PROFESSIONAL DEVELOPMENT AND TEACHER CHANGE: TEACHERS' PRACTICES AND BELIEFS ABOUT USING MULTIPLE REPRESENTATIONS IN TEACHING MATHEMATICS

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TAMARA MOORE, Co-adviser
TERRENCE WYBERG, Co-adviser

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Dedication

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Abstract

This study explores mathematics teachers' beliefs about using multiple representations, such as manipulative, pictorial, symbolic, language, and realistic representations that are described the Lesh translation model. Using a qualitative approach, the teachers' beliefs about the helpfulness and hindrances of using each representation described in the Lesh translation model were clarified. Teachers' belief changes were also explored, before and after participating in a yearlong professional development program. Furthermore, this study investigated the teachers' reflections on teaching mathematics based on their beliefs during the program. Results indicated that the teachers' beliefs were mostly in line with the Lesh translation model. Changes in the teachers' beliefs were also observed after participating in the program. These changed beliefs led teachers to change their practices in the classroom. The teachers mostly used pictorial, symbolic, language, and realistic representations during the three observations periods. Using translations within and between these types, especially pictorial and realistic representations, were varied in the three classroom observations. On the other hand, the teachers used few physical and virtual manipulatives, even though they claimed to have positive beliefs about using them in the classroom. That is, the teachers' beliefs concerning the hindrances of using manipulatives prevented them from using these representations in the classroom. Furthermore, changing teachers' practices required more time after their beliefs were changed. In future studies, researchers could focus on a longitudinal study of the interactions between beliefs and practices. In addition, research could explore more effective activities that involve physical manipulatives or virtual

manipulatives, which reduce any hindrances that were identified in this study for K-12th grade students.

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Chapter 1. Introduction

Over the past twenty years, U.S. students have consistently shown a weak conceptual understanding of mathematics, relative to their international peers in high-achieving countries such as Singapore, South Korea, and Japan (Tatsuoka & Corter, 2004). Research has shown a critical impact on using representations and interacting within and among representations for students' conceptual understanding in mathematics (Cramer, 2003; Goldin & Shteingold, 2001; Kamii, Kirkland, & Lewis, 2001; Perry & Atkins, 2002). Thus, various instructional approaches have been introduced by using representations, such as manipulatives and visualizations. However, researchers still stress the need for teachers to understand multiple representations in order to implement them in their classrooms (Hill, Ball, & Schilling, 2008; Neiss, 2005). Researchers have also argued that instructional change is required to help students gain a conceptual understanding, since research has shown that many teachers still teach using in a traditional instructional approach (Ball, 1990; Cuban, 1993; Steele, 2001) involving the directed flow of information from the teacher as sage to the student as passive learner. Furthermore, the Trends in International Mathematics and Science Study (TIMSS) indicates that over 96% of student work time in U.S. middle-school classrooms is spent on practicing routine procedures using a traditional instructional approach (Hiebert & Stigler, 2000). Given that students from Minnesota and many other states are now required to take first-year algebra in the eighth grade, it is imperative that they have a much deeper conceptual understanding in mathematics, in addition to solid procedural skills. A major instructional change is required in order to help all students acquire the conceptual understanding in mathematics necessary to succeed at algebra in the eighth grade. Research indicates that one important change is the adoption of instructional approaches using multiple representations; indeed, there is evidence that this approach can close the gap between U.S. students and those in high-achieving countries (Wang & Lin, 2005; Vernille, 2002).

Two decades of research emphasize the importance of using various mathematical representations to enhance students' conceptual understanding in mathematics classrooms (Lesh & Doerr, 2003). These representations include manipulatives (e.g., hands-on models), pictorials (e.g., graphs, diagrams, or pictures), symbols, language (e.g., talking or writing about mathematics), and real-world situations. In addition, recent research is considering virtual manipulative representations (e.g., interactive, web-based tools) as one of the multiple representations, as well as physical manipulative ones. Recent studies have found that using virtual manipulatives facilitates students to construct mathematical understanding and knowledge (Suh & Moyer-Packenham, 2007). Through teaching with those multiple representations, teachers can understand students' learning styles and their mathematical understanding (Suh, 2007; Suh, Moyer, & Heo, 2005). This understanding helps teachers create effective ways of teaching all students to develop conceptual understanding and procedural skills.

A theory of multiple representations, the Lesh translation model, explains how teachers can use these multiple representations and translations among representations to help students acquire more conceptual understanding. Research has shown that interaction among multiple representations plays a critical role in developing students' mathematical conceptual understanding (Cramer, 2003). That is, the Lesh translation model helps inform and differentiate classroom instruction for students' mathematical conceptual understanding because teachers can understand students' personal method of thinking while using multiple representations (Suh, 2007). However, many teachers often teach in a traditional instructional way that emphasizes the use of symbols (Steele, 2001) because they have little experience using multiple representations in their classes. An instructional change for developing students' conceptual understanding is required in the way of teaching mathematics and in teacher beliefs about what using multiple representations means to learn and do mathematics (Schram, Wilcox, Lanier, & Lappan, 1988; Wilcox, Schram, Lappan, & Lanier, 1991).

Researchers indicate that a greater understanding of teachers' beliefs is essential to the improvement of instructional practices (Lumpe, Haney, & Czerniak, 2000); this is because instructional practices are dependent on what individuals believe effective teaching entails (Speer, 2005). Research also claims that changes in teachers' beliefs precede changes in their teaching practices in implementing a particular innovation that does not initially conform to their prior beliefs (Czerniak & Lumpe, 1996; Pajares, 1992; Richards, Gallo, & Renandya, 2001). Therefore, teacher beliefs shape the way in which a teaching method is implemented (Lumpe et al., 2000). When teachers have negative beliefs about an instructional approach, that approach does not work for their students (Handal & Herrington, 2003). Research suggests that one obstacle to wide-scale implementation of multiple representations is due to many teachers' negative beliefs about them because they are unfamiliar with them (Lloyd, 2003). However, it is still unclear as to how teachers think about using multiple representations in their classes, how teachers' beliefs about teaching can be changed if they have negative beliefs about them, and how belief changes can impact teachers' practices in teaching mathematics. Borko (2004) suggests that a professional development program can help teachers learn about new ways to instruct, using multiple representations as well as changing their beliefs about how they teach. Research also calls for studies to help teachers articulate their beliefs about instructional approaches and to use them in order to reflect on their teaching (Richards et al., 2001).

The Purpose of the Study

Using multiple representations, such as manipulatives, pictorial, symbolic, language, and realistic representations, and translating within and among them can allow teachers to help students build their mathematical conceptual understanding. Mathematics teachers' beliefs about teaching mathematics affect their practice and students' learning. It is therefore critical to assess what beliefs teachers have with respect to using multiple representations for organizing instruction so as to understand how they affect their practice, and thus students' mathematics learning.

The focus of this study is to investigate whether mathematics teachers' beliefs for teaching mathematics align with a theory of multiple representations, the Lesh translation model, in teaching mathematics. The study also investigates how the teachers' beliefs and their practices are changed as a result of a yearlong professional development program. The professional development program includes various activities using multiple representations, such as activities from the Rational Number Project, which has used the

Lesh translation model to develop curricular materials, activities, and classroom instruction; moreover, this program has provided teachers with virtual manipulatives via technology.

The professional development program focused on sixth- through eighth-grade mathematics teachers. This program provided a total of 30 hours for content training (five face-to face days trainings) during the 2011-12 school year, and a total of 16 hours for professional learning community (PLC) meeting (four times between trainings) with their mathematics team, who participated in the training at each school. Six seventh-grade mathematics teachers from two schools in a Midwestern metropolitan area participated in this study, and all of them participated in the trainings and PLC meetings. Two of the teachers were from one school (School A), and the rest were from the other school (School B).

Research Questions

Three research questions are investigated in this study:

1. How do the beliefs of mathematics teachers participating in a yearlong professional development program align with the teaching and learning aspects in the Lesh translation model?
2. How do teachers' beliefs about teaching mathematics change through participating in a yearlong professional development program designed by using the Lesh translation model as a theoretical foundation?
3. How are teacher beliefs about the Lesh translation model reflected in their classroom practices?

To answer these questions, a multiple-case study method (Yin, 2003) was used to investigate mathematics teachers' beliefs about using a theory of multiple representations, the Lesh translation model and the impact of a sixth- through eighth-grade algebra professional development program, which has the Lesh translation model as its theoretical foundation. Pre- and post-teacher belief interviews (see Appendix A) were used to investigate whether the beliefs of teachers participating in the professional development program became more in line with the Lesh translation model at the end of the training. Three classroom observations, with each observation consisting of a three-day period, were conducted. I used the Oregon Teacher Observation Protocol (OTOP; Morrell, Wainwright, & Flick, 2004) and a checklist and descriptions about the use of multiple representations (see Appendix B). For analyzing the classroom observations, I illustrated how teachers used multiple representations such as manipulatives (physical and virtual ones), pictorial, symbolic, language, and realistic representations that were described in the Lesh translation model. Some teachers in this study considered virtual manipulatives as distinct from physical ones in their beliefs and practices. Interviews and classroom observations were used to probe teacher beliefs about teaching and the practices that have affected their beliefs.

The following chapter is a literature review about multiple representations, teacher beliefs, and the relationship between teacher beliefs and their practices in the classroom.

Chapter 2: Review of the Literature

The aim of this literature review is to provide a coherent summary of the research on topics relevant to this study: multiple representations in mathematics education and the impacts of teachers' beliefs on teaching. The Principles and Standards for School Mathematics (NCTM, 2000) emphasize the role of representation in learning mathematics. It asserts that students should "create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; and use representations to model and interpret physical social and mathematical phenomena" (p. 67). Researchers have argued that meaningful learning and teaching of mathematics can be accomplished when a variety of representations have been developed and translated (Bayazit, 2011; Goldin & Shteingold, 2001). Furthermore, teachers' beliefs play a critical role in their teaching and students' learning (Speer, 2005; Stuart & Thurlow, 2000). Teachers' beliefs toward new curricula and instructional models affect their practice in the classroom (Handal & Herrington, 2003). Research involving teachers' beliefs is required to explore how teachers effectively implement new instructional models such as using mathematical representations in the classroom.

Research on Multiple Representations

Theorists on multiple representations have recommended that curriculum activities and materials be presented in multiple representations in order to develop effective learning environments for each student (Lesh & Doerr, 2003; Shu, 2007). In early works about multiple representations within mathematics education, Dienes (1960) emphasized the significance of the perceptual variability principle or multiple embodiment principle, which suggests that students' learning is maximized when they are exposed to a concept through a variety of embodiments. Using a variety of materials is designed to promote the use of abstract thinking (Gningue, 2006). For instance, when teaching fractions, teachers can teach fractions with various representations such as fraction circles, chips, pattern blocks, or dot paper models. These concrete representations help in building abstract conceptual understanding (Pape & Tchoshanov, 2001). In this teaching strategy, students can make connections between those representations and the concepts of fractions with conceptual understanding. Bruner (1966) also expressed multiple representations theory with three modes of presenting understanding (representation system): enactive (action), iconic (pictorial), and symbolic (abstract) representation. According to Bruner, these three representational modes are developmental. The development of each mode depends on the previous mode, and after long-term practice with one mode, one can make a transition to the next mode (Bruner, 1966). Students enhance their knowledge from concrete situations and problems to abstract ideas through various manipulatives (Bruner, 1960; Post, 1988). These studies have emphasized using various representations and translating among them to develop students' mathematical knowledge.

