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Master's Thesis

Exploring a simple model for Pleistocene glacial variability

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Abstract

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Exploring a simple model for Pleistocene glacial variability

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We explore a simple model of Huybers that attempts to justify the hypothesis that obliquity paced glacial cycles throughout the Pleistocene. We find that, for some fixed parameters, the model has a small number of trajectories over all initial conditions. We also test the predictability of the model after adding noise. We then compare these findings with those of another well-known model for Pleistocene glacial variability.

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Abbreviations

MPT	**m**id-**P**leistocene **t**ransition
[M/K]yr	[**M**illion/**T**housand] **y**ears
[M/K]ya	[**M**illion/**T**housand] **y**ears **a**go

Chapter 1

Background

1.1 An Unsolved Mystery: the mid-Pleistocene transition

In the early Pleistocene ($\sim$2–1 Mya), glacial cycles had a period of 41 Ky. Then, 1.2 Mya this period became 100 Ky. This event, separating the early and late Pleistocene (1 Mya–10 Kya), is known as the mid-Pleistocene transition (MPT). The cause of the MPT is an open question in paleoclimate known as the 100,000 Year Problem.

The amount of incoming solar radiation to Earth is largely driven by three orbital variations: obliquity, the angle between Earth's rotational and orbital axes; precession, the movement of Earth's rotational axis; and the eccentricity of Earth's orbit around the Sun (Hays et al. [1976]). Obliquity, precession, and eccentricity—forces known as Milanković cycles—are periodic with period 41, 26, and 100 Ky, respectively. Thus, one hypothesis of the MPT is that the primary driver of the glacial cycles transitioned from obliquity to eccentricity as a result of a bifurcation (Maasch and Saltzman [1990]).

Huybers [2007] offers an alterative explanation to the theory that the MPT was a bifurcation, and argues that obliquity paced the glacial cycles throughout the Pleistocene. His argument is as follows: in the early Pleistocene, deglaciations occurred almost every obliquity cycle, or about every 41 Kyr. Then, around 1.2 Mya, deglaciations began to skip one or two periods of the obliquity signal, giving the appearance of 100 Kyr period. Huybers uses phase angle statistics to show that it is not the case that deglaciations occur independently of the phase of obliquity. He then gives a simple mathematical model, to be introduced in the follow section, that shows how obliquity might have paced the glacial cycles throughout the Pleistocene.

In this thesis, we will investigate Huybers' model from an abstract, mathematical point of view, and ask questions related to how trajectories respond to change in initial condition over the parameter space. We will then use these results to compare Huybers' model to a periodically forced oscillator that has also been used to study the MPT.

1.2 Huybers' model

Huybers [2007] argues that the transition from 41 Kyr to 100 Kyr glacial cycles was gradual, rather than instantaneous. To support his theory, Huybers gives the following model:

$$V_t = V_{t-1} + \eta_t \quad \text{and if } V_t \geq T_t\text{, linearly reset } V_t \text{ to zero over 10 timesteps.}$$
$$T_t = at + b + c\theta'_t$$

According to this model, the ice volume V_t grows some amount η_t until it reaches the threshold T_t, at which point V_t is linearly reset to zero over 10 timesteps (10 Ky). The threshold T_t depends on obliquity θ_t (here the prime means that θ_t has been normalized to have mean zero and unit variance) and parameters a, b, and c. Choosing a slope of $a = 0.05$ Ky^{-1}, an intercept of $b = 126$, and an obliquity amplitude of $c = 20$ simulates most of the deglaciations of the past 2 My (Figure 1.1).

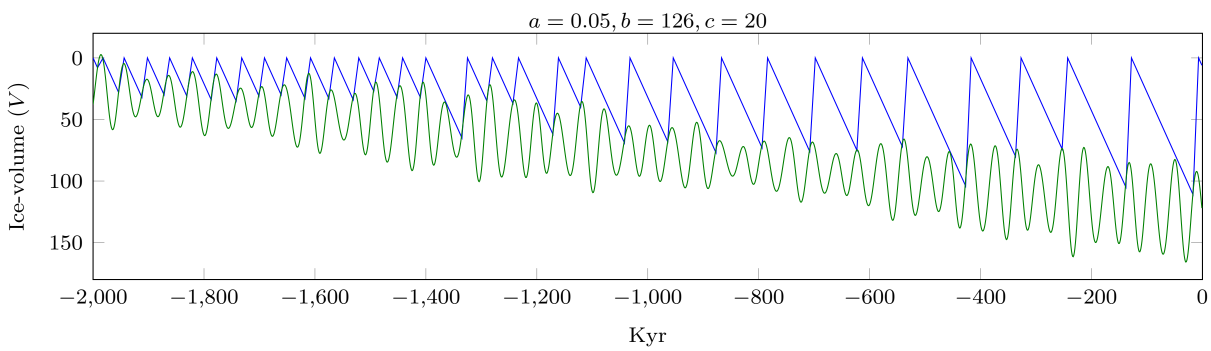


FIGURE 1.1: Huybers [2007] finds that a suitable choice of parameters "reproduces the timing of most deglaciations over the last 2 Ma."

To follow the conventions of Huybers [2007], the vertical axis has been inverted and time goes from -2000 Kya to the present. In the above figure, $\eta_t \equiv 1$. We shall also investigate the above model with η_t a random variable with unit mean (Figure 1.2).

