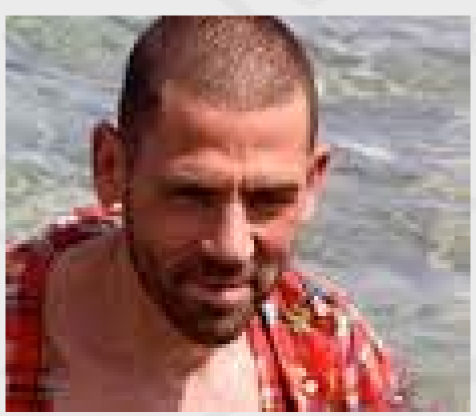
TOPOLOGY + QUANTUM CRITICALITY

1D TOPOLOGICAL ANDERSON INSULATORS

Dima Bagrets

U of Cologne



Alex Altland

Alex Kamenev

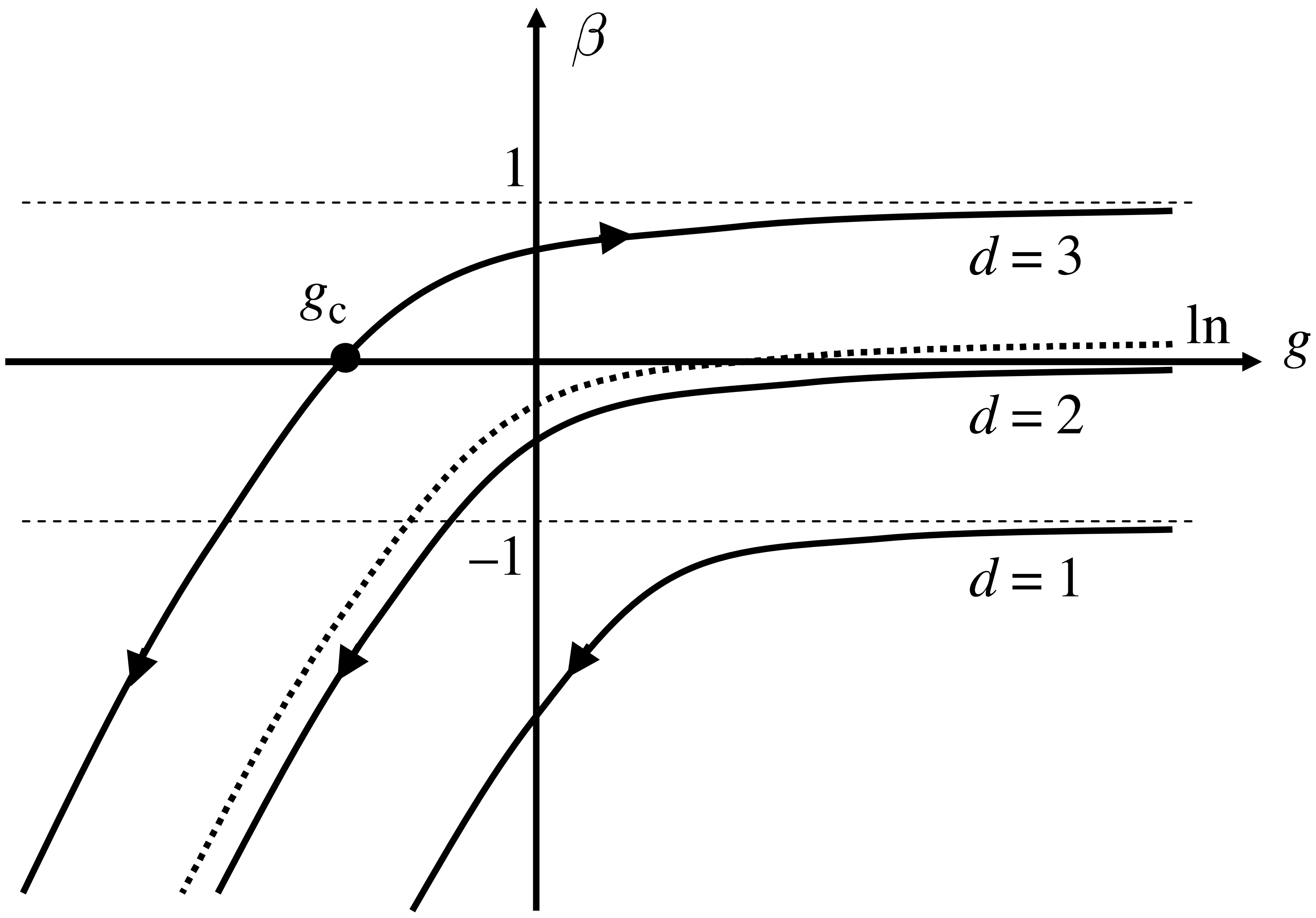
PRL **112**, 206602 (2014)

PRB **91**, 085429 (2015)

Thanks to Nick Read for inspiring critique

Scaling Theory of Localization

Abrahams, Anderson, Licciardello and Ramakrishnan, 1979



one parameter

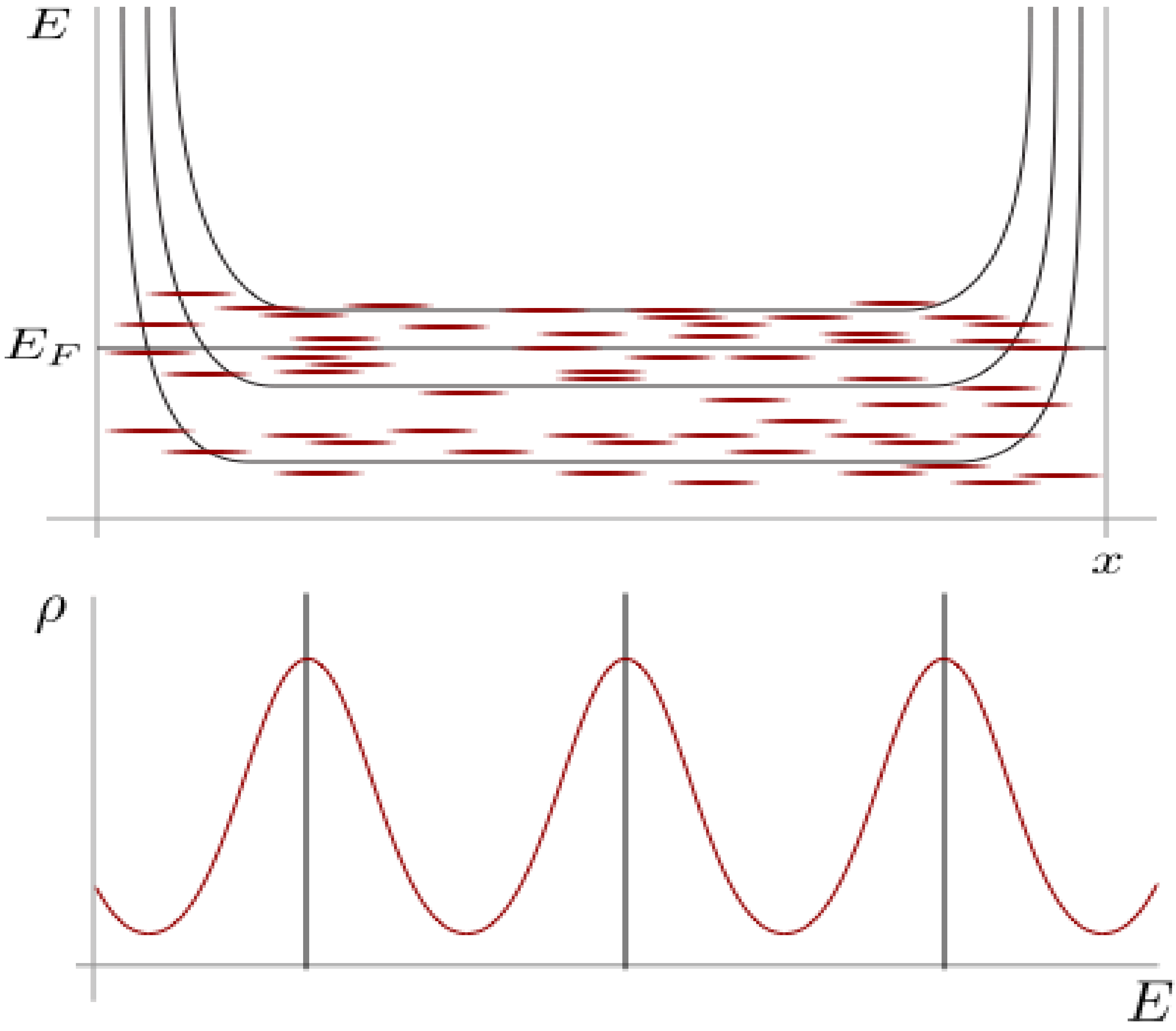
(conductance)