Recently, Janvier (1987) said that using representations in different ways are pedagogical strategies to clarify mathematical concepts, with the strengths and limitations that are explored. Thus, he explained two types of different translations, such as direct and indirect translations. Direct translations might be carried out from one representational mode to another one without using any other kind of representational mode between this translation; for instance, from an equation to a representation of a table. On the other hand, a translation from an equation to a table can be conducted by making translations from an equation to a graph, and then to a table. In this case, this kind of translation process is called an indirect translation (Janvier, 1998). Furthermore, Gagatsis and Shiakalli (2004) have found that these translation abilities are an important factor to improve students' problem-solving skills, and these abilities are related to student performance in mathematics. Therefore, these researchers have emphasized enhancing students' abilities to translate within and among mathematical representations.

The Lesh translation model presents a clear picture of representational fluency (Lesh, 1979). The model emphasizes student conceptual understanding through representing concepts in multiple forms and being able to translate within and among representations. That is, a student who has a deep understanding of a mathematical concept can easily provide many different forms of representations of the concept (manipulative, symbolic, pictorial, language, and realistic) and be able to describe how these representations relate to one another (translation). The Lesh translation model has been used to develop curriculum materials and classroom activities in order to help both teachers and students build a conceptual understanding of important mathematical ideas in the school mathematics curriculum (Cramer, 2003). In this study, the Lesh translation model was used as a theoretical framework to analyze teachers' beliefs about using multiple representations, and their use of them.

A theory of multiple representation-the Lesh translation model. The Lesh translation model (Figure 2.1) is a framework to represent the understanding of conceptual mathematical knowledge (Lesh & Doerr, 2003). It consists of multiple modes of representation: (1) manipulatives (concrete, hands-on models), (2) symbolic, (3) language, (4) pictorial, and (5) realistic (real-world, or experienced contexts) representations. The Lesh translation model emphasizes that the understanding of concepts lies in the ability of students to represent mathematical concepts through the five different categories of representation, and in the ability to translate between and within multiple modes of representation (Cramer, 2003). This type of translation can support students' relational thinking and mathematical conceptual understanding. Although distinct types of representational systems are important, the ability to translate among different modes of representation indicates deeper conceptual understanding within the system (Shu & Moyer-Packenham, 2007). Figure 2.1 shows the Lesh translation model in the diagram.

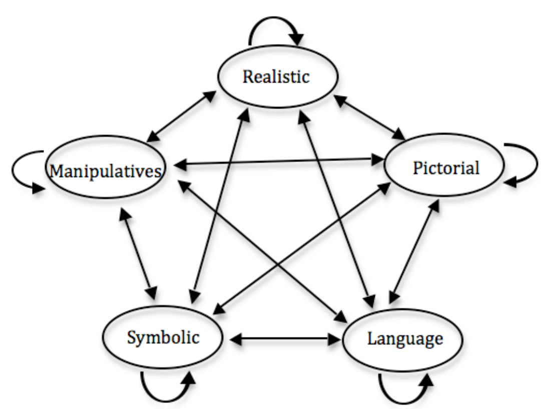


Figure 2.1. The Lesh translation model.

A translation is the reinterpretation of a mathematical concept within the same representation or between different representations (Cramer, Monson, Wyberg, Leavitt,

& Whitney, 2009). For example, when teachers or students model the fraction $\frac{4}{5}$ using an area model, such as one circle as the unit, they are translating from the symbolic to the pictorial representation. Then they may model the same fraction using a length model, such as a number line. At that time, they are translating within pictorial representations. Researchers have argued that teaching via translating within and among multiple representations has a critical impact on developing students' abilities to understand mathematical concepts and constructs, and on developing more proficient problem-solving skills (Cleaves, 2008; Cramer, 2003; Goldin & Shteingold, 2001; Kamii, Kirkland, & Lewis, 2001; Perry & Atkins, 2002). The Lesh translation model has been used as a framework to develop an effective learning environment. For example, the Rational Number Project has used the Lesh translation model to develop curricular materials and classroom instruction. Its main focus is to explore the role of various models as facilitators of the acquisition and use of mathematical concepts as learners move from the concrete to the abstract. Rational Number Project research has recognized that different representations have been useful for modeling different rational-number subconstructs (i.e., part-whole fractions, ratios, operators, or proportions) (Behr, Lesh, Post, & Silver, 1983). Students begin to construct a deeper understanding of fractions when they are represented in a variety of ways or when there are explicit linkages to everyday life and familiar situations involving the use of fractions (Cramer, 2003). Cramer (2003) also argues that it is important for teachers to experience using multiple representations and translate among these different modes of representations. The experience helps teachers become more aware of the weaknesses in the curricula they use, and thus supplement students' needs to learn mathematics. Implementing the Lesh translation model, in this way, has proven to be useful as a guide to develop the instruction of mathematics. This model promotes more teaching and learning through understanding concepts in the different representational forms and translating from one form to another.

Furthermore, recent research has divided manipulative representations into physical and virtual manipulative ones. They have explored the impacts of using virtual manipulatives with physical ones in the classroom.

Virtual manipulatives. Virtual manipulatives can be defined as interactive, web-based visual representations of a dynamic object (Moyer, Bolyard, & Spikell, 2002), while physical manipulatives can be defined as concrete objects. That is, virtual manipulatives can be interactive computer-based simulations or computer software that emulates physical manipulatives by keyboard operation instead of physical action on three-dimensional objects (Kim, 1993; Suh 2005; Terry, 1996). While virtual manipulatives are often computerized versions of familiar physical manipulatives, there is evidence that virtual tools offer additional benefits over the same tool in its concrete model (West, 2011). Virtual manipulatives can be considered as a type of manipulative, but they are different from physical manipulatives that have been called manipulatives in previous studies up to now. Virtual manipulatives might provide further advantages over physical manipulatives by eliminating some of the constraints they impose on the task. Some computer manipulatives may be more beneficial than any physical manipulative (Durmus & Karakirik, 2006). However, virtual manipulative representations should be used to give students opportunities to build mathematical knowledge, and these representations are often modeled after physical manipulative ones (Bouck & Flanagan, 2009; Moyer et al., 2002). For example, virtual manipulatives include a virtual geoboard with virtual rubber bands, Cuisenaire rods, virtual Base 10 Blocks, and algebra tiles. These virtual manipulatives are similar to physical manipulatives and often have the same names, but are presented in an interactive manner through an online format or a software environment (Bouck & Flanagan, 2009).

The use of virtual manipulatives in mathematics education is fairly recent (Suh et al., 2005), and there have been few studies of their effectiveness in learning (Mildenhall, Swan, Northcote, & Marshall, 2008; Steen, Brooks, & Lyon, 2006). These studies have found that using virtual manipulatives helps students improve their conceptual knowledge in mathematics (Reimer & Moyer, 2005; Steen et al., 2006; Suh et al., 2005; Suh & Moyer-Packenham, 2007). In addition, virtual manipulatives are designed to include multiple representations (Moyer-Packenham & Suh, 2011) such as pictures, drawings, letters, numbers, arithmetic operation signs, real-world contexts, and other representations. Suh, Moyer, and Heo (2005) found that using virtual manipulatives helps students make connections between visual depictions and symbolic models. However, many teachers may not be using virtual manipulatives because they do not have an understanding of how to use them for mathematics instruction; additionally, they may not understand how to use the technology (Mildenhall et al., 2008; Reimer & Moyer, 2005). Research recommends that more effective virtual manipulatives be developed for teaching and learning (Steen et al., 2006). Furthermore, teachers need to determine which activities are appropriate for their curriculum and their students' skill levels. Both virtual and physical manipulatives are only beneficial tools if teachers know how to integrate them into their teaching (Bouck & Flanagan, 2009).

In this study, I considered a model of multiple representations such as five representations that were described in the Lesh translation model, manipulative, pictorial, symbolic, language, and realistic representations, but manipulatives were considered as both physical and virtual manipulative representations to fully analyze teachers' practices in the classroom.

Definition of representations. The Lesh translation model includes five representations, such as manipulative, pictorial, symbolic, language, and realistic representations. In this study, manipulatives are considered to be two types of manipulatives–physical and virtual manipulatives – to explore the classroom observations. The following paragraphs explain definitions of each representation that is used to investigate teacher beliefs or practices in using multiple representations for this study.

- Physical manipulative representations are concrete objects, which students use to explore mathematical ideas through the students' visual and tactile senses (McNeil & Jarvin, 2007). These representations are objects to be handled and arranged by students and teachers that are used to convey abstract ideas or concepts by modeling or representing their ideas concretely (NCTM, 2000). Manipulatives include an array of items such as tens blocks, number cubes, 3-D models, and fraction circles.

- Virtual manipulative representations are an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge (Moyer et al., 2002). These representations are presented in an interactive manner through an online format or software environment (Bouck & Flanagan, 2009).
- Pictorial representations are visual representations such as drawings, graphs, tables, diagrams, and charts. These representations are drawn or are provided for students to read and interpret.
- Symbolic representations are written with numbers or letters that students write or interpret to demonstrate an understanding of mathematical concepts or problems such as algebra formulas. In these representations, students and teachers symbolize language to express mathematical thoughts, including how to do problems.
- Language representations are written or spoken language to explain or describe mathematical concepts, mathematical thoughts, or ways of solving problems without the use of a context. In these representations, students and teachers can talk and write about mathematics using language that describes the concept.
- Realistic representations are real-life stories or situations, or experienced contexts that are related with mathematical concepts or problems. Teachers and students use their knowledge of real-world contexts and experiences from their real lives to explain or understand mathematical concepts or mathematics problems.

As an example of multiple representations, consider the notion of one-half. The symbolic representations include ½, 0.5, 50%, and 1:2. The language representation is, "A whole is divided into two equal parts, consider only one of those parts. That is one-half of the whole." Physical manipulative representations could include items (e.g., counters) divided into two equal parts or a paper circle cut into two equal parts. There are virtual manipulative representations that have the same names as the physical manipulative ones, which are manipulated by a mouse, keyboard or touchscreen. Pictorial representations include drawings of number lines (with the ½ space marked) or a circle that has one shaded part out of two equal parts. Finally, the real-world situation representation might be a contextual example where the idea of ½ is useful (e.g., one pizza equally shared between two people).

