SOLITON'S REBUILDING IN ONE-DIMENSIONAL SCHRÖDINGER MODEL WITH POLYNOMIAL NONLINEARITY

By

Valery E. Grikurov

IMA Preprint Series # 1320

(July 1995)

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370
URL: http://www.ima.umn.edu

SOLITON'S REBUILDING IN ONE-DIMENSIONAL SCHRÖDINGER MODEL WITH POLYNOMIAL NONLINEARITY*

VALERY E. GRIKUROV †

Abstract. The solitons of the one-dimensional NLS equation are considered. We suppose that the nonlinearity 1) provides the existence of both stable and unstable solitons and 2) limits the possible solitonic parameters by some critical relation (the simplest representative is a two-terms polynomial). The first part of the paper is devoted to the perturbation of an unstable soliton, and the solution describing its rebuilding is proposed. In the second part the collision of solitons is numerically simulated, and the rebuilding of solitons is established: if solitonic parameters are close to the critical ones, then in a short time after the collision the new stuff of solitons is formed.

Key words. nonlinear Schrödinger (NLS) equation, nonintegrable, polynomial nonlinearity, soliton, stability, perturbation series, collisions, long-time behaviour

AMS(MOS) subject classifications. 35Q40, 35Q51, 35Q55, 35Q60, 65C20, 65Mxx

1. Introduction . The nonlinear Schrödinger (NLS) equation

$$\imath\psi_t = -\Delta\psi + F\left(|\,\psi\,|^2\right)\psi \tag{1.1}$$

arises in the mathematical description of the propagation of lasers through nonlinear optic materials and Langmuir waves in a plasma. Under certain conditions on the nonlinearity function $F(\xi)$ the equation (1.1) admits a one-parametric family of localized, finite energy travelling wave solutions — solitory waves or, simply, solitons. These solutions were extensively studied in recent years since its are of special importance due to the distinguished role they sometimes play in the long-time asymptotics of the initial value problem for nonlinear evolution equations.

The brief review of the known results should be started from the famous "cubic" one-dimensional NLS $F(\xi) = \pm\xi$ which is a completely integrable system. This implies, in particular, that, for the focusing case (the minus sign), the long-time asymptotics of the Cauchy problem with rapidly vanishing initial data consists of a finite number of solitons and a dispersive radiation which is governed by the free linear equation. For example, if the initial function equals to the sum of well separated solitons, when the radiation is (almost) zero and the long-time solitons (possibly) differ from the initial ones only in phase shifts (in this context one can talk about *superstability* of solitons). Among a lot of monographs and papers on this subject we refer to [2], [1].

* This work was made possible in part due to the support of Russian Foundation of Fundamental Investigations and International Science Foundation, grant R 40000. The part of this work was performed at Dept. of Math., Lund Inst. of Technology, Sweden, due to the support of Swedish Institute.

† Dept. Comp. & Math. Phys., Inst. on Phys., St.Petersburg Univ. 1 Uljanovskaia Ave., Petrodvoretz 198904 Russia
grikurov@phim.niif.spb.su

The main question arising in connection with solitory waves is their stability. The orbital stability theorem [3], that is "if the solution is close (in an appropriate norm, normally $\|\cdot\|_{H^1}$) to some soliton at $t = 0$, then for all time it remains close to the same soliton with, possibly, parameters being slowly varied in time", was proved by several authors under different assumptions (see [4] for the review). It was shown [5] that the condition

$$\frac{d}{d\alpha}\|\phi(\cdot;\alpha)\|_{L_2} > 0, \tag{1.2}$$

where $\phi(\cdot;\alpha)$ is the soliton which norm depends on the parameter α, yields the stability. Moreover, the detailed investigation of the dispersive residue ("scattered wave"), which appears to be described by the linear Schrödinger equation, was performed [6], [7].

The instability may imply the blow-up, or collapse, of the solution, that is, an infinite growth of the amplitude in a finite time. The homogeneous power nonlinearity $F(\xi) = -\xi^p$ with $p \geq 2/N$ in $N-$dimensional case was normally the object of blow-up studies both analytically [8],[9] and numerically ([10] and references therein). Since the smallest "cubic" nonlinearity ($p = 1$) is critical or supercritical in dimensions $N \geq 2$, the examination of more general nonlinearities, e.g. $F(\xi) = -\xi^p + \delta\xi^q$, where the additional defocusing term $\delta\xi^q$ ($\delta > 0,\ q > p$) prevents the collapse, seems to be natural.

Of course, the nonlinear evolution equations were also extensively studied numerically by many researchers. Some papers regarding the context of the given paper will be mentioned below.

In order to understand the stability properties of solitons and keeping in mind the above motivation, we concentrate on the one-dimensional NLS equation with the nonlinearity providing the existence of the global solution of the initial problem. Our goal is to present some hints, mostly by means of numerical experiments, to understand the following questions which are naturally separated into two parts.

1. The nature of soliton's instability is the spectral properties of the operator arising in linearization of the equation (1.1) around a soliton. Under the condition $\frac{d}{d\alpha}\|\phi(\cdot;\alpha)\|_{L_2} < 0$ the spectral representation of this operator contains exponents $e^{\lambda t}$ with some positive λ-s, which lead to (linear) instability. The question is: what is the long-time behaviour of the solution which is represented by the whole perturbation series? If the inequality above holds for all α-s (e.g., the supercritical power nonlinearity implies this), generally, the answer is a blow-up. However, what one can expect if the inequality (1.2) changes at some point α_*? (That is, if the system admits both stable and unstable solitons.)

   To answer this question, we construct an exact but particular solution U_α of the NLS equation in the form of a perturbation series around the soliton with the parameter α from an unstable region $\alpha < \alpha_*$. Actually, this solution describes two possible scenarios: an unstable soliton either rebuilds to another (stable) soliton or totally disperses; these two cases are distinguished by the sign of the perturbation's projection on some eigenstate associated with the linearization of the NLS equation. (As an exception, soliton perturbed by the function which has no projection on the aforementioned eigenstate behaves in accordance with the *stability theorem.*) We simulate various perturbations and compare the results with this solution. It appears that in the first and more interesting case (the rebuilding to a stable soliton) the long-time behaviour of the simulated solution looks like $U_{\tilde{\alpha}}$, where $\tilde{\alpha}$ is close to α.

The simplest nonlinearity to provide the existence of both stable and unstable solitons is a two-term polynomial with $p > 2$ in the one-dimensional case. We realize that $p = 3$ in one-dimension may hardly be connected with some applications. However, we believe that observations described above are of general interest and may be reproduced in higher dimensions. For instance, some nonlinearity models which provide the existence of both stable and unstable solitons in three dimensions were examined numerically in the paper [11], and the rebuilding behaviour of solitons was observed.

2. Generally, the soliton's superstability for nonintegrable case can hardly be expected, although it sometimes is observed [11] in numerical simulations. However, the whole destruction of colliding solitons is very likely the possible scenario. So the following questions are open: a) what kinds of the long-time behaviour are possible, provided the initial function is the sum of colliding solitons? b) what are the initial solitonic properties, if any, to provide the solution which long-time asymptotics may be described by means of solitory waves?

   We claim that, under certain conditions on the function $F(\xi)$ and the parameters of colliding solitons, the trapping of energy into a few distinguishing channels is possible. To observe such an effect one needs two points. First, the nonlinearity has to provide the *critical* limitation of the solitonic parameter: $\alpha < \alpha_{cr}$, * details to be explained below (of course, the "cubic" nonlinearity doesn't satisfy this condition). Second, the α-s of the colliding solitons must be near α_{cr}: $\varepsilon \equiv (\alpha_{cr} - \alpha)/\alpha_{cr} \ll 1$.

   Under these conditions, the typical picture is the "rebuilding of solitons": in a short time after the collision the *new* (in contrary to the completely integrable case) stuff of solitons is formed. The process of rebuilding is concomitant by small but rapid radiation — the dispersive residue. However, the behaviour

* The reader should not be confused by the double-usage of the term *critical* (here and in connection with the blow-up).

of the long-time solution depends on the energy of incident solitons and the smallness of ε, and may be more complicated. As far as we know, the phenomenon of the soliton's rebuilding was not noticed for the NLS equation. Since the conditions of the existence of solitons in higher dimensions [12] also admit α_{cr}, we believe that this phenomenon is peculiar not only for the one spacial dimension but for higher dimensions as well. †

The paper is organized as follows. In the next section we list the known facts connected with the solitons of the one-dimensional NLS equation. The Section 3 is devoted to the construction of the particular solution in a form of a perturbation series, which is explored in the Section 4 to study the evolution of an unstable soliton. The numerical results on the collision of solitons is presented in the Section 5. Section 6 is a brief conclusion, and some technical details of the numerical computations are collected in the Appendix.

2. Background . In the sequel the one-dimensional nonlinear Schrödinger equation

$$iu_t = -u_{xx} + F\left(|u|^2\right)u. \tag{2.1}$$

with rapidly vanishing initial data is considered. The nonlinearity function $F(\xi)$ is supposed to be an arbitrary smooth real function, $F(0) = 0$ for the convenience, restricted by two conditions, providing the global bounded solution and the existence of the solitons, respectively:

1. $F(\xi) \geq -C\xi^{\mu}$ for large positive ξ, $\quad \mu < 2, \quad C > 0.$
2. Equation

$$V(\xi;\alpha) \equiv \alpha\xi^2 + \int_0^{\xi^2} F(\xi)\, d\xi = 0 \tag{2.2}$$

has positive root ξ_α for α belonging to some positive interval Δ and, at the same time, $V(\xi;\alpha) > 0$ for $\xi \in (0, \xi_\alpha)$ and

$$\left.\frac{\partial V}{\partial \xi}\right|_{\xi_\alpha} \neq 0. \tag{2.3}$$

The bounds of the interval Δ are either the point $\alpha = 0$ or *critical* points α_{cr} at which the derivative (2.3) is equal to zero.