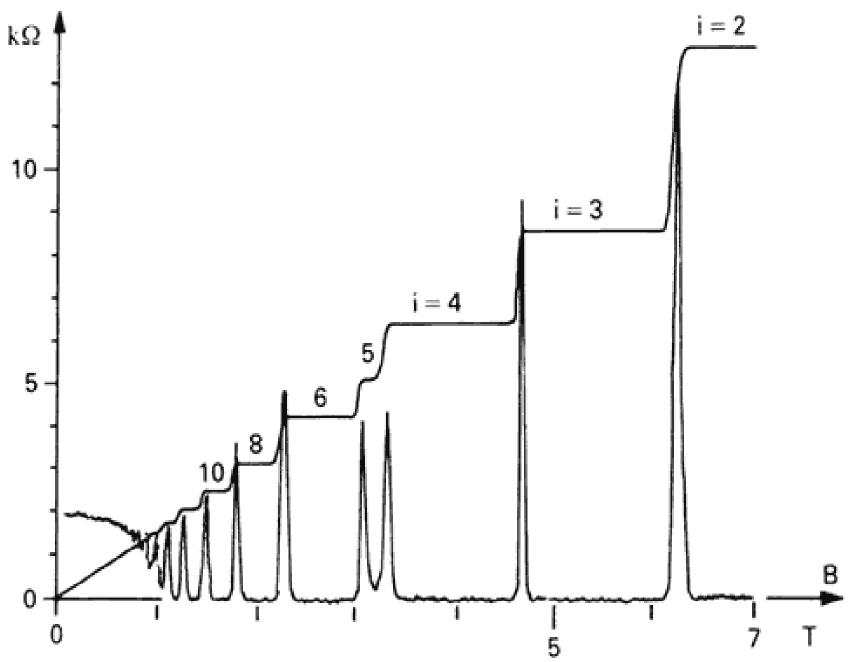
scaling

Exact results in 1D for 3 Wigner-Dyson symmetry classes (no transitions):

Efetov, Larkin 1986, Zirnbauer, Mirlin 1994, Brouwer 1996

Quantum Hall Effect (class A)





1998 Nobel prize press release

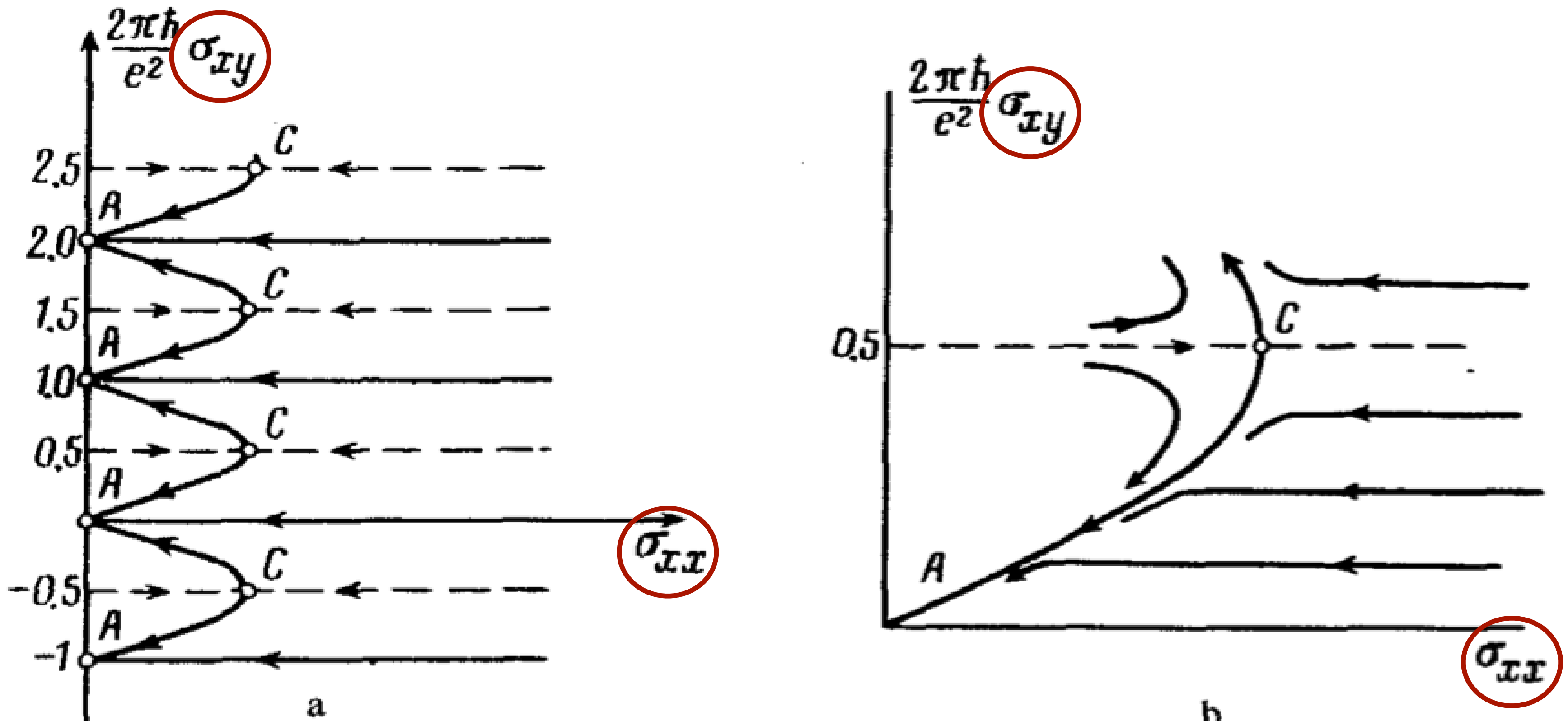
Scaling Theory of the QHE

universal interplay disorder/topology described by two parameters

σ_{xx} : average longitudinal transport coefficient

σ_{xy} : average topological index

- two parameter criticality



Khmelnitskii, 1983, Pruisken 1984

Periodic table

complex case:

Cartan\d	0	1	2	3	4	5	6	7	8	9	10	11	⋯
A	$\mathbb{Z}$	0	$\mathbb{Z}$	[illegible]	[illegible]	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	⋯
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	⋯

IQH

SSH

real case:

Cartan\d	0	1	2	3	4	5	6	7	8	9	10	11	⋯
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	⋯
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	⋯
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[illegible]	[illegible]	[illegible]	[illegible]	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	⋯
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	⋯
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	⋯
CII	0	$2\mathbb{Z}$	[illegible]	[illegible]	[illegible]	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	⋯
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	⋯
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	⋯

'Kitaev chain'

topolgical insulator (BiSe)

spin QH

1d delocalization phenomena

- delocalization in quasi-one dimensional geometries
  - 1998 AIII quantum wire (Brouwer, Mudry, Simons, Altland)
  - 1999 D quantum wire (Brouwer, Mudry, Furusaki)
  - 2004 AIII, D, BDI, DIII, CII (Read, Gruzberg, Vishveshwara)

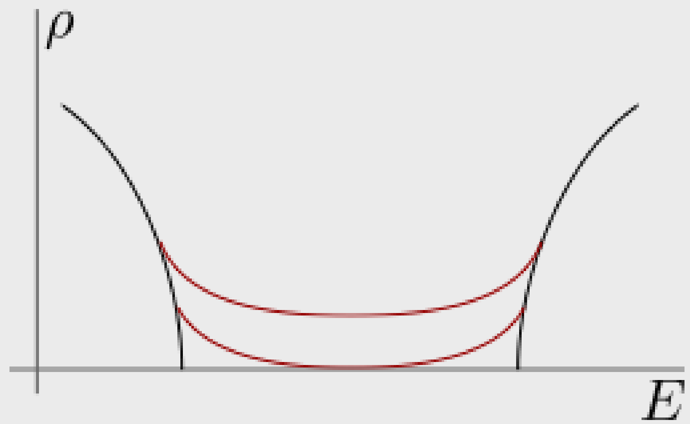
Unconventional criticality in 2d

  - 2001 Class C spin quantum Hall effect (Chalker et al., Gruzberg et al.)
  - 2001 Class D quantum criticality (Fisher et al., Read et al.)