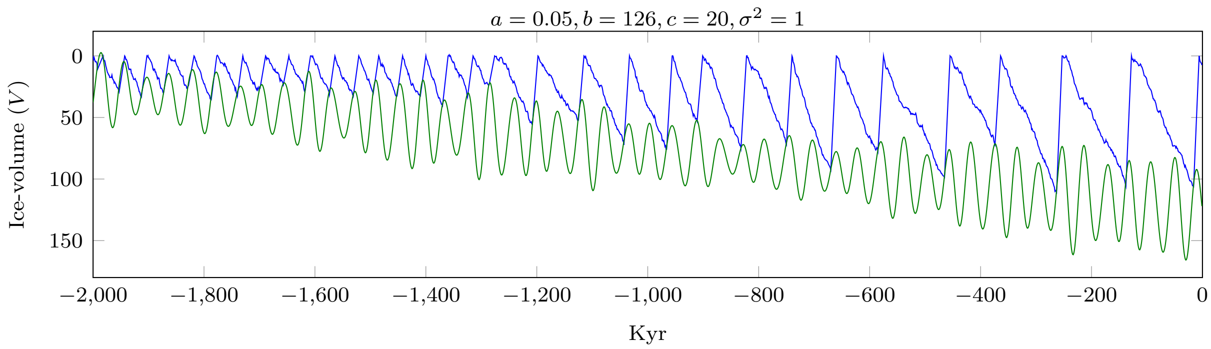


FIGURE 1.2: An example with η_t random.

We may also consider what happens when we alter the threshold to be a piecewise constant function that transitions linearly from a value b_{low} to another b_{hi}. Figure 1.3 shows us that, in this case, the period of V_t is approximately to the height of the threshold plus 10. That is, when $T_t = b_{\text{low}}$, the period of V_t is $b_{\text{low}} + 10$, and the same is true for b_{hi}.

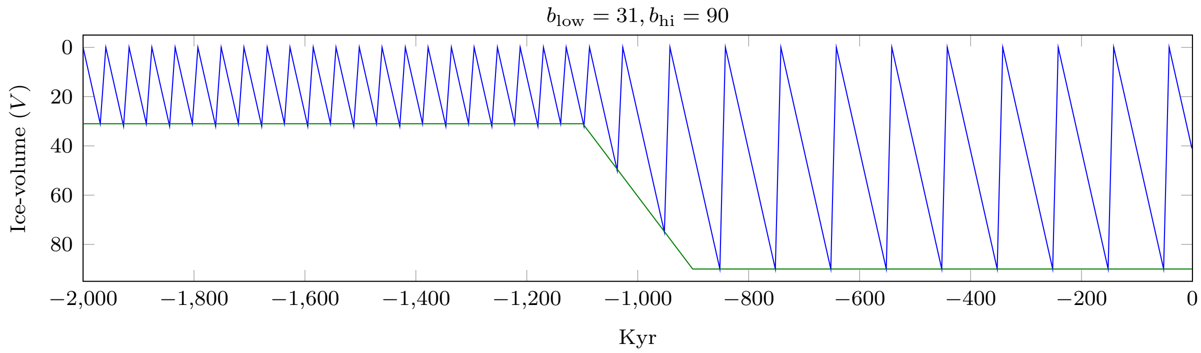


FIGURE 1.3: The threshold as a piecewise constant function.

Adding the obliquity term $c\theta'_t$ to the threshold gives a "piecewise oscillator," shown in Figure 1.4. The relevance of this adapted model will be explained in our discussion the periodically forced oscillator.

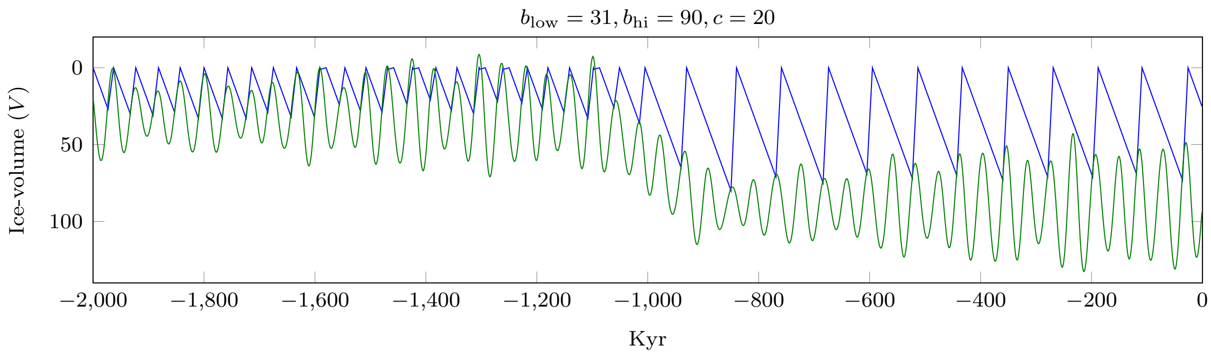


FIGURE 1.4: The threshold as a piecewise oscillator.

Chapter 2

Exploring Huybers' model

We investigate Huybers' model with the standard threshold $T_t = at + b + c\theta'_t$ and the piecewise oscillator, both with constant and stochastic case ice accumulation η_t. In the case of the standard, deterministic threshold, we ask **for fixed b and c, how does varying the initial condition effect the trajectories? Specifically, under what conditions do trajectories converge to each other, regardless of initial condition?** In the stochastic case, we investigate, **for fixed $\mathbf{b}, \mathbf{c}$, and $\mathbf{v_o}$, do deglaciations occur at a consistent phase of obliquity?** We ask analogous questions in the case of the piecewise oscillator while varying c.

2.1 Standard threshold, constant ice accumulation

We fix $a = 0.05, b \in [0, 200]$, and $c \in [0, 30]$[1]. For these fixed parameter values, we run the model for each initial condition $v_o \in [0, 100]$ and determine the number of timesteps it takes for all 101 trajectories to converge to a single trajectory. We can see that for $b \in [0, 100]$ and $c \in [0, 30]$, $T_{-2000} \leq V_{-2000} = v_o$; thus, in this parameter regime, any solutions $(V_t, v_o; a, b, c)$ will be identical after 10 Ky (Figure 2.1).

For $b > 100$ and c small, each v_o uniquely determines the phase of V_t, as demonstrated in Figure 2.2.

[1]Throughout this discussion, an interval $[a, b]$ will denote $[a, b] \cap \mathbb{Z}$.