Throughout the given paper, the following nonlinearity is used for all computations:

$$F(\xi) = -2(p+1)\xi^p + \frac{1}{\alpha_{cr}}(q+1)\xi^q, \quad 1 \leq p < q, \quad \alpha_{cr} > 0. \tag{2.4}$$

† The trapping of energy due to kink-antikink collisions in Klein-Gordon model was discussed in the series of papers [13],[14],[15],[16]. However, the resonance structure of this trapping is qualitatively unsimilar to the effects described in given paper.

This polynomial satisfy the conditions above and, at the same time, is a natural generalization of the cubic nonlinearity if $p = 1,\ q = 2$. ‡

The soliton is the two-parametric (up to translations in the space coordinate and the phase) solution of the equation (2.1), which rapidly vanishes if $|\, x \,| \to \infty$. It exists provided the parameters ω (frequency) and v (velocity) in the combination $\alpha = \omega + v^2/4$ satisfy the conditions (2.2)–(2.3). When the function

$$\phi(t, x; \omega, v) = \exp\left(\imath\omega t + \imath v x/2\right) u_0(s; \alpha), \tag{2.5}$$

where $s = x - vt$, is a soliton. Its amplitude $u_0(s; \alpha)$ is an even function of s which for $s > 0$ is implicitly determined by the relation

$$s = \int_{u_0}^{\xi_\alpha} d\xi / \sqrt{V(\xi; \alpha)}. \tag{2.6}$$

This integral converges while (2.3) holds.

The integrals

$$N = \int_{-\infty}^{\infty} |\, u \,|^2\, dx, \qquad Q = \int_{-\infty}^{\infty} \imath \bar{u} u_x dx, \tag{2.7}$$

$$H = \int_{-\infty}^{\infty} \left(|\, u_x \,|^2\, dx + \int_0^{|u|^2} F(\xi)\, d\xi \right)$$

are known to be conserved in time for any solution of the equation (2.1). Being computed for the soliton $u(t, x) = \phi(t, x; \omega, v)$, its depend only on the parameter α.

Consider the linearization of the equation (2.1) around the soliton (2.5):

$$u(t, x) = \exp(\imath\omega t + \imath \frac{v}{2} x) \left(u_0(s; \alpha) + \delta u(s; \alpha) + \ldots\right). \tag{2.8}$$

The pair of functions $(\delta u, \delta \bar{u})$ satisfies the equation

$$\imath \partial_t \begin{pmatrix} \delta u \\ \delta \bar{u} \end{pmatrix} = \hat{L}(s; \alpha) \begin{pmatrix} \delta u \\ \delta \bar{u} \end{pmatrix}, \tag{2.9}$$

where

$$\hat{L}(s; \alpha) = \begin{pmatrix} -\partial_s^2 + \alpha + F(u_0^2) + F'(u_0^2) u_0^2, & F'(u_0^2) u_0^2 \\ -F'(u_0^2) u_0^2, & \partial_s^2 - \alpha - F(u_0^2) - F'(u_0^2) u_0^2 \end{pmatrix},$$

where $u_0 = u_0(s; \alpha)$.

‡ It can be shown by simple scaling transformations that the coefficients of this polynomial may be fixed without loss of generality.

Keeping in mind a further convenience, we rewrite the equation (2.9) in a basis of real functions $\delta u_1, \delta u_2$ such that $\delta u = \delta u_1 + \imath\delta u_2$. In a new basis the equation (2.9) becomes

$$(2.10) \qquad \partial_t \begin{pmatrix} \delta u_1 \\ \delta u_2 \end{pmatrix} = L(s;\alpha) \begin{pmatrix} \delta u_1 \\ \delta u_2 \end{pmatrix},$$

where now

$$L(s;\alpha) = \begin{pmatrix} 0 & , & -\partial_s^2 + \alpha + F(u_0^2) \\ \partial_s^2 - \alpha - F(u_0^2) - 2F'(u_0^2)u_0^2 \, , & & 0 \end{pmatrix}.$$

The equations (2.9) and (2.10) are considered in the space of rapidly vanishing functions. It is known that the operator L has the continuous spectrum $(-\imath\infty, -\imath\alpha] \cup [\imath\alpha, \imath\infty)$ (points $\pm\imath\alpha$ may be resonances) and a finite number of eigenvalues on real and imaginary axes, $\lambda = 0$ always belongs to the spectrum, and the kernel space is at least four-dimensional. It is important (see [5],[6]) that the condition $N'(\alpha) < 0$ is necessary and sufficient that some eigenvalues to be on real axis of the spectral plane, symmetrically around the origin [5],[6].

3. Perturbation series and recurrent relations . In this and next sections the behaviour of an unstable soliton under perturbations is discussed.

Suppose that the nonlinearity function $F(\xi)$ provides a minimum to the function $N(\alpha)$ at some point α_* and consider the whole perturbation series:

$$(3.1) \qquad \psi(t,x) = \exp(\imath\omega t + \imath\frac{v}{2}x)\left(u_0(s;\alpha) + \epsilon u_1(t,s;\alpha) + \ldots\right),$$

where ϵ is a formal small parameter. By substitution of the expansion (3.1) into the equation (2.1) and putting to zero the coefficients of the powers of ϵ one obtains the recurrent partial differential equations

$$(3.2) \qquad \imath\partial_t \begin{pmatrix} u_n \\ \bar{u}_n \end{pmatrix} = \hat{L}(s;\alpha) \begin{pmatrix} u_n \\ \bar{u}_n \end{pmatrix} +$$

$$+ \sum_{k=1}^{n-1} F_{n-k} \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix} \begin{pmatrix} u_k \\ \bar{u}_k \end{pmatrix} + G_{n-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} u_0, \quad n = 1, 2, \ldots .$$

Here the operator $\hat{L}$ is the same as in the equation (2.9),

$$(3.3) \qquad F_k(u_0, u_1, \ldots, u_k) = \frac{1}{k!}\frac{d^k}{d\epsilon^k} F\left(|u_0 + \epsilon u_1 + \ldots|^2\right)\Big|_{\epsilon=0}.$$

These coefficients are evidently real. They depend on the function u_k in the following way:

$$(3.4) \; F_k(u_0, u_1, \ldots, u_k) = F'(u_0^2)u_0(u_k + \bar{u}_k) + G_{k-1}(u_0, u_1, \ldots, u_{k-1}), \quad G_0 \equiv 0.$$

The last relation defines the function G_{n-1} in (3.2).

The equation (3.2) in the real basis u_{n1}, u_{n2}, $u_n = u_{n1} + \imath u_{n2}$, has the form

$$\partial_t \begin{pmatrix} u_{n1} \\ u_{n2} \end{pmatrix} = L(s;\alpha) \begin{pmatrix} u_{n1} \\ u_{n2} \end{pmatrix} + \tag{3.5}$$

$$+\sum_{k=1}^{n-1} F_{n-k} \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix} \begin{pmatrix} u_{k1} \\ u_{k2} \end{pmatrix} + G_{n-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} u_0, \quad n = 1, 2, \dots .$$

Now we discuss a way of computing the coefficients (3.3) by a recurrent procedure. Note that these coefficients can be written as

$$F_k(u_0, u_1, \dots, u_k) = \frac{1}{k!}\frac{d^k}{d\epsilon^k} F\left(w_0 + \epsilon w_1 + \dots\right)\Big|_{\epsilon=0} = \tag{3.6}$$

$$= \frac{1}{k!}\frac{d^{k-1}}{d\epsilon^{k-1}} \left[F'\left(w_0 + \epsilon w_1 + \dots\right)\left(w_1 + 2\epsilon w_2 + \dots\right)\right]\Big|_{\epsilon=0} =$$

$$= \frac{1}{k!}\sum_{j=0}^{k-1} f_j^{(1)} j! C_{k-1}^j (k-j)! w_{k-j} = \sum_{j=0}^{k-1} f_j^{(1)} \frac{k-j}{k} w_{k-j},$$

where $|u_0 + \epsilon u_1 + \dots|^2 = w_0 + \epsilon w_1 + \dots$, that is,

$$w_j = \sum_{i=0}^{j} (u_{i1} u_{j-i,1} + u_{i2} u_{j-i,2}), \tag{3.7}$$

and

$$f_j^{(1)} = \frac{1}{j!}\frac{d^j}{d\epsilon^j} F'\left(w_0 + \epsilon w_1 + \dots\right)\Big|_{\epsilon=0}, \quad j = 0, 1, \dots, k-1 .$$

While $f_0^{(1)}$ is simply equal to $F'(u_0^2)$, we can apply, in turn, the formula (3.6) to $f_j^{(1)}$ for $j = 1, 2, \dots, k-1$. It leads to a new series of coefficients $f_j^{(2)}$, $j = 0, 1, \dots, k-2$, among which $f_0^{(2)} = F''(u_0^2)$. Then we again apply the formula (3.6) to $f_j^{(2)}$, etc. This process is interrupted after $k-1$ steps, and finally $f_0^{(k)} = F^{(k)}(u_0^2)$. Thus, the recurrent procedure to obtain k coefficients $f_j^{(1)}$ looks as follows:

$$f_j^{(l)} = \sum_{i=0}^{j-1} f_i^{(l+1)} \frac{j-i}{j} w_{j-i}, \quad 1 \le j \le k-l, \quad l = k-1, \dots, 1, \tag{3.8}$$

$$f_0^{(l)} = F^{(l)}(u_0^2).$$

When the coefficients $f_j^{(1)}$ are known, the function G_{n-1} can also be computed:

$$G_{n-1} = \sum_{j=1}^{n-1} \left(f_j^{(1)} \frac{n-j}{n} w_{n-j} + F'(u_0^2)(u_{j1}u_{n-j,1} + u_{j2}u_{n-j,2}) \right). \tag{3.9}$$

Suppose that solitonic parameter α is fixed in such a way that $N'(\alpha) < 0$, so the operator $L(s;\alpha)$ in (3.5) has a finite number of eigenvalues on real axis which are located symmetrically around the origin [5],[6]. Let $\lambda_*(\alpha)$ denotes the largest positive eigenvalue.