Impacts of multiple representations on learning and teaching mathematics. Research has suggested that using multiple representations on learning mathematical concepts has a positive impact on students' conceptual understanding (Pape & Tchoshanov, 2001). Thus, researchers have emphasized the importance of using mathematical representations to build mathematical knowledge and understanding. For example, Greenes and Findell (1999) stated that students were able to interpret algebraic equations in various representations, such as pictorial, graphical or symbolic representations so that they developed mathematical reasoning in algebra. Meyer (2001) argued in favor of the effectiveness of realistic mathematics education, which promotes the use of representation in middle-school algebra and progresses through levels of abstractions. Moreover, he argued that the first stage of mathematics activity should attach meaning to abstract ideas by using concrete experiences. Meyer (2001) stated that the bridge between concrete and abstract is through students' creation and the use of models, drawings, diagrams, tables or symbolic notation.

How Do Mathematical Representations Help or Hinder Student Learning

A role of mathematics teachers is to help students internalize abstract mathematical concepts (Cobb, Yackel, & Wood, 1992; Puchner, Taylor, O'Donnell, & Fick, 2008). Using translations within or among mathematical representations could be the best way to understanding abstract mathematical concepts (Cramer, 2003; Goldin & Shteingold, 2001; Kamii, Kirkland, & Lewis, 2001; Perry & Atkins, 2002). Using manipulatives helps students create their own pictures of mathematical concepts, which allows them to build mathematical structures in their minds (Boulton-Lewis, 1998; Dienes, 1960; Keller, 1993; Meria, 1998). Researchers have argued that experience with a plenty of manipulatives can make students effectively translate from their experience to the abstract mathematical concepts (Kennedy, 1986). In addition, using both physical and virtual manipulatives helps student visualize mathematical concepts (Bouck & Flanagan, 2009; Lamberty & Kolodner, 2002). Using visual representations helps to students in building and strengthening their conceptual understanding in abstract mathematical concepts (Keller, 1993; Suh & Moyer, 2007). Using language is a key to develop mathematical understanding (Daniels & Anghileri, 1995). Students can build their own understanding using written and spoken communication about mathematical concepts (Topping, Campbell, Douglas, & Smith, 2003; Yackel, Cobb, & Wood, 1991). Burns (1985) suggested that using language by asking questions promoted students' deeper mathematical thinking. Borko and Livingston (1989) said that teachers should ask questions that can help students explain and recognize the key procedures and mathematical concepts in the problem solutions they solve. In addition, researchers have argued that students can construct mathematical concepts if they are provided with realistic representations offering familiarity and relevance with their own experiences and lives (Boaler, 1993). Treffers (1987) argued that using realistic representations could increase students' abilities to use mathematical concepts.

On the other hand, there are a few studies that discuss the hindrances of using representations. For example, Uttal & DeLoache (1997) said that without instruction, students may treat manipulatives as interesting objects that have little or no connection to the mathematical concepts that the teacher hopes the students will learn. They also argued that some manipulatives that students may already know about some manipulatives as objects (e.g., toys, food, etc.), which may interfere with their understanding of the symbolic nature of such manipulatives. Other researchers have also stated that one of most common hindrances to using manipulatives in mathematics is classroom management (Midenhall et al., 2008). By using virtual manipulatives, some of the classroom management issues may be overcome; however, there still exist other issues in using virtual ones, such as a lack of understanding how to use them for mathematics instruction and a lack of knowledge in terms of how to use the technology (Midenhall et al., 2008). In addition, Boaler (1993) suggested that students often failed to engage in real world-contexts embedded in mathematical problems, which were not familiar to students (Boaler, 1993). However, we still need to better understand what mostly helps and/or hinders student learning when using representations, so that teachers can create effective learning environments using these representations.

In sum, research has emphasized the importance of using multiple representations and interacting within and among the representations for students' mathematical conceptual understanding. Studies have found the effectiveness of using mathematical representations and translating within and among them in order to build mathematical conceptual understanding. However, there are a few studies about hindrances of using the representations for teaching and learning mathematics. Thus, teachers need to have positive beliefs about using the representations in order to implement them successfully in their classrooms. Research indicates that a greater understanding of teachers' beliefs is essential to improving instructional practices (Lumpe et al., 1998); this is because instructional practices are dependent on what individuals believe effective teaching entails (Speer, 2005). Research has also argued that changes in teachers' beliefs precede changes in their teaching practices in implementing a particular innovation that does not initially conform to their prior beliefs (Richards et al., 2001). However, few studies exist about teachers' beliefs with respect to using multiple representations, and their practices in the classroom based on these beliefs.

In the following section, I review previous studies on teacher beliefs and the relationships among practice, curriculum, and beliefs in mathematics education.

Research on Teachers' Beliefs

Research on teacher beliefs on teaching and learning mathematics has become one of the most important domains in mathematics education research. Teacher beliefs shape the way in which a teaching method is implemented. Moreover, when teachers have negative beliefs about a new instructional approach, the approach is generally not effective for their students (Handal & Herrington, 2003). Research suggests that one obstacle to wide-scale implementation of multiple representations is many teachers' negative beliefs due to their unfamiliarity with them (Lloyd, 2003). However, it is still unclear as to how teachers' beliefs about teaching can be changed, and how belief changes can impact teachers' practices in teaching mathematics.

Definition of beliefs. Teachers' beliefs about teaching and learning mathematics are related to students' mathematical learning. There is a considerable debate in the definition of teachers' beliefs. The mathematics education literature defines teachers' beliefs as how teachers understand the nature of mathematics, teaching, and learning (Aguirre & Speer, 2000; Pajares, 1992). Sigel (1985) defines beliefs as "mental constructions of experience-often condensed and integrated into schemata or concepts" (p. 351) that are held to be true and that guide actions (Pajares, 1992). Harvey (1986) defines beliefs as an individual's representation of reality that has enough validity to guide thought and actions. According to the definitions of beliefs above, in this study, beliefs are defined as mental constructions of experience that guide people's thinking and actions. Beliefs can also be reorganized, rearranged, and changed by an individual's experience.

Teachers' beliefs play a critical role in predicting their thinking, intentions, and practices in their classroom (Speer, 2005). Instructional practices are dependent on what teachers bring to the classroom, and what they bring is dependent upon what they believe effective pedagogy entails. The assertion that people act upon what they believe has been supported by studies tracing the relationship of beliefs to practice (Crawley & Koballa, 1992). The connection between teacher beliefs and practice has been well established (Cohen, 1990). Since new research-based perspectives often require radical changes in teachers' beliefs, research on teaching must consider teachers' beliefs in relation to the practice of new perspectives in their classroom (Crawley & Koballa, 1992; Roehrig & Kruse, 2005; Speer, 2005). Other researchers argue that changes in teachers' beliefs precede changes in their teaching practices in implementing a particular innovation that does not initially conform to their prior beliefs (Richards et al., 2001). Therefore, research about changes in teachers' beliefs is required to successfully implement new instructional perspectives in the classroom.

Practice and teachers' beliefs. Establishing educational goals involves multiple considerations (Clements, 2007), and research provides models to build the goals in several directions, such as the constructivist perspective (Forman, 1993). One important goal of education is that students acquire a conceptual understanding of learning (NCTM, 2000). The beliefs of teachers are vital to achieving this goal (Aguirre & Speer, 2000), because teachers' actions and thinking are influenced by their beliefs (Handal & Herrington, 2003; Wilkins & Ma, 2003). Central to the effort for newly built goals is the premise that teachers' beliefs and their classroom practices are the most essential components for progress (Bybee, 1993). Several researchers (Czerniak & Lumpe, 1996) support the notion that teacher beliefs are precursors to change, and that the teacher is the crucial change agent in paving the way for achieving educational goals. However, teacher beliefs are largely ignored from most research reports. Therefore, investigations into the structure of teachers' beliefs, in the context of teaching practice aimed toward achieving educational goals, are required to guide current educational research efforts.

Curriculum and teachers' beliefs. The National Council of Teachers of Mathematics (NCTM, 2000) describes the framework for effective ways to structure mathematics classroom environments. These standards-based learning environments are based on communication, real-world relevance, cooperative learning, problem solving, discipline and subject connections, and an integration of technology. This has been widely accepted as an important structure for equitable and effective mathematics classrooms. Multiple representation instructional models and teaching frameworks that were described in the NCTM standards are different from traditional teaching models. These instructional models have been implemented and have shown success in improving student achievement (Riordan & Noyce, 2001). Research shows the effectiveness of these instructional models and teaching frameworks in teaching and learning mathematics (Cramer, 2003). For example, the Rational Number Project has used the theory of multiple representations, in particular the Lesh translation model, to develop curricular materials and classroom instruction (Cramer, 2003). Educators have found positive impacts of using multiple representations in teaching mathematics.

Previous studies about relationships between curricula, such as a standards-based curriculum, and teachers' beliefs have found that teacher beliefs influence effective learning environments and instructional practices (Handal & Herrington, 2003; Thompson, 1992; Wilkins & Ma, 2003). Thus, more traditional beliefs have been associated with more traditional learning environments and practices (Stipek, Givvin, Salmon, & MacGyvers, 2001). Furthermore, studies of the standards-based curriculum have found that when teachers have beliefs that are compatible with curriculum change, the implemented curriculum is successful in their classrooms. However, when teachers have negative beliefs about the curriculum, the goals of the curriculum will be diluted (Handal & Herrington, 2003). The gap between the goal of the curriculum and teachers' beliefs seems to cause failure of curricular change in their classrooms (Cheung & Ng, 2000; Handal & Herrington, 2003). Moreover, teachers' beliefs are not always consistent with the way they teach (Cooney, 1985) because teachers do not have rich experiences in new teaching theories and approaches to implement in their classes. Research implies that a teacher's experience in new teaching theories or approaches is required to help change their beliefs, and therefore achieve successful implementation of new teaching approaches in their classrooms. In sum, changing teacher beliefs is essential to changing traditional educational perspectives in teaching. The reason why beliefs need to be considered is that they affect the teachers' classroom practices, which in turn, accomplish new educational perspectives, such as using multiple representations in their classroom. That is, teachers' beliefs should be associated with effective teaching to implement new educational perspectives in teaching; yet, there are few studies examining whether or not their beliefs are compatible with instructional perspectives. Therefore, the research regarding teachers' beliefs and changes in teachers' beliefs are required to successfully implement instructional approaches, such as using multiple representations in the classroom.

In this study, a professional development program was used as an intervention to observe teachers' belief change. Previous researchers have argued that teachers' experience through effective professional development programs helps them change their beliefs toward new instructional models that are provided in the programs (Guskey, 2002). In the following sections, studies about professional development programs are described. These descriptions provide a research lens for effective professional development programs that help teachers develop their beliefs and practices in the classroom.

