Integrable Planar Curve Flows and the Vortex Membrane Flow in Euclidean 4-Space Using Moving Frames and the Variational Bicomplex

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Joseph Jonathan Benson

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

PETER OLVER

October, 2014

Acknowledgements

There are many people I would like to thank for their help, encouragement, and support during my time in graduate school. First I want to thank my advisor, Peter Olver, for accepting me as a student, for introducing me to moving frames and the variational bicomplex, for his direction and guidance when I was stuck, and for suggesting the problems that were to become the content of this thesis. Secondly, I want to thank Stephen Doty for accepting me into the Master's program at Loyola University in Chicago, for reigniting my passion for mathematics, for teaching me representation theory and overseeing my introduction to Lie groups. Without my experience at Loyola, I would not be at the University of Minnesota and in the position I currently hold.

I also want to thank Williard Miller, Dan Spirn, Doug Arnold, Bob Gulliver, Rick Moekel, and Fernando Reitich, for broadening my mathematical outlook, for providing enlightening discussion and helpful advice, for serving on my oral and final committees, for teaching the best courses I took at the University of Minnesota.

I want to thank my colleagues and friends, David Morrissey, Derek Olson, Allana Hoyer-Leitzel, Denis Bashkirov, Madeline Schrier, and Rob Thompson, for keeping me sane, for conversation, for laughs, for support.

Thank you to my family too, who often had to deal with a stressed and grumpy version of me. Thank you for your unwavering support.

Lastly and most importantly, I want to thank my wife, Emily Mohl, who was with me for every step of this incredible journey. Without her love and support, none of this would have been possible.

Dedication

To Bug.

Abstract

Evolution equations with infinite hierarchies of symmetries have been shown to naturally arise within the context of geometric, arc-length preserving flows of curves in the plane and in $\mathbb{R}^3$. In the following work, a systematic investigation into this phenomenon is conducted for the case of group actions on planar curves. The techniques of moving frames as developed by Fels and Olver, [16, 17], and the invariant variational bicomplex as developed by Kogan and Olver, [28], are used. A catalog of results is produced, connecting many invariant curve flows with integrable equations such as Burgers', KdV, mKdV, and Sawada-Kotera. In the last chapter, the techniques are extended to an investigation of the evolution of curvature of 2-dimensional surfaces in 4-dimensional Euclidean space under the Skew-Mean-Curvature flow.

Contents

List of Tables

Chapter 1

Introduction

Let G be a transformation group that acts smoothly on a manifold M. An invariant submanifold flow is a G-invariant evolutionary partial differential equation

$$\frac{\partial S}{\partial t} = \Phi[S]$$

governing the motion of submanifolds $S \subset M$. Invariance requires that G is a symmetry group of the equation; if $S(t)$ is any solution, then $g \cdot S(t)$ is also a solution for any fixed element $g \in G$. Invariant curve and surface flows arise in a number of applications, including geometric optics, computer vision, vortex dynamics, and elsewhere.

As a submanifold evolves according to an invariant submanifold flow, its differential invariants also evolve. We focus on the case of curves and their invariants. Plane curves have a lowest order differential invariant by which all other invariants can be written as functional combinations of this invariant and its invariant derivatives. Under certain invariant curve flows, the corresponding evolution of this lowest order invariant is often found to be given by a well-known integrable equation such as the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, or Burgers' equation. This phenomenon was first noticed in [23], where Hasimoto demonstrated a one-to-one correspondence between the integrable cubic nonlinear Schrödinger equation and the vortex filament flow, an (Euclidean) invariant curve flow governing the evolution of a thin vortex filament in an inviscid, incompressible fluid. Later in [29], Lamb brought attention to the seemingly quite common relationship between invariant flows and integrable equations. It should be noted that though many people agree on which equations should be deemed integrable, there is no universally accepted definition for this term. The sense that we will use it in this paper is that of possessing an infinite hierarchy of symmetries and a recursion operator. A recursion operator is an operator associated to the equation by which new symmetries are produced from old.

The classification of Lie algebras of two dimensional vector fields was done first by Sophus Lie in [33] for real complex vector fields and supplemented by González-López, Kamran, and Olver in [20] for real vector fields. In [11, 10], Chou and Qu performed a systematic investigation of planar curve flows and integrable equations using these classifications, discovering many new relationships between planar Klein geometries and integrable hierarchies. In the following work, I performed a similar investigation into the relationship between integrable equations and invariant planar curve flows. The approach is different, however, as I use the recent, powerful tools of the equivariant method of moving frames and the invariant version of the variational bicomplex. Many of the results of Chou and Qu were replicated, but in the current setting certain relationships became more evident. In particular, associated to any invariant is an invariant differential operator dubbed the *characteristic operator*. It is seen that in most instances, the characteristic operator is either the corresponding recursion operator, or appears as part of a factorization of the recursion operator.

In Chapter 2, we build the machinery necessary to investigate and describe the problems at hand. In Section 2.1.1, we introduce equivariant moving frames, as first formulated by Fels and Olver in [16, 17]. We then show how group actions on a manifold can be prolonged to act on the jets of submanifolds, which produce higher order moving frames. With these in hand, we demonstrate how a moving frame leads to an algorithmic construction of differential invariants of the action. In Section 2.2, we lay the foundations of the variational bicomplex, introducing jet space and contact forms. With the addition of a prolonged group action on jet space, we use a moving frame to "invariantize" the differential forms producing the invariant variational bicomplex and an invariant coframe by with which to take the exterior derivative with. The exterior derivative has an invariant tri-grading, which in the case of projectable group actions, reduces to a bigrading. It this invariant bigrading which will play a key role in all subsequent computations. We next introduce the recurrence formula, which allows us to implement a fully symbolic calculus of differential invariants and their invariant derivatives with respect to our invariant coframe. In Section 2.3, we take the invariant bicomplex constructions and apply them to invariant submanifold flows. Here, we restrict our presentation to curves in the plane, and thus develop the ideas of an invariant curve flow. We pay particular attention to the development of formulas that determine the evolution of the differential invariants of the curve according to the invariant flow. In the event that the invariant flow preserves the parametrization, the evolution of the differential invariants is given by an often non-local partial differential equation. Under certain invariant curve flows, this PDE is frequently given by a well-known local, integrable, and often nonlinear, equation. It is this phenomenon that is investigated in the next chapter.

In Chapter 3, we investigate the phenomenon mentioned above in detail. First we define symmetries of a differential equation. The existence of an infinite hierarchy of (generalized) symmetries is what we take to be the defining property of an integrable equation. Differential equations with this property often come equipped with a recursion operator, a integral-differential operator which effectively takes a symmetry and produces a new one, thus producing the infinite hierarchy. Once these ideas are established, we turn to the problem at hand. In [33], Sophus Lie gave a complete classification of independent group actions on the complex 2-plane. Using the techniques developed in Chapter 2, we systematically go through Lie's catalog and find integrable equations in nearly each case. Furthermore, associated to any differential invariant is an integral-differential operator termed the characteristic operator. It is used to determine the equation governing the evolution of the differential invariant. The surprising occurrence is that in many instances, the characteristic operator agrees with the recursion operator for our integrable equation. The body of this chapter presents these results with a discussion at the end. Unfortunately, no definitive pattern between the different group actions and the appearance of integrable equations and recursion operators is determined.

In Chapter 4, the focus is shifted to investigate the evolution of invariants of surfaces in four dimensions under a generalization of the vortex filament flow in three dimensions. Some background is given on the differential geometric approach to fluids, with the goal of describing the vortex filament flow. The appropriate generalization of the vortex filament flow requires viewing the vorticity as a differential two-form supported on a co-dimension two submanifold. Next, we establish the mean-curvature vector on the submanifold, and then we define the skew-mean-curvature flow on the submanifold. The skew-mean-curvature flow is a generalization of the vortex filament flow which can be defined for codimension two submanifolds in $\mathbb{R}^n$. At this point, we supply the necessary extensions of the moving frame and invariant variational bicomplex techniques developed in Chapter 2 applied to surfaces in $\mathbb{R}^4$. Note that the group action is that of the four-dimensional Euclidean group. We also frame some of the classical results of surfaces in $\mathbb{R}^3$ in our language of moving coframes. This is done in order to find parallels in these constructions when applied to our surfaces in $\mathbb{R}^4$. Lastly, we end with computations determining the evolution of the mean curvatures of the surface under the skew-mean-curvature flow.

Chapter 2

Background

2.1 Equivariant Moving Frames

2.1.1 Moving Frames

Let G be an r-dimensional Lie group acting on an m-dimensional manifold M.

Definition 2.1.1. A *moving frame* a smooth G equivariant map $\rho : M \to G$.

Left and right moving frames are determined by the corresponding action of G on itself. For $g, h \in G$, the left and right actions are $L_g(h) = gh$ and $R_g(h) = hg^{-1}$, respectively. Thus a right moving frame is defined by the equivariant relation $\rho(g \cdot z) = R_g(\rho(z)) = \rho(z)g^{-1}$, and similarly for a left moving frame. The two are related by the following:

Proposition 2.1.2. *If $\tilde{\rho}(z)$ is a left moving frame, then $\rho(z) = \tilde{\rho}(z)^{-1}$ is a right moving frame.*

In order to give the conditions for the existence of a moving frame, we need the following definitions.

Definition 2.1.3. The action of G on M is *free* if the isotropy subgroup for all $z \in M$ is trivial:

$$G_z = \{g \in G \mid g \cdot z = z\} = \{e\} \qquad \text{for all} \quad z \in M$$

Definition 2.1.4. The action is *semi-regular* if all its orbits have the same dimension. The action is *regular* if it is semi-regular, and if there is a regular foliation of M by the orbits of G.

Theorem 2.1.5. *Let G act on M. Then a moving frame $\rho : M \to G$ exists if and only if the action is free and regular on M.*

Proof. First, let $\tilde{\rho} : M \to G$ be a left moving frame on M. Let $z \in M$ and let $g \in G_z$, the isotropy subgroup of z. Since $g \cdot z = z$ and by left equivariance,

$$\tilde{\rho}(z) = \tilde{\rho}(g \cdot z) = g\tilde{\rho}(z)$$

which implies $g = e$. Hence, the action is free. To show regularity, let $z \in M$ and let $z_k = g_k \cdot z$ be a sequence of points in the orbit of z such that $z_k \to z$ as $k \to \infty$. Then

$$\tilde{\rho}(z_k) = \tilde{\rho}(g_k \cdot z) = g_k\tilde{\rho}(z) \to \tilde{\rho}(z), \quad \text{as} \quad k \to \infty$$

by equivariance and continuity of the moving frame map $(\tilde{\rho}(z_k) \to \tilde{\rho}(z))$. Hence, $g_k \to e$, which implies regularity of the action (as the $z_k = g_k \cdot z$'s near z, the g_k's must near the identity).

Now suppose the action is free and regular. Then choose local flat coordinates $z = (x, y) \in G \times X$, where if dim $M = m$, and dim $G = r$, then $X \simeq \mathbb{R}^{m-r}$. Locally, $M \simeq G \times X$, where the first r coordinates represent elements in G. Hence, a local cross-section K to the group orbits is given as the graph of a smooth map $a : X \to G,\ x = a(y)$. Note now that the map $\tilde{\rho}(x, y) = x \cdot a(y)$ is a smooth, G-equivariant map from M to G, and thus defines a left moving frame on M. □

From the above proof, we see that the choice of cross-section is intimately connected to the construction of the moving frame. In practice, coordinate cross-sections are used, though by no means is this required. Once a cross-section is chosen, one solves the resulting normalization equations to find the moving frame.

Geometrically, a right moving frame assigns to a point $z \in M$ the unique group element $g = \rho(z) \in G$ that maps it to the cross-section. We will illustrate the construction of a moving frame with the following example.

Example 2.1.6. Consider the following action of the Euclidean group $SE(2) = SO(2) \ltimes \mathbb{R}^2$ on $\mathbb{R}^4$, with local coordinates on $\mathbb{R}^4$ given by x, u, p, q, and group parameters ϕ, a, b:

$$x \mapsto x\cos\phi - u\sin\phi + a, \qquad u \mapsto x\sin\phi + u\cos\phi + b,$$
$$p \mapsto \frac{\sin\phi + p\cos\phi}{\cos\phi - p\sin\phi}, \qquad q \mapsto \frac{q}{(\cos\phi - p\sin\phi)^3}.$$

Incidentally, this action is only locally free. Indeed, the points $(x, u, p, q) = (0, 0, p, 0)$ are fixed by the group element $(\phi, a, b) = (\pi, 0, 0)$. The general practice is to ignore this discrete ambiguity, which we will also follow here.

We choose the cross-section $x = u = p = 0$ resulting in the following normalization equations:

$$\begin{aligned} x: &\qquad 0 = x\cos\phi - u\sin\phi + a \\ u: &\qquad 0 = x\sin\phi + u\cos\phi + b \\ p: &\qquad 0 = \frac{\sin\phi + p\cos\phi}{\cos\phi - p\sin\phi}. \end{aligned}$$

Solving this system for the three group parameters yields

$$\phi = -\tan^{-1} p, \qquad a = -\frac{x + up}{\sqrt{1+p^2}}, \qquad b = \frac{xp - u}{\sqrt{1+p^2}}$$

which defines the right moving frame $\rho : \mathbb{R}^4 \to SE(2)$. Here, we take the branch of $\phi = -\tan^{-1} p$ containing 0. Inverting this group element gives the left (classical) moving frame

$$\tilde{\phi} = \tan^{-1} p, \quad \tilde{a} = x, \quad \tilde{b} = u$$

2.1.2 Prolonged Transformation Groups

We now consider the induced action of G on p-dimensional submanifolds $S \subset M$. As G transforms the space M, there is an induced transformation of submanifolds of M. To simplify the following presentation, we assume $M = \mathbb{R}^2$ to be the real plane with "independent" variable x and "dependent" variable u. Submanifolds of M can be locally realized as functions of the form $u = f(x)$. As G acts on x and u, it correspondingly acts on the derivatives, or jet coordinates, u_x, u_{xx}, etc. The induced action of G on the jet coordinates is found simply by the chain rule. We write

$$y = \Xi_g(x, u), \qquad v = \Phi_g(x, u)$$

for the group transformation, where the subscript g indicates the transformation depends on the element $g \in G$. Differentiation with respect to $\tilde{x}$ is given by

$$D_y = \frac{1}{D_x \Xi} D_x \tag{2.1}$$

where

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \dots \tag{2.2}$$

is the total derivative with respect to x. Hence,

$$v_y = D_y v = \frac{1}{D_x \Xi} D_x \Phi, \qquad v_{yy} = D_y v_y, \qquad \text{etc..} \tag{2.3}$$

In the case of planar curves, we will denote the space of jets by J^n, where n refers to the order of the jets. Higher order moving frames are constructed using prolonged transformation groups:

$$\rho : J^n \to G, \qquad \rho(g \cdot z) = \rho(z) g^{-1}$$

where $z = (x, u, u_x, \dots, u_n),\ u_n := (D_x)^n\, u$.

Example 2.1.7. Consider the action of $SE(2)$ on the plane:

$$y = x \cos\phi - u \sin\phi + a, \qquad v = x \sin\phi + u \cos\phi + b.$$

Using equation (2.1), we find

$$D_y = \frac{1}{D_x(x\cos\phi - u\sin\phi + a)} D_x(x \sin\phi + u\cos\phi + b) = \frac{1}{\cos\phi - u_x \sin\phi}.$$

Application of equation (2.3) produces the prolonged transformation of the first and second order jet coordinates:

$$v_y = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x \sin\phi}, \qquad v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

The action of $SE(2)$ on J^2 is the same as the action presented in Example 2.1.6, where $p = u_x,\ q = u_{xx}$.

As before, we found the first order moving frame by setting $\tilde{x} = \tilde{u} = \tilde{u}_{\tilde{x}} = 0$ and solving for the group parameters ϕ, a and b resulting in the first order moving frame

$$\phi = -\tan^{-1} u_x, \qquad a = -\frac{x + u u_x}{\sqrt{1+u_x^2}}, \qquad b = \frac{x u_x - u}{\sqrt{1+u_x^2}}. \tag{2.4}$$

Higher order moving frames are constructed on jet bundles using the prolonged group transformations. For theoretical details, see [17]. The moving frame algorithm for prolonged group transformations is effectively the same as the algorithm described for a free and regular action of G on M. For computational ease, we continue the use of coordinate cross-sections to compute the moving frame, where we now incorporate the jet coordinates.

2.1.3 Differential Invariants

Moving frames provide an algorithm with which to generate invariants to the group action. A function $F : M \to \mathbb{R}$ is an *invariant* with respect to a group G if

$$F(g \cdot z) = F(z) \qquad \text{for all} \quad z \in M,\ g \in G.$$

Global invariants are generally not easy to find, so we work with local invariants instead. Local invariants are defined on some open subset $U \subset M$ with the property that $F(g \cdot z) = F(z)$ for all $z \in U$ and all $g \in V_z \subset G$, where V_z is a neighborhood of the identity in G, possibly depending on z.

In most cases of interest, the group action is transitive and there are no non-constant invariant functions. By prolonging the group action to jet space, we introduce the idea of differential invariants. A function $F : J^n(M) \to \mathbb{R}$ is a *differential invariant* if

$$F(g \cdot z) = F(z) \qquad \text{for all} \quad z \in J^n(M),\ g \in G.$$

The point here is that $F = F(x, u, u_x, \dots u_n) = F[u]$ depends on the the jet coordinates.

Invariants are easily constructed using our moving frame algorithm via a process called *invariantization.* Using the moving frame, arbitrary differential functions are projected to a unique invariant.

Definition 2.1.8. Let $\rho : J^n \to G$ be a right moving frame. The *invariantization* of a differential function $F : J^n \to \mathbb{R}$ is the function $\iota(F)(z)$ defined by

$$\iota(F)(z) = F(\rho(z) \cdot z).$$

The invariant function $\iota(F)(z)$ is the unique invariant that agrees with F on the cross-section, thus invariantization preserves all algebraic operations.

Proposition 2.1.9. *The invariantization of a differential function is a differential invariant.*

Proof. Let $\rho : J^n \to G$ be a moving frame and let $g \in G$. Then by right equivariance of the moving frame and by the definition of ι,

$$\iota(F)(g \cdot z) = F(\rho(g \cdot z) \cdot (g \cdot z)) = F(\rho(z)g^{-1}g \cdot z) = F(\rho(z) \cdot z) = \iota(F)(z).$$

$\square$

Example 2.1.10. Returning to Example 2.1.6 again, we see that

$$\iota(x) = 0, \quad \iota(u) = 0, \quad \text{and} \quad \iota(u_x) = 0.$$

Indeed, these invariants reflect the values of the cross-section used to construct the moving frame. The invariant functions coming from the normalization process are known as the *phantom invariants*. However, looking at the second derivative coordinate we see

$$\iota(u_{xx}) = \left. \frac{u_{xx}}{(\cos\phi - u_x \sin\phi)^3} \right|_{\phi = -\tan^{-1} u_x} = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} = \kappa$$

which is the Euclidean curvature of a planar curve.

2.2 Variational Bicomplex

The variational bicomplex has its modern origins in the works of Vinogradov [52, 53] and Tulczyjew [51]. Later contributions by Tsujishita [50] and Anderson [2, 3] demonstrated its power and efficacy in geometric approaches to studying local and global problems in partial differential equations and the calculus of variations. The following presentation will use a local coordinate approach, however an intrinsic, coordinate-free approach is possible using filtrations. See [7] for details on this approach.

We work on the infinite jet bundle, J^∞, which is defined as the inverse limit under the jet projections $\pi^n_k : J^n \to J^k$. We identify $F : J^k \to \mathbb{R}$ with $F \circ \pi^n_k : J^n \to \mathbb{R}$ for $n \geq k$. Thus any $F : J^\infty \to \mathbb{R}$ is given by $F \circ \pi^\infty_k$ for some finite k. Intuitively, we allow arbitrarily high derivative coordinates with the knowledge that functions and differential forms on J^∞ only depends on derivatives (jet coordinates) up to some finite order. Again, for ease of presentation, we tailor the following constructions to the jets of planar curves. In local coordinates, where all computations occur, the "independent" variable is x and the "dependent" variable is u. The coordinates on J^∞ are $(x, u, u_x, u_{xx}, \ldots)$.

The bicomplex construction relies on a natural splitting of the space of differential forms on the infinite jet bundle into horizontal and contact components, thus endowing the usual de Rham complex with the structure of a bicomplex.

A smooth function $u = f(x)$ has a natural prolongation to J^∞:

$$\begin{aligned} f^{(\infty)} : X &\longrightarrow J^\infty \\ x &\longmapsto (x, f(x), f'(x), \ldots). \end{aligned}$$

A differential form on J^∞ is defined to be a contact form θ if and only if $f^{(\infty)*}\theta = 0$ for any prolonged smooth function $u = f(x)$. It can be easily proved that any contact one-form is a linear combination of the basic contact one-forms,

$$\theta_j = du_j - u_{j+1}dx, \qquad j = 0, 1, 2, \ldots.$$

On T^*J^∞, these forms span the contact or vertical subbundle $\mathcal{C}$. With the coordinate one form dx which spans the *horizontal subbundle* denoted by H, the cotangent bundle can be split into horizontal and contact components, T^*J^∞ =H $\oplus\, \mathcal{C}$. The exterior derivative naturally splits into horizontal and contact or vertical components, $d = d_H + d_V$, where d_H increases horizontal degree (wedges on dx) and d_V increases contact degree (wedges on θ_j's). Closure, $d \circ d = 0$ implies the following relations:

$$d_H \circ d_H = 0 = d_V \circ d_V, \quad \text{and} \quad d_H \circ d_V + d_V \circ d_H = 0.$$

Local coordinate expressions for the exterior derivative of a differential function $F : J^\infty \to \mathbb{R}$ demonstrate the splitting. We use the notation $F = F(x, u, u_x, \ldots, u_n) := F[u]$ to imply F depends upon x, u and the derivatives of u with respect to x up to a finite order n. By a strategic rearranging of terms, we see

$$\begin{aligned} dF[u] &= \frac{\partial F}{\partial x}dx + \sum_{j=0}^{n} \frac{\partial F}{\partial u_j}du_j \\ &= D_x F\ dx + \sum_{j=0}^{n} \frac{\partial F}{\partial u_j}\theta_j = d_H F + d_V F \end{aligned}$$

where we recall equation (2.2) for the total derivative with respect to x. Extension of the horizontal and vertical differentials to differential k-forms come from the usual properties of the exterior derivative. In particular, for any basic contact 1-form θ_j, the following identity holds,

$$d\theta_j = d_H\theta_j = dx \wedge D_x\theta_j = dx \wedge \theta_{j+1}, \tag{2.5}$$

where $D_x\theta_j$ is the Lie derivative of θ_j with respect to the total derivative D_x.