- universal interpretation: topological insulators at quantum critical point

Addition of disorder to a clean band insulator …

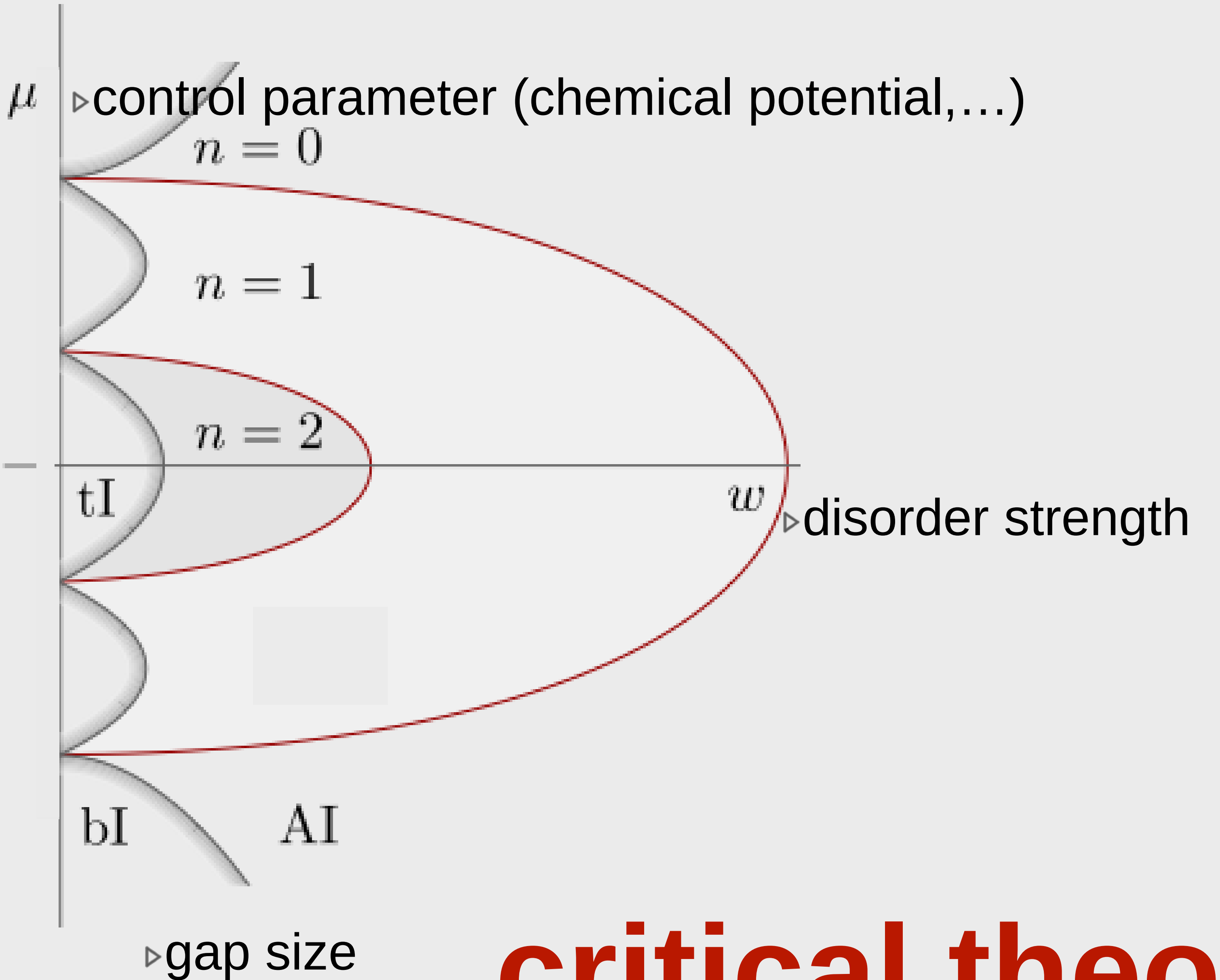
- invalidate k-theory of band gaps/topological indices
- destroys band gaps and
- turns band insulator into a nominal metal
- renders topological indices statistically distributed



	clean	disordered
k-theory	+	-
band gap	+	-
index	+	(+)

topological Anderson insulator

cf. Motrunich, Damle and Huse 2001, Groth et al. 2010



critical theory ?

Universal theory of disordered Top Insulators

- describe topological invariants without reference to k-space
- describe criticality in terms of two observables:

g : transport coefficient
χ : configurational average of index

- bare values $g = g(\mu, w, \dots),\ \chi = \chi(\mu, w, \dots)$

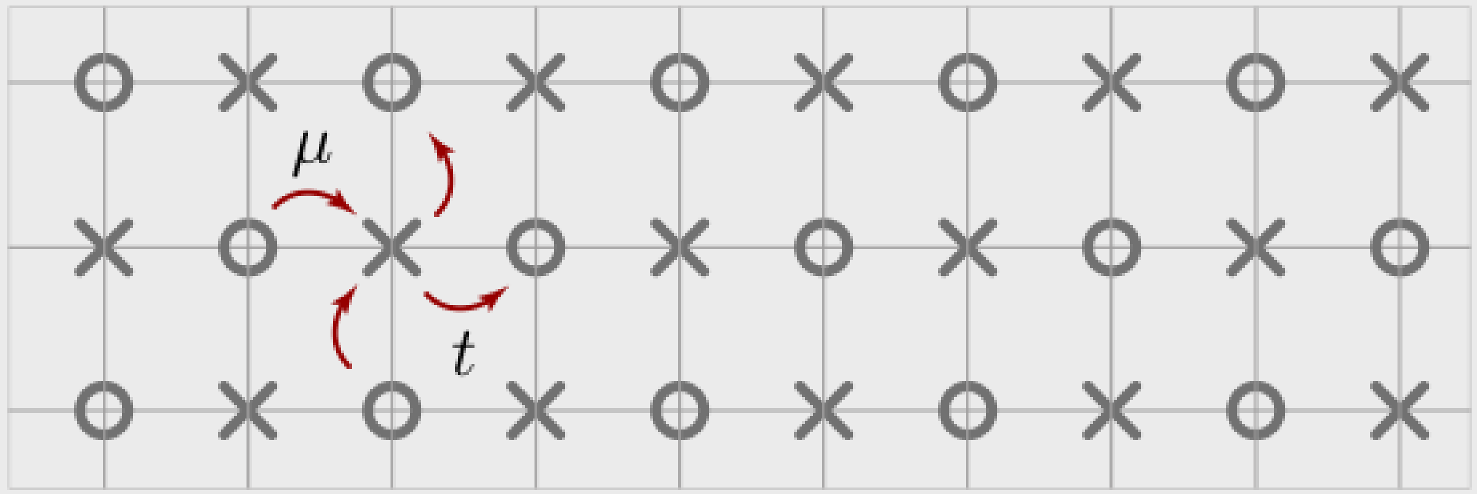
generically: $(g, \chi) \overset{L\to\infty}{\longrightarrow} (0, n)$

critical: $(g, \chi) \overset{L\to\infty}{\longrightarrow} (g_{\text{crit.}}, n + 1/2)$

- describe edge state formation

- similar architecture in all symmetry classes, d=1,2

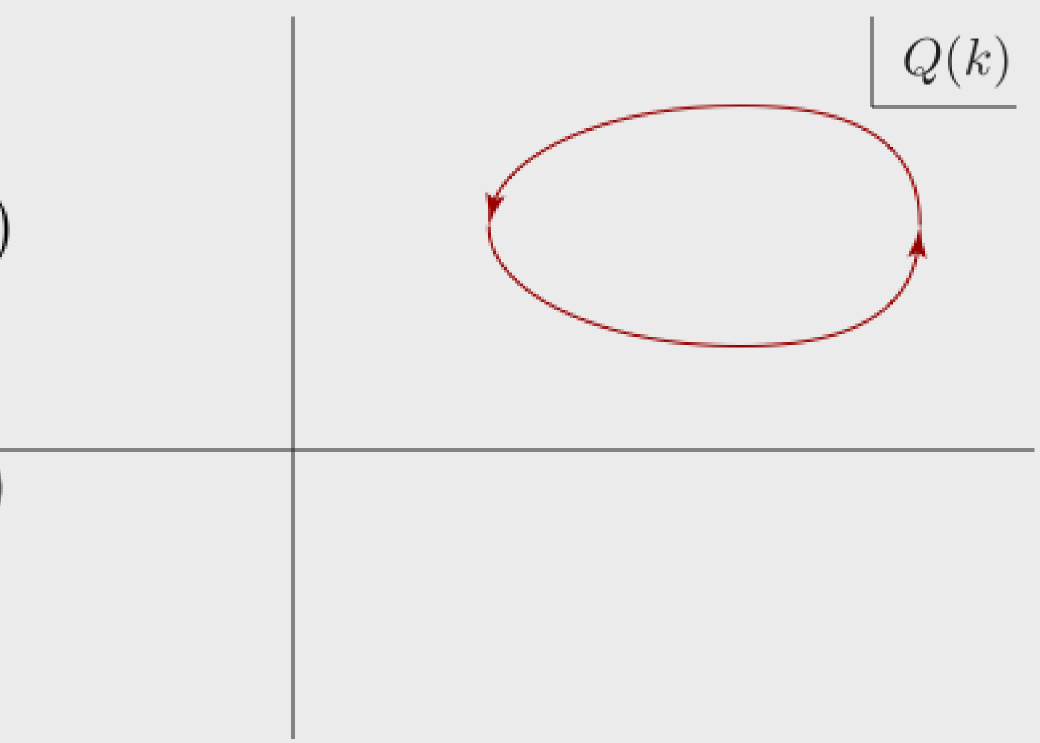
Z classes: AIII quantum wire (SSH model)



$$\hat{H} = \begin{array}{cc} \begin{pmatrix} & \hat{Q} \\ \hat{Q}^\dagger & \end{pmatrix} & \begin{array}{c} \times \\ \circ \end{array} \end{array}$$