Professional Development Program

Professional development is defined as any activity that is intended 1) to develop teachers' knowledge, skills and expertise; and 2) to prepare teachers for improved educational performance in present or future roles within a school setting (Desimone, 2009; OECD, 2009). Teachers can consider professional development as a vehicle for improving teaching practice, and in turn, improving student achievement. Professional development is critical to helping teachers become familiar with new methods or technologies of teaching and learning in their content areas, and keeping up with the changes in standards and assessments at the district, state, and national levels.

Core features based on previous research. As research begins to illuminate high-quality professional development, the following core features for effective professional development have been underscored (Bransford, Brown, & Cocking, 2000; Cohen & Hill, 2000; Desimone, 2009; Desimone, 2011; Garet, Porter, Desimone, Birman, & Yoon, 2001; Loucks-Horsley & Matsumoto; 1999):

- *Content focus*: Teachers need opportunities to develop well-organized bodies of content knowledge and pedagogical content knowledge of their disciplines. Thus, professional development activities should focus on subject matter content and deepen teachers' content skills, and the ways students learn that content.
- *Active learning:* Professional development activities should actively engage teachers in meaningful discussion with other teachers or training specialists about the goal of a lesson, tasks for students, teaching strategies, and student thinking or work, and practice.
- *Fostering coherence*: Professional development should be consistent with teachers' goals of what they are expecting from the professional development, and alignment with the standards and assessments at the district, state, and national levels.
- *Duration:* Professional development should have sufficient duration, including both the span of time over which the activity is spread (e.g., one semester) and hours of contact time (e.g., 20 hours).
- *Collective participants:* Teachers need time to work with peers together to improve their content and pedagogical content knowledge. Professional development should involve groups of teachers from the same school, grade level, or subject in order to build an interactive learning community.

These features are recommended for effective professional development programs by previous research. In this study, the professional development program involved all of these suggested features. Furthermore, the program involved other features that might bring about the development of teachers' belief changes and practices, such as research-based models and data driven by students.

Additional, but important features for effective professional development. Research has investigated the influence of teaching. In this section, we mention three additional components for developing effective professional development programs.

Data Driven by Students. Teaching should begin with an understanding of students' prior knowledge of mathematics and their mathematical needs to learn mathematics (Shulman, 1987). Through activities such as interviewing their students, teachers can understand their students' mathematical abilities and misconceptions in their learning of mathematics (Borko, 2004). These understandings about students help teachers gain insight into effective teaching approaches for their own students (Lappan, 1997). Professional development necessarily helps teachers understand how their own students best learn, and which tasks or materials best engage students. This knowledge can help teachers gather data about their own students in order to create appropriate curricula for their classes. Effective professional development entails creating, implementing, reflecting on, and modifying teaching approaches or activities teachers learn during professional development for their own students (Stein, Smith, & Silver, 1999).

Research-based models. There is no simple recipe for successfully teaching all students to improve their conceptual understanding (Graham & Fennell, 2001; NCTM, 2000). Teachers often bring their own experiences, backgrounds, and cultures into their classrooms, and these are one of the most significant factors in improving students' learning (Ball & Cohen, 1999; Loucks-Horsley & Matsumoto; 1999). It is essential for teachers to have experience or knowledge about new teaching skills and ways of students' thinking and understanding when working on mathematics (Ball & Cohen, 1999). To support teachers with this experience and knowledge, professional development should involve research-based models and should present rationales for using new teaching strategies or applying teaching theories. Research has created various models for engaging teachers and students with subject matter contents, which focus on students' thinking and its relationship to teaching (Wilson & Berne, 1999). These ideas, supported by empirical research, show evidence of improving student achievement. Teachers must build their understanding of the relationships among research-based models involving students' thinking, as well as new strategies, teaching and learning theories, and their classroom practices for their students (Rhine, 1998). A professional development program that involved the above-mentioned features was used in this study. The professional development program provided a variety of activities that highlight important ideas of the Lesh translation model, such as translating within or between mathematical representations, to build mathematical conceptual understanding.

Therefore, this study explores teachers' beliefs about using multiple representations, their belief changes, and their reflections on teaching mathematics based on their beliefs through a multiple-case study. I used a professional development program as an intervention to explore teachers' belief changes about using multiple representations. The teachers had experience in multiple representations through various activities in this program. The professional development program adopted the ideas of the

Lesh translation model to create the activities, and it used some of the Rational Number Project curricula so as to give teachers opportunities to experience using multiple representations. Through this study, we can understand teachers' beliefs about how using representations helps or hinders student learning and how teachers develop their beliefs toward using them through a professional development program. In addition, this study examines whether or not teachers' beliefs are reflected in instructional approaches, such as using multiple representations in the classroom.

The following chapter focuses on the methodology used to design this study, as well as on the collection and analysis of the data.

Chapter 3: Methodology

The aim of this chapter is to present the intended procedures for the study. It includes descriptions of the overall research design, the participants and the settings, the design of the professional development program, the data collection instruments, and the data analysis approaches used to address each research question.

Research has argued that professional development improves teacher beliefs and knowledge, and enhances their classroom practice, which holds the promise to promote students' achievement. Professional development programs should involve research-based models and should present rationales for using teaching models. In order to successfully implement teaching models by teachers, it is important for teachers to have beliefs aligned with the models, and these models should be used in their practices.

A multiple-case study method (Yin, 2003) was used to explore changes in teachers' beliefs about the impact of the Lesh translation model in mathematics classrooms while participating in the 2011-12 yearlong professional development program that focused on Rational Number Project curricula with multiple representations, which were designed after the Lesh translation model as a theoretical foundation. This study also described how changes in teacher beliefs potentially lead to changes in teachers' practices with respect to using multiple representations. The three research questions guide this study:

1. How do the beliefs of mathematics teachers participating in a yearlong professional development program align with the teaching and learning aspects in

the Lesh translation model?

2. How do teachers' beliefs about teaching mathematics change through participating in a yearlong professional development program designed by using the Lesh translation model as a theoretical foundation?
3. How are teachers' beliefs about the Lesh translation model reflected in their classroom practices?

This chapter delineates the procedures used to determine teachers' beliefs, the belief changes and their practice while participating in the program. The participants, setting, and data sources are described first. Then the professional development programs and the data-collection instruments are described. Finally, how the data were analyzed is discussed.

Participants and Settings

123 teachers from 24 schools participated in a yearlong professional development program that is a part of a Mathematics and Science Teacher Partnership (MSTP) project in a Midwestern metropolitan area. The researcher contacted 10 schools with three or more seventh-grade mathematics teachers who participated in the program and who were teaching regular mathematics classes. The researcher asked the teachers if they were interested in the multiple representation study. Six teachers from two schools demonstrated high interest. The researcher recruited these six teachers for this study (see Table 3.1). Two seventh-grade mathematics teachers were from one school (School A), and four seventh-grade mathematics teachers were from the other school (School B). School A uses Connected Mathematics Project curricula for middle-school teachers and

students, which were developed with funding from the National Science Foundation (NSF). The students in School A identify as White (77%), Asian (12%), Black (10%), Hispanic (1%), and Native American (0.2%). School B uses McDougal Littell Curricula, which were commercially designed for middle-school teachers and students. The students in School B identify as White (74%), Asian (7%), Black (13%), Hispanic (5%), and Native American (0.7%). Table 3.1 shows information about the participants.

Table 3.1

Information About the Participants

School	Type of school	Curriculum	Teacher	Teaching experience	Class size (number of students)*	Level of students*
A	Public	Connected Mathematics (7^{th} grade)	Ben	14 years	32	High achievement
			Sara	3 years	25	Low achievement
B	Public	McDougal Littell Curriculum (7^{th} grade)	Nick	34 years	32	High achievement
			Holly	12 years	37	High achievement
			Mary	21 years	30	Low achievement
			Pam	6 years	33	High achievement

Note. Classroom size and level of students represent the classes observed for this study. Classroom size represents the number of students the teacher taught in the observed class, and level of students represents students' level of achievement in mathematics.

Description of the Professional Development Used in This Study. A professional development program was used as an intervention to observe teachers' belief change. The professional development program has the seven features that were described in the literature review for effective teacher professional development programs, such as content focus, active learning, fostering coherence, appropriate duration, collective participants, data driven by students, and a research-based model. In this study, this program focused on algebra, especially rational numbers, and provided meaningful discussions and various activities using mathematical representations. It also focused on NCTM standards and Minnesota state standards. It was a yearlong professional development program focused on sixth- through eighth-grade mathematics teachers. In the professional development program, five face-to-face days were interspersed throughout the year with four hours of PLC meetings between each face-to-face meeting. This program provided teachers with students' work samples, and used many of the rational number project curricula, including technology based on the Lesh translation model. Table 3.2 summarizes a description of the professional development in this study.

Table 3.2

Description of the Professional Development in This Study

Features	In this study
Content focus	• Algebra-Rational numbers
Active learning	• Meaningful discussion about students, teaching strategies, student thinking, and practices • Activities (e.g., hands-on & modeling activities)
Fostering coherence	• Consistent with teachers' goals • Alignment with the standards and assessment at the district and state levels (Minnesota State & NCTM standards)
Duration	• One academic year (30 hours for training & 16 hours for PLC)
Collective participants	• Sixth- through eighth-grade mathematics teachers
Data driven by students	• Providing teachers with students' works and students' interviews
Research-based models	• Rational Number Project curricula based on the Lesh translation model

Five-day trainings. Based on the Rational Number Project that used the Lesh translation model as the theoretical framework, the module for the five-day professional development program was created for the regional sixth- through eighth-grade mathematics teachers as they prepare their students to think more algebraically in the middle grades. The professional development provided a total of 30 hours for content training during the school year, and a total of 16 hours for professional learning community (PLC) meetings with their mathematics team, who participated in the training at each school. The PLC meetings were led by each school facilitator, who attended three full days (nine hours) of customized training sessions specifically focused on Mathematical Reasoning PLC development, strategies, and management.

The module was created to enhance teachers' abilities: 1) to identify key learning points for various algebraic ideas; 2) to enhance content knowledge and pedagogical content knowledge to help guide student learning; 3) to understand multiple mathematical representations to help students' conceptual understanding; and 3) to improve the way that they listen to assess student thinking. The facilitators for the professional development were mathematics education specialists. They provided the teachers with the benchmarks, curricula and experiences needed for students to develop algebraic reasoning and thinking through working on algebraic problems, using multiple mathematical representations, viewing samples of students work problems using mathematical representations, and analyzing student work. The teachers who participated in the training spent most of their time solving algebraic problems using multiple representations to enhance their knowledge of algebraic concepts. They also looked at the different ways that students solve algebraic problems to develop their ability to sequence examples and lead classroom discussions that focus on the development of algebraic ideas. That is, the teachers were exposed to a variety of activities that provided them with ideas as to how they could use a translation within the same representation or between different representations. Table 3.3 shows the outline of the five-day professional development program. The following sections provide brief descriptions for each day of training.