The equation (3.5) for $n = 1$ is a uniform one. We will be interested in the solutions of the recurrent equations which are generated by the eigenfunction of the operator L corresponding to λ_*. So we fix the solution of (3.5) for $n = 1$ as

$$\begin{pmatrix} u_{11}(t,s) \\ u_{12}(t,s) \end{pmatrix} = Ce^{\lambda_* t} \begin{pmatrix} \varphi_1^{(1)}(s) \\ \varphi_2^{(1)}(s) \end{pmatrix}, \tag{3.10}$$

where

$$L \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \end{pmatrix} = \lambda_* \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \end{pmatrix} \tag{3.11}$$

and C is a some normalizing constant.

By the examination of the expressions (3.6)–(3.9) one can verify that functions F_k and G_{n-1} in (3.5) hold the following property:

$$F_k(u_0, \gamma u_1, \ldots, \gamma^k u_k) = \gamma^k F_k(u_0, u_1, \ldots, u_k), \tag{3.12}$$

$$G_{n-1}(u_0, \gamma u_1, \ldots, \gamma^{n-1} u_{n-1}) = \gamma^n G_{n-1}(u_0, u_1, \ldots, u_{n-1}).$$

Together with (3.10), that yields that recurrent equations (3.5) have the series of solutions

$$\begin{pmatrix} u_{n1}(t,s) \\ u_{n2}(t,s) \end{pmatrix} = C_n e^{\lambda_* t} \begin{pmatrix} \varphi_1^{(1)}(s) \\ \varphi_2^{(1)}(s) \end{pmatrix} + C^n e^{n\lambda_* t} \begin{pmatrix} \varphi_1^{(n)}(s) \\ \varphi_2^{(n)}(s) \end{pmatrix}, \tag{3.13}$$

where $\begin{pmatrix} \varphi_1^{(n)}(s) \\ \varphi_2^{(n)}(s) \end{pmatrix}$, $n \geq 2$, have to satisfy the equations

$$(L(s;\alpha) - n\lambda_0) \begin{pmatrix} \varphi_1^{(n)} \\ \varphi_2^{(n)} \end{pmatrix} + \sum_{k=1}^{n-1} F_{n-k}(u_0, \varphi^{(1)}, \ldots, \varphi^{(n-k-1)}) \times \tag{3.14}$$

$$\times\begin{pmatrix}0, & 1\\ -1, & 0\end{pmatrix}\begin{pmatrix}\varphi_1^{(k)}\\ \varphi_2^{(k)}\end{pmatrix}+G_{n-1}(u_0,\varphi^{(1)},\dots,\varphi^{(n-1)})\begin{pmatrix}0\\ -1\end{pmatrix}u_0=0,$$

$\varphi^{(k)}\equiv\varphi_1^{(k)}+\imath\varphi_2^{(k)}$. Since λ_* is a maximum eigenvalue of the operator $L(s;\alpha)$, operator $\left(L(s;\alpha)-n\lambda_*\right)^{-1}$ exists for $n\geq 2$ and functions $\varphi^{(n)}$ can be found.

Finally, putting all constants C_n in (3.13) to be equal to zero, we obtain the *particular* solution of the equation (2.1)

$$U_\alpha(z;s)\equiv u_0(s;\alpha)+\sum_{k\geq 1}z^k\left(C^k\varphi^{(k)}(s)\right),\quad z=\epsilon e^{\lambda_* t}. \tag{3.15}$$

It is well time now to recall that $U_\alpha(z;s)$ depends on α via λ_* and $\varphi^{(k)}(s)$.

4. Numerical computation of function (3.15) and long-time behaviour of unstable solitons . Without loss of generality we can now considered only ground solitons: $v=0,\ s\equiv x$.

First of all let us note that the parameter ϵC can be fixed by an appropriate shift in the t-variable. [§] Below we remove this constant ($\epsilon C=1$), keeping also in mind that eigenfunction $\varphi^{(1)}(x)$ (3.10) is fixed in such a way that $\mathrm{re}\{\varphi^{(1)}(0)u_0(0;\alpha)\}>0$. However, there are two different solutions, $U_\alpha(+z;x)$ and $U_\alpha(-z;x)$, $z=e^{\lambda_* t}$, which cannot be transformed one to another by a finite shift in time. In fact, these two solutions are the basis for the understanding of the long time behaviour of an unstable soliton.

Generally, $U_\alpha(\pm z;x)$ may be computed by means of the series (3.15) of the previous section. Some technical details of such a computation are collected in the Appendix. However, one hardly should expect that the series can be utilized for computations if z is large. In case of functions $U_\alpha(\pm z;x)$ the threshold of trustable computations has a clear physical explanation.

As shown below, the function $U_\alpha(+z;x)$ describes the rebuilding soliton. The essential transformation of this solution takes place on a short time interval and is concomitant by a rapid radiation. Unfortunately, it appears to be very difficult to handle this radiation numerically while solving the recurrent equations (3.14). The accumulation of errors due to the reflection of a radiation from the edges of the numerical grid yields that the numerical summation of the series (3.15) exhibits a *false* pole at some positive z corresponding to a time of essential rebuilding. Actually, we believe that the function $U_\alpha(z;x)$ has singularities somewhere in the complex plane z, but the question of the analytical properties of this function remains open now. [¶]

[§] By the way, this implies that the integrals of motion (2.7) does not depend on ϵC, and this observation was used for the control of the numerical computation of the function $U_\alpha(z;x)$.

[¶] An unexpected bonus of manipulations with the series (3.15) is that one can numerically estimate the blow-up time for *supercritical* case: $N'(\alpha)<0$ for all α. In this case the solution $U_\alpha(z;x)$ is expected to have a *true* real positive singular point which can be extracted from the computation of the series (3.15). The *critical* case $N'(\alpha)\equiv 0$ cannot be handled in the same way since $\lambda_*=0$ and the operator in (3.14) is not invertible.

From the other hand, since functions $U_\alpha(\pm z; x)$ are exact solutions of the NLS equation, one can compute its for any z if: 1) compute these functions for sufficiently small z and 2) use these data as the initial function for the simulation of the equation (2.1). The results presented below are obtained by this way.

Now we begin the discussion of the numerical results. The nonlinearity is supposed to be the polynomial (2.4). If $p \geq 2$ there exists some α_* such that $N'(\alpha) < 0$ for $\alpha < \alpha_*$. All computations of the given section performed for $p = 3,\ q = 6,\ \alpha_{cr} = 10$; for this example, $\alpha_* \approx 5.714$ (see Fig. 4.1).

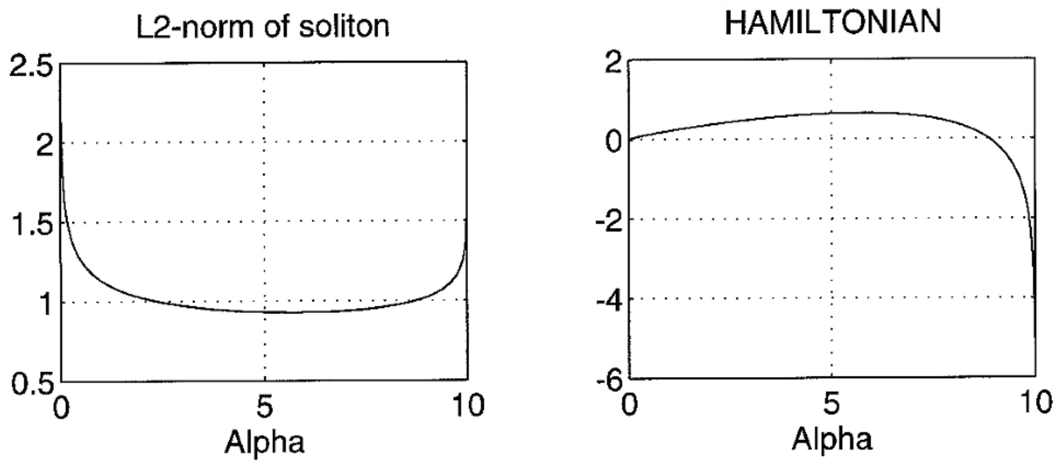

FIG. 4.1. L_2-norm of soliton and Hamiltonian versus α

First, the eigenvalues of the operator L are computed. We concentrate on the accurate computation of real eigenvalues, and the results for complex eigenvalues should not be trusted (see Appendix). Nevertheless, we do not observe any disagreements with the theory, that is: 1) all eigenvalues lay on real or imaginary axes; 2) most of them are located at the lines which are occupied by the continuous spectrum; 3) the four eigenvalues close to zero are always observed. For all examined cases only the one (up to the reflection) real eigenvalue λ_* is appeared. The Table 4.1 demonstrates the dynamics of this eigenvalue versus α, the last column in this table will be explained lately. The behaviour of few first functions $\varphi^{(k)}(x)$, including the eigenfunction $\varphi^{(1)}(x)$, is shown in the Fig. 4.2.