The one-forms dx and $\theta_j,\ j \geq 0$, form a *coframe*, or basis for the cotangent space T^*J^∞ at each point of J^∞, and thus any differential 1-form on J^∞ can be written uniquely as a linear combination of the coframe elements.

Lastly, it is worth mentioning that the variational aspect of the variational bicomplex comes from its role in the geometric formulation of problems in the calculus of variations. A variational problem has the form

$$\mathcal{L}[u] = \int L(x, u, u_x, \ldots)\, dx.$$

where traditionally, the integrand is known as the Lagrangian. In the geometric formulation, we take the Lagrangian form $\omega = L\ dx$. Computing $d_V\omega$ is akin to taking the variational derivative. The result is a 2-form (or more generally, a $p+1$-form where p is the number of independent variables) consisting of the wedge product of a contact 1-form and dx. Working modulo contact forms and using equation (2.5), we perform an integration by parts by taking derivatives off of the θ_j's. The result is an expression of the form $E(L)\theta \wedge dx$, where $E(L)$ is the Euler-Lagrange expression from the calculus of variations. For a much more detailed treatment, see [2].

2.2.1 Invariant Variational Bicomplex

With the addition of a group action on our base manifold M, we would like to build group invariance into the bicomplex structure. The primary motivation for this construction is the formulation of the group invariant form of the Euler-Lagrange equations associated to a variational problem with a symmetry group. Indeed, any variational problem with an associated symmetry group can be written in terms of the differential invariants of the group action. The Euler-Lagrange equations inherit the symmetries of the variational problem (but not conversely), and they can be written in terms of the invariants of the symmetry group with an extra non-invariant multiplier. Until [28], the explicit formulae for the group invariant Euler-Lagrange equations were only found through ad-hoc methods. In [27], Kogan and Olver introduce group invariance into the variational bicomplex, and in [28], they solve the aforementioned problem. For our purposes, we do not need the form of the invariant Euler-Lagrange equations associated to an invariant variational problem, however the group invariance built into the complex will be used extensively as will certain invariant differential operators arising from this construction. Most importantly, we will use the powerful *recurrence formula* to determine an invariant bigrading of the exterior derivative. The following presentation will again be restricted to the case of curves in the plane, and only relevant details will be presented. For construction in full generality, one should consult [28].

Once again, let G be a transformation group acting on the (complex or real) plane. Prolong the group action to 1-dimensional submanifolds (curves) so that G acts on J^∞. To construct our invariant bicomplex, we use a moving frame to invariantize differential forms. The precise definition of the invariantization of a differential form requires consideration of a corresponded *lifted action* on the right principle bundle

$$\pi : \boldsymbol{\mathcal{B}} = G \times J^\infty \to J^\infty, \qquad g \cdot (h, z) = (hg^{-1}, g \cdot z).$$

This action is always free and regular on any open subset of J^∞. Freeness is easily seen since $g \cdot (h, z) = (h, z)$ immediately implies that $g = e$. Similarly, any sequence of points $g_k \cdot (h, z)$ converging to (h, z) implies that g_k converges to the identity element e, thus demonstrating regularity.

We define the evaluation map $w : \boldsymbol{\mathcal{B}} \to J^\infty$ by

$$w(g, z) = g \cdot z,$$

which incidentally can be viewed as the target map for a groupoid structure on $\boldsymbol{\mathcal{B}}$.

A moving frame $\rho : J^\infty \to G$ defines a G-equivariant section of the lifted bundle, namely

$$\sigma : J^\infty \to \boldsymbol{\mathcal{B}}, \qquad \sigma(z) = (\rho(z), z)$$

which will be called the moving frame section. In this context, the invariantization of a differential function F as defined in Definition 2.1.8 is reformulated as

$$\iota(F)(z) = \sigma^* w^* F(z) = F \circ w \circ \sigma(z) = F(\rho(z) \cdot z).$$

The invariantization of a differential form requires a bit more care. First, we recall that a differential form ω is invariant if $g^*\omega = \omega$ for all $g \in G$. Alternatively, ω is invariant if $g^*\omega|_{g\cdot z} = \omega|_z$.

Because of the Cartesian product structure on the lifted bundle, a coframe is provided by the horizontal, dx, and contact forms, θ_j, on J^∞, as well as the Maurer-Cartan forms
$\mu^i,\ i = 1, \ldots, r$ on G. The exterior derivative on $\boldsymbol{\mathcal{B}}$ differentiates with respect to the jet coordinates *and* the group parameters, and thus splits $d = d_J + d_G$, where d_J is the exterior derivative on the jet coordinates and d_G is likewise on the group parameters. Denote by π_J and π_G the projections onto jet and Maurer-Cartan forms, respectively, e.g. if Ω is a differential form on $\boldsymbol{\mathcal{B}}$, then $\pi_J\Omega$ formally sets all Maurer-Cartan forms to 0 (however the resulting expression may still contain group parameters). Note the following identities:

$$\pi_J \circ d = d_J = d_J \circ \pi_J, \qquad \pi_J \circ d_G = 0, \qquad \text{but} \qquad d_G \circ \pi_G \neq 0. \tag{2.6}$$

Definition 2.2.1. The *invariantization* of a differential form Ω on J^∞ is the G-invariant differential form

$$\iota(\Omega) = \sigma^* \pi_J w^*(\Omega). \tag{2.7}$$

Note the process: the form is lifted to $\boldsymbol{\mathcal{B}}$ via the evaluation map, the Maurer-Cartan forms are then set to zero, and then we pull back via the moving frame section. This choice of invariantization operator agrees with Definition 2.1.8 when restricted to functions. Though the introduction of the jet projection creates complications that will be addressed shortly, this definition of invariantization allows one to invariantize a coframe, producing an invariant coframe. The following simple example demonstrates the process, as well as the necessity of the jet projection.

Example 2.2.2. Consider the scaling action of $\mathbb{R}^+ = \{\lambda > 0\}$ on $\mathbb{R}^2$ given by

$$(x, u) \longmapsto (\lambda^{-1}x, \lambda u) = w(\lambda; x, u)$$

We construct a moving frame using the cross-section $x = 1, u > 0$. The normalization equation is

$$1 = \lambda^{-1}x$$

which produces the moving frame $\lambda = \rho(x,u) = x$ and moving frame section $\sigma(x,u) = (\rho(x,u);x,u) = (x;x,u)$. Invariantizing the coordinate function u gives the basic invariant

$$\iota(u) = \sigma^* w^*(u) = w^*(x;u) = xu.$$

The prolonged group action on jets of curves in the plane is

$$u_x \longmapsto \lambda^2 u_x, \quad u_{xx} \longmapsto \lambda^3 u_{xx}, \quad \ldots.$$

Invariantizing the jet coordinates produces differential invariants:

$$\iota(u_x) = x^2 u_x, \quad \iota(u_{xx}) = x^3 u_{xx}, \quad \ldots$$

The basic coframe on J^∞ for curves in the plane is given by dx and $\theta = du - u_x dx$, $\theta_1 = du_x - u_{xx}dx, \ldots$. Invariantizing this coframe produces an invariant coframe. We apply equation (2.7).

$$\begin{aligned}\iota(dx) &= \sigma^* \pi_J w^*(dx) = \sigma^* \pi_J d(\lambda^{-1} x)\\ &= \sigma^* \pi_J(\lambda^{-1}dx - \lambda^{-2} x d\lambda) = \sigma^*(\lambda^{-1}dx) = \frac{1}{x}dx\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\iota(\theta) &= \rho^* \pi_J w^*(du - u_x dx) = \rho^*\big(d_J(\lambda u) - \lambda^2 u_x d_J(\lambda^{-1}x)\big)\\ &= x du - x u_x dx = x\theta.\end{aligned}$$

Hence $\iota(dx) = \dfrac{1}{x}dx,\ \iota(\theta) = x\theta,\ \iota(\theta_1) = x^2\theta_1, \ldots$ produces an invariant coframe on J^∞ for curves in the plane. Note that the jet projection is needed in order to produce an invariant coframe. Indeed in (2.8), without π_J,

$$\rho^*(\lambda^{-1}dx - \lambda^{-2}xd\lambda) = \frac{1}{x}dx - \frac{1}{x}dx = 0$$

which fails to produce a coframe (however the resulting differential form is nonetheless still invariant).

By invariantizing the basic coframe elements on J^∞, we produce an invariant coframe and the resulting *invariant complex*. If the action is projectable, the invariantized variational bicomplex produces an invariant bicomplex, however if the action is non-projectable, the bicomplex structure is lost due to an anomalous splitting of the differential. This will be discussed in further detail shortly.

Example 2.2.3. Let's return to our running example of $SE(2)$ acting on the plane, Example 2.1.6. To construct an $SE(2)$ invariant coframe on planar curves, we use our moving frame from equations (2.4) and apply equation (2.7):

$$\begin{aligned}\varpi = \iota(dx) = \sigma^* \pi_J w^* dx &= w^* \pi_J d(x\cos\phi - u\sin\phi + a)\\ &= w^*((\cos\phi - u_x \sin\phi)dx - (\sin\phi)\theta)\\ &= \sqrt{1+u_x^2}dx + \frac{u_x}{\sqrt{1+u_x^2}}\theta\end{aligned}$$

Note that the resulting fully invariant form is not purely horizontal, but instead contains a contact correction. The horizontal component is the usually Euclidean arc-length element.

In a similar manner, we compute

$$\begin{aligned}\vartheta = \iota(\theta) &= \frac{\theta}{\sqrt{1+u_x^2}},\\ \vartheta_1 = \iota(\theta_1) &= \frac{1}{1+u_x^2}\theta_1 - \frac{u_x u_{xx}}{(1+u_x^2)^2}\theta,\end{aligned}$$

etc. Note that these are contact forms, with no horizontal components.

2.2.2 Recurrence Formula

The invariantization operator respects all basic algebraic operations as well as function composition, however all complications arising in studies of differential invariants derive from the fact that in general, invariantization does not commute with exterior differentiation:

$$d\iota(\Omega) \neq \iota(d\Omega).$$

The obstruction lies in the intervening jet projection π_J in equation (2.7); indeed, ι is not a genuine pull-back. Fortunately, the correction terms $d\iota - \iota d$ can be algorithmically constructed using only linear algebra and the (prolonged) infinitesimal generators of the group action.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathfrak{g}$ be a basis for the Lie algebra to our group G. By local effectiveness, we can identify the Lie algebra elements and their corresponding vector fields on M, ([43] Theorem 2.62). The vector fields are then prolonged to J^∞, and we denote by $\mathbf{v}_k(\Omega)$ the Lie derivative of the differential form Ω on J^∞ with respect to the prolonged infinitesimal generator $\mathbf{v}_k$. In local coordinates, a vector field on the plane has the form

$$\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \varphi(x,u)\frac{\partial}{\partial u}. \tag{2.9}$$

Prolonged vector fields can be identified with their non-prolonged counterparts, and thus we use the same notation:

$$\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \sum_{j\geq 0}\varphi_{j+1}[u]\frac{\partial}{\partial u_j}. \tag{2.10}$$

where $\varphi_0 = \varphi$ and the higher order coefficients φ_j are constructed using the prolongation formula

$$\varphi_j = D_x\varphi_j - u_{j+1}D_x\xi, \qquad j = 0,1,2,\ldots. \tag{2.11}$$

See [43, 42] for a proof as well as the non-recursive version of the formula.

The Maurer-Cartan forms $\mu^1,\ldots,\mu^r$ are dual to the infinitesimal generators $\mathbf{v}_1,\ldots,\mathbf{v}_r$. The following lemma exploits this duality.

Lemma 2.2.4. *If $\widehat{\Omega} = \pi_J w^*\Omega$ is a lifted jet form on $\boldsymbol{\mathcal{B}}$, then*

$$d_G\widehat{\Omega} = \sum_{k=1}^{r}\mu^k \wedge \pi_J w^*[\mathbf{v}_k(\Omega)].$$

Proof.

$$\begin{aligned} d_G\widehat{\Omega} &= \sum_k \mu^k \wedge \mathbf{v}_k(\widehat{\Omega}) = \sum_k \mu^k \wedge \mathbf{v}_k(g^*\Omega) \\ &= \sum_k \mu^k \wedge g^*[\mathbf{v}_k(\Omega)] = \sum_k \mu^k \wedge \pi_J w^*[\mathbf{v}_k(\Omega)] \end{aligned}$$

where we used the duality of the μ^k's and $\mathbf{v}_k$'s, and the identification of $\pi_J w^*$ with the pull-back by a group element g. $\square$

With this lemma, we are now able to state the recurrence formula.

Lemma 2.2.5 (Recurrence Formula)**.** *If Ω is any differential form on J^∞, then*

$$d\iota(\Omega) = \iota(d\Omega) + \sum_{k=1}^{r}\nu^k \wedge \iota[\mathbf{v}_k(\Omega)] \tag{2.12}$$

where the ν^k's are the pull-backs of the Maurer-Cartan forms μ^k via the moving frame section σ.

Proof. We compute.

$$\begin{aligned} d\iota(\Omega) &= d\sigma^*\pi_J w^*\Omega = \sigma^* d\pi_J w^*\Omega = \sigma^*(d_J + d_G)\pi_J w^*\Omega \\ &= \sigma^*(\pi_J dw^*\Omega + d_G\pi_J w^*\Omega) = \sigma^*\pi_J w^* d\Omega + \sigma^*\left(\sum_{k=1}^{r} \mu^k \wedge \pi_J w^*[\mathbf{v}_k(\Omega)]\right) \\ &= \iota(d\Omega) + \sum_{k=1}^{r} \nu^k \wedge \iota[\mathbf{v}_k(\Omega)] \end{aligned}$$

where we used equations (2.6) and (2.7) and Lemma 2.2.4. □

Invariantization takes a form of bigrade (r, s), (wedge products of r horizontal forms and s contact forms) to a form of invariant bigrade (r, s) (wedge products of r invariant horizontal forms and s invariant contact forms). However, the Lie derivative operation does not in general preserve the bigrading of the complex. Whereas $\mathbf{v}_k$ does map contact forms to contact forms, we see that

$$\mathbf{v}(dx) = d\xi = d_H\xi + d_V\xi$$

gives a combination of horizontal and zeroth order contact forms (in the non-projectable case when ξ depends on u. In the case of a projectable action, $\xi = \xi(x)$, $d_V\xi = 0$, and the bigrading of the complex is preserved.

In the general setting, the exterior derivative has *invariant* decomposition

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}} \tag{2.13}$$

where $d_{\mathcal{H}}$ increases invariant horizontal degree by 1, $d_{\mathcal{V}}$ increases invariant contact degree by 1, and $d_{\mathcal{W}}$ decreases invariant horizontal degree by 1 and increases invariant contact degree by 2.

Remark 2.2.6. The anamolous $d_{\mathcal{W}}$ component plays no significant role in this paper, nor in any applications that the author is aware of. Furthermore, $d_{\mathcal{W}}$ returns zero when applied to differential functions, and is identically zero in the event that the group action is projectable. In this latter event, the term invariant variational bicomplex is accurate. However, in the general case, the $d_{\mathcal{W}}$ term prevents us from properly applying this label. Closure, $d^2 = 0$ results in the relations,

$$d_{\mathcal{H}}^2 = 0, \qquad d_{\mathcal{W}}^2 = 0, \qquad d_{\mathcal{V}}^2 + d_{\mathcal{H}}d_{\mathcal{W}} + d_{\mathcal{W}}d_{\mathcal{H}} = 0$$

$$d_{\mathcal{H}}d_{\mathcal{V}} + d_{\mathcal{V}}d_{\mathcal{H}} = 0, \qquad d_{\mathcal{V}}d_{\mathcal{W}} + d_{\mathcal{W}}d_{\mathcal{V}} = 0$$

which fails to retain the a genuine bicomplex structure. In [28], the authors refer to this structure as a *quasi-tricomplex*.

The invariant decomposition in equation (2.13) is determined by the recurrence formula, but to use it one needs the expressions for the pulled-back Maurer-Cartan forms. Fortunately, these are easily solved using the cross-section which produces our moving frame. Though the methods extend to more general cross-sections, coordinate cross-sections greatly simplify the computations. Indeed, by setting certain coordinate functions to be constant, the left hand sides of equation (2.12) are equal to 0, and the right hand sides result in a linear system of equations for the $\nu^k,\ k = 1, \ldots, r$, by which we recover expressions of the pulled-back Maurer-Cartan forms. With these in hand, we can now unleash the power of the recurrence formula. First, a quick example.

Example 2.2.7. Let us apply the recurrence formula to our running example of the standard action of $SE(2)$ on the plane. The (prolonged) infinitesimal generators of the action are

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x} + 3u_x u_{xx}\partial_{u_{xx}} + \ldots.$$

The cross-section used in Example 2.1.6 was $x = u = u_x = 0$, and hence our phantom invariants are

$$\iota(x) = \iota(u) = \iota(u_x) = 0.$$

We use these values in the recurrence formula, (2.12), to compute the pulled-back Maurer-Cartan forms:

$$\begin{aligned}
\Omega = x: &\quad 0 = d\iota(x) = \iota(dx) + \nu^1\iota[\mathbf{v}_1(x)] + \nu^2\iota[\mathbf{v}_2(x)] + \nu^3\iota[\mathbf{v}_3(x)] = \varpi + \nu^1\\
\Omega = u: &\quad 0 = d\iota(u) = \iota(du) + \nu^1\iota[\mathbf{v}_1(u)] + \nu^2\iota[\mathbf{v}_2(u)] + \nu^3\iota[\mathbf{v}_3(u)] = \vartheta + \nu^2\\
\Omega = u_x: &\quad 0 = d\iota(u_x) = \iota(du_x) + \nu^1[\mathbf{v}_1(u_x)] + \nu^2\iota[\mathbf{v}_2(u_x)] + \nu^3[\mathbf{v}_3(u_x)] = \kappa\varpi + \vartheta_1 + \nu^3
\end{aligned}$$

where we recall that $\iota(du) = \iota(u_x dx + \theta)$, $\iota(du_x) = \iota(u_{xx}dx + \theta_1)$, and $\kappa = \iota(u_{xx})$ is the Euclidean curvature.

The resulting linear system is easily solved for ν^1, ν^2 and ν^3:

$$\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\varpi - \vartheta_1.$$

The coefficients of the invariant horizontal components of the Maurer-Cartan forms are called the *Maurer-Cartan invariants*. These are the entries of the Frenet-Serret matrix from the classical moving frame approach, see [22]. In this example, they are

$$R^1 = -1, \qquad R^2 = 0, \qquad R^3 = -\kappa.$$

With the Maurer-Cartan forms computed, the power of the recurrence formula is at our fingertips. The quantities that we will be most interested in will be $d_{\mathcal{H}}\theta_j, d_{\mathcal{V}}\kappa$, and $d_{\mathcal{V}}\varpi$. We compute the latter two quantities first. To find $d_{\mathcal{V}}\kappa$, we first note that $\kappa = \iota(u_{xx})$, so we let $\Omega = u_{xx}$ in the recurrence formula:

$$\begin{aligned} d\kappa = d\iota(u_{xx}) &= \iota(du_{xx}) + \sum_{j=1}^{3} \nu^j \iota[\mathbf{v}_j(u_{xx})] \\ &= \iota(u_{xxx}dx + \theta_2) + \nu^3 \iota[u_x u_{xx}] \\ &= I_3\varpi + \vartheta_2 \end{aligned}$$

where $I_3 = \iota(u_{xxx})$. Hence,

$$d_{\mathcal{V}}\kappa = \vartheta_2. \tag{2.14}$$

Next we let $\Omega = dx$ in the recurrence formula:

$$\begin{aligned} d\varpi = d\iota(dx) &= \iota(ddx) + \sum_{j=1}^{3} \nu^j \wedge \iota[\mathbf{v}_j(dx)] \\ &= \nu^1 \wedge \iota[d(1)] + \nu^3 \wedge \iota[d(-u)] \\ &= -\nu^3 \wedge \iota[u_x dx + \theta] = (\kappa\varpi + \vartheta_1) \wedge \vartheta \\ &= -\kappa\vartheta \wedge \varpi + \vartheta_1 \wedge \vartheta. \end{aligned} \tag{2.15}$$

The second term in (2.15) is $d_{\mathcal{W}}\varpi$, c.f. equation (2.13). The first term then is

$$d_{\mathcal{V}}\varpi = -\kappa\vartheta \wedge \varpi. \tag{2.16}$$

Lastly, we compute the invariant horizontal differentials of the basic invariant contact forms. We will let $\Omega = \theta, \theta_1$ in the recurrence formula, however we will do the preliminary computation of the corresponding Lie derivatives of the basic contact forms with respect the the $\mathbf{v}_i$. Note that $\mathbf{v}_i(\theta_j) = 0$ for $i = 1, 2$ and all $j \geq 0$. Using the properties of derivation and commutation with the differential, we see

$$\begin{aligned}
\mathbf{v}_3(\theta) &= \mathbf{v}_3(du - u_x dx) = d(\mathbf{v}_3(u)) - \mathbf{v}_3(u_x)dx - u_x d(\mathbf{v}_3(x)) \\
&= dx - (1 + u_x^2)dx - u_x d(-u) = -u_x(du - u_x dx) = -u_x\theta \\
\mathbf{v}_3(\theta_1) &= \mathbf{v}_3(du_x - u_{xx}dx) = d(\mathbf{v}_3(u_x)) - \mathbf{v}_3(u_{xx})dx - u_{xx}d(\mathbf{v}_3(x)) \\
&= 2u_x\theta_1 + u_{xx}\theta
\end{aligned}$$

etc. Hence,

$$\begin{aligned}
d\vartheta = d\iota(\theta) &= \iota(d\theta) + \nu^3 \wedge \iota[-u_x\theta] \\
&= -\vartheta_1 \wedge \varpi = \varpi \wedge \vartheta_1 \\
d\vartheta_1 = d\iota(\theta_1) &= \iota(d\theta_1) + \nu^3 \wedge \iota[2u_x\theta_1 + u_{xx}\theta] \\
&= -\vartheta_2 \wedge \varpi - (\kappa\varpi + \vartheta_1) \wedge \kappa\vartheta = \varpi \wedge (\vartheta_2 - \kappa^2\vartheta) - \kappa\vartheta_1 \wedge \vartheta.
\end{aligned}$$

Thus,

$$d_{\mathcal{H}}\vartheta = \varpi \wedge \vartheta_1, \qquad d_{\mathcal{H}}\vartheta_1 = \varpi \wedge (\vartheta_2 - \kappa^2\vartheta) \tag{2.17}$$

and incidentally, $d_{\mathcal{V}}\vartheta = 0, d_{\mathcal{V}}\vartheta_1 = -\kappa\vartheta_1 \wedge \vartheta$.