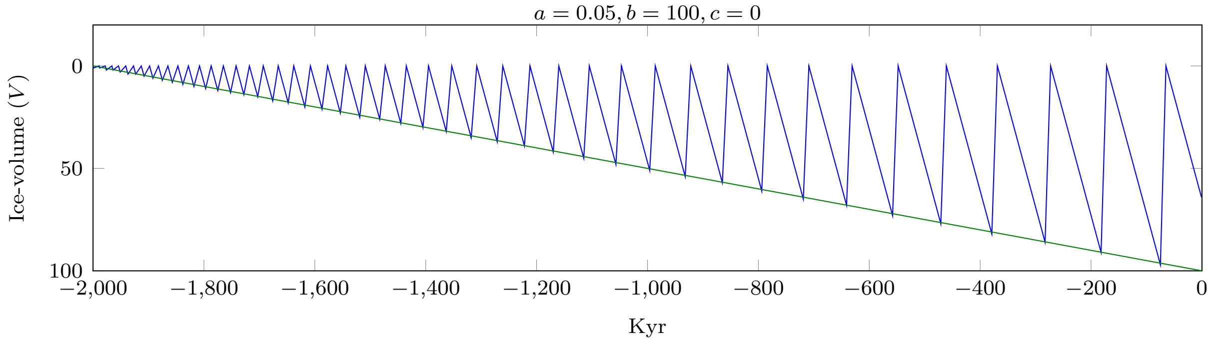


FIGURE 2.1: For $b < 100$, all trajectories will be the same after 10 Ky.

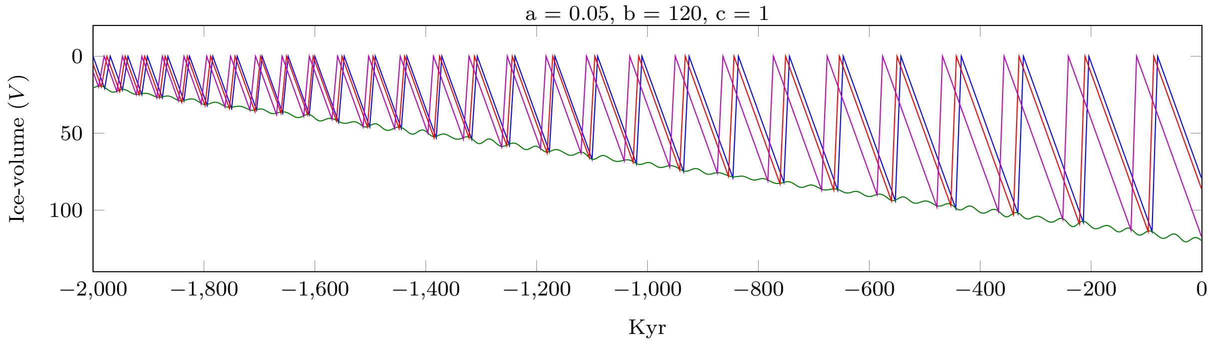


FIGURE 2.2: When the threshold resembles a line, the initial condition determines the phase of V_t. Here we have initial conditions 0, 5, and 10.

As the obliquity amplitude c increases, a bottleneck occurs near the initial position between the threshold and $V = 0$, thus increasing the likelihood that two solutions will intersect and become the same (Figure 2.3).

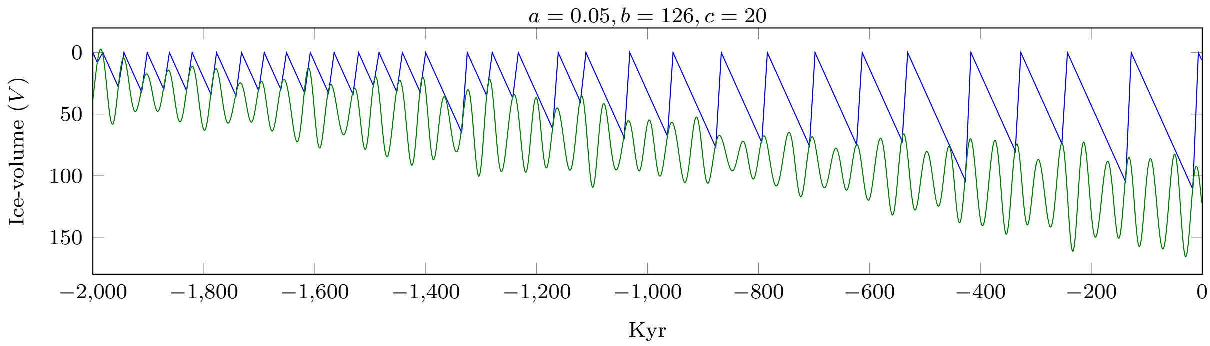


FIGURE 2.3: A bottleneck occurs near −2000 Ky as c increases.

One would suspect that as b increases, c must also increase in order for trajectories to converge to one another. Indeed, the result of our test confirms this intuition (Figure 2.4). There are, however, some outliers: trajectories with $b = 143, c = 8$ all converge to each other (Figure 2.5), though $(143, 8)$ lies in a region of the parameter space that is dominated by (b, c) pairs such that solutions do not converge to one another (e.g.

Figure 2.6). It is worth noting that when $b = 126, c = 20$, so that the glaciations of the Pleistocene are reproduced as in Huybers [2007], there is only one trajectory.