The main numerical results of this section are as follows: Fig. 4.3 and Fig. 4.4, in which the simulation of $U_\alpha(+e^{\lambda_* t}; x)$ and $U_\alpha(-e^{\lambda_* t}; x)$, respectively, is presented, show two essentially different scenarios of the evolution of an unstable soliton. While the solution $U_\alpha(-e^{\lambda_* t}; x)$ corresponds to the whole dispersion of the soliton, the first one, $U_\alpha(+e^{\lambda_* t}; x)$, exhibits the short-time process of the rebuilding of the initial soliton ($\alpha < \alpha_*$) to a new soliton with the parameter α_∞ belongs to the stable region ($\alpha_\infty > \alpha_*$). Some part of the energy is taken away by the dispersive radiation which

α	$\lambda_*(\alpha)$	α_∞
0.1	0.2671	9.3466
1.0	2.6094	8.9067
2.0	4.5577	8.3653
3.0	5.7070	7.7553
4.0	5.8392	6.9256
5.0	4.4697	6.1671
5.50	2.5637	5.7620
5.60	1.7484	5.8214
5.70	0.9807	5.8296
5.71	0.7741	5.8297
5.75	0.8961$\imath$	—
5.80	1.5893$\imath$	—
6.0	3 0751$\imath$	—
7.0	6.6699$\imath$	—
7.35	7.3259$\imath$	—

TABLE 4.1
λ_* and its complex continuation versus α. Right column: long-time soliton's parameter

is concomitant to the process of rebuilding.

The "mapping of rebuilding" $\alpha \to \alpha_\infty$ is summarized in the last column of the Table 4.1.

Next results demonstrate that the solution $U_\alpha(+e^{\lambda_* t};x)$ provides the long-time behaviour of an arbitrary, under some condition (see below), perturbation of an unstable soliton. Let $U_\alpha^{(-)}(+e^{\lambda_* t};x)$ denotes the another exact particular solution of (2.1) which is constructed by the same way as $U_\alpha(\pm e^{\lambda_* t};x)$ but associated with the eigenvalue $-\lambda_*$ (now we fix the corresponding eigenfunction $\varphi_{(-)}^{(1)}(x)$ by the condition $\left(\varphi_{(-)}^{(1)},\varphi^{(1)}\right)u_0(0;\alpha)>0$).

We compare the simulation of $U_\alpha^{(-)}(+e^{\lambda_* t};x)$ and of the Gaussian perturbation

$$u|_{t=0} = u_0(x;\alpha) + \nu\exp(-\gamma x^2) \tag{4.1}$$

with $U_\alpha(+e^{\lambda_* t};x)$ in the Fig. 4.5. One can see that in all cases the stable soliton dominating in the final stage has slightly different α_∞-s. "Mappings of rebuilding" for $U_\alpha^{(-)}(e^{\lambda_* t};x)$ and $U_\alpha(e^{\lambda_* t};x)$ are identical with the neglect able difference. At the same time, for other perturbations the deviation of α_∞ is small but noticeable. To explain this difference, let us note that eigenfunctions $\varphi_{(-)}^{(1)}$ and $\varphi^{(1)}$ are orthogonal to the other spectral subspaces of the operator L, which are responsible for the *"stable"* evolution of the initial soliton (however, these functions don't orthogonal one to another). Thus, except the case of $U_\alpha^{(-)}(e^{\lambda_* t};x)$, the small and slow variations of soliton are exited before the rebuilding.

The above examples give us an idea how to distinguish the perturbation which yields the rebuilding of the soliton (recall that the alternative is the dispersion). If

$\delta u(x)$ denotes the perturbation of the soliton $u_0(x;\alpha)$, the inequality

$$\left(\delta u, \varphi^{(1)}\right) u_0(0;\alpha) > 0 \tag{4.2}$$

implies the rebuilding. The explanation of the condition (4.2) is as follows: to give rise for the rebuilding, $\delta u(x)$ has to have non-zero projection on the eigenstate $\varphi^{(1)}(x)$ and, at the same time, tends to increase the amplitude of the initial soliton $u_0(x;\alpha)$. The decreasing of the amplitude of $u_0(x;\alpha)$ by the perturbation shifts the soliton more into the unstable region and yields the whole dispersion. As an exception, any soliton is stable under the perturbation which has no projection on the eigenstate $\varphi^{(1)}(x)$.

5. Rebuilding near critical solitons due to collisions . In this section we present the results of the numerical simulation of soliton's collisions. We consider the initial data of the form

$$u(0,x) = \phi(0, x - x_0; \omega, v) + \phi(0, x + x_0; \omega, -v), \tag{5.1}$$

where x_0 is sufficiently large to neglect the interaction of the initial solitons with a computer accuracy. So only the collision of synchronized in phase solitons of the same frequency ω and velocity v is examined. As a matter of fact, the desynchronization and the variation of parameters ω and v does not provide any other scenario different from the ones listed below and will not be discussed here.

To specify the further restrictions on the function $F(\xi)$, we suppose that for some α_{cr} two equations

$$V(\xi_{\alpha_{cr}};\alpha_{cr}) = 0, \qquad \left.\frac{\partial V(\xi;\alpha_{cr})}{\partial \xi}\right|_{\xi=\xi_{\alpha_{cr}}} = 0,$$

where V is determined by (2.2), are satisfied simultaneously. The integral (2.6) diverges for $\alpha = \alpha_{cr}$, hence soliton exists only if $\alpha \equiv \omega + v^2/4 < \alpha_{cr}$. So α_{cr} is the *critical* value for soliton solutions. The simplest representative of the nonlinearity with such a property is, again, the polynomial (2.4), but now with no restrictions on its powers. Unless mot marked especially, $p = 1, q = 2$ and $\alpha_{cr} = 10$ are used in numerical examples.

Most interesting effects are observed for small and very small values of $\varepsilon \equiv (\alpha_{cr} - \alpha)/\alpha_{cr}$ of colliding solitons (5.1). We also will not discuss here the case $\varepsilon \approx 1$.

Some possible scenarios of the collision are presented by Fig. 5.1 —Fig. 5.3. The main inference from this presentation that the energy of colliding solitons in a short time after the collision distributes into few channels. The propagation inside each of these channels exhibits the soliton-like behaviour with, possibly, modulated parameters. By another words, a new stuff of solitons appears due to the collision. The dispersive radiation is burn during the process of soliton's rebuilding. Also a

ε	v	$\varepsilon_{0\infty}$	$\varepsilon_{v\infty}$, v
0.025	0.5	0.00100	—
0.025	2	0.00124	—
0.025	8	0.02118	—
0.1	2	0.01330	—
0.1	8	0.00672	—
0.001	2	0.00476	—
0.001	8	0.00566	—
0.00001	2	0.00396	—
0.00001	5	0.77348	0.00070, 1.0000
0.00001	6	0.32548	0.00084, 1.1048
0.00001	8	0.29494	0.00252, 1.3662
0.00001	10	0.47414	0.00541, 1.5893

TABLE 5.1
"Mapping of rebuilding" of solitons due to collision

small radiation is concomitant to the propagation of a soliton with the modulation of the amplitude and the frequency.

The kind of the trapping depends on the smallness of ε. For *not* very small values of ε ($\varepsilon > 0.01 - 0.1$) one can see the rebuilding "two to one": the final stage of the process is a single ground soliton with a dispersive radiation far away from it (Fig. 5.1). However, the picture is changed if ε becomes more smaller, and the final behaviour strongly depends on the energy of the incident solitons. Fig. 5.2 corresponds to the moderate velocity and demonstrates the formation of still single ground soliton but with slowly decaying variations of the amplitude. This modulated soliton radiates periodically. At last, the most interesting picture arises in the case of fast colliding solitons: besides the ground (and modulated) soliton, the long-time solution contains another solitons travelling with different velocities (Fig. 5.3). The Table 5.1 contains the summary of "mappings of rebuilding" $(\varepsilon, v) \to (\varepsilon_{0\infty}, \varepsilon_{v\infty})$, where $\varepsilon_{0\infty}$ and $\varepsilon_{v\infty}$ corresponds, respectively, to an average amplitude of the ground and travelling (if available) solitons. In the bottom portion of this table one can see that the trapped energy moves from the ground soliton to the travelling ones when the velocity v is increased.

In case of the more power nonlinearity the rebuilding "two to one" is the typical case, that is, it is observed for the wide range of ε-s (see Table 5.2). We believe that the phenomenon of multiplying of solitons after the collision is peculiar to the powers $p \geq 2$ as well but for significantly larger energies. The simulation of such highly oscillating solutions requires the modification of the numerical technique (e.g., as was applied in [17]), which is not carried out in the given work.

ε	v	$\varepsilon_{0\infty}$
0.5	2	0.06494
0.1	2	0.00078
0.001	2	$< 10^{-5}$
0.0001	2	$< 10^{-5}$

TABLE 5.2
The same as in the Table 5.1 but for $p = 2$, $q = 4$ in (2.4)

6. Conclusion . In the given paper we discussed the numerical simulation of some solutions of the nonintegrable one-dimensional NLS equation with the polynomial nonlinearity. These solutions may be characterized as describing the rebuilding of solitons. One of such solutions, $U_\alpha(e^{\lambda_* t}; x)$, arises as a perturbation series around the unstable soliton $u_0(x;\alpha)$, $\alpha < \alpha_*$, and entirely determined by the initial soliton. At a long-time, this solution is a new soliton with the parameter $\alpha_\infty > \alpha_*$ plus the dispersive residue. We formulate now the following conjecture based on the numerical experiments with the solution $U_\alpha(e^{\lambda_* t}; x)$: *if $u|_{t=0} = u_0(x;\alpha) \pm \delta u(x)$, where $\delta u(x)$ is small in the appropriate norm, then, for the some sign of the perturbation, $u(x,t) = U_{\tilde\alpha}(e^{\lambda_* t}; x) + o(1)$ when $t \to \infty$, where $\tilde\alpha$ is close to α.*

We also found out the complicated rebuilding processes which arise due to collisions of solitons with such parameters that $\varepsilon \equiv (\alpha_{cr} - \alpha)/\alpha_{cr} \ll 1$, where α_{cr} is the critical limit for the solitonic parameter α. After the collision, the solution consists of few solitons, which parameters differ from the initial ones, and dispersive radiation. The velocities and the frequencies of new solitons strongly depend on the ε and on the energy of colliding solitons. The understanding of this dependence is a challenge for the further researches.