Table 3.3

The Module for the Professional Development Program

GOAL of Module: To create a professional development experience for mathematics teachers as they prepare their students for success in eighth-grade algebra.		
Time	Topic	Key ideas
Day 1 (Sep/ Oct)	Making sense of rational numbers	• Five interpretations of fractions -Participants solved problems involving fractions using a variety of materials. -Materials: fraction circles, paper strips, chips, and number lines Fraction-ordering strategies -Participants used manipulatives, mental images, and student work to investigate the stages that students pass through as they compare and order fractions. -Materials: fraction circles, paper strips, chips, and number lines. -NCTM Process Standards: Problem Solving, Representation, and Communication
Day 2 (Nov)	Reasoning with rational numbers using quotient interpretation	• Equal sharing problems to understand rational numbers. -Participants learned how to write problems, assess work, and planned for future teaching related to equal sharing problems. -Participants focused attention on the quotient interpretation of rational numbers. -Materials: student work, pictures, fraction circles Multiple group problems that are a set of problems involving key mathematical ideas of fractions, and that are solved using strategies including direct modeling, repeated addition, grouping and combining, and multiplicatives. -Participants learned how to use different types of problems in their classrooms to further understand learning benchmarks around fractions and the development of algebraic ideas. -Materials: student work, fraction circles, and patty paper, number lines -NCTM Process Standards: Problem Solving, Reasoning and Proof, Representation, and Communication

(Continued)

Day 3 (Jan)	Multiplication and division with rational numbers	• Fraction multiplication -Participants learned how number selection and model choice develop the concept of multiplication and the operator interpretation of rational numbers. -Participants learned how fraction multiplication is developed in the Rational Number Project (RNP) curriculum. -Materials: student work, fraction circles, patty paper, and number lines Fraction division -Participants learned how to write measurement division problems that encourage the development of the same denominator algorithm for the division of fractions. -Participants learned how fraction division was developed in the RNP curriculum. -Materials: student work, fraction circles, and number lines -NCTM Process Standards: Problem Solving, Connection, and Reasoning and Proof
Day 4 (March)	Unit rate and algebraic solutions	• Model Eliciting Activity (MEA)-Ratios and proportions -Participants learned how to use the Bigfoot MEA to develop proportional reasoning skills among their students. -Materials: Bigfoot MEA, student work, and measuring tools Developing the ration interpretation of rational number -Participants learned how students reason proportionally and how to increase student understanding of proportionality with rational numbers. -NCTM Process Standards: Problem Solving, Reasoning and Proof, Representation, and Communication
Day 5 (April/ May)	Making sense of fractions, decimals, and percentages	• Naming, ordering and operations involving decimals. -Participants learned how to name, order and perform operations involving decimals using a variety of formal and informal approaches. -Materials: number lines, +/- decimal grids, crayons, base-ten blocks, student work Working with percentages -Participants learned to use percent grids to make sense of situations involving percentages. -Materials: percent grids, number lines, and graphs -NCTM Process Standards: Problem Solving, Reasoning and Proof, and Representation

Data Collection

Data collection for this study occurred during fall 2011 through spring 2012. These data included teachers' interviews, classroom observations, and classroom artifacts. Figure 3.1 shows timeline of data collections for this study.

Dates		Sep/Oct 2011		Nov 2011		Jan 2012		March 2012		May 2012	
PD		Day 1	PLC	Day 2	PLC	Day 3	PLC	Day 4	PLC	Day 5	
Interview	Pre										Post
Observation			Ob 1				Ob 2				Ob 3

Note: PD = professional development program, PLC=professional learning community, and Ob 1, 2, and 3 = Observation 1, 2, and 3.

Figure 3.1. The timeline of data collections for this study.

Teachers' Interviews. I conducted pre- and post-semi-structured interviews with each teacher to capture his or her beliefs about teaching mathematics and using multiple representations, before and after the training. The first three interview questions were adapted from the "teacher beliefs interview" (Luft & Roehrig, 2007) that was used in the open-ended teacher beliefs survey. The other seven questions were created in order to discover teachers' beliefs about the Lesh translation model. For example, what do you think about using manipulative, pictorial, symbolic, language, or realistic representations in your class? In addition, how do you think about the connections of multiple representations to represent a mathematical idea in your mathematics class? I included all interview questions in Appendix A. The interviews were audiotaped and then transcribed.

Classroom observation. To investigate how the teachers taught mathematics in their classrooms, I conducted classroom observations. Two seventh-grade teachers from School A and four seventh-grade teachers from School B were observed three times during one academic year, and each observation consisted of three consecutive days. The first classroom observations were between the first and second training sessions (October), and the second observations were between the third and fourth training sessions (February). The third observations were conducted after the last training session

(May). The classroom observation protocol used the Oregon Teacher Observation Protocol (OTOP; Morrell et al., 2004). To observe the use of multiple representations, a checklist of five categories was added to the original protocol: (1) Representation through Realistic, Real-World, or Experienced Contexts (R), (2) Symbolic Representation (S), (3) Language Representation (L), (4) Pictorial Representation (P), and (5) Representation with physical or virtual manipulatives (M) (Appendix B). In the category of manipulatives, I described whether they used either physical manipulatives or virtual ones.

Each category on the checklist documented whether a teacher utilized the aspects of the Lesh translation model during their classroom practice. During the fieldwork, I also recorded details about what materials and activities the teachers used, which mathematics problems were provided, how they organized their classes, and how they facilitated their classroom discussions. It was important to understand how teachers used the Lesh translation model in their classrooms because class organization or facilitation was related to how or what materials and activities involving different representations were used. I videotaped all of the observations. As a non-participant observer, I also recorded field notes that identified the type of classroom instruction, representations that were used in the instruction, and translations among the representations. From each classroom observation, I collected the student worksheets that students used during their class.

The following section involves the data analysis for teacher beliefs, belief changes, and classroom observations.

Data Analysis

The Lesh translation model was used to analyze the data from the teachers' interviews and classroom observations, which described translations within and between multiple representations, such as manipulatives (physical or virtual ones), pictorial, symbolic, language, and realistic representations (Figure 2.1). The process of "constant comparative analysis" (Corbin & Strauss, 2008) was used to complete a cross-case analysis of the data. Rubrics were used to describe each of the teachers' beliefs about teaching through the five different categories of representation and through translating within and among multiple modes of representations. A comparison of the teachers' beliefs across time was then created in order to investigate changes in their beliefs. Conceptualization and categorization across each teacher's data occurred to discover what each teacher believed about the Lesh translation model in teaching mathematics, and how they practiced the model in their classroom. That is, in the analysis of the classroom observations, I distinguished virtual manipulatives from physical ones when the teachers did so; thus, I reported both physical and virtual representations in their practices.

Matrices and diagrams were created to display and organize the teacher beliefs. That is, in the beginning of the analysis, I used a form of open coding to categorize and classify the data in order to consider all possible meanings based on different representations and to examine each piece of data up close. The results were useful in generating several rubrics about the helpfulness and hindrances using manipulative, pictorial, symbolic, language, and realistic representations, as well as using translations among multiple representations. The results were also helpful in reaching conclusions about what teachers' beliefs are about using representations, and how their beliefs changed after having experience using multiple representations through the professional development program for this study.

For example, in the three general belief questions about teaching, such as the role of the mathematics teacher, what to teach, and how to maximize student learning, I described their beliefs about using translating within and among representations, even though the belief questions were not related to mathematical representations. That is, I coded their beliefs as translations "within representations" or "between representations" from those three questions. For the interview questions about each representation, I found themes as to how they believed using each representation and translations among multiple representations helped or hindered student learning in their teaching. That is, first, I decided which responses pertained to the helpfulness or hindrances of using them. Second, I gleaned main ideas regarding the helpfulness and hindrances of using such representations from the teachers' answers. Finally, I created common themes if the teachers answered similarly such as responses of "deeper understanding," "visualization," "overgeneralizations," etc. I also compared between the pre- and post interviews based on the themes about the helpfulness or hindrances of using representations. The codes communicated a clear understanding of teachers' beliefs about using representations and translating among them. I described each participant's beliefs about them, and his or her belief changes about them. I also analyzed them across time and participants to examine their beliefs and belief changes. In addition, to further enhance the credibility of the study, I conducted member checks by contacting participant teachers to ask them if the themes and interpretations about their beliefs were accurate.

In the classroom observations, I coded whether the teachers strongly or weakly used translating within and among representations that were described in the Lesh translation model, where manipulatives included both physical and virtual ones. For an example of coding the translations, consider two equivalent fractions, $\frac{4}{6}$ and $\frac{6}{9}$. If a teacher drew fraction circles to explain that $\frac{6}{9}$ is equivalent to $\frac{4}{6}$, then I coded the teacher as strongly using translation between pictorial and symbolic representations. However, if a teacher did not tie these representations closely together during class (e.g., briefly showing the fraction circles for $\frac{4}{6}$ and $\frac{6}{9}$, and not enough explanation to connect the fraction circle with the written symbols, $\frac{4}{6}$ and $\frac{6}{9}$), I coded the teacher as weakly using translation between representations. On the other hand, when a teacher explained the equivalent fractions using the fraction circles and showed the same equivalent fractions using a number line, I coded the teacher as strongly using translation between representations. However, if they did not significantly tie the fraction circle and number line together to teach the same equivalent fractions, I coded the teacher as weakly using translation between representations. I also used separate coding when they used translations within and among them during warm-ups, reviewing the previous class or checking homework. I analyzed across time and participants to observe changes in their practices.

In the following chapter, each participant's beliefs, belief changes, and practices are described.

Chapter 4: Single Case Analysis

This study could divided into three main parts: 1) teachers' beliefs about using multiple representations that were described in the Lesh translation model; 2) teachers' belief changes after participating in a professional development program that was designed based on the Lesh translation model as a theoretical foundation; and 3) teachers' use of multiple representations in the classroom. In this chapter, I analyze each teacher's beliefs and belief changes about using multiple representations, and his or her practices regarding how to use mathematical representations in the classroom. Therefore, I will first describe each teacher's beliefs about teaching. Their responses will be reported from the three interview questions related to the following: (1) what he or she believes the role of the mathematics teacher is; (2) how he or she decides what or what not to teach; and (3) how he or she maximizes student learning in his or her classroom. These questions were intended to investigate their general beliefs about teaching, and not to ask about their specific beliefs regarding the Lesh translation model in their teaching. These general beliefs will help us understand how these teachers' beliefs reflect the Lesh translation model, and how their beliefs are reflected in their classroom practices. Second, I will describe the teachers' beliefs about the Lesh translation model. I will use the first four interview questions, such as the role of the mathematics teacher, what to teach, how to maximize students' learning, and how to use mathematical representations to describe these beliefs. I will also use other interview questions to clarify their beliefs and give examples of how they represent mathematical ideas in multiple representations. Third, I will describe their belief changes while participating in the professional development program. Finally, I will explore how they used the Lesh translation model in their classroom practices.

Ben's Case

Ben had 14 years of teaching experience in middle schools, and he was teaching high-achieving seventh-grade students using the connected mathematics curriculum in School A. His class size was about 32 students, and he always changed the students' groups to discuss mathematical concepts and solve problems in each observation.