- winding number (clean)

$$n = -\frac{i}{2\pi}\int_0^{2\pi} dk\,\mathrm{tr}\,(\hat{Q}^{-1}\partial_k \hat{Q})$$



single channel: $n = \Theta(t-\mu)$

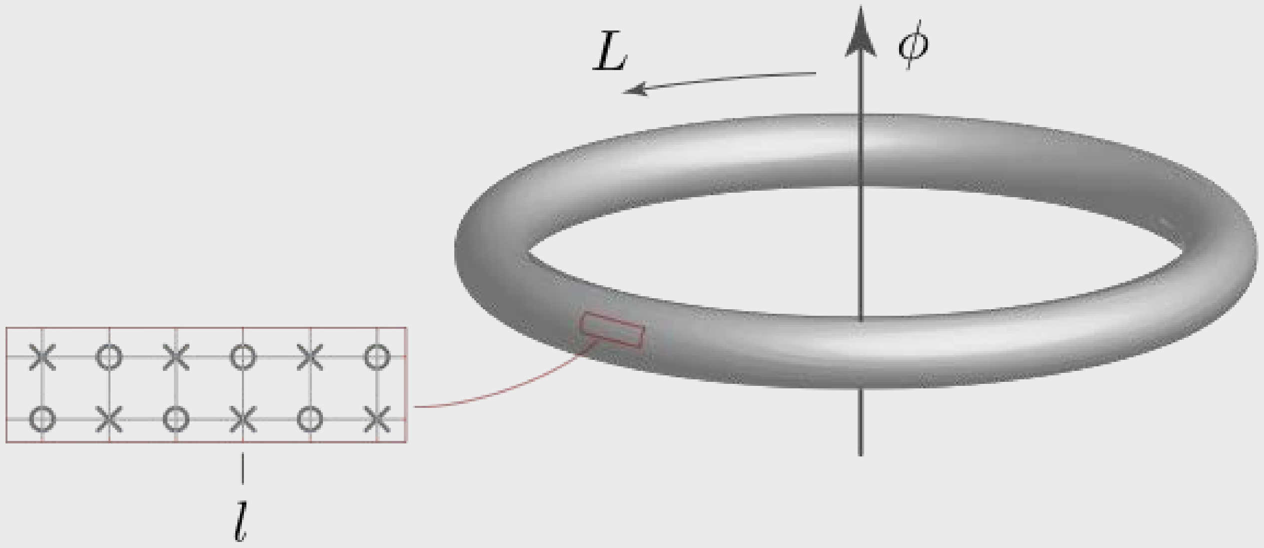
Nchannels: $n \in [0, N]$

generalized definition of winding number

$$n = -\frac{i}{2\pi}\int_0^{2\pi} dk\,\mathrm{tr}\,(\hat{Q}^{-1}\partial_k \hat{Q})$$

- generalized definition

$$\boxed{\begin{aligned} |\circ_l\rangle &\to e^{+i\frac{\phi l}{L}}\,|\circ_l\rangle \\ |\times_l\rangle &\to e^{-i\frac{\phi l}{L}}\,|\times_l\rangle \end{aligned}}$$



$$Z(\phi) \equiv \frac{\det \hat{G}_0(\phi_0)}{\det \hat{G}_0(i\phi_1)}, \qquad G_0(\theta) \equiv (i0 - \hat{H}(\theta))^{-1}, \qquad \phi = (\phi_0, \phi_1)^T,$$

cf. Nazarov, 94

$$n \equiv \chi = -\frac{1}{2\pi}\int_0^{2\pi} d\phi_0\, Z(\phi)\big|_{\phi_1=0},$$

$$g = (\partial^2_{\phi_0} + \partial^2_{\phi_1})\big|_{\phi=0} Z(\phi)$$

field integral

$$Z(\phi) = \int \mathcal{D}T \, \exp(-S[T])$$

$$S[T] = \int_0^L dx \left[\frac{\tilde{\xi}}{4} \, \mathrm{str}(\partial_x T \partial_x T^{-1}) + \tilde{\chi} \, \mathrm{str}(T^{-1}\partial_x T) \right]$$

- matrix fields $T = U \begin{pmatrix} e^{y_1} & \\ & e^{iy_0} \end{pmatrix} U^{-1}$
- boundary conditions $T(L) = T(0) \begin{pmatrix} e^{\phi_1} & \\ & e^{i\phi_0} \end{pmatrix}$
- good picture: path integral of quantum point particle in time L.

Understanding the path integral

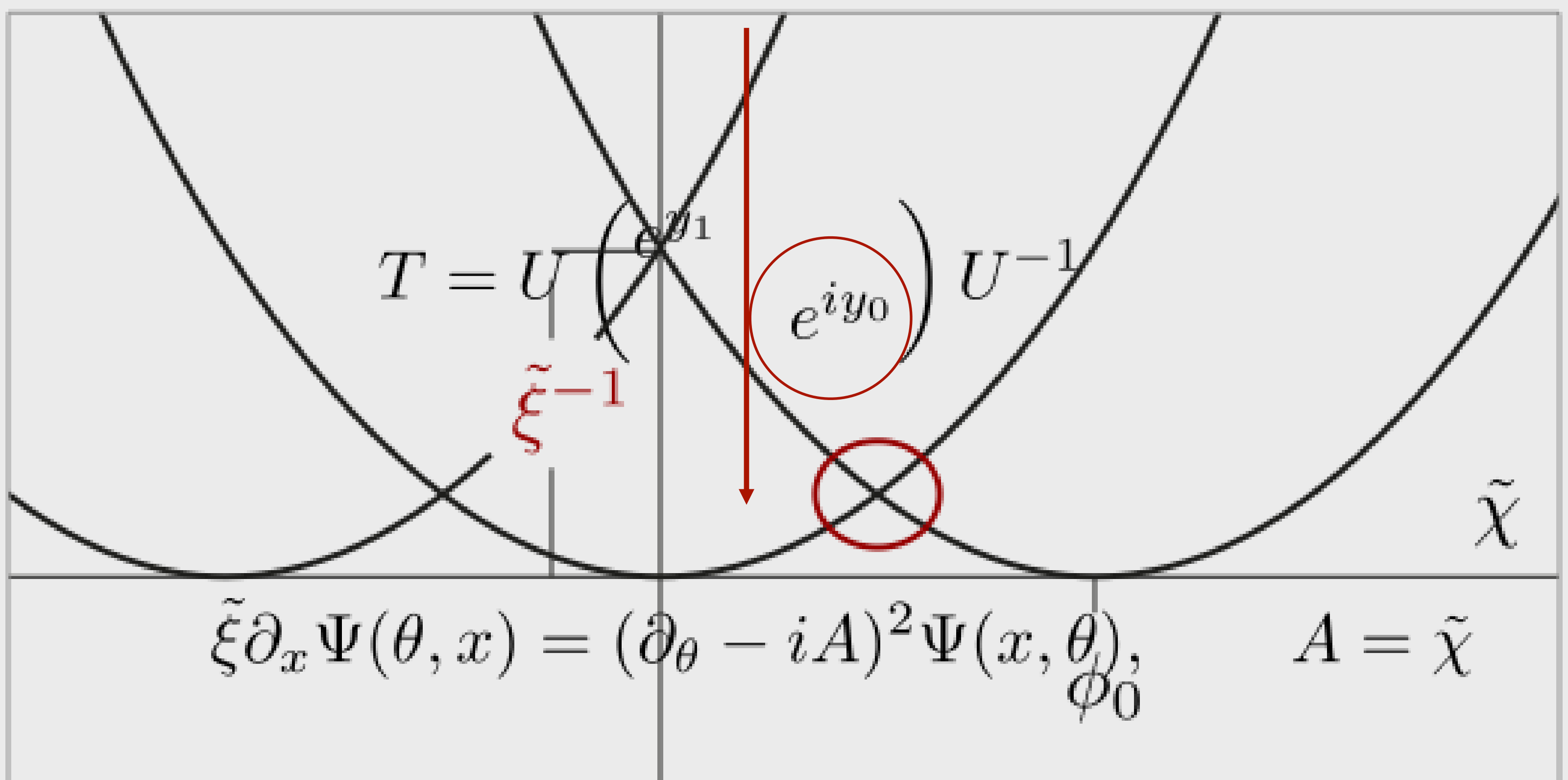
- focus on compact sector: $y_0 = \theta$
- theory reduces to QM on a ring with twisted boundary conditions and subject to magnetic flux

$$Z(\phi) = \int \mathcal{D}T \; \exp(-S[T])$$

$$Z(\phi) = \int \mathcal{D}\theta \; \exp(-S[\theta])$$

$$S[T] = \int_0^L dx \left[\frac{\tilde{\xi}}{4} \mathrm{str}(\partial_x T \partial_x T^{-1}) + \tilde{\chi}\, \mathrm{str}(T^{-1}\partial_x T) \right]$$

$$S[\theta] = \int_0^L dx \left[\frac{\tilde{\xi}}{4} \partial_x \theta \partial_x \theta + i\tilde{\chi}\, \partial_x \theta \right]$$



$$T = U \begin{pmatrix} e^{y_1} & \\ & e^{iy_0} \end{pmatrix} U^{-1}$$

$$\tilde{\xi}\partial_x \Psi(\theta, x) = (\partial_\theta - iA)^2 \Psi(x, \theta), \qquad A = \tilde{\chi}$$

transfer matrix solution of full problem

$$\tilde{\xi}\,\partial_x \Psi(y,x) = \frac{1}{J(y)}(\partial_\alpha - iA_\alpha)J(y)(\partial_\alpha - iA_\alpha)\Psi(y,x),$$

$$J(y) = \sinh^{-2}\left(\frac{1}{2}(y_1 - iy_0)\right) \qquad A = \tilde{\chi}(1,i)^T$$

