objects of the spectral sequence, are of the stated form. □

Proposition 4.1.2. *Let the surjection $\pi : R = k[[x_1,\dots,x_n]] \to A$ (and the associated objects I,X,Y) be as above.*

(a) For all p and q, the k-space $E_2^{p,q} = H_{dR}^p(H_I^q(R))$ is finite-dimensional.

(b) Suppose we have another surjection of k-algebras $\pi' : R' = k[[x_1,\dots,x_{n'}]] \to A$ with kernel I'. Write $\{E_{r,R}^{p,q}\}$ (resp. $\{\mathbf{E}_{r,R'}^{p,q}\}$) for the Hodge-de Rham spectral sequence for homology defined using π (resp. π'). Then for all p and q, the k-spaces $E_{2,R}^{p,q}$ and $\mathbf{E}_{2,R'}^{p+n'-n,q+n-n'}$ are isomorphic. (That is, the E_2-term is independent of R and π, up to a bidegree shift.)

Proof of part (a). For all q, the $\mathscr{D}$-module $H_I^q(R)$ is holonomic [8, 2.2(d)], so its de Rham cohomology spaces are finite-dimensional by Theorem 1.2.4. This proves part (a). □

A proof of part (b) is considerably longer. We first reduce it to Lemma 4.1.3 below and then prove Lemma 4.1.3.

Write X' for the spectrum of R'. The surjection π' induces a closed immersion $Y \hookrightarrow X'$. Form the complete tensor product $R'' = R\widehat{\otimes}_k R'$ [43, V.B.2], again a complete regular k-algebra, and let $\pi'' : R'' \to A$ be the induced surjection $\pi\widehat{\otimes}_k\pi'$ of k-algebras, which gives rise to a third closed immersion $Y \hookrightarrow X'' = \mathrm{Spec}(R'')$ and a third Hodge-de Rham spectral sequence $\{E^{p,q}_{r,R''}\}$. It suffices to show that both $E^{p,q}_{2,R}$ and $\mathbf{E}^{p+n'-n,q+n-n'}_{2,R'}$ are isomorphic to $E^{p+n',q+n'}_{2,R''}$. Replacing R' by R'' and using symmetry, we reduce to the case in which the two surjections $\pi : R \to A$ and $\pi' : R' \to A$ satisfy $\pi' = \pi \circ g$ for some surjection $g : R' \to R$ of k-algebras. Let $I = \ker \pi$, $I' = \ker \pi'$, and $I'' = \ker g$, and suppose the dimensions of R and R' are n and n' respectively. As $R'/I'' \simeq R$ is regular, I'' is generated by $n'-n$ elements that form part of a regular system of parameters for R'. By induction on $n'-n$, we reduce further to the case $n'-n = 1$, since we can factor the closed immersion $X \hookrightarrow X'$ into a sequence of codimension-one immersions, and the isomorphisms on E_2-terms compose while the bidegree shifts add. Therefore we assume $\ker g$ is a principal ideal, of the form (f) where $x_1, \dots, x_n, f$ is a regular system of parameters for R'. By Cohen's structure theorem, the complete regular local k-algebra R' takes the form $k[[x_1, \dots, x_n, z]]$; making a change of variables if necessary, we may assume $f = z$. By Proposition 1.2.5, this change of variables does not affect de Rham cohomology. Thus $R = k[[x_1, \dots, x_n]]$, $R' = R[[z]]$, and g is the surjection carrying z to 0, so that $I' = IR' + (z)$. We state this special case, to which we have reduced the proposition, in the form of a lemma:

Lemma 4.1.3. *Let $R = k[[x_1, \dots, x_n]]$ and let I be an ideal of R. Let $R' = R[[z]]$ and $I' = IR' + (z)$. Then for all p and q, we have an isomorphism*

$$H^p_{dR}(H^q_I(R)) \simeq H^{p+1}_{dR}(H^{q+1}_{I'}(R'))$$

of k-spaces, where the de Rham cohomology is computed by regarding $H^q_I(R)$ as a $\mathrm{Diff}(R,k)$*-module and $H^{q+1}_{I'}(R')$ as a* $\mathrm{Diff}(R',k)$*-module.*

We first give a definition.

Definition 4.1.4. Let M be any k-space. Then $M_+ = \oplus_{i>0} M \cdot z^{-i}$, a $\mathrm{Diff}(k[[z]],k)$-module, whose elements are finite sums $\sum_i \frac{m_i}{z^i}$ where $m_i \in M$. If M is an R-module (resp. a $\mathrm{Diff}(R,k)$-module), then M_+ defined in this way is an R'-module (resp. a $\mathrm{Diff}(R',k)$-module), with ∂_z-action defined by the quotient rule: $\partial_z(\frac{m}{z^\alpha}) = \frac{-\alpha m}{z^{\alpha+1}}$.

We will frequently refer to this functor as the "+-operation" on R-modules or k-spaces. In the case of an R-module M, this definition coincides with the "key functor" $G(M) = M \otimes_R R'_z/R'$ of Núñez-Betancourt and Witt [44, §3]. A special case of one of their results will be useful for us:

Proposition 4.1.5. *[44, Lemma 3.9] With R, R', I, and I' as in the statement of Lemma 4.1.3, we have isomorphisms*

$$(H^p_I(R))_+ \simeq H^{p+1}_{I'}(R')$$

of R'-modules, for all p (these isomorphisms are functorial in R).

We need one final ingredient before giving the proof of Lemma 4.1.3: a short exact sequence relating the "full" de Rham complex $\Omega^\bullet_{R'}$ of R' with its "partial" de Rham complex $R' \otimes \Omega^\bullet_R$ defined using the derivations $\partial_1, \dots, \partial_n$ but omitting $\partial_z = \frac{\partial}{\partial z}$.

Definition 4.1.6. With R and R' as in the statement of Lemma 4.1.3, we define a short exact sequence of complexes

$$0 \to R' \otimes \Omega^\bullet_R[-1] \xrightarrow{\iota} \Omega^\bullet_{R'} \xrightarrow{\pi} R' \otimes \Omega^\bullet_R \to 0$$

where the map ι is simply the wedge product with dz, and so its image is precisely the direct sum of those summands of $\Omega^\bullet_{R'}$ with a dz wedge factor (thus π corresponds to setting $dz = 0$). The sheaf-theoretic analogue, a short exact sequence of complexes of sheaves on X', is constructed similarly.

This short exact sequence of complexes gives rise to a long exact sequence of cohomology:

$$\cdots \to h^{p-1}(R' \otimes \Omega^\bullet_R) \xrightarrow{c^{p-1}} h^p(R' \otimes \Omega^\bullet_R[-1]) \to$$
$$h^p(\Omega^\bullet_{R'}) \to h^p(R' \otimes \Omega^\bullet_R) \xrightarrow{c^p} h^{p+1}(R' \otimes \Omega^\bullet_R[-1]) \to \cdots,$$

where c denotes the connecting homomorphism. After accounting for the shift of -1, we see that c^p is a map from the $\mathrm{Diff}(k[[z]],k)$-module $h^p(R' \otimes \Omega^\bullet_R)$ to itself. We will need precisely to identify the maps c^p.

Lemma 4.1.7. *With the notation of the preceding paragraph, we have $c^p = (-1)^p \partial_z$ as maps from the $\mathrm{Diff}(k[[z]],k)$-module $h^p(R' \otimes \Omega^\bullet_R)$ to itself. (The same holds if R' is replaced by any $\mathrm{Diff}(R',k)$-module.)*

Proof. We use the explicit construction of the connecting homomorphism given, for example, in [13, Prop. 6.9]. Denote by $d^\bullet_{R'}$ the differentials in the de Rham complex $\Omega^\bullet_{R'}$. Given an element of $h^p(R' \otimes \Omega^\bullet_R)$, which is the cohomology class of some cocycle $\omega \in R' \otimes \Omega^p_R$, the image under c^p of this class is taken to be the class of the cocycle $(\iota^{p+1})^{-1}(d^p_{R'}((\pi^p)^{-1}(\omega)))$, where the superscript -1 means "choose *any* preimage": this definition is independent of all choices made. We proceed to calculate this composite, making convenient choices for the preimages.

We can write ω as a sum

$$\omega = \sum_{i_1,\dots,i_p} \rho_{i_1\cdots i_p}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

where all $\rho_{i_1\cdots i_p} \in R'$. One choice of preimage $(\pi^p)^{-1}(\omega)$ is ω itself, since none of its terms contain dz wedge factors and hence all are left fixed by π^p. Therefore $d^p_{R'}((\pi^p)^{-1}(\omega)) = d^p_{R'}(\omega)$. Since ω is a cocycle in $R' \otimes \Omega^p_R$, its image under the pth de Rham differential with respect to $dx_1,\dots,dx_n$ is zero, and so the only terms in the definition of $d^p_{R'}(\omega)$ that survive are those involving dz. That is, we have

$$d^p_{R'}(\omega) = \sum_{i_1,\dots,i_p} \partial_z(\rho_{i_1\cdots i_p})\, dz \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

which, by rearranging the wedge terms, is equal to

$$\sum_{i_1,\dots,i_p} (-1)^p \partial_z(\rho_{i_1\cdots i_p})\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dz.$$

Finally, a choice of preimage under ι^{p+1} (which is simply the map $\wedge dz$) for the above sum is

$$(\iota^{p+1})^{-1}(d^p_{R'}((\pi^p)^{-1}(\omega))) = \sum_{i_1,\dots,i_p} (-1)^p \partial_z(\rho_{i_1\cdots i_p})\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} = (-1)^p \partial_z(\omega),$$

from which the lemma follows. (The same calculation works for arbitrary $\mathrm{Diff}(R',k)$-modules M, replacing each $\rho_{i_1\cdots i_p}$ with an element $m_{i_1\cdots i_p}$ of M.) □

We can now prove Lemma 4.1.3.