Acknowledgments
The author is indebted to Prof V.S.Buslaev for stimulative discussions.

Appendix
Here we discuss the computation scheme and the accuracy of numerical results presented in the given paper.

Our simulation approach is the discretization of equation (2.1) by means of an explicit difference scheme. Let τ, h denote the steps of discretization in time and space variables, respectively. The standard explicit scheme is of $O(\tau) + O(h^2)$ accuracy. It turns out that the simulation of soliton-like solutions with large derivatives requires too small discretization that reasonable accuracy be achieved in a standard explicit scheme. As a sequence, the computation fails due to the accumulation of rounding errors. So we apply the $O(h^4)$ approximation of the second spatial derivative and the generalization of an explicit scheme normally called the method of (long) lines.

That means the substitution of the partial differential equation (2.1) by the set of ordinary equations

$$(.1)\quad \imath\frac{du_j}{dt} = -\frac{-u_{j-2}+16u_{j-1}-30u_j+16u_{j+1}-u_{j+2}}{12h^2} + F\left(\mid u_j \mid^2\right)u_j + O(h^4),$$

$$j = -M, \ldots, M,$$

where M is sufficiently large. Standard Runge-Kutta approach is applied to the system (.1), so the discretization is of $O(\tau^4)$ accuracy. We found that the results converged for the step sizes $\tau = 0.001, \quad h = 0.05$ (the stability condition of the applied scheme is $\tau \le \frac{2}{5}h^2$). The convergency of the results is estimated by the following inequalities:

$$\frac{d}{dt}\|\cdot\|_C \le 0.001, \quad \frac{d}{dt}\|\cdot\|_{L_2} \le 0.005, \quad \frac{d}{dt}\|\cdot\|_{H^1} \le 0.01,$$

where $\|\cdot\|$ denotes $\|\psi_{\tau,h} - \psi_{\tau/4,h/2}\|$.

Since the simulated processes are concomitant by the rapid radiation, the special care should be included to eliminate the reflection of the energy from edges of the numerical grid. The method used in this work is simply to extended the numerical grid together with the propagating radiation. The Fig. .1 gives us an idea about the necessary speed of such an extension.

The good tool to control the accuracy of simulation is the integrals of motion (2.7). The typical relative deviations of N and H are shown in the Fig. .2, peaks correspond to the regions of the fastest variation of the solution.

Now we came to the discussion regarding the numerical spectral evolution of the operator (2.10) and the inversion of $(L(x;\alpha) - n\lambda_*)$ in recurrent relations (3.14). Again, the approach is the approximation of the operator L by the finite difference scheme (.1) of the accuracy $O(h^4)$. To restrict the problem on a finite interval, one needs the estimation of the exponential decaying of the solution to be computed. This estimation for the eigenfunction (3.10) is

$$(.2)\qquad \varphi^{(1)}(x) \sim \mathrm{const}\exp\left(-\mathrm{re}\sqrt{\alpha \pm \imath\lambda_*}\mid x \mid\right), \quad \mid x \mid \to \infty.$$

Since only the real eigenvalues (with eigenfunctions (.2) decaying more rapidly than soliton u_0 itself) are of interest, one can restrict the numerical grid by the interval corresponding to the decay of the soliton u_0. Note that if the estimate $\mid \lambda \mid \ge \mu > 0$ is someway determined *a priory* (e.g., by means of a draft computation), the solution may be considered on a shorter interval.

After some experiments we concluded that each of the eigenfunctions $\varphi_1^{(1)}$, $\varphi_2^{(1)}$ must be represented by its values at approximately 400 points, with a grid step h

is about 0.05. That is, operator L is approximated by the matrix of a size about 800×800.

However, the computation of the whole series to obtain the solution $U_\alpha(e^{\lambda_* t}; x)$ faces some additional differences. The fact is this solution contains, besides the rebuilding soliton, the rapid radiation. To handle this radiation properly, it is more convenient to work in the space of functions which decay not so fast as a soliton. The idea is to use the substitution

$$\varphi^{(k)}(x) = u_0(x;\alpha)\psi^{(k)}(x).$$

In the new functional space operator L has the form

$$L\begin{pmatrix} u_0\psi_1^{(k)} \\ u_0\psi_2^{(k)} \end{pmatrix} = u_0\mathcal{L}\begin{pmatrix} \psi_1^{(k)} \\ \psi_2^{(k)} \end{pmatrix}, \quad \psi^{(k)} = \psi_1^{(k)} + \imath\psi_2^{(k)},$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & , & -\partial_x^2 - 2\frac{u_0'}{u_0}\partial_x \\ \partial_s^2 + 2\frac{u_0'}{u_0}\partial_x - 2F'(u_0^2)u_0^2 \, , & & 0 \end{pmatrix}.$$

Being considered as an operator in the space of decaying functions (that means functions $\varphi^{(k)}(x)$ vanish faster than soliton $u_0(x;\alpha)$), $\mathcal{L}$ has the continuous spectrum on the imaginary axis, except the origin, and the same real eigenvalues, if any, as the operator L. Due to the last observation the operator $\mathcal{L}$ can be used for the computation of the series (3.15).

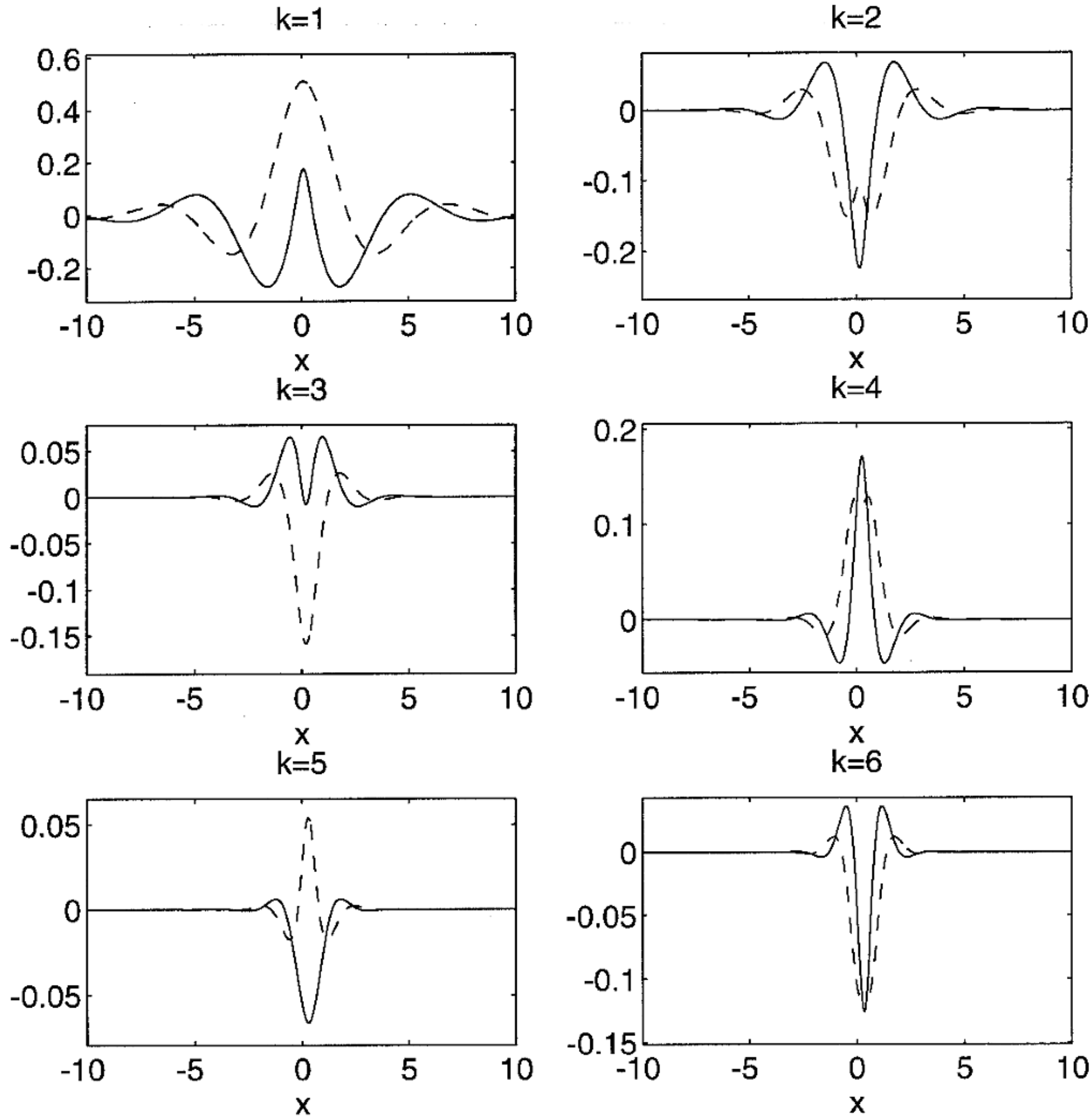

FIG. 4.2. *Six first coefficients of the series (3.15);* $\alpha = 1$. *Solid line —* $re\{\varphi^{(k)}(x)\}/u_0(x;\alpha)$, *dashed line —* $im\{\varphi^{(k)}(x)\}/u_0(x;\alpha)$