At this point we note that there is an invariant analog of equation (2.5). For any invariant contact form η, its invariant horizontal differential is written

$$d_{\mathcal{H}}\eta = \varpi \wedge \mathcal{D}\eta. \tag{2.18}$$

We can recursively construct the invariant derivatives of the invariant contact forms using the recurrence formula, thus writing any higher order invariant contact form as an invariant differential operator applied to the zeroth order invariant contact form. In particular, we can apply this to $d_{\mathcal{V}}\kappa$ and $d_{\mathcal{V}}\varpi$. Since $d_{\mathcal{V}}\kappa$ is an invariant contact form, we write $d_{\mathcal{V}}\kappa = \mathcal{A}_\kappa(\vartheta) = \mathcal{A}(\vartheta)$, where we will often drop the subscript when the context is clear. The operator $\mathcal{A}$ is called the *invariant linearization* of κ. Similarly, $d_{\mathcal{V}}\varpi = \mathcal{B}(\vartheta) \wedge \varpi$, where $\mathcal{B}$ is called the *invariant Hamiltonian operator*, [28].

Example 2.2.8. In the previous example, we see from equation (2.17) that

$$\mathcal{D}\vartheta = \vartheta_1 \qquad\qquad \vartheta_1 = \mathcal{D}\vartheta \tag{2.19}$$

$$\mathcal{D}\vartheta_1 = \vartheta_2 - \kappa^2\vartheta \qquad\qquad \vartheta_2 = \mathcal{D}\vartheta_1 + \kappa^2\vartheta = (\mathcal{D}^2 + \kappa^2)\vartheta \tag{2.20}$$

From the above relations and equations (2.14) and (2.16),

$$d_{\mathcal{V}}\kappa = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\vartheta, \qquad \text{i.e.,} \qquad \mathcal{A} = \mathcal{D}^2 + \kappa^2. \tag{2.21}$$

is the invariant linearization of κ and

$$d_{\mathcal{V}}\varpi = -\kappa\vartheta \wedge \varpi, \qquad \text{i.e.} \qquad \mathcal{B} = -\kappa \tag{2.22}$$

is the invariant Hamiltonian operator.

2.3 Invariant Submanifold Flows

We continue with our *modus operandi* and consider the simple case of curves in the plane. Let $S \subset M$ be a 1-dimensional submanifold. The invariant horizontal and zeroth order invariant contact forms ϖ, ϑ form a G-equivariant coframe for T^*M along S, i.e. a section of the vector bundle $T^*M \to S$. We denote by $\mathbf{t}, \mathbf{n}$ the corresponding duall tangent vectors which form a G-equivariant frame for the bundle $TM \to S$. Contact forms annihilate the tangent space to S, and thus $\mathbf{t}$ forms a basis of the tangent bundle $TS \to S$, whereas $\mathbf{n}$ forms a basis for the G-equivariant normal bundle $NS \to S$.

Let

$$\mathbf{V} = \mathbf{V}_S = \mathbf{V}_T + \mathbf{V}_N = I\mathbf{t} + J\mathbf{n}$$

be a section of the bundle $TM \to S$ where $\mathbf{V}_T$ and $\mathbf{V}_N$ denote the tangential and normal components, respectively. We will refer to $\mathbf{V}$ as a vector field, however the term is technically imprecise. Any such vector field generates a submanifold flow:

$$\frac{\partial S}{\partial t} = \mathbf{V}_{S(t)}. \tag{2.23}$$

This is an nth order system of partial differential equations, where n is the maximum order of our moving frame and of the coefficients I, J.

Lemma 2.3.1. *Equation* (2.23) *represents an invariant submanifold flow if and only if the coefficients* $I = \langle \mathbf{V}; \varpi \rangle$, $J = \langle \mathbf{V}; \vartheta \rangle$ *are differential invariants. Here,* $\langle \cdot, \cdot \rangle$ *is the pairing between tangent vectors and forms.*

The tangential component $\mathbf{V}_T$ serves only to reparameterize S, and thus if we are only interested in the image of S under the flow we can set $\mathbf{V}_T = 0$. Thus $\mathbf{V}$ and

$\mathbf{V}_N$ effectively define the same submanifold flow, modulo reparameterization. The flow generated by $\mathbf{V}_N$ is called a *normal flow.*

Example 2.3.2. Euclidean invariant planar curve flows are among the most well-studied flows. Examples include

$\mathbf{V} = \mathbf{n}$: This is the grassfire flow, describing the evolution of the boundary of a grassfire in a homogeneous medium, [46]. This flow is also used in geometric optics.

$\mathbf{V} = \kappa\mathbf{n}$: This is the curve shortening flow, [46, 41] , which can also be characterized as the gradient flow associated to the arc-length functional.

$\mathbf{V} = \kappa_s\mathbf{n}$: This flow induces the integrable modified Korteweg-deVries equation for the evolution of curvature, [10, 19].

Flows that preserve arc-length are another important class of geometric curve flows. We have one independent invariant horizontal one-form

$$\varpi = \omega + \eta = ds + \eta$$

where $ds = \omega$ is identified with the contact invariant arc-length element. Invariance under the flow of $\mathbf{V}$ means that $\mathbf{V}(\omega)$ vanishes on the submanifold, i.e. $\mathbf{V}(\omega)$ is a contact form.

Each of the above examples are normal flows, and therefore do not preserve the Euclidean arc-length. Generally, there are non-local tangential coefficients that make the associated flow arc-length preserving, however, in the third example there is a *local* tangential reparameterization coefficient that preserves arc-length: $\mathbf{V} = \frac{1}{2}\kappa^2\mathbf{t} + \kappa_s\mathbf{n}$.

Lemma 2.3.3. *The curve flow induced by* $\mathbf{V} = I\mathbf{t} + J\mathbf{n}$ *preserves the* G*-invariant arc-length if and only if the Lie derivative* $\mathbf{V}(\varpi)$ *is a contact form.*

For a proof, one can consult [42], Theorem 1.65. We will use the notation $\alpha \equiv \beta$ if $\alpha - \beta$ is a contact form.

Lemma 2.3.4. *If the vector field* $\mathbf{V}$ *defines an arc-length preserving flow, then it commutes with the invariant differentiations:* $[\mathbf{V}, \mathcal{D}] = 0$.

Proof. Let F be a differential function. Then

$$\begin{aligned}\mathcal{D}(\mathbf{V}(F))\varpi = d_{\mathcal{H}}(\mathbf{V}(F)) \equiv d(\mathbf{V}(F)) = \mathbf{V}(dF) \equiv \mathbf{V}(d_{\mathcal{H}}F) = \mathbf{V}(\mathcal{D}F\varpi)\\ = \mathbf{V}(\mathcal{D}F)\varpi + \mathcal{D}F\mathbf{V}(\varpi) = \mathbf{V}(\mathcal{D}F)\varpi\end{aligned}$$

where we assumed $\mathbf{V}$ to be arc-length preserving, i.e. $\mathbf{V}(\varpi) \equiv 0$. □

Lemma 2.3.5. *If $\mathcal{C}$ is any invariant differential operator, then $\mathbf{V} \lrcorner \mathcal{C}(\vartheta) = \mathcal{C}(\mathbf{V} \lrcorner \vartheta)$ for any invariant contact form ϑ.*

Proof. $\mathbf{V}$ preserves the contact ideal. Using Cartan's magic formula (see for example [42], Proposition 1.66), we find

$$\begin{aligned}0 \equiv \mathbf{V}(\vartheta) = \mathbf{V} \lrcorner d\vartheta + d(\mathbf{V} \lrcorner \vartheta) \equiv \mathbf{V} \lrcorner (\varpi \wedge \mathcal{D}\vartheta) + \mathcal{D}(\mathbf{V} \lrcorner \vartheta)\varpi\\ \equiv \big(- \mathbf{V} \lrcorner \mathcal{D}\vartheta + \mathcal{D}(\mathbf{V} \lrcorner \vartheta)\big)\, \varpi\end{aligned}$$

i.e. $\mathcal{D}(\mathbf{V} \lrcorner \vartheta) = \mathbf{V} \lrcorner \mathcal{D}\vartheta$. The result follows by iteration of this expression. □

With these lemmas in hand, we are now able to derive the condition that $\mathbf{V} = I\mathbf{t}+J\mathbf{n}$ be arc-length preserving. We start with Cartan's formula.

$$\mathbf{V}(\varpi) = \mathbf{V} \lrcorner d\varpi + d(\mathbf{V} \lrcorner \varpi) \equiv \mathbf{V} \lrcorner (\mathcal{B}(\vartheta) \wedge \varpi) + \mathcal{D}I\varpi = (\mathcal{B}(J) + \mathcal{D}I)\, \varpi$$

where we used Lemma 2.3.5 in the second line. We have just proven the following:

Theorem 2.3.6. *The flow induced by the vector field $\mathbf{V} = I\mathbf{t} + J\mathbf{n}$ is arc-length preserving if and only if*

$$\mathcal{D}I + \mathcal{B}(J) = 0$$

If $\mathbf{V}_N$ is a normal flow, we can construct its arc-length preserving counterpart by introducing the generally non-local tangential coefficient,

$$I = \mathcal{D}^{-1}\mathcal{B}(J). \tag{2.24}$$

Example 2.3.7. With our running example of the $SE(2)$ action on the plane, the arc-length preserving condition is

$$\mathcal{D}I = \mathcal{B}(J) = \kappa J.$$

Most of the flows listed in Example 2.3.2 have non-local intrinsic counterparts. One exception is the mKdV flow where $J = \kappa_s$ and $I = \frac{1}{2}\kappa^2$.

Evolution of Invariants

An invariant curve flow induces a corresponding evolution of the differential invariants of the curve. For normal flows (with zero tangential component), the evolution of the differential invariants is given by an infinite-dimensional dynamical system of coupled ordinary differential equations. This is a result of the fact that in general, time differentiation and arc-length differentiation do not commute, i.e. the arc-length parameter will vary in time. This infinite hierarchy of differential equations can often be closed off at finite order by the use of signatures, see [44] for details.

We will restrict our focus to arc-length preserving flows $\mathbf{V} = I\mathbf{t} + J\mathbf{n}$, where I is given by equation (2.24). The evolution of a differential invariant κ under this flow is given by the Lie derivative $\mathbf{V}(\kappa)$:

$$\frac{\partial \kappa}{\partial t} = \mathbf{V}(\kappa) = \mathbf{V} \lrcorner\, d\kappa = \mathbf{V} \lrcorner (d_{\mathcal{H}}\kappa + d_{\mathcal{V}}\kappa) = \mathbf{V} \lrcorner (\kappa_s \varpi + \mathcal{A}_\kappa(\vartheta)) = \kappa_s I + \mathcal{A}_\kappa(J)$$

where we used Lemma 2.3.5 in the last equality. Applying Theorem 2.3.6 and using (2.24) in the above, we obtain the following theorem:

Theorem 2.3.8. *If $\mathbf{V} = I\mathbf{t} + J\mathbf{n}$ is an arc-length preserving flow, then the evolution of any differential invariant K is given by*

$$\frac{\partial \kappa}{\partial t} = \Big(\mathcal{A}_\kappa - \kappa_s \mathcal{D}^{-1}\mathcal{B}\Big)(J) = \mathcal{P}_\kappa(J). \tag{2.25}$$

Definition 2.3.9. The integro-differential operator

$$\mathcal{P}_\kappa := \mathcal{A}_\kappa - \kappa_s \mathcal{D}^{-1}\mathcal{B} \tag{2.26}$$

is called the *characteristic operator* for the differential invariant κ.

The subscripts on $\mathcal{P}$ and $\mathcal{A}$ will generally be omitted when the context is clear. Note that (2.25) says that the time evolution of a differential invariant κ under an arc-length preserving flow is determined by an intrinsic integro-differential operator associated to κ, and the normal component of the flow.

Remark 2.3.10. Since arc-length is preserved, $\mathcal{D}$ and $\frac{\partial}{\partial t}$ commute, however forcing this relationship induces the generally non-local term

$$\kappa_s \mathcal{D}^{-1}\mathcal{B}(J) = \kappa_s \int \mathcal{B}(J)\, ds.$$

Hence for most choices of normal component J, (2.25) will be an integro-differential equation. Additionally, any constant of integration will just produce a multiple of κ_s, which represents the arc-length preserving tangential flow $\kappa_t = \kappa_s$, and just serves to translate the arc-length parameter. We can effectively ignore this and thus take the integration constant to be zero.

Example 2.3.11. We return to our running example the the $SE(2)$ action on the plane. The characteristic operator for the Euclidean curvature κ is given by

$$\mathcal{P} = \mathcal{A} - \kappa_s \mathcal{D}^{-1}\mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

where we recall equations (2.21) and (2.22). The $\cdot$ occurring after $\mathcal{D}^{-1}$ indicates to first multiply by $-\kappa$ and then apply $\mathcal{D}^{-1}$.

For the flows in Example 2.3.2, we have

$$\begin{aligned}
\text{Grassfire Flow}: &\quad J = 1 &\qquad& \kappa_t = \kappa^2 + \kappa_s \mathcal{D}^{-1}\kappa \\
\text{Curve Shortening Flow}: &\quad J = \kappa &\qquad& \kappa_t = \kappa_{ss} + \kappa^3 + \kappa_s \mathcal{D}^{-1}\kappa^2 \\
\text{mKdV Flow}: &\quad J = \kappa_s &\qquad& \kappa_t = \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s
\end{aligned}$$

Note that the first two flows induce nonlocal evolutions of curvature. However, the mKdV flow with $J = \kappa_s$ produces the integrable mKdV equation. Furthermore, the characteristic operator $\mathcal{P}$ is the associated recursion operator for the mKdV equation, [42]. The next chapter will discuss the frequency of the occurrence of integrable equations and recusion operators arising in the evolution of differential invariants under different planar group actions.

Chapter 3

Integrable Equations and Characteristic Operators

3.1 Integrable Equations and Characteristic Operators

3.1.1 Introduction

In [23], Hasimoto showed that the evolution of a thin vortex filament in an inviscid, incompressible fluid is transformable to the cubic nonlinear Schrödinger equation, which is known to have many of the properties now generally associated to so-called completely integrable equations, i.e. evolution equations possessing soliton solutions, bihamiltonian structures, solution by inverse-scattering, auto-Bäcklund transformations, and infinite hierarchies of generalized symmetries and conservation laws. Many equations having the above properties also have associated to them a recursion operator, by which infinite hierarchies of symmetries are produced. It is in this sense that we will use the term integrable, i.e. an equation is integrable if it has a recursion operator. In [29], Lamb showed that the motion of helical space curves were related to the sine-Gordon and Hirota equations, which can be related to the nonlinear Schödinger and modified Korteweg-deVries equations, respectively. What followed was a proliferation of papers investigating the relationship between geometric, arc-length preserving (non-stretching) flows and integrable equations, [19, 24, 32], including studies into associated Poisson structures [34], and recursion operators [30]. In [38], the authors demonstrate a connection between the Serret-Frenet equations and the AKNS scattering problem at zero eigenvalue. In [1], the author uses moving frames to derive bi-Hamiltonian operators from group-invariant flows of non-stretching curves.

Notably, in [11, 10], Chou and Qu produced a detailed investigation into the relationship between arc-length preserving planar curve flows in various Klein geometries and integrable equations describing the evolution of minimal order differential invariants. The KdV, mKdV, Burgers', and Sawada-Kotera [47] equations, among others naturally arise in this fashion. Some of these relationships provide geometric explanations of the Miura transformation, [37], relating the KdV and mKdV equations, and the Hopf-Cole transformation relating Burgers' equation with the heat equation.

Further investigations into integrable dynamics arising from the evolution of curves in $\mathbb{R}^3$ have also been performed. In [31], Langer and Perline showed that via the Hasimoto transform, the dynamics of a non-stretching vortex filament gives rise to the recursion operator of the cubic non-linear Schrödinger equation. In [12], Chou and Qu demonstrate connections between motions of curves in affine and centro-affine geometry and integrable equations such as the KdV, Harry Dym, Sawada-Kotera, Kaup-Kupershmidt, Boussinesq, and Hirota-Satsuma equations.

In [44], Olver uses the techniques of the equivariant method of moving frames [16, 17] and the invariant variational bicomplex [28] to describe an algorithmic process of performing investigations into invariant submanifold flows and the associated evolutions of their differential invariants. The advantages of using this framework are the algorithmic nature of the constructions, and the "invariant calculus" which allows for all relevant computations to be done symbolically, i.e. the local coordinate representation of all invariant functions, forms, and operators are not needed. Rather than work with potentially complicated group actions, and their even more complicated prolongations, we take an infinitesimal approach and effectively work with the Lie algebra associated to our Lie group. The infinitesimal generators of the group action are employed along with the powerful recurrence formula, which then allows for a systematic production of the *characteristic operator*, an integro-differential operator which produces the curvature evolution under (group invariant) arc-length preserving flows. The algorithms presented in this paper also account for (non-arc-length preserving) normal flows, which play prominent roles in engineering, computer vision, and geometric applications, [18, 21, 46, 41, 5, 19, 35, 8]. Furthermore, because of the algorithmic nature of the construction, the process can be implemented in symbolic computation software such as MATHEMATICA or MAPLE. Frequently, recursion operators arise naturally within the context of the invariant variational bicomplex, and in many instances coincide with the characteristic operator. This observation motivated a return to the results of Chou and Qu, with the newfound optimism that by using the aforementioned tools, a stronger connection between group invariant flows and integrable evolution equations might be established.

3.1.2 Symmetries and Recursion Operators

This section will include a brief summary of the necessary definitions to proceed further. The notation follows that in [42], in which much more detailed treatment can be found. Let

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{3.1}$$

be a vector field on the space $M = X \times U$ divided into "independent", x^i, and "dependent", u^α, variables. We prolong the action to act on jet space coordinates u^α_J where J is a given multi-index notation. From equations (2.10), (2.11),

$$\mathrm{pr}^{(n)}\,\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{j=\#J\geq 0}^{n} \varphi^\alpha_J(x,u^{(j)})\frac{\partial}{\partial u^\alpha_J}. \tag{3.2}$$

where φ^α_J is constructed recursively with $\varphi^\alpha_0 = \varphi^\alpha$ by

$$\varphi^\alpha_{Ji} = D_i\varphi^\alpha_J - \sum_{j=1}^{p} u^\alpha_{Jj} D_i \xi^j,$$

and

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{j=\#J\geq 0} u^\alpha_{Ji}\frac{\partial}{\partial u^\alpha_J}$$

is the total derivative with respect to x^i.

Remark 3.1.1. One can also use the explicit formula to compute the higher order prolongation coefficients, a proof of which can be found in [42]:

$$\varphi_J^\alpha = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha, \qquad \text{where} \qquad Q^\alpha = \varphi^\alpha - \sum_{i=1}^{p} \xi^i u_{i^\alpha}.$$

Before stating the infinitesimal symmetry criterion, we need a couple of technical definitions. A system $\Delta^\nu(x, u^{(n)}) = 0$, $\nu = 1, \dots, l$, of differential equations is *locally solvable* if at each point $(x_0, u_0^{(n)}) \in \{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\}$, there exists a smooth solution $u = f(x)$ of the system, defined for x in a neighborhood of x_0. The system is said to be of *maximal rank* if the associated Jacobian matrix

$$J_\Delta(x, u^{(n)}) = \left(\frac{\partial \Delta^\nu}{\partial x^i}, \frac{\partial \Delta^\nu}{\partial u_J^\alpha} \right)$$

is of rank l whenever $\Delta = 0$. Furthermore, a system of differential equations is called *nondegenerate* if it is of maximal rank and locally solvable.

Proposition 3.1.2. *If $\Delta^\nu = 0$, $\nu = 1, \dots, l$ represents an n-th order system of nondegenerate differential equations, then $\mathbf{v}$ is an infinitesimal symmetry of the system of the system if and only if*

$$pr^{(n)}\mathbf{v}(\Delta^\nu) = 0, \quad \textit{for all} \quad \nu = 1, \dots, k$$

Following Emmy Noether, [40], we expand our definition of symmetry to allow for jet variables in the vector field coordinates. We denote by $\xi[u] := \xi(x^i, u_J^\alpha)$ and $\varphi[u] := \varphi(x^i, u_J^\alpha)$, and where it is assumed that ξ and ϕ depend on derivatives of u up to some finite order, $\#J \leq n$. A *generalized vector field* on M takes the form

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha[u] \frac{\partial}{\partial u^\alpha}.$$

The prolongation formula and infinitesimal symmetry criterion carry over as before to generalized symmetries.

An important class of generalized vector fields are those in which the coefficients of the $\partial/\partial x^i$ are zero.

Definition 3.1.3. An *evolutionary vector field* is a generalized vector field of the form

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha}.$$

The q-tuple $Q = (Q_1[u], \ldots, Q_q[u])$ is called its *characteristic.*

Note that the prolongation of evolutionary vector fields takes a particularly simple form:

$$\mathrm{pr}\ \mathbf{v}_Q = \sum_{\alpha,J} D_J Q_\alpha[u] \frac{\partial}{\partial u_J^\alpha}.$$

Any generalized vector field of the form (3.1) has an associated *evolutionary representative*:

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} (\varphi^\alpha - \xi^i u_i^\alpha) \frac{\partial}{\partial u^\alpha} = \sum_\alpha Q^\alpha \frac{\partial}{\partial u^\alpha}.$$

which essentially has all the same properties.

Proposition 3.1.4. *A generalized vector field* $\mathbf{v}$ *is a symmetry of a system of differential equations if and only if its evolutionary representative is.*

Proof. Note that the prolongation formula, (3.2), can alternatively be written as

$$\mathrm{pr}\ \mathbf{v} = \mathrm{pr}\ \mathbf{v}_Q + \sum_{i=1}^{p} \xi^i D_i$$

Hence, when applied to a system of equations $\Delta^\nu = 0$, the second term on the righthand side is always zero when evaluated on solutions to the system and

$$\mathrm{pr}\ \mathbf{v}(\Delta^\nu) = \mathrm{pr}\ \mathbf{v}_Q(\Delta^\nu).$$

The result then follows by the infinitesimal symmetry criterion, Proposition 3.1.2. □

A *differential function* is a smooth function $P(x, u^{(n)})$ of x, u, and the derivatives of u with respect to x up to some finite order. We denote by A^q the space of q-tuples of differential functions. We are now in position to define recursion operators.

Definition 3.1.5. Let $\Delta = 0$ be a system of differential equations. A *recursion operator* for Δ is a linear operator $\mathcal{R} : A^q \to A^q$ with the property that whenever Q is a characteristic of an evolutionary symmetry of Δ, so is $\widetilde{Q} = \mathcal{R}Q$, i.e. if $\mathbf{v}_Q$ is an evolutionary symmetry of Δ, so is $\mathbf{v}_{\mathcal{R}Q} = \mathbf{v}_{\widetilde{Q}}$.