Proof of Lemma 4.1.3. The differentials in the complexes of Definition 4.1.6 are merely k-linear, but in every degree p, the short exact sequence

$$0 \to R' \otimes \Omega_R^{p-1} \xrightarrow{\iota^p} \Omega_{R'}^p \to R' \otimes \Omega_R^p \to 0$$

is a *split* exact sequence of finite free R'-modules. As local cohomology commutes with direct sums, this sequence remains split exact after applying the functor $H_{I'}^q$ for any q:

$$0 \to H_{I'}^q(R' \otimes \Omega_R^{p-1}) \xrightarrow{\iota_q^p} H_{I'}^q(\Omega_{R'}^p) \to H_{I'}^q(R' \otimes \Omega_R^p) \to 0.$$

Fixing q but varying p, we obtain a short exact sequence of complexes of k-spaces

$$0 \to H_{I'}^q(R' \otimes \Omega_R^\bullet[-1]) \xrightarrow{\iota_q^\bullet} H_{I'}^q(\Omega_{R'}^\bullet) \to H_{I'}^q(R' \otimes \Omega_R^\bullet) \to 0$$

which we can rewrite (and replacing q by $q+1$) as

$$0 \to H_{I'}^{q+1}(R') \otimes \Omega_R^\bullet[-1] \xrightarrow{\iota_{q+1}^\bullet} H_{I'}^{q+1}(R') \otimes \Omega_{R'}^\bullet \to H_{I'}^{q+1}(R') \otimes \Omega_R^\bullet \to 0$$

since Ω_R^i (resp. $\Omega_{R'}^i$) is a finite free R- (resp. R'-) module. This short exact sequence of complexes gives rise to a long exact sequence of cohomology which takes the form

$$\begin{aligned}\cdots \to h^p(H_{I'}^{q+1}(R') \otimes \Omega_R^\bullet) \xrightarrow{\partial_z} h^p(H_{I'}^{q+1}(R') \otimes \Omega_R^\bullet) \xrightarrow{\bar{\iota}_{q+1}^p} \\ h^{p+1}(H_{I'}^{q+1}(R') \otimes \Omega_{R'}^\bullet) \to h^{p+1}(H_{I'}^{q+1}(R') \otimes \Omega_R^\bullet) \xrightarrow{\partial_z} h^{p+1}(H_{I'}^{q+1}(R') \otimes \Omega_R^\bullet) \to \cdots,\end{aligned}$$

where we know by Lemma 4.1.7 that, up to a sign, the connecting homomorphism is ∂_z. Now by Lemma 4.1.5, we know that $H_{I'}^{q+1}(R') \simeq (H_I^q(R))_+$ as R'-modules. The differentials in the complex $H_{I'}^{q+1}(R') \otimes \Omega_R^\bullet$ do not involve z or dz, and so the $+$-operation passes to its cohomology, since cohomology commutes with direct sums: we have

$$h^p(H_{I'}^{q+1}(R') \otimes \Omega_R^\bullet) \simeq (h^p(H_I^q(R) \otimes \Omega_R^\bullet))_+$$

as k-spaces for all p. For any k-space M, the action of ∂_z on M_+ is given in Definition 4.1.4, and it is clear from this definition (since $\mathrm{char}(k) = 0$) that $\ker(\partial_z : M_+ \to M_+) = 0$ and $\mathrm{coker}(\partial_z : M_+ \to M_+) \simeq M$, the latter corresponding to the $\frac{1}{z}$-component of M_+. Returning to the displayed portion of the long exact sequence, the second ∂_z is injective, and so by exactness the unlabeled arrow is the zero map; this implies that $\bar{\iota}^p_{q+1}$ is surjective, inducing an isomorphism between $h^{p+1}(H^{q+1}_{I'}(R') \otimes \Omega^\bullet_{R'}) = H^{p+1}_{dR}(H^{q+1}_{I'}(R'))$ and the cokernel of the first ∂_z. Since this cokernel is isomorphic to $h^p(H^q_I(R) \otimes \Omega^\bullet_R) = H^p_{dR}(H^q_I(R))$, the proof of Lemma 4.1.3, and hence of Proposition 4.1.2 (b), is complete. $\square$

Proposition 4.1.2 gives a new set of invariants for complete local rings in equicharacteristic zero, namely, the (finite) dimensions of the E_2-objects, with the bidegree shift taken into account:

Definition 4.1.8. For all $p, q \geq 0$, let $\rho_{p,q} = \dim_k(H^{n-p}_{dR}(H^{n-q}_I(R)))$.

By Proposition 4.1.2, $\rho_{p,q}$ is finite and depends only on A and a choice of coefficient field $k \subset A$. We note the similarity of the definition of the invariants $\rho_{p,q}$ to the Lyubeznik numbers $\lambda_{p,q}$ [8, Thm.-Def. 4.1], although our $\rho_{p,q}$ appear to be well-defined only in the characteristic zero case. One way to define $\lambda_{p,q}$ is as the dimension of the socle of $H^p_{\mathfrak{m}}(H^{n-q}_I(R))$, where $\mathfrak{m} \subset R$ is the maximal ideal [24, Lemma 2.2]. To define the $\rho_{p,q}$, we use de Rham cohomology instead of iterated local cohomology; furthermore, note the difference in the indices.

Remark 4.1.9. If $H^{n-q}_I(R)$ is supported only at $\mathfrak{m}$, so that $H^{n-q}_I(R) \simeq E^{\oplus \lambda_q}$ for some $\lambda_q \geq 0$ ([8, Thm. 3.4]; here E is the Matlis dualizing module), then the de Rham cohomology of $H^{n-q}_I(R)$ is easy to calculate: we have $\rho_{p,q} = \lambda_q$ if $p = 0$ and 0 otherwise. Therefore, in this case, $\rho_{p,q} = \lambda_{p,q}$ for all p and q.

We now prove the full statement of Theorem 4.0.7(a) (we have already proved part (b) above). Our goal is to construct a bidegree-shifted morphism between the Hodge-de Rham spectral sequences arising from two surjections $R \to A$ and $R' \to A$ of k-algebras which, at the level of E_2-objects, consists of the isomorphisms of Lemma 4.1.3: by Proposition 1.3.7, this is enough. The preliminary reductions given in the paragraph before Lemma 4.1.3 remain valid when considering the spectral sequences, so we need only address the case in which R, R', I, and I' are as in the statement of that lemma. The basic strategy is the same: we use the short exact sequence 4.1.6 of complexes as well as the $+$-operation, which will now be applied to double complex resolutions of the de Rham complexes $\Omega^\bullet_X$ and $\Omega^\bullet_{X'}$. Our first task is to construct these; we work first at the level of R- (resp. R'-) modules and k-linear maps, and then sheafify the results.

Lemma 4.1.10. *Let $\mathscr{I}^\bullet$ be the minimal injective resolution of R as an R-module. For all p, $\mathscr{I}^p$ has a structure of* $\mathrm{Diff}(R,k)$*-module, and $R \to \mathscr{I}^\bullet$ is a complex in the category of* $\mathrm{Diff}(R,k)$*-modules.*

Proof. We use the Cousin complex, for which [35, IV.2] is the original reference. Recall that the Cousin complex $C^\bullet(R)$ of R is constructed recursively in the following way: $C^{-2}(R) = 0$, $C^{-1}(R) = R$, and for $i \geq 0$, $C^i(R) = \oplus(\operatorname{coker} d^{i-2})_{\mathfrak{p}}$, the direct sum extending over all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{ht}\mathfrak{p} = i$. The differentials in the complex are simply the natural localization maps. It is immediate from the definition of the Cousin complex that it is a complex of $\mathrm{Diff}(R,k)$-modules, since localizations of $\mathrm{Diff}(R,k)$-modules are again $\mathrm{Diff}(R,k)$-modules and natural localization maps are $\mathrm{Diff}(R,k)$-linear [8, Example 2.1]. However, since R is a Gorenstein local ring, its minimal injective resolution and its Cousin complex coincide [45, Thm. 5.4]. □

Likewise, if we let $\mathscr{J}^\bullet$ be the minimal injective resolution of R' as an R'-module, Lemma 4.1.10 implies that $R' \to \mathscr{J}^\bullet$ is a complex in the category of $\mathrm{Diff}(R',k)$-modules. By taking finite direct sums of the resolutions $\mathscr{I}^\bullet$ and $\mathscr{J}^\bullet$, we construct three double complexes:

Definition 4.1.11. Let $\mathscr{I}^{\bullet,\bullet}$ be the double complex $\mathscr{I}^{p,q} = \mathscr{I}^q \otimes_R \Omega^p_R$ whose vertical differentials are induced by the differentials in the complex $\mathscr{I}^\bullet$ and whose horizontal differentials are those in the de Rham complexes $\mathscr{I}^q \otimes_R \Omega^\bullet_R$ of the $\mathrm{Diff}(R,k)$-modules $\mathscr{I}^q$. Similarly, let $\mathscr{J}_0^{\bullet,\bullet}$ be the double complex $\mathscr{J}_0^{p,q} = \mathscr{J}^q \otimes_R \Omega^p_R$ and let $\mathscr{J}^{\bullet,\bullet}$ be the double complex $\mathscr{J}^{p,q} = \mathscr{J}^q \otimes_{R'} \Omega^p_{R'}$.

Note that these double complexes have exact sequences of R- (or R'-) modules for columns, but merely complexes in the category of k-spaces for rows. In the case of $\mathscr{I}^{\bullet,\bullet}$, the rows are the de Rham complexes of the $\mathrm{Diff}(R,k)$-modules $\mathscr{I}^q$; in the case of $\mathscr{J}_0^{\bullet,\bullet}$, the rows are the de Rham complexes of the $\mathscr{J}^q$ regarded as $\mathrm{Diff}(R,k)$-modules; and in the case of $\mathscr{J}^{\bullet,\bullet}$, the rows are the de Rham complexes of the $\mathscr{J}^q$ regarded as $\mathrm{Diff}(R',k)$-modules. (We recall again Convention 1.2.3: if we write $\otimes\Omega_R^\bullet$, the tensor products of objects are being taken over R, but if we write $\otimes\Omega_{R'}^\bullet$, the tensor products of objects are being taken over R'.)

