The BV formalism for homotopy Lie algebras

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Denis Aleksandrovich Bashkirov

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Alexander A Voronov, advisor

November, 2014

Acknowledgements

First and foremost, I am indebted to my advisor, Professor Alexander Voronov, for all the immense help, support and patience that he provided during the project as well as for his deep mathematical erudition and passion for the subject that he keeps generously sharing with me and with others. I am grateful to Janko Latschev for a helpful advice and to Jim Stasheff for his feedback on an earlier version of the paper. I would like to thank Maxim Kontsevich, Yvette Kosmann-Schwarzbach, and Luca Vitagliano for some helpful remarks on the project communicated to my advisor. Many thanks to the faculty of the mathematics and physics departments at the University of Minnesota and fellow graduate students, who made my years of study so enjoyable.

Abstract

The present work concerns certain aspects of homotopy Lie and homotopy Batalin-Vilkovisky structures. In the first part we characterize a class of homotopy BV-algebras canonically associated to strongly homotopy Lie algebras and show that a category of strongly homotopy Lie algebras embeds into a certain subcategory of particularly simple homotopy Batalin-Vilkovisky algebras. In the second part of the work we introduce the notions of coboundary and triangular homotopy Lie bialgebras and discuss a possible a framework for quantization of such bialgebras.

Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation and summary of the results

The results presented in the thesis appeared during a study aimed at the geometric meaning of solutions of the generalized Maurer-Cartan and the quantum master equation set up in certain strongly homotopy Lie (or L_∞-algebras) and Batalin-Vilkovisky algebras. The L_∞-structure can be regarded as a homotopy version of the usual Lie algebra structure, where the Jacobi identity (or rather a series of identities) is required to hold not 'on the nose', but rather up to a coherent homotopy (or 'up to a BRST exact term' in physics terminology). A typical way in which such algebras arise is the following: for a Lie algebra $\mathfrak{g}$, a differential-graded vector space V quasi-isomorphic to $\mathfrak{g}$ naturally acquires an L_∞-structure; the original $\mathfrak{g}$ can be recovered as $H^\bullet(V)$. On the other hand, a number of interesting L_∞-algebras of a different origin have been discovered in geometry and physics as well. We list some of them in chapter 2.

One way to approach the notion of a strongly homotopy Lie algebra is via the language of formal geometry. Namely, the data of an L_∞-algebra $\mathfrak{g}$ is equivalent to that of a formal pointed differential graded manifold $\mathfrak{g}[1]$. The corresponding L_∞ structure is encoded in the cofree dg cocommutative coalgebra $S(\mathfrak{g}[1])$ of distributions on $\mathfrak{g}[1]$ supported at the basepoint. The idea of Batalin-Vilkovisky (BV) formalism in physics suggests that it might be useful to study what the L_∞ structure looks like from the point of view of the standard differential graded commutative algebra structure on $S(\mathfrak{g}[1])$. In fact, we elaborate how an L_∞ structure on $\mathfrak{g}$ translates into a commutative homotopy

Batalin-Vilkovisky (BV_∞) structure on $S(\mathfrak{g}[-1])$ and characterize BV_∞-algebras arising in this way (Theorem 3.2.5). Furthermore, we show that this construction is functorial, thus embedding the category of L_∞-algebras (with possibly non-strict morphisms) into the category of BV_∞-algebras (Theorem 3.3.7).

The correspondence between L_∞ and BV_∞ structures that we establish is to a large extent motivated by the technique of higher derived brackets. The origins of the latter can be traced back to the iterated commutators of A. Grothendieck, see Exposé VII_A by P. Gabriel in [SGA70], and J.-L. Koszul [Kos85], used in the algebraic study of differential operators, though the subject really flourished later in physics under the name of higher "antibrackets" in the works of J. Alfaro, I. A. Batalin, K. Bering, P. H. Damgaard and R. Marnelius [AD96, BBD97, BM98, BM99a, BM99b, BDA96, Ber07] on the BV formalism. A mathematically friendly approach was developed by F. Akman's [Akm97, Akm00] and generalized further by T. Voronov [Vor05a, Vor05b], who described L_∞ brackets derived by iterating a binary Lie bracket not necessarily given by the commutator. The notion of a homotopy BV algebra was studied by K. Bering and T. Lada [BL09], K. Cieliebak and J. Latschev [CL07], O. Kravchenko [Kra00], and D. Tamarkin and B. Tsygan [TT00].

The second part of the thesis concerns quantization of L_∞-bialgebras and was partially motivated by the quest of searching a structure that can be regarded as a homotopy analog of Poisson-Lie structures and quantum groups. Quantum groups made their appearance in mathematics in the seminal works of V.Drinfel'd [Dri87] and M.Jimbo [Jim86] on algebraic structures behind the inverse scattering method in integrable systems [FST79],[Skl82] and underlying certain types of integrable lattice models in statistical physics [Bax72] (alternative approaches were also taken later by Yu.Manin[Man91] and S.Woronowicz[Wor89]). A quantum group in the sense of Drinfel'd and Jimbo is a non-commutative Hopf algebra A subject to the condition of being *quasitriangular* [Dri87]. The latter implies, in particular, the existence of a solution $\mathcal{R}$ to the quantum Yang-Baxter equation $\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}$ set up in A, thus providing one with a way of constructing quantum integrable systems by passing to representations of A. More conceptually, the quasitriangular condition provides the data needed to put a braided structure on the monoidal category of A-representations. Eventually, quantum groups found their applications in low-dimensional topology[RT91], representation theory, number theory as well ([CP94] and references therein).

The most important examples of quantum groups arise as deformations (in the sense of Hopf algebras) of universal enveloping algebras and algebras of functions on groups. In the first case, starting with a Lie algebra $\mathfrak{g}$ over a field of characteristic zero and a Hopf algebra deformation $U_h(\mathfrak{g})$ of its universal enveloping algebra $U(\mathfrak{g})$, one can pass to the "classical limit" $\delta(x) := \frac{\Delta_h(x)-\Delta_h^{op}(x)}{h}$ thus equipping $U(\mathfrak{g})$ with a co-Poisson-Hopf structure, where $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is a co-Poisson cobracket. Furthermore, the restriction $\delta|_{\mathfrak{g}}$ becomes a well-defined cobracket on $\mathfrak{g}$ turning it into a Lie bialgebra.

The above process can be reversed: as it was shown in [Res92], any finite dimensional complex Lie bialgebra $(\mathfrak{g}, \delta)$ can be *quantized*, meaning that one can always come up with a Hopf algebra deformation $U_h(\mathfrak{g})$ whose classical limit, in the sense of the above formula, agrees with δ. While a priori $U_h(\mathfrak{g})$ is just a Hopf algebra, one would be interested in having a quasitriangular structure on it. It was shown in [Dri83] that such a structure always exists in a special case when $\mathfrak{g}$ is a *triangular* Lie bialgebra. This class of Lie bialgebras is defined as follows: let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$ ("*a classical r-matrix*") be a skew-symmetric element satisfying the classical Yang-Baxter equation $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{23}, r_{13}] = 0$. Equivalently, this can be restated in the form of the Maurer-Cartan equation $\{r, r\} = 0$ taking place in $S(\mathfrak{g}[-1])$, where $\{,\}$ is the Schouten bracket. One can verify, that the Chevalley-Eilenberg coboundary $\partial_{CE}(r) : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ of r (considered as a 0-cocycle) with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$ is a well-defined Lie cobracket. The compatibility with the Lie algebra structure on $\mathfrak{g}$ is packed into the relation $\partial^2_{CE}(r) = 0$. Now, the statement is that there exists a quantization $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$, which is a *triangular* Hopf algebra. This condition is stronger than being quasitriangular. In particular, the category of representations of a triangular Hopf algebra is symmetric monoidal rather than just braided.

The upshot of the above construction is that there is a source of quantum groups coming from the data of a Lie algebra $\mathfrak{g}$ and a solution of the classical master equation in $S(\mathfrak{g}[-1])$. The goal of our project is to promote this construction to the world of strongly homotopy Lie algebras. In particular, this is supposed to generalize the work [BSZ13] done for the case of Lie 2-bialgebras. One of the tools that we employ is the *big bracket algebra*, which is essentially a canonical Poisson algebra of functions on a symplectic supermanifold $T^*(V^*[-1])$. While being a relatively simple object, it is quite useful for describing various Lie and homotopy Lie related algebraic structures. In particular, for a given strongly homotopy Lie algebra $\mathfrak{g}$ we show (Proposition 4.2.3), using a standard deformation-theoretic argument, how a solution of the generalized Maurer-Cartan equation set up in $S(\mathfrak{g}[-1])[1]$ (with a homotopy Lie structure given by the derived brackets) gives rise to a homotopy Lie bialgebra structure on $\mathfrak{g}$.

We finish by reviewing the notion of a universal enveloping algebra of a strongly homotopy Lie algebra and discussing a possible approach to a homotopy co-Poisson Hopf structure on it.

1.2 Conventions and notations

The ground field k is assumed to be of characteristic zero.

A graded vector space will mean a $\mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$-graded k-vector space. By a *differential graded* (dg) vector space V we will mean a complex of k-vector spaces with a differential of degree $+1$. The degree of a homogeneous element $v \in V$ will be denoted by $|v|$.

For a n-tuple $x = (x_1, \ldots, x_n)$ of elements of a graded vector space V and a permutation $\sigma \in S_n$, the *Koszul sign* $(-1)^{|x_\sigma|}$ is defined by the equality $x_1 \wedge \cdots \wedge x_n = (-1)^{|x_\sigma|} x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}$. The sign of a permutation $\sigma \in S_n$ is denoted by $(-1)^\sigma$.

For any integer n, we define a *translation* (or *n-fold desuspension*) $V[n]$ of V: $V[n]^k := V^{k+n}$ for each $k \in \mathbb{Z}$.

For a dg vector space V, we will also consider the dg $k[[\hbar]]$-module $V[[\hbar]]$ of formal power series in a variable $\hbar$ of degree 2.

By a slight abuse of terminology, we will sometimes refer to differential operators of order $\leq n$ as to differential operators of order n.

Chapter 2

Algebraic preliminaries

In this chapter we introduce the basic ingredients of our work. Specifically, we review the notions of strongly homotopy Lie algebras, Batalin-Vilkovisky algebras and present some examples of such structures arising in geometry and physics.

2.1 Strongly homotopy Lie algebras

Strongly homotopy Lie algebras, also called L_∞-algebras, first appeared in the context of deformation theory [SS85] generalizing the standard Lie and dg-Lie structures.

Definition 2.1.1. An L_∞*-algebra* is a graded vector space V endowed with a collection

$$l_k : V^{\otimes n} \to V, \quad k \in \mathbb{N}$$

of skew-symmetric linear maps of degree $2-k$ subject to the relation

$$\sum_{i+j=k+1} (-1)^{i(j-1)} \sum_{\sigma \in \mathrm{Sh}_{i,k-i}} (-1)^{|x_\sigma|} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots x_{\sigma(k)}) = 0 \qquad (2.1)$$

holding for each $k \geq 1$.

Here, $\mathrm{Sh}_{n,m-n}$ is the set of $(n, m-n)$ *shuffles*, that is, permutations σ on $\{1, 2, \dots, m\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ and $\sigma(n+1) < \sigma(n+2) < \cdots < \sigma(m)$.

The mappings l_k for $k > 2$ are to be understood as *higher Lie brackets*, while equations (2.1) can be regarded as *higher Jacobi identities*. More precisely, for $k = 1$

the equation (2.1) reads

$$l_1 \circ l_1 = 0.$$

Hence, $d = l_1$ is a degree 1 differential on V. For $k = 2$, we have

$$-l_2(l(x_1), x_2) + (-1)^{|x_1||x_2|} l_2(l_1(x_2), x_1) + l_1(l_2(x_1, x_2)) = 0.$$

Taking into account the (graded) skew-symmetry of l_2 and denoting the bilinear mapping $l_2 : V^{\otimes 2} \to V$ by the brackets $[\,,\,]$, the above equation can be written as

$$d[x_1, x_2] = [dx_1, x_2] + (-1)^{|x_1|}[x_1, dx_2].$$

That is, the differential d is also a derivation of the bracket. While this bracket is skew-symmetric by definition, it does not necessary satisfy the Jacobi identity. However, the deviation from the Jacobi identity is controlled by the ternary term $l_3 : V^{\otimes 3} \to V$. Namely, the equation (2.1) yields

$$(-1)^{|x_1||x_3|+1}(dl_3(x_1, x_2, x_3) + l_3(dx_1, x_2, x_3) + (-1)^{|x_1|} l_3(x_1, dx_2, x_3) + (-1)^{|x_1|+|x_2|} l_3(x_1, x_2, x_3))$$

$$= \underbrace{(-1)^{|x_1||x_3|}[[x_1, x_2], x_3] + (-1)^{|x_2||x_1|}[[x_2, x_3], x_1] + (-1)^{|x_3||x_2|}[[x_3, x_1], x_2]}_{Jac_3(x_1, x_2, x_3)}.$$

Up to a sign, the left-hand side is $d \circ l_3 + l_3 \circ d$ computed on $x_1 \otimes x_2 \otimes x_3$. Hence, l_3 is a homotopy between the chain map $Jac_3 : V^{\otimes 3} \to V$ and the zero map. Thus, an L_∞-algebra V with $l_k = 0$ for $k > 2$ is nothing but a differential graded Lie algebra, and if, in addition to that, $l_1 = 0$ and V is concentrated in degree zero, then the resulting structure is just an ordinary Lie algebra. Analogously, higher Jacobi identities coherently control higher homotopy properties of higher brackets l_k on V.

Remark. 1. More conceptually, the origins of the L_∞-structure and the passage from Lie algebras to their homotopy counterparts can be understood in the uniformizing framework of operads [MSS02, GJ94, LV12]. Namely, a strongly homotopy Lie algebra should be regarded as an algebra over a cofibrant model of the Lie operad (in the appropriate closed model structure on the category of differential graded operads). Such a cofibrant model can be constructed by means of Koszul duality theory[GK94].

2. There is a version of the L_∞-structure, where the k-th bracket l_k is assumed to have degree $k-2$ for each $k \geq 1$. The higher Jacobi identities (2.1) remain the same.

The data of a L_∞-algebra V can be conveniently packed into a *codifferential* on the cofree graded-cocommutative coalgebra $S(V[1])$ endowed with the *shuffle comultiplication*:

$$\delta(x_1 \dots x_m) := \sum_{n=0}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} (x_{\sigma(1)} \dots x_{\sigma(n)}) \otimes (x_{\sigma(n+1)} \dots x_{\sigma(m)}),$$

where $x_1, \dots, x_m \in V[1]$. Here, a *codifferential* D is a square-zero coderivation of degree $+1$, that is a linear map $D : S(V[1]) \to S(V[1])$ of degree $+1$ subject to the co-Leibniz rule $\delta \circ D = (D \otimes id + id \otimes D) \circ \delta$ and such that $D^2 = 0, D(1) = 0$. Since a coderivation is determined by its projection to the cogenerators, we can write

$$D = D_1 + D_2 + D_3 + \dots,$$

where D_n is the extension as a coderivation of the nth symmetric component $l_n : S^n(V[1]) \to V[1]$ of the projection $S(V[1]) \xrightarrow{D} S(V[1]) \to V[1]$. More explicitly, for $x_1, \dots, x_m \in V[1]$

$$D_n(x_1 \dots x_m) = \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} l_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) x_{\sigma(n+1)} \dots x_{\sigma(m)}, \tag{2.2}$$

if $m \geq n$, and $D_n(x_1 \dots x_m) = 0$ otherwise. The condition $D^2 = 0$ is equivalent to the entire series of the higher Jacobi identities.

Remark. This description happens to be just a special case of a more general story: given a Koszul operad $\mathcal{P}$, the data of a homotopy $\mathcal{P}$-algebra (i.e. an algebra over the minimal cofibrant replacement of $\mathcal{P}$) on V is equivalent to a square-zero differential on the cofree $\mathcal{P}^!$-coalgebra, where $\mathcal{P}^!$ is the quadratic dual of $\mathcal{P}$[GK94].

In these terms, the appropriate notion of a morphism between L_∞-algebras takes the following form

Definition 2.1.2. A morphism $\phi : V \to W$ of L_∞-algebras *is* a morphism of the corresponding cofree dg-cocommutative coalgebras $(S(V[1]), D_V) \to (S(W[1]), D_W)$.

One should notice that this condition is weaker than a mere preservation of brackets under a linear mapping $V \to W$.