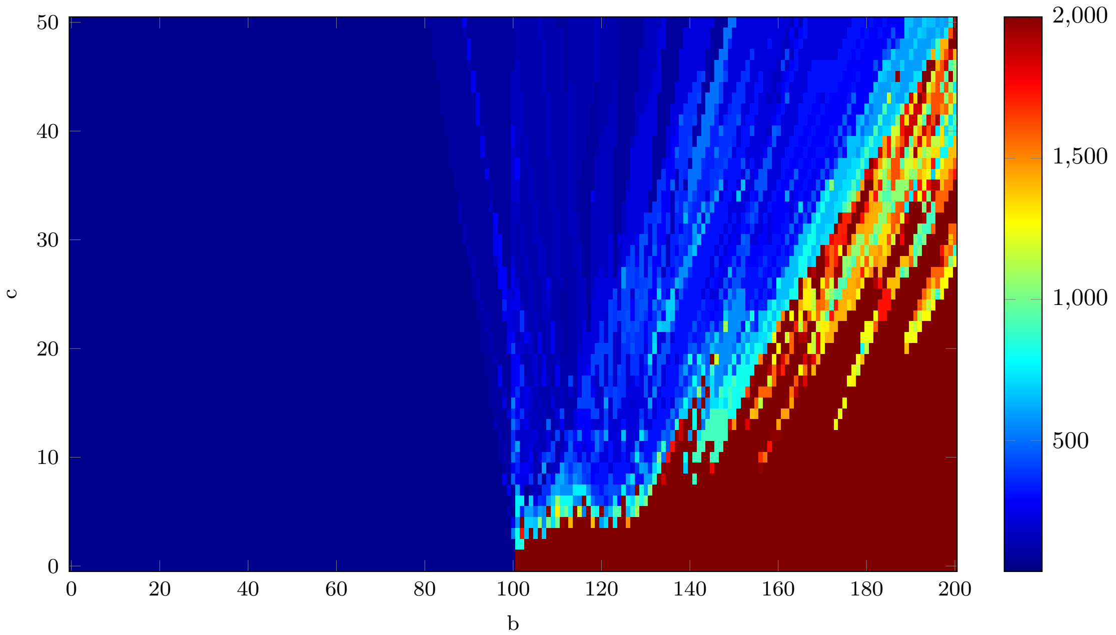


FIGURE 2.4: Colormap showing the number of timesteps after which the 101 trajectories $(V_t, v_o), v_o \in [0, 100]$ converge to each other for a given b and c.

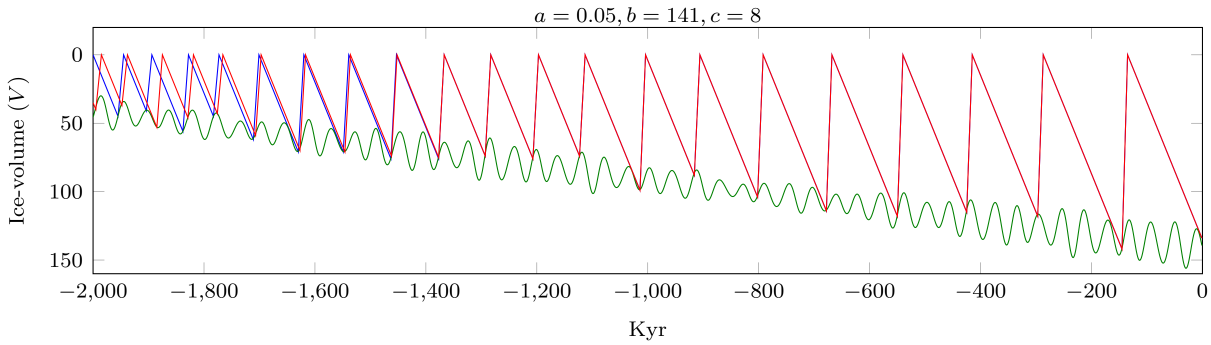


FIGURE 2.5: Solutions with initial conditions $v_o = 0$ and 35 converging.

Figure 2.6 also shows the models sensitivity to initial conditions in particular parameter regimes. One can see that the blue line ($v_o = 12.74$) misses the initial peak of the threshold, whereas the red line ($v_o = 12.76$) hits it, causing the two trajectories to be subsequently out of phase.

Of course, the test depicted in Figure 2.4 is a crude one, giving almost zero information in the case that any two trajectories $(V_t, v_o; a, b, c)$ and $(V_t, v_1; a, b, c)$ do not converge to one another. Figures 2.7,2.8 reveal that for most b and c, there are few trajectories.

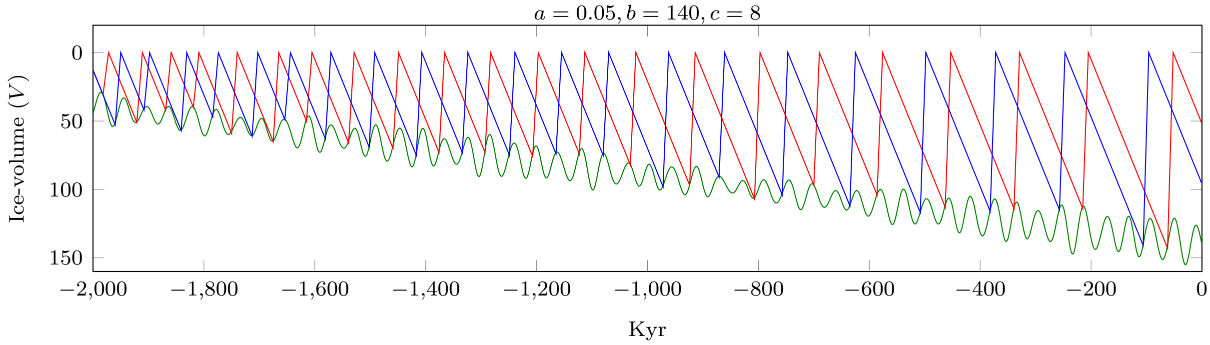


FIGURE 2.6: Sensitivity to initial conditions: $v_o = 12.74$ (blue) versus $v_o = 12.76$ (red).