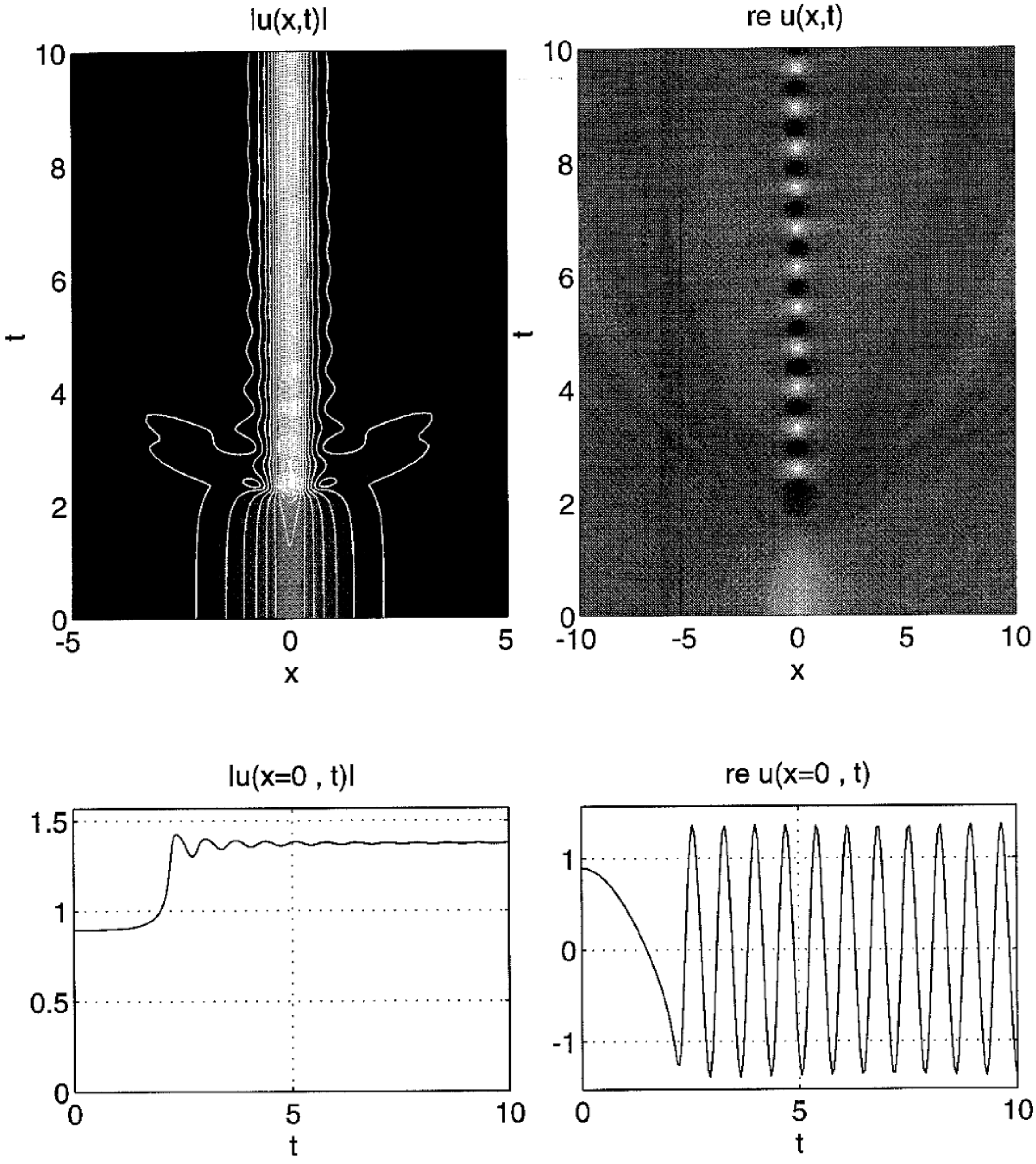

FIG. 4.3. *Simulation of the solution* $U_\alpha(+e^{\lambda_* t}; x)$; $\alpha = 1$

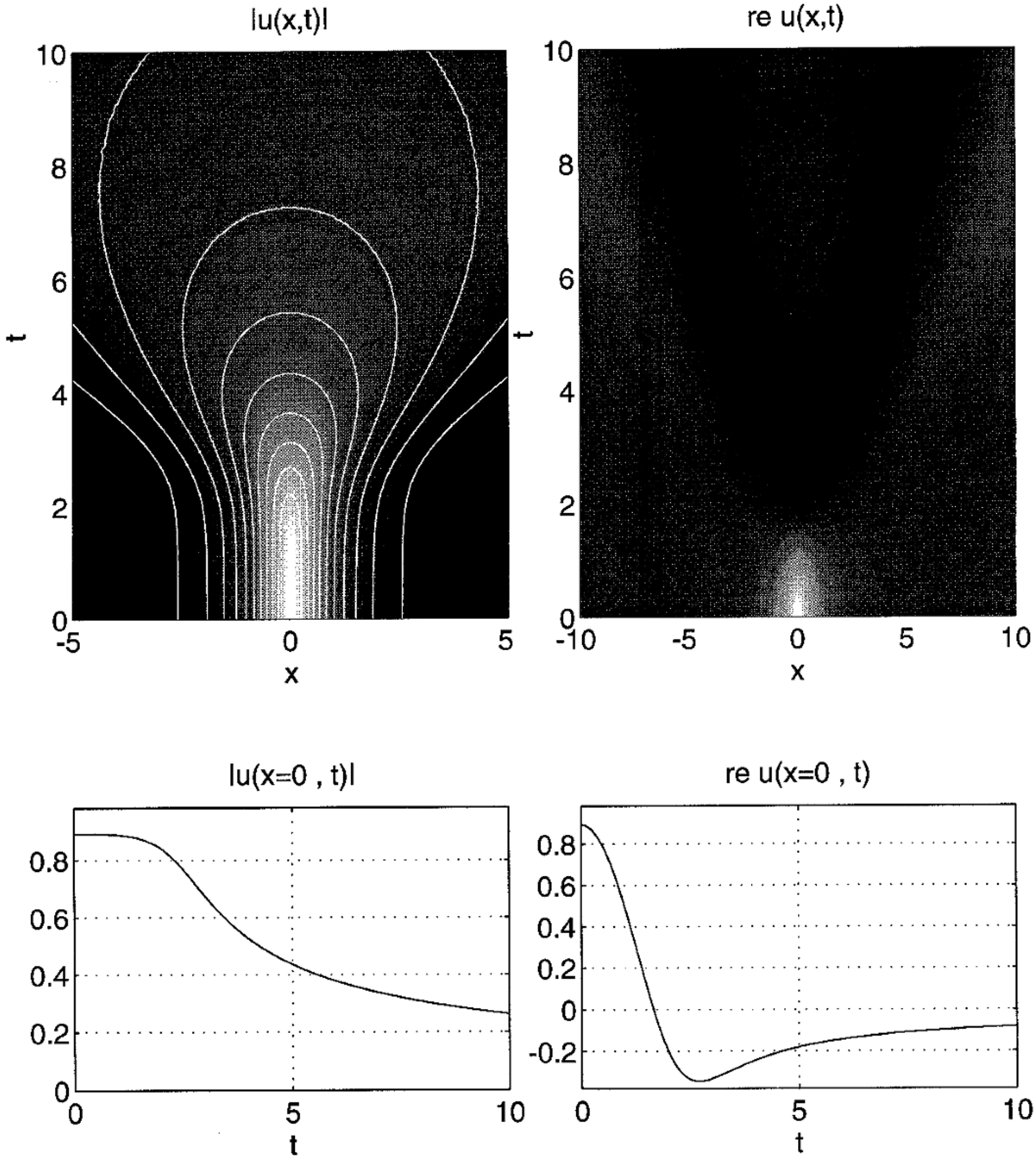

FIG. 4.4. *Simulation of the solution $U_\alpha(-e^{\lambda_* t}; x)$; $\alpha = 1$*