2.1.1 Examples

Example 2.1.3. ([Rog12, BHR10]) While symplectic manifolds appear as phase spaces of mechanical systems (that is, 0+1-dimensional field theories), the analogous geometric notion for higher-dimensional field theories is represented by *multisymplectic* or n*-plectic* manifolds [KT79]. An n*-plectic structure* on a smooth manifold M is a $(n+1)$-form ω on M which is closed and non-degenerate ($i_v\omega = 0$ iff $v = 0$, where i_v is the usual interior product). The symplectic case correponds to $n = 1$. Given an n-plectic manifold (M, ω), we define the space of *Hamiltonian* $(n-1)$*-forms* as

$$\Omega^{n-1}_{Ham}(M) = \{\alpha \in \Omega^{n-1}(M) | \exists \text{ a vector field } v \text{ such that } d\alpha = -i_v\omega\}.$$

A vector field v from this definition is called a *Hamiltonian vector field* corresponding to α. Due to non-degeneracy of ω, such a field, if it exists, is unique.

The algebra of functions of a symplectic manifold comes with a canonical Lie bracket (the Poisson bracket). This generalizes to the n-plectic case as follows. Given an n-plectic manifold (M, ω), let $L_0 = \Omega^{n-1}_{Ham}(M)$ and $L_i = \Omega^{n-i-1}(M)$ for $0 < i \leq n-1$. For $k \geq 1$, define maps $l_k : L^{\otimes k} \to L$ on $L = \bigoplus_{i=0}^{n-1} L_i$ by

$$l_1(\alpha) = \begin{cases} d\alpha, & \alpha \in L_{>0} \\ 0, & \alpha \in L_0 \end{cases}$$

$$l_k(\alpha_1, \dots, \alpha_k) = \begin{cases} (-1)^{k/2+1} i(v_{\alpha_1}, \dots, v_{\alpha_k})\omega, & \alpha_1, \dots, \alpha_k \in L_0, k \text{ is even} \\ (-1)^{(k-1)/2} i(v_{\alpha_1}, \dots, v_{\alpha_k})\omega, & \alpha_1, \dots, \alpha_k \in L_0, k \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \quad \text{for } k > 1,$$

where $v_{\alpha_1}, \dots, v_{\alpha_k}$ are the Hamiltonian fields associated to $\alpha_1, \dots, \alpha_k$ respectively, and $i(v_{\alpha_1}, \dots, v_{\alpha_k}) = i_{\alpha_1} \dots i_{\alpha_k}$. As shown in [Rog12], $(L, \{l_k\})$ is an L_∞-algebra concentrated in degrees less than n and with brackets l_k having degree $k-2$ for $1 \leq k \leq n$.

Example 2.1.4. ([Roy02]) Let M be a symplectic or, more generally, a Poisson manifold representing a phase space of a mechanical system. Imposing physical constraints on the system amounts to restricting ourselves to a submanifold C of M. Recall that a Poisson structure on M induces a (possibly singular) foliation of M into symplectic leaves. Now, given a leaf $\mathcal{O}$ of such a foliation, the pullback of the symplectic form on $\mathcal{O}$

along the inclusion $\mathcal{O} \cap C \to \mathcal{O}$ endows $\mathcal{O} \cap C$ with a presymplectic structure. Thus, a submanifold C naturally carries a foliation by *pre*symplectic leaves. A natural question to ask here is what kind of geometric structure characterizes a foliation by presymplectic leaves (just as Poisson structure determines a foliation into symplectic leaves)? This question leads to the notions of *Dirac structure*, *Courant brackets*[Cou90] and *Courant algebroid*[LWX97]. The latter remarkably gives rise to a certain "small" L_∞-algebra.

A *Courant algebroid* is a vector bundle $E \to M$ equipped with a non-degenerate symmetric bilinear form $\langle , \rangle$, a bracket $[,]$ on $\Gamma(E)$ and a bundle map (*an anchor*) $\rho : E \to TM$ such that for any $e_1, e_2, e_3 \in \Gamma(E)$, $f \in C^\infty(M)$,

$$
\begin{aligned}
&[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]] \\
&[e_1, f \cdot e_2] = \rho(e_1) f \cdot e_2 + f \cdot [e_1, e_2] \\
&[e_1, e_1] = \frac{1}{2} \mathcal{D} \langle e_1, e_1 \rangle \\
&\rho(e_1) \langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle,
\end{aligned}
$$

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is determined by $\langle e, Df \rangle = \rho(e) f$. Skew-symmetric Courant brackets are defined by $[\![e_1, e_2]\!] = \frac{1}{2}([e_1, e_2] - [e_2, e_1])$. A standard example of such a structure is obtained by taking the bundle $TM \oplus T^*M \to M$, the anchor map being a projection onto M, $\mathcal{D}$ being the deRham differential and the bracket defined by

$$
[X_1 + \xi_1, X_2 + \xi_2] = [X_1, X_2] + (L_{X_1} \xi_2 - L_{X_2} \xi_1 + \frac{1}{2} d(\xi_1(X_2) - \xi_2(X_1)).
$$

Given a Courant algebroid $(E \to M, \rho, [,], D)$, let $L_0 = \Gamma(E)$, $L_1 = C^\infty(M)$. Define $l_k : L^{\otimes k} \to L$ on $L = L_0 \oplus L_1$ by setting

$$
\begin{aligned}
&l_1 = \mathcal{D} \\
&l_2(e_1, e_2) = [\![e_1, e_2]\!], \quad e_1, e_2 \in L_0 \\
&l_2(e_1, f) = [\![e, f]\!], \quad e \in L_0, f \in L_1 \\
&l_2 = 0, \quad \text{in other cases} \\
&l_3(e_1, e_2, e_3) = -\frac{1}{3} \langle [\![e_1, e_2]\!], e_3 \rangle + \text{cyclic permutations}, \quad e_1, e_2, e_3 \in L_1 \\
&l_3 = 0, \quad \text{in other cases.}
\end{aligned}
$$

Then (L, l_1, l_2, l_3) is an L_∞-algebra with $|l_k| = k - 2$[Roy02].

Example 2.1.5. ([Zei10, Zei07]) Another example of an L_∞-algebra concentrated in small degrees is the *Yang-Mills L_∞-algebra.* Let M be a Riemannian n-manifold with a principal G-bundle on it. Consider the complex of $\mathfrak{g} = Lie(G)$-valued forms

$$0 \to \underbrace{\Omega^0(M;\mathfrak{g})}_{L_0} \xrightarrow{d} \underbrace{\Omega^1(M;\mathfrak{g})}_{L_1} \xrightarrow{d*d} \underbrace{\Omega^{n-1}(M;\mathfrak{g})}_{L_2} \xrightarrow{d} \underbrace{\Omega^n(M;\mathfrak{g})}_{L_3} \to 0$$

and take $L = \bigoplus_{i=0}^{3} L_i$. It can be equipped with a L_∞-structure (with brackets of order greater than 3 being trivial) as follows. Take $l_1 : L \to L$ to be the differential on the corresponding component of the comples. While the binary bracket l_2 is quite non-trivial and we would refer the reader to the original works for its definition, the triple braket l_3 is

$$l_3(\alpha,\beta,\gamma) = [\alpha, *[\beta,\gamma]] + [\beta, *[\gamma,\alpha]] + [\gamma, *[\alpha,\beta]]$$

for $\alpha,\beta,\gamma \in L_1 = \Omega^1(M)$, and $l_3 = 0$ for all other combinations of the arguments. Here, $[,]$ is the standard bracket on the Lie algebra-valued forms.

As an application, it is argued in [Zei07] that solutions of the (generalized) Maurer-Cartan equation

$$l_1(\alpha) + \frac{1}{2!}l_2(\alpha,\alpha) + \frac{1}{3!}l_3(\alpha,\alpha,\alpha) = 0$$

in the sheaf of Yang-Mills L_∞-algebras on M are in one-to-one correspondense with the solutions of the Yang-Mills equation $d_\alpha * F_\alpha = 0$, where $d_\alpha = d + [\alpha, -]$ and F_α is the curvature of α.

Example 2.1.6. ([Zwi93],[Mar01]) There is an example of an L_∞-algebra arising in the context of closed (bosonic) string field theory. Let $\mathcal{H}$ be a graded Hilbert space representing the state space of a combined conformal field theory containing matter and ghost fields. Consider the subspace $\mathcal{H}_{rel}$ consisting of states annihilated by $b_0 - \bar{b_0}$ and $L_0 - \bar{L_0}$, where $b_0^\pm, L_0^\pm$ are the zero modes of the antighost and stress-energy fields respectively. In [Zwi93] B.Zwiebach constructs for each genus $g \geq 0$ and $n \geq 0$ multilinear and graded-commutative *string products* $\mathcal{H}_{rel}^{\otimes n} \to \mathcal{H}_{rel}$, $B_1,\dots,B_n \mapsto [B_1,\dots,B_n]_g$. For $n = 1$, $g = 0$, the string product $B \mapsto [B]_0$ is identified with the BRST differential Q of the theory. The main identity that these string products satisfy for fixed g, n is

$$\sum_{i+j=n+1} \sum_{\sigma\in \mathrm{Sh}_{i,n-i}, g_1+g_2=g} \pm[[B_{\sigma(1)},\dots,B_{\sigma(i)}]_{g_1}, B_{\sigma(i+1)},\dots,B_{\sigma(n)}]_{g_2} + \frac{1}{2}\sum_s \pm[\Phi_s,\Phi^s,B_1,\dots,B_n],$$

where $\{\Phi_s\}$ is a basis of $\mathcal{H}$ and $\{\Phi^s\}$ is a dual basis in the sense that $(-1)^{|\Phi_r|}\langle \Phi_r, \Phi^s\rangle = \delta_s^t$. For $g = 0$, the above identity becomes just a higher Jacobi identity(2.1). Thus, genus zero string products give $\mathcal{H}_{rel}$ an L_∞-structure. The algebraic structure underlying the general (arbitrary genus) case is known as a *loop homotopy Lie algebra*[Mar01].

2.2 Batalin-Vilkovisky algebras

Definition 2.2.1. A *Batalin-Vilkovisky* (*BV*, for short) algebra is a unital associative graded-commutative algebra V equipped with

1. a skew-symmetric bracket $\{,\} : V \otimes V \to V$ of degree $+1$;
2. a degree $+1$ linear map $\Delta : V \to V$ squaring to zero $\Delta^2 = 0$

subject to the following conditions:

a) the bracket $\{,\}$ satisfies the (graded) Jacobi identity;

b) for all $a \in V$, $\{a, -\}$ is a derivation: $\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{(|a|+1)|b|} b \cdot \{a, c\}$;

c) for $a, b \in V$, $\Delta(\{a, b\}) = \{\Delta(a), b\} + (-1)^{|a|+1}\{a, \Delta(b)\}$;

d) the bracket and the operator Δ are related via

$$\{a, b\} = (-1)^{|a|}\Delta(a \cdot b) - (-1)^{|a|}\Delta a \cdot b - a \cdot \Delta b$$

for $a, b \in V$.

To get a more concise description of a BV-structure well-suited for our needs, recall the following definition:

Definition 2.2.2. A k-linear operator $D : V \to V$ on a graded commutative algebra V is a said to be a *differential operator of order* $\leq n$ if for any $n+1$ elements $a_0, \ldots, a_n \in V$, we have

$$[[\ldots[D, L_{a_0}], \ldots], L_{a_n}] = 0,$$

where the L_a is the left-multiplication operator

$$L_a(b) := ab$$

on V and the bracket $[-, -]$ is the graded commutator of two k-linear operators.

Lemma 2.2.3. *Let V be an associative graded-commutative algebra and Δ be a degree $+1$, second-order differential operator $\Delta : V \to V$ such that $\Delta^2 = 0, \Delta(1) = 0$. Then the bracket $\{,\} : V \otimes V \to V$ defined by $\{a,b\} := [[\Delta, L_a], L_b](1)$ satisfies conditions of definition 2.2.1. That is, endows V with a BV-structure.*

The nilpotence condition on Δ yields 2.2.1a) and 2.2.1.c), condition 2.2.1 b) follows from Δ being of order two.

Remark. 1. Brackets defined in such fashion are said to be *derived brackets* and Δ is referred to as a *generator*. A generator Δ for a given BV bracket is not unique: adding a first-order differential operator to Δ makes no effect on the derived brackets.

2. Associative graded-commutative algebras V equipped with a skew-symmetric bracket of degree $+1$ satisfying conditions 2.2.1a) and b) are known as *Gerstenhaber algebras.*

2.2.1 Homotopy Batalin-Vilkovisky algebras

We will utilize a strictly commutative version of the notion of a *homotopy BV algebra*, which is due to Kravchenko [Kra00]. It is less general than the full-fledged homotopy versions of [TT00] and [GCTV12]. Nevertheless, we will take the liberty to use the term BV_∞-algebra, following a trend set by several authors [CL07, TTW11, BL13].

Definition 2.2.4. Let $\hbar$ be a formal variable of degree 2. A BV_∞*-algebra* is a graded commutative algebra V over k with a k-linear map $\Delta : V \to V[[\hbar]]$ of degree $+1$ satisfying the following properties:

$$\Delta = \frac{1}{\hbar}\sum_{n=1}^{\infty} \hbar^n \Delta_n,$$

where Δ_n is a differential operator of order (at most) n on V,

$$\Delta^2 = 0, \qquad \text{and} \qquad \Delta(1) = 0,$$

The continuous (in the $\hbar$-adic topology), $k[[\hbar]]$-linear extension of Δ to $V[[\hbar]]$ will also be denoted $\Delta : V[[\hbar]] \to V[[\hbar]]$ and called a BV_∞ *operator*.

Since $|\hbar| = 2$, then $|\Delta_1| = 1$, $|\Delta_2| = -1$, and in general, $|\Delta_n| = 3 - 2n$ for $n \geq 1$. Moreover, note that Δ_1 satisfies $\Delta_1^2 = 0$, and turning V into a differential graded commutative algebra.

If $\Delta_n = 0$ for $n \geq 3$, the structure we obtain is known as a *differential graded BV algebra* [Akm97, BK97, Man99, Kra00, TT00]. If moreover $\Delta_1 = 0$, we recover the usual notion of a BV algebra.

2.2.2 Examples

The origins of the BV-structure can be traced back to works of I.Batalin and G.Vilkovisky of gauge fixing in quantum field theory [BV81]. More mathematically-oriented treatments of their construction can be found in [Sch93],[Get94]. A canonical example of a BV-algebra is the following.

Example 2.2.5. Let M be an odd symplectic supermanifold, i.e. a supermanifold equipped with a closed, non-degenerate, odd two-form ω. For $f \in C^\infty(M)$, the Hamiltonian vector field X_f is defined by $i_{X_f}\omega = -df$. Then the algebra of functions $C^\infty(M)$ with the odd Poisson bracket $\{f, g\} := (-1)^{|f|-1} X_f(g)$ is a Gerstenhaber algebra. Now, as a special case take $M = T^*[-1]N$ for a graded manifold N. If, in addition, N has volume form v, then there is an induced volume form (a section of the Berezinian bundle) μ_v on M. The divergence operator $div_\mu : \Gamma(M) \to C^\infty(M)$ is defined via $\int_M (div_\mu X) f \, \mu_v = -\int_M X(f)$ for all $f \in C^\infty(M)$. One can show that $div_\mu(f \cdot X) = f div_\mu(X) + (-1)^{|f||X|} X(f)$ and this identity implies that Δ_v defined by $\Delta_v(f) := div_v(X_f)$ is actually a BV-operator generating the odd Poisson bracket.