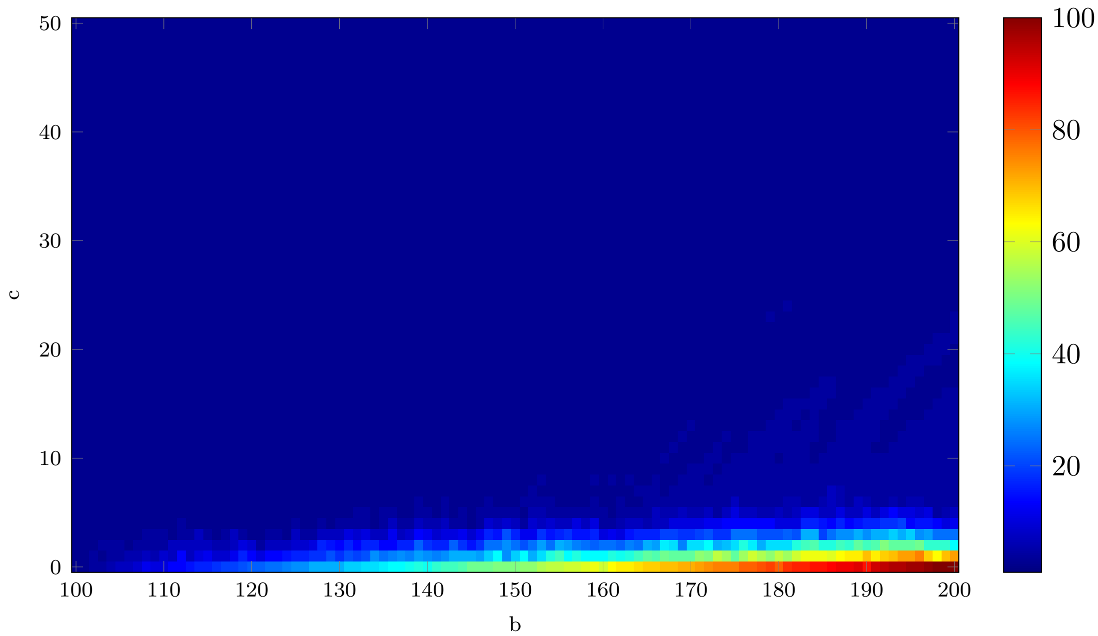


FIGURE 2.7: Number of distinct trajectories at time 0 Ky over the (b, c) parameter space (error tolerance 1×10^{10}).

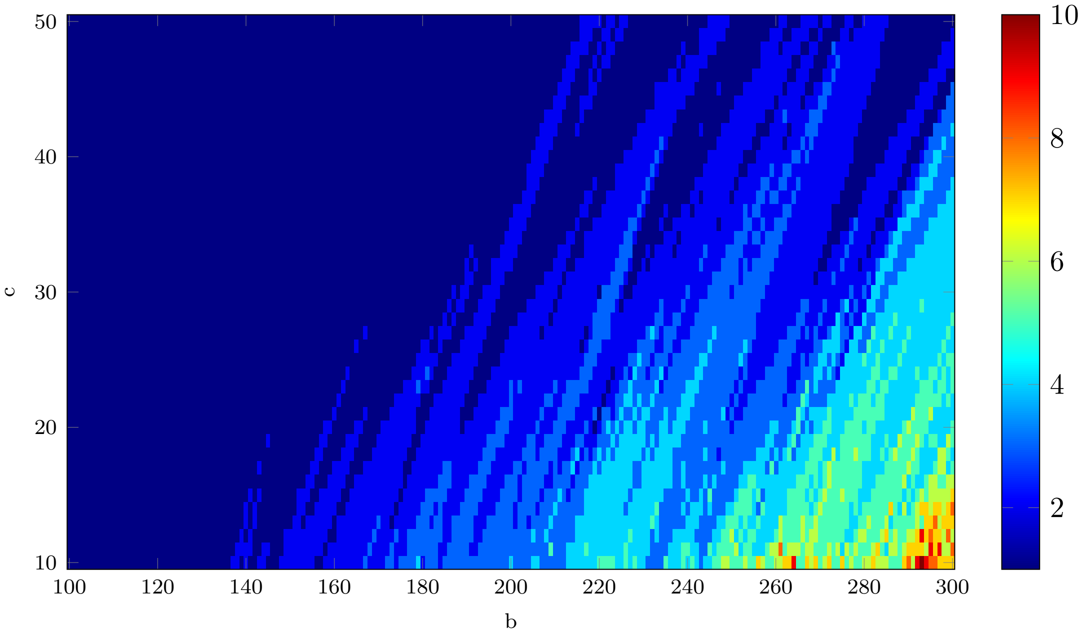


FIGURE 2.8: Number of distinct trajectories at time 0 Ky in the parameter space $[100, 300] \times [10, 50]$.

2.2 Piecewise oscillator, constant ice accumulation

We examine the piecewise oscillator with $b_{\text{low}} = 31, b_{\text{hi}} = 90$, and variable c. As in the case of the standard threshold from the previous section, we fix c, vary the initial condition v_o and compare trajectories. Because T_{-2000} is approximately 30, it suffices to consider initial conditions $v_o \in [0, 30]$. As before, we first determine for which c do the 31 different trajectories converge to each other.

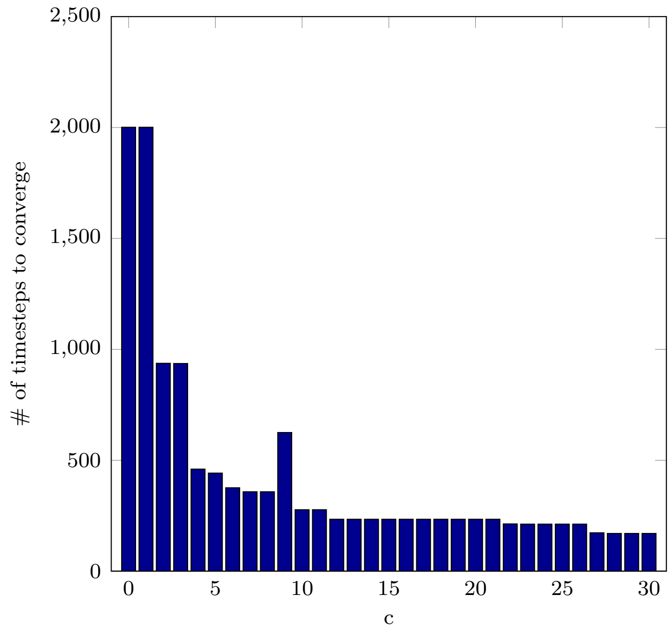


FIGURE 2.9: For c different from 0 and 1, all trajectories converge to each other in less than 2000 Ky.