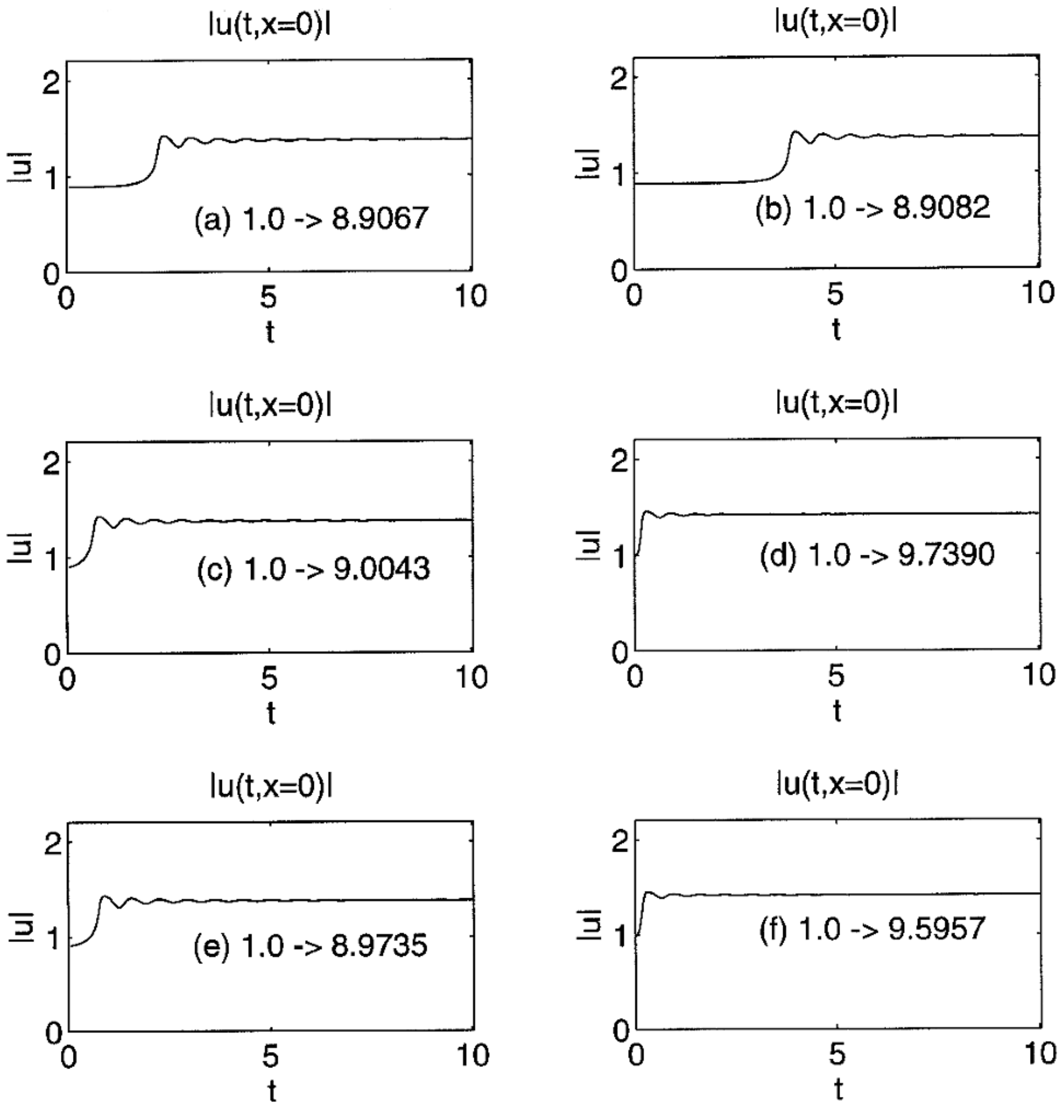

FIG. 4.5. *Different perturbations of the soliton* $\alpha = 1$. *(a) Solution* $U_\alpha(+e^{\lambda_* t}; x)$; *(b) Solution* $U_\alpha^{(-)}(+e^{\lambda_* t}; x)$; *Gaussian perturbation (4.1): (c)* $\nu = 0.01$, $\gamma = 0.1$; *(d)* $\nu = 0.1$, $\gamma = 0.1$; *(e)* $\nu = 0.01$, $\gamma = 1$; *(f)* $\nu = 0.1$, $\gamma = 1$