Remark. In physics, this particular BV-algebra appears in the following context. Let N be a space of fields [1] , $S : N \to \mathbb{R}$ be an action functional and G acting on N be a symmetry group (gauge group) of the action: $S[g \cdot \Psi] = \Psi$. In the path integral formulation, quantization of a physical system is achieved by computing expectation values of observables $\mathcal{O} : N \to \mathbb{R}$ via $\langle \mathcal{O} \rangle = \int_N e^{\frac{i}{\hbar} S[\Psi]} \mathcal{O}(\Psi)$. In presense of symmetries, the relevant quantity to compute is actually $\langle \mathcal{O} \rangle = \int_{N/G} e^{\frac{i}{\hbar} \bar{S}[\Psi]} \bar{\mathcal{O}}(\Psi)$, where $\bar{S}$,$\bar{\mathcal{O}}$ are restrictions of S and $\mathcal{O}$ onto N/G. The orbit space N/G can be hard to work with,

[1] Generally, N is an infinite-dimensional manifold (typically, the space of functions or sections of a vector bundle on a space-time manifold). In such a case Feynman's path integral is notoriously ill-defined. So in our exposition we will be assuming that $\dim(N) < \infty$ (a toy case).

but there seems to be a way around. Instead of integrating over the orbit space, one can try to integrate over a submanifold transversal to the orbits. To do so, assume that symmetries of the system are given by the Lie algebra $\mathfrak{g} = Lie(G)$ action $\mathfrak{g} \to Vect(N)$, $\gamma \mapsto X_\gamma$ and choose $F : M \to \mathfrak{g}$ ('gauge fixing function'). The zero locus of F is the submanifold we would like to compute our integral over. The integral can be expressed as $\int_N \mathcal{O} e^{\frac{i}{\hbar}S} \delta_0(F) \det(A)$, where δ_0 is the delta-function at $0 \in \mathfrak{g}$ and $\det(A)$ is the Jacobian, $A(p)\gamma = dF_p(X_\gamma(p))$. Consider a supermanifold $\bar{N} = \Pi\mathfrak{g} \times \Pi\mathfrak{g}^* \times N \times \mathfrak{g}^*$, whose function algebra is $C^\infty(\bar{N}) = \wedge(\mathfrak{g} \oplus \mathfrak{g}^*) \oplus C^\infty(N \times \mathfrak{g}^*)$. One can show that the previous integral is equivalent to $\int_{\bar{N}} \mathcal{O} e^{\frac{i}{\hbar}S_F}$ for a new action $S_F = S + \frac{\hbar}{i}(F + A)$, where A is considered as an element of $End(\mathfrak{g}) \subset \wedge(\mathfrak{g} \oplus \mathfrak{g}^*)$.

Furthermore, we pass from $\bar{N}$ to $\tilde{N} = T^*[-1]\bar{N}$. The action S extends to $\tilde{S}$, the gauge fixing condition now specifies a Lagrangian submanifold L_F of $\tilde{N}$ and the algebra of functions $C^\infty(\tilde{N})$ acquires a BV-operator Δ. The expectation value of $\Psi \in C^\infty(\tilde{N})$ is now $\int_{L_F} \Psi e^{\frac{i}{\hbar}\tilde{S}}$. We would like to retrieve physically meaningful observables, that is, such $\Psi \in C^\infty(\tilde{N})$ that are invariant with respect to the choice of a gauge-fixing submanifold L_F. It turns out that this is guaranteed by the condition $\Delta(\Psi e^{\frac{i}{\hbar}\tilde{S}}) = 0$. This is equivalent to $\Omega(\Psi) = 0$, where $\Omega : C^\infty(\tilde{N}) \otimes \mathbb{C} \to C^\infty(\tilde{N}) \otimes \mathbb{C}$ is a square-zero operator defined by $\Omega(f) = \{\Sigma, f\} - i\hbar\Delta(f)$. Here, Σ is a solution of a *quantum master equation* $i\hbar\Delta\Sigma - \frac{1}{2}\{\Sigma, \Sigma\} = 0$. Thus *quantum observables* are non-trivial cocycles of $(C^\infty(\tilde{N}) \otimes \mathbb{C}, \Omega)$ The actual computation of $\langle\Psi\rangle$ can be attempted by standard perturbative techniques.

Example 2.2.6. Let $\mathfrak{g}$ be a Lie algebra. The free graded-commutative algebra $S(\mathfrak{g}[-1])$ is a BV-algebra with the standard Chevalley-Eilenberg differential as a BV-operator.

Example 2.2.7. ([CS99],[CV06]) Let M be a closed, oriented manifold of dimension d and LM be the free loop space on M. M.Chas and D.Sullivan constructed a BV-structure on $H_*(LM)[d]$ with the *loop product* as a multiplicative operation, and a BV-operator $\Delta : H_k(LM) \to H_{k+1}(LM)$ defined as $\alpha \mapsto \rho_*(e \otimes \alpha)$, where $\rho : S^1 \times LM \to LM$ is the obvious circle action and e is a generator of $H_1(S^1)$.

Example 2.2.8. ([Xu99]) Let $A \to M$ be a vector bundle of rank n over a smooth manifold M. Recall that A is called a *Lie algebroid*, if $\Gamma(A)$ is equipped with a Lie bracket and a bundle morphism $a : A \to TM$ (an *anchor map*) such that the associated map $\Gamma(A) \to \Gamma(TM) = Vect(M)$ is a Lie algebra morphism and the Leibniz identity $[\xi, f \cdot \eta] = f \cdot [\xi, \eta] + L_{a(\xi)} f \cdot \eta$ holds for any $\xi, \eta \in \Gamma(A), f \in C^\infty(M)$. For a Lie algebroid $A \to M$ and a vector bundle $E \to M$ there are a corrsponding notions of an *A-connection* $\nabla : \Gamma(A) \otimes \Gamma(E) \to \Gamma(E)$ and its curvature generalizing their standard differential-geometric couterparts. As shown in [Xu99], Gerstenhaber algebra structures on $\mathcal{A} = \bigoplus_{i=0}^{n} \Gamma(\wedge^i A)$ are in one-to-one correspondence with A-connections. Moreover, under this correspondence BV-algebra structures are precisely those coming from *flat* A-connections.

Example 2.2.9. Let M be a Poisson manifold and P be the corresponding Poisson bivector field. The de Rham complex $\Omega(M)$ is a BV-algebra with a BV-operator being $d_P = [d, i_P]$, where d is the de Rham differential.

A generalization of this construction yields and example of a BV_∞-algebra.

Example 2.2.10. ([BL13, KV08]) Let M be a smooth graded manifold and $C^\infty(M, S(T[-1]M)[1])$ be the graded Lie algebra of multivector fields on M with respect to the Schouten bracket. When M is a usual, ungraded manifold, $S(T[-1]M)[1]$ is the exterior-algebra bundle $\bigwedge TM$, in which a k-vector field, or a section of $\bigwedge^k TM$, has degree $k-1$. A *generalized Poisson structure* on a graded manifold M is a multivector field P of degree one such that $[P, P]_{SN} = 0$. A generalized Poisson structure on M may be expanded as $P = P_0 + P_1 + \dots$ with $P_n \in C^\infty(M, S^n(T[-1]M)[1])$. For $n \geq 1$, the generalized Lie derivative $\Delta_n = [d, i_{P_n}]$, where $i_{(-)}$ is the interior product, defines an nth-order differential operator of degree $3 - 2n$ on the de Rham algebra $(\Omega(M), d)$, where $\Omega(M) := C^\infty(M, S(T^*[-1]M))$. If we assume that $P_0 = 0$ to avoid differential operators of order zero, then $\Delta = \Delta_1 + \Delta_2 \hbar + \dots + \Delta_n \hbar^{n-1} + \dots : \Omega(M) \to \Omega(M)[[\hbar]]$ defines a BV_∞ structure on $\Omega(M)$, known as the *de Rham-Koszul* BV_∞ structure.

Chapter 3

Higher derived brackets and BV_∞-algebras

3.1 From L_∞-algebras to BV_∞-algebras

The construction of this section belongs essentially to C. Braun and A. Lazarev, see [BL13, Example 3.12].

Theorem 3.1.1 (C. Braun and A. Lazarev). *Given an L_∞-algebra $\mathfrak{g}$, the free graded commutative algebra $S(\mathfrak{g}[-1])$ becomes a BV_∞-algebra under the BV_∞ operator*

$$\Delta := \frac{1}{\hbar}\sum_{n=1}^{\infty} \hbar^n D_n. \tag{3.1}$$

Proof. Since $D_n : S^m(\mathfrak{g}[1]) \to S^{m-n+1}(\mathfrak{g}[1])$ is a degree one map, it turns into a map $D_n : S^m(\mathfrak{g}[-1]) \to S^{m-n+1}(\mathfrak{g}[-1])$ of degree $3-2n$ under the new grading.[1] For each n,

$$\sum_{i+j=n} D_i D_j = 0,$$

because this sum is exactly the component of D^2 which maps $S^m(\mathfrak{g}[1])$ to $S^{m-n+2}(\mathfrak{g}[1])$. The map D_n will also be a differential operator of order n, because of the following proposition, which may be observed directly from Equation (2.2).

[1] Strictly speaking, the use of D_n to denote the two maps is abuse of notation, because they differ by powers of the double suspension operator $\mathfrak{g}[1] \to \mathfrak{g}[-1]$, but we prefer to keep it this way, because double suspension does not affect signs.

Proposition 3.1.2. *The coderivation of the coalgebra $S(\mathfrak{g}[1])$ extending a linear map $S^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ becomes a differential operator of order $\leq n$ on the algebra $S(\mathfrak{g}[-1])$.*

The statement follows from the following lemma.

Lemma 3.1.3. *Let $S(U)$ be the graded symmetric algebra of a graded vector space U. If an operator $P : S(U) \to S(U)$ is a coderivation whose projection to U is a linear map $p : S^n(U) \to U$ for some $n \geq 1$ then for any $a \in S(U)$ the commutator $[P, L_a]$ is a linear combination $\sum_i \pm L_{c_i} Q_i$, where $c_i \in S(U)$ and Q_i is a coderivation whose projection to U is a linear map $S^{<n}(U) \to U$ for each i.*

Proof of Lemma. This is a simple computation, using an equation between P and p similar to (2.2), for which we will use Sweedler's notation $P(a) = p(a_{(1)})a_{(2)}$, where $a_{(1)} \otimes a_{(2)} := \sum_{(a)} a_{(1)} \otimes a_{(2)} := \delta(a)$ for $a \in S(U)$. We have

$$\begin{aligned} [P, L_a]b &= P(ab) - (-1)^{|a|\cdot|P|} aP(b) \\ &= (-1)^{|a_{(2)}|\cdot|b_{(1)}|} p(a_{(1)}b_{(1)})a_{(2)}b_{(2)} - (-1)^{|a|\cdot|P|} ap(b_{(1)})b_{(2)}. \end{aligned}$$

The part of the first term for which $a_{(1)} = 1$ and $a_{(2)} = a$ will cancel the last term $ap(b_{(1)})b_{(2)}$. Therefore

$$\begin{aligned} [P, L_a]b &= \sum_{(a_{(1)}) \in S^{>0}(U)} (-1)^{|a_{(2)}|\cdot|b_{(1)}|} p(a_{(1)}b_{(1)})a_{(2)}b_{(2)} \\ &= \sum_{(a_{(1)}) \in S^{>0}(U)} (-1)^{|a_{(2)}|\cdot(|P|+|a_{(1)}|)} a_{(2)} p(a_{(1)}b_{(1)})b_{(2)}. \end{aligned}$$

Grouping the remaining terms into $c := a_{(2)}$ and $Q(b) := p(a_{(1)}b_{(1)})b_{(2)}$, we obtain a required expansion. $\square$

The construction of this section seems to be math-physics folklore in the case when $(\mathfrak{g}, d, [-,-])$ is a dg Lie algebra: the differential $\Delta = D_1 + \hbar D_2$ defines a dg BV algebra structure on $S(\mathfrak{g}[-1])$. The operator Δ is essentially the homological Chevalley-Eilenberg differential:

$$\begin{aligned} \Delta(x_1 \dots x_m) &= \sum_{i=1}^{m} (-1)^{|x_1 \dots x_{i-1}|} x_1 \dots dx_i \dots x_m \\ &+ \hbar \sum_{1 \leq i < j \leq m} (-1)^{|x_{\sigma(i,j)}|+|x_i|} [x_i, x_j] x_1 \dots \widehat{x_i} \dots \widehat{x_j} \dots x_m, \end{aligned}$$

where $\sigma(i,j)$ is the corresponding shuffle, the x_i's in $\mathfrak{g}$ are treated as elements of $\mathfrak{g}[-1]$, and, following standard conventions, $d = l_1$ and $[x_i, x_j] = (-1)^{|x_i|} l_2(x_i, x_j)$.

Remark. An A_∞-analog of the above construction has been proposed by J. Terilla, T. Tradler, and S. Wilson in [TTW11]: for an A_∞-algebra V, the tensor algebra $T(V[-1])$ (considered with the shuffle product) is equipped with a BV_∞-structure.

Remark. Later we will also need a certain $\hbar$-enhancement of the construction of a BV_∞-algebra from an L_∞-algebra. Suppose the graded $k[[\hbar]]$-module $\mathfrak{g}[[\hbar]]$ for a graded vector space $\mathfrak{g}$ is provided with the structure of a topological L_∞-algebra over $k[[\hbar]]$ with respect to $\hbar$-adic topology. Then the same formula (3.1) defines a BV_∞-structure on $S(\mathfrak{g}[-1])$ over k. There is a subtlety, though: each operator D_n is a formal power series in $\hbar$ now, and in the $\hbar$-expansion

$$\Delta = \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \Delta_n,$$

there are contributions to Δ_n from D_1, D_2, $\dots$, and D_n. This still guarantees that Δ_n is a differential operator of order at most n on $S(\mathfrak{g}[-1])$ satisfying the conditions of Definition 2.2.4.

Thus, for given an L_∞-algebra $\mathfrak{g}$, we obtain a canonical BV_∞-algebra structure on $S(\mathfrak{g}[-1])$. There is also a construction going in the opposite direction.

3.2 From BV_∞-algebras to L_∞-algebras

Suppose we have a BV_∞-algebra V. Then for each $n \geq 1$, the following *higher brackets*

$$\begin{aligned} l_n^\hbar(a_1, \dots, a_n) &:= [[\dots[\Delta, L_{a_1}], \dots], L_{a_n}]1 \\ &= \sum_{k=n}^{\infty} \hbar^{k-1} [[\dots[\Delta_k, L_{a_1}], \dots], L_{a_n}]1 \end{aligned} \tag{3.2}$$

on $V[[\hbar]]$, their $\hbar$-modification

$$L_n := \frac{1}{\hbar^{n-1}} l_n^\hbar, \tag{3.3}$$

and their "semiclassical limit"

$$\begin{aligned} l_n(a_1, \dots, a_n) &:= \lim_{\hbar \to 0} \frac{1}{\hbar^{n-1}} l_n^\hbar(a_1, \dots, a_n) \\ &= \lim_{\hbar \to 0} \frac{1}{\hbar^{n-1}} [[\dots[\Delta, L_{a_1}], \dots], L_{a_n}]1 \\ &= [[\dots[\Delta_n, L_{a_1}], \dots], L_{a_n}]1 \end{aligned} \tag{3.4}$$

on V turn out to be L_∞ brackets, according to the results stated in this section below. Observe also that we have a linear (or strict) L_∞-morphism

$$\begin{array}{rcl}(V[[\hbar]][-1], l_n^\hbar) & \to & (V[[\hbar]][1], L_n),\\ v & \mapsto & \hbar v,\end{array}$$

which becomes an L_∞-isomorphism after localization in $\hbar$. Thus, we can think of the L_∞ structure given by the brackets L_n as an $\hbar$-translation of the L_∞ structure given by $l_n^\hbar$.

One can express Δ through $l_n^\hbar$'s via the following useful formula

$$\Delta(a_1 \dots a_n) = \sum_{j=1}^{n} \sum_{\sigma \in \mathrm{Sh}_{j,n-j}} (-1)^{|a_\sigma|} l_j^\hbar(a_{\sigma(1)}, \dots, a_{\sigma(j)}) a_{\sigma(j+1)} \dots a_{\sigma(n)} \tag{3.5}$$

for $a_1, \dots, a_n \in V$, which is easy to prove by induction on n using Equation (3.6) below, starting with $n = 1$ for $l_1^\hbar = \Delta$.

Theorem 3.2.1 (Bering-Damgaard-Alfaro)**.** *For a* BV_∞*-algebra* V*, the higher brackets* $l_n^\hbar$*,* $n \geq 1$*, defined by* (3.2) *endow the suspension* $V[[\hbar]][-1]$ *with the structure of an* L_∞*-algebra over* $k[[\hbar]]$*. Moreover, the bracket* $l_{n+1}^\hbar$ *measures the deviation of* $l_n^\hbar$ *from being a multiderivation with respect to multiplication.*

Remark. This result was first observed by the physicists [BDA96] and proven in a more general context of *higher derived brackets* by T. Voronov [Vor05a, Vor05b]. The fact about the L_∞-algebra was also rediscovered by O. Kravchenko in [Kra00].