Each of these three double complexes can be sheafified. Consider first the double complex $\mathscr{I}^{\bullet,\bullet}$. For all p and q, let $\widetilde{\mathscr{I}^{p,q}}$ denote the associated quasi-coherent sheaf on X. The vertical differentials of $\mathscr{I}^{\bullet,\bullet}$ are R-linear, and so induce $\mathscr{O}_X$-linear morphisms between the associated sheaves, and the horizontal differentials induce k-linear morphisms on the associated sheaves in the same way that the de Rham complex of $\mathscr{O}_X$ is constructed. For all p and q, $\mathscr{I}^{p,q}$ is an injective R-module, and so the sheaf $\widetilde{\mathscr{I}^{p,q}}$ is flasque [3, Prop. III.3.4], and hence acyclic for the functor Γ_Y on the category of sheaves of k-spaces on X [34, Prop. 1.10]. Therefore we have a double complex $\widetilde{\mathscr{I}^{\bullet,\bullet}}$ whose objects are all Γ_Y-acyclic sheaves of k-spaces on X and whose columns are acyclic resolutions of the Ω_X^p (because the associated sheaf functor is exact when applied to complexes of R-modules). In the same way, we sheafify the double complexes $\mathscr{J}_0^{\bullet,\bullet}$ and $\mathscr{J}^{\bullet,\bullet}$, obtaining double complexes $\widetilde{\mathscr{J}_0^{\bullet,\bullet}}$ and $\widetilde{\mathscr{J}^{\bullet,\bullet}}$ of sheaves of k-spaces on X' which are Γ_Y-acyclic.

Definition 4.1.12. Let $E_{\bullet,R}^{\bullet;\bullet}$ be the column-filtered spectral sequence associated with the double complex $\Gamma_Y(X,\widetilde{\mathscr{I}^{\bullet,\bullet}})$ of k-spaces. Similarly, let $\mathbf{E}_{\bullet,R'}^{\bullet;\bullet}$ be the column-filtered spectral sequence associated with the double complex $\Gamma_Y(X',\widetilde{\mathscr{J}^{\bullet,\bullet}})$, and let $\mathscr{E}_\bullet^{\bullet;\bullet}$ be the column-filtered spectral sequence associated with the double complex $\Gamma_Y(X',\widetilde{\mathscr{J}_0^{\bullet,\bullet}})$.

By Lemma 1.3.8, we know that $E_{\bullet,R}^{\bullet;\bullet}$ coincides with the Hodge-de Rham spectral sequence for the complex $\Omega_X^\bullet$, and that $\mathbf{E}_{\bullet,R'}^{\bullet;\bullet}$ coincides with the Hodge-de Rham spectral sequence for the complex $\Omega_{X'}^\bullet$, so there is no ambiguity of notation. The “intermediate” spectral sequence $\mathscr{E}_\bullet^{\bullet;\bullet}$, which by Lemma 1.3.8 coincides with the hypercohomology spectral sequence for the complex

$\mathscr{O}_{X'} \otimes \Omega^\bullet_X$, will be used to relate the Hodge-de Rham spectral sequences for X and X' via the $+$-operation. The first step in this process is the following lemma:

Lemma 4.1.13. *Let R, R', I, and I' be as in the statement of Lemma 4.1.3, and let $\mathscr{I}^\bullet$ (resp. $\mathscr{J}^\bullet$) be the minimal injective resolution of R (resp. R') in the category of R-modules (resp. R'-modules) as above. Then for all q, we have an isomorphism*

$$(\Gamma_I(\mathscr{I}^{q-1}))_+ \simeq \Gamma_{I'}(\mathscr{J}^q)$$

of $\mathrm{Diff}(R',k)$*-modules.*

Proof. Fix $q \geq 0$. As R' is a Gorenstein local ring, the structure of its minimal injective resolution $\mathscr{J}^\bullet$ is well-known [1, Thm. 18.8]: $\mathscr{J}^q = \oplus_{\mathrm{ht}\,\mathfrak{p}=q} E(R'/\mathfrak{p})$, where $E(R'/\mathfrak{p})$ is the R'-injective hull of $R'/\mathfrak{p}$. In particular, $\mathscr{J}^q = 0$ for $q > n+1$ and $\mathscr{J}^{n+1}$ is the Matlis dualizing module $E_{R'}$. Applying the functor $\Gamma_{I'}$ amounts to discarding those summands corresponding to prime ideals outside the closed subscheme $V(I') \subset X'$ [4, Ex. 10.1.11]: that is, we have the equality

$$\Gamma_{I'}(\mathscr{J}^q) = \oplus_{\mathrm{ht}\,\mathfrak{p}=q, I'\subset\mathfrak{p}} E_{R'}(R'/\mathfrak{p}),$$

where again only prime ideals of height q appear in the decomposition. Since $I' = IR' + (z)$, there is a one-to-one correspondence between prime ideals $\mathfrak{p}$ of R' containing I' and prime ideals $\mathfrak{q}$ of $R = R'/(z)$ containing I (indeed, any such $\mathfrak{p}$ takes the form $\mathfrak{q}R' + (z)$). If $\mathrm{ht}\,\mathfrak{p} = q$, then $\mathrm{ht}\,\mathfrak{q} = q-1$, so we have the decomposition

$$\Gamma_{I'}(\mathscr{J}^q) = \oplus_{I\subset\mathfrak{q}\in\mathrm{Spec}(R), \mathrm{ht}\,\mathfrak{q}=q-1} E_{R'}(R'/(\mathfrak{q}R' + (z)))$$

as R'-modules. Note that we have R'-module isomorphisms

$$R'/(\mathfrak{q}R' + (z)) \simeq (R'/(z))/(((\mathfrak{q}R' + (z))/(z)) \simeq R/\mathfrak{q},$$

where $R/\mathfrak{q}$ is viewed as an R-module upon which $z \in R'$ acts trivially. But by [44, Prop. 3.11], the R'-module $E_{R'}(R'/(\mathfrak{q}R' + (z)))$ is obtained from the R-module $E_R(R/\mathfrak{q})$ by the $+$-operation. (This identification holds at the level of $\mathrm{Diff}(R',k)$-modules.) We then have isomorphisms

$$\Gamma_{I'}(\mathscr{J}^q) \simeq \oplus_{I\subset\mathfrak{q}\in\mathrm{Spec}(R), \mathrm{ht}\,\mathfrak{q}=q-1} (E_R(R/\mathfrak{q}))_+ = (\Gamma_I(\mathscr{I}^{q-1}))_+,$$

of $\mathrm{Diff}(R',k)$-modules, where we have again used [4, Ex. 10.1.11]. $\square$

Lemma 4.1.14. *Let $E_{\bullet,R}^{\bullet,\bullet}$ and $\mathscr{E}_{\bullet}^{\bullet,\bullet}$ be the spectral sequences of Definition 4.1.12. There is an isomorphism*

$$(E_{\bullet,R}^{\bullet,\bullet})_{+} \xrightarrow{\sim} \mathscr{E}_{\bullet}^{\bullet,\bullet}[(0,1)],$$

where the object on the left-hand side is obtained by applying the $+$-operation to all the objects and differentials of the spectral sequence $E_{\bullet,R}^{\bullet,\bullet}$ (this notation means that the morphism of spectral sequences has bidegree $(0,1)$, as in Definition 1.3.6).

Proof. We consider the objects of the double complexes giving rise to these spectral sequences. For all p and q, we have

$$\Gamma_Y(X', \widetilde{\mathscr{J}_0^{p,q+1}}) \simeq \Gamma_{I'}(\mathscr{J}_0^{p,q+1}) = \Gamma_{I'}(\mathscr{J}^{q+1} \otimes \Omega_R^p) \simeq \Gamma_{I'}(\mathscr{J}^{q+1}) \otimes \Omega_R^p$$

and similarly

$$(\Gamma_Y(X, \widetilde{\mathscr{I}^{p,q}}))_{+} \simeq (\Gamma_I(\mathscr{I}^{p,q}))_{+} = (\Gamma_I(\mathscr{I}^q \otimes \Omega_R^p))_{+} \simeq (\Gamma_I(\mathscr{I}^q) \otimes \Omega_R^p)_{+}$$

by Lemma 4.1.1 and the fact that $\Gamma_I = H_I^0$ and $\Gamma_{I'} = H_{I'}^0$ commute with direct sums. By Lemma 4.1.13, we have $\Gamma_{I'}(\mathscr{J}^{q+1}) \simeq (\Gamma_I(\mathscr{I}^q))_{+}$ for all q. Therefore, for all p and q, we have

$$\Gamma_{I'}(\mathscr{J}^{q+1}) \otimes \Omega_R^p \simeq (\Gamma_I(\mathscr{I}^q))_{+} \otimes \Omega_R^p \simeq (\Gamma_I(\mathscr{I}^q) \otimes \Omega_R^p)_{+},$$

so the *objects* of the double complexes are isomorphic with the indicated bidegree shift. Finally, we observe that the differentials in the complex $\Omega_R^{\bullet}$ do not involve z or dz, so the isomorphisms $(\Gamma_I(\mathscr{I}^q))_{+} \otimes \Omega_R^p \simeq (\Gamma_I(\mathscr{I}^q) \otimes \Omega_R^p)_{+}$ commute with the differentials of the double complex and thus assemble to an isomorphism of double complexes. An isomorphism of double complexes induces an isomorphism between the corresponding column-filtered spectral sequences, and the lemma follows. $\square$

We are now ready to complete the proof of Theorem 4.0.7.