Figure 2.9 shows that for c different from zero and one, all trajectories converge to each other in less than 2000 Ky. When $c = 0$, T_t is a piecewise linear function and, as before, we have that each initial condition uniquely determines the phase of V_t. When $c = 1$, however, there are only 5 distinct trajectories at time 0 Ky (Figure 2.10).

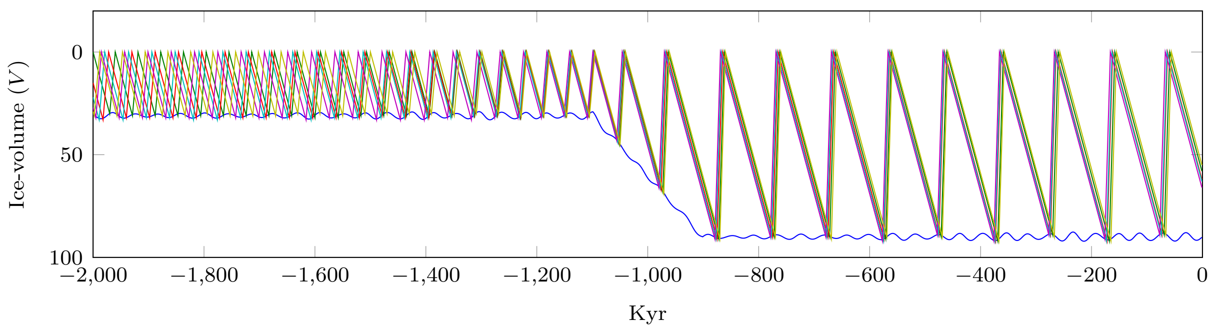
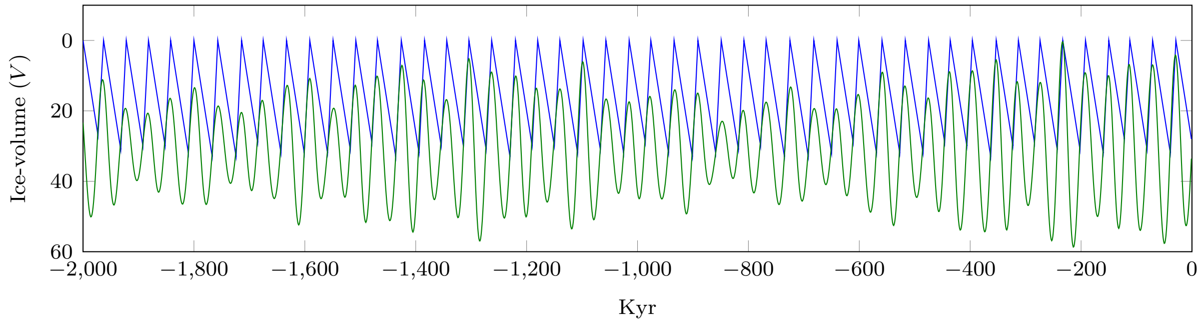


FIGURE 2.10: For $c = 1$, there are five distinct trajectories at time 0 Ky.

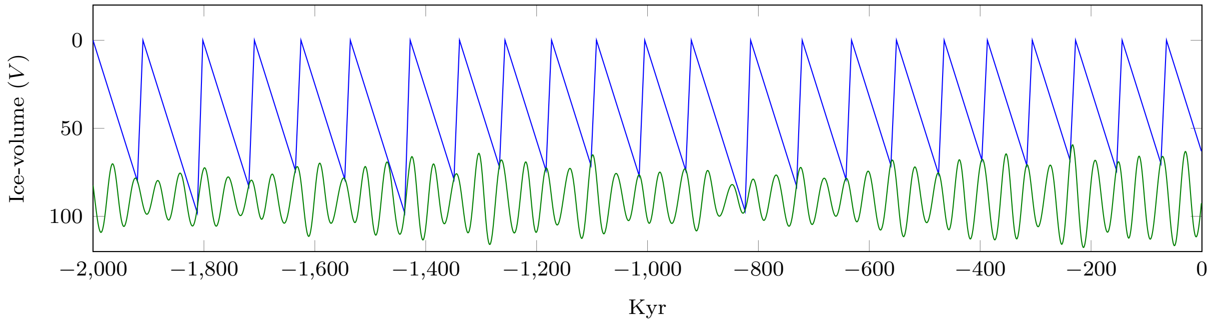
Upon inspection of these trajectories, one can see that when the threshold is at a height of $b_{\text{low}} = 31$, the trajectories appear to be converging to each other. Then, after the transition to $b_{\text{hi}} = 90$, the trajectories appear to separate from one another. To investigate this, we inspect the convergence of the trajectories in the case where

$$T_t = b + c\theta'_t,$$

that is, when the threshold is an oscillation having amplitude c plus some constant b (Figure 2.11).



(A) $b = 31, c = 13$

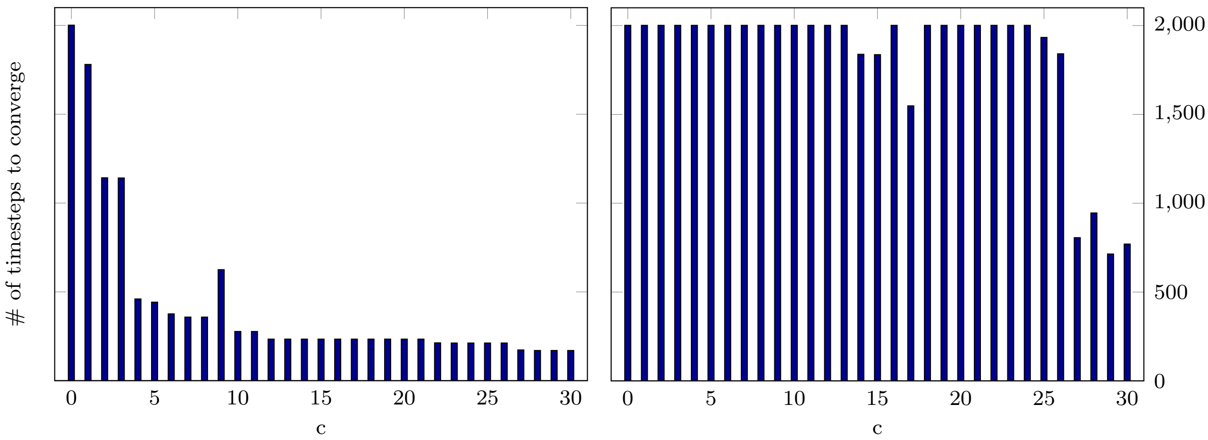


(B) $b = 90, c = 13$

FIGURE 2.11: The "constant oscillator".

