Construction, Operations, and Applications of the Surreal Numbers

A Thesis

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Abstract

The surreal numbers are a unifying, non-Archimedean ordered field, the existence of which was speculated at by Hahn and Hausdorff and later formalized by Alling and Conway. In this expository thesis we see in Section 1.1 how the surreals can be constructed via Dedekind-like cuts, as well as how they arise naturally in quantifying game positions in Section 1.2. Section 2 describes the operations unique to the surreals, allowing us to manipulate strange numbers such as $\frac{5}{\sqrt[3]{\omega}} - \varepsilon$. The predominant application of surreal numbers thus far has been to combinatorial game theory, and to that end in Section 3.2 we will evaluate a Go game using the program developed by Berlekamp and Wolfe. In Section 4 we see the strengthening currents of a cultural shift; where many have chided the surreals for lacking hard applications, in the past decade alone it has been shown that the surreals are isomorphic to the maximal realization of the hyperreals, that surreal analysis is a field in its own right riddled with open questions, and excitingly that the surreals have a transseries structure with a derivation.

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Introduction

Before I began researching the contents of this thesis I was studying Go. I read *Hikaru no Go*, carried Lessons in the Fundamentals of Go with me everywhere, puzzled over *tsumego.* I studied old *kifu*, wrote papers on the state-space and computational complexity of Go, the evolution of Go AI . And I learned about surreal numbers.

In his 1962 paper "On the existence of real-closed fields that are η_α-sets of power $\aleph_\alpha$," Alling constructed an ordered field, a class, of ordinals isomorphic to the surreal numbers. Later, a simpler construction arose from the study of Go endgames by Conway, presented by Knuth in his 1974 novel *Surreal Numbers*; in his 1976 monograph, *On Numbers and Games*, Conway used the construction to delve into operations over the surreals, as well as introducing *nimbers*, or Grundy numbers. Neither Go nor any reduction of Go is mentioned in *ONAG*, and it was not until nearly 20 years later, with the publication of *Mathematical Go* by Berlekamp and Wolfe, that surreal numbers were used to analyze any game that people actually played.

Surreal numbers have been summarily dismissed - whether that is due to their supposed progenitor, the oft hostile academic clime, or because everyone has something better to do is beyond this introduction - I will try to accomplish two things and champion my cause: explain what the surreal numbers are and how they can be (and are) being applied. In Section 1 we construct the surreals using two natural and intuitive methods. Section 2 details the set properties of the surreal numbers. In Section 3, we will examine a single game of Go and how a rudimentary knowledge of surreal numbers can turn the outcome to one's favor. Section 4 details the connection between surreal numbers and non-standard analysis, the emergent field of surreal analysis, and recent developments in the connection between surreal numbers and transseries.

1 Constructing the Surreal Numbers

Here we will examine two ways of constructing the surreal numbers. We will begin more formally in Section 1.1 with an inductive construction on pairs of sets and forming equivalence classes. In Section 1.2 we will see how surreal numbers arise in a natural (and equivalent) manner while studying games.

1.1 Dedekind-like Cuts

Let L and R be two sets such that $\forall l \in L$ and $\forall r \in R$, $l < r$ and $\exists x = \{L \mid R\}$. In addition to this, we have the following rules and relations (as well as operations covered in Section 2.1):

Let $x = \{x^L \mid x^R\}$ and $y = \{y^L \mid y^R\}$

- $x \geq y \iff (x^R > y \wedge x > y^L)$. $x \leq y \iff y \geq x$
  - $x \geq x$
  - If $x \geq y$ and $y \geq z$, $x \geq z$
  - For any two numbers x and y, we must have $x \geq y$ or $y \geq x$
- $x = y \iff (x \geq y \wedge y \geq x)$
  - $x = x$
- $x > y \iff (x \geq y \wedge y \ngeq x)$. $x < y \iff y > x$
  - $x^L < x < x^R$

1.1.1 The Integers and Dyadic Rationals

When we begin our construction, we are only given the empty set, $\varnothing$, which may be in either L or R: $\{\varnothing \mid \varnothing\} = \{\,\mid\,\}$, and we call this 0. Given 0, we can place it in *either* L or R.[1]. We will call these numbers 1 and -1. Continuing in this manner, we construct the integers.

[1]Since $0 \leq 0$, by definition $\{0 \mid 0\}$ cannot be a number. It is, however, a game 3

- $\{0\,|\,\} = 1$
- $\{\,|\,0\} = -1$
- $\{1\,|\,\} = 2$
- $\{\,|-1\,\} = -2$
- $\{n\,|\,\} = n+1$ and $\{\,|\,-n\} = -(n+1)$ for $n \in \mathbb{N}$

Now that we have a few more numbers to work with, let's see what happens.

Suppose we bulked up L or R.What would $\{0,1,2\,|\,\}$ or $\{\,|\,1,0,4\}$ be? Recall that $\forall l \in L$ and $\forall r \in R$, $l < r$. That is, we need only take the maximal element of the left set and the minimal element of the right set to the to determine the given number. So $\{0,1,2\,|\,\} = 3$ and $\{\,|\,1,0,4\} = -1$

What is $\{0\,|\,1\}$? The easiest way to interpret this number is as the number that occurs halfway between 0 and 1, $\frac{1}{2}$. Similarly, $\{-1\,|\,0\} = -\frac{1}{2}$ In many cases, we can just take the arithmetic mean of the maximal and minimal elements. In Section 2 we will see that these names are justified.

1.1.2 What about $\frac{1}{3}$?

We have so far found the numbers that can be constructed with a finite number of elements in L and R, precisely the integers and dyadic rationals.[2]. What about, well, everything else? When in our construction does the remained of $\mathbb{R}$ pop up? It is not until after all of these numbers have been constructed that we find the true beauty in the surreals.

To construct $\mathbb{Z}$ we just iterated on L and R. Continuing that process indefinitely, we eventually reach the first ordinal, $\{0,1,2,3,\cdots\,|\,\} = \omega$,

[2]The surreals admit field operations (2.1)

and $\{\, | \cdots -3, -2, -1, 0\} = -\omega$.]

Given all of the dyadic rationals, we can rest assured that convergent infinite series exist for L and R that will define the remainder of $\mathbb{R}$

- $\{0, \frac{3}{16}, \cdots \,|\, \frac{1}{4}, \frac{13}{64}\} = \frac{1}{5}$ [3]

- $\{\frac{x}{2y} : 3x < 2y \,|\, \frac{x}{2y} : 3x > 2y\} = \frac{1}{3}$

- $\{3, \frac{25}{8}, \frac{201}{64}, \cdots \,|\, 4, \frac{7}{2}. \frac{13}{4}, \frac{51}{16}, \cdots\} = \pi$

We also have redundancies in that the integers and dyadics are constructed again. Since we have all of them now, consider $\{0 \,|\, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots\}$, a number greater that 0 but less than all of the positive rationals that we have been able to construct, an infinitesimal number $\varepsilon = \frac{1}{\omega}$, and $-\varepsilon$.
With each number we construct we have more fodder for L and R.

- $\{0, 1, 2, 3, \cdots, \omega \,|\, \} = \omega + 1$

- $\{0, 1, 2, 3, \cdots \,|\, \omega\} = \omega - 1$

- $\{n \,|\, \omega - n\} = \frac{\omega}{2}$

- $\{0, 1, 2, 3, \cdots \,|\, \omega, \frac{\omega}{2}, \frac{\omega}{4}, \frac{\omega}{8}\} = \sqrt{\omega}$ [8, p. 13,14]

The surreals continue to grow, and we can continue to construct and manipulate and talk about the reals, infinites, and infinitesimals in a cohesive way. The surreals form an arithmetic continuum, a proper class, the largest totally ordered field. All other ordered fields[4] can be realized as subfields[5] of the surreals, which will be discussed in further detail in Section 2.2.

By embellishing Dedekind cuts, we have incorporated infinite and infinitesimal quantities, extending $\mathbb{R}$, akin to the hyperreals (more on the exact relation

[3] Division is defined by recursion on an initial value (2.1)
[4] We can define *surcomplex* numbers of the form $\alpha + i\beta$ where α and β are surreal
[5] Isomorphic to subtrees 1.3

in Section 4). However, there is more than one way to skin a cat. While the above classic approach gives an atmosphere of familiarity, a more game theoretic approach will further enlighten us. The surreals are more than just numbers. They are, in fact, games.

1.2 Hackenbush

Hackenbush is a two player, partisan, combinatorial game[6] played on a graph composed of colored edges attached to a "ground" ("wall," "ceiling," etc), typically indicated by a dotted line. It, and its many variants, were introduced by Conway in *ONAG* [8].

In Blue-Red Hackenbush (BR), each edge is colored blue or red. The first/left/Blue player, is only allowed to cut blue edges; the second/right/Red player, is only allowed to cut red edges. The normal play convention (win condition) is that the last player able to move wins (or the player who is unable to cut an edge loses). Any number of edges connected by their endpoints may make up the graph, so long as some chain of edges connects them all to the ground. When an edge is cut, any edges no longer connected to the ground are also removed.

In Blue-Red Green Hackenbush (BRG), the rules as stated above apply with additional green edges which can be cut by either player. Blue-Red Hackenbush is a special case of Blue-Red-Green Hackenbush. Any conclusions drawn from BRG can be drawn from BR and Green Hackenbush (G).