Proof of Theorem 4.0.7(a). As already described, we need only prove the result in the special case of Lemma 4.1.3. We retain the notation of that lemma. Consider again the short exact sequence of Definition 4.1.6 and its sheafified version

$$0 \to \mathscr{O}_{X'} \otimes \Omega_X^\bullet[-1] \xrightarrow{\iota} \Omega_{X'}^\bullet \to \mathscr{O}_{X'} \otimes \Omega_X^\bullet \to 0.$$

As described in section 1.3, the morphism of complexes ι induces a morphism between the corresponding spectral sequences for hypercohomology supported at Y. These spectral sequences were identified as $\mathscr{E}_\bullet^{\bullet,\bullet}[(-1,0)]$ (respectively, $\mathbf{E}_{\bullet,R'}^{\bullet,\bullet}$) in the paragraph following Definition 4.1.12. Accounting for the shift of -1, we see that this induced morphism has the following form:

$$\iota_\bullet^{\bullet,\bullet} : \mathscr{E}_\bullet^{\bullet,\bullet} \to \mathbf{E}_{\bullet,R'}^{\bullet,\bullet}[(1,0)]$$

Identifying first $\mathscr{E}_\bullet^{\bullet,\bullet}[(0,1)]$ with $(E_{\bullet,R}^{\bullet,\bullet})_+$ (by Lemma 4.1.14) and then $E_{\bullet,R}^{\bullet,\bullet}$ with the $\frac{1}{z}$-component of $(E_{\bullet,R}^{\bullet,\bullet})_+$, we see that this further induces a morphism

$$\phi_\bullet^{\bullet,\bullet} : E_{\bullet,R}^{\bullet,\bullet} \to \mathbf{E}_{\bullet,R'}^{\bullet,\bullet}[(1,1)],$$

given in every degree by the inclusion of $E_{r,R}^{\bullet,\bullet}$ as the $\frac{1}{z}$-component of $(E_{r,R}^{\bullet,\bullet})_+ \simeq \mathscr{E}_r^{\bullet,\bullet}[(0,1)]$ followed by $\iota_\bullet^{\bullet,\bullet}$. If $r = 2$, the maps $\phi_2^{p,q}$ are precisely the isomorphisms $H_{dR}^p(H_I^q(R)) \xrightarrow{\sim} H_{dR}^{p+1}(H_{I'}^{q+1}(R'))$ appearing in the proof of Lemma 4.1.3, which were induced by the morphism of complexes ι and the inclusion of $E_{2,R}^{p,q} = H_{dR}^p(H_I^q(R))$ as the $\frac{1}{z}$-component of $(E_{2,R}^{p,q})_+$. Therefore the morphism $\phi_\bullet^{\bullet,\bullet}$ of spectral sequences is an isomorphism at the E_2-level. By Proposition 1.3.7, it follows that ϕ is an isomorphism at all later levels, including the abutments. The proof is complete. □

Remark 4.1.15. *A priori*, the isomorphism class of the spectral sequence $\{E_{r,R}^{p,q}\}$ and the integers $\rho_{p,q}$ of Definition 4.1.8 depend on the choice of coefficient field $k \subset A$. It is an open problem whether they are independent of this choice. In the case where the local cohomology modules $H_I^q(R)$ are supported only at $\mathfrak{m}$, they are indeed independent of this choice by Remark 4.1.9 (since the $\lambda_{p,q}$ are known to be independent), which provides supporting evidence for a positive answer to this open question.

4.2 The de Rham cohomology of a complete local ring

In this final section, we turn our attention to de Rham cohomology, and prove Theorems 4.0.8 and 4.0.10. We prove the latter theorem (and Theorem 4.0.8(b) as an immediate corollary) first, by synthesizing our results from the previous chapter on Matlis duality and local cohomology of formal schemes. The proof of Theorem 4.0.8(a), which we give next, proceeds similarly to that of Theorem 4.0.7(a). As in section 4.1, we will first give the proof of Theorem 4.0.8(a) for the E_2-terms of the spectral sequences, to bring out the main ideas of the proof. There is a technical complication that will make the E_2 case more difficult than the E_2 case of Theorem 4.0.7(a). (We will need to verify various Mittag-Leffler hypotheses for inverse systems, for which Proposition 3.3.19 will be necessary.)

Our notation is the same as in section 4.1. Thus A is again a complete local ring with a coefficient field k of characteristic zero and $\pi : R \to A$ is a surjection of k-algebras with $R = k[[x_1, \dots, x_n]]$ for some n. The surjection π induces a closed immersion of spectra $\operatorname{Spec}(A) = Y \hookrightarrow X = \operatorname{Spec}(R)$, defined by the coherent sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ associated with the ideal $I = \ker \pi \subset R$. Let $\widehat{X}$ be the formal completion of Y in X, and let $\widehat{\Omega}^\bullet_X$ be the formal completion of the (continuous) de Rham complex $\Omega^\bullet_X$. Since X is the spectrum of a complete regular local ring and so the sheaves Ω^i_X are finite free $\mathscr{O}_X$-modules, we have an alternative description of the complex $\widehat{\Omega}^\bullet_X$. As formal completion commutes with finite direct sums, the sheaf $\widehat{\Omega}^i_X$ is a direct sum of copies of $\mathscr{O}_{\widehat{X}}$. All of the derivations $\partial_{x_j} : R \to R$ induce $\mathscr{I}$-adically continuous maps $\mathscr{O}_{\widehat{X}} \to \mathscr{O}_{\widehat{X}}$, and if we form the de Rham complex of $\mathscr{O}_{\widehat{X}}$ with respect to these derivations, we recover precisely the complex $\widehat{\Omega}^\bullet_X$.

The *(local) de Rham cohomology* of the local scheme Y is defined as $H^i_{P,dR}(Y) = \mathbb{H}^i_P(\widehat{X}, \widehat{\Omega}^\bullet_X))$, the hypercohomology (supported at the closed point P of Y) of the completed de Rham complex. These k-spaces are known to be finite-dimensional and independent of the choice of R and π, just as in the case of homology. The Hodge-de Rham spectral sequence for cohomology begins $\tilde{E}_1^{p,q} = H^q_P(\widehat{X}, \widehat{\Omega}^p_X)$ and abuts to $H^{p+q}_{P,dR}(Y)$. We also recall from section 4.1 that the Hodge-de Rham spectral sequence for homology has E_1-term given by $E_1^{n-p,n-q} = H^{n-q}_Y(X, \Omega^{n-p}_X)$

and abuts to $H^{dR}_{p+q}(Y)$. The assertion of Theorem 4.0.10 is that, for all p and q, the k-spaces $E_2^{n-p,n-q}$ and $\tilde{E}_2^{p,q}$ are dual. Since we know by Theorem 4.0.7(b) that the former k-space is finite-dimensional, this duality implies that the latter is as well, which will prove Theorem 4.0.8(b). Our work in sections 3.3 and 3.4 is nearly enough to establish this duality; all that remains is to identify the rows of the E_1-term of the *cohomology* spectral sequence with de Rham complexes of $\mathscr{D}$-modules.

Lemma 4.2.1. *Let the surjection $\pi : R = k[[x_1,\dots,x_n]] \to A$ and the associated objects I, X, $\widehat{X}$, and Y be as above, and let $\{\tilde{E}_r^{p,q}\}$ the corresponding Hodge-de Rham spectral sequence for cohomology. For all q, the qth row of the term $\{\tilde{E}_1^{\bullet,q}\}$ is isomorphic, as a complex of k-spaces, to the de Rham complex $H^q_P(\widehat{X},\mathscr{O}_{\widehat{X}})\otimes\Omega^\bullet_R$, where the left* Diff$(R,k)$*-structure on $H^q_P(\widehat{X},\mathscr{O}_{\widehat{X}})$ is the* inverse limit action *of Definition 3.4.4.*

Proof. Let q be fixed. Both formal completion along Y and local cohomology H^q_P commute with finite direct sums, so for all p, $\tilde{E}_1^{p,q} = H^q_P(\widehat{X},\widehat{\Omega}^p_X)$ is a direct sum of copies of the R-module $H^q_P(\widehat{X},\mathscr{O}_{\widehat{X}})$, and the complex $\{\tilde{E}_1^{\bullet,q}\}$ has differentials induced by the differentials in the complex $\Omega^\bullet_X$ by first passing to $\mathscr{I}$-adic completions and then applying the functor H^q_P. This is exactly the de Rham complex of $H^q_P(\widehat{X},\mathscr{O}_{\widehat{X}})$ with respect to the inverse limit action, since by Proposition 3.4.3 we have an isomorphism $H^q_P(\widehat{X},\mathscr{O}_{\widehat{X}}) \simeq \varprojlim H^q_P(\widehat{X},\mathscr{O}_X/\mathscr{I}^l)$. □

Proof of Theorems 4.0.10 and 4.0.8(b). Fix q. By Lemma 4.2.1, the qth row of the E_1-term of the cohomology spectral sequence, $\tilde{E}_1^{\bullet,q}$, is the de Rham complex of the left Diff(R,k)-module $H^q_P(\widehat{X},\mathscr{O}_{\widehat{X}})$. By Theorem 3.4.5 and Remark 3.4.6, this complex is isomorphic (as a complex of k-spaces) to the de Rham complex of the left Diff(R,k)-module $D(H_I^{n-q}(R))$. We obtain the E_2-term by taking cohomology, so for all p, $\tilde{E}_2^{p,q} = H^p_{dR}(D(H_I^{n-q}(R)))$. Since $H_I^{n-q}(R)$ is a holonomic Diff(R,k)-module, Theorem 3.3.1 applies: we have an isomorphism

$$H^p_{dR}(D(H_I^{n-q}(R))) \simeq (H^{n-p}_{dR}(H_I^{n-q}(R)))^*.$$

The right-hand side of this isomorphism is nothing but $(E_2^{n-p,n-q})^*$ (by Lemma 4.1.1), completing the proof. □

We will now begin working toward the proof of Theorem 4.0.8(a). The reductions immediately preceding Lemma 4.1.3 are equally valid here, so for the remainder of the section, we assume that $R = k[[x_1,\dots,x_n]]$ and $R' = R[[z]]$, and it suffices to compare the Hodge-de Rham spectral sequences for cohomology corresponding to an arbitrary ideal $I \subset R$ (defining a closed immersion $Y = \mathrm{Spec}(R/I) \hookrightarrow X = \mathrm{Spec}(R)$ and the ideal $I' = IR' + (z) \subset R'$ (defining a closed immersion $Y \hookrightarrow X' = \mathrm{Spec}(R')$).

In the proof of Theorem 4.0.7(a), we made use of the $+$-operation on k-spaces. Its replacement in this section is the following operation:

Definition 4.2.2. Let M be any k-space. We define $M^+ = M[[z]]$, the $\mathrm{Diff}(k[[z]],k)$-module of formal power series with coefficients in M. If M is an R-module (resp. a $\mathrm{Diff}(R,k)$-module), then M^+ defined in this way is an R'-module (resp. a $\mathrm{Diff}(R',k)$-module), with ∂_z-action defined by $\partial_z(\sum m_l z^l) = \sum (l+1)m_{l+1}z^l$.