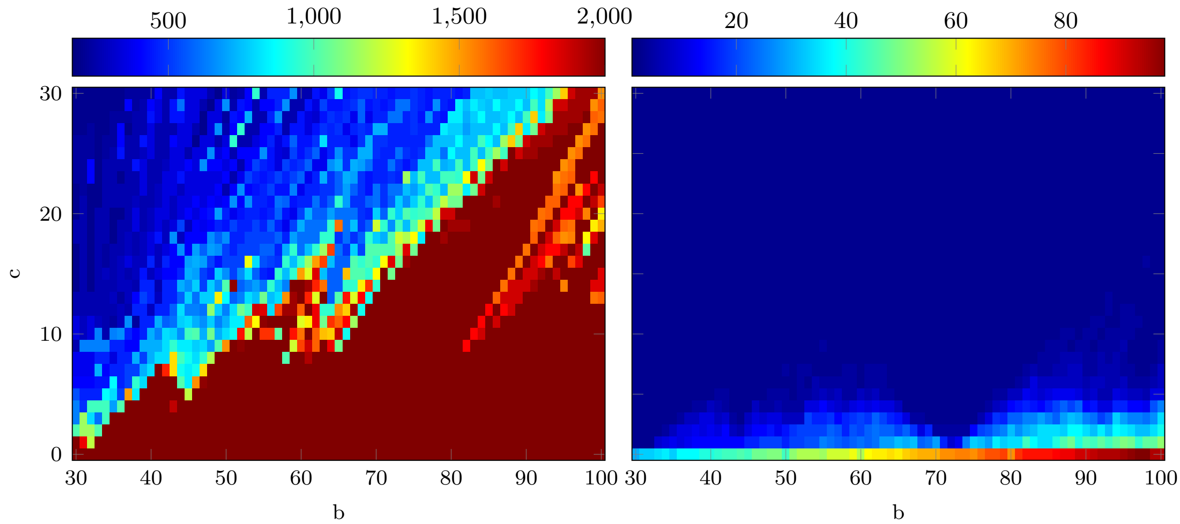
The results of this inspection (Figure 2.12) confirm the hypothesis above: for $b = 31$, all trajectories converge to each other for every value of $c \neq 0$; for $b = 90$, there are multiple trajectories for most c.

To see what happens between these parameter values, we generate a colormap of the number of timesteps for all trajectories to converge to one another for the "constant oscillator" and, as before, a colormap of the number of trajectories at 0 Ky (Figure 2.13). We see a similar result as in the previous section: given b, all trajectories will converge to one another so long as c is large enough (Figure 2.13a). A majority of the parameter space has all trajectories converging to one another. Moreover, Figure 2.13b shows that throughout the rest of the parameter space, the number of distinct trajectories is relatively small: for $c \geq 6$, the number of trajectories is at most 10.



(A) $b = 31$ (B) $b = 90$

FIGURE 2.12: Constant oscillator: convergence time against c.



(A) The number of timesteps for all trajectories to converge to each other.

(B) The number of distinct trajectories at time 0 Ky.

FIGURE 2.13: Constant oscillator: a closer look.

2.3 Randomizing ice accumulation

We now let the ice accumulation η_t be a random variable with unit mean and variation σ^2. In this case, of course, one cannot expect trajectories to converge to each other. So, instead we track when the model deglaciates, or when $V_t \geq T_t$, and attempt to study if any patterns appear. To do this, we number the "valleys" of the threshold function (Figure 2.14), then track which valleys are hit by a particular trajectory. Then, for

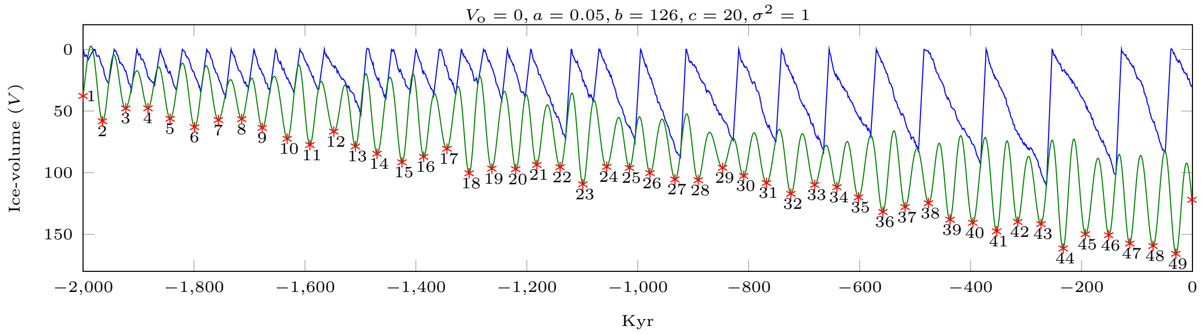
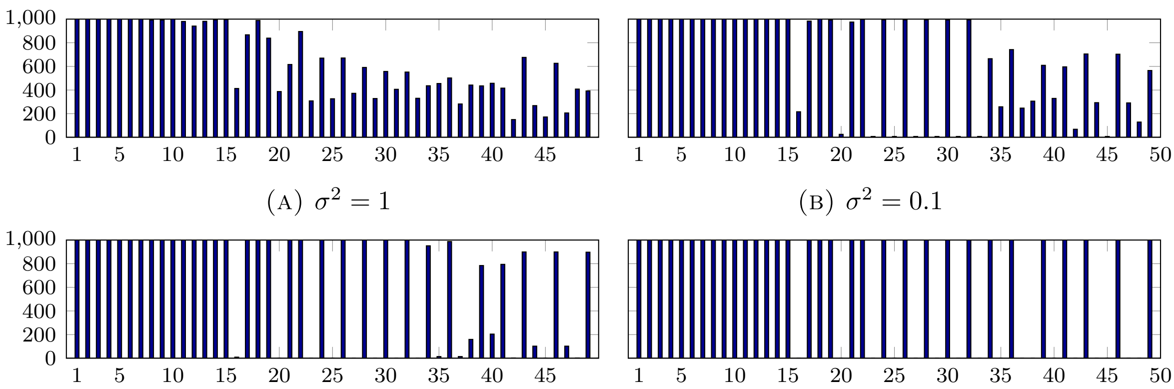