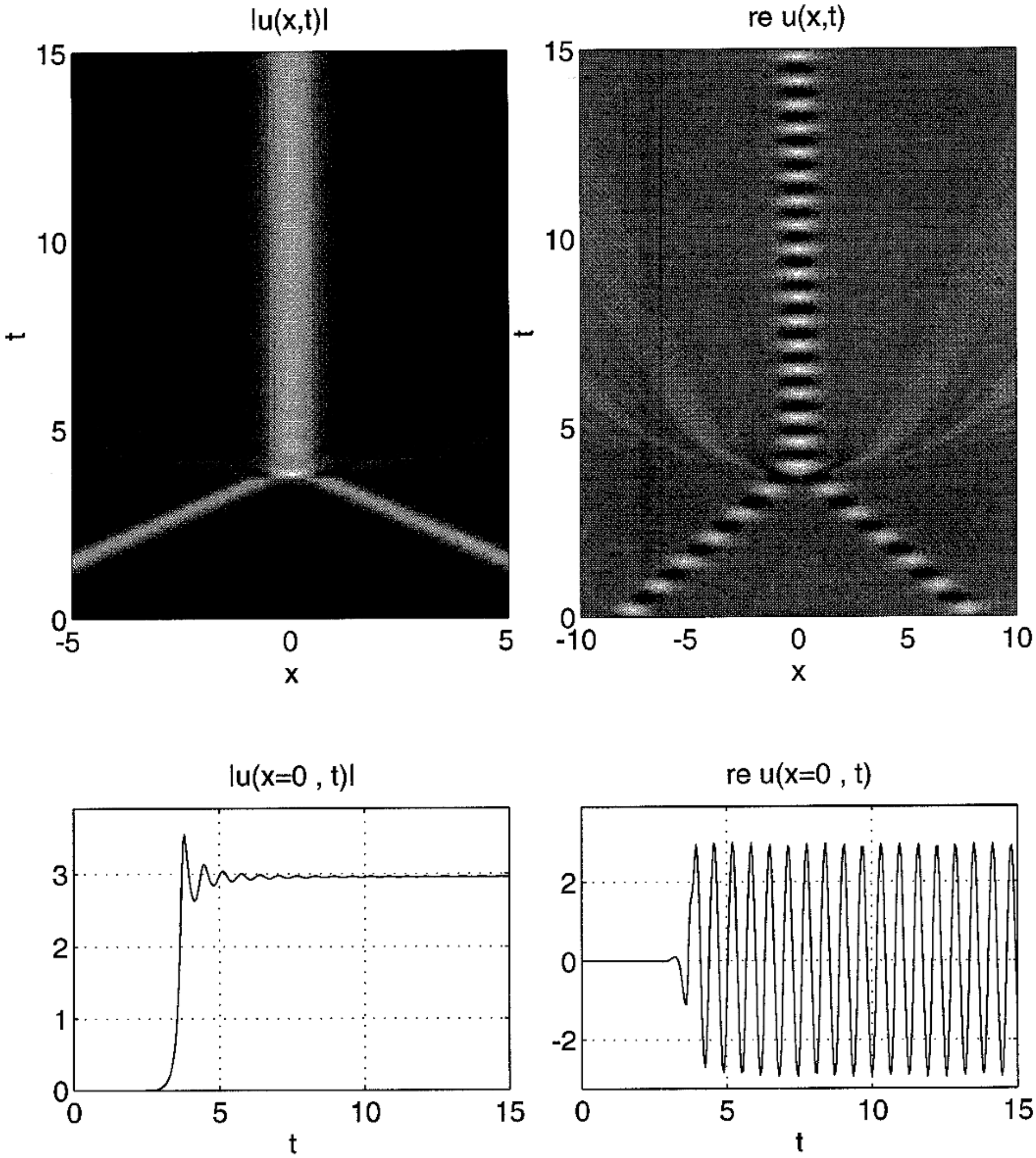

FIG. 5.1. *Simulation of soliton's collision:* $\varepsilon = 0.1$, $v = 2$

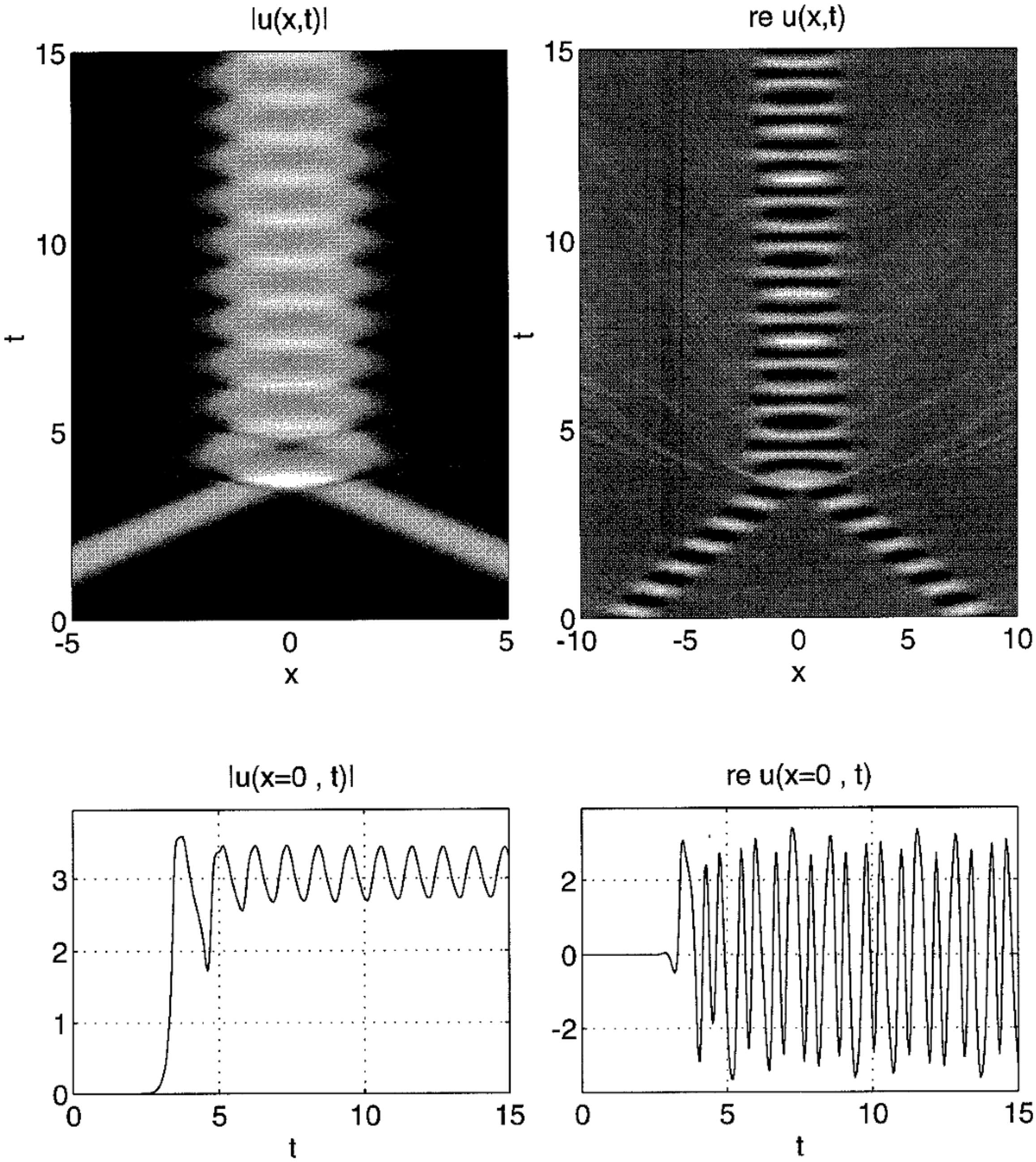

FIG. 5.2. *Simulation of soliton's collision:* $\epsilon = 0.0001$, $v = 2$

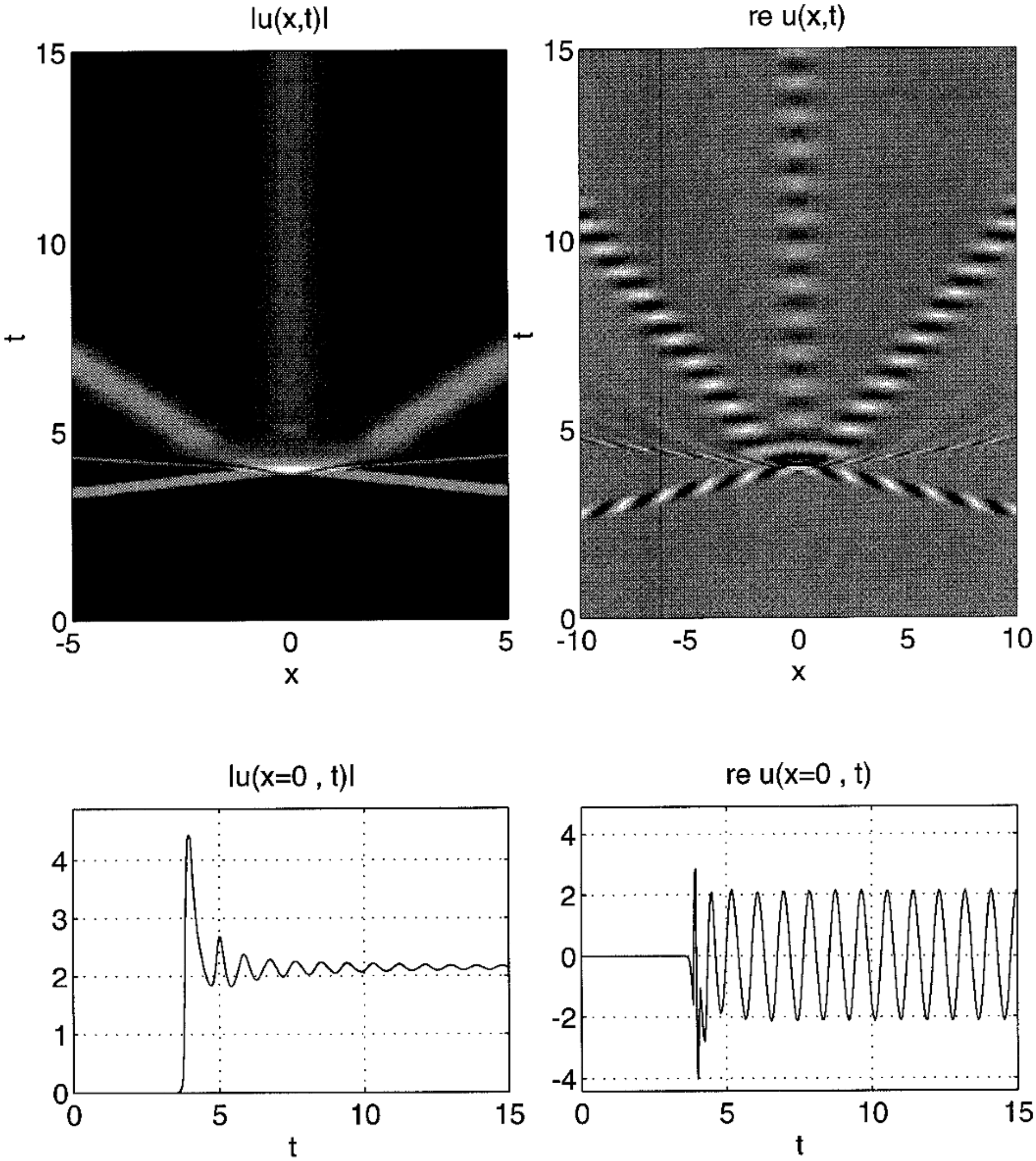

FIG. 5.3. *Simulation of soliton's collision:* $\varepsilon = 0.00001$, $v = 8$

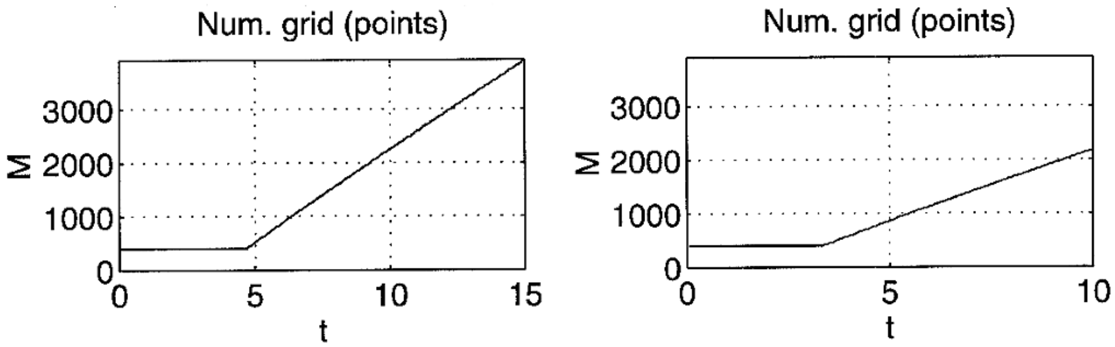

FIG. .1. *Typical number of grid points on the half of x-axis. Left: Simulation of soliton's collisions. Right: Simulation of the solution* $U_\alpha(e^{\lambda_* t}; x)$

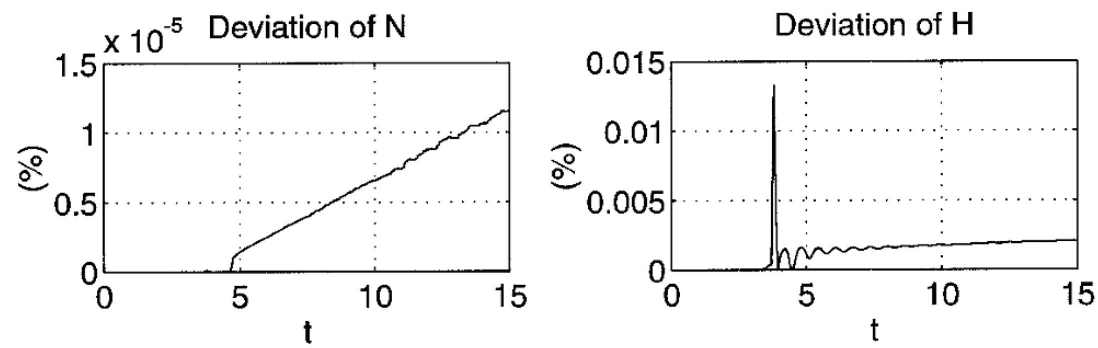

FIG. .2. *Typical relative error of the integrals of motion (2.7)*

REFERENCES

[1] M.J.Ablowitz, D.J.Kaup, A.C.Newell and H.Segur. The inverse scattering transform — Fourier analysis for nonlinear problems. 1974, *Stud.Appl.Math.*, **53**, 249–315

[2] S. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zaharov. Theory of Solitons. The Inverse Scattering Method. 1984, Consultants Bureau, NY

[3] T.Cazenave and P.L.Lions. Orbital stability of standing waves for some nonlinear Schrödinger equation. 1982, *Comm.Math.Phys.*, **85**, 549–561

[4] J.L. Bona and A. Soyer. On the stability of solitory-wave solutions of model equations for long waves. 1994, *J.Nonlinear Sci.*, **4**, 449–470

[5] M.I. Weinstain. Lyapunov stability of ground states of nonlinear dispersive evolution equations. 1986, *Comm. Pure Appl. Math.*, **29**, 51–67

[6] V.S. Buslaev and G.S. Perelman. Scattering for the nonlinear Schrödinger equation: states that are close to a soliton. 1992, *Algebra and Analysis*, **4**, 63–102

[7] A.Soffer and M.I.Weinstain. Multichannel nonlinear scattering for nonintegrable equations. 1990, *Comm. Math. Phys.*, **133**, 119-146

[8] R.T.Glassey. On the blowing up of the solutions to the Cauchy problem for the nonlinear Schrödinger equation. 1977, *J.Math.Phys.*, **18**,1794–1797

[9] M.I.Weinstain. On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations. 1986, *Comm.Part.Diff.Eq.*, **11**,545–565.

[10] N.E.Kosmatov, V.F.Shevts and V.E.Zakharov. Computer simulation of wave collapses in the nonlinear Schrödinger equation. 1991,*Physica D*, **52**, 16–35

[11] D.E. Edmondson and R.N.Enns. Robust bistable light bullets. 1992, *Optics Lett.*, **17**, 586–588

[12] W.A.Strauss. Existence of solitory waves in higher dimensions. 1977, *Comm.Math.Phys.*, **55**, 149–162

[13] D.K.Campbell, J.F.Schonfeld and C.A.Wingate. Resonance structure in kink-antikink interaction in ϕ^4 theory. 1983, *Physica D*, **9**, 1–32

[14] M.Peyrard and D.K.Campbell. Kink-antikink interaction in a modified *sin*-Gordon model. 1983, *Physica D*, **9**, 33–51

[15] M.Peyrard and D.K.Campbell. Solitory wave collisions revisited. 1986, *Physica D*, **18**, 47–5

[16] D.K.Campbell, M.Peyrard and P.Sadamo. Kink-antikink interaction in the double *sin*-Gordon equation. 1986, *Physica D*, **19**, 165–205

[17] S.Jin, C.D.Levermore and D.W.McLaughlin. The behaviour of solitons of NLS equation in the semiclassical limit. In: *Singular limits of dispersive Waves*, 1994, NATO ASI Ser., vol.320, 235–255