If M is an R-module, the R'-module M^+ is always z-adically complete. What is more, k-linear maps between k-spaces and R-linear maps between R-modules extend to the corresponding formal power series objects: if $f : M \to N$ is a k-linear map between k-spaces (or an R-linear map between R-modules), $f^+ : M^+ \to N^+$ is defined by $f^+(\sum m_l z^l) = \sum f(m_l)z^l$.

Remark 4.2.3. If M is an R-module, the R'-module $M^+ = M[[z]]$ is usually not isomorphic to $M \otimes_R R'$. This is an example of the failure of inverse limits to commute with tensor products. We do have

$$M^+ \simeq \varprojlim M[[z]]/z^l \simeq \varprojlim (M \otimes_R R'/z^l),$$

where every $M \otimes_R R'/z^l$ is regarded as an R'-module via the projection $R' \to R'/z^l$, but this inverse limit need not be isomorphic to $M \otimes_R (\varprojlim R'/z^l)$.

In the proofs below, we will often take two inverse limits simultaneously in the process of forming the module M^+. The general principle is the following:

Lemma 4.2.4. *Let $\{M_l\}$ be an inverse system of R-modules, indexed by $\mathbb{N}$, and let $M = \varprojlim M_l$. Then $\varprojlim (M_l \otimes_R R'/z^l) \simeq M^+$ as R'-modules.*

Proof. As an R'-module, $M^+ \simeq \varprojlim_l((\varprojlim_s M_s) \otimes_R R'/z^l)$. Consider the inverse system $\{M_s \otimes_R R'/z^l\}$, indexed by $\mathbb{N} \times \mathbb{N}$ where $(s,l) \leq (s',l')$ if and only if $s \leq s'$ and $l \leq l'$. Then

$$\varprojlim_l((\varprojlim_s M_s) \otimes_R R'/z^l) \simeq \varprojlim_{(s,l)} M_s \otimes_R R'/z^l.$$

As the "diagonal" inverse system $\{M_l \otimes_R R'/z^l\}$ is cofinal with $\{M_s \otimes_R R'/z^l\}$, their inverse limits are isomorphic, as desired. □

We have defined the formal power series operation both for k-spaces and for R-modules. In what follows, we will frequently apply the operation to an entire complex whose objects are R-modules but whose differentials are merely k-linear. We will still (abusively) use the notation $\otimes R'/z^l$ for the lth truncation of the formal power series operation, a convention which we record here:

Definition 4.2.5. Let $C^\bullet$ be a complex whose objects are R-modules and whose differentials are merely k-linear. The complex $C^\bullet \otimes R'/z^l$ is the direct sum of l copies of $C^\bullet$, indexed by z^i for $i = 0,\ldots,l-1$.

Note that the complex $(C^\bullet)^+$, obtained by applying the formal power series operation to all objects and differentials of $C^\bullet$, is the inverse limit (in the category of complexes of k-spaces) of $C^\bullet \otimes R'/z^l$.

Remark 4.2.6. We note one important difference between the formal power series operation defined above and the $+$-operation of section 4.1. The $+$-operation is defined using an infinite direct sum. Therefore, the question of whether it commutes with cohomology reduces to the question of whether the underlying category satisfies Grothendieck's axiom $AB4$ [46], that is, whether direct sums are exact. This is true for the categories of R-modules, of k-spaces, and of sheaves of Abelian groups on a topological space [2, p. 80]. By contrast, the formal power series operation is defined using an infinite direct product, which commutes with cohomology if the underlying category satisfies axiom $AB4^*$. This is true for the categories of R-modules and k-spaces but *not* sheaves [2, p. 80].

The main technical preliminary result we need concerns the interaction of the formal power series operation with the de Rham complexes of local cohomology modules on formal schemes. For reference, we repeat in the statement of this proposition the notation we will use for the remainder of this section.

Lemma 4.2.7. *Let $R = k[[x_1,\dots,x_n]]$ and let I be an ideal of R. Let $R' = R[[z]]$ and $I' = IR' + (z)$. Let $\mathscr{I}$ (resp. $\mathscr{I}'$) be the associated sheaf of ideals of $\mathscr{O}_X$ (resp. $\mathscr{O}_{X'}$) where $X = \operatorname{Spec}(R)$ and $X' = \operatorname{Spec}(R')$. Let Y be the closed subscheme of X defined by $\mathscr{I}$; via the natural closed immersion $X \hookrightarrow X'$, we identify Y with the closed subscheme of X' defined by $\mathscr{I}'$, which we also denote Y. Let $P \in Y$ be the closed point. Then for all q, we have*

$$H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'}) \otimes \Omega^\bullet_R \simeq (H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}}) \otimes \Omega^\bullet_R)^+$$

as complexes of k-spaces, where the $\operatorname{Diff}(R,k)$*-structure on both local cohomology modules is the* inverse limit action *of Definition 3.4.4.*

The proof of Lemma 4.2.7 involves several ideas, so we begin with a lemma focusing on a single local cohomology module, with no reference to its de Rham complex. We will appeal below not only to this lemma but also to its proof.

Lemma 4.2.8. *All notation is the same as in Lemma 4.2.7. For all q, we have*

$$H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'}) \simeq (H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}}))^+$$

as R'-modules.

Proof. For all l, let J_l be the ideal $I^l R' + (z^l)$. The families $\{J_l\}$ and $\{(I')^l\}$ of ideals of R' are cofinal, and we have isomorphisms

$$R'/J_l = R'/(I^l R' + (z^l)) \simeq R/I^l \otimes_R R'/z^l$$

as R'-modules for all l (the R'-module structure on $R/I^l \otimes_R R'/z^l$ being defined via the canonical projection $R' \to R'/z^l$).

Denote by $\mathfrak{n}$ (resp. $\mathfrak{m}$) the maximal ideal of R' (resp. R). For all q, we have isomorphisms

$$H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'}) \simeq \varprojlim H^q_{\mathfrak{n}}(R'/(I')^l) \simeq \varprojlim H^q_{\mathfrak{n}}(R'/J_l),$$

the first isomorphism by Proposition 3.4.3 and the second by the cofinality of $\{J_l\}$ and $\{(I')^l\}$. We saw above that $R'/J_l \simeq R/I^l \otimes_R R'/z^l$ as R'-modules, and therefore

$$H^q_{\mathfrak{n}}(R'/J_l) \simeq H^q_{\mathfrak{n}}(R/I^l \otimes_R R'/z^l)$$

as R'-modules. We claim that the right-hand side is isomorphic to $H^q_{\mathfrak{m}}(R/I^l) \otimes_R R'/z^l$ as an R'-module.

The R'-module $R/I^l \otimes_R R'/z^l$ is annihilated by a power of z, and so

$$H^i_{(z)}(R/I^l \otimes_R R'/z^l) = R/I^l \otimes_R R'/z^l$$

if $i = 0$, and is zero otherwise. The spectral sequence of Example 1.3.10 corresponding to the composite functor $\Gamma_{\mathfrak{n}} = \Gamma_{\mathfrak{m}R'} \circ \Gamma_{(z)}$ therefore degenerates at E_2, and we have isomorphisms

$$H^q_{\mathfrak{n}}(R/I^l \otimes_R R'/z^l) \simeq H^q_{\mathfrak{m}R'}(R/I^l \otimes_R R'/z^l)$$

as R'-modules. By the change of ring principle [4, Thm. 4.2.1], it does not matter whether we compute this last local cohomology module over R' or over R'/z^l, so in fact

$$H^q_{\mathfrak{m}R'}(R/I^l \otimes_R R'/z^l) \simeq H^q_{\mathfrak{m}(R'/z^l)}(R/I^l \otimes_R R'/z^l)$$

as R'/z^l-modules. Finally, since R'/z^l is flat over R, the flat base change theorem [4, Thm. 4.3.2] implies that

$$H^q_{\mathfrak{m}(R'/z^l)}(R/I^l \otimes_R R'/z^l) \simeq H^q_{\mathfrak{m}}(R/I^l) \otimes_R R'/z^l$$

as R'/z^l-modules. The previous two isomorphisms of R'/z^l-modules are isomorphisms of R'-modules, as well, since the R'-structures are defined using the projection $R' \to R'/z^l$. Putting these isomorphisms together, we see that

$$H^q_{\mathfrak{n}}(R'/J_l) \simeq H^q_{\mathfrak{n}}(R/I^l \otimes_R R'/z^l) \simeq H^q_{\mathfrak{m}}(R/I^l) \otimes_R R'/z^l$$

as R'-modules, for all l. As the isomorphisms of [4, Thm. 4.2.1, Thm. 4.3.2] are functorial, the isomorphisms above form a compatible system, and passing to the inverse limit, we have

$$H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'}) \simeq \varprojlim H^q_{\mathfrak{n}}(R'/J_l) \simeq \varprojlim H^q_{\mathfrak{m}}(R/I^l) \otimes_R R'/z^l,$$

and since $\varprojlim H^q_{\mathfrak{m}}(R/I^l) \simeq H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ (again by Proposition 3.4.3), the rightmost module above is isomorphic as an R'-module to $(H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}}))^+$ by Lemma 4.2.4, as desired. □

We now introduce de Rham complexes and prove Lemma 4.2.7.