Green Hackenbush is an impartial game where both players can cut any edge. By the Sprague - Grundy theorem, any impartial game with perfect information under the normal play convention is equivalent to a nimber, i.e. Green Hackenbush is essentially nim.

[6]Please refer to Appendix B for unfamiliar game theory terms

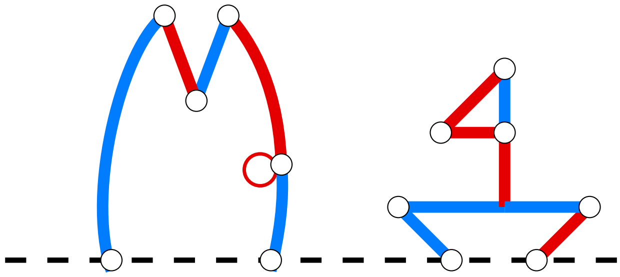


Figure 1: A Hackentuna with a Hackenschooner

A Hackenbush game is any configuration of nodes and line segments such that there exists at least one path from every node to the ground. A Hackenbush tree is a game such that there exists a *unique* path from every node to the ground. A Hackenbush stalk or string is a game such that each node connects to at most two edges (and at least one). A Hackenbush loop is a stalk such that both ends are connected to the ground, or a stalk that connects to itself at some node.

1.2.1 Blue-Red Hackenbush

While Hackenbush is a pretty boring game to play, in subjecting ourselves to its tedium we can generate the surreal numbers (BR) and nimbers (G). First we will look at some simple Hackenbush games and find, amazingly enough,that integers exist.

We can explicitly represent any given Hackenbush game in terms of the *value* of the plays remaining to Blue after either players' term. We could similarly describe a game in terms of Red's remaining moves, but no one bothers since the two methods are equivalent (sort of like how the left set of a Dedekind cut determines the right). So, we can write the game where Blue's options are $a, b, c, ...$ after her own turn and $d, e, f, ...$ after Red's turn as $G = \{a, b, c, ... \mid d, e, f, ...\}$. For an arbitrary game, such as
There isn't a general rule to find the associated value without actually playing the game. However, we can find the values of finite and infinite Hackenbush

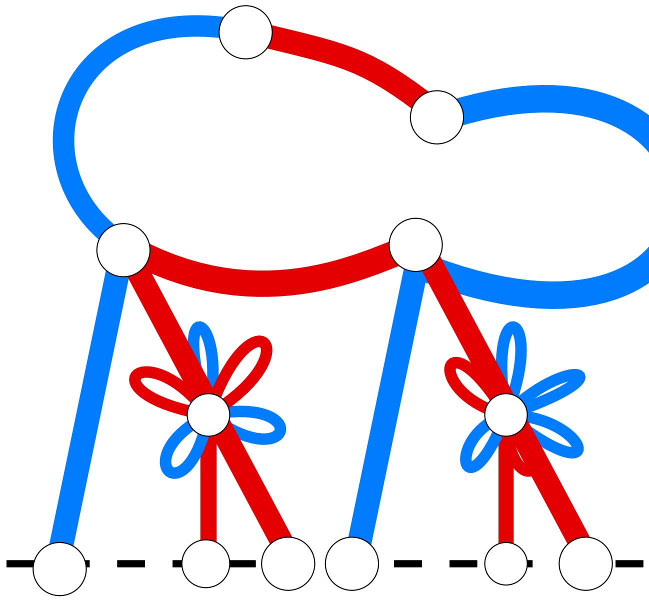


Figure 2: A Hacken...capybara?

trees, loops, and stalks.

There is no game

What is the game - - - - ?

Neither Blue nor Red can make a move, so we have the game $\{\,|\,\}$, which we will call 0.

Placing an edge of either color on the ground gets us 1 and -1: lets look at

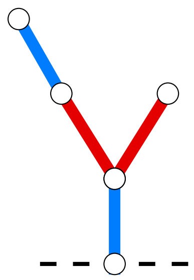
For (a), Blue can make one move which leads to the 0 position for her whereas

(a) (b)

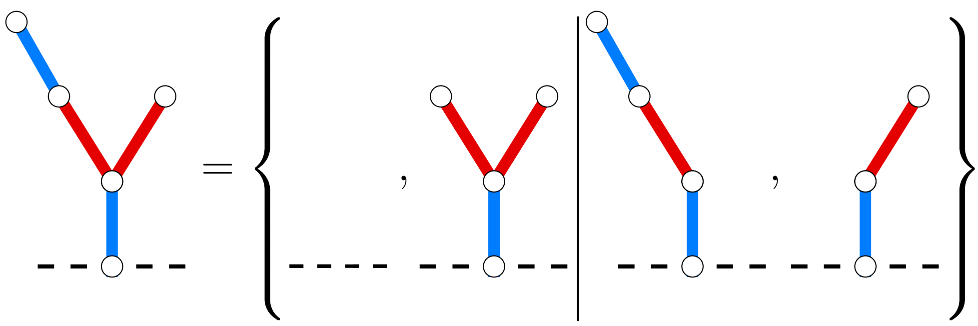
Red can more no moves, giving us $\{0\,|\,\} = 1$. (b) Shows us the exact opposite situation, $\{\,|\,0\} = -1$.

In general, a blue stalk (B) is valued at 1 and a red stalk (R) is value at -1. Any monochromatic chain $\overbrace{BBBB\cdots}^{n} = n$ and $\overbrace{RRRR\cdots}^{n} = -n$.

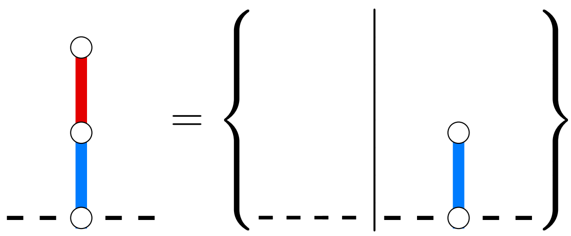
To see how this plays out with more complex trees, let's look at



Blue and Red both have only two choices; Blue's moves lead to the games in the left set, Red's in the right.



What happens in these three non-zero subgames? As we will later see, the other games invariably depend on the outcome of the simplest game, BR.

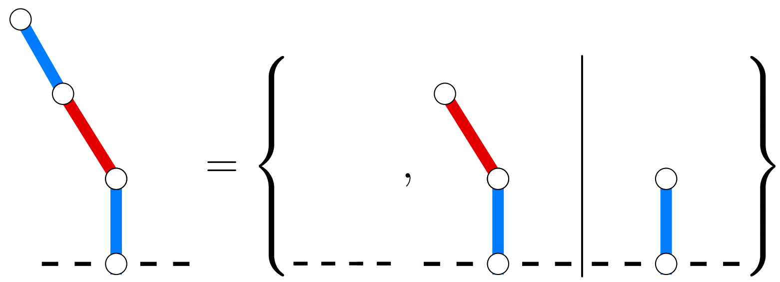


$$= \{0\,|\,1\} = \frac{1}{2}$$

This equality can easily be proven by demonstrating that the sum of the games

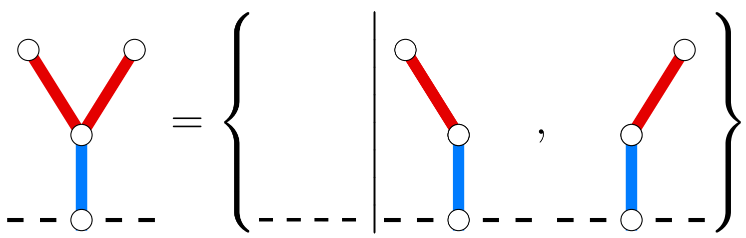
$\{0\,|\,1\}$, $\{0\,|\,1\}$, and $\{\,|\,0\}$ is the zero game.

Now,



$$= \left\{0, \frac{1}{2} \,\middle|\, 1\right\} = \frac{3}{4}$$

and



$$= \left\{ 0 \,|\, \frac{1}{2}, \frac{1}{2} \right\} = \left\{ 0 \,|\, \frac{1}{2} \right\} = \frac{1}{4}$$

By using just these finite Hackenbush trees and stalks (non-branching), we have generated the integers and dyadic rationals.

Often finding the value of a game is a matter of finding the arithmetic mean of the left and right sets. But what are the games $\{5 \,|\, 28\}$, or $\{\pi \,|\, 4\}$? To answer these questions, we must appeal to what is known as the *simplicity hierarchy* (see Section 1.3), which will be discussed in further detail later on. For out Hackenbush dealings, we just need to know what the simplest games is:

If $G = \{G_L \,|\, G_R\}$, where G_L and G_R are the options for Blue and Red respectively, then G is the *simplest* game that satisfies the condition that $G_L < G$ and $G_R < G$. Simplest here means "earliest generated", or "smallest whole number," or the number that satisfies the following rules given by Berlekamp et al. in *Winning Ways*:[7]

- $0 = \{\,|\,\}$
- $n + 1 = \{n \,|\,\}$
- $\dfrac{2p+1}{2^{q+1}} = \left\{ \dfrac{p}{2^q} \,\middle|\, \dfrac{p+1}{2^q} \right\}$
- $-(n+1) = \{\,|\, -n\}$

1.2.2 Hackenbush Stalks and Berlekamp's Rule

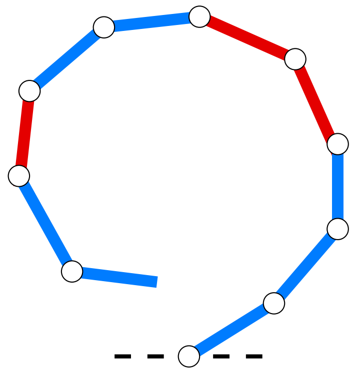
Given any stalk, we can find its associate value using the following method:

[7] "If there's *any* number that fits, the answer's the simplest number that fits." [6, p.22].