It is worth noting that in most instances, the recursion operators associated to nonlinear differential equations are actually integro-differential operators. The essential property of producing new symmetries from old is maintained, however in some instances, they may be non-local.

The criteria for being a recursion operator requires the following definition:

Definition 3.1.6. Let $P \in A^r$. The *Fréchet derivative* of P is the differential operator $\mathrm{D}_P : A^q \to A^r$ defined so that

$$d_V P = \mathrm{D}_P(\theta). \tag{3.3}$$

In local coordinates, D_P is a $q \times r$ matrix differential operator with entries

$$(\mathrm{D}_P)_{\mu\nu} = \sum_J \frac{\partial P_\mu}{\partial u_J^\nu} D_J. \tag{3.4}$$

For example, if $P = u_{xxx} + u^2 u_x$, then

$$d_V P = \theta_3 + u^2\theta_1 + 2uu_x\theta = (D_x^3 + u^2 D_x + 2uu_x)(\theta) = \mathrm{D}_P(\theta).$$

Theorem 3.1.7. *Suppose $\Delta[u] = 0$ is a system of nondegenerate differential equations. If $\mathcal{R} : A^q \to A^q$ is a linear operator such that $\mathrm{D}_\Delta \cdot \mathcal{R} = \widetilde{\mathcal{R}} \cdot \mathrm{D}_\Delta$ for all solutions u to Δ and for some $\widetilde{\mathcal{R}} : A^q \to A^q$, then $\mathcal{R}$ is a recursion operator for the system.*

The proof is found in [42], Theorem 5.29. Our focus will be in the case of evolution equations, where $\Delta = u_t - K[u] = 0$, where K is a differential function depending on x, u and the derivatives of u with respect to x, but not on t and derivatives with respect to t. In this event, the operator $\widetilde{\mathcal{R}}$ must be the same as $\mathcal{R}$ and the condition reduces to

$$D_t\mathcal{R} = [\mathrm{D}_K, \mathcal{R}]. \tag{3.5}$$

Here, D_t acts on $\mathcal{R}$ by acting only on the coefficients of the total derivative operators D_x in $\mathcal{R}$. For example, if $\mathcal{R} = D_x + u + u_x D_x^{-1}$, then

$$D_t\mathcal{R} = u_t + u_{xt} D^{-1}.$$

Example 3.1.8. We demonstrate that the operator $\mathcal{R} = D_x + u + u_x D_x^{-1}$ is a recursion operator for Burgers' equation, $u_t = u_{xx} + 2uu_x$. We want to show that $\mathcal{R}$ satisfies (3.5). First note that $K = u_{xx} + 2uu_x$, and using (3.4) we find

$$\mathrm{D}_K = D_x^2 + 2uD_x + 2u_x.$$

We compute the commutator $[\mathrm{D}_K, \mathcal{R}] = \mathrm{D}_K \cdot \mathcal{R} - \mathcal{R} \cdot \mathrm{D}_K$ first.

$$\begin{aligned}
\mathrm{D}_K \cdot \mathcal{R} &= (D_x^2 + 2uD_x + 2u_x)(D_x + u + u_x D_x^{-1}) \\
&= D_x^3 + uD_x^2 + 2u_x D_x + u_{xx} + u_x D_x + 2u_{xx} + u_{xxx} D_x^{-1} \\
&\quad + 2u(D_x^2 + uD_x + u_x + u_x + u_{xx} D_x^{-1}) \\
&\quad + 2u_x D_x + 2uu_x + 2u_x^2 D_x^{-1} \\
&= D_x^3 + 3uD_x^2 + (5u_x + 2u^2)D_x + 3u_{xx} + 6uu_x + (u_{xxx} + 2uu_{xx} + 2u_x^2)D_x^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R} \cdot \mathrm{D}_K &= (D_x + u + u_x D_x^{-1})(D_x^2 + 2uD_x + 2u_x) \\
&= D_x^3 + 2uD_x^2 + 2u_x D_x + 2u_x D_x + 2u_{xx} \\
&\quad + u(D_x^2 + 2uD_x + 2u_x) + u_x D_x^{-1} \cdot (D_x^2 + 2uD_x + 2u_x).
\end{aligned}$$

We can single out the last term involving the inverse (integral) operator D_x^{-1} and note that $\mathrm{D}_K = D_x \cdot (D_x + 2u)$. Hence $u_x D_x^{-1} \cdot (D_x^2 + 2uD_x + 2u_x) = u_x(D_x + 2u)$. Combining like terms and then subtracting gives us

$$\begin{aligned}
\mathcal{R} \cdot \mathrm{D}_K &= D_x^3 + 3uD_x^2 + (5u_x + 2u^2)D_x + 2u_{xx} + 4uu_x \\
[\mathrm{D}_K, \mathcal{R}] &= u_{xx} + 2uu_x + (u_{xxx} + 2uu_{xx} + 2u_x^2)D_x^{-1}
\end{aligned}$$

On the other hand

$$D_t \mathcal{R} = D_t(D_x + u + u_x D_x^{-1}) = u_t + u_{xt} D_x^{-1} = u_{xx} + 2uu_x + (u_{xxx} + 2uu_{xx} + 2u_x^2)D_x^{-1}$$

from which we see (3.5) is satisfied. Note that in computing $D_t \mathcal{R}$, the total t and x derivatives are assumed to commute, and all t-derivatives are eliminated using Burgers' equation.

3.1.3 Examples of the Process

In the following section, we present the catalog of results for our problem at hand. Sophus Lie's classification, [33], of independent actions on the complex 2-plane is analyzed case by case, and in almost every instance, integrable equations arise. The method is generally as follows. The infinitesimal generators of the action are displayed. A minimal cross-section is chosen, and the techniques outlined in Chapter 1 are implemented.

Particular use is made of the recurrence formula to determine the $\mathcal{A}$ and $\mathcal{B}$ operators. From these operators, the characteristic operator $\mathcal{P}$ is constructed using formula (2.26), and thus we obtain the general expression for the time evolution of the minimal order differential invariant, κ, under an intrinsic flow with normal component J, so that $\kappa_t = \mathcal{P}(J)$.

At this point, some guesswork is required. One wants to choose J so that the resulting evolution equation is both local and integrable. In the event that the choice of J results in an integrable equation, we check to see if $\mathcal{P}$ is an associated recursion operator. If it is not, we often find that $\mathcal{P}$ can be "amended" with an extra factor to produce a correct recursion operator. This will be explained below.

We start with two examples that will demonstrate the general computations needed to perform the investigation.

Example 3.1.9. Our first example comes from the 3-dimensional action

$$(x, u) \mapsto \left(\frac{e^b x}{1 - ax} + c, e^{-b} u(1 - ax)^2 \right), \qquad a, b, c \in \mathbb{R}$$

which is given by the following prolonged infinitesimal generators:

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = x\partial_x - u\partial_u - 2u_x\partial_{u_x} - 3u_{xx}\partial_{u_{xx}} - \dots$$

$$\mathbf{v}_3 = x^2\partial_x - 2xu\partial_u - (2u + 4xu_x)\partial_{u_x} - (6u_x + 6xu_{xx})\partial_{u_{xx}} - \dots .$$

This is Case 1.1 from Table 3.1 below.

For future reference, we include the following Lie derivatives of the basic contact forms with respect to these generators:

$$\mathbf{v}_1(\theta_j) = 0, \qquad \mathbf{v}_2(\theta_j) = -(j+1)\theta_j, \qquad \mathbf{v}_3(\theta_j) = -2(j+1)x\theta_j - j(j+1)\theta_{j-1} \quad (3.6)$$

for $j = 0, 1, \dots$.

We use the cross-section $x = 0,\ u = 1,\ u_x = 0$ coupled with the recurrence formula, equation (2.12) to produce the pulled-back Maurer-Cartan forms, $\nu^i,\ i = 1, 2, 3$:

$$\begin{aligned}
&\Omega = x: && 0 = \varpi + \nu^1 && \nu^1 = -\varpi^1 \\
&\Omega = u: && 0 = \vartheta - \nu^2 && \nu^2 = \vartheta \\
&\Omega = u_x: && 0 = I_2\varpi + \vartheta_1 - 2\nu^3 && \nu^3 = \frac{1}{2}\kappa\varpi^1 + \frac{1}{2}\vartheta_1,
\end{aligned} \qquad (3.7)$$

where we recall the notation $\varpi := \iota(dx)$ and $\vartheta_j := \iota(\theta_j)$ and write $\kappa := \iota(u_{xx}) = I_2$.

Remark 3.1.10. It is worth emphasizing that all computations here are done symbolically, and we do not need the local coordinate expressions for $\kappa, \mathcal{D}, \varpi$, etc. However, in this example, these are easily calculated:

$$\kappa = \frac{2uu_{xx} - 3u_x^2}{2u^4}, \qquad \mathcal{D} = \frac{1}{u} D_x, \qquad \varpi = u\,dx, \qquad \vartheta = \frac{1}{u}\theta, \ldots$$

Note that κ is the Schwarzian derivative, which is invariant under all linear fractional transformations of the complex plane. See [45] for a general overview.

Next, we use the recurrence formula to compute $d_{\mathcal{V}}\kappa$ and $d_{\mathcal{V}}\varpi$:

$$\begin{aligned} d\kappa = d\iota(u_{xx}) &= I_3\varpi + \vartheta_2 - 3\kappa\nu^2 & d\varpi = d\iota(dx) &= \nu^2 \wedge \varpi \\ &= I_3\varpi + \vartheta_2 - 3\kappa\vartheta & &= \vartheta \wedge \varpi \\ &= d_{\mathcal{H}}\kappa \ + \ d_{\mathcal{V}}\kappa & &= d_{\mathcal{V}}\varpi. \end{aligned}$$

Hence,

$$d_{\mathcal{V}}\kappa = \vartheta_2 - 3\kappa\vartheta \qquad \text{and} \qquad d_{\mathcal{V}}\varpi = \vartheta \wedge \varpi. \tag{3.8}$$

To compute the operators $\mathcal{A}$ and $\mathcal{B}$, we compute the invariant derivatives of the invariant basic contact forms. The full recurrence formula provides these derivatives, however the following formula gives us a shortcut:

$$\mathcal{D}\vartheta_j = \vartheta_{j+1} + \sum_{k=1}^{r} R^k \iota(\mathbf{v}_k(\theta_j)).$$

where the R^k are the Maurer-Cartan invariants (see Example 2.2.7). Here,

$$R^1 = -1, \qquad R^2 = 0, \qquad R^3 = \frac{1}{2}\kappa,$$

c.f. (3.7).

Using (3.6), we compute the following invariant derivatives of the invariant contact forms, and recursively construct each basic invariant contact form as an invariant derivative of the zeroth order invariant contact form:

$$\begin{aligned} \mathcal{D}\vartheta &= \vartheta_1 & \vartheta_1 &= \mathcal{D}\vartheta \\ \mathcal{D}\vartheta_1 &= \vartheta_2 - \kappa\vartheta & \vartheta_2 &= \mathcal{D}\vartheta_1 + \kappa\vartheta = (\mathcal{D}^2 + \kappa)\vartheta \\ \mathcal{D}\vartheta_2 &= \vartheta_3 - 3\kappa\vartheta_1 & \vartheta_3 &= \mathcal{D}\vartheta_2 + 3\kappa\vartheta_1 = (\mathcal{D}^3 + 4\kappa\mathcal{D} + \kappa_s)\vartheta. \end{aligned}$$

Recalling equation (3.8), we have

$$d_{\mathcal{V}}\kappa = \vartheta_2 - 3\kappa\vartheta = (\mathcal{D}^2 - 2\kappa)\vartheta, \qquad\qquad d_{\mathcal{V}}\varpi = \vartheta \wedge \varpi,$$

i.e.

$$\mathcal{A} = \mathcal{D}^2 - 2\kappa \quad \text{and} \quad \mathcal{B} = 1.$$

The characteristic operator, given in Definition (2.3.9), is

$$\mathcal{P} = \mathcal{D}^2 - 2\kappa - \kappa_s\mathcal{D}^{-1}. \tag{3.9}$$

Under an intrinsic flow with normal component J, the differential invariant κ evolves according to

$$\kappa_t = \mathcal{P}(J).$$

Note that not all choices of J produce a local PDE. However the choice of $J = \kappa_s$ gives

$$\kappa_t = \mathcal{P}(\kappa_s) = \kappa_{sss} - 3\kappa\kappa_s$$

which is the KdV equation. Furthermore, we observe that $\mathcal{P}$ is the associated recursion operator, see for example [42].

Example 3.1.11. To provide some contrast, we present an alternate example. This is Case 3.3 from Table 3.1, below. The infinitesimal generators are

$$\mathbf{v}_1 = \partial_u, \qquad \mathbf{v}_2 = u\partial_u, \qquad \mathbf{v}_3 = u^2\partial_u.$$

We use the cross-section $u = 0, u_x = 1, u_{xx} = 0$. Note that because x is invariant, we cannot use the x-coordinate to normalize. We label $\kappa =: \iota(u_{xxx})$. Incidentally,

$$\kappa = \frac{2u_x u_{xxx} - 3u_{xx}^2}{2u_x^2}.$$

Without presenting all the computation details, it is found that

$$\mathcal{A} = \mathcal{D}^3 + 2\kappa\mathcal{D} + \kappa_s, \qquad \text{and} \qquad \mathcal{B} = 0.$$

The characteristic operator is

$$\mathcal{P} = \mathcal{D}^3 + 2\kappa\mathcal{D} + \kappa_s$$

By choosing $J = \kappa$, we see that

$$\kappa_t = \kappa_{sss} + 3\kappa\kappa_s,$$

which is the mKdV equation. However, the characteristic operator is not the associated recursion operator. Rather, $\mathcal{R}_{KdV} = \mathcal{P}\mathcal{D}^{-1}$. As we will see below, finding the correct recursion operator by attaching a $\mathcal{D}^{-1}$ onto $\mathcal{P}$ is surprisingly common.

3.2 The Catalog

	Generators	Dim	Structure
1.1	$p,\ xp - uq,\ x^2p - 2xuq$	3	$\mathfrak{sl}(2)$
1.2	$p,\ xp - uq,\ x^2p - (2xu+1)q$	3	$\mathfrak{sl}(2)$
1.3	$p,\ xp,\ uq,\ x^2p - xuq$	4	$\mathfrak{gl}(2)$
1.4	$p,\ xp,\ x^2p,\ q,\ uq,\ u^2q$	6	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$
1.5	$p,\ \eta_1(x)q, \ldots, \eta_k(x)q$	$k+1$	$\mathbb{C} \ltimes \mathbb{C}^k$
1.6	$p,\ uq,\ \eta_1(x)q, \ldots, \eta_k(x)q$	$k+2$	$\mathbb{C}^2 \ltimes \mathbb{C}^k$
1.7	$p,\ xp + \alpha uq,\ q,\ xq, \ldots, x^{k-1}q$	$k+2$	$\mathfrak{a}(1) \ltimes \mathbb{C}^k$
1.8	$p,\ xp + (ku + x^k)q,\ q,\ xq, \ldots, x^{k-1}q$	$k+2$	$\mathbb{C} \ltimes (\mathbb{C} \ltimes \mathbb{C}^k)$
1.9	$p,\ xp,\ uq,\ q,\ xq, \ldots, x^{k-1}q$	$k+3$	$(\mathfrak{a}(1) \oplus \mathbb{C}) \ltimes \mathbb{C}^k$
1.10	$p,\ 2xp + (k-1)uq,\ x^2p + (k-1)xuq,$		
	$q,\ xq, \ldots, x^{k-1}q$	$k+3$	$\mathfrak{sl}(2) \ltimes \mathbb{C}^k$
1.11	$p,\ xp,\ x^2p + (k-1)xuq,\ uq$		
	$q,\ xq, \ldots, x^{k-1}q$	$k+4$	$\mathfrak{gl}(2) \ltimes \mathbb{C}^k$
2.1	$p,\ q,\ xp - uq,\ uq,\ xq$	5	$\mathfrak{sa}(2)$
2.2	$p,\ q,\ xp,\ uq,\ uq,\ xq$	6	$\mathfrak{a}(2)$
2.3	$p,\ q,\ xp,\ up,\ xq,\ uq,\ x^2p + xuq,\ xup + u^2q$	8	$\mathfrak{sl}(3)$
3.1	$\eta_1(x)q, \ldots, \eta_k(x)q$	k	$\mathbb{C}^k$
3.2	$uq,\ \eta_1(x)q, \ldots, \eta_k(x)q$	$k+1$	$\mathbb{C} \ltimes \mathbb{C}^k$
3.3	$q,\ uq,\ u^2q$	3	$\mathfrak{sl}(2)$

Table 3.1: List of Lie Algebras of Vector Fields in $\mathbb{C}^2$

Table 3.1 reproduces the tables found in [43], pages 472-3. The table gives a complete classification of Lie algebras of vector fields on $\mathbb{C}^2$, first compiled by Sophus Lie himself.

The vector fields in the first category are transitive and imprimitive. Those in the second category are transitive and primitive, and those in the third are intransitive. An imprimitive action is one that admits an invariant foliation of the manifold. For example, case 1.7 admits two invariant foliations, by horizontal and vertical lines. The local action induced by these generators preserve this foliation, i.e. horizontal (vertical) lines are mapped to horizontal (vertical) lines.

We employ the following notation for the partial derivatives in the x and u directions: $p := \frac{\partial}{\partial x}$ and $q := \frac{\partial}{\partial u}$. In cases 1.5, 1.6, 3.1 and 3.2, the functions $\eta_1(x), \ldots, \eta_k(x)$ satisfy a kth order constant coefficient homogeneous linear ordinary differential equation. For cases 1.5-1.11, 3.1 and 3.2, we require $k \geq 1$. The situations where $k = 0$ are equivalent to other cases already listed. The numbering above will be referenced in the following analysis, however it is merely a convenience, and no claim is made on the benefits of this particular ordering.

In the tables below, the following data is given: the reference number from Table 3.1, the cross-section used in the computations, the definition of κ, the $\mathcal{A}, \mathcal{B}$ and $\mathcal{P}$ operators, the normal component J, the evolution equation $\kappa_t = \mathcal{P}(J)$, and the identification of the evolution equation when applicable.

The notation for the cross-section is $\{x^0, u^0, u^0_x, u^0_{xx}, \ldots, u^0_n\}$, where the entries are the cross-section constants, i.e. $u_k = u^0_k$. For example $\{0, 1, 0\}$ implies the cross-section $x = 0, u = 1, u_x = 0$. In all cases, coordinate cross-sections were used. This was done for ease of computation. The moving frame algorithm and recurrence formula allow for more complicated cross-sections, and there is a technique for writing the invariants given by one cross-section in terms of the invariants given by another. Thus, we strive for computational simplicity in using only coordinate cross-sections. In general, I use 0 and 1 for cross-sectional coordinates, again for computational ease. In some instances, values other than 1 are used to give a nicer form for the resulting evolution of κ.

Cases 1.1, 1.2, 3.3, and 1.4

The cases presented in Table 3.2 represent the three inequivalent actions of $\mathfrak{sl}(2)$ on the plane. Note that the two transitive cases (1.1 and 1.2) produce the integrable KdV and mKdV equations where $\mathcal{P}$ is the associated recursion operator. The third case (3.3) also produces the KdV equation with normal component $J = \kappa$, however the KdV recursion

1.1	$\{0,1,0\}$	$\kappa = \iota(u_{xx})$
$\mathcal{A} = \mathcal{D}^2 - 2\kappa$	$\mathcal{B} = 1$	$\mathcal{P} = \mathcal{D}^2 - 2\kappa - \kappa_s \mathcal{D}^{-1}$
$J = \kappa_s$	$\kappa_t = \kappa_{sss} - 3\kappa\kappa_s$	KdV
1.2	$\{0,0,1\}$	$\kappa = \iota(u_{xx})$
$\mathcal{A} = \mathcal{D}^2 - 4 - \frac{1}{4}\kappa^2 - \frac{1}{2}\kappa_s$	$\mathcal{B} = \frac{1}{2}\mathcal{D} + \frac{1}{4}\kappa$	$\mathcal{P} = \mathcal{D}^2 - \frac{1}{4}\kappa^2 - 4 - \kappa_s \mathcal{D}^{-1} \cdot \kappa$
$J = \kappa_s$	$\kappa_t = \kappa_{sss} - \frac{3}{8}\kappa^2\kappa_s - 4\kappa_s$	mKdV
3.3	$\{x,0,1,0\}$	$\kappa = \iota(u_{xxx})$
$\mathcal{A} = \mathcal{D}^3 + 2\kappa\mathcal{D} + \kappa_s$	$\mathcal{B} = 0$	$\mathcal{P} = \mathcal{D}^3 + 2\kappa\mathcal{D} + \kappa_s$
$J = \kappa$	$\kappa_t = \kappa_{sss} + 3\kappa\kappa_s$	KdV

Table 3.2: The three inequivalent actions of $\mathfrak{sl}(2)$

operator is given by $\mathcal{R}_{\text{KdV}} = \mathcal{P}\mathcal{D}^{-1}$. This represents the first of many instances where $J = \kappa$ produces an integrable equation in which the associated recursion operator $\mathcal{R}$ is written $\mathcal{R} = \mathcal{P}\mathcal{D}^{-1}$.

It is also worth mentioning that there is a connection between these three inequivalent actions, as noted in [13]. If we write case 1.1 using (y, v) coordinates, then the transformation defined by

$$y = x, \qquad v = 1/u_x$$

transforms the prolonged vector fields of case 3.3 to case 1.1. Writing case 1.2 in (z, w) coordinates, we have the transformation

$$z = y, \qquad w = v_y/2v$$

which transforms the prolonged vector fields of case 1.1 to case 1.2. It is interesting to note that the extra $\mathcal{D}^{-1}$ term appearing as the obstruction to $\mathcal{P}_{3.3}$ being a recursion operator arises from this change of coordinates taking case 3.3 to case 1.1. Similarly, the mKdV equation from case 1.2 transforms to an integrable equation written in terms of invariants from the action of case 1.1, which itself is transformable back to the mKdV equation, from which we can find the KdV equation via the Miura transformation. Complete details of these transformations would take us too far astray, but will merit further investigation.