Proof of Lemma 4.2.7. We retain the notation introduced in the proof of Lemma 4.2.8. For all l and s, let $J_{l,s} = I^{l+s}R' + (z^l)$. (Note that $J_{l,0} = J_l$.) The families $\{J_{l,s}\}$ (with s fixed) and $\{(I')^l\}$ of ideals of R' are cofinal. For all l and s, the derivations $\partial_1, \dots, \partial_n$ induce (by the Leibniz rule) k-linear maps $R'/J_{l,s} \to R'/J_{l,s-1}$, as all of these derivations fix z. In turn, these maps induce k-linear maps on local cohomology as described in section 3.4. We can therefore construct, for every l, a "de Rham-like" complex

$$0 \to H^q_{\mathfrak{n}}(R'/J_{l,n}) \to \oplus_{1 \le i \le n} H^q_{\mathfrak{n}}(R'/J_{l,n-1}) \to \cdots \to H^q_{\mathfrak{n}}(R'/J_{l,0}) \to 0$$

using the derivations $\partial_1, \dots, \partial_n$. We write $H^q_{\mathfrak{n}}(\mathscr{C}^\bullet_l)$ for this complex (*cf.* Definition 4.2.10 below). The argument of Lemma 4.2.8 applies to all terms of this complex, and using the fact that the differentials in this complex do not involve z or dz, we see that this complex is isomorphic to the complex

$$(0 \to H^q_{\mathfrak{m}}(R/I^{l+n}) \to \oplus_{1 \le i \le n} H^q_{\mathfrak{m}}(R/I^{l+n-1}) \to \cdots \to H^q_{\mathfrak{m}}(R/I^l) \to 0) \otimes R'/z^l,$$

which we write $H^q_{\mathfrak{m}}(C^\bullet_l) \otimes R'/z^l$. We now pass to the inverse limit of both systems of complexes. For all s, we have

$$\varprojlim H^q_{\mathfrak{n}}(R'/J_{l,s}) \simeq H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'})$$

as R'-modules, by Proposition 3.4.3 and the cofinality of $\{J_{l,s}\}$ and $\{(I')^l\}$. Moreover, by the definition of the inverse limit action, the differentials in the complex $H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'}) \otimes \Omega^\bullet_R$ are given by taking the inverse limit of the differentials in the complexes $H^q_{\mathfrak{n}}(\mathscr{C}^\bullet_l)$. That is, we have

$$H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'}) \otimes \Omega^\bullet_R \simeq \varprojlim H^q_{\mathfrak{n}}(\mathscr{C}^\bullet_l),$$

for all l, as complexes of k-spaces. On the other hand, again using Proposition 3.4.3, we have

$$\varprojlim H^q_{\mathfrak{m}}(R/I^l) \simeq H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$$

as R-modules, and by the definition of the inverse limit action,

$$H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}}) \otimes \Omega^\bullet_R \simeq \varprojlim H^q_{\mathfrak{m}}(C^\bullet_l),$$

for all l, as complexes of k-spaces. To conclude the proposition, it suffices to show that

$$\varprojlim H^q_{\mathfrak{n}}(\mathscr{C}^\bullet_l) \simeq (\varprojlim H^q_{\mathfrak{m}}(C^\bullet_l))^+$$

as complexes of k-spaces. We have already shown that we have isomorphisms $H^q_{\mathfrak{n}}(\mathscr{C}^\bullet_l) \simeq H^q_{\mathfrak{m}}(C^\bullet_l) \otimes R'/z^l$ that clearly form a compatible system, which implies

$$\varprojlim H^q_{\mathfrak{n}}(\mathscr{C}^\bullet_l) \simeq \varprojlim H^q_{\mathfrak{m}}(C^\bullet_l) \otimes R'/z^l,$$

and since $\varprojlim H^q_{\mathfrak{m}}(C^\bullet_l) \otimes R'/z^l \simeq (\varprojlim H^q_{\mathfrak{m}}(C^\bullet_l))^+$ (by applying Lemma 4.2.4 to all objects in the complex), the proposition follows. □

We can now begin the proof of Theorem 4.0.8(a). We have already reduced ourselves to the hypotheses of Lemma 4.2.7. With R and R' as in the statement of that lemma, our goal is to compare the spectral sequence $\{\tilde{E}^{p,q}_{r,R}\}$, arising from the surjection $R \to A$, to the spectral sequence $\{\tilde{\mathbf{E}}^{p,q}_{r,R'}\}$, arising from the surjection $R' \to A$. We will first prove that the E_2-objects of these spectral sequences are isomorphic. As we have identified these E_2-objects in Lemma 4.2.1, this claim is equivalent to the following proposition:

Proposition 4.2.9. *All notation is the same as in Lemma 4.2.7. For all p and q, we have*

$$\tilde{E}^{p,q}_{2,R} = H^p_{dR}(H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}})) \simeq H^p_{dR}(H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'})) = \tilde{\mathbf{E}}^{p,q}_{2,R'}$$

as k-spaces, where the de Rham cohomology is computed by regarding $H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ as a $\mathrm{Diff}(R,k)$*-module and $H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'})$ as a* $\mathrm{Diff}(R',k)$*-module.*

Proof. The argument closely parallels that of Lemma 4.1.3. Consider the short exact sequence

$$0 \to \mathscr{O}_{\widehat{X'}} \otimes \widehat{\Omega}^\bullet_X[-1] \xrightarrow{\iota} \widehat{\Omega}^\bullet_{X'} \xrightarrow{\pi} \mathscr{O}_{\widehat{X'}} \otimes \widehat{\Omega}^\bullet_X \to 0$$

of complexes of sheaves of k-spaces on $\widehat{X'}$, where ι is simply $\wedge dz$. Thus this sequence is the analogue, for completed sheaves, of the short exact sequence given in Definition 4.1.6. This sequence consists of split exact sequences of finite free $\mathscr{O}_{\widehat{X'}}$-modules. Apply H^q_P to the entire sequence of complexes: as formal completion and local cohomology commute with finite direct sums, we obtain a short exact sequence

$$0 \to H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_R[-1] \to H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_{R'} \xrightarrow{\pi^\bullet_q} H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_R \to 0$$

of complexes of k-spaces. The corresponding long exact sequence in cohomology (accounting for the shift of -1) is

$$\begin{aligned} \cdots \to h^{p-1}(H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_R) \xrightarrow{\partial_z} h^{p-1}(H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_R) \to \\ h^p(H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_{R'}) \xrightarrow{\overline{\pi}^p_q} h^p(H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_R) \xrightarrow{\partial_z} h^p(H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_R) \to \cdots, \end{aligned}$$

where we know by Lemma 4.1.7 that, up to a sign, the connecting homomorphism is ∂_z. By Lemma 4.2.7, we have

$$H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_R \simeq (H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}}) \otimes \Omega^\bullet_R)^+$$

as complexes of k-spaces. By Remark 4.2.6, the formal power series operation on the right-hand side commutes with cohomology, so taking the cohomology of both sides, we find

$$h^p(H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}) \otimes \Omega^\bullet_R) \simeq (h^p(H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}}) \otimes \Omega^\bullet_R))^+$$

as k-spaces for all p. For any k-space M, the action of ∂_z on M^+ is given in Definition 4.2.2, and it is clear from this definition (since $\mathrm{char}(k) = 0$) that $\mathrm{coker}(\partial_z : M^+ \to M^+) = 0$ and $\ker(\partial_z : M^+ \to M^+) \simeq M$, the latter corresponding to the "constant term" component of M^+. Returning to the displayed portion of the long exact sequence, the first ∂_z is surjective, and so by exactness the unlabeled arrow is the zero map; this implies that $\overline{\pi}_q^p$ is injective, inducing an isomorphism between the kernel of the second ∂_z and

$$h^p(H_P^q(\widehat{X}', \mathscr{O}_{\widehat{X}'}) \otimes \Omega^\bullet_{R'}) = H^p_{dR}(H^q_P(\widehat{X}', \mathscr{O}_{\widehat{X}'})).$$

Since this kernel is isomorphic to $h^p(H_P^q(\widehat{X}', \mathscr{O}_{\widehat{X}}) \otimes \Omega^\bullet_R) = H^p_{dR}(H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}}))$, the proof is complete. □

We next work toward the general case of Theorem 4.0.8(a). Our goal is to construct a morphism between the Hodge-de Rham spectral sequences $\{\tilde{E}^{p,q}_{r,R}\}$ and $\{\tilde{\mathbf{E}}^{p,q}_{r,R'}\}$ arising from the two surjections $R \to A$ and $R' \to A$ which, at the level of E_2-objects, consists of the isomorphisms of Proposition 4.2.9: by Proposition 1.3.5, this is enough. As in the proof of Theorem 4.0.7(a), we will construct an "intermediate" spectral sequence $\{\tilde{\mathscr{E}}^{p,q}_r\}$. The analogue of Lemma 4.1.14, however, will be significantly harder to prove, since we are working with inverse limits and must therefore check various Mittag-Leffler conditions.

We begin with definitions of several complexes, collecting pieces of notation introduced in the course of the preceding proofs together with some obvious variations.

Definition 4.2.10. All notation is the same as in Lemma 4.2.7.

(a) Let $C_l^\bullet$ be the complex

$$0 \to R/I^{l+n} \to \oplus_{1\leq 1\leq n} R/I^{l+n-1} \to \cdots \to R/I^l \to 0$$

defined using the derivations $\partial_1, \ldots, \partial_n$. Note that this is a complex of R-modules with k-linear differentials.

(b) Let $\mathscr{C}_l^\bullet$ be the complex

$$0 \to R'/J_{l,n} \to \oplus_{1\leq 1\leq n} R'/J_{l,n-1} \to \cdots \to R'/J_{l,0} \to 0$$

defined using the derivations $\partial_1, \ldots, \partial_n$, where for all l and s, $J_{l,s} = I^{l+s}R' + (z^l)$. This is a complex of R'-modules with k-linear differentials, and we have $\mathscr{C}_l^\bullet \simeq C_l^\bullet \otimes R'/z^l$.

(c) Let $H^q_{\mathfrak{m}}(C^\bullet_l)$ (resp. $H^q_{\mathfrak{n}}(\mathscr{C}^\bullet_l)$) be the complex obtained by applying local cohomology functors to the previous two complexes, in the manner described in section 3.4. In the course of the proof of Lemma 4.2.7, we saw that $H^q_{\mathfrak{n}}(\mathscr{C}^\bullet_l) \simeq H^q_{\mathfrak{m}}(C^\bullet_l) \otimes R'/z^l$.

(d) Let

$$\widetilde{C^\bullet_l} = (0 \to \mathscr{O}_X/\mathscr{I}^{l+n} \to \oplus_{1\le 1\le n}\mathscr{O}_X/\mathscr{I}^{l+n-1} \to \cdots \to \mathscr{O}_X/\mathscr{I}^l \to 0)$$

and

$$\widetilde{\mathscr{C}^\bullet_l} = (0 \to \mathscr{O}_{X'}/\mathscr{J}_{l,n} \to \oplus_{1\le 1\le n}\mathscr{O}_{X'}/\mathscr{J}_{l,n-1} \to \cdots \to \mathscr{O}_{X'}/\mathscr{J}_{l,0} \to 0)$$

be the sheafified versions of the first two complexes, where $\mathscr{J}_{l,s} = \widetilde{J_{l,s}}$ for all l and s. These can be viewed as complexes of sheaves of k-spaces on $\widehat{X}$ (resp. $\widehat{X'}$). We have $\widehat{\Omega}^\bullet_X \simeq \varprojlim \widetilde{C^\bullet_l}$ and $\mathscr{O}_{\widehat{X'}} \otimes \widehat{\Omega}^\bullet_X \simeq \varprojlim \widetilde{\mathscr{C}^\bullet_l}$ in the respective categories of complexes of sheaves.