A string containing n B's is the number n. A string containing n R's is $-n$.

Suppose a string starts with n B's. Then $n-1$ is taken as the integer part, the first BR as the decimal point, and writing the remaining B's as 1's and the R's as 0's, leaving off the last B if there is one.

Consider



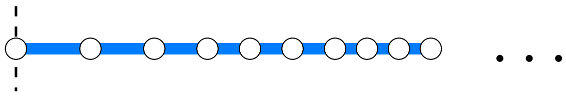
$= BBBRRBBRBB = BB \quad BR \quad RBBRBB$

Per Berlekamp's rule, this stalk is 2.01101. Now we need to convert .01101 into decimal: $(.01101)_2 = (0\cdot 2^{-1}+1\cdot 2^{-2}+1\cdot 2^{-3}+0\cdot 2^{-4}+1\cdot 2^{-5})_{10} = \frac{1}{4}+\frac{1}{8}+\frac{1}{32} = \frac{15}{32} = .40625$. Our stalk is equal to 2.40625.

What about numbers like $\frac{1}{3}$, π, and ω?. We can (as one may intuit) get these, and the rest of the real numbers, using infinite strings.

ω, the first infinite ordinal, comes after all of the natural numbers. As a surreal number, we can express it as $\{\mathbb{N} \mid \}$, and as a Hackenbush stalk consisting of an $\omega - 1$'s worth of blue edges

If we try to convert $\frac{1}{3}$ to binary, we get something nonterminating as expected. We choose to start with B as $\frac{1}{3}$ is positive.



$$\frac{1}{3} = \left(\frac{1}{11}\right)_{10} = 0.\overline{01} = BR \quad \overline{RB}$$

In short, *it can be done.*

To end this section, a garish, technicolor landscape of various Hackenbush games and their values has been provided.

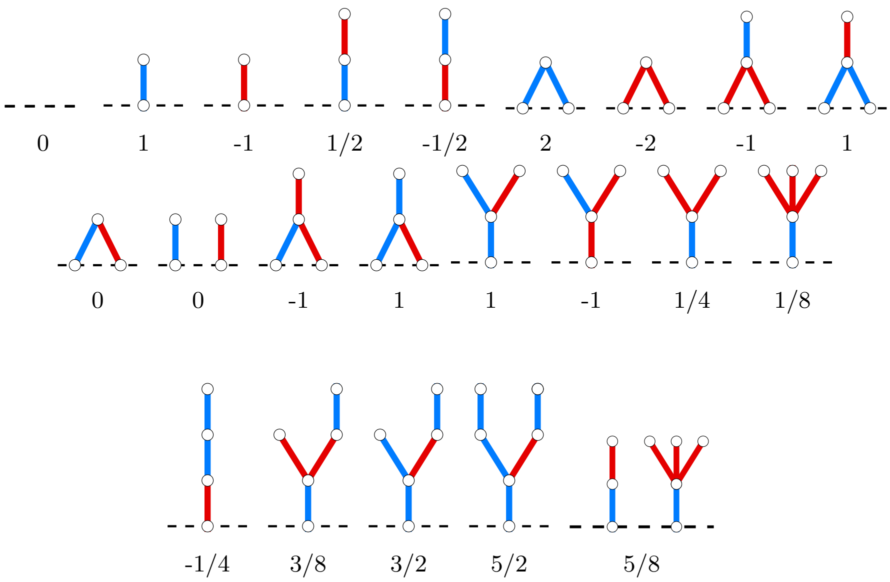
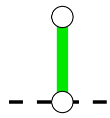


Figure 3: A Surreal Forest

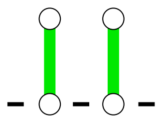
1.2.3 Green Hackenbush

In Green Hackenbush, all of the edges are green and may be cut by either player. As in other impartial games, the only difference between the players is who goes first and the available moves depend only on the current configuration of the game.

Let's start by again considering the game of a single edge.



No matter who goes first, the outcome is the zero game, so this game is equal to $\{0\,|\,0\}$.This...thing doesn't satisfy our criteria for a number (in particular $0 \not< 0$, and the simplicity rule fails). Now consider



If Blue goes first she gives up a point to Red, and if Red goes first the point goes to Blue, giving us the game $\{-1\,|\,1\} = 0$ (by the simplicity rule). We call the object $\{0\,|\,0\}$ $\star$ (star). Since someone [8, p. 72] bothered to give it a name, we will anticipate its return in the future (Section 3).

1.3 The Simplicity Hierarchy

Given any Hackenbush game, and more generally, given any surreal number $\{L \mid R\}$ how do we know what that number is? Our constructions allow us to visualize the surreal number tree, showing us the progression of our constructions.

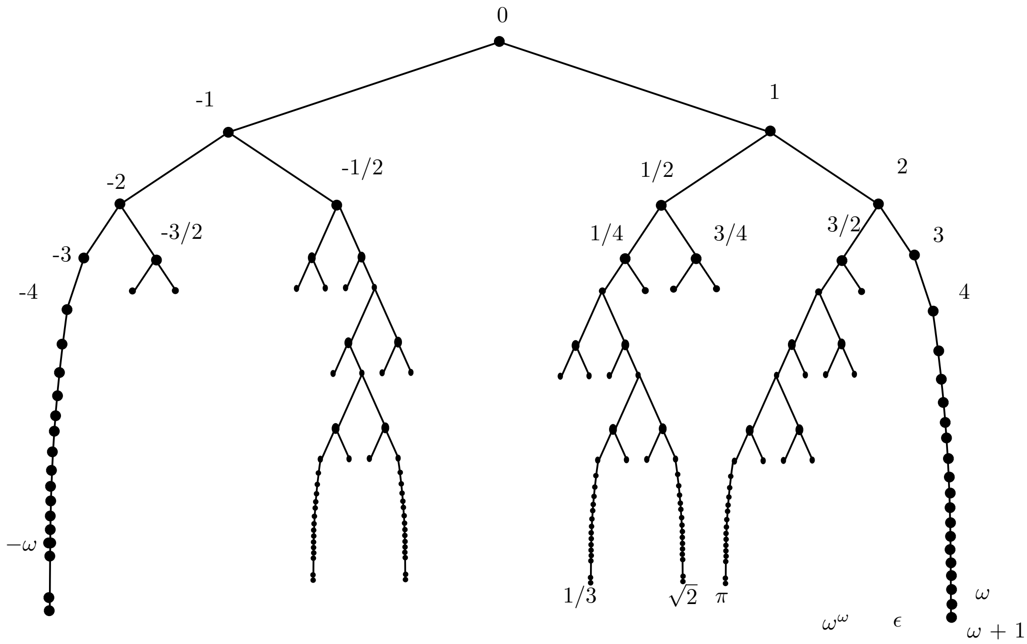


Figure 4: The Surreal Number Tree

From each number branches other numbers, giving the surreals a hierarchal nature. Ehrlich [12] refers to this as the *simplicity hierarchy* of the surreal numbers; x is simpler than y if x occurs earlier in the construction. 3 is simpler than π, $\frac{1}{4}$ is simpler than $\frac{13}{8}$. We now see why taking the mean does not always work. $\{3 \mid 17\}$ isn't 10, nor is it 5, 13, e^2, nor the myriad other number occurring between 3 and 17. It is, simply, 4.

2 Properties

While it is obviously more convenient to use real and ordinal operations when necessary and possible, it is worth knowing that, in their least pretty form, we can do operations with the surreals. More importantly, under these arithmetic operations, the surreals form a field. Proofs of each are given by Conway [8, p. 17-22].

2.1 Operations

Let $x = \{x^L \,|\, x^R\}$, $y = \{y^L \,|\, y^R\}$, $z = \{z^L \,|\, z^R\}$

Addition

$x + y = \{x^L + y, x + y^L \,|\, x^R + y, x + y^R\}$

- $x + 0 \equiv x$
- $x + y \equiv y + x$
- $(x + y) + z \equiv x + (y + z)$

Negation

$-x = \{-x^R \,|\, -x^L\}$

- $-(x + y) \equiv -x + -y$
- $-(-x) \equiv x$
- $x + -x = 0$

Multiplication

$xy = \{x^Ly+xy^L-x^Ly^L, x^Ry+xy^R-x^Ry^R \,|\, x^Ly+xy^R-x^Ly^R, x^Ry+xy^L-x^Ry^L\}$
$= \{xy-(x-x^L)(y-y^L),\ xy-(x^R-x)(y^R-y) \,|\, xy+(x-x^L)(y^R-y),\ xy+ (x^R-x)(y-y^L)\}$

- $xy = \{x^L y, xy^L \mid x^R y, xy^R\}$ fails

- $x - x^L > 0 \wedge y - y^L > 0 \implies (x - x^L)(y - y^L) < 0 \implies xy > x^L y + xy^L - x^L y^L$

- $x0 \equiv 0x \equiv 0$

- $x1 \equiv 1x \equiv x$

- $xy \equiv yx$

- $(-x)y \equiv x(-y) \equiv -xy$

- $(x+y)z \equiv xz + yz$

- $0 < x, y \implies 0 < xy$

Division

For every positive x, there exists a y such that $xy = 1$.