Ben's beliefs about teaching

In the following sections, I explore Ben's general beliefs about teaching mathematics, such as (1) what he believes the role of the mathematics teacher is; (2) how he decides what or what not to teach; and (3) how he maximizes student learning in the classroom. These beliefs are related to his beliefs about using multiple representations. Thus, this analysis helps investigate research questions about teachers' beliefs in using representations and their practice in this study.

The role of the mathematics teacher. In both the pre- and post-interviews, Ben described the role of a mathematics teacher as exposing mathematical representations to help students understand mathematics. In the pre-interview, he emphasized making the connections between mathematics and the real-world, such as "...not necessarily always doing algorithms, but making the connections between the real world and mathematics." In the post-interview, he described the role of the mathematics teacher as "fostering understanding of mathematics in many different ways." He also said, "Students need someone who shows them why it works, not just the answer."

What or what not to teach. Ben explained that he considered students' prior knowledge for deciding what and what not to teach in both the pre- and post-interviews. He looked at "what the students had done last year and how in-depth they went into last year's scenarios" (Ben's pre-interview). Understanding students' ability levels helped him make sure what things needed to be considered when he planned a lesson (Ben's post-interview).

How to maximize students learning. Ben highlighted in both the pre- and post-interviews the importance of discussing mathematical conceptions and the processes of problem solving with students. In the pre-interview, he gave an example of how to lead his class by asking questions to the students. He said that when students come up an answer, such as x=5, he asks follow-up questions for the answer, such as, "Why is 5 correct?", "What does 5 represent?", and so on. In the post-interview, Ben said he needed to plan lessons in different ways after understanding students' ability levels. He said that it is important to "figure out how to help students learn." I summarized Ben's beliefs about teaching mathematics below in Table 4.1.

Table 4.1

Ben – Summary of His Beliefs About Teaching Mathematics

Ben	**1. The role of the mathematics teacher**
	• Make the connections between mathematics and the real-world (pre) • Help students understand mathematics (pre & post) • Foster an understanding of mathematics in many different ways (post) • Show students why mathematics works (post)
	2. What to teach
	• Consider students' prior knowledge (pre & post)
	3. How to maximize
	• Discuss mathematical conceptions (pre & post) • Talk about the processes of problem solving (pre & post) • Plan lessons in different ways after understanding students' ability levels (post)

Ben's beliefs about the Lesh translation model in the first three interview questions about teaching. The first three interview questions were about (1) what the role of the mathematics teacher is; (2) what or what not to teach; and (3) how to maximize student learning in the classroom. In this section, I described how Ben talked about the Lesh translation model in the three questions about teaching. Ben elaborated on the teaching aspects described by the Lesh translation model while responding to questions about teachers' beliefs about teaching. He said that showing realistic representations is one role of the mathematics teacher in the pre-interview. In the post-interview, he also talked about helping students translate within languages, and among multiple representations as a role of the mathematics teacher. The following excerpt illustrates his beliefs about using translations within or between representations in the pre- and post-interviews:

I would say to expose the students to the real-world connection between math and *themselves* ... (Ben's pre-interview)…*I think the role of the math teacher is to foster understanding of mathematics in many different ways, whether it be hands on or arithmetic-based…I think the kids need to have someone who shows/explains to them why it works, not just the answer…* (Ben's post-interview).

Ben believed that using language helps him understand students' ability levels so that he can plan lessons in order to maximize students' learning. He also said that using language is very helpful for students to deepen their conceptual understanding. That is, Ben described the translations within realistic or language representations within the question of the role of the mathematics teacher, and as the question of a way to maximize students' learning. Ben explained his belief about using mathematical representations in the interview question about how he used mathematical representations in his teaching. Ben said that he used "many real-world scenarios" (Ben's post-interview), and he discussed real-world situations in order to provide students with good examples to improve their conceptual understanding. In addition, Ben thought of the translation within manipulatives in his post-interview. He said that he tried to use as many manipulatives as he could. Table 4.2 summarizes how Ben discussed the translations between or within representations.

Table 4.2

Ben's Case From Interview Questions 1-3

Ben	Instances of representational fluency	Pre	Post
The role of the mathematics teacher	Representations	R	L
	Translation within representations	R	L
	Translation between representations		T
What to teach	Representations		
	Translation within representations		
	Translation between representations		
How to maximize	Representations	L	L
	Translation within representations	L	L
	Translation between representations		

Note. L: Language, R: Realistic representations, and T: Translation among multiple representations (representations not identified).

Ben's beliefs about representations

The following sections describe Ben's beliefs about how each representation helped and hindered his teaching of mathematics. The examples of using each representation are described. I also elaborate on his beliefs about the translations between and within multiple representations.

How representations help student learning. In the interview questions about each representation, Ben elaborated on his beliefs about how each representation helped him with teaching mathematics. He said that manipulatives were helpful in building students' conceptual understanding in his post-interview. In addition, he elaborated that manipulative representations could be thought of as visual representations, and pictorial representations could encourage students to use language representations. He connected manipulatives with visual representation: "We used manipulatives to make that connection easier for the visual" (Ben's pre-interview). Ben claimed that he has talked with his students about “how to find the rate by tables and how to find the rate by graphs” (Ben’s post-interview) when they worked with pictorial representations. Thus, he also connected pictorials with language. That is, Ben described using a representation in connecting it with another representation. He said that both language and realistic representations are one way to connect something with mathematics. He also stated that pictorial, symbolic, language, and realistic representations are useful in developing students’ deeper understanding in mathematics. Table 4.3 summarizes how Ben discussed the helpfulness of using each representation for student learning.

Table 4.3

Ben – Helpfulness of Using Representations From Interview Questions 5-9

Help \ Ben	Manipulatives		Pictorial		Symbolic		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Deeper understanding		✓	✓	✓	✓	✓	✓	✓	✓	✓
Visualization	✓	✓								
A means of communication				✓						
Connection with mathematics							✓	✓	✓	✓

How representations hinder student learning. Ben talked about the hindrance of representations in his pre-interview. For manipulatives, he stated that when students overgeneralized a manipulative to understand mathematics concepts, they could become confused. He also considered the hindrances of pictorial representations, even though he strongly believed in their positive contributions. The following excerpt illustrates Ben’s description about the hindrance of pictorial representations in his pre-interview:

I use so many pictorial representations. With the hindering part, I have trouble with saying it hinders, other than if a student has too much data in front of them *and they might be a little bit slower of a learner…*(Ben's pre-interview).

Ben also explained that students need to understand the meanings of mathematical symbols so that they can improve their understanding. He added that if students do not understand the meaning of symbols, it can be a considerable hindrance for them. Ben also described the hindrance of language when some students lose mathematical concepts while explaining their thinking. Table 4.4 summarizes how Ben discussed the hindrances of using each representation for student learning.

Table 4.4

Ben – Hindrances of Using Representations From Interview Questions 5-9

Ben / Hindrance	Manipulatives		Pictorial		Symbolic		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Over-generalization	✓									
No connection			✓							
Too much data			✓							
Slow progress for some			✓							
May lack understanding					✓					
May become lost in mathematical concepts							✓			
No hindrance		✓		✓		✓		✓	✓	✓

Ben's examples of using representations in his classrooms. Ben gave specific examples as to how he represented mathematical ideas in the five different modes from the interview questions about each representation. For manipulatives, he said that he used red chips and blue chips to teach positive and negative integers (Ben's pre-interview), and for symbols, he explained the understanding of symbols about positive and negative integers to learn algebra (Ben's post-interview). He also used algebra tiles, counting blocks, and hands-on activities with cubic, quadratic, and exponential relationships (Ben's post-interview) as examples of manipulatives. In the pictorial representation, he described the connection between tables and graphs to teach linear algebra, and he also said, "They talked about how to find the rate by tables and how to find the rate by graphs" (Ben's post-interview) in both the pre- and post-interviews. Thus, he used pictorial representations as a way to communicate mathematical ideas. For the language representation, he also described how he discussed teaching linear algebra with his students. The following excerpt illustrates Ben's descriptions involving how he used language representations in the classroom with his students:

> *When I taught linear algebra at the very beginning, we talked about the rate. And some kid kept on referring to that as the coefficient of x. And some kid said, 'Oh, it's just the number in front of the x.' Yes, but it also represents rate and how quickly or slowly something is moving. Their understanding orally that, 'Oh, I know that value tells us that for every increase of 6'…* (Ben's post-interview).

Ben used realistic representations to teach linear relationships. He showed examples of using realistic, symbolic, and language representations at the same time in his examples. Moreover, he showed different contexts in teaching a concept. The following excerpt illustrates Ben's description about connecting realistic representations with symbolic ones in his pre-interview:

> *We made a connection with shadow length and the student's height length, and we figured out the ratio to find the missing value of the students. The day before, we did the same thing with a mirror* (Ben's pre-interview).

He also explained his ideas to use realistic, symbolic, and language representations together in his post-interview. The following excerpt shows his descriptions regarding examples in his post-interview:

> *Teachers are getting away from the idea of shoving equations down kids' throats, but really understanding them, putting in real-life scenarios about why something works…*(Ben's post-interview).

How translations among multiple representations help or hinder student learning, Ben said that the translation among multiple representations around the same concept helped him interpret mathematical concepts in multiple ways and improve student learning. As a result, he gave examples about using the translations within or between representations. The following excerpt shows Ben's descriptions concerning examples in his pre-interview:

> *It also helps students' understanding if a student could explain why this problem represents a linear relationship, if a student could look at graphs, if he could also prove it with a table, … and if he could do all those things…*(Ben's pre-interview).

He also explained examples of using translations among representations in his post-interview. The following excerpt illustrates his descriptions about examples in his post-interview:

> *If they can see a value or an answer or an equation in different scenarios, whether it be a table or a graph, then that completely helps them truly understand what that represents…*(Ben's post-interview).

Ben also elaborated that it helps students make a connection among mathematical concepts. He thought that using multiple representations did not hinder any students' learning. Table 4.5 summarizes how Ben discussed translations among multiple representations for student learning:

Table 4.5

Ben – Translations Among Multiple Representations

Ben	Help			Hindrance	Translation example	
	Deeper understanding	Connection	Multiple interpretations	No hindrance	Within representations	Between representations
Pre	✓	✓	✓	✓	✓	
Post	✓		✓	✓		✓

In sum, Ben believed that using all multiple representations described in the Lesh translation model helps deepen students' mathematical understanding, in both the pre- and post-interview. He said that using translations between the representations helps develop students' conceptual understanding in mathematics. He especially emphasized the translations within language or realistic representations, and the translations between language and pictorial representations in the overall interviews. I will describe the changes of his beliefs in the following section. I only describe when there were changes in his beliefs.