(e) Finally, we consider a sheaf-theoretic variant of Definition 4.2.5. If $\mathscr{F}^\bullet$ is a complex whose objects are sheaves of $\mathscr{O}_X$-modules and whose differentials are merely k-linear, the complex $\mathscr{F}^\bullet \otimes \mathscr{O}_{X'}/z^l$ is the direct sum of l copies of $\mathscr{F}^\bullet$, indexed by z^i for $i = 0, \ldots, l-1$. (At the level of objects, $\mathscr{F} \otimes \mathscr{O}_{X'}/z^l$ is shorthand for the $\mathscr{O}_X$-module $\mathscr{F} \otimes_{\mathscr{O}_X} i^*(\mathscr{O}_{X'}/\mathscr{Z}^l)$, where i is the closed immersion $X \hookrightarrow X'$ and $\mathscr{Z}$ is the sheaf of ideals defining this immersion.) As an example, we have $\widetilde{\mathscr{C}^\bullet_l} \simeq \widetilde{C^\bullet_l} \otimes \mathscr{O}_{X'}/z^l$.

For the complexes of *sheaves* in Definition 4.2.10, we have corresponding spectral sequences for local hypercohomology, and we will need to work with all of these.

Definition 4.2.11. If $\mathscr{F}^\bullet$ is a complex of sheaves of k-spaces on $\widehat{X}$ (or $\widehat{X'}$, which has the same underlying space), the *local hypercohomology spectral sequence* for $\mathscr{F}^\bullet$ is the spectral sequence defined in section 1.3 with respect to the functor Γ_P of sections supported at the closed point. If $\mathscr{L}^{\bullet,\bullet}$ is any Cartan-Eilenberg resolution of $\mathscr{F}^\bullet$ (or, more generally, a double complex resolution satisfying the conditions of Lemma 1.3.8), this spectral sequence is the column-filtered spectral sequence associated with the double complex $\Gamma_P(\widehat{X}, \mathscr{L}^{\bullet,\bullet})$. It begins $E_1^{p,q} = H^q_P(\widehat{X}, \mathscr{F}^p)$ and has abutment $\mathbb{H}^{p+q}_P(\widehat{X}, \mathscr{F}^\bullet)$. We introduce the following notation for the specific hypercohomology spectral sequences we will consider below:

(a) $\tilde{E}^{\bullet;\bullet}_{\bullet,R}$ is the local hypercohomology spectral sequence for $\widehat{\Omega}^{\bullet}_{X}$. (This is precisely the Hodge-de Rham spectral sequence arising from the surjection $R \to A$.)

(b) $\tilde{\mathbf{E}}^{\bullet;\bullet}_{\bullet,R'}$ is the local hypercohomology spectral sequence for $\widehat{\Omega}^{\bullet}_{X'}$. (This is precisely the Hodge-de Rham spectral sequence arising from the surjection $R' \to A$.)

(c) $\tilde{\mathscr{E}}^{\bullet;\bullet}_{\bullet}$ is the local hypercohomology spectral sequence for $\mathscr{O}_{\widehat{X'}} \otimes \widehat{\Omega}^{\bullet}_{X}$.

(d) For all l, $(\tilde{E}_l)^{\bullet;\bullet}_{\bullet,R}$ is the local hypercohomology spectral sequence for $\widetilde{C_l^{\bullet}}$.

(e) For all l, $(\tilde{\mathscr{E}}_l)^{\bullet;\bullet}_{\bullet}$ is the local hypercohomology spectral sequence for $\widetilde{\mathscr{C}_l^{\bullet}}$.

The key to the proof of Theorem 4.0.8(a) is the following analogue of Lemma 4.1.14. Once this lemma is established, the rest of the proof will closely parallel our work in section 4.1.

Lemma 4.2.12. *Let $\tilde{E}^{\bullet;\bullet}_{\bullet,R}$ and $\tilde{\mathscr{E}}^{\bullet;\bullet}_{\bullet}$ be the spectral sequences of Definition 4.2.11. There is an isomorphism*

$$(\tilde{E}^{\bullet;\bullet}_{\bullet,R})^{+} \xrightarrow{\sim} \tilde{\mathscr{E}}^{\bullet;\bullet}_{\bullet},$$

where the object on the left-hand side is obtained by applying the formal power series operation to all the objects and differentials of the spectral sequence $\tilde{E}^{\bullet;\bullet}_{\bullet,R}$.

Lemma 4.1.14 was a consequence of the fact that the double complexes giving rise to the two spectral sequences considered there were related by the $+$-operation. We were therefore able to prove that lemma by working entirely at the level of double complexes. The analogous reasoning fails here for reasons alluded to in Remark 4.2.6: the formal power series operation is an infinite direct *product*. The obvious extension of this operation to sheaves need not commute with the functor Γ_P, because taking stalks of sheaves (a direct limit) and direct products of sheaves (an inverse limit) need not commute. The proof of Lemma 4.2.12 will take place at the level of spectral sequences, not merely double complexes.

We will give the proof of Lemma 4.2.12 in four steps, as follows:

1. $\varprojlim (\tilde{E}_l)^{\bullet;\bullet}_{\bullet,R}$, defined by taking the "term-by-term" inverse limit of all objects and differentials in the spectral sequences, is again a spectral sequence: each term is derived from its

predecessor by taking cohomology. (An identical proof shows that $\varprojlim(\tilde{\mathscr{E}}_l)_\bullet^{\bullet;\bullet}$ is a spectral sequence.)

2. The spectral sequences $\tilde{E}_{\bullet,R}^{\bullet;\bullet}$ and $\varprojlim(\tilde{E}_l)_{\bullet,R}^{\bullet;\bullet}$ are isomorphic. (An identical proof works for $\tilde{\mathscr{E}}_\bullet^{\bullet;\bullet}$ and $\varprojlim(\tilde{\mathscr{E}}_l)_\bullet^{\bullet;\bullet}$.)

3. For all l, the spectral sequences $(\tilde{\mathscr{E}}_l)_\bullet^{\bullet;\bullet}$ and $(\tilde{E}_l)_{\bullet,R}^{\bullet;\bullet} \otimes R'/z^l$ are isomorphic, via isomorphisms that form a compatible system (the latter spectral sequence, defined by the natural extension of Definition 4.2.5 to spectral sequences, is a direct sum of l copies of $(\tilde{E}_l)_{\bullet,R}^{\bullet;\bullet}$).

4. We have an isomorphism $(\tilde{E}_{\bullet,R}^{\bullet;\bullet})^+ \xrightarrow{\sim} \tilde{\mathscr{E}}_\bullet^{\bullet;\bullet}$ of spectral sequences (the general statement).

Step (4) follows immediately from step (3) by applying Lemma 4.2.4 to all objects of the spectral sequences, so we will not give it a separate proof below. Steps (2) and (3) are not difficult. Step (1), which is necessary for the later statements to be well-defined, is more difficult and depends on our work in section 3.3.

Proof of step (1). We show first that the E_2-term of the "term-by-term" inverse limit, $\varprojlim(\tilde{E}_l)_{2,R}^{\bullet;\bullet}$, is obtained from the E_1-term by taking cohomology: that is, we show that for all p and q, we have isomorphisms

$$\varprojlim(\tilde{E}_l)_{2,R}^{p;q} = \varprojlim h^p(H_{\mathfrak{m}}^q(C_l^\bullet)) \simeq h^p(\varprojlim H_{\mathfrak{m}}^q(C_l^\bullet)) = h^p(\varprojlim(\tilde{E}_l)_{1,R}^{\bullet;q})$$

of k-spaces. This is the assertion that for the inverse system $\{H_{\mathfrak{m}}^q(C_l^\bullet)\}$ of complexes of k-spaces, taking cohomology commutes with inverse limits. By [11, Prop. 13.2.3], it suffices to check that for all p and q, the inverse systems $\{H_{\mathfrak{m}}^q(C_l^p)\}$ and $\{h^p(H_{\mathfrak{m}}^q(C_l^\bullet))\}$ both satisfy the Mittag-Leffler condition. By [4, Thm. 7.1.3], each $H_{\mathfrak{m}}^q(C_l^p)$ is an Artinian R-module; as the transition maps in this inverse system are R-linear, induced by the canonical R-linear maps $C_{l+1}^p \to C_l^p$, the Mittag-Leffler condition for this first system is immediate.

In order to verify the Mittag-Leffler condition for the second system, we consider the Matlis dual of the first system. For all q and l, the differentials in the complex $H_{\mathfrak{m}}^q(C_l^\bullet)$ have Matlis duals by Corollary 3.1.16, since they are k-linear maps between Artinian R-modules. The transition maps in the *direct* system $\{D(H^q_{\mathfrak{m}}(C^\bullet_l))\}$ of complexes are R-linear in each degree, and each $D(H^q_{\mathfrak{m}}(C^p_l))$ is a finitely generated R-module (since it is the Matlis dual of an Artinian R-module). In fact, by Theorem 3.4.1, $D(H^q_{\mathfrak{m}}(C^p_l))$ is a direct sum of $\binom{n}{p}$ copies of the R-module $\mathrm{Ext}^{n-q}_R(R/I^{l+n-p}, R)$, and so $\varinjlim D(H^q_{\mathfrak{m}}(C^p_l))$ is a direct sum of $\binom{n}{p}$ copies of

$$\varinjlim \mathrm{Ext}^{n-q}_R(R/I^{l+n-p}, R) = H^{n-q}_I(R).$$

Finally, the de Rham complex of the $\mathrm{Diff}(R,k)$-module $H^{n-q}_I(R)$ is the direct limit of the complexes $D(H^q_{\mathfrak{m}}(C^\bullet_l))$, since the inverse limit of their Matlis duals is the de Rham complex of $D(H^{n-q}_I(R)) = H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$ by Theorem 3.4.5 and the definition of the inverse limit action on $H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$.