Proof. Using the Jacobi identity for the commutator of linear operators along with the fact that L_a and L_b (graded) commute, it is easy to check that the higher brackets $l_n^\hbar$ are symmetric on $V[[\hbar]]$:

$$l_n^\hbar(a_{\pi(1)}, \dots, a_{\pi(n)}) = (-1)^{|a_\pi|} l_n^\hbar(a_1, \dots, a_n)$$

for all $a_1, \dots, a_n \in V[[\hbar]]$, where $(-1)^{|a_\pi|}$ is the Koszul sign, see Section 3.1. Since $|\Delta| = 1$, the degree of $l_n^\hbar$ as a bracket on $V[[\hbar]]$ will be the same. We can extend the $k[[\hbar]]$-linear operators $l_n^\hbar : S^n(V)[[\hbar]] \to V[[\hbar]]$ to coderivations $D_n : S(V)[[\hbar]] \to S(V)[[\hbar]]$ and consider the total coderivation

$$D = D_1 + D_2 + \dots .$$

on $S(V)[[\hbar]]$. The differential property $D^2 = 0$ for this coderivation is equivalent to the series of *higher Jacobi identities*:

$$\sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|a_\sigma|} l^{\hbar}_{m-n+1}(l^{\hbar}_n(a_{\sigma(1)}, \dots, a_{\sigma(n)}), a_{\sigma(n+1)}, \dots, a_{\sigma(m)}) = 0$$

for all $a_1, \dots, a_m \in V[[\hbar]]$, $m \geq 1$. The physicists [BDA96] and T. Voronov [Vor05a] in a more general situation checked these identities using the following key observation for an arbitrary odd operator Δ on $V[[\hbar]]$, not necessarily squaring to zero:

$$\sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|a_\sigma|} l^{\hbar}_{m-n+1}(l^{\hbar}_n(a_{\sigma(1)}, \dots, a_{\sigma(n)}), a_{\sigma(n+1)}, \dots, a_{\sigma(m)})$$
$$= [[\dots[\Delta^2, L_{a_1}], \dots], L_{a_m}]1.$$

Given that $\Delta^2 = 0$, the higher Jacobi identities follow.

The deviated multiderivation property, more precisely,

$$\begin{aligned} l^{\hbar}_{n+1}(a_1, \dots, a_i, a_{i+1}, \dots, a_{n+1}) = l^{\hbar}_n(a_1, \dots, a_i \cdot a_{i+1}, \dots, a_{n+1}) \\ - (-1)^{(1+|a_1|+\dots|a_{i-1}|)|a_i|} a_i l^{\hbar}_n(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}) \\ - (-1)^{(1+|a_1|+\dots|a_i|)|a_{i+1}|} a_{i+1} l^{\hbar}_n(a_1, \dots, a_i, a_{i+2}, \dots, a_{n+1}) \end{aligned} \quad (3.6)$$

of the higher brackets may be derived from the identity

$$[Q, L_{ab}] = [[Q, L_a], L_b] + (-1)^{|Q| \cdot |a|} L_a[Q, L_b] + (-1)^{(|Q|+|a|)|b|} L_b[Q, L_a]$$

for an arbitrary (homogeneous) linear operator Q on $V[[\hbar]]$. Applying this to $Q = [[\dots[\Delta, L_{a_1}], \dots], L_{a_n}]$, we see that $l^{\hbar}_{n+1}$ measures the deviation of $l^{\hbar}_n$ from being a derivation in the last variable. Since the higher brackets are symmetric, we obtain the same property in each variable. $\square$

Corollary 3.2.2 (T. Voronov [Vor05a])**.** *Given a* BV_∞*-algebra* V*, the brackets* L_n*,* $n \geq 1$*, defined by* (3.3) *endow the graded* $k[[\hbar]]$*-module* $V[[\hbar]][1]$ *with the structure of an* L_∞*-algebra over* $k[[\hbar]]$*. Likewise, the brackets* l_n*,* $n \geq 1$*, defined by* (3.4) *endow the graded vector space* $V[1]$ *with the structure of an* L_∞*-algebra over* k*. Moreover, the brackets* l_n *are multiderivations of the graded commutative algebra structure.*

Proof. The statements of the corollary are obtained as the "semiclassical limit" of the statements of Theorem 3.2.1, and so is the proof. Note the change of suspension to desuspension from the theorem to the corollary. This corresponds intuitively to the statement that the semiclassical limit of the space $V[[\hbar]][-1]$ is $\hbar V[-1] = V[1]$. Concretely, the desuspension guarantees that the degree of the nth higher bracket l_n on $V[2]$ is still one: indeed $|\Delta_n| = 3-2n$, when Δ_n is considered as an operator on V; therefore, the degree of l_n as a multilinear operation on $V[2]$ will be $3 - 2n + 2(n-1) = 1$.

The multiderivation property is obtained by dividing (3.6) by $\hbar^{n-1}$ and noticing that the left-hand side will not survive the limit as $\hbar \to 0$, because it has $\hbar$ as a factor. $\square$

Remark. This algebraic structure, which nicely combines a graded commutative multiplication with an L_∞ structure, is a particular case of the G_∞-algebra structure, see [GJ94, Tam98, Tam99, Vor00].

Example 3.2.3. The L_∞ structure in Example 2.2.10 is known as the de Rham-Koszul L_∞ structure and generalizes the Koszul brackets on the de Rham complex of a manifold, [KV08, BL13].

We would like to characterize those BV_∞-algebras which come from L_∞-algebras as in Section 3.1. Note that such a BV_∞-algebra is free as a graded commutative algebra by construction: $V = S(U)$, and that for each $n \geq 1$, the nth component Δ_n of the BV_∞ operator maps $S^m(U)$ to $S^{m-n+1}(U)$ for $m \geq n$ and to 0 for $0 \leq m < n$, because of Equation (2.2). Since an nth-order differential operator on a free algebra $S(U)$ is determined by its restriction to $S^{\leq n}(U)$, this condition on Δ_n is equivalent to the condition that Δ_n maps $S^n(U)$ to U and $S^{<n}(U)$ to 0. Interpreting differential operators on $S(U)$ as linear compbinations of partial derivatives with polynomial coefficients, differential operators of the above type may also be characterized as *differential operators of order n with linear coefficients.*

Definition 3.2.4. A *pure* BV_∞-algebra is the free graded commutative algebra $S(U)$ on a graded vector space U with a BV_∞ operator $\Delta : S(U) \to S(U)[[\hbar]]$ such that, for each $n \geq 1$, Δ_n maps $S^n(U)$ to U and $S^{<n}(U)$ to 0.

The following theorem (Parts (1) and (2)) shows that freeness and purity are not only necessary but also sufficient conditions for a BV_∞-algebra to arise from an L_∞-algebra.

Theorem 3.2.5. *1. Given a pure* BV_∞ *algebra* $(V = S(U), \Delta)$*, the restriction of the brackets* (3.4) *to* $U[1] \subset S(U)[1]$ *provides* $U[1]$ *with the structure of an* L_∞*-subalgebra.*

2. The original pure BV_∞ *structure on* $S(U)$ *coincides with the* BV_∞ *structure* (3.1) *of Section* 3.1 *coming from the derived* L_∞ *structure on* $U[1]$.

3. If we start with an L_∞ *structure on a graded vector space* $U[1]$ *and construct the* BV_∞*-algebra* $S(U)$ *as in Section* 3.1*, then the derived brackets* (3.4) *on* $U[1] \subset S(U)[1]$ *return the original* L_∞ *structure on* $U[1]$.

Proof. The first statement we need to check is that $l_n(x_1, \dots, x_n)$ is in U whenever $x_1, \dots, x_n \in U$ and $n \geq 1$, as a priori all we know is that $l_n(x_1, \dots, x_n) \in S(U)$. The condition that Δ_n maps $S^m(U)$ to 0 for $0 \leq m < n$ implies by (3.4) that $l_n(x_1, \dots, x_n) = \Delta_n(x_1 \dots x_n)$, which must be in U, because of the condition $\Delta_n : S^n(U) \to S^1(U) = U$.

For the second statement, we need to check that the nth-order differential operator Δ_n, the nth component of the given BV_∞ structure, is equal to the coderivation D_n defined by (2.2). Recall that on the free algebra $S(U)$, an nth-order differential operator is determined by its restriction to $S^{\leq n}(U)$. Given the assumption that Δ_n vanishes on $S^{<n}(U)$, it follows that Δ_n on $S(U)$ is determined by its restriction to $S^n(U)$. By the previous paragraph, its restriction to $S^n(U)$ is equal to l_n. On the other hand, this is also the restriction of the coderivation D_n to $S^n(U)$, as per formula (2.2). Lemma 3.1.2 shows that the coderivation D_n is also an nth-order differential operator. Thus, it is also determined by its restriction to $S^n(U)$.

Finally, let l_n be the L_∞-brackets on an L_∞-algebra $U[1]$ and $\tilde{l}_n$ be the higher derived brackets produced on the pure BV_∞-algebra $S(U)$ by formula (3.4) for $n = 1, 2, \dots$ We claim that $\tilde{l}_n(a_1, \dots, a_n) = l_n(a_1, \dots, a_n)$ for all n and $a_1, \dots, a_n \in U$. Indeed,

$$\tilde{l}_n(a_1, \dots, a_n) = [[\dots[D_n, L_{a_1}], \dots], L_{a_n}]1,$$

where D_n is the extension of l_n to $S(U)$ as a coderivation, see Equation (2.2). The same equation implies that $D_n : S(U) \to S(U)$ is zero on $S^{<n}(U)$. Hence all but one term $(D_n \circ L_{a_1} \circ \dots \circ L_{a_n})(1)$ of this iterated commutator vanish. It remains to observe that by (2.2) this is nothing but $l_n(a_1, \dots, a_n)$. $\square$

Remark. A general, not necessarily pure BV_∞-structure on $S(U)$ leads to an interesting algebraic structure on $U[1]$, called an *involutive* L_∞*-bialgebra*. From the properadic, rather than BV prospective, this structure is described in [Val07] and [DCTT08].

3.3 Functoriality

We are going to prove that the correspondence between BV_∞-algebras and L_∞-algebras established above is functorial.

Recall the definition of a morphism between L_∞-algebras.

Definition 3.3.1. An L_∞*-morphism* $\mathfrak{g} \to \mathfrak{g}'$ between L_∞-algebras is a morphism $S(\mathfrak{g}[1]) \to S(\mathfrak{g}'[1])$ of codifferential graded coalgebras, *i.e.*, a morphism of graded coalgebras commuting with the structure codifferentials, such that $1 \in S^0(\mathfrak{g}[1])$ maps to $1 \in S^0(\mathfrak{g}'[1])$.

Remark. Since we deal with counital coalgebras, we assume that L_∞-morphisms respect the counits. The extra condition $1 \mapsto 1$ means that we are talking about "pointed" morphisms, if we invoke the interpretation of L_∞-morphisms as morphisms between formal pointed dg manifolds, see [KS].

Here comes the corresponding notion of a morphism between BV_∞-algebras. We will only need this notion for BV_∞-algebras of Theorem 3.2.5, that is to say, BV_∞-algebras which are pure. Somewhat more generally, we will give a definition in the case when the source BV_∞-algebra is just free. A more general notion of a BV_∞-morphism for more general BV_∞-algebras can be found in [TT00]. We use the definition of a BV_∞-morphism by Cieliebak-Latchev [CL07].

Before giving the definition, we need to recall a few more notions. Fix a morphism $f : A \to A'$ between graded commutative algebras. We say that a k-linear map $D : A \to A'$ is a *differential operator of order* $\leq n$ *over* $f : A \to A'$ or simply a *relative differential operator of order* $\leq n$ if for any $n+1$ elements $a_0, \dots, a_n \in A$, we have

$$[[\dots[D, L_{a_0}], \dots], L_{a_n}] = 0,$$

where $[D, L_a]$ is understood as the map $A \to A'$ defined by

$$[D, L_a](b) := D(ab) - (-1)^{|a| \cdot |D|} f(a) D(b).$$

For $f = \mathrm{id}$ we recover the standard definition Def. 2.2.2 of a differential operator on a graded commutative algebra.

Let $V = S(U)$ be a free graded commutative algebra and V' an arbitrary graded commutative algebra. Given a k-linear map $\varphi : S(U) \to V'[[\hbar]]$ of degree zero such that $\varphi(1) = 0$, define a degree-zero, continuous $k[[\hbar]]$-linear map $\exp(\varphi) : S(U)[[\hbar]] \to V'[[\hbar]]$, called the *exponential*, by

$$\exp(\varphi)(x_1 \dots x_m) := \\ \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1+\dots+i_k=m}}^{m} \frac{1}{i_1! \dots i_k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \varphi(x_{\sigma(1)} \dots x_{\sigma(i_1)}) \dots \varphi(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)}),$$

where S_m denotes the symmetric group, $x_1, \dots, x_m$ are in U, and $(-1)^{|x_\sigma|}$ is the Koszul sign of the permutation of $x_1 \dots x_m$ to $x_{\sigma(1)} \dots x_{\sigma(m)}$ in $S(U)$. By convention, we set $\exp(\varphi)(1) := 1$. The reason for the exponential notation, introduced by Cieliebak and Latschev [CL07], is, perhaps, the following statement, which they might have been aware of. The proof is a straightforward computation.

Lemma 3.3.2. *If $S \in \lambda U[[\hbar]]^0[[\lambda]]$ or $\lambda U((\hbar))^0[[\lambda]]$, where λ is another, degree-zero formal variable, then*

$$\exp(\varphi)(e^S) = e^{\varphi(e^S)}.$$

Here we have extended φ and $\exp(\varphi)$ to $\lambda S(U)((\hbar))[[\lambda]]$ by $\hbar^{-1}$- and λ-linearity and continuity.

Remark. The extra formal variable λ in the lemma guarantees "convergence" of the exponential e^S. We could have achieved the same goal, if we considered completions of our algebras or assumed that λ was a nilpotent variable, varying over the maximal ideals of finite-dimensional local Artin algebras. Informally speaking, given the way the space $S(U)[\lambda, \hbar, \hbar^{-1}]$ of $S(U)$-valued polynomials in λ and Laurent polynomials in $\hbar$ is completed: $S(U)((\hbar))[[\lambda]]$, we could think of λ as being "much smaller" than $\hbar$.

The exponential, not surprisingly, has an inverse, called the *logarithm*, which we will use a little later. Given a k-linear map $\Phi : S(U) \to V'[[\hbar]]$ of degree zero such that $\Phi(1) = 1$, define a degree-zero, continuous $k[[\hbar]]$-linear map $\log(\Phi) : S(U)[[\hbar]] \to V'[[\hbar]]$

by

$$\log(\Phi)(x_1 \dots x_m)$$
$$:= \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1+\dots+i_k=m}}^{m} \frac{1}{i_1! \dots i_k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \Phi(x_{\sigma(1)} \dots x_{\sigma(i_1)}) \dots$$
$$\Phi(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)})$$

under the same notation as for the exponential. By convention, we set $\log(\Phi)(1) := 0$. The formula for $\log(\Phi)$ can be obtained by recursively solving $\exp(\log \Phi) = \Phi$ for the coefficients of log.

Definition 3.3.3 (Cieliebak-Latchev [CL07])**.** A BV_∞-*morphism* from a BV_∞-algebra $(V = S(U), \Delta)$ to a BV_∞-algebra (V', Δ') is a k-linear map $\varphi : V \to V'[[\hbar]]$ of degree zero satisfying the following properties:

1. $\varphi(1) = 0$,
2. $\exp(\varphi)\Delta = \Delta' \exp(\varphi)$, and
3. φ admits an expansion
$$\varphi = \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \varphi_n,$$
where $\varphi_n : V \to V'$ is a differential operator of order $\leq n$ over the morphism $S(U) \to V'$ induced by the zero linear map $U \xrightarrow{0} V'$, *i.e.*, φ_n maps $S^{>n}(U)$ to 0.

We will use the same notation for the continuous, $k[[\hbar]]$-linear extension $\varphi : V[[\hbar]] \to V'[[\hbar]]$ of the k-linear map $\varphi : V \to V'[[\hbar]]$, as well as for the corresponding BV_∞-morphism $\varphi : (V, \Delta) \to (V', \Delta')$.

Example 3.3.4. An example of a BV_∞-morphism $S(V) \to V$ may be obtained from the projection $p_1 : S(V) \to V$ of the symmetric algebra $S(V)$ to its linear component $V = S^1(V)$ for any BV_∞-algebra V. Before talking about morphisms, we need to provide $S(V)$ with the structure of a BV_∞-algebra. To do that, we take the L_∞ structure on $V[[\hbar]][1]$ over $k[[\hbar]]$ given by the brackets L_n, see (3.3), and then the BV_∞ structure on $S(V)$ from the remark at the end of Section 3.1. To regard p_1 as a BV_∞-morphism, we compose it with the inclusion $V \subset V[[\hbar]]$ and get a k-linear map $\varphi = \varphi_1 : S(V) \to V[[\hbar]]$. By construction, $\varphi(1) = 0$. One can easily check that $\exp(\varphi) = m$, the multiplication operator $S(V) \to V$. To see that $\exp(\varphi)$ commutes with the BV_∞ operators, we observe that, for $a_1, \ldots, a_n \in V$, the value of the BV_∞ operator coming from the brackets L_j on the product $a_1 \otimes \cdots \otimes a_n \in S(V)$ is equal to

$$\sum_{j=1}^{n} \hbar^{j-1} \sum_{\sigma \in \mathrm{Sh}_{j,n-j}} (-1)^{|a_\sigma|} L_j(a_{\sigma(1)}, \ldots, a_{\sigma(j)}) \otimes a_{\sigma(j+1)} \otimes \cdots \otimes a_{\sigma(n)},$$

because of Equations (2.2) and (3.1). When we apply m to that, the tensor product (multipliciation in $S(V)$) will change to multiplication in V. The result will just be equal to $(\Delta m)(a_1 \otimes \cdots \otimes a_n)$ in view of Equation (3.5).