- $xy^L < 0 < xy^R$ for all y^L, y^R

- y is a number

- $(xy)^L < 1 < (xy)^R$ for all $(xy)^L$, $(xy)^R$

- $xy = 1$

$y = \left\{0, \frac{1+(x^R-x)y^L}{x^R}, \frac{1+(x^L-x)y^R}{x^L} \mid \frac{1+(x^L-x)y^L}{x^L}, \frac{1+(x^R-x)y^R}{x^R}\right\}$
Recursion on $y_{n+1} = \frac{1+(x_1-x)y_n}{x_1}$, where one begins with an initial value for y_0 and find subsequent terms for y^L's and y^R's, and x_1 is some non-zero option for x.

$n^{\mathbf{th}}$ Roots

$\sqrt{x} = y = \{\sqrt{x^L}, \frac{x+y^L y^R}{y^L+y^R} \mid \sqrt{x^R}, \frac{x+y^L y^{L*}}{y^L+y^{L*}}, \frac{x+y^R y^{R*}}{y^R+y^{R*}}\}$

Where x^L and x^R are non-negative, y^L, y^{L_*}, y^R, and y^{R_*} chosen such that the denominators or non-zero.

Does our hasty assignment of $\{0\,|\,1\} = \frac{1}{2}$ align with these definitions? Yes. We can show that, surprisingly, once again, $\frac{1}{2} + \frac{1}{2} = 1$:

$$\frac{1}{2} + \frac{1}{2} = \{0 + \tfrac{1}{2}, \tfrac{1}{2} + 0 \,|\, \tfrac{1}{2} + 1, 1 + \tfrac{1}{2}\} = \{\tfrac{1}{2} \,|\, \tfrac{3}{2}\}$$

Is $\{\frac{1}{2} \,|\, \frac{3}{2}\} = \{0\,|\,\}$ true? Recall from Section 1.1, for surreal numbers to be equal, $x \geq y$ and $y \geq x$, meaning $x^R > y$, $x > y^L$, $y^R > x$, and $y > x^L$. We can see that $1 < \frac{3}{2}$, $0 < \frac{1}{2}$, $\frac{1}{2} < 1$, and $\varnothing$ is incomparable.

2.2 Set Theoretic Properties of the Surreals

Every surreal number x is a set, and given x we can define another surreal number $\{x\,|\,\}$ and continue this process indefinitely. As such, the collection of surreals forms a *class*. In Zermelo-Fraenkel the properties of a class, a collection of sets, are not axiomatized. Here we will use the language of von Neumann-Bernays-Gödel set theory[8].

The surreals are actually a *proper class*, a class that is not a set, as there exists a bijection from them to the von Neumann definition of ordinals,[9]. We can define an ordinal α by its predecessors $\beta < \alpha$ and associate it with a surreal number $\hat{\alpha} = \{\hat{\beta} : \beta < \alpha\,|\,\}$.

Since the surreals are a proper class and not a set, they are not exactly a field. Conway made this distinction by referring to the surreals as a Field, a proper class with the properties of a field. The field structure and ordering on the surreals create what Ehrlich calls an "absolute arithmetic continuum," [13], making

[8] Assuming axiom of global choice

[9] See Appendix A – Burali-Forti paradox and related concepts

the surreals to non-Archimedean ordered fields what $\mathbb{R}$ is to Archimedean ordered fields. That is, the surreals are, up to isomorphism, the largest ordered field, and every real-closed ordered field is isomorphic to an initial subfield (a subtree) of the surreals [13, p. 8] [12, p.1253]. These include $\mathbb{Q}$, $\mathbb{R}$, certain classes of hyperreals and superreals, the Levi-Civita field, transfinite ordinals, and cardinals (being identified with their initial ordinals). The surreals give us a massive, unifying structure to work with.

We can go on to define *surcomplex* numbers of the form $\alpha + i\beta$ where α and β are complex.[10] By adjoining i, the surcomplexes form an algebraically closed Field/proper class. They form the algebraic closure of the rationals extended by what Conway calls a "University" of transcendental elements [8, p.42], University indicating elements which are algebraically independent over the surreals.

Remarkably, is has recently been shown by Mantova and Berarducci that surreal numbers have a natural transseries structure [5]. Their paper is currently under review, with more work forthcoming. It builds upon work by Kruskal, L. van der Dries, and others who have conjectured at the correspondence. The unifying structure of the surreals hints at great simplifications, further elucidated in Section 4.3.

3 Applications to Game Theory

As we have seen, unto themselves the surreals form a proper class, and in turn are a subclass of two player, combinatorial games with perfect information games. [9] Classic and popular examples of such games are Konane, Chess, and Go. In these games, board positions can be assigned values based on the moves available to each player. The game associated with that value indicates what advantage, or disadvantage, the player with the first move has in that position.

[10] See Section 4.2 for why anyone would want to do this besides it being cool

3.1 Comparing Games to 0

Unlike the surreals games do not have a total ordering, nor do they have all of the structural benefits of a field that lends the surreals some form of analysis. However, games can be compared to the surreal number 0, and inherent their additive property, which allows us to determine the game's outcome. [9, p. 3]. The first such game, $\{0\,|\,0\} = *$, arose when we looked at Green Hackenbush. There we saw that the players, which we will call could only move to the zero game. This means that the the player with the first move automatically wins. $*$ is an infinitesimal, and gives us a host of other infinitesimal quantities that, in the context of a game, are rounded in the player's favor.

$*$ cannot be compared to 0. It is a quantity less than all positive numbers, and greater than all negative numbers. It is not equivalent to 0 since the game $\{\,|\,\}$ is an automatic first player loss. As such, $* + * = 0$. For any number x, we have the game $\{x\,|\,x\} = x + * = x*$. [6, p. 39] The numbers $x*$ are known as *nimbers* and are the values of nim heaps[11]

Given $*$, we have the games $\{0\,|\,*\} = \uparrow$ and $\{*\,|\,0\} = \downarrow$, with the property $\uparrow = -\downarrow$. [6, p. 64–6] [8, p.77–8]. In the game $\{0\,|\,*\}$, L can win by moving to 0, while R moving to $*$ loses due to L's response by moving to 0. This means that $\{0\,|\,*\}$ is a positive game, so $0 < \uparrow$, and $\downarrow < 0$. Neither $\uparrow$ nor $\downarrow$ can be compared to $*$, but other games such as $\uparrow + \uparrow = \uparrow\uparrow > *$.

$\{\,\|\,\}$	$= 0$	$\{\uparrow\,\|\,\downarrow\}$	$= *$
$\{0\,\|*\}$	$= \uparrow$	$\{0, *\,\|\,0\}$	$= \uparrow *$
$\{*\,\|\,0\}$	$= \downarrow$	$\{0\,\|\,0, *\}$	$= \downarrow *$
$\{0\,\|\uparrow *\}$	$= \uparrow\uparrow$	$\{0\,\|\uparrow\}$	$= \uparrow\uparrow *$
$\{\downarrow * \,\|\,0\}$	$= \downarrow\downarrow$	$\{\downarrow\,\|\,0\}$	$= \downarrow\downarrow *$

Table 1: Other Infinitesimal Games [6, p. 72]

[11]See Appendix B

3.2 Go

We can break board positions of certain games into *disjunctive sums*, non-interacting subgames. [8, p. 74]. Hackenbush trees can be chopped into simpler branches and these branches analyzed individually. A Go board can be divided into smaller boards.[12]. Moves can be made in these positions without altering the rest of the game.[13]

The technique discussed in the following section, a program established by Berlekamp and Wolfe in *Mathematical Go* in 1994, builds on Conway's work. It's primary application is in Go endgames, where a player may seek to push for a one point victory in a draw. A seasoned Go player may add it to her endgame arsenal, while an amateur may use this technique in life-or-death problems to find the correct, or best, solution.

A standard game of Go is played on a 19 by 19 grid, where played Black and White take turns placing stones on points on intersection. The goal of the game is to surround the largest area. Points are counted as totally area surrounded and number of opponent's stones captured (by surrounding them on all sides). Stones may not be moved unless captured. Board positions cannot be immediately repeated.[14] The game when both players pass (choose not to move), or when one player resigns.

3.2.1 The Meaning of a Position

When using surreal numbers to evaluate game positions in Go, there are several changes in scoring convention. Points gained by White are given a negative value and can be thought of as points not taken by Black, whereas points gained by Black are given a positive value. When breaking the board into disjunctive

[12]As an experienced player will point out, local moves have global impact (for example, a novice may neglect to tenuki) early and mid game. However, in late game local fights can often be broken into disjunctive sums

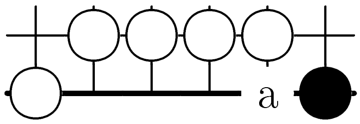
[13]In terms of score, position of pieces, or play order.

[14]This is known as the *ko* rule. See Appendix D for an example

sums, we ignore the previous scoring rules;. We will then be playing what Berlekamp dubbed the *chilled* game.