FIGURE 2.14: Numbered "valleys" of the threshold function.

variation $\sigma^2 \in \{1, 0.1, 0.01, 0\}$, we iterate this process over 1000 trials with v_o, a, b, c fixed, and generate a cumulative count of the valleys hit (Figure 2.15). These statistics show us that for $\sigma^2 \geq 0.1$, the behavior of the model is unpredictable one the threshold is sufficiently large and V_t begins skipping obliquity cycles. However, it should be noted that $\sigma^2 \geq 0.1$ is a relatively large amount of noise in the system, and that the system is resilient to smaller perturbations ($\sigma^2 \leq 0.01$), which is more realistic in the physical sense.



(A) $\sigma^2 = 1$

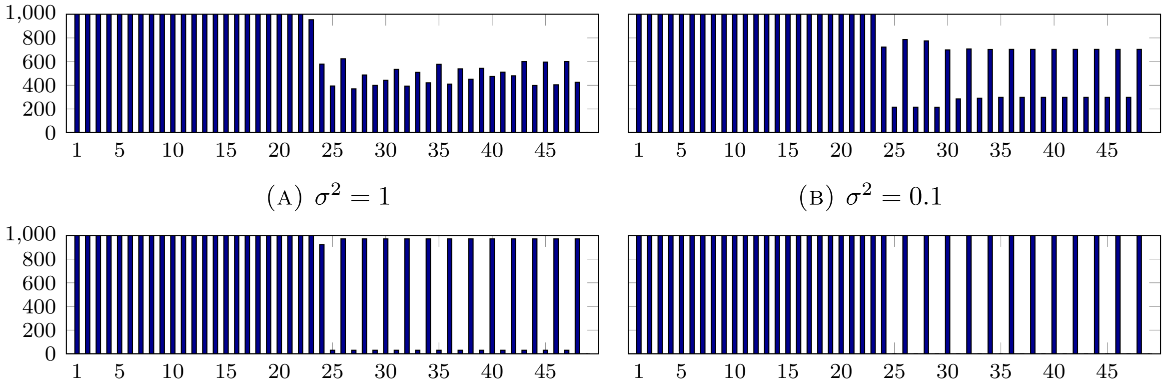
(B) $\sigma^2 = 0.1$

(C) $\sigma^2 = 0.01$

(D) $\sigma^2 = 0$

FIGURE 2.15: Distribution of deglaciations with the standard threshold function ($v_o = 0, a = 0.05, b = 126, c = 20$).

Calculating these statistics in the case of the piecewise oscillator shows a similar result (Figure 2.16). When the threshold peaks (minima) are close to zero, the model is fairly predictable as V_t is forced to hit every obliquity cycle. Thus, one might guess that this behavior could be replicated by increasing c as b increases. It appears that the amount of noise that the system can tolerate decreases as the height of the threshold increases, though we do not test these hypotheses.



(A) $\sigma^2 = 1$ (B) $\sigma^2 = 0.1$

(C) $\sigma^2 = 0.01$ (D) $\sigma^2 = 0$

FIGURE 2.16: Distribution of deglaciations with the piecewise oscillator ($v_o = 0, a = 0.05, b_{\text{low}} = 31, b_{\text{hi}} = 90, c = 20$).

Chapter 3

Concluding remarks

3.1 Comparing Huybers' model to the forced oscillator

At the beginning of this thesis, we mentioned another model used to study glacial variability in the Pleistocene: a periodically forced oscillator studied by Maasch and Saltzman [1990]. This model, before adding the forcing term, contains a Hopf bifurcation in which a stable fixed point bifurcates to a stable limit cycle with 100 Kyr period. Then, 41 Kyr forcing is added to the system, and the bifurcation transitions the system from 41 Kyr oscillations to 100 Kyr. In contrast, the period of Huybers' model, which is internal (unforced), depends largely on the height of the threshold, as we saw in the case of the piecewise oscillator threshold. It is the piecewise oscillator threshold that allows us to mimic the Maasch & Saltzman model: we hold the system at 41 Kyr cycles for some period of time before transitioning (bifurcating) to 100 Kyr cycles.

Huybers' model enjoys many desirable properties: for most parameter values, we saw that the initial condition is an irrelevant piece of data. In the cases that initial condition does change the ultimate trajectory of the model output, there were few trajectories. Thus, if we believed the model to be a valid explanation of the glacial cycles, it could be used somewhat reliably to predict future glacial cycles without the need for high precision in determining the ice-volume 2 Ma (i.e., knowing the initial condition). In addition, this remains true even after adding a small amount of noise to the system.

In the case of Maasch & Saltzman's model, the deterministic system is similar to Huybers' model: Oestreicher [2014] shows that the forced oscillator has precisely five different trajectories, which are predictable based on initial condition. Adding a tiny noise amount of noise to the model, however, destroys any predictability. So, in terms of using a model to predict the future behavior of a system, Huybers' model appears to be the clear "winner," assuming there is any stochasticity in the climate system. Whether paleoclimate research will continue to support Huybers' hypothesis on Pleistocene glacial variability, however, is the true test of his model.

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