Table 3.3 demonstrates the appearance of the KdV equation within the six-dimensional

1.4 $\{0,0,1,1,1,1\}$	$\kappa = \iota(u_5)$
$\mathcal{A} = \mathcal{D}^5 + (4\kappa - 5)\mathcal{D}^3 + 5\kappa_s\mathcal{D}^2$	$\mathcal{P} = \mathcal{D}^5 + (4\kappa - 5)\mathcal{D}^3 + 6\kappa_s\mathcal{D}^2$
$+(6 - 10\kappa + 4\kappa^2 + 4\kappa_{ss})\mathcal{D}$	$+(6 - 10\kappa + 4\kappa^2 + 4\kappa_{ss})\mathcal{D}$
$-2\kappa_s + 2\kappa\kappa_s + \kappa_{sss}$	$-5\kappa_s + 4\kappa\kappa_s + \kappa_{sss} - \kappa_s\mathcal{D}^{-1} \cdot \kappa_s$
$\mathcal{B} = -\mathcal{D}^3 + (3 - 2\kappa)\mathcal{D} - \kappa_s$	
$J = 1$ $\kappa_t = \kappa_{sss} + 3\kappa\kappa_s - 5\kappa_s$	KdV
$J = \kappa$ $\kappa_t = (\kappa_5 + 5\kappa\kappa_{sss} + 10\kappa_s\kappa_{ss} + \frac{15}{2}\kappa^2\kappa_s)$	
$-5(\kappa_{sss} + 3\kappa\kappa_s) + 6\kappa_s$	KdV

Table 3.3: The case of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$

action of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Note that with $J = 1$, we find the KdV equation with a translational term, but with $J = \kappa$, we find a combination of the KdV equation and the second member of its hierarchy. The operator $\mathcal{P}$ is not the associated recursion operator, but similar to case 3.3, $\hat{\mathcal{R}} = \mathcal{P}\mathcal{D}^{-1}$ can be viewed as a fourth order recursion operator for the KdV equation. More precisely, the KdV hierarchy is usually generated by the second order recursion operator $\mathcal{R}_{\text{KdV}} = \mathcal{D}^2 + 2\kappa + \kappa_s\mathcal{D}^{-1}$, hence we can write $\hat{\mathcal{R}} = \mathcal{R}_{\text{KdV}}^2 - 5\mathcal{R}_{\text{KdV}} + 6 = \mathcal{P}\mathcal{D}^{-1}$.

Case 1.10

Table 3.4 shows results of the action of case 1.10 for $k = 1, 2$ and 3. We see that when $k = 1$ and $k = 3$, we have $J = \kappa$ producing the KdV equation and the recursion operator appears as $\mathcal{R} = \mathcal{P}\mathcal{D}^{-1}$ in each case. In the latter case we again find the second member of the KdV hierarchy.

When $k = 2$, we find the Sawada-Kotera equation, but $\mathcal{P}$ is not the recursion operator in this case. The recursion operator is given as

$$\mathcal{R}_{\text{SW}} = \mathcal{P}(\mathcal{D}^2 + \tfrac{1}{3}\kappa + \tfrac{1}{3}\kappa_s\mathcal{D}^{-1}),$$

see [42]. The Sawada-Kotera equation, like most of the integrable evolution equations we encounter, start with the "seed" function κ_s. In most cases, we apply the associated recursion operator to κ_s to produce our integrable equation. The Sawada-Kotera

$k=1$ $\quad\{0,1,-3,0\}$	$\kappa=\iota(u_{xxx})$
$\mathcal{A}=\mathcal{D}^3+\frac{2}{3}\kappa\mathcal{D}$ $\quad\mathcal{B}=-\frac{1}{3}\mathcal{D}$	$\mathcal{P}=\mathcal{D}^3+\frac{2}{3}\kappa\mathcal{D}+\frac{1}{3}\kappa_s$
$J=\kappa$ $\quad\kappa_t=\kappa_{sss}+\kappa\kappa_s$	KdV
$k=2$ $\quad\{0,0,0,-1,0\}$	$\kappa=\iota(u_{xxxx})$
$\mathcal{A}=\mathcal{D}^4+\frac{5}{3}\kappa\mathcal{D}^2+\frac{2}{3}\kappa_s\mathcal{D}+\frac{1}{3}\kappa_{ss}+\frac{4}{9}\kappa^2$	$\mathcal{P}=\mathcal{D}^4+\frac{5}{3}\kappa\mathcal{D}^2+\frac{4}{3}\kappa_s\mathcal{D}$
$\mathcal{B}=-\frac{2}{3}\mathcal{D}^2-\frac{2}{9}\kappa$	$+\frac{4}{9}\kappa^2+\frac{1}{3}\kappa_{ss}+\frac{2}{9}\kappa_s\mathcal{D}^{-1}\cdot\kappa$
$J=\kappa_s$ $\quad\kappa_t=\kappa_5+\frac{5}{3}\kappa\kappa_{sss}+\frac{5}{3}\kappa_s\kappa_{ss}+\frac{5}{9}\kappa^2\kappa_s$	Sawada-Kotera
$k=3$ $\quad\{0,0,0,0,-\frac{3}{2},0\}$	$\kappa=\iota(u_5)$
$\mathcal{P}=\mathcal{D}^5+\frac{4}{3}\kappa\mathcal{D}^3+2\kappa_s\mathcal{D}^2+\left(\frac{4}{9}\kappa^2+\frac{4}{3}\kappa_{ss}\right)\mathcal{D}+\frac{4}{9}\kappa\kappa_s+\frac{1}{3}\kappa_{sss}-\frac{1}{9}\kappa_s\mathcal{D}^{-1}\cdot\kappa_s$	
$J=1$ $\quad\kappa_t=\frac{1}{3}(\kappa_{sss}+\kappa\kappa_s)$	KdV
$J=\kappa$ $\quad\kappa_t=\kappa_5+\frac{5}{3}\kappa\kappa_{ss}+\frac{10}{3}\kappa_s\kappa_{ss}+\frac{5}{6}\kappa^2\kappa_s$	KdV
$k=4$ $\quad\{0,0,0,0,0,0,-\frac{1}{5},0\}$	$\kappa=\iota(u_6)$
$\mathcal{P}=\mathcal{D}^6+\frac{10}{3}\kappa\mathcal{D}^4+\frac{20}{3}\kappa_s\mathcal{D}^3+\left(\frac{8}{3}\kappa^2+9\kappa_{ss}\right)\mathcal{D}^2$	
$\quad+\left(5\kappa_{sss}+\frac{10}{3}\kappa\kappa_s\right)\mathcal{D}-\frac{1}{3}\kappa_s^2+\frac{4}{3}\kappa\kappa_{ss}+\kappa_{ssss}$	
$J=\kappa_s$	
$\kappa_t=\kappa_7+\frac{7}{3}\kappa\kappa_5+\frac{14}{3}\kappa_s\kappa_4+7\kappa_{ss}\kappa_{sss}+\frac{14}{9}\kappa^2\kappa_{sss}+\frac{14}{3}\kappa\kappa_s\kappa_{ss}+\frac{7}{9}\kappa_s^3+\frac{28}{81}\kappa^3\kappa_s$	
	Sawada-Kotera

Table 3.4: The case of $\mathfrak{sl}(2)\ltimes\mathbb{C}^k$ for $k=1,2,3,4$

equation is different in that we apply $\mathcal{P}$ to κ_s to produce the equation, yet $\mathcal{P}$ is not the recursion operator. In fact, the "next in line" in the hierarchy is found by applying $\mathcal{R}_{\mathrm{SW}}$ to κ_s to produce a 7th order equation. Hence, there are two sequences in the Sawada-Kotera hierarchy. They can be written

$$\kappa_t=\mathcal{R}_{\mathrm{SW}}^{n-1}\mathcal{P}(\kappa_s),\qquad\text{and}\qquad\kappa_t=\mathcal{R}_{\mathrm{SW}}^{n}(\kappa_s),\qquad n=1,2,\ldots.$$

Note the orders of the first sequence are $5,11,17,\ldots$ and the orders of the second sequence are $7,13,19,\ldots$. When $k=4$, we recover the 7th order Sawada-Kotera equation. However, the characteristic operator $\mathcal{P}$ is not the associated recursion operator. As of this writing, I am unsure if and how the recursion operator for the Sawada-Kotera equation relates to $\mathcal{P}$ in this instance.

Lastly, if we write the variables in case 1.2 ($\mathfrak{sl}(2)$) as (y, v), then the "change of variables"

$$y = x, \qquad v = \frac{u_{xxx}}{u_{xx}}$$

transforms the generators from case 1.10, $k = 2$ to case 1.2. Details are omitted.

Cases 1.3 and 1.11

1.3	{0,1,1,1}	$\kappa = \iota(u_{xxx})$
$\mathcal{A} = \mathcal{D}^3 + (\frac{7}{4} - \frac{3}{2}\kappa - \frac{1}{4}\kappa^2 - \frac{1}{2}\kappa_s)\mathcal{D}$		$\mathcal{P} = \mathcal{D}^3 + (\frac{7}{4} - \frac{3}{2}\kappa - \frac{1}{4}\kappa^2)\mathcal{D}$
$\mathcal{B} = -\frac{1}{2}\mathcal{D}^2 + (\frac{3}{4} + \frac{1}{4}\kappa)\mathcal{D}$		$-\frac{3}{4}\kappa_s - \frac{1}{4}\kappa\kappa_s + \frac{1}{4}\kappa_s\mathcal{D}^{-1} \cdot \kappa_s$
$J = \kappa$	$\kappa_t = \kappa_{sss} - \frac{9}{4}\kappa\kappa_s - \frac{3}{8}\kappa^2\kappa_s + \frac{7}{4}\kappa_s$	Gardner
1.11 $k = 1$	$\{0, 0, 1, 1, 1\}$	$\kappa = \iota(u_{xxxx})$
$\mathcal{P} = \mathcal{D}^4 + \kappa\mathcal{D}^3 + (1 - \kappa^2 + 3\kappa_s)\mathcal{D}^2 + (\kappa - \kappa^3 - \kappa\kappa_s + 3\kappa_{ss})\mathcal{D}$		
$+\kappa_s - 2\kappa^2\kappa_s + \kappa_s^2 + \kappa_{sss} + \kappa_s\mathcal{D}^{-1} \cdot (\kappa\kappa_s - \kappa_{ss})$		
$J = 1$	$\kappa_t = \kappa_{sss} - \frac{3}{2}\kappa^2\kappa_s$	mKdV
$J = \kappa_s - \frac{1}{2}\kappa^2$	$\kappa_t = (\kappa_{sssss} - 10\kappa\kappa_s\kappa_{ss} - \frac{5}{2}\kappa^2\kappa_{sss} - \frac{5}{2}\kappa_s^3 + \frac{15}{8}\kappa^4\kappa_s)$	mKdV
	$+\kappa_{sss} - \frac{3}{2}\kappa^2\kappa_s$	

Table 3.5: The cases of $\mathfrak{gl}(2)$ and $\mathfrak{gl}(2) \ltimes \mathbb{C}^k$

In Table 3.5, we have the actions of cases 1.3 and 1.11 $k = 1$, which have the Lie algebra structure of $\mathfrak{gl}(2)$ and $\mathfrak{gl}(2) \ltimes \mathbb{C}$, respectively. Looking first at case 1.3 we see that the chosen cross-section and J coefficient produce a KdV-mKdV hybrid equation, which has been referred to as the Gardner equation in the literature, [14]. The Gardner equation arises via a generalization of the Miura transform relating solutions of the mKdV equation to solutions of the KdV equation. Indeed, if v satisfies the mKdV equation, then $u = v^2 + v_x$ satisfies the KdV equation. The generalization, referred to as the Gardner transformation in [14], uses $v = \frac{1}{2}\varepsilon^{-1} + \varepsilon w$ instead of just v. The Miura transform thus becomes

$$u = \frac{1}{4}\varepsilon^{-2} + w + \varepsilon w_x + \varepsilon^2 w^2.$$

The constant can be absorbed into the solution for u, and so we have

$$u = w + \varepsilon w_x + \varepsilon^2 w^2.$$

This is the Gardner transformation. Solutions w of the Gardner equation determine solutions of the KdV equation through this transformation. Furthermore, the Gardner equation is integrable.

Looking back at Table 3.5, the recursion operator for the Gardner equation found in case 1.3 is given by $\mathcal{R}_G = \mathcal{P}\mathcal{D}^{-1}$. We remark here that using the alternate cross-section $\{0, 1, 0, 1/2\}$, the algorithm produces the mKdV equation:

$$\kappa_t = \mathcal{P}(\kappa) = \kappa_{sss} - \tfrac{3}{2}\kappa^2\kappa_s - 2\kappa_s$$

where similarly, $\mathcal{R}_{mKdV} = \mathcal{P}\mathcal{D}^{-1}$.

Lastly, we point out that the 4-dimensional action of $\mathfrak{gl}(2)$ from case 1.3 can be reduced to the 3-dimensional action of $\mathfrak{sl}(2)$ given in Case 1.2. Writing the new variables as (y, v), the "change of variable" given by

$$y = x, \qquad v = \frac{u_x}{u}$$

produces the generators from Case 1.2. Note that this transformation is very reminiscent of the Hopf-Cole transformation.

Looking now at case 1.11 in Table 3.5, we see that $J = 1$ and $J = \kappa_s - \frac{1}{2}\kappa^2$ gives us, respectively the mKdV equation, and a combination of the mKdV equation with the second member of its hierarchy. However, there does not seem to be an obvious way to relate the recursion operator for the mKdV equation and $\mathcal{P}$. Perhaps its worth remarking that

$$\mathcal{P}(\kappa_s - \tfrac{1}{2}\kappa^2) = (\mathcal{R}^2_{mKdV} + \mathcal{R}_{mKdV})(\kappa_s).$$

Case 1.7

Case 1.7 is presented in Table 3.6. Notice that this case contains an arbitrary parameter α. If $\alpha \neq 1$, the general formula for $\mathcal{P}$ is given, however this will not generally produce integrable equations. For the particular values of α listed though, we do find some of the equations we are searching for. When $\alpha = 0$, we find Burgers' equation. The

1.7 $k=1,\alpha\neq 1$	$\{0,0,1\}$	$\kappa=\iota(u_{xx})$
$\mathcal{A}=\mathcal{D}^2-\frac{1+\alpha}{1-\alpha}\kappa\mathcal{D}-\frac{\alpha}{1-\alpha}\kappa_s+\frac{\alpha}{(1-\alpha)^2}\kappa^2$		$\mathcal{P}=\mathcal{D}^2-\frac{1+\alpha}{1-\alpha}\kappa\mathcal{D}-\frac{1+\alpha}{1-\alpha}\kappa_s$
$\mathcal{B}=\frac{1}{1-\alpha}\mathcal{D}-\frac{\alpha}{(1-\alpha)^2}\kappa$		$+\frac{\alpha}{(1-\alpha)^2}\kappa^2+\frac{\alpha}{(1-\alpha)^2}\kappa_s\mathcal{D}^{-1}\cdot\kappa$
$\alpha=2,1/2$ $J=\kappa_s$	$\kappa_t=\kappa_{sss}\pm 3\kappa\kappa_{ss}\pm 3\kappa\kappa_s^2+3\kappa^2\kappa_s$	Burgers
$\alpha=0$ $J=\kappa$	$\kappa_t=\kappa_{ss}-2\kappa\kappa_s$	Burgers
$\alpha=-1$ $J=\kappa_s$	$\kappa_t=\kappa_{sss}-\frac{3}{8}\kappa^2\kappa_s$	mKdV
$k=1,\alpha=1$	{0,0,*,1}	$\kappa=\iota(u_{xxx})$
$\mathcal{A}=\mathcal{D}^3-2\kappa\mathcal{D}^2+(\kappa^2-2\kappa_s)\mathcal{D}-\kappa_{ss}+\kappa\kappa_s$		$\mathcal{P}=\mathcal{D}^3-2\kappa\mathcal{D}^2+(\kappa^2-3\kappa_s)\mathcal{D}$
$\mathcal{B}=\mathcal{D}^2-\kappa\mathcal{D}-\kappa_s$		$-\kappa_{ss}+2\kappa\kappa_s$
$J=\kappa$	$\kappa_t=\kappa_{sss}-3\kappa\kappa_{ss}-3\kappa_s^2+3\kappa^2\kappa_s$	Burgers
$k=2,\ \alpha\neq 2$	{0,0,0,1}	$\kappa=\iota(u_{xxx})$
$\mathcal{P}=\mathcal{D}^3+\frac{2\alpha}{\alpha-2}\kappa\mathcal{D}^2+\left(\frac{2\alpha-1}{(\alpha-2)^2}\kappa^2+\frac{3\alpha}{\alpha-2}\kappa_s\right)\mathcal{D}+\frac{\alpha}{\alpha-2}\kappa_{ss}+\frac{3\alpha-1}{(\alpha-2)^2}\kappa\kappa_s$		
$-\kappa_s\mathcal{D}^{-1}\cdot\left(\frac{3\alpha-1}{(\alpha-2)^2}\kappa_s-\frac{\alpha(\alpha-1)}{(\alpha-2)^3}\kappa^2\right)$		
$\alpha=0,\ J=\kappa$	$\kappa_t=\kappa_{sss}-\frac{3}{8}\kappa^2\kappa_s$	mKdV
$\alpha=1,\ J=\kappa$	$\kappa_t=\kappa_{sss}-3\kappa\kappa_{ss}-3\kappa_s^2+3\kappa^2\kappa_s$	Burgers
$\alpha=2$	{0,0,0,*,1}	$\kappa=\iota(u_{xxxx})$
$J=-\frac{1}{2}$	$\kappa_t=\kappa_{sss}-3\kappa\kappa_{ss}-3\kappa_s^2+3\kappa^2\kappa_s$	Burgers

Table 3.6: The case of $\mathfrak{a}(1)\ltimes\mathbb{C}^k$

corresponding recursion operator is $\mathcal{R}_B=\mathcal{P}\mathcal{D}^{-1}$. When $\alpha=-1$, we find the mKdV equation with the corresponding recursion operator matching exactly, $\mathcal{R}_{mKdV}=\mathcal{P}$.

When $\alpha=1/2$ or 2 we find the second member of the Burgers' hierarchy. In this instance, the recursion operator is given as

$$\mathcal{R}_B=\mathcal{D}+\kappa+\kappa_s\mathcal{D}^{-1}.$$

We focus on the $\alpha=2$ case here to simplify the presentation, omitting the $\pm$'s. Hence, $\mathcal{P}(\kappa)=\mathcal{R}_B^2(\kappa_s)$. However, the factorization of $\mathcal{P}$ is as follows:

$$\mathcal{P}=\mathcal{D}^2+3\kappa\mathcal{D}+\kappa_s+2\kappa^2+2\kappa_s\mathcal{D}^{-1}\cdot\kappa=(\mathcal{D}+\kappa+\kappa_s\mathcal{D}^{-1})(\mathcal{D}+2\kappa)=\mathcal{R}_B(\mathcal{D}+2\kappa).$$

So $\mathcal{P}=\mathcal{R}_B(\mathcal{D}+2\kappa)$. Note that the operator $\mathcal{D}+2\kappa$ produces the right-hand side of Burgers' equation when applied to κ_s, but is not a recursion operator:

$$(\mathcal{D}+2\kappa)(\kappa_s)=\kappa_{ss}+2\kappa\kappa_s=(\mathcal{D}+\kappa+\kappa_s\mathcal{D}^{-1})(\kappa_s).$$

In the instance where $\alpha = 1$, we have that u_x is invariant, and cannot be used in the construction of a coordinate cross-section. The cross-section used is $x = u = 0, u_{xx} = 1$, and the curvature is taken to be $\kappa = \iota(u_{xxx})$. With $J = \kappa$, we have the second equation in Burgers' hierarchy and another instance of $\mathcal{R}_B = \mathcal{P}\mathcal{D}^{-1}$.

I second half of Table 3.6 looks at the case when $k = 2$. In the instances when $\alpha = 0$ and $\alpha = 1$, we find the mKdV and Burgers' equations, respectively. Furthermore, we have another instance of the associated recursion operators for these equations being given by $\mathcal{R} = \mathcal{P}\mathcal{D}^{-1}$. The $k = 2, \alpha = 2$ instance is also documented. Once again, u_{xx} is invariant, and we cannot use it to construct a coordinate cross-section. With $J = -1/2$, we find the second member in Burgers' hierarchy.

Cases 1.5, 1.6, 3.1, 3.2

1.5 $k = 1$	{0,0}	$\kappa = \iota(u_x)$
$\mathcal{A} = \mathcal{D} + \frac{\eta'(0)}{\eta(0)}$	$\mathcal{B} = 0$	$\mathcal{P} = \mathcal{D} + \frac{\eta'(0)}{\eta(0)}$
$J = \kappa_s$	$\kappa_t = \kappa_{ss} + \frac{\eta'(0)}{\eta(0)}$	heat
3.1 $k = 1$	{*,0}	$\kappa = \iota(u_x)$
$\mathcal{A} = \mathcal{D} - \frac{\eta'(x)}{\eta(x)}$	$\mathcal{B} = 0$	$\mathcal{P} = \mathcal{D} - \frac{\eta'(x)}{\eta(x)}$
$J = \kappa_s$	$\kappa_t = \kappa_{ss} - \frac{\eta'(x)}{\eta(x)}\kappa_s$	heat
1.6 $k = 1$	{0,0,1}	$\kappa = \iota(u_{xx})$
$\mathcal{A} = \mathcal{D}^2 + (\kappa + 2c)\mathcal{D} + \kappa_s$	$\mathcal{B} = 0$	$\mathcal{P} = \mathcal{D}^2 + (\kappa + 2c)\mathcal{D} + \kappa_s$
$J = \kappa$	$\kappa_t = \kappa_{ss} + 2\kappa\kappa_s + 2c\kappa_s$	Burgers
3.2 $k = 1$	{*,0,1}	$\kappa = \iota(u_{xx})$
$\mathcal{A} = \mathcal{D}^2 + (\kappa + 2c)\mathcal{D} + \kappa_s$	$\mathcal{B} = 0$	$\mathcal{P} = \mathcal{D}^2 + (\kappa + 2c)\mathcal{D} + \kappa_s$
$J = \kappa$	$\kappa_t = \kappa_{ss} + 2\kappa\kappa_s + 2c\kappa_s$	Burgers

Table 3.7: The cases involving constant coefficient linear homogeneous ODEs

The cases presented in Table 3.7 are the four cases from Table 3.1 where the coefficients of the infinitesimal generators come from a constant coefficient linear homogeneous ordinary differential equation. The cases labeled 3.1 and 3.2 are intransitive and

x is invariant. In 1.6 and 3.2, the constant c appearing comes from the first order ODE, $\eta'(x) + c\eta(x) = 0$.

Notice the similarity between the results of cases 1.5 and 3.1. Both instances give the heat equation with $J = \kappa_s$, where the associated recursion operator and the characteristic operator agree. The infinitesimal generators of case 1.5 are the same as those in case 3.1 with the addition of translation in x. Once x is normalized via the moving frame algorithm, we find very nearly identical results.

The cases of 1.6 and 3.2 are similarly related. In each of those instances, the recursion operator for Burgers' equation $\mathcal{R}_B$, and the characteristic operator $\mathcal{P}$, have the now familiar relationship $\mathcal{R}_B = \mathcal{P}\mathcal{D}^{-1}$.