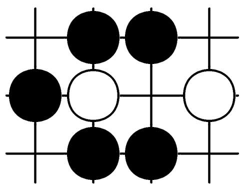
Territories are given a fractional value $\pm\frac{1}{2^n}$ where n is the number of points White or Black gets when moving first. In Figure 5, if White moves first at a, this corridor would be valued at $-\frac{1}{8}$. If Black blocks, it would be $-\frac{1}{4}$

Figure 5: A corridor



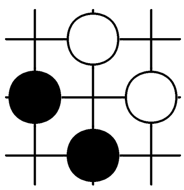
We will also enforce a one point tax on all player moves and for *sente*, taking initiative or having advantage in an area.

Figure 6: Another star is born



In Figure 6, Blacks move would gain 2 points, while White's move would gain 0, giving the game $\{2\,|\,0\}$. However, under our taxation system, Black would lose a point to both *sente* and placing a stone, making this (as a chilled game, equivalent to $\{0\,|\,0\} = \ *$, as in Figure 7.

Figure 7: A *dame*

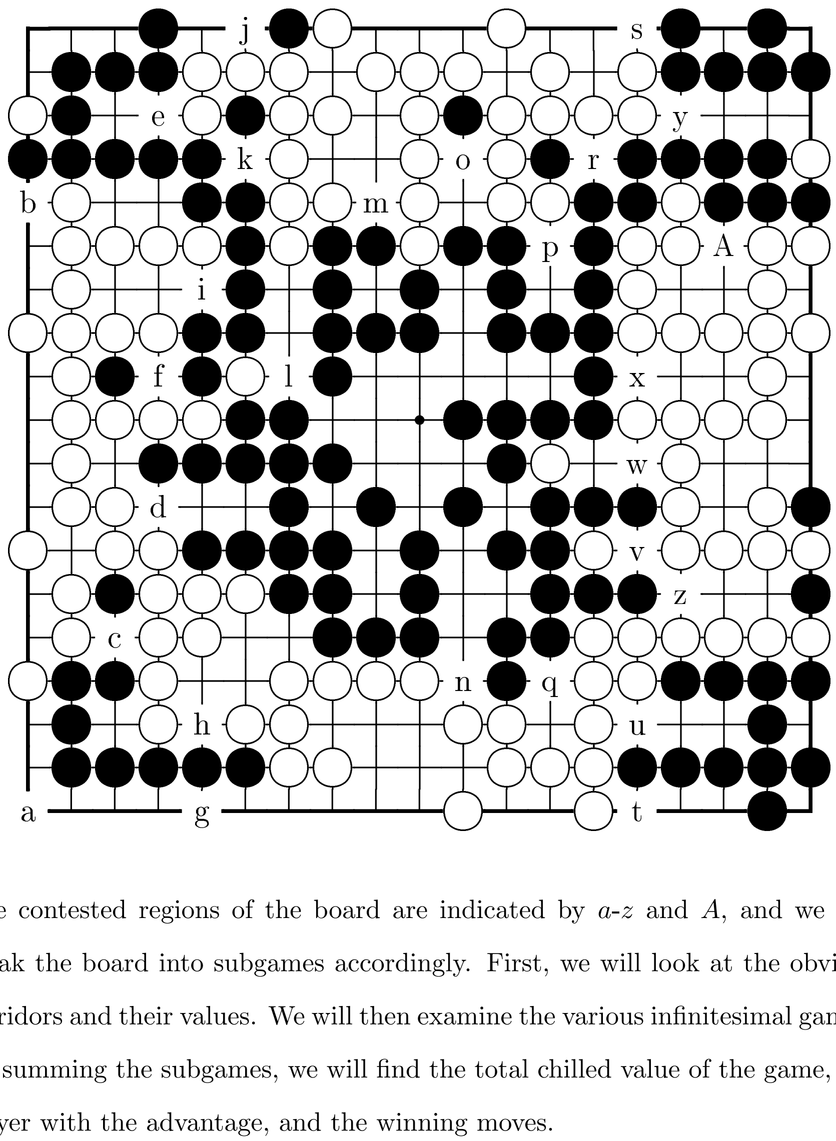


Fractional and infinitesimal values are rounded in favor of the player who makes the last move. $*$ is rounded to -1 or 1, $\uparrow$ is rounded to 0 or 1, and $\downarrow$ is rounded to -1 or 0 (the lower numbers being in favor of White, the higher in favor of Black).

3.2.2 Endgame

To best see how this method plays out is *in media res.* Here we have a problem from *Mathematical Go* [7, p. 173]

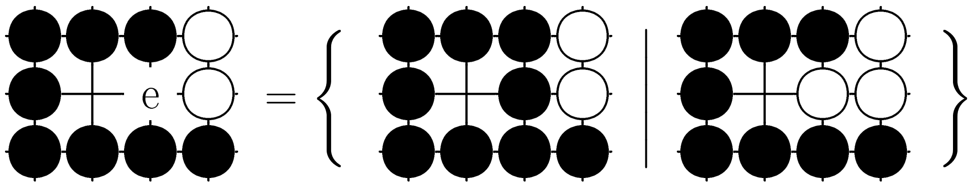
Figure 8: Find the winning moves



The contested regions of the board are indicated by *a-z* and *A*, and we will break the board into subgames accordingly. First, we will look at the obvious corridors and their values. We will then examine the various infinitesimal games. By summing the subgames, we will find the total chilled value of the game, the player with the advantage, and the winning moves.

3.2.2.1 Corridors

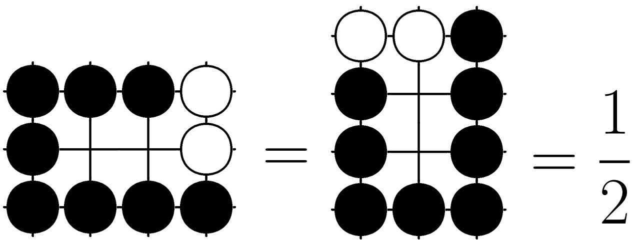
Corridors are easy to evaluate and easy to choose from: longer corridors get you more points.The regions around e, p, and q are equivalent in value for Black and White. Now, suppose we played the chilled game on these corridors. Let's look at what happens if they play at e.



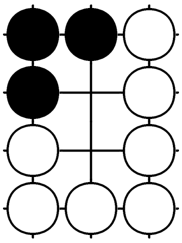
$$= \quad \{1 \mid 0\}$$

This becomes the game $\{-1 \mid -1\} = -1*$ if Black has *sente*, meaning that Black can at best round this to a 0 and White to a -2. When totaling the subgames, we will see more on why playing these corridors is a bad idea.

Without chilling, the remaining corridors (p, q, d, t, u, b, i, h, A, and x) are also easily evaluated to fractional values, as given below.

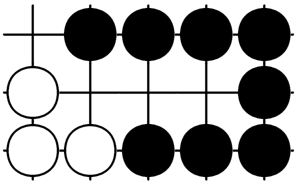


Plays at e and p

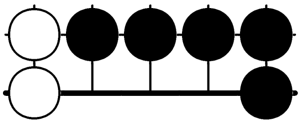


$$= -\frac{1}{2}$$

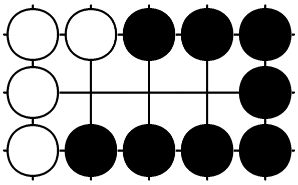
Play at q



$$=$$

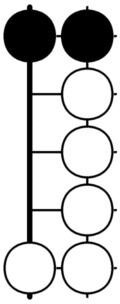


$$=$$

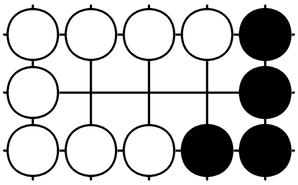


$$= \frac{1}{4}$$

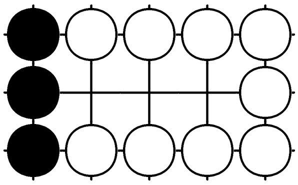
Plays at d, t, and u



$$=$$

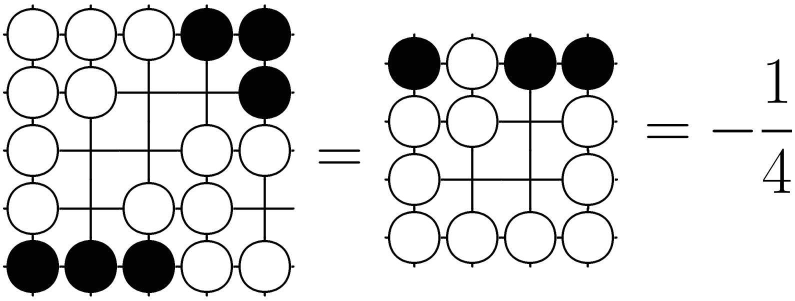


$$=$$



$$= -\frac{1}{4}$$

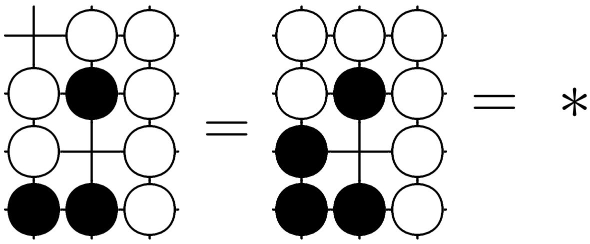
Plays at b, i, and x



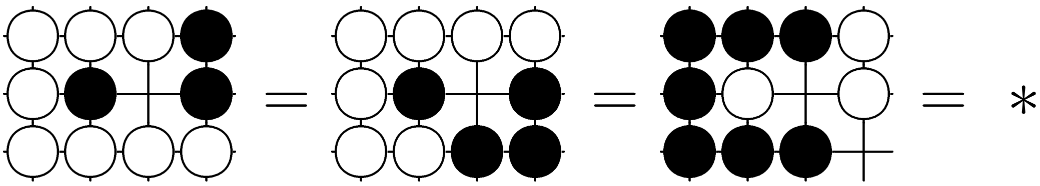
Plays at h and A

3.2.2.2 Stars

The plays at c, k, f, r, and v result in the game $*$. In the first four subgames, (for instance, c) if White plays first, she captures a black stone, but this chills to 0 due to *sente* and the tax for playing. Black has nothing. This gives $\{0 \,|\, 0\}$, which is indeed $*$. The situation at v is the same with Black having *sente*.