- solvable problem:
  - eigenfunctions: $\Psi_l(y) = \sinh\left(\frac{1}{2}(y_1 - iy_0)\right) e^{il_\alpha y_\alpha} \qquad (l_0, l_1) \in (\mathbb{Z} + \frac{1}{2}, \mathbb{R})$
  - eigenvalues: $\epsilon_l = (l_0 - \tilde{\chi})^2 + (l_1 - i\tilde{\chi})^2$
  - solution by spectral sum:

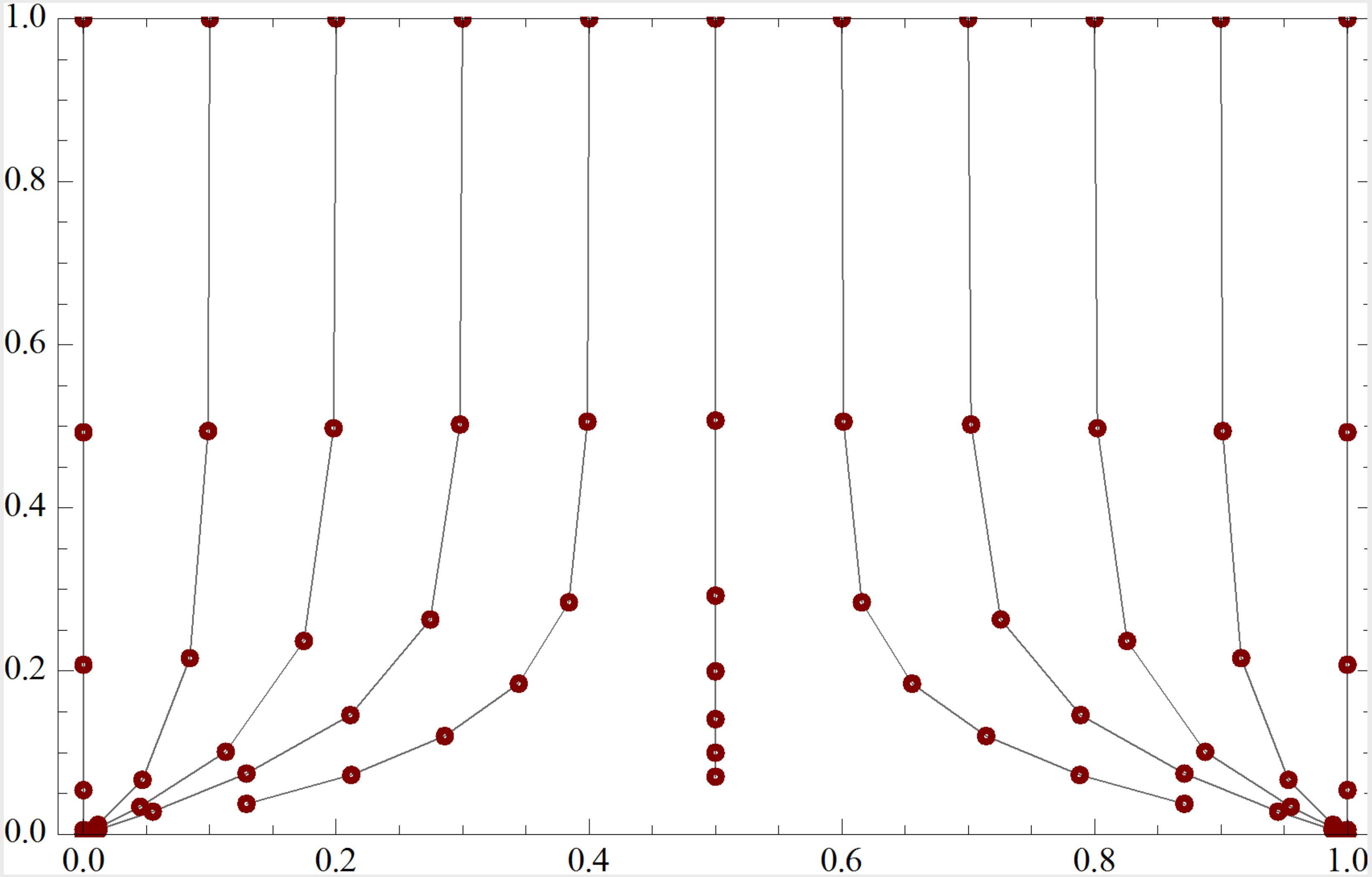
$$\Psi(\phi, L) = 1 + \frac{1}{\pi} \sum_{l_0 \in \mathbb{Z}+\frac{1}{2}} \int dl_1\, \frac{\Psi_l(\phi)}{l_0 + il_1}\, e^{-\epsilon_l L/\tilde{\xi}}$$

results

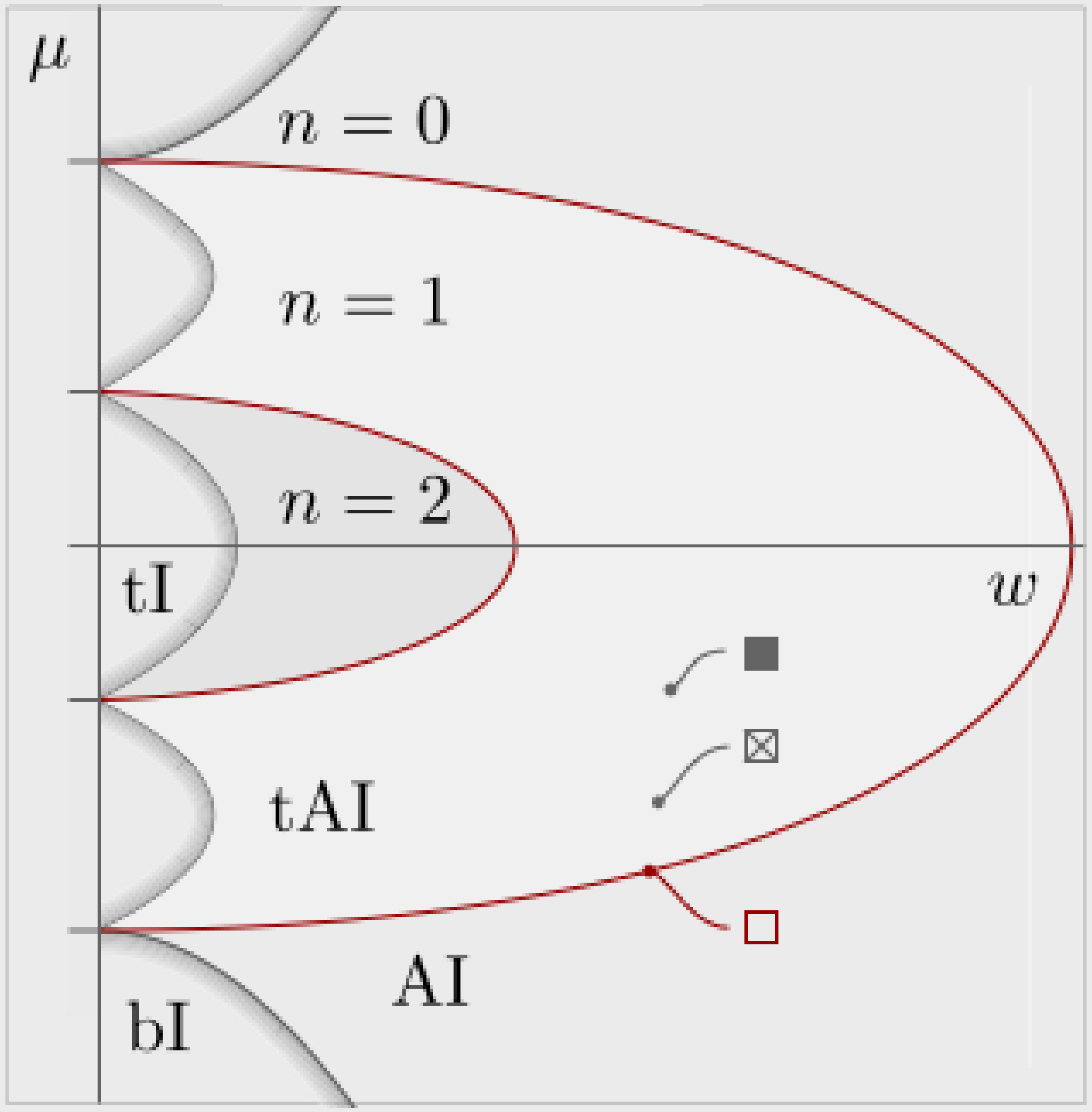
$$g = \sqrt{\frac{\tilde{\xi}}{\pi L}} \sum_{l_0 \in \mathbb{Z}+1/2} e^{-(l_0-\tilde{\chi})^2 L/\tilde{\xi}},$$