Ben's belief changes

In his pre-interview, Ben described manipulatives as a visual representation, and he also said that it could be a hindrance if students overgeneralized a manipulative representation to understand different mathematical concepts in his pre-interview. However, he emphasized that using manipulatives helps students' conceptual understanding, as well as the visualization of mathematical concepts in his post-interview. I summarized Ben's belief changes about using manipulatives in Figure 4.1.

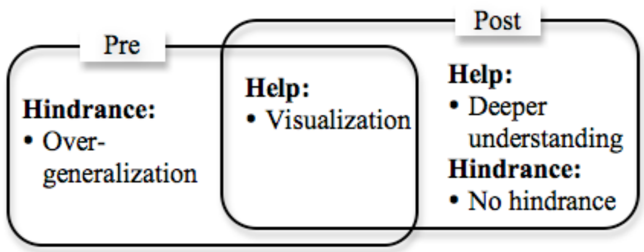


Figure 4.1. Ben's belief changes about manipulatives.

Ben also explained that he tried to use as many hands-on materials as possible in his post-interview. The following illustrates an excerpt about using manipulatives from his post-interview transcript:

> *I try to use as much hands-on stuff as possible…I try to use as many different ways of teaching, whether it be hands on, getting the kids to act out scenarios. I do as much as I can…* (Ben's post-interview).

For the pictorial representations, Ben said in both the pre- and post-interviews that this representation helps deepen students' understanding in mathematics. He added in his post-interview that this helps students explain mathematical concepts. In the pre-interview, he mostly described the hindrances of pictorial representations rather than its benefit. For example, he said that there could be hindrances if students were not able to connect the pictorial representations with mathematical concepts or other representations, or if there were too much data to use. He also said in his pre-interview that some slower learners might not need to show all different pictorial representations in order to explain a concept. However, he did not describe any hindrances in his post-interview. I summarized Ben's belief changes about using pictorials in Figure 4.2.

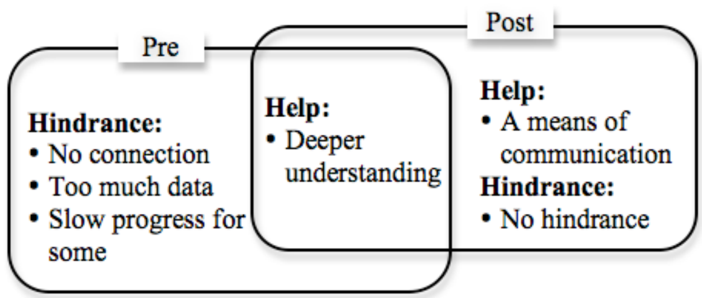


Figure 4.2. Ben's belief changes about pictorials.

For symbols and language, Ben said that using those representations helps students facilitate deeper mathematical understanding in the pre- and post-interviews. For language, he said that it was used to connect something with mathematics when explaining their thinking, in both pre- and post-interviews. He elaborated that there would be hindrances if students did not understand the meaning of symbols during classes. He said that if students lost mathematical concepts while explaining their thinking, it could hinder their learning in mathematics. However, Ben said that there were no hindrances in using symbolic and language representations during his post-interview. I summarized Ben's belief changes about using symbols and language in Figure 4.3.

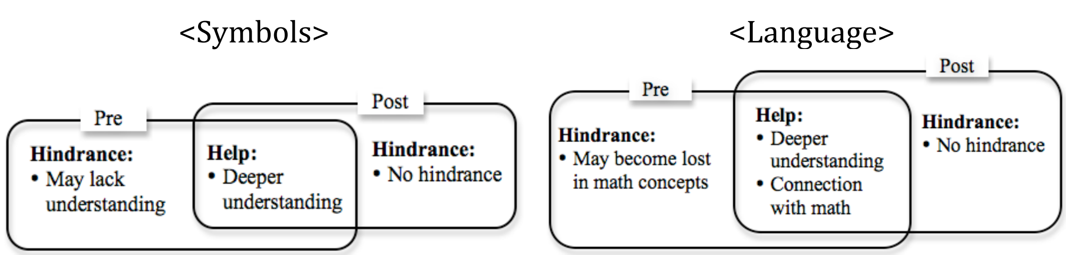


Figure 4.3. Ben's belief changes about symbols and language.

In sum, a marked change in Ben's beliefs was about using manipulatives. He showed more positive beliefs toward using manipulatives for his 7th-grade class. For language, he described positive rather than negative things in his post-interview. In addition, Ben elaborated in his post-interview that the translations between pictorial and language representations, as well as the translations within pictorial representations were effective for deepening students' understanding.

Ben's practices

Observation 1. Ben's classes in the first observation (October, 2011) were algebra classes for teaching the expanded forms of quadratic equations, which is $ax^2 + bx + c$. On Day 1 of the first observation, he emphasized visual representation such as tables, graphs, and area models. He also used a word problem to teach how to change between factored and expanded forms. He discussed the key terms in the problem with the students, and he let students draw a picture, and make an expanded form for this problem. On Day 2, he focused more on finding values a, b, or c in the form $ax^2 + bx + c$. Therefore, he used symbolic, language, and pictorial representations. For Day 3, he reviewed changing from expanded to factored forms of quadratic equations using symbols.

Using the Lesh translation model in the first observation. On the first day of the first observation, Ben used area models as a pictorial representation to teach quadratic equations focused on the switch between the factored and expanded forms. He first showed an area model to find the expanded form. He also had students change a factored form into the expanded form in symbolic representations. At the end of class, he used two real-world problems to find the quadratic equations, and he showed the area models for the problems. Students found the expanded forms using the area models. Thus, Ben mostly used pictorial, symbolic, and language representations with some realistic representations. In addition, Ben mostly used translation from pictorials to symbols, but little translation from symbols to pictorials. Ben used translating from realistic to symbolic or pictorial representations, but little symbolic or pictorial to realistic representation.

On Day 2, he used pictorial representations to review the day 1 class. He also asked how to draw the area models from factored forms of the quadratic equations, and then he asked the students how to draw the expanded forms using the area models. However, he mostly used the translations between symbolic and language representations, such as finding the missing values b and c in the form $x^2 + bx + c$. Thus, he explained that in the form, $x^2 + bx + c$, students could find two numbers that would not only multiply to equal the constant term "c," but would also add up to equal "b," the coefficient on the x-term. He also taught how to find the factor forms of the quadratic equations $x^2 + bx + c$. In addition, when a student asked about using area models to solve problems, he drew the area model for a problem. On Day 3, he used the same pictorial representation in checking homework. For the topic of the day, he used plenty of different problems to switch expanded to factored forms of quadratic equations, and he asked students how they solved the problems. As a result, he mostly used the translation within symbols, but he only checked the answers from the students. The diagrams in Figure 4.4 show the use of mathematical representations in the first classroom observation.

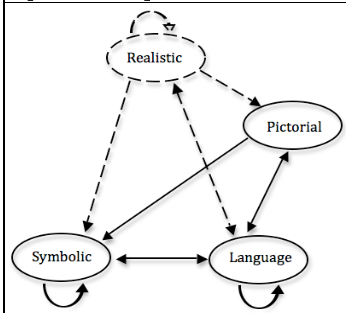
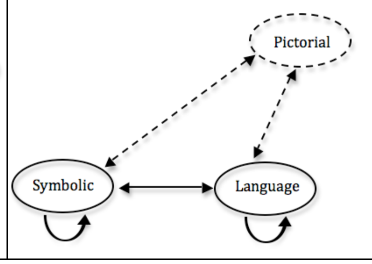
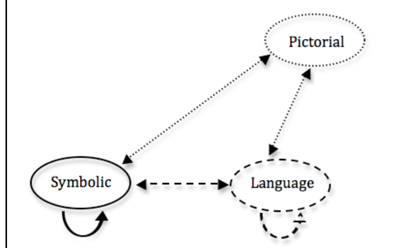
<table>
<tr><td>Day 1:
Expanded forms of quadric equations</td><td>Day 2:
Expanded forms of quadric equations</td><td>Day 3:
Review factored forms of quadric equations</td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td colspan="3">Note. Type of line:
Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class</td></tr>
</table>

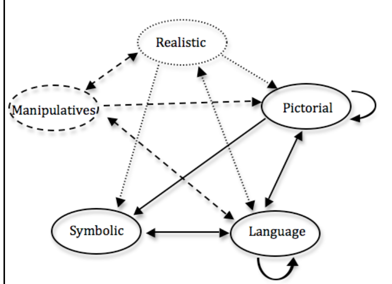
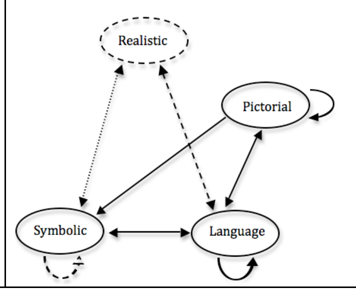
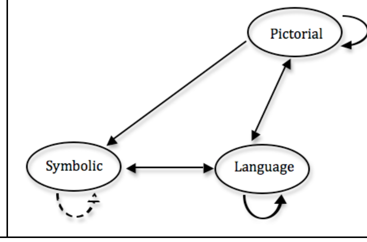
Figure 4.4. The first classroom observation in Ben's algebra class for three days.

Observation 2. In the second observation (February, 2012), he taught squares, rectangles, quadrilaterals, and polygons in geometry classes. On Day 1 of the second observation, he talked about finding slopes on graphs to connect with the properties of squares and rectangles in the beginning of class. He also used a real-world problem to find the slope on a graph. For example, he provided coordinate planes to set up the location of two houses for a warm-up problem. Then his students found the slope through the two points on the coordinate planes. Finally they talked about how to figure out perpendicular lines using graphs, and then they found a formula for them, such as the multiplication of two slopes equals to -1($m_1 \bullet m_2 = -1$). When he talked about the concepts of perpendicular lines, he used right triangles and squares. He also used a piece of copy paper to check out perpendicular lines as a short classroom activity after talking about the right angle of each corner of the paper.

On Day 2, he started out by comparing between Celsius and Fahrenheit. He showed a formula, $F = \frac{9}{5}C + 32$, and he asked what this formula represented. Then they solved problems in order to convert the current Fahrenheit degrees to Celsius. In checking assignments, most problems in the assignment were real-world word problems, and they used coordinate planes to solve the problems. For the topic of the day, they talked about the different types of quadrilaterals such as squares, rectangles, parallelograms, rhombuses, or trapezoids. They used pictures on geo-board dot paper, and they discussed how to clarify the different types of quadrilaterals using the terms of perpendicular and parallel. To prove the perpendicular and parallel lines of each quadrilateral on the pictures, the students used the slopes of sides and the formula, $m_1 \bullet m_2 = -1$ in the pictures. On Day 3, he taught the definitions of polygons, such as triangles, squares, rectangles, parallelograms, rhombuses, trapezoids, and so on. They defined each polygon in written language, and he shared different types of polygon pictures that were drawn on geo-board dot paper. He also discussed how to prove 90 degrees to figure out the types of quadrilaterals.