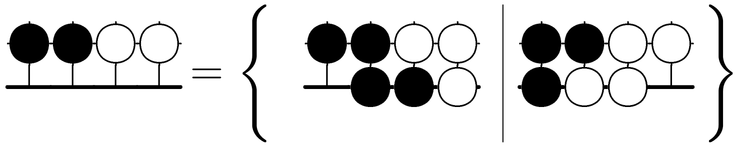


Plays at c and k

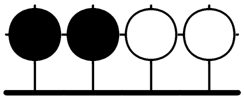


Plays at f, r, and v

The play in the region around g is more interesting. Here we have both White and Black trying to expand their territory through reversible moves. Depending on who has *sente*, they attack, and the other player is forced to respond to protect their territory. In this position the sequence of moves is reflected.



This is the game $\{1 \mid -1\}$, which chills to $*$, so

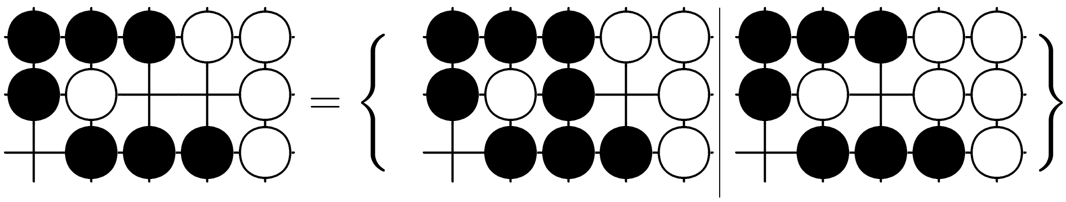


$$= \quad *$$

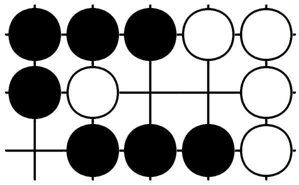
Play at g

3.2.2.3 Ups and Downs

The obvious move for Black around w is to capture the lone White stone. White can't save her stone, and has nothing other than moving to a *dame*.

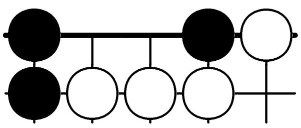


This subgame is $\{0 \mid *\} = \uparrow$. Similar games play out in a, j, m, o, and s.

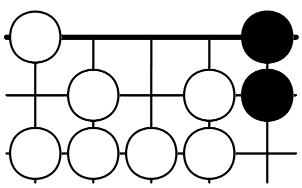


$$= \uparrow$$

Play at w

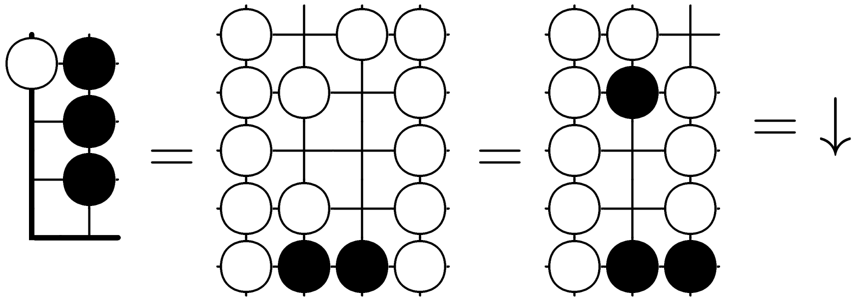


$$=$$



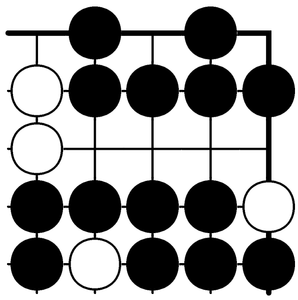
$$= \downarrow$$

Plays at j and s



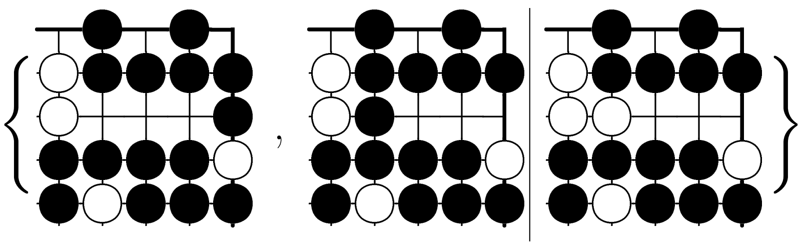
Plays at a, m and o

The remaining subgames are more complicated. We have to take more than one move into consideration in larger regions that aren't as nicely bonded by one stone at the end. Take, for example, y

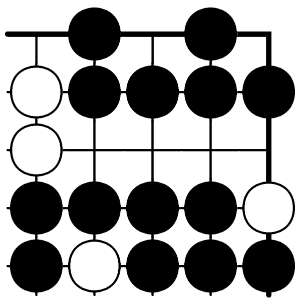


Play at y

Here we have White imposing on Black from two ends of a corridor-like shape, and Black can defend at either end. White can choose to extend her group on the left.

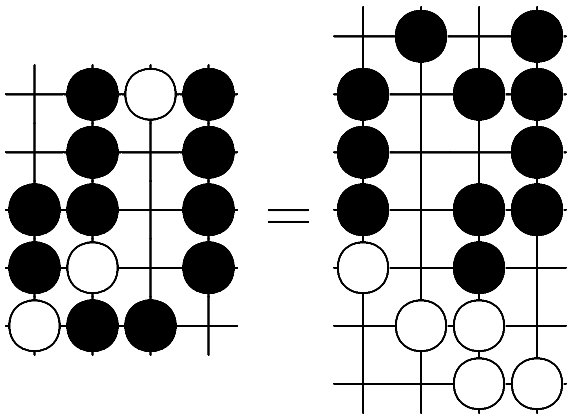


In this situation, White can make the are more of a *dame*, and Black can capture that White stone and have his points taxed away, in three variations. All in all, three $\uparrow$'s, thus



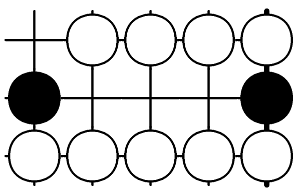
$$= \uparrow\uparrow\uparrow$$

3.2.2.4 More Ups, Downs, and Stars



$$= \uparrow\uparrow *$$

Plays at l and n



$$= \downarrow\downarrow *$$

Play at z

3.2.2.5 The winning moves

Now we may take the summand of this game.

The corridors:

$$\frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = 0$$

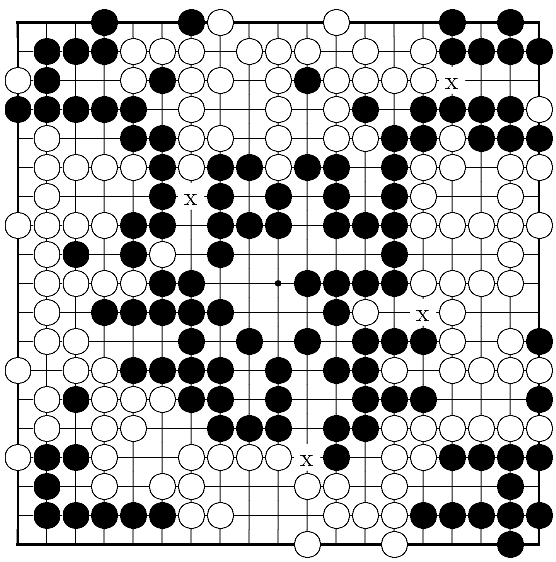
Recalling that $* + * = 0$ and $\uparrow = -\downarrow$ we can simplify the infinitesimals

$$\overbrace{* + * + * + * + * + *}^{0} + \overbrace{\uparrow + \downarrow}^{0} + \downarrow + \overbrace{\downarrow + \downarrow + \downarrow + \uparrow\uparrow\uparrow}^{0} + \uparrow\uparrow * + \overbrace{\uparrow\uparrow * + \downarrow\downarrow *}^{0}$$

What we are left with is $\downarrow + \uparrow\uparrow * = \uparrow *$. Like $*$, $\uparrow *$ is incomparable to 0 and is a first player win game. Playing on any of the corridors gives the second player the advantage: suppose Black played at d. Then the game would be $-\frac{1}{4} \uparrow *$, and White could gain back the lost points and round the game in her favor.