Remark. A BV_∞-morphism can be regarded as a quantization of a morphism of dg commutative algebras. Indeed, φ_1 must be nonzero only on $U = S^1(U) \subset S(U)$ and by construction $\exp(\varphi_1)$ will be a graded algebra morphism. The equation $\exp(\varphi)\Delta = \Delta' \exp(\varphi)$ at $\hbar = 0$ reduces to $\exp(\varphi_1)\Delta_1 = \Delta'_1 \exp(\varphi_1)$, which implies that $\exp(\varphi_1)$ is a morphism of dg algebras with respect to the "classical limits" Δ_1 and Δ'_1 of the BV_∞ operators.

Another feature of a BV_∞-morphism $\varphi : S(U)[[\hbar]] \to V'[[\hbar]]$ is that it propagates solutions $S \in \lambda U((\hbar))^2[[\lambda]]$ of the *Quantum Master Equation* (*QME*)

$$\Delta e^{S/\hbar} = 0 \tag{3.7}$$

to solutions of the QME in $\lambda V'((\hbar))^2[[\lambda]]$.

Proposition 3.3.5. *If* $\varphi : S(U) \to V'$ *is a* BV_∞*-morphism and* $S \in \lambda U((\hbar))^2[[\lambda]]$ *is a solution of the QME* (3.7)*, then*

$$S' := \hbar\varphi(e^{S/\hbar}) \in \lambda V'((\hbar))^2[[\lambda]]$$

is a solution of the QME

$$\Delta' e^{S'/\hbar} = 0.$$

Proof. By Lemma 3.3.2 we have $e^{\varphi(e^{S/\hbar})} = \exp(\varphi)(e^{S/\hbar})$. Since $\exp(\varphi)$ must respect the BV_∞ operators, we get

$$\Delta' e^{\varphi(e^{S/\hbar})} = \Delta' \exp(\varphi)(e^{S/\hbar}) = \exp(\varphi)\Delta(e^{S/\hbar}) = 0.$$

$\square$

Now we are ready to study functorial properties of the correspondence between L_∞-algebras and BV_∞-algebras from Theorem 3.2.5. Since the BV_∞-algebra corresponding to an L_∞-algebra is pure, we would like to concentrate on BV_∞-morphisms between such BV_∞-algebras. Among these BV_∞-morphisms, those of the following type turn out to form an interesting category.

Definition 3.3.6. We will call a BV_∞-morphism $\varphi : S(U) \to S(U')$ between BV_∞-algebras which are free as graded commutative algebras *pure*, if φ_n maps $S^n(U)$ to U' and all other symmetric powers $S^k(U)$ to 0. In other words, one can say that φ_n is a *differential operator of order n with linear coefficients, relative* with respect to the morphism $S(U) \to S(U')$ induced by the zero map $U \xrightarrow{0} S(U')$.

BV_∞*-algebras which are free as graded commutative algebras* form a *category* under pure BV_∞-morphisms in the following way. Given pure BV_∞-morphisms $V \xrightarrow{\varphi} V' \xrightarrow{\psi} V''$, their composition $\psi \diamond \varphi : V \to V''$ is defined by composing their exponentials:

$$\psi \diamond \varphi := \log(\exp(\psi) \circ \exp(\varphi)).$$

Under this composition, the role of identity morphism on $S(U)$ is played by $\varphi = \varphi_1 = \mathrm{id}_U$: in this case, $\exp(\varphi) = \mathrm{id}_{S(U)}$.

Proposition 3.3.7. *The composition $\psi \diamond \varphi$ of any pure* BV_∞*-morphisms is a pure* BV_∞*-morphism.*

Proof. First of all, we need to see that the properties (1)-(3) of a BV_∞-morphism are satisfied. Property (1) is satisfied because of our conventions on the values of exponentials and logarithms of maps at 1. Property (2) is obvious by construction. Property (3) may be established from the formula

$$\begin{aligned}(\psi \diamond \varphi)(x_1 \dots x_m) &= \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1\\ i_1+\dots+i_k=m}}^{m} \frac{1}{i_1! \dots i_k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \psi(\varphi(x_{\sigma(1)} \dots x_{\sigma(i_1)}) \dots \\ &\qquad \varphi(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)})), \qquad (3.8)\end{aligned}$$

which is easily verified by exponentiating it and comparing it to $\exp(\psi) \circ \exp(\varphi)$. Indeed, the coefficient $(\psi \diamond \varphi)_n(x_1 \dots x_m)$ by $\hbar^{n-1}$ on the right-hand side will be coming from terms

$$\psi_j(\varphi_{j_1}(x_{\sigma(1)}\dots x_{\sigma(i_1)})\dots\varphi_{j_k}(x_{\sigma(m-i_k+1)}\dots x_{\sigma(m)}))$$

with $j-1+\sum_{p=1}^k(j_p-1) = j-1+\sum_{p=1}^k j_p - k = n-1$. Observe that because of Property (3) for ψ and φ, for such a term not to vanish, it is necessary that $j \geq k$ and $j_p \geq i_p$ for each p. Thus, we will have $n = j+\sum_{p=1}^k j_p - k \geq k + \sum_{p=1}^k i_p - k = m$, which is Property (3) for $\psi \diamond \varphi$. The fact that the composite BV_∞-morphism is pure is obvious from Eq. (3.8) and purity of ψ. □

Theorem 3.3.8. *The correspondence $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ of Section 3.1 from L_∞-algebras to BV_∞-algebras is functorial. This functor establishes an equivalence between the category of L_∞-algebras and the full subcategory of pure BV_∞-algebras of the category of BV_∞-algebras free as graded commutative algebras with pure morphisms. The functor $V = S(U) \mapsto U[1]$ of Theorem 3.2.5(1) provides a weak inverse to this equivalence.*

Restricting this to the case of dg Lie algebras and dg BV algebras, we obtain the following corrolaries.

Corollary 3.3.9. *The functor $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ from dg Lie algebras to dg BV algebras establishes an equivalence between the category of dg Lie algebras with L_∞-morphisms and the category of dg BV algebras (V, Δ_1, Δ_2), free as graded commutative algebras $V = S(U)$ and whose BV structure is pure: Δ_2 maps U to 0 and $S^2(U)$ to U, with BV_∞-morphisms $S(U) \to S(U')$ satisfying the purity condition.*

Corollary 3.3.10. *The functor $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ from dg Lie algebras to dg BV algebras establishes an equivalence between the category of dg Lie algebras (with strict, or linear, morphisms) and the category of dg BV algebras (V, Δ_1, Δ_2), free as graded commutative algebras $V = S(U)$ and whose BV structure is pure: Δ_2 maps U to 0 and $S^2(U)$ to U, with morphisms defined as morphisms $\Phi : S(U) \to S(U')$ of graded algebras respecting the differentials Δ_1 and Δ_2 and satisfying the purity condition: Φ maps U to U'.*

Now let us prove the theorem.

Proof. We need to see that an L_∞-morphism $\mathfrak{g} \to \mathfrak{g}'$ induces a BV_∞-morphism $S(\mathfrak{g}[-1]) \to S(\mathfrak{g}'[-1])$. By definition an L_∞-morphism is graded coalgebra morphism $\Phi : S(\mathfrak{g}[1]) \to S(\mathfrak{g}'[1])$ respecting the codifferentials and such that $\Phi(1) = 1$. As a coalgebra morphism,

Φ is determined by its projection $\varphi : S(\mathfrak{g}[1]) \to \mathfrak{g}'[1]$ to the cogenerators $\mathfrak{g}'[1]$ via the following formula:

$$\Phi(x_1 \dots x_m) = \\ \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1\\ i_1+\dots+i_k=m}}^{m} \sum_{\sigma \in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_\sigma|} \varphi(x_{\sigma(1)} \dots x_{\sigma(i_1)}) \dots \varphi(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)}), \quad (3.9)$$

where $\mathrm{Sh}_{i_1,\dots,i_k}$ denotes the set of $(i_1,\dots,i_k)$ shuffles, $x_1,\dots,x_m$ are in $\mathfrak{g}[1]$, and $(-1)^{|x_\sigma|}$ is the Koszul sign of the permutation of $x_1 \dots x_m$ to $x_{\sigma(1)} \dots x_{\sigma(m)}$ in $S(\mathfrak{g}[1])$. (For $m = 0$, we just have $\Phi(1) = 1$ and $\varphi(1) = 0$.) The above formula follows from iteration of the coalgebra morphism property:

$$\delta^{k-1}\Phi = \Phi^{\otimes k}\delta^{k-1}$$

along with its projection to $(\mathfrak{g}'[1])^{\otimes k}$ for each $k = 1,\dots,m$. To turn φ into a BV_∞-morphism, we need to rewrite it as a power series in $\hbar$:

$$\varphi_\hbar := \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \varphi_n, \quad (3.10)$$

where $\varphi_n : S(\mathfrak{g}[-1]) \to S(\mathfrak{g}'[-1])$ maps all symmetric powers to 0, except for $S^n(\mathfrak{g}[-1])$, on which φ_n is the restriction of $\varphi : S(\mathfrak{g}[1]) \to \mathfrak{g}'[1]$ to $S^n(\mathfrak{g}[1])$ along with an appropriate shift in degree to make it into a linear map $S^n(\mathfrak{g}[-1]) \to \mathfrak{g}'[-1]$. Note that the degree of φ was supposed to be zero, as it was a projection of the morphism Φ of graded coalgebras. In terms of grading on $S^n(\mathfrak{g}[-1])$ and $\mathfrak{g}'[-1]$, the degree of shifted φ_n is $2-2n$. Multiplication by $\hbar^{n-1}$ shifts that degree back to 0, thus we see that the degree of $\varphi_\hbar$ is zero as well.

Note that by construction, the purity condition on $\varphi_\hbar$ is satisfied, and thereby we have

$$\exp(\varphi_\hbar)(x_1 \dots x_m) \\ = \sum_{k=1}^{m} \frac{\hbar^{m-k}}{k!} \sum_{\substack{i_1,\dots,i_k=1\\ i_1+\dots+i_k=m}}^{m} \frac{1}{i_1! \dots i_k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \varphi(x_{\sigma(1)} \dots x_{\sigma(i_1)}) \dots \\ \varphi(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)}),$$

whence, comparing this to the right-hand side of (3.9), we get

$$\exp(\varphi_\hbar) = \sum_{m=0}^{\infty} \hbar^m \Phi_m,$$

where Φ_m is the component of Φ of degree $-m$ in the grading given by the symmetric power, so as

$$\Phi = \sum_{m=0}^{\infty} \Phi_m.$$

We know that Φ is compatible with the structure codifferentials D and D' of $\mathfrak{g}$ and $\mathfrak{g}'$: $\Phi D = D'\Phi$. The BV$_\infty$ operator on $S(\mathfrak{g}[-1])$ was defined as $\Delta = \sum_{m=1}^{\infty} \hbar^{m-1} D_m$, where D_m maps each $S^n(\mathfrak{g}[1])$ to $S^{n-m+1}(\mathfrak{g}[1])$; likewise for $S(\mathfrak{g}'[-1])$, see (3.1). Thus, the equation $\exp(\varphi_\hbar)\Delta = \Delta' \exp(\varphi_\hbar)$ is satisfied, being just a weighted sum of the components of the equation $\Phi D = D'\Phi$, where the component shifting the symmetric power down by $n \geq 0$ is being multiplied by $\hbar^n$. This completes verification of the fact that $\varphi_\hbar$ is a pure BV$_\infty$-morphism.

Conversely, we need to see that every pure BV$_\infty$-morphism comes from an L_∞-morphism. By Theorem 3.2.5 we can assume that the source and the target of this BV$_\infty$-morphism are the BV$_\infty$-algebras $S(\mathfrak{g}[-1])$ and $S(\mathfrak{g}'[-1])$ coming from some L_∞-algebras $\mathfrak{g}$ and $\mathfrak{g}'$. Every BV$_\infty$-morphism is given by a formal $\hbar$-series like (3.10) satisfying the three conditions of Definition 3.3.3. Since the morphism is pure, we can "drop" the $\hbar$ from $\varphi_\hbar$ and note that the formal series

$$\varphi := \sum_{n=1}^{\infty} \varphi_n$$

will produce a well-defined linear map $S(\mathfrak{g}[1]) \to \mathfrak{g}'[1]$. Dropping the $\hbar$ results in this map also having degree zero. Now we can generate a unique morphism $\Phi : S(\mathfrak{g}[1]) \to S(\mathfrak{g}'[1])$ of coalgebras by the linear map φ. This morphism Φ will be given by formula (3.9). Since $\varphi_\hbar(1) = 0$, we get $\varphi(1) = 0$ and $\Phi(1) = 1$ by the same formula. We just need to check that this morphism Φ respects the codifferentials D and D' on these two coalgebras, respectively. As in the first part of the proof, we see that the equation $\exp(\varphi_\hbar)\Delta = \Delta' \exp(\varphi_\hbar)$ implies $\Phi D = D'\Phi$. Thus, Φ is an L_∞-morphism.

We also need to check the functoriality properties of the correspondence $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$. The fact that $\mathrm{id}_{\mathfrak{g}}$ maps to the identity morphism is obvious. Now, if we have two L_∞-morphisms $\mathfrak{g} \to \mathfrak{g}' \to \mathfrak{g}''$ given by dg coalgebra morphisms $S(\mathfrak{g}[1]) \xrightarrow{\Phi} S(\mathfrak{g}'[1]) \xrightarrow{\Psi}$

$S(\mathfrak{g}''[1])$ with $\Phi = \sum_{m=0}^{\infty} \Phi_m$ and $\Psi = \sum_{m=0}^{\infty} \Psi_m$, we note that the exponentials $\exp(\varphi_\hbar) = \sum_{m=0}^{\infty} \hbar^m \Phi_m$ and $\exp(\psi_\hbar) = \sum_{m=0}^{\infty} \hbar^m \Psi_m$ of the respective BV_∞-morphisms will compose in the same way as Φ and Ψ, the only difference being that the component decreasing the symmetric power by m gets multiplied by $\hbar^m$. $\square$

3.4 Adjunction

In this section, we establish a certain bijection on . The quotation marks are due to the fact that in our setting, arbitrary BV_∞-algebras do not even make up a category. However, the theorem below makes sense for arbitrary BV_∞-algebras and BV_∞-morphisms.

Recall that given an L_∞-algebra $\mathfrak{g}$, we have constructed a BV_∞-algebra $S(\mathfrak{g}[-1])$ in Section 3.1. Conversely, given a BV_∞-algebra V, we have used the higher derived brackets L_n to induce an L_∞-stucture on $V[[\hbar]][1]$ over $k[[\hbar]]$ as in Corollary 3.2.2.

Note that both constructions are functorial. The fact that $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ defines a functor is the first statement of Theorem 3.3.8. We need to see that the construction assigning to a BV_∞-algebra V the L_∞-algebra $(V[[\hbar]][1], L_n)$ is also functorial. Given a BV_∞-morphism $\varphi : V = S(U) \to V'$, we need to construct an L_∞-morphism $V[[\hbar]][1] \to V'[[\hbar]][1]$. This construction will be accomplished in two steps.

Step 1. Compose the BV_∞-morphism $\varphi : V \to V'$ with the BV_∞-morphism $p_1 : S(V) \to V$ of Example 3.3.4 to get a BV_∞-morphism $\varphi \diamond p_1 : S(V) \to V'$.

Step 2. Given an L_∞-algebra $\mathfrak{g}[[\hbar]]$ over $k[[\hbar]]$ and a BV_∞-morphism $\psi : S(\mathfrak{g}[-1]) \to V'$, where $S(\mathfrak{g}[-1])$ is provided with the BV_∞ structure of the remark at the end of Section 3.1, we will construct a canonical L_∞-morphism $\mathfrak{g}[[\hbar]] \to V'[[\hbar]][1]$. Then we will just apply this construction to the BV_∞-morphism $S(V) \to V'$ of Step 1.