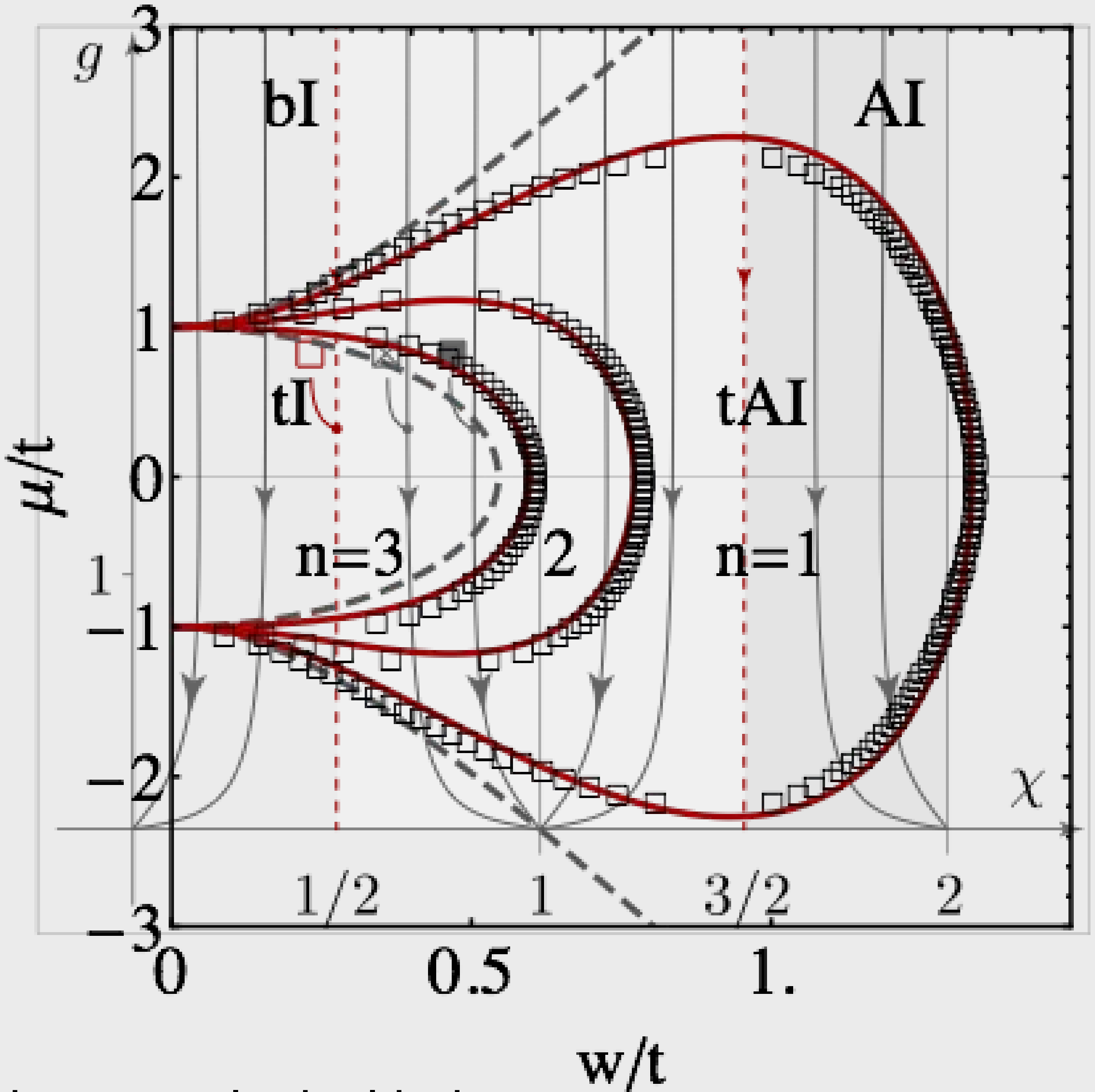
$$\chi = n - \frac{1}{4} \sum_{l_0 \in \mathbb{Z}+1/2} \left[\mathrm{erf}\left(\sqrt{\frac{L}{\tilde{\xi}}} (l_0 - \delta\tilde{\chi}) \right) - (\delta\tilde{\chi} \leftrightarrow -\delta\tilde{\chi}) \right],$$

where $\chi = n + \tilde{\chi}$



phase diagram



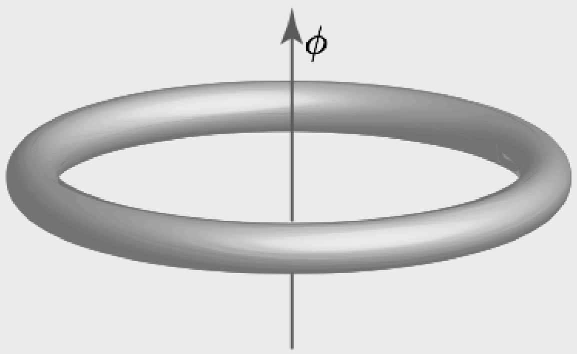


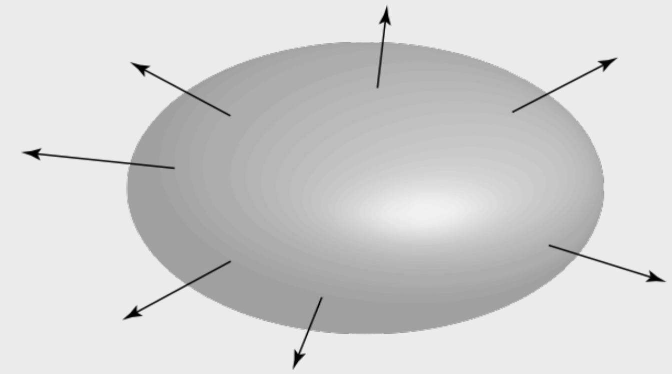
- phase boundaries: lines of half integer bare topological index
- flow describes stabilization of self-averaging topological phase/boundary state generation