If White moves on a $*$ regions makes the game total $\uparrow > 0$, setting the game up for a Black win.

Figure 9: Possible winning moves



4 Applications to Analysis

Almost from their discovery, surreal numbers have been viewed as some childish, gimmicky thing. Where some concepts, such as the p-adics, remained untouched for years due to the zealotry of their authors or their level of abstraction, others suffer from attempted commercialization.[15] Conway's admonition, "..the Field **No** [the surreals] is really irrelevant to non-standard analysis," [8, p. 44], and expressions of disappointment in the surreals lacking the unification and applications he initially wanted [23, p. 4] have created an air of the surreals being an old toy we're tired of playing with. In addition to Knuth's baffling and, frankly, disturbing presentation of the construction of the surreals packaged as almost-but-not-quite lewd romantic vacation[16], it's really a wonder the surreals have been seen as anything other than borderline recreational mathematics pursued by pseudo-eccentrics.

As it stands, sentiments regarding the frivolousness of surreal numbers are misguided,[17] and those lamenting the absence of applications are categorically wrong. Here we will survey several recent advancements regarding surreals in association with non-standard analysis, surreal analysis, and transseries.

4.1 Non-standard Analysis

In various works[18] Ehrlich has proven that the surreals, in addition to generalizing the ordinals [13, p. 1253] [12, p. 8], are the largest ordered proper class and incorporate other systems[19] dealing with infinites and infinitesimals, with particular focus on Robinson's hyperreals, ${}^*\mathbb{R}$. Over the centuries mathematicians of various renown have attempted to both incorporate infinitesimals into analysis and banish them outright; happily, in his 1966 book *Non-standard analysis*,

[15] Hensel stuck with it and look where we are now
[16] Complete with Conway in the mix as "J.H.W.H. Conway"
[17] Strategies are morphisms on games and that is all I'm going to say on the matter.
[18] [13, p. 1253] [12, p. 8]
[19] Ehrlich mentions Veronese, Levi-Vicita, Hilbert, and Hahn

Robinson gave a full treatment of the subject, utilizing the transfer principle as proven by Jerzy Łoś in 1955.[20]. Succinctly, the transfer states that first order statements valid in $\mathbb{R}$ are valid in ${}^*\mathbb{R}$, allowing for a rigorous treatment of infinite and infinitesimal quantities in analysis, in early examples side stepping what we hold to be "traditional" $\epsilon - \delta$ proofs.

There is not a unique ordered field containing $\mathbb{R}$ known as *the* hyperreals. Usually, the ultrafilter construction on $\mathbb{R}^{\mathbb{N}}$ is taken to be ${}^*\mathbb{R}$. [15]. The axiomatic approach taken by Keisler (see Appendix C), as commented upon by himself [19, p. 59] and in correspondences between him and Ehrlich [13, p. 36–37], suggest there is a unique model of hyperreals in NBG such that they form a proper class isomorphic to the surreals. Ehrlich notes that it immediately follows such a relation that the surreals admit an extension to non-standard analysis, and that the transfer principle holds for the surreals as well.

4.2 Surreal Analysis

The surreal numbers do not easily lend themselves to an analysis given their generally unpleasant operations where we cannot appeal to real or ordinal operations. There have been difficulty (and sporadic attempts) at defining surreal analogues of specific real functions, such as in Section 2.1 and Gonshor's reproduction of Kruskal's definition of exponentiation [16] (the later of which has been used in Rubinstein-Salzedo and Swaminathan's 2014 paper to define the logarithm and arctangent functions), and Conway's technique of using "genetic" definitions of surreal functions $f(x) = \{f_L(x, x_L, x_R) \mid f_R(x, x_L, x_R)\}$.

In his 2004 thesis, Fornasiero generalizes indefinite Riemann integration for

[20]Debates regarding the legitimacy of NSA, whether it is worthwhile to pursue, and so on, are beyond the interest of this paper. Here we merely wish to show a connection between constructions of the hyperreals and the surreals

recursively defined, genetic surreal functions. [14, p. 25, 45]:

$$
\begin{aligned}
F(x) &= \int_0^x f(t)dt \\
&= \Big\{ F(x_L) + \int_{x_L}^{x} f_P^L(t, t^0, f(t^0)), \{F(x_R) - \int_{x}^{x^R} f_P^R(t, t^0, f(t^0)) \Big| \\
&\qquad\qquad F(x_L) + \int_{x_L}^{x} f_P^R(t, t^0, f(t^0)), \{F(x_R) - \int_{x}^{x^R} f_P^L(t, t^0, f(t^0)) \Big\}
\end{aligned}
$$

where P varies on partitions of (x_L, x) or (x, x_R) depending on the context. This is a marked improvement on the formula given by Conway in 2000 in his epilogue to *ONAG* 2^{nd} ed, attributed to Norton and Kruskal. It addresses Conway's issue of translation invariance, but encounters some of the same problems. In particular, $\exp(x)$ as defined by Kruskal and Gonshor, and other recursive definitions provided by Fornasiero, fails to give the correct antiderivative over $[0, \omega]$, producing e^{ω} and not $e^{\omega} - 1$.

In their joint paper *Analysis on Surreal Numbers*, Rubinstein-Salzedo and Swaminathan present several new achievements in surreal analysis: a formula for the limit of a sequence and a complete characterization of convergent sequences, consequently allowing for the evaluation of limits and derivatives of surreal functions, Maclaurin expansions are used in the style of Gonshor to provide genetic definitions of $\arctan(x)$ and $-\log(1 - x)$. [24, p. 8], the Intermediate Value Theorem is proven for surreal functions despite the surreals not being Cauchy complete, and finally they show that if a consistent definition of integration exists, the Fundamental Theorem of Calculus will hold over the surreals.

A host of open questions remain in surreal analysis. At the forefront are the development of a consistent integration formula for both definite and indefinite integrals, and finding formula for transcendental functions. Other ares in need of development are differential equations, evaluation of series, α-partial derivatives for α an ordinal, and searching for greater generalization as a whole. There are grave doubts on the part of the author that "surcomplex analysis" is anything other than a bad French pun.

4.3 Transseries

The field of transseries $\mathbb{T}$ forms another extension of $\mathbb{R}$. To do this, we extend the differential field of Laurent series $\mathbb{R}((x^{-1}))$, ordered by making x infinite and requiring $x' = 1$, to *transmonomials* of the form $x^b e^L$, where $b \in \mathbb{R}$ and L is log-free. The term was coined by Écalle in solving Dulac's conjecture.[21] Transseries occur as asymptotic solutions of differential and functional equations, such as

$$\begin{aligned} \frac{1}{1-x^{-1}} &= 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots \\ \zeta(x) &= 1 + 2^{-x} + 3^{-x} + 4^{-x} + \cdots \\ \Gamma(x) &= \frac{\sqrt{2\pi}e^{x(\log x - 1)}}{x^{1/2}} + \frac{\sqrt{2\pi}e^{x(\log x - 1)}}{12x^{3/2}} + \frac{\sqrt{2\pi}e^{x(\log x - 1)}}{288x^{5/2}} + \cdots \end{aligned}$$

Transseries arise naturally in asymptotic analysis, model theory, computer algebra (computing asymptotic expansions), and show promise in non-Archimedean geometry. [28, p. 6]

It has been conjectured by van der Dries, van der Hoeven, Kuhlmann, and Matusinsk [5, p. 1] that the surreals, with the Gonshor exponential, have a transseries structure. In March 2015, Berarducci and Mantova showed that "the surreal numbers have a natural transseries structure...and a compatible Hardy-type derivation." [5, p. 2]. This powerful correlation will prove to be an invaluable tool, leading to correspondences between growth rate functions and numbers [28, p. 6], and in general a simplified approach to dealing with transseries.

[21] As stated by Aschenbrenner et al, "a polynomial vector field in the plane can only have finitely many limit cycles," [4, p. 3] which is related to Hilbert's 16th problem.

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A Set Theory Terminology

Dedekind Cut - Dedekind constructed the real numbers from the rationals by dividing the rationals into two sets, L and R, such that no element of L was greater than any element of R, defining a new number $\{L|R\}$ (in the case where nether L nor R has an extreme point, i.e L contains no greatest element, R contains no least element).

Ordinal number - Order type of a well ordered set.

> For an infinite set, the order type determines the cardinality, but well-ordered sets of a particular cardinality can have many different order types. For a countable infinite set, the set of possible order types is uncountable.

Cardinal number - Generalization of natural numbers, used to measure the cardinality of sets.

Total order - A binary relationship on some set which is transitive, anti-symmetric, and total.

Anti-symmetric - For R a relation, $\forall a, b \in X,\ R(a, b) \wedge R(b, a) \implies a = b$.

Reflexive - $a \leq a$.

Total order - $\forall a, b, \in X, aRb \vee bRa$.