Case 1.9

1.9 $k = 1$ {0,0,1,1}		$\kappa = \iota(u_{xxx})$
$\mathcal{A} = \mathcal{D}^3 + (3 - 2\kappa)\mathcal{D}^2 + (2 - 3\kappa + \kappa^2 - 2\kappa_s)\mathcal{D} - \kappa_s + \kappa\kappa_s - \kappa_{ss}$		
$\mathcal{B} = \mathcal{D}^2 + (2 - \kappa)\mathcal{D} - \kappa_s$		
$\mathcal{P} = \mathcal{D}^3 + (3 - 2\kappa)\mathcal{D}^2 + (2 - 3\kappa + \kappa^2 - 3\kappa_s)\mathcal{D} - 3\kappa_s + 2\kappa\kappa_s - \kappa_{ss}$		
$J = -1$	$\kappa_t = \kappa_{ss} - 2\kappa\kappa_s + 3\kappa_s$	Burgers
$J = \kappa$	$\kappa_t = (\kappa_{sss} - 3\kappa\kappa_{ss} - 3\kappa_s^2 + 3\kappa^2\kappa_s) + 3(\kappa_{ss} - 2\kappa\kappa_s) + 2\kappa_s$	Burgers

Table 3.8: The action of $(\mathfrak{a}(1) \oplus \mathbb{C}) \ltimes \mathbb{C}^k$

The computations for Case 1.9 are shown in Table 3.8. We find Burgers' equation for $J = -1$ and $J = \kappa$. In the latter instance, we can write $\mathcal{R}_B = \mathcal{P}\mathcal{D}^{-1}$, where $\mathcal{R}_B$ is the second order recursion operator associated to Burgers' equation as displayed.

Note that the characteristic operator here is a genuine differential operator, with no $\mathcal{D}^{-1}$ term. Thus, any choice of J produces a local formula for the evolution of κ.

Case 1.8

The evolution equation in Table 3.9 is not an integrable equation. It resembles the third order member of the Burgers' hierarchy, but the coefficients are incorrect. Interestingly,

1.8 $k=1$	{0,0,0}	$\kappa=\iota(u_{xx})$
$\mathcal{A}=\mathcal{D}^2+2\kappa\mathcal{D}+\kappa^2+\kappa_s$	$\mathcal{P}=\mathcal{D}^2+2\kappa\mathcal{D}+\kappa^2+2\kappa_s+\kappa_s\mathcal{D}^{-1}\cdot\kappa$	
$\mathcal{B}=-\mathcal{D}-\kappa$		
$J=\kappa_s$	$\kappa_t=\kappa_{sss}+2\kappa\kappa_{ss}+\frac{3}{2}\kappa^2\kappa_s+2\kappa_s^2$	

Table 3.9: The action of $\mathbb{C}\ltimes(\mathbb{C}\ltimes\mathbb{C}^k)$

$\mathcal{P}$ can be factored as

$$\mathcal{P}=(\mathcal{D}+\kappa+\kappa_s D^{-1})(\mathcal{D}+\kappa),$$

where the first term in this factorization is the recursion operator for Burgers equation,

$$\kappa_t=\kappa_{ss}+2\kappa\kappa_s.$$

One more interesting thing in this case is that any coordinate cross-section produces the preceding $\mathcal{A}$ and $\mathcal{B}$ operators.

Case 2.1

2.1	{0,0,0,1,0}	$\kappa=\iota(u_{xxxx})$
$\mathcal{A}=\mathcal{D}^4+\frac{5}{3}\kappa\mathcal{D}^2+\frac{5}{3}\kappa_s\mathcal{D}+\frac{4}{9}\kappa^2+\frac{1}{3}\kappa_{ss}$		$\mathcal{P}=\mathcal{D}^4+\frac{5}{3}\kappa\mathcal{D}^2+\frac{4}{3}\kappa_s\mathcal{D}$
$\mathcal{B}=\frac{1}{3}\mathcal{D}^2-\frac{2}{9}\kappa$		$+\frac{4}{9}\kappa^2+\frac{1}{3}\kappa_{ss}+\frac{2}{9}\kappa_s\mathcal{D}^{-1}\cdot\kappa$
$J=\kappa_s$	$\kappa_t=\kappa_{sssss}+\frac{5}{3}\kappa\kappa_{sss}+\frac{5}{3}\kappa_s\kappa_{ss}+\frac{5}{9}\kappa^2\kappa_s$	Sawada-Kotera

Table 3.10: The action of $\mathfrak{sa}(2)$

This computation appears in [44] in Example 5.6. As in Case 1.7, $k=2$ from Table 3.6, discussed above, we find the Sawada-Kotera equation, with the same relationship between the recursion operator and $\mathcal{P}$.

Discussion

Here we make a few further observations about some of the patterns that arise. First, we see from Table 3.11 that instances in which the characteristic operator equals an associated recursion operator ($\mathcal{R}=\mathcal{P}$) is in fact not entirely common. The dimension of the Lie algebra plays some role here. For instance, the dimension of Case 1.1 is 3, so the lowest order invariant has order 2. This implies that the characteristic operator for this invariant will be a second order differential operator, which matches the order of the associated recursion operator for the KdV equation. Alternatively, we see that the dimension of the Lie algebra coincides with the order of the integrable equation. This pattern holds for the first 4 cases listed. Case 3.1 is a primitive action, written locally in terms of evolutionary vector fields, and thus x is an invariant. In this case, $\kappa = \iota(u_x)$, which produces a first order differential operator which matches the recursion operator of the heat equation. It should also be mentioned that $J = \kappa_s$ is the "seed" symmetry for each of the KdV, mKdV and heat equations.

Contrast this dimensional analysis to the much more common situation in the second part of Table 3.11, where $\mathcal{R} = \mathcal{P}\mathcal{D}^{-1}$. In these instances, the order of the characteristic operator is one more than the order of the recursion operators of the equations we find. Attaching a $\mathcal{D}^{-1}$ in front of these characteristic operators produces the desired recursion operators in all these situations. Of course, we recognize the reoccurring pattern:

$$\kappa_t = \mathcal{P}(\kappa) = \mathcal{P}\mathcal{D}^{-1}\mathcal{D}(\kappa) = \mathcal{P}\mathcal{D}^{-1}(\kappa_s).$$

In all instances listed in the second part of Table 3.11, we find the relationship that $\mathcal{R} = \mathcal{P}\mathcal{D}^{-1}$. Again, we can alternatively see that the dimension of the Lie algebra is one more than the order of the integrable equation.

The Sawada-Kotera equation makes a few appearances on the list, which are singled out. Since the hierarchy for this equation is a little non-standard, the relationship between the recursion operator and the characteristic operator doesn't fit into the patterns already demonstrated. Furthermore, the third of the Sawada-Kotera instances shown gives the 7th order equation in the hierarchy, and there's no clear relationship between the characteristic operator and the associated recursion operator.

The last section in Table 3.11 lists a few of the cases which don't seem to fall into an obvious pattern. Case 1.7 gives the 3rd order Burgers' equation, but $\mathcal{P}$ decomposes as the first order Burgers' recursion operator with an extra factor. Case 1.8 doesn't give an integrable equation, but the characteristic operator similarly decomposes as Burgers' first order recursion operator with an extra factor. Case 1.11 gives the fifth order mKdV equation, but $J = \kappa_s - \frac{1}{2}\kappa^2$ is different than all other J choices. Further, the relationship between the characteristic operator and the recursion operator is still unclear.

Case	Structure	Dim	J	κ_t
		$\mathcal{R}=\mathcal{P}$		
1.1	$\mathfrak{sl}(2)$	3	κ_s	KdV
1.2	$\mathfrak{sl}(2)$	3	κ_s	mKdV
1.7 $k=1,\ \alpha=-1$	$\mathfrak{a}\ltimes\mathbb{C}$	3	κ_s	mKdV
1.5 $k=1$	$\mathbb{C}\ltimes\mathbb{C}$	2	κ_s	heat
3.1 $k=1$	$\mathbb{C}$	1	κ_s	heat
		$\mathcal{R}=\mathcal{P}\mathcal{D}^{-1}$		
1.10 $k=1$	$\mathfrak{sl}(2)\ltimes\mathbb{C}$	4	κ	KdV
3.3	$\mathfrak{sl}(2)$	3	κ	KdV
1.4	$\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$	6	κ	KdV-2
1.10 $k=3$	$\mathfrak{sl}(2)\ltimes\mathbb{C}^3$	6	κ	KdV-2
1.3	$\mathfrak{gl}(2)$	4	κ	mKdV
1.7 $k=2,\ \alpha=0$	$\mathfrak{a}\ltimes\mathbb{C}^2$	4	κ	mKdV
1.6 $k=1$	$\mathbb{C}^2\ltimes\mathbb{C}$	3	κ	Burgers
1.7 $k=1,\ \alpha=0$	$\mathfrak{a}\ltimes\mathbb{C}$	3	κ	Burgers
1.7 $k=1,\ \alpha=1$	$\mathfrak{a}\ltimes\mathbb{C}$	3	κ	Burgers
3.2 $k=1$	$\mathbb{C}\ltimes\mathbb{C}$	2	κ	Burgers
1.7 $k=2,\ \alpha=1$	$\mathfrak{a}\ltimes\mathbb{C}^2$	4	κ	Burgers-2
1.9 $k=1$	$(\mathfrak{a}(1)\oplus\mathbb{C})\ltimes\mathbb{C}$	4	κ	Burgers-2

Sawada-Kotera

Case	Structure	Dim	J	Recursion
1.10 $k=2$	$\mathfrak{sl}(2)\ltimes\mathbb{C}^2$	5	κ_s	$\mathcal{R}=\mathcal{P}(\mathcal{D}^2+\frac{\kappa}{3}+\frac{\kappa_s}{3}\mathcal{D}^{-1})$
2.1	$\mathfrak{sa}(2)$	5	κ_s	$\mathcal{R}=\mathcal{P}(\mathcal{D}^2+\frac{\kappa}{3}+\frac{\kappa_s}{3}\mathcal{D}^{-1})$
1.10 $k=4$	$\mathfrak{sl}(2)\ltimes\mathbb{C}^4$	7	κ_s	$\mathcal{R}=$???

Cases not falling into a clear pattern

1.7 $k=1$, $\alpha=2,\frac{1}{2}$	$\mathfrak{a}\ltimes\mathbb{C}$	3	κ_s	Burgers-2	$\mathcal{P}=\mathcal{R}_B(\mathcal{D}+2\kappa)$
1.8	$\mathbb{C}\ltimes(\mathbb{C}\ltimes\mathbb{C})$	3	κ_s	Burgers-2-like	$\mathcal{P}=\mathcal{R}_B(\mathcal{D}+\kappa)$
1.11 $k=1$	$\mathfrak{gl}(2)\ltimes\mathbb{C}$	5	$\kappa_s-\frac{1}{2}\kappa^2$	mKdV-2	$\mathcal{P}=$???

Table 3.11: Summary of results

Chapter 4

Motion of Surfaces in $\mathbb{R}^4$

4.1 Motion of Surfaces in $\mathbb{R}^4$

4.1.1 Introduction

In [23], Hasimoto discoverd a remarkable connection between the vortex filament equation and the integrable nonlinear cubic Schrödinger equation. Shortly thereafter, Lamb, [29], further demonstrated examples of the relationship between geometric curve flows and integrable equations. As explained in the previous chapter, this generated a lot of active research into this subject to understand the relationship and its degree of generality.

In [48] for $\mathbb{R}^4$, and [26] for $\mathbb{R}^n$, the authors present a Hamiltonian framework for higher dimensional vortex membranes of co-dimension 2. This chapter will describe the vortex membrane flow, a Poisson structure and associated Hamiltonian, and the generalization of these objects to co-dimension 2 submanifolds of $\mathbb{R}^n$. Within this framework, the skew-mean curvature flow will be introduced, which is thought of as a generalization of the vortex filament flow. In the case of two dimensional surfaces in $\mathbb{R}^4$, I will apply the tools of moving frames and the variational bicomplex to compute the formulas describing the evolution of the quantities of interest under the skew-mean-curvature flow. With these formulas, a connection to higher dimensional analogues of the Schrödinger equation may be possible.

4.1.2 Vortex Filament Flow

The starting point is an inviscid (zero viscosity), incompressible fluid in an open (connected) subset of $\Omega \subset \mathbb{R}^3$. The motion of the fluid is described by the evolution of the fluid velocity field v which is governed by the Euler equation:

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p, & \nabla \cdot v = 0, \\ v \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where v is assumed to be divergence free and tangent to the boundary of Ω. The pressure p is uniquely defined up to an additive constant by these restrictions on v. Further details can be found in most texts on fluid mechanics, see for example [9].

The vorticity of a velocity field is $\xi = \nabla \times v$. The vorticity form of Euler's equation is

$$\partial_t \xi + v(\xi) = 0$$

where $v(\xi)$ represents the Lie derivative of ξ along the vector field v. Note that in 2D the vorticity is viewed as a scalar field and in 3D as a vector field. However, to generalize to velocity fields in higher dimensions, v is identified with a differential 1-form, and the vorticity ξ is the exterior derivative of v. Hence, the vorticity is appropriately viewed as a differential 2-form.

The Euler equation has a Hamiltonian formulation. Complete details of the following presentation can be found in [6, 36]. First, let M be an n-dimensional manifold with volume form μ. Let G be the infinite dimensional Lie group of volume preserving diffeomorphisms of M, and let $\mathfrak{g}$ be its Lie algebra consisting of smooth, divergence-free vector fields on M tangent to ∂M. The dual space to this Lie algebra is the space of cosets of smooth 1-forms on M modulo exact 1-forms, $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$. The pairing between $[\eta] \in \mathfrak{g}^*$ and $V \in \mathfrak{g}$ is given by

$$\langle [\eta], V \rangle = \int_M (V \lrcorner \eta)\mu$$

where $V \lrcorner \eta$ is the interior product between V and η.

The Euler equation becomes

$$\partial_t[\eta] + v[\eta] = 0 \tag{4.1}$$

where $[\eta] \in \mathfrak{g}^*$ is the coset of the 1-form associated to v via the Riemannian metric on M. Instead of cosets of 1-forms, we deal with their differentials. The *vorticity 2-form* $\xi := d\eta$ is the differential of the 1-form η associated to the vector field v. Since 1-forms in the same coset have equal vorticities, the vorticity 2-form is well-defined. Futhermore, using this definition the Euler equation

$$\partial_t(d\eta) + v(d\eta) = 0 \tag{4.2}$$

means that the vorticity 2-form is transported by the fluid flow, thus allowing for higher dimensional generalizations.

If we assume our manifold M is simply connected ($H^1(M) = 0$), then the space of vorticities coincides with $\mathfrak{g}^*$. In this context, the space of vorticities has the natural Lie-Poisson structure. The symplectic leaves are coadjoint orbits of the group of volume preserving diffeomorphisms. See [36] for details. The Euler equation thus defines a Hamiltonian evolution on these orbits.

Let V, W be two divergence-free vector fields in M, regarded as a pair of variations of $\xi \in \mathfrak{g}^*$. The Lie-Poisson symplectic structure on coadjoint orbits is the following:

$$\omega_\xi^{LP}(V,W) := \langle d^{-1}\xi, [V,W]\rangle = \langle \eta, [V,W]\rangle = \int_M \eta \wedge ([V,W] \lrcorner \mu) = \int_M \xi \wedge (V \lrcorner (W \lrcorner \mu))$$

where $d\eta = \xi$ and the divergence-free condition on V and W imply $[V,W] \lrcorner \mu = d(V \lrcorner (W \lrcorner \mu))$.

Our interest now lies in singular vorticities, i.e. vorticities whose support is not of full dimension. In $\mathbb{R}^3$, a vortex filament is one-dimensional (codimension 2) support of a vortex field, i.e. the vortex field only takes non-zero values along the filament. In $\mathbb{R}^4$, we consider a *vortex membrane*, a vorticity 2-form with two-dimensional support. This latter case is our primary interest.

4.1.3 Skew Mean Curvature Flow

In $\mathbb{R}^2$, vorticity fields with 0-dimensional support are point vortices. We write the vorticity as $\xi = \sum_{j=1}^N C_j \delta_{z_j}$ where $z_j = (x_j, y_j)$ are the coordinates of the jth point vortex, C_j its strength, and δ_{z_j} is the Kronecker-delta which takes a unit value at z_j and is zero otherwise. The evolution of the vortices according to the Euler equation is given by

$$C_j \dot{x}_j = \frac{\partial H}{\partial y_j}, \qquad C_j \dot{y}_j = -\frac{\partial H}{\partial x_j}, \quad 1 \leq j \leq N, \tag{4.3}$$

where

$$H = -\frac{1}{4\pi} \sum_{j<k}^{N} C_j C_k \ln |z_j - z_k|^2$$

is the Hamiltonian function. In the case of $N = 2$ and $N = 3$ point vortices, the equations (4.3) are integrable, but for $N \geq 4$ they are not, see [39] for details.

In $\mathbb{R}^3$, we move from point vortices to vortex filaments. The vorticity is given by $\xi = \nabla \times v$, where v is the fluid of the velocity and ξ has support on a curve $\gamma \subset \mathbb{R}^3$. The vortex filament equation

$$\frac{\partial \gamma}{\partial t} = \gamma_s \times \gamma_{ss}$$

describes the evolution of γ as governed by the Euler equations, where the subscript s denotes differentiation with respect to arc-length. The localized induction approximation (LIA) of the vorticity motion is a procedure which allows one to keep only the local terms in the vorticity Euler equation, and by which the vortex filament equation is derived. See [26] for further details. It is not hard to see that the vortex filament equation takes the alternate form,

$$\frac{\partial \gamma}{\partial t} = \kappa \mathbf{b},$$

where κ represents the Euclidean curvature of γ and $\mathbf{b}$ is the binormal direction. This equation is sometimes also referred to as the binormal equation. The binormal equation is Hamiltonian with respect to the Marsden-Weinstein symplectic structure given by

$$\omega_\gamma^{MW}(V, W) := \int_\gamma V \lrcorner (W \lrcorner \mu).$$

The corresponding Hamiltonian function for equation (4.2) is given by the kinetic energy:

$$H(v) = \tfrac{1}{2} \int_M v \cdot v \; \mu = \tfrac{1}{2} \int_M (\mathrm{curl}^{-1} \xi) \cdot (\mathrm{curl}^{-1} \xi) \; \mu.$$

Note, the energy is local in terms of velocities, but non-local in terms of vorticities. When the LIA is applied, the filament equation remains Hamiltonian, but with new Hamiltonian functional

$$H(\gamma) = \mathrm{length}(\gamma) = \int_\gamma ds.$$

We further remark that the variational derivative, or the gradient, of this functional is given by

$$\frac{\delta H}{\delta \gamma} = -\kappa \mathbf{n}$$

where κ is the curvature of γ and $\mathbf{n}$ is the unit normal direction from classical Euclidean curve geometry. The dynamics are given by the corresponding skew-gradient $\kappa \mathbf{b}$, where $\mathbf{b}$ is the binormal direction. The skew-gradient vector is obtained from the gradient vector by a rotation of the plane orthogonal to $\mathbf{t}$ by $\pi/2$.

We now turn our attention to vortex membranes in $\mathbb{R}^4$, which are codimension 2 supports of vorticity 2-forms. First, the second fundamental form on surfaces in $M \subset \mathbb{R}^4$ is a vector-valued quadratic form mapping $T_pM \otimes T_pM \to N_pM$. Let $\mathbf{n}_1(p), \mathbf{n}_2(p)$ be a basis for N_pM. Then the second fundamental form can be written

$$I\!I(v, w) = I\!I_1(v, w)\mathbf{n}_1(p) + I\!I_2(v, w)\mathbf{n}_2(p).$$

Alternatively, the second fundamental form M in $\mathbb{R}^4$ can be viewed as a section of the bundle $S^2T^*M \otimes NM \to M$ where S^2T^*M is the space of symmetric 2-tensors of T^*M and $NM = (TM)^{\perp} \subset \mathbb{R}^4$ is the normal bundle to M. This means that the second fundamental form is a vector bundle valued differential invariant. See [25] for details.

Each $I\!I_i$ is a symmetric quadratic form and so each can be represented by a symmetric 2×2 matrix. The *mean curvature vector* $\mathbf{MC}(p) \in N_pM$ is the normalized trace of the second fundamental form, i.e. the trace of the coefficient matrices divided by 2:

$$\mathbf{MC}(p) = \tfrac{1}{2}\mathrm{Tr}(I\!I_1)\mathbf{n}_1 + \tfrac{1}{2}\mathrm{Tr}(I\!I_2)\mathbf{n}_2.$$

Now, the Marsden-Weinstein symplectic structure can be extended to codimension 2 submanifolds $M \subset \mathbb{R}^4$ in the obvious way:

$$\omega_M^{MW}(V, W) := \int_M V \lrcorner (W \lrcorner \mu).$$

The Hamiltonian function on vortex membranes M is their volume:

$$H(P) = \mathrm{volume}(M) = \int_M \mu_M,$$

where μ_M is the volume form of the metric induced from $\mathbb{R}^n$.

Theorem 4.1.1. *The Hamiltonian vector field associated to H and ω_M^{MW} on codimension 2 membranes $M \subset \mathbb{R}^4$ is*

$$v_H(p) = -4J(\mathbf{MC}(p)),$$

where J is the operator of positive $\pi/2$ rotation in the normal space N_pM to M.

See [48] for the proof, and see [26] for a generalization of this result to codimension 2 submanifolds in $\mathbb{R}^n$. Since the Hamiltonian vector field is found from the skew-gradient field, we need to show that the the first variation of the volume functional is a constant times the mean curvature vector. Indeed, the volume functional is represented by the two-form $\iota(dx \wedge dy) = \varpi^1 \wedge \varpi^2$. Using the techniques developed in [28], we apply $d_{\mathcal{V}}$ and "integrate by parts" to compute the variational derivative of the volume form:

$$d_{\mathcal{V}}(\varpi^1 \wedge \varpi^2) \equiv -(2H^1\vartheta^1 + 2H^2\vartheta^2) \wedge \varpi^1 \wedge \varpi^2,$$

where the the quantities H^1 and H^2 are the mean curvatures in each of the normal directions. Thus, the mean curvature vector is written

$$\mathbf{MC} = H^1\mathbf{n}_1 + H^2\mathbf{n}_2$$

and the skew-mean-curvature vector is

$$-J(\mathbf{MC}) = -H^2\mathbf{n}_1 + H^1\mathbf{n}_2.$$

See Section 4.1.5 for the complete discussion.

Definition 4.1.2. The *higher vortex filament equation* on submanifolds of codimension 2 in $\mathbb{R}^4$ is given by the *skew-mean-curvature flow*:

$$\frac{\partial M}{\partial t}(p) = -J(\mathbf{MC})(p). \tag{4.4}$$

Remark 4.1.3. For $\mathbb{R}^3$, the mean curvature vector is $\kappa\mathbf{n}$, and the skew-mean-curvature flow, (4.4), reduces to the binormal equation: $\partial_t\gamma = -J(\kappa\mathbf{n}) = \kappa\mathbf{b}$.