Z-universality

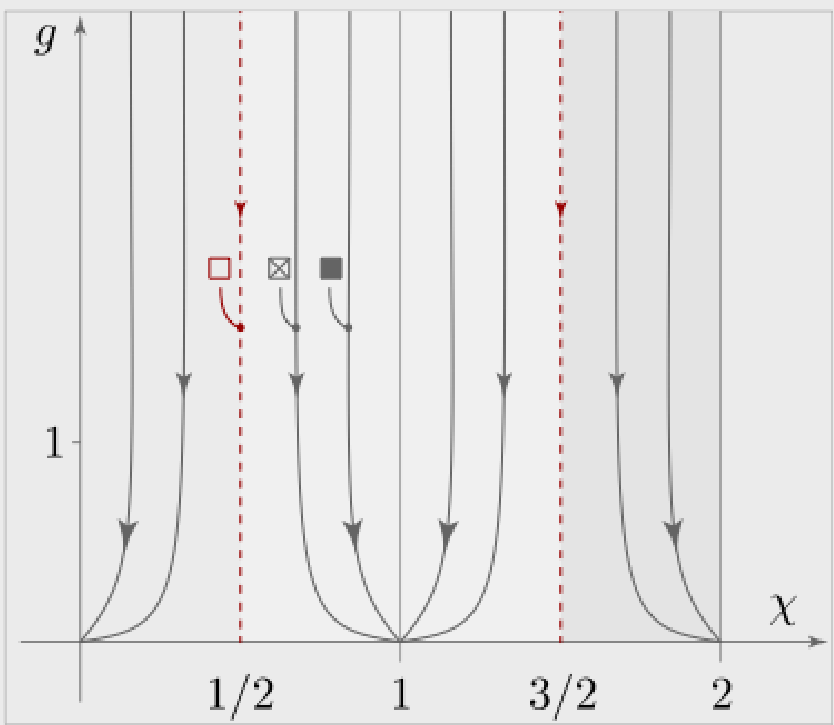
1d AIII, BDI, CII

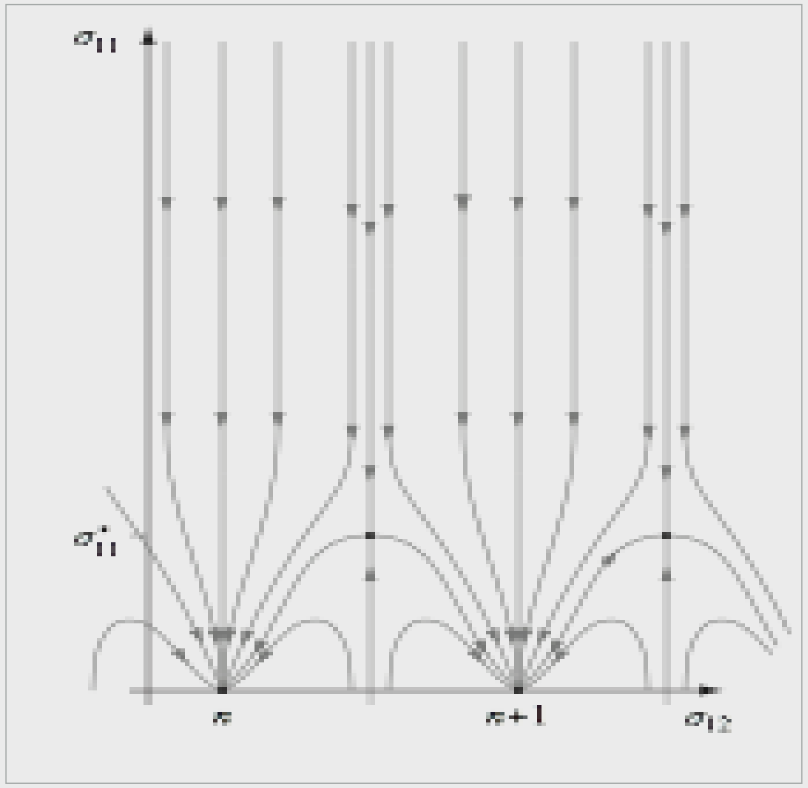
2d A, C, (D)





$$S[M] = \tilde{g}\, S_{\mathrm{diff}}[M] + \tilde{\chi}\, S_{\mathrm{top}}[M]$$



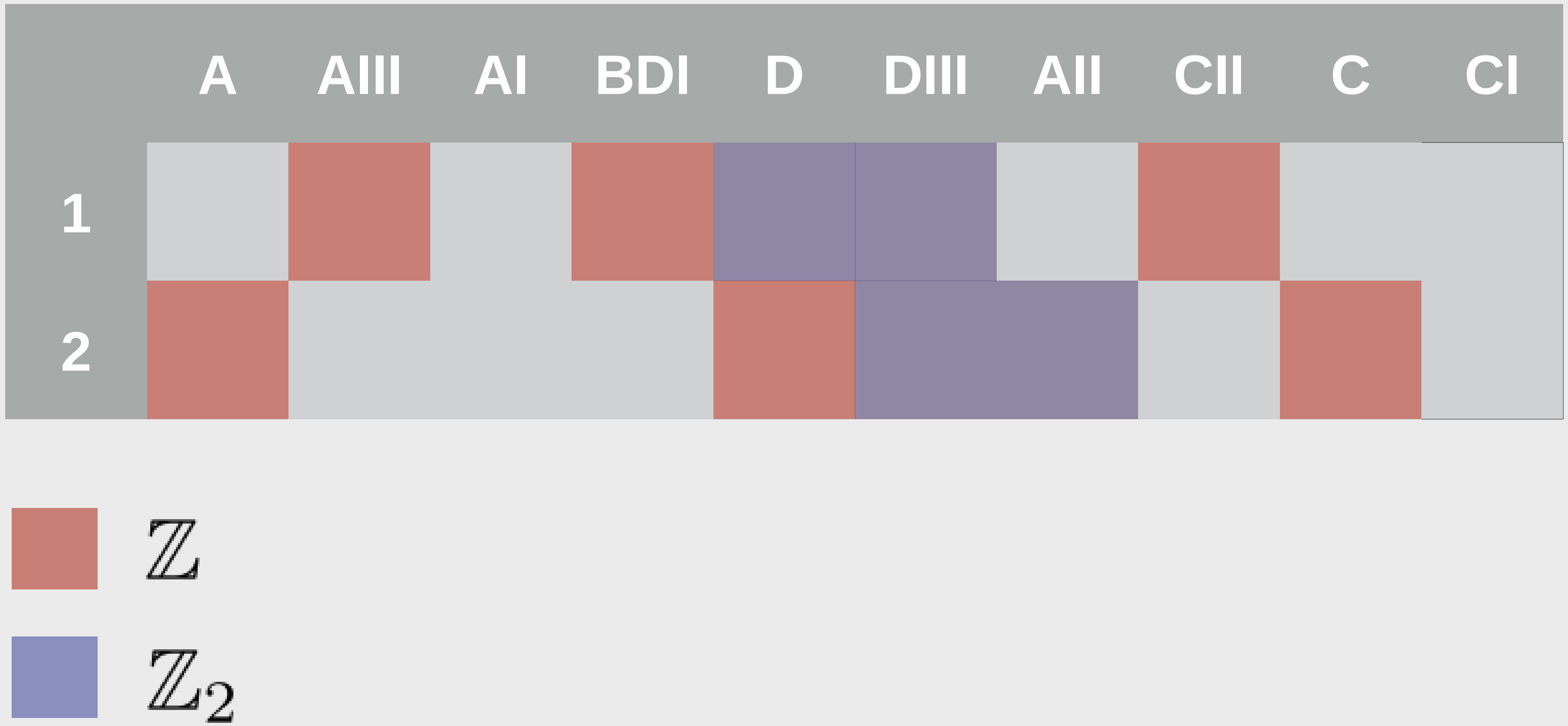


generic flow $(g, \chi) \xrightarrow{L \to \infty} (0, n)$

$$n S_{\mathrm{top}}[M] \longrightarrow n\, S_{\mathrm{boundary}}[T]$$

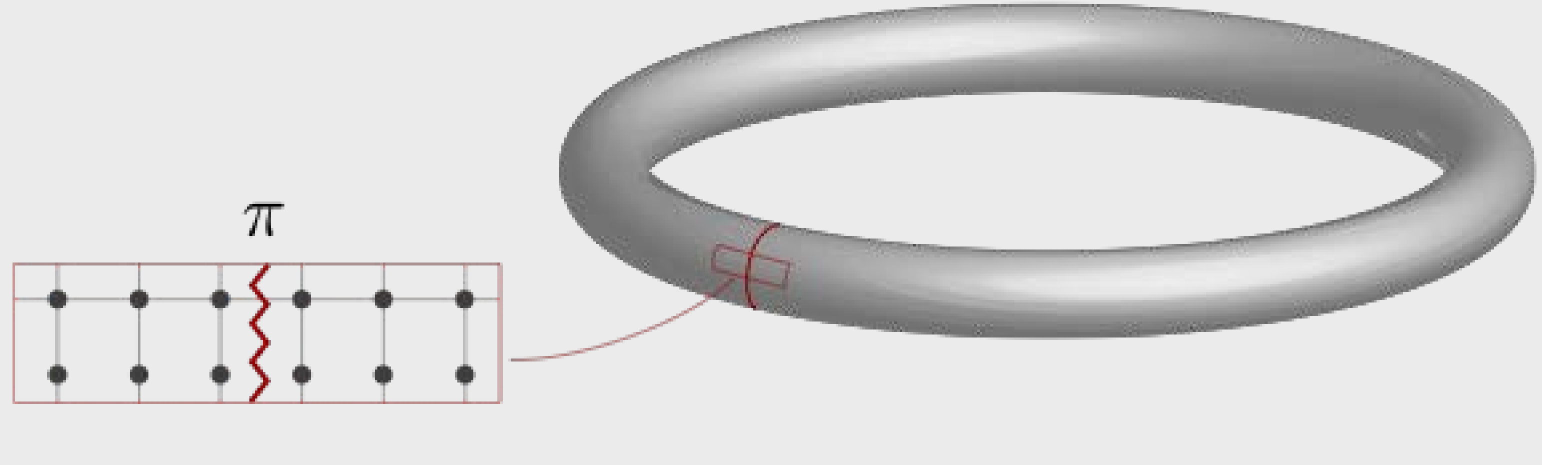
Z2

generalization to Z2



1d Z2 — D and DIII

$$H = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}$$

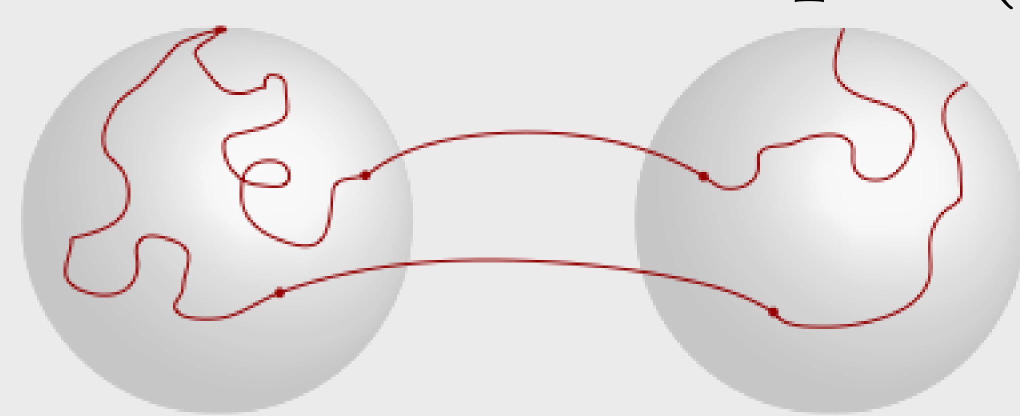


- field theory for granular array

$$S(Q_{x+1}, Q_x) = -\frac{1}{4}\sum_{x=1}^{L}\sum_{n=1}^{2N} \mathrm{Str}\ln\left[\mathbb{1} - \frac{t_n^2}{4}(Q_x - Q_{x+1})^2\right]$$