In order to construct an L_∞-morphism $\mathfrak{g}[[\hbar]] \to V'[[\hbar]][1]$, take the graded $k[[\hbar]]$-coalgebra morphism, continuous in the $\hbar$-adic topology,

$$F : S(\mathfrak{g}[1])[[\hbar]] \to S(V'[2])[[\hbar]]$$

induced by the $k[[\hbar]]$-linear map

$$f : S(\mathfrak{g}[1])[[\hbar]] \to V'[[\hbar]][2]$$

whose restriction $f|_{S^k(\mathfrak{g}[1])[[\hbar]]} : S^k(\mathfrak{g}[1])[[\hbar]] \to V'[[\hbar]][2]$ is the restriction of $\hbar^{1-k}\psi$ to $S^k(\mathfrak{g}[1])[[\hbar]$ for $k \geq 0$:

$$f|_{S^k(\mathfrak{g}[1])[[\hbar]]} = \hbar^{1-k}\psi|_{S^k(\mathfrak{g}[1])[[\hbar]]}.$$

This map takes values in $V'[[\hbar]][2]$, despite the division by a power of $\hbar$, because the restriction of ψ to $S^k(\mathfrak{g}[-1])$ is in fact equal to $\sum_{n=k}^{\infty} \hbar^{n-1}\psi_n = \hbar^{k-1}\sum_{n=0}^{\infty} \hbar^n \psi_{n+k}$. Note that since ψ is of degree zero, f will also have degree zero.

We need to check that F defines an L_∞-morphism. It is easy to see that $F(1) = 1$, because $\psi(1) = 0$. What is far less trivial is the fact that F respects the codifferentials, the structure codifferential D on $S(\mathfrak{g}[1])[[\hbar]]$ and the codifferential D' on $S(V'[2])[[\hbar]]$ induced as a continuous coderivation, see (2.2), by the sum of the brackets (3.3):

$$L_n : S^n(V'[2])[[\hbar]] \to V'[[\hbar]][2].$$

What we know is $\Delta' \exp(\psi) = \exp(\psi)\Delta$, where Δ' is the BV_∞ operator on V' and Δ is the structure codifferential D on $S(\mathfrak{g}[1])[[\hbar]]$ enhanced by $\hbar$, as in the remark at the end of Section 3.1. To see that this implies the equation $D'F = FD$, we need to develop some BV calculus and compare it to colagebra calculus.

Let us start with coalgebra calculus. Each side of the equation is a continuous coderivation over the coalgebra morphism F and as such determined by the projection $p_1 : S(V'[2])[[\hbar]] \to V'[[\hbar]][2]$ to the cogenerators $V'[[\hbar]][2]$ of the range. Thus, all we need to show is that $p_1 D'F = p_1 FD$, after projecting to the cogenerators. Now, for a monomial $x_1 \dots x_m \in S^m(\mathfrak{g}[1])$, we have

$$\begin{aligned} p_1 D'F(x_1 \dots x_m) \\ = \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1+\dots+i_k=m}}^{m} \sum_{\sigma\in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_\sigma|} L_k(f(x_{\sigma(1)} \dots x_{\sigma(i_1)}), \dots, \\ f(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)})), \end{aligned} \tag{3.11}$$

using the shuffle notation, see Equation (3.9), as well as

$$p_1 FD(x_1 \dots x_m) = f(D(x_1 \dots x_m)). \tag{3.12}$$

We need to show that the right-hand sides of these equations are equal, based on the equation $\Delta' \exp(\psi) = \exp(\psi)\Delta$. We will do that after we develop some BV calculus.

Turning to BV calculus, we have

$$\Delta(x_1 \dots x_m) = \sum_{k=1}^{m} \hbar^{k-1} \sum_{\tau \in \mathrm{Sh}_{k,m-k}} (-1)^{|x_\tau|} l_k(x_{\tau(1)}, \dots, x_{\tau(k)}) x_{\tau(k+1)} \dots x_{\tau(m)}, \quad (3.13)$$

where l_k's are the L_∞ brackets on $\mathfrak{g}$, because of Equation (2.2). Now apply $\exp(\psi)$ to both sides, reassemble products of ψ's not containing l_k's into $\exp(\psi)$, and use (3.13) again to pass from l_k's back to Δ and get

$$\exp(\psi)\Delta(x_1 \dots x_m) = \sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} \psi(\Delta(x_{\sigma(1)} \dots x_{\sigma(n)})) \exp(\psi)(x_{\sigma(n+1)} \dots x_{\sigma(m)}). \quad (3.14)$$

Move on to computation of $\Delta' \exp(\psi)$:

$$\Delta' \exp(\psi)(x_1 \dots x_m) = \sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1+\dots+i_k=n}}^{n} \sum_{\tau \in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_{\tau\sigma}|} l_k^{\hbar}(\psi(x_{\tau\sigma(1)} \dots x_{\tau\sigma(i_1)}), \dots, \psi(x_{\tau\sigma(n-i_k+1)} \dots x_{\tau\sigma(n)})) \exp(\psi)(x_{\sigma(n+1)} \dots x_{\sigma(m)}), \quad (3.15)$$

which follows from the definition of $\exp(\psi)$ and the identity (3.5).

Now let us compare (3.14) with (3.15), which are equal by assumption. One can show by induction on m that the top, $n = m$ terms of the two formulas must also be equal:

$$\psi(\Delta(x_1 \dots x_m)) = \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1+\dots+i_k=m}}^{m} \sum_{\tau \in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_\tau|} l_k^{\hbar}(\psi(x_{\tau(1)} \dots x_{\tau(i_1)}), \dots, \psi(x_{\tau(m-i_k+1)} \dots x_{\tau(m)})).$$

It remains to pass from ψ, Δ, and $l_k^\hbar$ to f, D, and L_k, respectively, in this equation, with appropriate powers of $\hbar$, resulting in the equation

$$\begin{aligned}
&\hbar^{m-1} f(D(x_1 \dots x_m)) \\
&\qquad = \hbar^{m-1} \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1\\ i_1+\dots+i_k=m}}^{m} \sum_{\tau \in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_\tau|} L_k(f(x_{\tau(1)} \dots x_{\tau(i_1)}), \dots, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f(x_{\tau(m-i_k+1)} \dots x_{\tau(m)})).
\end{aligned}$$

In view of (3.11) and (3.12), we see that $D'F = FD$. This completes Step 2.

Theorem 3.4.1. *Suppose $\mathfrak{g}$ is an L_∞-algebra and V is a* BV_∞*-algebra. There exists a canonical bijection*

$$\mathrm{Hom}_{\mathrm{BV}_\infty}(S(\mathfrak{g}[-1]), V) \cong \mathrm{Hom}_{L_\infty}(\mathfrak{g}, V[[\hbar]][1]),$$

where the L_∞-structure on $V[[\hbar]][1]$ is given by the modified brackets L_n. This bijection is natural in the L_∞-algebra $\mathfrak{g}$ and in the BV_∞*-algebra V.*

Proof. A correspondence from the BV_∞-morphisms on the left-hand side to the L_∞-morphisms on the right-hand side was constructed in Step 2 before the theorem in a more general case of an L_∞-algebra over $k[[\hbar]]$.

Conversely, given an L_∞-morphism $F : S(\mathfrak{g}[1]) \to S(V[2])[[\hbar]]$, we use the same conversion formula

$$\varphi|_{S^k(\mathfrak{g}[1])[[\hbar]]} = \hbar^{k-1} f|_{S^k(\mathfrak{g}[1])[[\hbar]]}, \tag{3.16}$$

f being the projection of F to the cogenerators $V[2][[\hbar]]$, for $k \geq 0$, as in Step 2 before the theorem, to get a BV_∞-morphism $\varphi : S(\mathfrak{g}[-1]) \to V$. Tracing the argument there backward, we see that φ is indeed a BV_∞-morphism. This establishes a bijection in the adjunction formula.

The naturality of the construction follows from the fact that, in view of (3.16), F and $\exp(\varphi)$ are given by almost identical formulas, with the only difference coming from insertion of powers of $\hbar$, which plays the role of grading shift. $\square$

Corollary 3.4.2. *The functor $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ of Section 3.1 from the category of L_∞-algebras to the category of* BV_∞*-algebras free as graded commutative algebras with pure* *morphisms has a right adjoint, which is given by the functor of modified higher derived brackets* L_n.

Chapter 4

On quantization of L_∞-bialgebras

In this chapter we develop the notion of a *triangular L_∞-bialgebra* and discuss the notion of its *quantization*. We begin by recalling the basic definition and facts concerning quantization of ordinary Lie bialgebras and then then propose a generalization of these constructions to the homotopy Lie case.

Definition 4.0.3. A *Lie bialgebra* is a vector space $\mathfrak{g}$ equipped with a Lie bracket $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and a Lie *cobracket* $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$; that means, δ is skew-symmetric linear map such that

1. $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on the dual space $\mathfrak{g}^*$;

2. δ is a 1-cocycle in the (cohomological) Chevalley-Eilenberg complex with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$, that is $\delta([x, y]) = [x, \delta(y)] - [y, \delta(x)]$. This plays the role of a compatibility condition between the bracket and cobracket.

Example 4.0.4. Let X, Y, H be the standard generators of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$: $[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y$. A cobracket turning $\mathfrak{g}$ into a Lie bialgebra can be defined by $\delta(H) = 0$, $\delta(X) = H \wedge X$, $\delta(Y) = H \wedge Y$.

Example 4.0.5. Recall that a Lie group G is said to be *Poisson-Lie* if it has a Poisson structure and the multiplication map $G \times G \to G$ is a morphism of Poisson manifolds (a Poisson bivector π on G induces a canonical Poisson structure $\pi \oplus \pi$ on $G \times G$). Since tangent bundle of a Lie group is trivializable, we can write $TG = G \times \mathfrak{g}$, where $\mathfrak{g} = Lie(G)$ and a Poisson structure on G is $\pi : G \to \mathfrak{g} \otimes \mathfrak{g}$. Take $\delta := (d\pi)_e : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$.

Then $\mathfrak{g}$ equipped with δ is a Lie bialgebra. Moreover, all finite-dimensional Lie bialgebras (over $\mathbb{R}, \mathbb{C}$) appear in this way:

Theorem 4.0.6. *([Dri87]) Categories of finite-dimensional Lie bialgebras (over $\mathbb{R}, \mathbb{C}$) and connected, simply connected Poisson-Lie groups are equivalent.*

Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. We identify r with the zero cocycle $k \to \mathfrak{g} \otimes \mathfrak{g}, 1 \mapsto r$ in the cohomological Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$ and consider its coboundary $\varphi = d_{CE}(r) : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, $\varphi(X) = [X, r]$.

Proposition 4.0.7. *A cobracket $\varphi = d_{CE}(r)$ gives $\mathfrak{g}$ a Lie bialgebra structure if and only if*

1. *$r_{12} + r_{21}$ is $\mathfrak{g}$-invariant;*
2. *$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$ is $\mathfrak{g}$-invariant.*

We utilize the standard Sweedler's notation here. For instance, if $r = \sum_i a_i \otimes b_i$, then $r_{21} = \sum_i b_i \otimes a_i$, $r_{13} = \sum_i a_i \otimes 1 \otimes b_i \in U(\mathfrak{g})^{\otimes 3}$ etc.

Definition 4.0.8. Lie bialgebras with cobrackets of such form are called *coboundary bialgebras.* A coboundary Lie bialgebra $\mathfrak{g}$ is called *quasitriangular* if the *classical Yang-Baxter equation* $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ holds. If, in addition, r is skew-symmetric, i.e. $r_{12} + r_{21} = 0$, then $\mathfrak{g}$ is said to be *triangular.*

Solutions of the classical Yang-Baxter equation are traditionally called *r-matrices* (the terminology comes from the case $\mathfrak{g} = End(V)$).

The classical Yang-Baxter equation can be written in the form of the Maurer-Cartan equation

$$\{r, r\} = 0,$$

where $\{,\}$ is the Schouten bracket on $S(\mathfrak{g}[-1])$ defined by $\{a, b\} = [a, b]$ for $a, b \in \mathfrak{g}$ and extended further by the graded skew-symmetry and the graded Leibniz rule $\{a, b \wedge c\} = \{a, b\} \wedge c + (-1)^{(|a|+1)\cdot|b|} b \wedge \{a, c\}$.

Recall that for a Lie algebra $\mathfrak{g}$, its universal enveloping algebra $A = U(\mathfrak{g})$ is actually a *Hopf algebra.* That is, it is a unital associative algebra equipped with a coassociative comultiplication $\Delta : A \to A \otimes A$, counit $\epsilon : A \to k$ and an antipode map $S : A \to A$ such that comultiplication and counit are homomorphisms of algebras and the antipode S is the two-sided inverse to id_A with respect to the convolution product $f*g = m\circ(f\otimes g)\circ\Delta$, where $m : A\otimes A \to A$ is the multiplication; that is, $\eta\circ\epsilon = m\circ(S\otimes id_A)\circ\Delta$, $m\circ(id_A\otimes S)\circ\Delta$, where $\eta : k\to A$ is the unit of A.

In case of $U(\mathfrak{g})$, the comultiplication, antipode and counit are defined by $\Delta(x) = 1\otimes x + x\otimes 1$, $S(x) = -x$ and $\epsilon(x) = 0$. Furthermore, if $\mathfrak{g}$ is a Lie bialgebra, then $U(\mathfrak{g})$ acquires some additional structure:

Proposition 4.0.9. *Lie cobracket $\delta : \mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ extended to $\delta : U(\mathfrak{g})\to U(\mathfrak{g})\otimes U(\mathfrak{g})$ via $\delta(a_1a_2) = \delta(a_1)\Delta(a_2) + \Delta(a_1)\delta(a_2)$ is a co-Poisson bracket. That means, besides being a coderivation, it also satisfies the co-Leibniz identity*

$$(\Delta\otimes id)\circ\delta = (id\otimes\delta)\Delta + \sigma_{23}(\delta\otimes id)\Delta$$

and the co-Jacobi identity: the composition

$$A \xrightarrow{\delta} A\otimes A \xrightarrow{\delta\otimes id} A\otimes A\otimes A \xrightarrow{cyc.perm.} A\otimes A\otimes A$$

is zero.

Definition 4.0.10. A *deformation* of a Hopf algebra A over a field k is a topological Hopf algebra $A_\hbar$ over the ring $k[[\hbar]]$, where $\hbar$ is a formal variable, such that

1. $A_\hbar \simeq A[[\hbar]]$ as a $k[[\hbar]]$-module;

2. $m_\hbar = m\,(mod\,\hbar)$, $\Delta_\hbar = \Delta\,(mod\,\hbar)$, where m's and Δ's are multiplications and comultiplications of A and $A_\hbar$ respectively.

We are ready now to formulate what a quantization of a Lie bialgebra is.

Definition 4.0.11. A *quantization of a Lie bialgebra* $\mathfrak{g}$ is a deformation $U_\hbar(\mathfrak{g})$ of its universal enveloping algebra $U(\mathfrak{g})$ as a Hopf algebra such that the canonical co-Poisson bracket δ on $U(\mathfrak{g})$ is recovered in the "classical limit":

$$\delta(x) = \frac{\Delta(a) - \Delta^{op}(a)}{\hbar}\,(mod\,\hbar),$$

where $x\in U(\mathfrak{g})$ and $a\in U_\hbar(\mathfrak{g})$, $a = x\,(mod\,\hbar)$.

Theorem 4.0.12. *([Dri87, Dri83]) Let* $\mathfrak{g}$ *be a finite-dimensional real Lie algebra and* $r \in \mathfrak{g} \otimes \mathfrak{g}$ *be a skew-symmetric element satisfying* $\{r, r\} = 0$*. Then* $\mathfrak{g}$ *admits a quantization* $U_\hbar(\mathfrak{g})$ *with* $U_\hbar(\mathfrak{g}) \simeq U(\mathfrak{g})[[\hbar]]$ *as* $\mathbb{R}[[\hbar]]$*-algebras and, moreover,* $U_\hbar(\mathfrak{g})$ *is a triangular Hopf alebra.*

The latter means that $U_\hbar(\mathfrak{g})$ is a cocommutative (i.e. $\Delta = \Delta^{op}$) and there is an invertible element $R \in U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})$ such that

1. $(\Delta \otimes id)(R) = R_{13}R_{23}$, $(id \otimes \Delta)(R) = R_{13}R_{12}$

2. $R_{21} = R^{-1}$

If a Hopf algebra is not cocommutative, but rather satisfies $R\Delta R^{-1} = \Delta^{op}$ it is said to be *quasitriangular*. An element R satisfying condition 1 is called a *universal* r*-matrix*. It satisfies the *quantum Yang-Baxter equation* $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, which made its first appearance in statistical physics.