4.1.4 Review of relevant computations for Euclidean surfaces in $\mathbb{R}^3$

In this section, we review the computations needed to work with surfaces in $\mathbb{R}^3$, see [28] for complete details. In contrast to the previous chapters, the appearance of an additional independent variable creates new complications. We denote by x and y the independent variables and let u be the dependent variable, thus locally identifying surfaces as graphs of functions $u = u(x, y)$. We invariantize the horizontal coframe dx and dy producing the invariant horizontal coframe

$$\varpi^1 := \iota(dx), \qquad \varpi^2 := \iota(dy).$$

Hence, for a differential function F, its invariant horizontal differential is

$$d_{\mathcal{H}}F = \mathcal{D}_1 F \varpi^1 + \mathcal{D}_2 F \varpi^2.$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the dual invariant differential operators associated to the invariant horizontal coframe ϖ^1 and ϖ^2. Several complications arise from the introduction of a second independent variable. First, $\mathcal{D}_1$ and $\mathcal{D}_2$ do not in general commute, $[\mathcal{D}_1, \mathcal{D}_2] \neq 0$. However, since $\mathcal{D}_1$ and $\mathcal{D}_2$ are viewed as vector fields on our surface, their bracket should be a linear combination thereof. We compute

$$d_{\mathcal{H}}\varpi^1 = Y^1 \varpi^1 \wedge \varpi^2, \qquad d_{\mathcal{H}}\varpi^2 = Y^2 \varpi^1 \wedge \varpi^2,$$

where the Y^i are called the *commutator invariants*. A quick computation of $d_{\mathcal{H}}^2 F = 0$ implies the following commutation formula

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_2\mathcal{D}_1 - \mathcal{D}_1\mathcal{D}_2 = Y^1\mathcal{D}_1 + Y^2\mathcal{D}_2. \tag{4.5}$$

Definition 4.1.4. The *twisted invariant adjoints* of the invariant differential operators $\mathcal{D}_j$ are defined as

$$\mathcal{D}_1^\dagger = -(\mathcal{D}_1 + Y^2), \qquad \mathcal{D}_2^\dagger = -(\mathcal{D}_2 - Y^1)$$

Remark 4.1.5. The twisted invariant adjoints arise in the context of an invariant divergence from $d_{\mathcal{H}}$ applied to invariant horizontal 1-forms (or more generally, $(p-1)$-forms). If $\Omega = Q^1\varpi^2 - Q^2\varpi^1$ is an invariant 1-form, its invariant divergence is given by

$$d_{\mathcal{H}}\Omega = -(\mathcal{D}_1^\dagger Q^1 + \mathcal{D}_2^\dagger Q^2)\varpi^1 \wedge \varpi^2.$$

This is the invariant analog of the divergence operator as characterized by d_{p-1} from the de Rham complex, taking $(p-1)$-forms to p-forms. We will not explicitly use this particular characterization of the twisted invariant adjoints. See [28] for further details.

Example 4.1.6. The infinitesimal generators of the Euclidean group action on $\mathbb{R}^3$ are

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_y, \qquad \mathbf{v}_3 = \partial_u$$

$$\mathbf{v}_4 = -u\partial_x + x\partial_u, \qquad \mathbf{v}_5 = -u\partial_y + y\partial_u, \qquad \mathbf{v}_6 = -y\partial_x + x\partial_y,$$

which we prolong to act on surfaces $u = u(x, y)$ in $\mathbb{R}^3$. The cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0$$

gives the classical (Darboux) moving frame. Using the recurrence formula, we compute the pulled back Maurer-Cartan forms:

$$\nu^1 = -\varpi^1, \qquad \nu^2 = -\varpi^2, \qquad \nu^3 = -\vartheta$$

$$\nu^4 = -I_{20}\varpi^1 - \vartheta_{10}, \qquad \nu^5 = -I_{02}\varpi^2 - \vartheta_{01}.$$

$$\nu^6 = -\frac{1}{I_{20} - I_{02}}(I_{21}\varpi^1 + I_{12}\varpi^2 + \vartheta_{11})$$

Here, we write $I_{20} = \iota(u_{xx})$, $I_{21} = \iota(u_{xxy})$, $\vartheta_{10} = \iota(\theta_{10}) = \iota(du_x - u_{xx}dx - u_{xy}dy)$, etc. The invariants $\kappa^1 := I_{20}$ and $\kappa^2 := I_{02}$, are identified with the principal curvatures. Furthermore, using the recurrence formula we find the following relations:

$$\mathcal{D}_2\kappa^1 = \kappa^1_{,2} = I_{21} = \iota(u_{xxy}), \qquad \text{and} \qquad \mathcal{D}_1\kappa^2 = \kappa^2_{,1} = I_{12} = \iota(u_{xyy}).$$

Recall that the second fundamental form of a surface parameterized by x and y is

$$II = Ldx^2 + 2Mdx\,dy + Ndy^2$$

where

$$L = \frac{u_{xx}}{\sqrt{1 + u_x^2 + u_y^2}}, \quad M = \frac{u_{xy}}{\sqrt{1 + u_x^2 + u_y^2}}, \quad N = \frac{u_{yy}}{\sqrt{1 + u_x^2 + u_y^2}}$$

Invariantization diagonalizes the second fundamental form:

$$\iota(II) = \kappa^1(\varpi^1)^2 + \kappa^2(\varpi^2)^2 \tag{4.6}$$

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_3$	0	$\mathbf{v}_2$
$\mathbf{v}_2$	0	0	0	0	$\mathbf{v}_3$	$-\mathbf{v}_1$
$\mathbf{v}_3$	0	0	0	$-\mathbf{v}_1$	$-\mathbf{v}_2$	0
$\mathbf{v}_4$	$-\mathbf{v}_3$	0	$\mathbf{v}_1$	0	$\mathbf{v}_4$	$-\mathbf{v}_5$
$\mathbf{v}_5$	0	$-\mathbf{v}_3$	$\mathbf{v}_2$	$-\mathbf{v}_4$	0	$\mathbf{v}_6$
$\mathbf{v}_6$	$-\mathbf{v}_2$	$\mathbf{v}_1$	0	$\mathbf{v}_5$	$-\mathbf{v}_6$	0

Table 4.1: Commutator relations for $SE(3)$

The commutator relations for the $\mathbf{v}_i$ are given in Table 4.1, where the i, j entry is $[\mathbf{v}_i, \mathbf{v}_j]$.

The structure constants c_{ij}^k from the commutation relations $[\mathbf{v}_i, \mathbf{v}_j] = \sum_k c_{ij}^k \mathbf{v}_k$ satisfy the Maurer-Cartan equation $d\nu^k = -\frac{1}{2}\sum_{i,j} c_{ij}^k \nu^i \wedge \nu^j$. The first three Maurer-Cartan forms form an invariant coframe on the surface, and the latter three are called the connection forms, [15, 22].

The commutator invariants are found by computing $d_{\mathcal{H}}\varpi^i$ for $i = 1, 2$:

$$\begin{aligned}
d\varpi^1 = d\iota(dx) &= \iota(ddx) + \sum_{j=1}^{6} \nu^j \wedge \iota[\mathbf{v}_j(dx)] = \nu^4 \wedge \iota(-du) + \nu^6 \wedge \iota(-dy) \\
&= (-\kappa^1\varpi^1 - \vartheta_{10}) \wedge \vartheta - \frac{1}{\kappa^1 - \kappa^2}(\kappa^1_{,2}\varpi^1 + \kappa^2_{,1}\varpi^2 + \vartheta_{11}) \wedge -\varpi^2 \\
\implies \qquad d_{\mathcal{H}}\varpi^1 &= \frac{\kappa^1_{,2}}{\kappa^1 - \kappa^2}\varpi^1 \wedge \varpi^2
\end{aligned}$$

A similar computation shows that

$$d_{\mathcal{H}}\varpi^2 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}\varpi^1 \wedge \varpi^2$$

and hence the commutator invariants are

$$Y^1 = \frac{\kappa^1_{,2}}{\kappa^1 - \kappa^2}, \qquad \text{and} \qquad Y^2 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}.$$

It is now important to note that the coefficients of II in equation (4.6) are alternately computed from the structure equations applied to $\nu^3 = -\vartheta$. First, we briefly recall Cartan's lemma:

Lemma 4.1.7 (Cartan). *Suppose that ω^i, $i = 1, \ldots, n$ are linearly independent elements of a vector space V and that θ^j, $j = 1, \ldots, n$ satisfy $\sum_{i=1}^{n} \theta^i \wedge \omega^j = 0$. Then there exist* *scalars h_{ij} such that*

$$\theta^i = \sum_{j=1}^{n} h_{ij}\omega^j.$$

In our situation, when restricted to a submanifold, $\nu^3 = -\vartheta \equiv 0$. Hence,

$$0 \equiv d\nu^3 = \nu^4 \wedge \nu^1 + \nu^5 \wedge \nu^2$$

and so by Cartan's lemma,

$$\begin{bmatrix} \nu^4 \\ \nu^5 \end{bmatrix} = \begin{bmatrix} \kappa^1 & 0 \\ 0 & \kappa^2 \end{bmatrix} \begin{bmatrix} \nu^1 \\ \nu^2 \end{bmatrix}. \tag{4.7}$$

modulo contact forms. This technique of using the structure equations to compute the coefficients of II will be employed again in the next section. From (4.7), the mean curvature is found as the normalized trace of the second fundamental form:

$$H = \frac{1}{2}(\kappa^1 + \kappa^2).$$

According to Definition 4.1.4 the twisted invariant adjoints are

$$\mathcal{D}_1^\dagger = -\left(\mathcal{D}_1 + \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}\right), \qquad \mathcal{D}_2^\dagger = -\left(\mathcal{D}_2 - \frac{\kappa^1_{,2}}{\kappa^1 - \kappa^2}\right).$$

These forms are constructed on jet space, however when restricted to surfaces (pulled-back via an immersion), contact components are set to 0. Modulo contact forms, the $\varpi^1 \wedge \varpi^2$ components of $d\nu^4 = -\nu^5 \wedge \nu^6$ produces the *Gauss-Codazzi syzygy*:

$$\mathcal{D}_1^\dagger\left(\frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}\right) - \mathcal{D}_2^\dagger\left(\frac{\kappa^1_{,2}}{\kappa^1 - \kappa^2}\right) = \kappa^1\kappa^2 = K. \tag{4.8}$$

Note that this expresses the Gaussian curvature K as an invariant divergence, which is the source of the Gauss-Bonnet Theorem from classical Euclidean surface theory, see for example [22].

4.1.5 Computations for surfaces in $\mathbb{R}^4$

This section contains primarily computational results culminating in the expressions for the evolution of the mean curvatures under the skew-mean curvature flow. An added complication is the addition of a second "dependent" variable, which requires a second zeroth order basic contact form. As a result, higher order invariant contact forms are written as matrix differential operators applied to the zeroth order invariant contact forms. Since most of the theory has already been established, this section will be treated as a final working example.

We are looking at the Euclidean invariants of surfaces in $\mathbb{R}^4$. The infinitesimal generators of $SE(4)$ on $\mathbb{R}^4$ are

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_y, \qquad \mathbf{v}_3 = \partial_u, \qquad \mathbf{v}_4 = \partial_v,$$

$$\begin{aligned} &\mathbf{v}_5 = -u\partial_x + x\partial_u, && \mathbf{v}_6 = -u\partial_y + y\partial_u, && \mathbf{v}_7 = -v\partial_x + x\partial_v, \\ &\mathbf{v}_8 = -v\partial_y + y\partial_v, && \mathbf{v}_9 = y\partial_x - x\partial_y, && \mathbf{v}_{10} = v\partial_u - u\partial_v. \end{aligned}$$

These generators are prolonged to 2-dimensional submanifolds $\{u(x,y), v(x,y)\} \subset \mathbb{R}^4$. We choose the cross-section

$$x = y = u = v = u_x = u_y = v_x = v_y = 0,\ u_{xy} = v_{xy} = 1.$$

Application of the recurrence formula produces the pulled-back Maurer-Cartan forms:

$$\nu^1 = -\varpi^1, \qquad \nu^2 = -\varpi^2, \qquad \nu^3 = -\vartheta^1, \qquad \nu^4 = -\vartheta^2$$

$$\begin{aligned} &\nu^5 = -I^1_{20}\varpi^1 - \varpi^2 - \vartheta^1_{10}, && \nu^7 = -I^2_{20}\varpi^1 - \varpi^2 - \vartheta^2_{10}, \\ &\nu^6 = -\varpi^1 - I^1_{02}\varpi^2 - \vartheta^1_{01}, && \nu^8 = -\varpi^1 - I^2_{02}\varpi^2 - \vartheta^2_{01}, \end{aligned}$$

$$\nu^9 = \frac{1}{I^1_{20} - I^1_{02} + I^2_{20} - I^2_{02}} \left((I^1_{21} + I^2_{21})\varpi^1 + (I^1_{12} + I^2_{12})\varpi^2 + \vartheta^1_{11} + \vartheta^2_{11}\right),$$

$$\begin{aligned} \nu^{10} = \frac{1}{I^1_{20} - I^1_{02} + I^2_{20} - I^2_{02}} \Bigg(& \left(I^1_{21}(I^2_{02} - I^2_{20}) - I^2_{21}(I^1_{02} - I^1_{20})\right)\varpi^1 \\ & + \left(I^1_{12}(I^2_{02} - I^2_{20}) - I^2_{12}(I^1_{02} - I^1_{20})\right)\varpi^2 + (I^2_{02} - I^2_{20})\vartheta^1_{11} - (I^1_{02} - I^1_{20})\vartheta^2_{11} \Bigg), \end{aligned}$$

The superscript attached to the ϑ's indicates invariantization in the u and v variables, e.g. $\vartheta^1_{20} = \iota(\theta^1_{20}) = \iota(du_{xx} - u_{xxx}dx - u_{xxy}dy)$ and $\vartheta^2_{11} = \iota(\theta^2_{11}) = \iota(dv_{xy} - v_{xxy}dx - v_{xyy}dy)$.

In order to simplify some of the ensuing expressions, we write

$$\kappa^1 := I^1_{20}, \qquad \kappa^2 := I^1_{02}, \qquad \eta^1 := I^2_{20}, \qquad \eta^2 := I^2_{02}.$$

Also, the Maurer-Cartan invariants appearing in ν^9 and ν^{10} will be denoted

$$\nu^9: \quad \tau^1 := \frac{I^1_{21} + I^2_{21}}{\kappa^1 - \kappa^2 + \eta^1 - \eta^2}, \quad \tau^2 := \frac{I^1_{12} - I^2_{12}}{\kappa^1 - \kappa^2 + \eta^1 - \eta^2}, \quad \tau^3 := \frac{1}{\kappa^1 - \kappa^2 + \eta^1 - \eta^2}$$

$$\nu^{10}: \quad \alpha^1 := \Big(I^2_{21}(\kappa^1 - \kappa^2) - I^1_{21}(\eta^1 - \eta^2)\Big)\tau^3, \quad \alpha^2 := \Big(I^2_{12}(\kappa^1 - \kappa^2) - I^1_{12}(\eta^1 - \eta^2)\Big)\tau^3$$

so ν^9 and ν^{10} are written

$$\nu^9 = \tau^1\varpi^1 + \tau^2\varpi^2 + \tau^3(\vartheta^1_{11} + \vartheta^2_{11}), \quad \nu^{10} = \alpha^1\varpi^1 + \alpha^2\varpi^2 + (\kappa^1 - \kappa^2)\tau^3\vartheta^2_{11} - (\eta^1 - \eta^2)\tau^3\vartheta^1_{11}.$$

Computing the exterior derivatives of the ϖ^i's, we find

$$\begin{aligned} d\varpi^1 &= \nu^9 \wedge \varpi^2 + \nu^5 \wedge -\vartheta^1 + \nu^7 \wedge -\vartheta^2 \\ &= \tau^1\varpi^1 \wedge \varpi^2 - \Big(\kappa^1\vartheta^1 + \eta^1\vartheta^2\Big) \wedge \varpi^1 + [-\vartheta^1 - \vartheta^2 + \tau^3(\vartheta^1_{11} + \vartheta^2_{11})] \wedge \varpi^2 \\ &\quad + \vartheta^1_{10} \wedge \vartheta^1 + \vartheta^2_{10} \wedge \vartheta^2, \end{aligned}$$

and

$$\begin{aligned} d\varpi^2 &= \nu^9 \wedge -\varpi^1 + \nu^6 \wedge -\vartheta^1 + \nu^8 \wedge -\vartheta^2 \\ &= \tau^2\varpi^1 \wedge \varpi^2 + [-\vartheta^1 - \vartheta^2 - \tau^3(\vartheta^1_{11} + \vartheta^2_{11})] \wedge \varpi^1 - \Big(\kappa^2\vartheta^1 + \eta^2\vartheta^2\Big) \wedge \varpi^2 \\ &\quad + \vartheta^1_{01} \wedge \vartheta^1 + \vartheta^2_{01} \wedge \vartheta^2, \end{aligned}$$

producing the commutator invariants $Y^1 = \tau^1$ and $Y^2 = \tau^2$, and the twisted invariant adjoints:

$$\mathcal{D}^\dagger_1 = -(\mathcal{D}_1 + \tau^2), \qquad \mathcal{D}^\dagger_2 = -(\mathcal{D}_2 - \tau^1).$$

From the Maurer-Cartan structure equations, we have

$$d\nu^9 = -\nu^5 \wedge \nu^6 - \nu^7 \wedge \nu^8.$$

Comparing the coefficients of $\varpi^1 \wedge \varpi^2$ produces the syzygy

$$\tau^2_{,1} - \tau^1_{,2} + (\tau^1)^2 + (\tau^2)^2 = -(\kappa^1\kappa^2 + \eta^1\eta^2 - 2)$$

or, equivalently,

$$\begin{bmatrix} \mathcal{D}_1^\dagger & \mathcal{D}_2^\dagger \end{bmatrix} J \begin{bmatrix} \tau^1 \\ \tau^2 \end{bmatrix} = -(\kappa^1\kappa^2 + \eta^1\eta^2 - 2). \tag{4.9}$$

where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

We note here the similarity between the construction of this syzygy and the construction of the Gauss-Codazzi syzygy, (4.8), through the use the structure equations. A second syzygy comes from comparing the coefficients of $\varpi^1 \wedge \varpi^2$ in

$$d\nu^{10} = -\nu^5 \wedge \nu^7 - \nu^6 \wedge \nu^8.$$

It is similarly compactly written

$$\begin{bmatrix} \mathcal{D}_1^\dagger & \mathcal{D}_2^\dagger \end{bmatrix} J \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix} = -\Big((\kappa^1 - \kappa^2) - (\eta^1 - \eta^2)\Big). \tag{4.10}$$

Remark 4.1.8. As the right-hand side of the expressions in (4.9), (4.10) are expressed as invariant divergences, they are invariant null-Lagrangians, [28]. The presence of invariant null-Lagrangians is important to the study of cohomology classes of the invariant bicomplex, [4, 49]. Particularly, in [49], Valiquette and Thompson showed that the cohomology of the invariant edge complex is isomorphic to the Lie algebra cohomology of G. It is worth investigating what these null Lagrangians imply for surface geometry in $\mathbb{R}^4$, e.g. higher codimensional analogs of the Gauss-Bonnet theorem.

Proceeding as we did with surfaces in $\mathbb{R}^3$, we consider the coframe elements $\nu^3 = -\vartheta^1$ and $\nu^4 = -\vartheta^2$. Pulled back to our surface, these forms are 0, and so

$$\begin{aligned} 0 \equiv & d\nu^3 = \nu^5 \wedge \nu^1 + \nu^6 \wedge \nu^2, \\ 0 \equiv & d\nu^4 = \nu^7 \wedge \nu^1 + \nu^8 \wedge \nu^2, \end{aligned}$$

Applying Cartan's lemma gives

$$\begin{bmatrix} \nu^5 \\ \nu^6 \end{bmatrix} = \begin{bmatrix} \kappa^1 & 1 \\ 1 & \kappa^2 \end{bmatrix} \begin{bmatrix} \nu^1 \\ \nu^2 \end{bmatrix}, \qquad \begin{bmatrix} \nu^7 \\ \nu^8 \end{bmatrix} = \begin{bmatrix} \eta^1 & 1 \\ 1 & \eta^2 \end{bmatrix} \begin{bmatrix} \nu^1 \\ \nu^2 \end{bmatrix}. \tag{4.11}$$

Define the dual normal vectors $\mathbf{n}_i$ by $\langle \vartheta_i, \mathbf{n}_j \rangle = \delta^i_j,\ j = 1, 2$ where $\langle \cdot, \cdot \rangle$ is the pairing between forms and vectors and δ^i_j is the Kronecker delta. Then $\mathbf{n}_1$ and $\mathbf{n}_2$ form a basis for NM, and the second fundamental form has the form

$$II = [\varpi^1\ \varpi^2] \begin{bmatrix} \kappa^1 & 1 \\ 1 & \kappa^2 \end{bmatrix} \begin{bmatrix} \varpi^1 \\ \varpi^2 \end{bmatrix} \mathbf{n}_1 + [\varpi^1\ \varpi^2] \begin{bmatrix} \eta^1 & 1 \\ 1 & \eta^2 \end{bmatrix} \begin{bmatrix} \varpi^1 \\ \varpi^2 \end{bmatrix} \mathbf{n}_2. \tag{4.12}$$

The coefficient matrices of II come from (4.11), and we have omitted the tensor notation. The determinant and trace of these matrices are invariants. Indeed, one can take a "Gaussian curvature" in each of the normal directions

$$K^1 = \kappa^1\kappa^2 - 1, \qquad K^2 = \eta^1\eta^2 - 1$$

as well as the mean curvatures described earlier

$$H^1 = \tfrac{1}{2}(\kappa^1 + \kappa^2), \qquad H^2 = \tfrac{1}{2}(\eta^1 + \eta^2)$$

Remark 4.1.9. The right hand side of (4.9) is minus the sum of K^1 and K^2. The right-hand side of (4.10) is the obstruction to the simultaneous diagonalization of the coefficient matrices in (4.12). These matrices commute if and only if the right hand side of (4.10) is equal to 0. The κ^i's and η^j's should not be viewed as principal curvatures because they are not the eigenvalues of these matrices. However, they are elementary functions of the eigenvalues, and so could play the role of principal curvatures.