$$e^{-S_\mathrm{k}} \equiv \prod_{k=1}^{2N} r_k \equiv \tilde{\chi}$$

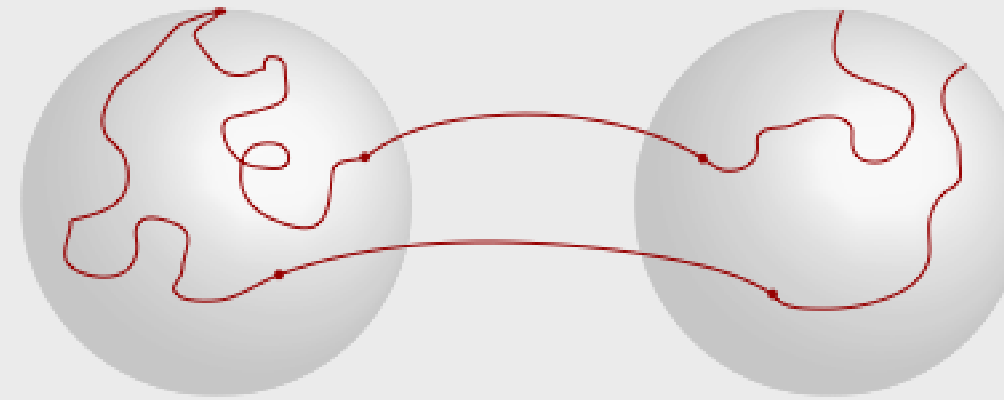
$\mathrm{SpO}(2|2)$



- amplitude (fugacity) of kinks

Transfer matrix with kinks: hidden supersymmetry.

$$\tilde{\xi}\,\partial_x \hat{\Phi} = \left[\frac{1}{2}\hat{\mathcal{D}}^2 + i\,\tilde{\chi}\,\hat{\mathcal{D}}\right]\hat{\Phi}$$



$$\hat{\mathcal{D}} = \begin{pmatrix} & D^\dagger \\ D & \end{pmatrix}$$

$$D^{(\dagger)} = -i\partial_y \pm i A(y)$$

class D $\quad A(y) = -1/\sinh 2y$

Pöschl-Teller operators $\quad -\partial_y^2 - \dfrac{\lambda(\lambda-1)}{\cosh^2 y} + \dfrac{\lambda(\lambda+1)}{\sinh^2 y}$

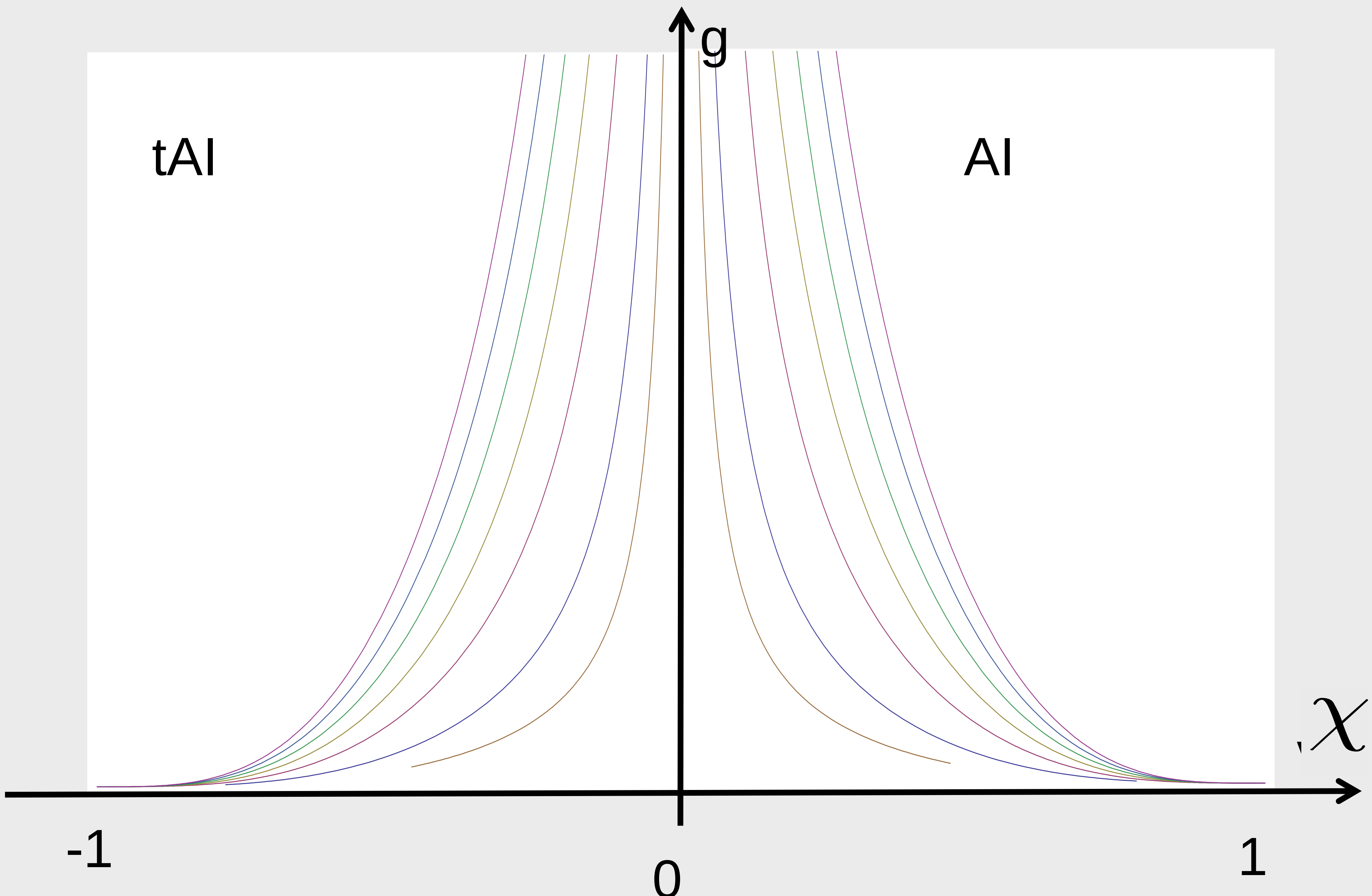
$$\epsilon(l) = \frac{1}{2}\,l^2 + i\,\tilde{\chi}\,l \qquad\qquad \lambda = \pm 1/2$$

▹cf. Grover, Vishwanath, Science 2014

Transfer matrix with kinks: results.

$$\chi(L) = \frac{1}{2}\int dl\ \coth\left(\pi l/2\right)\sin(\tilde{\chi} l L/\tilde{\xi}) e^{-l^2 L/2\tilde{\xi}}. = \left\langle \frac{\mathrm{Pf}(G_\pi^{-1})}{\mathrm{Pf}(G_0^{-1})} \right\rangle$$

$$g(L) = \frac{1}{8}\int dl\, l \coth\left(\pi l/2\right)\cos(\tilde{\chi} l L/\tilde{\xi}) e^{-l^2 L/2\tilde{\xi}} \quad \propto \frac{1}{\sqrt{L}}\, e^{-\tilde{\chi}^2 L/(2\tilde{\xi})}$$



Summary

- real space approach to translationally non-invariant topological insulators
  - 2-parameter field theory
  - probed by continuous (Z) or point-like (Z2) topological sources
  - universal scaling
  - stabilization of topology by localization

Thouless topology 2d class A

Khmelnitskii/Pruisken criticality 2d class A

Chalker et al. 2d class C

Ludwig et al. exact solution 2d class C

Fisher et al., Read et al., Zirnbauer/Serban 2d class D

Zirnbauer 1d super-Fourier analysis

Mirlin, Mudry, Gruzberg et al. disordered topological matter

Beenakker et al., Brouwer et al. scattering theory of topological matter

Read, Gruzberg, Vishveshwara 1d topological quantum criticality

Ludwig et al. disorder vs. bulk-boundary correspondence

Kane/Fu, 2d class AII