If X is **totally ordered** under $\leq$ (general binary relation), then $\forall a, b \in X$:

If $a \leq b$ and $b \leq a$, then $a = b$

If $a \leq b$ and $b \leq c$, then $a \leq c$

$a \leq b$ or $b \leq a$

ex. 1 Any subset of a totally ordered set is itself totally ordered (restricted by the order on the whole set)

ex. 2 Any set of cardinal or ordinal numbers (both are well ordered)

ex. 3 The lexicographical order on the Cartesian product of a set of totally ordered sets indexed by ordinals (such as words ordered alphabetically)

ex. 4 $\mathbb{R}$, and by (1) its subsets, under $<$ and $>$

ex. 5 If X is any set, f injective from X to a totally ordered set, then f induces a total ordering on X by setting $x_1 < x_2 \iff f(x_1) < f(x_2)$

ex. 6 In particular, $\mathbb{C}$ is not totally ordered: Suppose $0 < i$, but $i^2 = -1 < 0$. Suppose $0 < -i$, but $(-i)^2 = -1 < 0$...

Partial order - Reflexive, anti-symmetric, and transitive.

Well-order - A well-order on a set S is a total order with the property that every nonempty subset of S has a least element in the ordering.

Every well-ordered set is uniquely order isomorphic to a unique ordinal number, called the order type. The position of each element is also given by an ordinal number.

Lexicographical order - Given two partially ordered sets A and B, the lexicographical order on $A \times B$ is defined as $(a, b) \leq (a', b') \iff [a < a' \vee (a = a' \wedge b \leq b')]$. These appendices are not lexicographically ordered.

Ordered field - A field with a total ordering of its elements compatible with the field operations.

A finite field cannot be ordered.

A field $(F, +, \cdot)$ with a total order $\leq$ on F is an ordered field if:

whenever $a \leq b$, $a + c \leq b + c$

when $0 \leq a$ and $0 \leq b$, $0 \leq ab$

Every ordered field can be embedded into the surreal numbers

Set - a collection of (distinct) objects (elements). For sets A and B, $A = B \iff \forall a \in A \wedge \forall b \in B, b \in A \wedge a \in B$.

Power set - set of all subsets of a set S, denoted $\mathcal{P}(S), 2^S$, etc. If $|S| = n < \infty, |\mathcal{P}(S)| = 2^n$.

Class - a collection of objects that can be defined by a certain condition.

Proper class - a class which is not a set. The surreals form a proper class.

Russell's Paradox - Let $R = \{x|x \notin x\}$. Then $R \in R \implies R \notin R$.

Cantor's Paradox - There is no greatest cardinal number (well ordered so we can make this statement). Suppose C is the greatest cardinal. Then C is a set, and $|C| < |2^C|$.

> The collection of infinite cardinalities is itself infinite and larger than the other infinities
>
> Since the cardinal numbers are well ordered via indexing by the ordinals, there is no greatest ordinal
>
> The cardinal numbers are a proper class, i.e. do not form a single set
>
> If S is any set, then S cannot contain elements of all cardinalities. There is a strict upper bound on the cardinalities of the elements of S

Burali-Forti paradox - Constructing the set of ordinals leads to a contradiction as the set would have the properties of an ordinal and would interrupt the strict ordering of the set. Thus the ordinals are a proper class.

Axiom of choice - For every indexed family $(S_i)_{i \in I}$ of nonempty sets, there exists an indexed family $(x_i)_{i \in I}$ of elements such that $x_i \in S_i$ for every $i \in I$.

Axiom of global choice - A variant on the axiom of choice applying to proper classes and sets of sets, in particular the surreals and its subclasses. It states that there exists a global choice function τ such that for every non-empty set z, $\tau(z)$ is an element of z.

A.1 The construction of $\mathbb{R}$ from $\mathbb{Q}$

A subset $\alpha \in Q$ is a *cut* if

1. α is non-empty and $\alpha \neq \mathbb{Q}$

2. If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$

   (a) If $p \in \alpha$ and $q \notin \alpha$, then $p < q$

   (b) If $r \notin \alpha$ and $r < s$, then $s \notin \alpha$

3. If $p \in \alpha$ then $p < r$ for some $r \in \alpha$

α is the lower set L in the definition above; R is the complement of L, $L \cup R = \mathbb{Q}$, so one completely determines the other.
$\alpha < \beta$ means that α is a proper subset of β. Let $\mathbb{R}$ be the set of all cuts α, totally ordered under $<$, and has the least upper bound property/Dedekind completeness.

Addition: $\alpha + \beta = \{a + b \,|\, a \in \alpha \wedge b \in \beta\}$
Subtraction: $\alpha - \beta = \{a - b \,|\, a \in \alpha \wedge b \in \beta\}$
Negation: $-\beta = \{a - b \,|\, a < 0 \wedge b \in (\mathbb{Q} - \beta)\}$
Multiplication: $\alpha \cdot 0 = 0 \cdot \alpha = 0$, and for $\alpha, \beta \geq 0$

$$\alpha\beta = \{ab \,|\, (a \geq 0 \wedge a \in \alpha) \wedge (b \geq 0 \wedge b \in \beta)\} \cup \{x \in \mathbb{Q} \,|\, x < 0\}$$

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0, \beta < 0 \\ -[(-\alpha)\beta] & \text{if } \alpha < 0, \beta > 0 \\ -[\alpha(-\beta)] & \text{if } \alpha < 0, \beta < 0 \end{cases}$$

Division: For $\alpha \geq 0$, $\beta > 0$, $\frac{\alpha}{\beta} = \{\frac{a}{b} \,|\, a \in \alpha \wedge b \in \mathbb{Q} - \beta\}$. Playing around with negation gives us the other cases, simlar to above.
Roots: For $x \geq 0$, $\sqrt[n]{x} = \{y \in \mathbb{R}^+ \,|\, y^n < x\}$
It can be shown that every real number corresponds to a cut, and you can tediously find how a number like π or e or e^π or whatever suits your fancy is defined by a cut, but it isn't enlightening in the least. Perhaps more relevant, $\mathbb{R}$ can be constructed from the surreals (in essentially the same way, by "embedding" $\mathbb{R}$ in the surreals (apparently the number of copies of $\mathbb{R}$ in the surreals forms a proper class [13]).

B Game Theory Terminology

combinatorial game - a game with no chance, sequential play (players take turns), perfect information (all past moves are known), and only one winner (no ties)

partisan - some moves are available to one player and not the other (i.e. not impartial). For example, Go and Chess are partisan games. "**Misère**" games switch this.

normal play convention - in combinatorial game theory, the normal play convention (win condition), of an impartial game is that the last player able to move is the winner.

Sprague-Grundy theorem - Every impartial game under the normal play convention is equivalent to a nimber.

nim - a game where players take turns removing objects from heaps.

nimbers - the values of nim heaps, the nimbers are the ordinals under nimber arithmetic.

C Axioms of ${}^*\mathbb{R}$

From Keisler [19]

Axiom A

$\mathbb{R}$ is a complete ordered field

Axiom B

${}^*\mathbb{R}$ is an ordered field extension of $\mathbb{R}$

Axiom C

${}^*\mathbb{R}$ has a positive infinitesimal, that is, an element ε such that $0 < \varepsilon$ and $\varepsilon < r$ for every positive $r \in \mathbb{R}$.

Axiom D (Function Axiom)

For each real function f of n variables there is a correpsonding hyperreal function f^*, called the *natural extension* of f. The field opreations of $^*\mathbb{R}$ are the natural extensions of the field operations of $\mathbb{R}$.

Axiom E (Transfer Axiom)

Given two systems of formulas S, T with the same variables, if every real solution of S is a solution of T, then every hyperreal solution of S is a solution of T.

D Go Terminology

Atari - When a stone only has one liberty and may be captured in the next move if not given more

Dame - Neutral, unfilled points lying between stones.

Eyes - internal liberties; a group with at least two eyes is alive (cannot be captured)

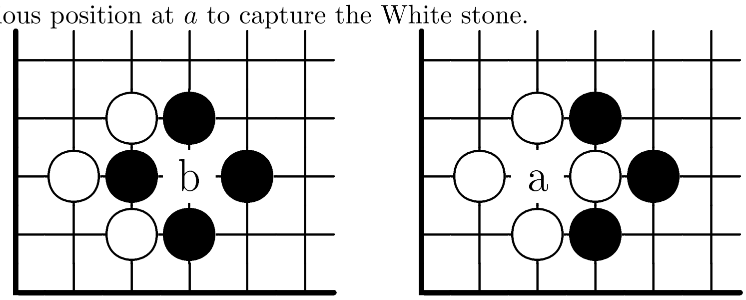
Gote and Sente - A move with *sente*/initiative compels the opponent to respond directly to that move, or have *gote*. *Gote* is generally seen as negative as you are essentially losing the edge (though this can be part of some larger gambit)

Joseki - Traditional opening and mid-game sequences of play.

Kifu - Go game records showing order of play

Komi - A point advantage given to white for going second. Depending on the rank of the players or the scoring rules, this can be anywhere from .5 to 7.5 points. This tradition did not begin until the 1930s, and the exact point advantage White should get for not having *sente* has evolved since, and has greatly influenced opening stragegies for Black and White.

Ko - At its most basic, *ko* rules prevent immediately repeating a board position so as to not have infinite loops of recapturing stones. A simple example would be if White captures the Black stone by playing at b, Black cannot play the previous position at a to capture the White stone.



Liberty - A open point adjacent to a stone. At least one liberty is necessary for a set of stones to survive.

Miai - Two points on the board equivalent in value; if Black takes one, White will take the other.

Tedomari - The last play. Applies to not only the last play in the game, but for different stages in the game and different parts of the board.

Tenuki - Playing somewhere else instead of responding to a move local. Often used to gain *sente*.