The mean curvatures arise as the trace of the second fundamental form, i.e. the trace of each of the coefficient matrices. The *mean curvature vector* is

$$\mathbf{MC} = H^1\mathbf{n}_1 + H^2\mathbf{n}_2$$

and the *skew-mean-curvature vector* is

$$\mathbf{V} = -H^2\mathbf{n}_1 + H^1\mathbf{n}_2 = J\mathbf{MC}$$

To determine the evolution of the mean curvatures under the skew-mean curvature flow, there are a few quantities that need computing. We start with

$$\begin{aligned}
d_{\mathcal{V}}\kappa^1 &= \vartheta^1_{20} + 2\tau^3(\vartheta^1_{11} + \vartheta^2_{11}) + \eta^1\tau^3((\kappa^1 - \kappa^2)\vartheta^2_{11} - (\eta^1 - \eta^2)\vartheta^1_{11}) \\
d_{\mathcal{V}}\kappa^2 &= \vartheta^1_{02} - 2\tau^3(\vartheta^1_{11} + \vartheta^2_{11}) + \eta^2\tau^3((\kappa^1 - \kappa^2)\vartheta^2_{11} - (\eta^1 - \eta^2)\vartheta^1_{11}) \\
d_{\mathcal{V}}\eta^1 &= \vartheta^2_{20} + 2\tau^3(\vartheta^1_{11} + \vartheta^2_{11}) - \kappa^1\tau^3((\kappa^1 - \kappa^2)\vartheta^2_{11} - (\eta^1 - \eta^2)\vartheta^1_{11}) \\
d_{\mathcal{V}}\eta^2 &= \vartheta^2_{02} - 2\tau^3(\vartheta^1_{11} + \vartheta^2_{11}) - \kappa^2\tau^3((\kappa^1 - \kappa^2)\vartheta^2_{11} - (\eta^1 - \eta^2)\vartheta^1_{11})
\end{aligned}$$

and so

$$d_{\mathcal{V}}H^1 = \tfrac{1}{2}(\vartheta^1_{20} + \vartheta^1_{02}) + H^2\tau^3\Big((\kappa^1-\kappa^2)\vartheta^2_{11} - (\eta^1-\eta^2)\vartheta^1_{11}\Big) \tag{4.13}$$

$$d_{\mathcal{V}}H^2 = \tfrac{1}{2}(\vartheta^2_{20} + \vartheta^2_{02}) - H^1\tau^3\Big((\kappa^1-\kappa^2)\vartheta^2_{11} - (\eta^1-\eta^2)\vartheta^1_{11}\Big) \tag{4.14}$$

where we recall that $\tau^3 = \dfrac{1}{\kappa^1-\kappa^2+\eta^1-\eta^2}$.

The invariant derivatives of the invariant contact forms are given as matrix operators acting on the zeroth order invariant contact forms. For example, $\mathcal{D}_1\vartheta^1 = \vartheta^1_{10} + \alpha^1\vartheta^2$, and so

$$\vartheta^1_{10} = \begin{bmatrix}\mathcal{D}_1 & -\alpha^1\end{bmatrix}\begin{bmatrix}\vartheta^1\\ \vartheta_2\end{bmatrix}, \qquad \text{or briefly,} \qquad \vartheta^1_{10} = \begin{bmatrix}\mathcal{D}_1 & -\alpha^1\end{bmatrix}.$$

Here are the invariant matrix operators associated to the higher order invariant contact forms.

$$\begin{aligned} \vartheta^1_{10} &= [\ \mathcal{D}_1 \quad -\alpha^1\ ], & \vartheta^2_{10} &= [\ \alpha^1 \quad \mathcal{D}_1\ ], \\ \vartheta^1_{01} &= [\ \mathcal{D}_2 \quad -\alpha^2\ ], & \vartheta^2_{01} &= [\ \alpha^2 \quad \mathcal{D}_2\ ]. \end{aligned} \tag{4.15}$$

$$\begin{aligned} \vartheta^1_{11} &= [\ \mathcal{D}_1\mathcal{D}_2 + \tau^1\mathcal{D}_1 + \kappa^1 + \kappa^2 - \alpha^1\alpha^2 \quad -\alpha^2\mathcal{D}_1 - \alpha^2_{,1} - \tau^1\alpha^1 + \eta^1 + \kappa^2 - \alpha^1\mathcal{D}_2\ ] \\ \vartheta^2_{11} &= [\ \alpha^2\mathcal{D}_1 + \alpha^1\mathcal{D}_2 + \tau^1\alpha^1 + \kappa^1 + \eta^2 + \alpha^2_{,1} \quad \mathcal{D}_1\mathcal{D}_2 + \tau^1\mathcal{D}_1 + \eta^1 + \eta^2 - \alpha^1\alpha^2\ ] \end{aligned}$$

Remark 4.1.10. Incidentally, the syzygy in equation (4.10) can be found be comparing these two expressions for ϑ^1_{11} found from $\mathcal{D}_1\vartheta^1_{01}$ and $\mathcal{D}_2\vartheta^1_{10}$. Similarly, it arises in comparing the expressions for ϑ^2_{11} from $\mathcal{D}_1\vartheta^2_{01}$ and $\mathcal{D}_2\vartheta^2_{10}$.

$$\begin{aligned} \vartheta^1_{20} &= [\ \mathcal{D}_1^2 - \tau^1\mathcal{D}_2 + (\kappa^1)^2 - (\alpha^1)^2 + 1 \quad -2\alpha^1\mathcal{D}_1 - \alpha^1_{,1} + \tau^1\alpha^2 + \kappa^1\eta^1 + 1\ ] \\ \vartheta^1_{02} &= [\ \mathcal{D}_2^2 + \tau^2\mathcal{D}_1 + (\kappa^2)^2 - (\alpha^2)^2 + 1 \quad -2\alpha^2\mathcal{D}_2 - \alpha^2_{,2} - \tau^2\alpha^1 + \kappa^2\eta^2 + 1\ ] \\ \vartheta^2_{20} &= [\ 2\alpha^1\mathcal{D}_1 + \alpha^1_{,1} - \tau^1\alpha^2 + \kappa^1\eta^1 + 1 \quad \mathcal{D}_1^2 - \tau^1\mathcal{D}_2 + (\eta^1)^2 - (\alpha^1)^2 + 1\ ] \\ \vartheta^2_{02} &= [\ 2\alpha^2\mathcal{D}_2 + \alpha^2_{,2} + \tau^2\alpha^1 + \kappa^2\eta^2 + 1 \quad \mathcal{D}_2^2 + \tau^2\mathcal{D}_1 + (\eta^2)^2 - (\alpha^2)^2 + 1\ ] \end{aligned}$$

From equation (4.13), we find

$$
\begin{aligned}
d_{\mathcal{V}}H^1 &= \tfrac{1}{2}(\vartheta^1_{20} + \vartheta^1_{02}) + H^2\tau^3\Big((\kappa^1-\kappa^2)\vartheta^2_{11} - (\eta^1-\eta^2)\vartheta^1_{11}\Big) \\
&= \tfrac{1}{2}\Big(\Big[\ \mathcal{D}_1^2 - \tau^1\mathcal{D}_2 + (\kappa^1)^2 - (\alpha^1)^2 + 1 \quad -2\alpha^1\mathcal{D}_1 - \alpha^1_{,1} + \tau^1\alpha^2 + \kappa^1\eta^1 + 1\ \Big] \\
&+ \Big[\ \mathcal{D}_2^2 + \tau^2\mathcal{D}_1 + (\kappa^2)^2 - (\alpha^2)^2 + 1 \quad -2\alpha^2\mathcal{D}_2 - \alpha^2_{,2} - \tau^2\alpha^1 + \kappa^2\eta^2 + 1\ \Big]\Big) \\
&+ H^2\tau^3\Bigg((\kappa^1-\kappa^2)\left[\alpha^2\mathcal{D}_1 + \alpha^1\mathcal{D}_2 + \tau^1\alpha^1 + \kappa^1 + \eta^2 + \alpha^2_{,1}\right. \\
&\qquad\qquad\qquad\qquad \left.\mathcal{D}_1\mathcal{D}_2 + \tau^1\mathcal{D}_1 + \eta^1 + \eta^2 - \alpha^1\alpha^2\right] \\
&\quad - (\eta^1-\eta^2)\left[\mathcal{D}_1\mathcal{D}_2 + \tau^1\mathcal{D}_1 + \kappa^1 + \kappa^2 - \alpha^1\alpha^2\right. \\
&\qquad\qquad\qquad\qquad \left.-\alpha^2\mathcal{D}_1 - \alpha^2_{,1} - \tau^1\alpha^1 + \eta^1 + \kappa^2 - \alpha^1\mathcal{D}_2\right]\Bigg)
\end{aligned}
$$

and from equation (4.14),

$$
\begin{aligned}
d_{\mathcal{V}}H^2 &= \tfrac{1}{2}(\vartheta^2_{20} + \vartheta^2_{02}) - H^1\tau^3\Big((\kappa^1-\kappa^2)\vartheta^2_{11} - (\eta^1-\eta^2)\vartheta^1_{11}\Big) \\
&= \tfrac{1}{2}\Big(\Big[\ 2\alpha^1\mathcal{D}_1 + \alpha^1_{,1} - \tau^1\alpha^2 + \kappa^1\eta^1 + 1 \quad \mathcal{D}_1^2 - \tau^1\mathcal{D}_2 + (\eta^1)^2 - (\alpha^1)^2 + 1\ \Big] \\
&+ \Big[\ 2\alpha^2\mathcal{D}_2 + \alpha^2_{,2} + \tau^2\alpha^1 + \kappa^2\eta^2 + 1 \quad \mathcal{D}_2^2 + \tau^2\mathcal{D}_1 + (\eta^2)^2 - (\alpha^2)^2 + 1\ \Big]\Big) \\
&- H^1\tau^3\Bigg((\kappa^1-\kappa^2)\left[\alpha^2\mathcal{D}_1 + \alpha^1\mathcal{D}_2 + \tau^1\alpha^1 + \kappa^1 + \eta^2 + \alpha^2_{,1}\right. \\
&\qquad\qquad\qquad\qquad \left.\mathcal{D}_1\mathcal{D}_2 + \tau^1\mathcal{D}_1 + \eta^1 + \eta^2 - \alpha^1\alpha^2\right] \\
&\quad - (\eta^1-\eta^2)\left[\mathcal{D}_1\mathcal{D}_2 + \tau^1\mathcal{D}_1 + \kappa^1 + \kappa^2 - \alpha^1\alpha^2\right. \\
&\qquad\qquad\qquad\qquad \left.-\alpha^2\mathcal{D}_1 - \alpha^2_{,1} - \tau^1\alpha^1 + \eta^1 + \kappa^2 - \alpha^1\mathcal{D}_2\right]\Bigg)
\end{aligned}
$$

I have not found a way to simplify these expressions further, however there are some possible directions which merit continued investigation. First, there is a strong indication that one should define an almost complex structure on the cotangent bundle. Indeed, the frequent appearance of the J operator from equation (4.9) and the ability to locally identify $T_p\mathbb{R}^4 = T_pM \oplus N_pM = \mathbb{C}^2$ with two copies of the complex plane support this claim. Furthermore, the relationships in equations (4.15) can be written in the following way:

$$
\begin{aligned}
\mathcal{D}_1(\vartheta^1 + i\vartheta^2) &= (\vartheta^1_{10} + i\vartheta^2_{10}) + i\alpha^1(\vartheta^1 + i\vartheta^2) \\
\mathcal{D}_2(\vartheta^1 + i\vartheta^2) &= (\vartheta^1_{01} + i\vartheta^2_{01}) - i\alpha^2(\vartheta^1 + i\vartheta^2).
\end{aligned}
$$

Alternatively, if we continue to view the equations in (4.15) as matrix operators, we find the relationships

$$\vartheta^2_{10} = \vartheta^1_{10} J^{-1}, \qquad\qquad \vartheta^2_{01} = \vartheta^1_{01} J^{-1}.$$

Lastly, the identification of the bundle $T^*\mathbb{R}^4 \to M$ with $\mathbb{C}^2 \to M$ indicates that it may be possible to apply some of the results of the previous chapter to the current situation. It may be possible to view the flow of the surface in four dimensions as a flow of a one-complex-dimensional curve in the two-complex-dimensional plane, which then might allow us to identify the skew-mean-curvature flow as some geometric flow studied in the previous chapter. As of this writing, it is not clear how to reconcile the 10-dimensional Euclidean action on $\mathbb{R}^4$ with one of the actions on $\mathbb{C}^2$.

Chapter 5

Conclusion and Discussion

In this work, moving frames and the variational bicomplex were used to investigate problems relating to the geometric flows of curves in the plane and surfaces in $\mathbb{R}^4$. Regrettably, both investigations that were pursued in this paper came to somewhat unsatisfactory conclusions. The connection between geometric curve flows and integrable equations remains unclear, though certain patterns did emerge. The evolution of the mean curvatures of a surface under the skew-mean-curvature flow in four dimensions was produced, however the formulae for this evolution is a bit unwieldy, and would be difficult to implement in any sort of practical manner. On the other hand, the door remains open for further investigations that would shed new light on these problems.

First, though a complete classification of local actions on three dimensional space has not been recorded, a similar analysis of the evolution of space curves and their connection to integrable equations is possible. Indeed, one of the motivations of the planar curve classification was the search for any insights that would assist the investigation of the relationship between space curves and integrable equations. The remarkable Hasimoto transformation, relating the vortex filament flow to the integrable, cubic nonlinear Schrödinger equation, is in some sense the primary motivator, and the search for similar type transformations and relationships continues to this day.

In a related, but separate direction, the search remains for a connection between the vortex membrane flow (or the skew-mean-curvature flow) for codimension two manifolds and a higher dimensional analogue of the nonlinear Schrödinger equation. Again, this search is motivated by the Hasimoto transformation. The appearance of an almost complex structure on the normal spaces to the manifold seem to give should give a nice structure to the flow which would potentially make it easier to analyze.

Certain syzygies arose in the study of surfaces in $\mathbb{R}^4$, expressing certain differential invariants as an invariant divergence. This is the source of the Gauss-Bonnet Theorem for surfaces in $\mathbb{R}^3$, and thus the extension to a higher dimensional analog is worth further investigation.

References

[1] Stephen C. Anco. Bi-Hamiltonian operators, integrable flows of curves using moving frames and geometric map equations. *J. Phys. A*, 39(9):2043–2072, 2006.

[2] I. Anderson. The variational bicomplex. Technical report, Utah State Technical Report,1989, 1989.

[3] Ian M. Anderson. Introduction to the variational bicomplex. In *Mathematical Aspects of Classical Field Theory (Seattle, WA, 1991)*, volume 132 of *Contemp. Math.*, pages 51–73. Amer. Math. Soc., Providence, RI, 1992.

[4] Ian M. Anderson and Juha Pohjanpelto. The cohomology of invariant variational bicomplexes. *Acta Appl. Math.*, 41(1-3):3–19, 1995. Geometric and algebraic structures in differential equations.

[5] Sigurd Angenent, Guillermo Sapiro, and Allen Tannenbaum. On the affine heat equation for non-convex curves. *J. Amer. Math. Soc.*, 11(3):601–634, 1998.

[6] V. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):319–361, 1966.

[7] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor′kova, I. S. Krasil′shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov. *Symmetries and Conservation Laws For Differential Equations of Mathematical Physics*, volume 182 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1999. Edited and with a preface

by Krasil′shchik and Vinogradov, Translated from the 1997 Russian original by Verbovetsky [A. M. Verbovetskiĭ] and Krasil′shchik.

[8] P. Broadbridge and P. Tritscher. An integrable fourth-order nonlinear evolution equation applied to thermal grooving of metal surfaces. *IMA J. Appl. Math.*, 53(3):249–265, 1994.

[9] Alexandre J. Chorin and Jerrold E. Marsden. *A Mathematical Introduction to Fluid Mechanics*, volume 4 of *Texts in Applied Mathematics*. Springer-Verlag, New York, third edition, 1993.

[10] K.-S. Chou and C.-Z. Qu. Integrable equations arising from motions of plane curves. II. *J. Nonlinear Sci.*, 13(5):487–517, 2003.

[11] Kai-Seng Chou and Changzheng Qu. Integrable equations arising from motions of plane curves. *Phys. D*, 162(1-2):9–33, 2002.

[12] Kai-Seng Chou and Changzheng Qu. Integrable motions of space curves in affine geometry. *Chaos Solitons Fractals*, 14(1):29–44, 2002.

[13] Peter A. Clarkson and Peter J. Olver. Symmetry and the Chazy equation. *J. Differential Equations*, 124(1):225–246, 1996.

[14] Lokenath Debnath. *Nonlinear Partial Differential Equations for Scientists and Engineers*. Birkhäuser Boston, Inc., Boston, MA, 1997.

[15] Manfredo P. do Carmo. *Differential Forms and Applications*. Universitext. Springer-Verlag, Berlin, 1994. Translated from the 1971 Portuguese original.

[16] Mark Fels and Peter J. Olver. Moving coframes. I. A practical algorithm. *Acta Appl. Math.*, 51(2):161–213, 1998.

[17] Mark Fels and Peter J. Olver. Moving coframes. II. Regularization and theoretical foundations. *Acta Appl. Math.*, 55(2):127–208, 1999.

[18] M. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *J. Differential Geom.*, 23(1):69–96, 1986.

[19] Raymond E. Goldstein and Dean M. Petrich. The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane. *Phys. Rev. Lett.*, 67(23):3203–3206, 1991.

[20] Artemio González-López, Niky Kamran, and Peter J. Olver. Lie algebras of vector fields in the real plane. *Proc. London Math. Soc. (3)*, 64(2):339–368, 1992.

[21] Matthew A. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differential Geom.*, 26(2):285–314, 1987.

[22] Heinrich W. Guggenheimer. *Differential Geometry.* Dover Publications, Inc., New York, 1977. Corrected reprint of the 1963 edition, Dover Books on Advanced Mathematics.

[23] H. Hasimoto. A soliton on a vortex filament. *Journal of Fluid Mechanics*, 51(03):477–485, 1972.

[24] Thomas A. Ivey. Integrable geometric evolution equations for curves. In *The Geometrical Study of Differential Equations (Washington, DC, 2000)*, volume 285 of *Contemp. Math.*, pages 71–84. Amer. Math. Soc., Providence, RI, 2001.

[25] Thomas A. Ivey and J. M. Landsberg. *Cartan for Beginners: Differential Geometry Via Moving Frames and Exterior Differential Systems*, volume 61 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2003.

[26] Boris Khesin. Symplectic structures and dynamics on vortex membranes. *Mosc. Math. J.*, 12(2):413–434, 461–462, 2012.

[27] Irina A. Kogan and Peter J. Olver. The invariant variational bicomplex. In *The Geometrical Study of Differential Equations (Washington, DC, 2000)*, volume 285 of *Contemp. Math.*, pages 131–144. Amer. Math. Soc., Providence, RI, 2001.

[28] Irina A. Kogan and Peter J. Olver. Invariant Euler-Lagrange equations and the invariant variational bicomplex. *Acta Appl. Math.*, 76(2):137–193, 2003.

[29] G. L. Lamb, Jr. Solitons on moving space curves. *J. Mathematical Phys.*, 18(8):1654–1661, 1977.

[30] Joel Langer. Recursion in curve geometry. *New York J. Math.*, 5:25–51 (electronic), 1999.

[31] Joel Langer and Ron Perline. Poisson geometry of the filament equation. *J. Nonlinear Sci.*, 1(1):71–93, 1991.

[32] Joel Langer and Ron Perline. Curve motion inducing modified Korteweg-de Vries systems. *Phys. Lett. A*, 239(1-2):36–40, 1998.

[33] Sophus Lie. *Gesammelte Abhandlungen. Band 5.* Johnson Reprint Corp., New York-London, 1973. Herausgegeben von Friedrich Engel und Poul Heegaard, Abhandlungen über die Theorie der Transformationsgruppen. Abteilung 1, Herausgegeben von Friedrich Engel, Reprint of the 1924 edition.

[34] G. Marí Beffa and P. J. Olver. Poisson structures for geometric curve flows in semi-simple homogeneous spaces. *Regul. Chaotic Dyn.*, 15(4-5):532–550, 2010.

[35] G. Marí Beffa, J. A. Sanders, and Jing Ping Wang. Integrable systems in three-dimensional Riemannian geometry. *J. Nonlinear Sci.*, 12(2):143–167, 2002.

[36] Jerrold Marsden and Alan Weinstein. Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. *Phys. D*, 7(1-3):305–323, 1983. Order in chaos (Los Alamos, N.M., 1982).

[37] Robert M. Miura. Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *J. Mathematical Phys.*, 9:1202–1204, 1968.

[38] Kazuaki Nakayama, Harvey Segur, and Miki Wadati. Integrability and the motion of curves. *Phys. Rev. Lett.*, 69(18):2603–2606, 1992.

[39] Paul K. Newton. *The N-Vortex Problem*, volume 145 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001. Analytical techniques.

[40] Emmy Noether. Invariant variation problems. *Transport Theory Statist. Phys.*, 1(3):186–207, 1971. Translated from the German (Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1918, 235–257).

[41] P. Olver, G. Sapiro, and A. Tannenbaum. *Differential invariant signatures and flows in computer vision: A symmetry group approach.* Springer, 1994.

[42] Peter J. Olver. *Applications of Lie Groups to Differential Equations*, volume 107 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1993.

[43] Peter J. Olver. *Equivalence, Invariants, and Symmetry.* Cambridge University Press, Cambridge, 1995.

[44] Peter J. Olver. Invariant submanifold flows. *J. Phys. A*, 41(34):344017, 22, 2008.

[45] Valentin Ovsienko and Sergei Tabachnikov. What is . . . the Schwarzian derivative? *Notices Amer. Math. Soc.*, 56(1):34–36, 2009.

[46] Guillermo Sapiro and Allen Tannenbaum. On affine plane curve evolution. *J. Funct. Anal.*, 119(1):79–120, 1994.

[47] Katuro Sawada and Takeyasu Kotera. A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-like equation. *Progr. Theoret. Phys.*, 51:1355–1367, 1974.

[48] Banavara N. Shashikanth. Vortex dynamics in $\mathbb{R}^4$. *J. Math. Phys.*, 53(1):013103, 21, 2012.

[49] Robert Thompson and Francis Valiquette. On the cohomology of the invariant Euler-Lagrange complex. *Acta Appl. Math.*, 116(2):199–226, 2011.

[50] Toru Tsujishita. On variation bicomplexes associated to differential equations. *Osaka J. Math.*, 19(2):311–363, 1982.

[51] Włodzimierz M. Tulczyjew. The Lagrange complex. *Bull. Soc. Math. France*, 105(4):419–431, 1977.

[52] A. M. Vinogradov. The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws. I. The linear theory. *J. Math. Anal. Appl.*, 100(1):1–40, 1984.

[53] A. M. Vinogradov. The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws. II. The nonlinear theory. *J. Math. Anal. Appl.*, 100(1):41–129, 1984.