We have thus verified the hypotheses of Corollary 3.3.20 for the direct system $\{D(H^q_{\mathfrak{m}}(C^\bullet_l))\}$: it is a direct system of complexes with k-linear differentials whose objects are finitely generated R-modules, the transition maps are R-linear in each degree, and the direct limit is the de Rham complex $H^{n-q}_I(R) \otimes \Omega^\bullet_R$ of a holonomic $\mathrm{Diff}(R,k)$-module. We conclude from that corollary that for all p and l, the images of $h^{n-p}(D(H^q_{\mathfrak{m}}(C^\bullet_l)))$ in $h^{n-p}(D(H^q_{\mathfrak{m}}(C^\bullet_{l+s})))$ stabilize in the strong sense as s varies, with finite-dimensional stable image. By Corollary 3.1.16, all the Matlis duals (of objects and differentials) in the direct system $\{D(H^q_{\mathfrak{m}}(C^\bullet_l))\}$ coincide with k-linear duals. Since k-linear dual is a contravariant, exact functor, we thus have $h^{n-p}(D(H^q_{\mathfrak{m}}(C^\bullet_l))) \simeq (h^p(H^q_{\mathfrak{m}}(C^\bullet_l)))^*$, and by Lemma 3.3.9, the inverse system $\{h^p(H^q_{\mathfrak{m}}(C^\bullet_l))\}$ satisfies the Mittag-Leffler condition, as desired.

Examining the proof of Lemma 3.3.9, we see that in fact we can conclude something stronger than the Mittag-Leffler condition, namely the following: for all p and q, given l, there exists s such that the image of $h^p(H^q_{\mathfrak{m}}(C^\bullet_{l+s}))$ in $h^p(H^q_{\mathfrak{m}}(C^\bullet_l))$ is a finite-dimensional k-space. That is, the inverse system $\{h^p(H^q_{\mathfrak{m}}(C^\bullet_l))\}$ is *eventually finite*. Since any descending chain of k-subspaces of a finite-dimensional k-space must terminate, it is clear that eventual finiteness implies the Mittag-Leffler condition. But what is more, eventual finiteness is inherited by cohomology: if, for any r, the inverse system $\{(\tilde{E}_l)^{p,q}_{r,R}\}$ is eventually finite, so is the inverse system

$\{(\tilde{E}_l)^{p,q}_{r+1,R}\}$, since the objects $(\tilde{E}_l)^{p,q}_{r+1,R}$ are subquotients of the objects $(\tilde{E}_l)^{p,q}_{r,R}$. By induction on r, we conclude that the E_{r+1}-term of the "term-by-term" inverse limit, $\varprojlim(\tilde{E}_l)^{\bullet,\bullet}_{r+1,R}$, is obtained from the E_r-term by taking cohomology, and so $\varprojlim(\tilde{E}_l)^{\bullet;\bullet}_{\bullet,R}$ is a well-defined spectral sequence. $\square$

Proof of step (2). Since $\widehat{\Omega}^\bullet_X \simeq \varprojlim \widetilde{C^\bullet_l}$ as complexes of sheaves on $\widehat{X}$, we have projection maps $\pi_l : \widehat{\Omega}^\bullet_X \to \widetilde{C^\bullet_l}$ for all l. As described in section 1.3, each such projection map induces a morphism between the corresponding local hypercohomology spectral sequences. By the universal property of inverse limits, this family of projection maps induces a morphism of spectral sequences $\tilde{E}^{\bullet;\bullet}_{\bullet,R} \to \varprojlim(\tilde{E}_l)^{\bullet;\bullet}_{\bullet,R}$ (where the right-hand side is a well-defined spectral sequence by step (1)). We claim this is an isomorphism; by Proposition 1.3.5, it suffices to check this on the E_1-objects of both sides. By definition, for all p and q, we have $\tilde{E}^{p,q}_{1,R} = H^q_P(\widehat{X}, \widehat{\Omega}^p_X)$, a direct sum of $\binom{n}{p}$ copies of $H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}})$, and for all l, we have $(\tilde{E}_l)^{p,q}_{1,R} = H^q_P(\widehat{X}, \widetilde{C^p_l})$, a direct sum of $\binom{n}{p}$ copies of $H^q_{\mathfrak{m}}(R/I^{l+n-p})$. The induced map $\tilde{E}^{p,q}_{1,R} \to \varprojlim(\tilde{E}_l)^{p,q}_{1,R}$ is therefore a finite direct sum of copies of the canonical map $H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}}) \to \varprojlim H^q_{\mathfrak{m}}(R/I^{l+n-p})$, which we already know to be an isomorphism by Proposition 3.4.3. $\square$

Proof of step (3). This is the only step in the proof where we can work entirely at the level of the double complexes giving rise to the spectral sequences. Let l be given. Choose a Cartan-Eilenberg resolution $\widetilde{C^\bullet_l} \to \mathscr{L}^{\bullet,\bullet}_l$ in the category of sheaves of k-spaces on $\widehat{X}$. By Remark 4.2.6, this category satisfies axiom $AB4$, and so direct sums are exact; furthermore, direct sums of injective sheaves are again injective, and the functor Γ_P commutes with direct sums. We conclude that the double complex $\mathscr{L}^{\bullet,\bullet}_l \otimes \mathscr{O}_{X'}/z^l$ (which is simply a finite direct sum of copies of $\mathscr{L}^{\bullet,\bullet}_l$) satisfies the conditions of Lemma 1.3.8, and so the double complex $\Gamma_P(\widehat{X}, \mathscr{L}^{\bullet,\bullet}_l \otimes \mathscr{O}_{X'}/z^l)$ gives rise to the local hypercohomology spectral sequence for the complex $\widetilde{C^\bullet_l} \otimes \mathscr{O}_{X'}/z^l \simeq \widetilde{\mathscr{C}^\bullet_l}$. On the one hand, this spectral sequence is $(\tilde{\mathscr{E}}_l)^{\bullet;\bullet}_{\bullet}$ by definition. On the other hand, since Γ_P commutes with direct sums, we have $\Gamma_P(\widehat{X}, \mathscr{L}^{\bullet,\bullet}_l \otimes \mathscr{O}_{X'}/z^l) \simeq \Gamma_P(\widehat{X}, \mathscr{L}^{\bullet,\bullet}_l) \otimes R'/z^l$ as double complexes of k-spaces, and since cohomology of k-spaces commutes with direct sums, this isomorphism induces an isomorphism $(\tilde{\mathscr{E}}_l)^{\bullet;\bullet}_{\bullet} \simeq (\tilde{E}_l)^{\bullet;\bullet}_{\bullet,R} \otimes R'/z^l$ at the level of spectral sequences, as desired. The fact that these isomorphisms form a compatible system follows from the fact that the isomorphisms $\widetilde{C_l^\bullet} \otimes \mathscr{O}_{X'}/z^l \simeq \widetilde{\mathscr{C}_l^\bullet}$ form a compatible system, since as discussed in section 1.3, the association of a local hypercohomology spectral sequence with a complex of sheaves is functorial in the complex. This completes the proof of Lemma 4.2.12. $\square$

We are now ready to complete the proof of Theorem 4.0.8.

Proof of Theorem 4.0.8(a). Consider again the short exact sequence from the proof of Proposition 4.2.9:

$$0 \to \mathscr{O}_{\widehat{X'}} \otimes \widehat{\Omega}^\bullet_X[-1] \xrightarrow{\iota} \widehat{\Omega}^\bullet_{X'} \xrightarrow{\pi} \mathscr{O}_{\widehat{X'}} \otimes \widehat{\Omega}^\bullet_X \to 0.$$

As described in section 1.3, the morphism of complexes π induces a morphism between the corresponding spectral sequences for local hypercohomology, given in Definition 4.2.11. That is, there is an induced morphism

$$\pi_\bullet^{\bullet,\bullet} : \tilde{\mathbf{E}}_{\bullet,R'}^{\bullet,\bullet} \to \tilde{\mathscr{E}}_\bullet^{\bullet,\bullet}.$$

Identifying first $\tilde{\mathscr{E}}_\bullet^{\bullet,\bullet}$ with $(\tilde{E}_{\bullet,R}^{\bullet,\bullet})^+$ (by Lemma 4.2.12) and then $\tilde{E}_{\bullet,R}^{\bullet,\bullet}$ with the "constant term" component of $(\tilde{E}_{\bullet,R}^{\bullet,\bullet})^+$, we see that this further induces a morphism

$$\psi_\bullet^{\bullet,\bullet} : \tilde{\mathbf{E}}_{\bullet,R'}^{\bullet,\bullet} \to \tilde{E}_{\bullet,R}^{\bullet,\bullet}$$

given in every degree by $\pi_\bullet^{\bullet,\bullet}$ followed by the projection of $\tilde{\mathscr{E}}_\bullet^{\bullet,\bullet} \simeq (\tilde{E}_{\bullet,R}^{\bullet,\bullet})^+$ on its constant term component. If $r = 2$, the maps $\psi_2^{p,q}$ are precisely the inverses of the isomorphisms $H^p_{dR}(H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}})) \simeq H^p_{dR}(H^q_P(\widehat{X'}, \mathscr{O}_{\widehat{X'}}))$ appearing in the proof of Proposition 4.2.9, which were induced by the morphism of complexes π and the projection of $(\tilde{E}_{2,R}^{p,q})^+ = (H^p_{dR}(H^q_P(\widehat{X}, \mathscr{O}_{\widehat{X}})))^+$ on its constant term component. Therefore the morphism $\psi_\bullet^{\bullet,\bullet}$ of spectral sequences is an isomorphism at the E_2-level. By Proposition 1.3.5, it follows that ψ is an isomorphism at all later levels, including the abutments. The proof is complete. $\square$

Remark 4.2.13. As we remarked at the end of section 4.1, the isomorphism class of the spectral sequence $\{\tilde{E}_{r,R}^{p,q}\}$ depends *a priori* on the choice of coefficient field $k \subset A$. It is an open problem whether it is independent of this choice.

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