Remark. Conditions of being triangular or quasitriangular are of representation-theoretic nature. Given a Hopf algebra A, the category of A-modules acquires a monoidal structure via $A \xrightarrow{\Delta} A \otimes A \to U \otimes_k V$ with a counit $A \to k$ as a unit object. If, in addition, we have a universal r-matrix $R \in A \otimes A$, then a braiding can be put on $A - mod$ via $c_{U,V} : U \otimes V \to V \otimes U$, $u \otimes v \mapsto R_2 v \otimes R_1 u$ with the hexagon identities following from the quantum Yang-Baxter equation. In the triangular case, $R^{-1} = R_{21}$ implies $c_{U,V} \circ c_{V,U} = id_{U \otimes V}$. Thus, $A - mod$ becomes symmetric monoidal.

Our goal is to find an analog of the construction

Lie algebra $\mathfrak{g}$ plus a solution of the MC equation $\{r, r\} = 0, r \in \mathfrak{g} \otimes \mathfrak{g}$ r is skew-symmetric	$\rightsquigarrow$	A triangular Lie bialgebra $(\mathfrak{g}, [,], \delta = d_{CE}(r))$	$\rightsquigarrow\rightsquigarrow$	Co-Poisson Hopf algebra $U(\mathfrak{g})$	$\rightsquigarrow$	Quantization $U_\hbar(\mathfrak{g})$

in the homotopy Lie setup.

4.1 The big bracket algebra

Let V be a graded vector space. We take γ to be the composition $S(V[1]) \stackrel{\delta}{\to} S(V[1]) \otimes S(V[1]) \stackrel{pr_1 \otimes id}{\to} V[1] \otimes S(V[1])$, where δ is the standard shuffle comultipication on the cofree cocommutative coalgebra $S(V[1])$. Invoking Sweedler's notation, we define for $f \in \mathrm{Hom}_k(S^m(V[1]), S^n(V[1]))$, $g \in \mathrm{Hom}_k(S^p(V[1]), S^q(V[1]))$ and $x \in S(V[1])$,

$$(f \circ_1 g)(x) := (-1)^{|x_{(1)}| \cdot |g|} f(x_{(1)} \cdot \gamma_{(1)}(g(x_{(2)})))\gamma_{(2)}(g(x_{(2)})), \quad \delta(x) = x_{(1)} \otimes x_{(2)}$$

whenever $q \geq 1$ and set $(f \circ_1 g)(x) = 0$ otherwise. Here, Hom's are assumed to be graded spaces of homogeneous k-linear mappings. The following input-output diagram illustrates this operation:

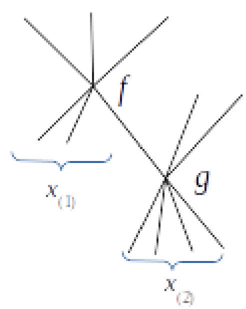

Figure 4.1: The $\circ_1$-operation.

The *big bracket algebra* $\mathcal{B}$ on V is defined on $\prod_{m,n \geq 0} \mathrm{Hom}_k(S^m(V[1]), S^n(V[1]))$ by giving it a degree zero (graded) Lie bracket

$$[\![f, g]\!] = f \circ_1 g - (-1)^{|f| \cdot |g|} g \circ_1 f.$$

Remark. The definition of the big bracket is due to Y.Kosmann-Schwarzbach [KS92, KS04], who, in particular, interpreted it as the canonical Poisson bracket on the algebra of functions on a symplectic supermanifold $T^*(V^*[-1])$.

The significance of this structure is due to the fact that a variety of Lie-related structures on V can be encoded in the form of Maurer-Cartan elements in $\mathcal{B}$:

Proposition 4.1.1. *([Kra07]) Let V be concentrated in degree zero.*

1. *A degree* $+1$ *element* $l \in Hom(S^2(V[1]), V[1])$ *such that* $[\![l, l]\!] = 0$ *determines a Lie algebra structure on* V *via* $\{x, y\} = [\![[\![l, x]\!], y]\!]$.

2. *A degree* -1 *elements* $c \in Hom(V[1], S^2(V[1]))$ *such that* $[\![c, c]\!] = 0$ *determines a Lie coalgebra structure on* V *via* $\delta(x) = [\![c, x]\!]$.

3. *Elements* $l, c \in \mathcal{B}$ *as above determine a Lie bialgebra structure on* V *provided that* $[\![c + l, c + l]\!] = 0$.

Proposition 4.1.2. *([KS92]) Let* $\mathfrak{g}$ *be a Lie algebra with an element* $l \in \mathcal{B}$ *determining its structure. For a fixed* p*, A the cohomological Chevalley-Eilgenberg complex with coefficients in* $S^p(\mathfrak{g}[-1])$ *is a subspace of* $\mathcal{B}$ *on* $\mathfrak{g}$ *and, moreover,* $d_{CE} = [\![l, -]\!]$.

An analogous characterization of homotopy Lie structures on a graded vector space V in terms of Maurer-Cartan elements in $\mathcal{B}$ is available:

Proposition 4.1.3. *([Kra07]) Let* $L = \sum\limits_{k \geq 1} l_k$*,* $l_k \in Hom(S^k(V[1]), V[1])$ *be such that* $|l_k| = 1$ *for all* k *and* $[\![L, L]\!] = 0$*. Then* V *acquires a* L_∞*-structure with the* k*-th bracket given by* $x_1, \ldots, x_k \mapsto [\![\ldots [\![[\![l_k, x_1]\!], x_2]\!], \ldots, x_k]\!]$

The case of a particular interest for us is a homotopy analog of Lie bialgebras. In the spirit of the above propositions such a structure can be introduced as follows: let $T = \sum\limits_{k,l \geq 1} t_{kl}$, where $t_{kl} \in \mathrm{Hom}(S^k(V[1]), S^l(V[1]))$, $|t_{kl}| = 1$, be such that $[\![T, T]\!] = 0$. Then V equipped with operations $\tau_{kl} : \wedge^k V \to \wedge^l V$ defined via

$$x_1, \ldots, x_k \mapsto [\![\ldots [\![[\![t_{kl}, x_1]\!], x_2]\!], \ldots, x_k]\!]$$

is called a L_∞*-bialgebra*. Here τ_{k1} correspond to L_∞-brackets, τ_{1l} correspond to L_∞-cobrackets and other τ_{k1} represent mixed operations. An analysis of the first few terms of the expansion

$$[\![T, T]\!] = [\![t_{11}, t_{11}]\!] + [\![t_{11}, t_{12}]\!] + [\![t_{11}, t_{21}]\!] + \cdots = 0$$

shows immediately that τ_{11} is a differential $d = [\![t_{11}, -]\!]$ (which is also a derivation of both bracket and cobracket). Moreover, a cobracket τ_{12} is a cocycle with respect to this differential up to homotopy controlled by τ_{22}. Both Jacobi and co-Jacobi identities for τ_{21}, τ_{12} also hold up to homotopy. Thus, the cohomology of V with respect to d is a standard Lie bialgebra.

4.2 r_∞-matrices, coboundary and triangular L_∞-bialgebras

Let V be a L_∞-algebra on a graded vector space V with $l = \sum_{i\geq 1} l_i \in \mathcal{B}$ determining its L_∞-structure. For $j \geq 2$, take $t_j \in \mathrm{Hom}(k, S^j(V[1])) \subset \mathcal{B}$ and let $t = \sum_{j\geq 2} t_j$.

Lemma 4.2.1. *If $|[\![l_i, t_j]\!]| = 1$ for all i, j and $[\![[\![l, t]\!], [\![l, t]\!]]\!] = 0$, then $T = l + [\![l, t]\!]$ determines a homotopy Lie bialgebra structure on $\mathfrak{g}$ with $[\![l, t]\!]$ giving the cobrackets and mixed operations part.*

Proof. The statement follows directly from the graded Jacobi identity that the big bracket obeys, proposition 4.1.3 and definition of the L_∞-bialgebra structure. □

We call homotopy Lie bialgebras of this type *coboundary* in accordance to definition 4.0.15, which is a special case of this construction for V concentrated in degree zero.

Let $\mathfrak{g}$ be a L_∞-algebra. Generalizing the construction of triangular Lie bialgebras outlined earlier in this chapter, we would like to put a L_∞-bialgebra structure on $\mathfrak{g}$ starting from the data of a solution r (an *r_∞-matrix*) of the (generalized) Maurer-Cartan equation

$$l_1(r) + \frac{1}{2!} l_2(r, r) + \frac{1}{3!} l_3(r, r, r) + \cdots = 0$$

in $S(\mathfrak{g}[-1])[1]$. The L_∞-structure that we use here is obtained by first passing from a L_∞-algebra $\mathfrak{g}$ to a BV$_\infty$-algebra $S(\mathfrak{g}[-1])$ (Theorem 3.1.1) and then applying the higher derived brackets method (Corollary 3.2.2).

Lemma 4.2.2. *Let V, W be L_∞-algebras and ϕ be a L_∞-morphism between, that is, a morphism of the corresponding dg cocommutative coalgebras $\phi : (S(V[1]), D_V) \to (S(W[1]), D_W)$. Then any solution $r \in \mathfrak{g}$ of the Maurer-Cartan equation*

$$l_1(r) + \frac{1}{2!} l_2(r, r) + \frac{1}{3!} l_3(r, r, r) + \cdots = 0$$

in V gets mapped to a solution in W.

Proof. Write ϕ as $\phi_1 + \phi_2 + \ldots$, where $\phi_n : S^n(V[1]) \to S(W[1]) \to W[1]$ is the restriction of ϕ onto $S^n(V[1])$ followed by a projection on the cogenerators W of $S(W[1])$. Let r be a solution of the MC equation in V. We denote by e^r the (formal) sum $r + \frac{r \cdot r}{2!} + \frac{r \cdot r \cdot r}{3!} + \ldots$ Since $S(V[1])$ is a cofree cocommutative coalgebra, the value of the coderivation D_V

is determined by its projection to the cogenerator $S(V[1]) \xrightarrow{D_V} S(V[1]) \to V[1]$. Under this projection, $D_V(e^r)$ is precisely the left-hand side of the MC equation (see 2.2). Thus, $D_V(e^r) = 0$. Now, since $D_W\phi = \phi D_V$, we get $D_W(\phi(e^r)) = 0$, where $\phi(e^r) = \phi_1(r) + \frac{\phi_2(r \cdot r)}{2!} + \frac{\phi_3(r \cdot r \cdot r)}{3!} + \dots$, which implies further that $\phi(e^r)$ is a MC element in W. □

In particular, given a L_∞-morphism ϕ from $S(\mathfrak{g}[-1])[1]$ to the big bracket algebra $\mathcal{B}$ on $\mathfrak{g}$, a MC element $r \in S^{\geq 2}(\mathfrak{g}[-1])[1]$ can be sent to $\phi(e^r)$ solving $[\![\phi(e^r), \phi(e^r)]\!] = 0$. To get such a morphism ϕ, we define $\phi_n : S^n(S(\mathfrak{g}[-1])[2]) \to \mathcal{B}$ for $n \geq 1$ via

$$(\phi_n(x_1, \dots, x_n))(x) = l_{n+|x|}(x_1 \otimes \cdots \otimes x_n \otimes \delta^{max}(x)),$$

where $|x| = p$ for $x \in S^p(\mathfrak{g}[1])$, l_k's are the derived brackets on $S(\mathfrak{g}[-1])[1]$ and δ^{max} is the maximal non-zero iteration of the standard reduced coproduct on the symmetric coalgebra $S(\mathfrak{g}[1])$ applied to $x \in S(\mathfrak{g}[1])$:

$$\delta^{max}(x) = \delta^{p-1}(x) \in \underbrace{\mathfrak{g}[1] \otimes \cdots \otimes \mathfrak{g}[1]}_{p}, \quad x \in S^p(\mathfrak{g}[1]).$$

The maps $\phi_1, \phi_2, \dots$ determine a map ϕ from $S(\mathfrak{g}[-1][2])$ into the dg coalgebra $S(\mathcal{B}[1])$, as projections onto the cogenerator $\mathcal{B}$.

Proposition 4.2.3. *ϕ is an L_∞-morphism $S(\mathfrak{g}[-1])[1] \to \mathcal{B}$.*

Proof. Due to the absense of higher-order brackets on $\mathcal{B}$, the condition that ϕ commutes with the coalgebra differentials 2.2 on $S(\mathfrak{g}[-1][2])$ and $S(\mathcal{B}[1])$ takes the form

$$\sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} \phi_{m-n+1}(l_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}), x_{\sigma(n+1)}, \dots, x_{\sigma(m)}) \tag{4.1}$$

$$= \sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} [\![\phi_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \phi_{m-n}(x_{\sigma(n+1)}, \dots, x_{\sigma(m)})]\!] \tag{4.2}$$

for all $m \geq 1$. After applying to $x \in S(\mathfrak{g}[1])$, the left-hand becomes

$$\sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} l_{m-n+1+|x|}(l_n(x_{\sigma(1)} \dots x_{\sigma(n)}) \otimes x_{\sigma(n+1)} \otimes \cdots \otimes x_{\sigma(m)} \otimes \delta^{max}(x)). \tag{4.3}$$

To write out the right-hand side, we observe first that

$$\begin{aligned} \phi_n(x_{\sigma(1)}\dots x_{\sigma(n)}) \circ_1 \phi_{m-n}(x_{\sigma(n+1)}\dots x_{\sigma(m)})(x) = \\ (-1)^{|x_{(1)}||\phi_{m-n}(\dots)|}\phi_n(x_{\sigma(1)}\dots x_{\sigma(n)})(x_{(1)},\gamma_{(1)}(\phi_{m-n}(x_{\sigma(n+1)}\dots x_{\sigma(m)})(x_{(2)}))) \\ \gamma_{(2)}(\phi_{m-n}(x_{\sigma(n+1)}\dots x_{\sigma(m)})(x_{(2)})) = \\ (-1)^{|x_{(1)}||\phi_{m-n}(\dots)|}l_{n+\dots}(x_{\sigma(1)}\otimes\dots\otimes x_{\sigma(n)}\otimes\delta^{max}(x_{(1)},\gamma_{(1)}(\phi_{m-n}(x_{\sigma(n+1)}\dots x_{\sigma(m)})(x_{(2)})))) \\ \gamma_{(2)}(\phi_{m-n}(x_{\sigma(n+1)}\dots x_{\sigma(m)})(x_{(2)}))) \end{aligned}$$

Due to the multiderivation property 3.6 of the derived brackets, the last expression absorbs into

$$\begin{aligned} (-1)^{|x_{(1)}||\phi_{m-n}(\dots)|}l_{n+\dots}(x_{\sigma(1)}\otimes\dots\otimes x_{\sigma(n)}\otimes\delta^{max}(x_{(1)})\otimes\phi_{m-n}(x_{\sigma(n+1)}\dots x_{\sigma(m)})(x_{(2)})) = \\ (-1)^{|x_{(1)}||\phi_{m-n}(\dots)|}l_{n+\dots}(x_{\sigma(1)}\otimes\dots\otimes x_{\sigma(n)}\otimes\delta^{max}(x_{(1)})\otimes l_{m-n+\dots}(x_{\sigma(n+1)}\otimes\dots\otimes x_{\sigma(m)}\otimes\delta^{max}(x_{(2)})) \end{aligned}$$

We get a similar expression for $\phi_{m-n}(x_{\sigma(n+1)}\dots x_{\sigma(m)})(x) \circ_1 \phi_n(x_{\sigma(1)}\dots x_{\sigma(n)})$ as well. Due to signs, the only terms that will survive on the right-hand side of 4.1 are those of the form

$$l_{|x_{(2)}|+1}(l_{m+|x_{(1)}|}(x_{\sigma(1)}\otimes\dots\otimes x_{\sigma(m)}\otimes\delta^{max}(x_{(1)}))\otimes\delta^{max}(x_{(2)}))$$

It remains to observe that together with 4.3 they constitute the Jacobi identity holding in $S(\mathfrak{g}[-1])[1]$ on the element $x_1\otimes\dots\otimes x_n\otimes\delta^{max}(x)$. □

Definition 4.2.4. Let $\mathfrak{g}$ be an L_∞-algebra and $l \in B$ be corresponding element of the big bracket algebra B. An element $r \in S^{\geq 2}(\mathfrak{g}[-1])[1]$ is called a r_∞*-matrix* if if satisfies the Maurer-Cartan equation

$$l_1(r) + \frac{1}{2!}l_2(r,r) + \frac{1}{3!}l_3(r,r,r) + \dots = 0,$$

$[\![l,\phi(e^r)]\!] = 0$ and $\phi(e^r)$ is of total degree 1.

An r_∞-matrix r gives rise to a L_∞-bialgebra structure (that we call *triangular*) on $\mathfrak{g}$. Indeed, since r is a MC element, then

$$[\![l+\phi(e^r), l+\phi(e^r)]\!] = [\![l,l]\!] + 2[\![l,\phi(e^r)]\!] + [\![\phi(e^r),\phi(e^r)]\!] = 0$$

.

4.3 Universal enveloping algebra and further questions

Recall that a *strongly homotopy associative* (or A_∞, for short) algebra structure on a graded vector space V consists of a collection of multilinear maps $m_k : V^{\otimes k} \to V$ of degree $2-k$ such that

$$\sum_{i+j=k} (-1)^{i+j} \sum_{l=0}^{i+j} (-1)^{l(i+1)+i(|x_1|+\cdots+|x_l|)} m_{j+1}(x_1, \ldots, x_j, m_i(x_l, \ldots, x_{l+i}), x_{l+i+1}, \ldots, x_{i+j}) = 0$$

In particular, if V is concentrated in degree zero, then this is a regular associative algebra structure; if V is concentrated in degrees $0, 1$, then the A_∞-structure is dg associative.

Remark. Just as for L_∞-algebras all this data can be packed into a degree $+1$ codifferential $D : T^c(V) \to T^c(V)$ on the tensor coalgebra $T^c(V) = V \oplus (V \otimes V) \oplus \ldots$ with the comultiplication $\Delta(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n-1} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n)$. A morphism of A_∞-algebras is defined as a morphism of the corresponding dg coalgebras.

There is a notion of the *universal enveloping algebra* $U(V)$ of a L_∞-algebra V generalizing the standard non-homotopy version of it. In terms of generators and relations it admits the following description. Let $(F(V), \{m_i\})$ be a *free* A_∞-algebra generated by the vector space V. Consider the ideal $I \subset F(V)$ generated by elements of the form

$$\sum_{\sigma \in S_n} (-1)^{\sigma} (-1)^{|x_\sigma|} m_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) - l_n(x_1, \ldots, x_n),$$

where $\{l_i\}$ are the brackets of a L_∞-algebra V.

Theorem 4.3.1. *[LM95] The correspondence $V \mapsto U(V)$ is a functor from the category of L_∞ to the category of A_∞ algebras with strict morphisms (those are maps of the underlying graded vector spaces preserving the operations "pointwise" rather than morphisms of the corresponding coalgebras). Functor U is left adjoint to the symmetrization functor L: given a A_∞-algebra $(V, \{m_i\})$, it equips V with a L_∞-structure*

$$l_n(x_1, \ldots, x_n) := \sum_{\sigma \in S_n} (-1)^{\sigma} (-1)^{|x_\sigma|} m_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

.

Let A, B be A_∞-algebras given in terms of presentations $A = F(X_A)/R_A$, $B = F(X_B)/R_B$. Then $A \Box B := F(X_A \oplus X_B)/(R_A, R_B, S_{AB})$, where S_{AB} is the ideal generated by

$$\sum_{\sigma \in S_n} (-1)^\sigma (-1)^{|x_\sigma|} m_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad x_i \in X_A \cup X_B,$$

gives a symmetric monoidal structure on the category of A_∞-algebas with strict morphisms. The unit object 1 is the trivial A_∞-algebra (k, m_2) with the only non-trivial operation being the multiplication in k. The universal enveloping algebra functor U respects this structure:

Theorem 4.3.2. *[LM95] For any two L_∞-algebras V_1, V_2, there is a natural isomorphism $U(V_1 \oplus V_2) \simeq U(V_1) \Box U(V_2)$.*

Moreover, the homomorphism $\delta : U(V) \to U(V \oplus V) \simeq U(V) \Box U(V)$ induced by the inclusion $V \to V \oplus V$, $x \mapsto x \oplus x$ turns $U(V)$ into a cocommutative, coassociative coalgebra object in the category of A_∞-algebras with strict morphisms. The counit is given by the augmentation map $U(V) \to 1 = k$.

The questions which remain for further investigation are the following:

- Does $U(\mathfrak{g})$ posess a natural homotopy Hopf structure? Rather than being just a coalgebra, one might expect that it would be a A_∞-bialgebra with an antipode and, possiby, with higher comultipications subject to some compatibility conditions. If so, how to get this structure from the big bracket algebra elements corresponding to $\mathfrak{g}$?

- Develop the notion of a deformation of a homotopy Hopf algebra and define quantization in the way analogous to the standard definition. A different approach to quantization of homotopy Lie bialgebras (using the framework of PROPs) was taken in [Mer06].

- Study representation-theoretic properties of $U(\mathfrak{g})$-modules, where $\mathfrak{g}$ is a triangular L_∞-bialgebra.

- There is a notion of a generalized (or homotopy) Poisson structure (see example 2.2.10). Related to that is the notion of a homotopy Poisson-Lie group [Meh11].

Is there an analog of theorem 4.0.13, that is, can we make sense of ”integration” of a homotopy Lie bialgebra to a Poisson-Lie group?

References

[AD96] J. Alfaro and P. H. Damgaard, *Non-abelian antibrackets*, Physics Letters B **369** (1996), no. 3, 289–294.

[Akm97] F. Akman, *On some generalizations of Batalin-Vilkovisky algebras*, Journal of Pure and Applied Algebra **120** (1997), no. 2, 105–141.

[Akm00] ———, *A master identity for homotopy Gerstenhaber algebras*, Comm. Math. Phys. **209** (2000), no. 1, 51–76.

[Bax72] Rodney J Baxter, *Partition function of the eight-vertex lattice model*, Annals of Physics **70** (1972), no. 1, 193–228.

[BBD97] I. Batalin, K. Bering, and P. H. Damgaard, *Second class constraints in a higher-order lagrangian formalism*, Physics Letters B **408** (1997), no. 1, 235–240.

[BDA96] K. Bering, P. H. Damgaard, and J. Alfaro, *Algebra of higher antibrackets*, Nuclear Physics B **478** (1996), no. 1, 459–503.

[Ber07] K. Bering, *Non-commutative Batalin-Vilkovisky algebras, homotopy Lie algebras and the Courant bracket*, Communications in mathematical physics **274** (2007), no. 2, 297–341.

[BHR10] John C. Baez, Alexander E. Hoffnung, and Christopher L. Rogers, *Categorified symplectic geometry and the classical string*, Comm. Math. Phys. **293** (2010), no. 3, 701–725. MR 2566161 (2011d:53192)

[BK97] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of lie algebras of polyvector fields*, International Mathematics Research Notices **1998** (1997), no. 4, 201–215.

[BL09] K. Bering and T. Lada, *Examples of homotopy lie algebras*, Archivum Mathematicum **45** (2009), no. 4, 265–277.

[BL13] C. Braun and A. Lazarev, *Homotopy BV algebras in Poisson geometry*, **74** (2013), no. 2, 265–277.

[BM98] I. Batalin and R. Marnelius, *Quantum antibrackets*, Physics Letters B **434** (1998), no. 3, 312–320.

[BM99a] ———, *Dualities between Poisson brackets and antibrackets*, Internat. J. Modern Phys. A **14** (1999), no. 32, 5049–5073.

[BM99b] I. A. Batalin and R. Marnelius, *General quantum antibrackets*, Theoretical and Mathematical Physics **120** (1999), no. 3, 1115–1132.

[BSZ13] Chengming Bai, Yunhe Sheng, and Chenchang Zhu, *Lie 2-bialgebras*, Comm. Math. Phys. **320** (2013), no. 1, 149–172. MR 3046993

[BV81] IA Batalin and GA Vilkovisky, *Gauge algebra and quantization*, Physics Letters B **102** (1981), no. 1, 27–31.

[CL07] K. Cieliebak and J. Latschev, *The role of string topology in symplectic field theory*, New perspectives and challenges in symplectic field theory, CRM Proc. Lecture Notes **49** (2007), 113–146.

[Cou90] Theodore James Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 631–661. MR 998124 (90m:58065)

[CP94] Vyjayanthi Chari and Andrew Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994. MR 1300632 (95j:17010)

[CS99] Moira Chas and Dennis Sullivan, *String topology*, arXiv preprint math/9911159 (1999).

[CV06] Ralph L. Cohen and Alexander A. Voronov, *Notes on string topology*, String topology and cyclic homology, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2006, pp. 1–95. MR 2240287

[DCTT08] Gabriel C Drummond-Cole, John Terilla, and Thomas Tradler, *Algebras over cobar (cofrob)*, arXiv preprint arXiv:0807.1241 (2008).

[Dri83] V. G. Drinfeld, *Constant quasiclassical solutions of the Yang-Baxter quantum equation*, Dokl. Akad. Nauk SSSR **273** (1983), no. 3, 531–535. MR 725930 (85d:58040)

[Dri87] ———, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 798–820. MR 934283 (89f:17017)

[FST79] Ludwig D Faddeev, Evgueni K Sklyanin, and Leon A Takhtajan, *Quantum inverse problem method i*, Theor. Math. Phys **40** (1979), no. 688-706, 86.

[GCTV12] Imma Gálvez-Carrillo, Andrew Tonks, and Bruno Vallette, *Homotopy Batalin-Vilkovisky algebras*, J. Noncommut. Geom. **6** (2012), no. 3, 539–602. MR 2956319

[Get94] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (1994), no. 2, 265–285. MR 1256989 (95h:81099)

[GJ94] E. Getzler and J.D. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, arXiv preprint hep-th/9403055 (1994).

[GK94] Victor Ginzburg and Mikhail Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), no. 1, 203–272. MR 1301191 (96a:18004)

[Jim86] Michio Jimbo, *Quantum R matrix related to the generalized Toda system: an algebraic approach*, Field theory, quantum gravity and strings (Meudon/Paris, 1984/1985), Lecture Notes in Phys., vol. 246, Springer, Berlin, 1986, pp. 335–361. MR 848629 (87j:17013)

[Kos85] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque (1985), 257–271, The mathematical heritage of Élie Cartan (Lyon, 1984).

[Kra00] O. Kravchenko, *Deformations of Batalin-Vilkovisky algebras*, Poisson geometry (Warsaw, 1998), Banach Center Publ., vol. 51, Polish Acad. Sci., Warsaw, 2000, pp. 131–139. MR 1764440 (2001e:17036)

[Kra07] Olga Kravchenko, *Strongly homotopy lie bialgebras and lie quasi-bialgebras*, Letters in Mathematical Physics **81** (2007), no. 1, 19–40.

[KS] M. Kontsevich and Y. Soibelman, *Deformation theory I, draft of the book.*

[KS92] Yvette Kosmann-Schwarzbach, *Jacobian quasi-bialgebras and quasi-Poisson Lie groups*, Mathematical aspects of classical field theory (Seattle, WA, 1991), Contemp. Math., vol. 132, Amer. Math. Soc., Providence, RI, 1992, pp. 459–489. MR 1188453 (94b:17025)

[KS04] ______, *Derived brackets*, Letters in Mathematical Physics **69** (2004), no. 1, 61–87.

[KT79] Jerzy Kijowski and Włodzimierz M. Tulczyjew, *A symplectic framework for field theories*, Lecture Notes in Physics, vol. 107, Springer-Verlag, Berlin-New York, 1979. MR 549772 (81m:70001)

[KV08] H. M. Khudaverdian and T. Voronov, *Higher Poisson brackets and differential forms*, Geometric methods in physics, AIP Conf. Proc., vol. 1079, Amer. Inst. Phys., Melville, NY, 2008, pp. 203–215.

[LM95] Tom Lada and Martin Markl, *Strongly homotopy lie algebras*, Communications in algebra **23** (1995), no. 6, 2147–2161.

[LV12] Jean-Louis Loday and Bruno Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012. MR 2954392

[LWX97] Zhang-Ju Liu, Alan Weinstein, and Ping Xu, *Manin triples for Lie bialgebroids*, J. Differential Geom. **45** (1997), no. 3, 547–574. MR 1472888 (98f:58203)

[Man91] Yu. I. Manin, *Quantum groups*, Nederl. Akad. Wetensch. Verslag Afd. Natuurk. **100** (1991), no. 5, 55–68. MR 1119249

[Man99] ———, *Frobenius manifolds, quantum cohomology, and moduli spaces*, vol. 47, American Mathematical Soc., 1999.

[Mar01] Martin Markl, *Loop homotopy algebras in closed string field theory*, Comm. Math. Phys. **221** (2001), no. 2, 367–384. MR 1845329 (2002h:55012)

[Meh11] Rajan Amit Mehta, *On homotopy poisson actions and reduction of symplectic q-manifolds*, Differential Geometry and its Applications **29** (2011), no. 3, 319–328.

[Mer06] SA Merkulov, *Quantization of strongly homotopy lie bialgebras*, MPI, 2006.

[MSS02] Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002. MR 1898414 (2003f:18011)

[Res92] N. Reshetikhin, *Quantization of Lie bialgebras*, Internat. Math. Res. Notices (1992), no. 7, 143–151. MR 1174619 (93h:17041)

[Rog12] Christopher L. Rogers, *L_∞-algebras from multisymplectic geometry*, Lett. Math. Phys. **100** (2012), no. 1, 29–50. MR 2892558

[Roy02] Dmitry Roytenberg, *On the structure of graded symplectic supermanifolds and Courant algebroids*, Quantization, Poisson brackets and beyond (Manchester, 2001), Contemp. Math., vol. 315, Amer. Math. Soc., Providence, RI, 2002, pp. 169–185. MR 1958835 (2004i:53116)

[RT91] Nicolai Reshetikhin and Vladimir G Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Inventiones mathematicae **103** (1991), no. 1, 547–597.

[Sch93] Albert Schwarz, *Geometry of Batalin-Vilkovisky quantization*, Comm. Math. Phys. **155** (1993), no. 2, 249–260. MR 1230027 (95f:81095)

[SGA70] *Schémas en groupes. I: Propriétés générales des schémas en groupes*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151, Springer-Verlag, Berlin-New York, 1970.

[Skl82] Evgeny Konstantinovich Sklyanin, *Some algebraic structures connected with the Yang?Baxter equation*, Functional Analysis and its Applications **16** (1982), no. 4, 263–270.

[SS85] Michael Schlessinger and James Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. Pure Appl. Algebra **38** (1985), no. 2-3, 313–322. MR 814187 (87e:13019)

[Tam98] D. E. Tamarkin, *Another proof of M. Kontsevich formality theorem*, arXiv preprint math.QA/9803025 **24** (1998).

[Tam99] ———, *Operadic proof of M. Kontsevich's formality theorem*, ProQuest LLC, Ann Arbor, MI, 1999, Thesis (Ph.D.)–The Pennsylvania State University.

[TT00] D. E. Tamarkin and B. Tsygan, *Noncommutative differential calculus, homotopy BV algebras and formality conjectures*, Methods Funct. Anal. Topology **6** (2000), no. 2, 85–100.

[TTW11] J. Terilla, T. Tradler, and S. O. Wilson, *Homotopy DG algebras induce homotopy BV algebras*, J. Homotopy Relat. Struct. **6** (2011), no. 1, 177–182.

[Val07] B. Vallette, *A Koszul duality for PROPs*, Trans. Amer. Math. Soc. **359** (2007), no. 10, 4865–4943.

[Vor00] A. A. Voronov, *Homotopy Gerstenhaber algebras*, Conférence Moshé Flato 1999, Springer, 2000, pp. 307–331.

[Vor05a] T. Voronov, *Higher derived brackets and homotopy algebras*, Journal of Pure and Applied Algebra **202** (2005), no. 1, 133–153.

[Vor05b] ______, *Higher derived brackets for arbitrary derivations. in travaux mathématiques. fasc*, Departamento de Matemáticas, Universidad de los Andes, Carrera 1 N. 18A - 10, 2005, p. 163.

[Wor89] Stanisław L Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups)*, Communications in Mathematical Physics **122** (1989), no. 1, 125–170.

[Xu99] Ping Xu, *Gerstenhaber algebras and bv-algebras in poisson geometry*, Communications in mathematical physics **200** (1999), no. 3, 545–560.

[Zei07] Anton M Zeitlin, *Homotopy lie superalgebra in yang-mills theory*, Journal of High Energy Physics **2007** (2007), no. 09, 068.

[Zei10] Anton M. Zeitlin, *Conformal field theory and algebraic structure of gauge theory*, J. High Energy Phys. (2010), no. 3, 056, 32. MR 2653468 (2011k:81235)

[Zwi93] Barton Zwiebach, *Closed string field theory: quantum action and the Batalin-Vilkovisky master equation*, Nuclear Phys. B **390** (1993), no. 1, 33–152. MR 1203476 (94i:81137)