Using the Lesh translation model in the second observation. On Day 1, he used a real-world situation to find the slope on a graph for a warm-up problem. He taught drawing squares using two points on coordinate planes, and he proved the perpendicular lines in the squares. He also explained the perpendicular lines using graphs, and using the algebraic concept such as the multiplication of two slopes equals to -1($m_1 \bullet m_2 = -1$). He also manipulated a piece of copy paper to check out the perpendicular lines. The students also practiced drawing right triangles and rectangles through two points on coordinate planes. That is, Ben mostly used translating between language, symbolic, and pictorial representations, and translating from pictorials to symbols. He used the translations within pictorial and language representations, and he used few manipulatives and realistic representations.

On Day 2, he used a real-world situation to switch Celsius to Fahrenheit in the warm-up problem, and he used pictorial, symbolic, and language representations for discussing the assignment from the last class. For the topic of the day, they talked about the different types of quadrilaterals using pictures on geo-board dot paper. He talked about flying kites to hear about the types of quadrilaterals from the students. The students proved why each quadrilateral could be a square, rectangle, parallelogram, rhombus, or trapezoid using the perpendicular lines and parallel lines of each quadrilateral in the pictures. Thus, they used the terms of the slopes of sides and the formula, $m_1 \bullet m_2 = -1$ to explain them. That is, Ben mostly used the translations between pictorials and language, and between symbols and language. He also used translating from pictorials to symbols. On Day 3, he defined polygons such as triangles, squares, rectangles, parallelograms, rhombuses, and trapezoids in written language. He also used different types of polygon pictures that were drawn on geo-board dot paper, and he proved each polygon using slopes. Therefore, he actively translated within and between pictorial and language representations. He also used symbolic representations. The diagrams in Figure 4.5 show the use of mathematical representations in the second classroom observation.

Day 1: Squares	Day 2: Quadrilaterals	Day 3: Polygons
		

Note.
1. Manipulatives = Physical manipulatives
2. Type of line:

Strongly used in teaching

Weakly used in teaching

Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

Figure 4.5. The second classroom observation in Ben's geometry class for three days.

Observation 3. In the third observation (May, 2012), Ben taught linear equations in algebra. On Day 1, they talked about the medians in a triangle during an assignment check-up. Then they talked about the equations of median lines in the triangle. He also used the ratio of the mid-point of the triangle. He connected geometric concepts with algebraic concepts. For the topic of the day, Ben taught linear equations using a real-world word problem from the textbook. Ben and his students talked about the system of linear equations (e.g., $y = -2x + 14$ and $y = 0.4x + 0.5$. Find $(x,\ y)$), and the students solved the problems during the rest of class time.

On Day 2, they solved the linear systems of equations in the textbook such as "$3x-2y=-2$ and $2x-3y=16$. Find (x, y)," or "$y = \frac{2}{3}x - 4$ and $2x + y = 4$. Find (x,y)." He also talked about solving the problems by graphs. The students solved the problems using another traditional algebra book. On Day 3, he taught linear systems of equations continuously. He used a real-world word problem to solve a linear system of equations. The students worked solving the linear system of equation problems focused on symbols.

Using the Lesh translation model in the third observation. On Day 1, Ben connected geometric concepts with algebraic concepts to talk about the equations of median lines in triangles. Thus, he used pictorial, symbolic, and language representations in checking the homework. For the topic of the day, he used a real-word problem to find a point to be able to meet two airplanes. The students solved problems to find the (x, y) values from two linear equations. As a result, they mostly used symbols in the problems, and they also used language representation to explain how to find them. Therefore, Ben actively used translating within and between symbolic and language representations. He also used translating from realistic to symbolic or pictorial representations, but little symbolic or pictorial to realistic representation.

On Day 2, he solved various problems about the linear systems of equations such as "$3x-2y=-2$ and $2x-3y=16$. Find (x, y)," or "$y = \frac{2}{3}x - 4$ and $2x + y = 4$. Find (x,y)." Day 3 was same as the day 2 in teaching the linear systems of equations. However, he used a real-world problem on Day 3. Therefore, he primarily used the translation between language and symbolic representations on both Days 2 and 3. The diagrams in Figure 4.6 show the use of mathematical representations during the third classroom observation.

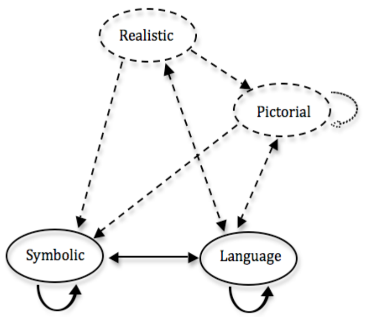
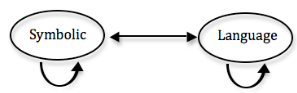
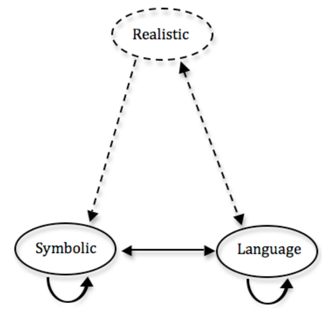
<table>
<tr><td>Day 1:
Linear systems of equations</td><td>Day 2:
Linear systems of equations</td><td>Day 3:
Linear systems of equations</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td colspan="3">Note. Type of line:
Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class</td></tr>
</table>

Figure 4.6. The third classroom observation in Ben's algebra class for three days.

In sum, Ben mostly used the translations within and between symbolic, and language representations in the overall classroom observations. In the first and second observations, he provided many pictorial representations to encourage mathematical thinking, and to translate to mathematical symbols. However, in the last section of the chapter for the linear system of equations (observation 3), he mainly used symbolic and language representations. He also used some real-world word problems from the textbooks. His use of the manipulative representation was missing or limited in all observations. He did not use the manipulative representation during observations 1 and 3 and he only used one type of manipulative representation during his geometry class in the second observation.

Sara's Case

Sara had three years of teaching experience in middle schools, and she was teaching low-achieving seventh-grade students using the connected mathematics curriculum in School A. Her class size was about 25 students, and she always changed the students' groups to discuss mathematical concepts and solve problems in each observation. She also actively used the interactive whiteboard to engage students participating in class.

Sara's beliefs about teaching

In the following sections, I explore Sara's general beliefs about teaching mathematics, such as (1) what she believes the role of the mathematics teacher is; (2) how she decides what or what not to teach; and (3) how she maximizes student learning in the classroom. These beliefs are related to her beliefs about using multiple representations. Thus, this analysis helps investigate research questions about teachers' beliefs in using representations and their practice in this study.

The role of the mathematics teacher. Sara emphasized creating the best learning environment as the role of the mathematics teacher. In the pre-interview, she said the role of the mathematics teacher is "not only to teach students the required contents in a variety of ways," but also to "engage them" in learning mathematics. She also mentioned that teachers should help students "encourage [students'] excitement about math." In order to create effective learning environments, she was trying to "incorporate different types of learning." In the post-interview, Sara said the role of the teacher is to create "a safe environment" in order to give students "opportunities to do cooperative learning." She also believed that the role of the teacher is to "show that math is relevant to their life." Thus, she believed that teachers could help students understand the importance of mathematics in their lives.

What or what not to teach. Sara said she worked with colleagues in order to decide what or what not to teach. In the professional learning community (PLC), they made decisions as to what they would cover in their classes (Sara's pre-interview). She also said that seventh-grade teachers, including both Sara and Ben, shared strategies for what worked for students. Sara said that the seventh-grade teachers, including herself also believed that "the curriculum and the standards drive what content needs to be taught" (Sara's post-interview) to their students. She said that "the seventh-grade teachers thought about how they could deliver those contents for their own students" (Sara's post-interview). In the pre-interview, she explained that she planned lessons based on her own creativity and by collaborating with her PLC members, or she researched on her own about how to teach. In the post-interview, she elaborated that she used different teaching strategies to meet all students' needs. She also said that students needed reading and comprehension abilities in order to learn mathematics.

How to maximize student learning. Sara said she maximized students' learning through differentiating students' work and homework based on their ability levels. In the pre-interview, she said, "If I explain it through an algorithm, that's half the kids. If I can explain it with a picture, that's the other half of the kids." Thus, she emphasized using pictorials, as well as mathematical procedures. On the post-tests, Sara maintained that she was reaching different learning styles in her classroom because "every child learns so differently from their culture to just the way that they learn." She also elaborated that her students needed to practice before their tests, so she encouraged students to review what they needed for their performance. I summarized Sara's beliefs about teaching mathematics in Table 4.6.

Table 4.6

Sara – Summary of Her Beliefs About Teaching Mathematics

Sara	**1. The role of the mathematics teacher**
	• Create a best learning environment (pre & post) • Engage students in learning mathematics (pre) • Encourage their excitement about mathematics (pre) • Incorporate different types of learning (pre) • Give students opportunities to practice cooperative learning (post) • Help students understand the importance of mathematics in their lives (post)
	2. What to teach
	• Work with colleagues to decide (pre & post) • Make lessons based on her own creativity (pre) • Consider the curriculum and the standards (post) • Use different teaching strategies to meet all students' needs (post)
	3. How to maximize
	• Differentiate students' works and homework (pre & post) • Using pictorials (pre) • Reach different learning styles (post) • Review & practice what students need for their performance (post)

Sara's beliefs about the Lesh translation model in the first three interview questions about teaching. The first three interview questions were about (1) what the role of the mathematics teacher is; (2) what or what not to teach; and (3) how to maximize student learning in the classroom. In this section, I described how Sara talked about the Lesh translation model in the three questions about teaching. Sara said that she used multiple representations actively in her classes when she answered the question about teaching. Sara stated in her pre- and post-interviews that using each representation and translating between representations are very good teaching strategies to help students' learning in mathematics. Sara elaborated that she was trying to "engage her students by using pictures, books, and hands-on things" (Sara's pre-interview) in the question about her role as the teacher. She also said that as her role of the teacher, "a huge part of middle-school mathematics is showing that mathematics is relevant to their life" (Sara's post-interview). That is, she described the mathematics teacher's role as using representations for students' engagement, and as making mathematics relevant for their motivation in learning mathematics. When she decided what to teach, she thought about the different ways of presenting information, such as verbal or written, and pictures in her pre-interview. Sara also said that she adopted various materials from the National Urban Alliance (NUA) to "balance in my classroom of hands-on… that they're getting the writing piece and the reading pieces" (Sara's post-interview). Sara explained that she tried to maximize student learning using mathematical representations in her classroom. The following excerpt shows Sara's descriptions about how to maximize student learning in her classroom:

> *If I explain it through an algorithm, that's half the kids. If I can explain it with a picture, that's the other half of the kids. So I feel like the more I can represent and show answers and show steps and show problems in different ways, I feel like that also maximizes the learning as well* (Sara's pre-interview).

She believed that by connecting multiple representations, students could overcome their limitation of understanding mathematical concepts presented by one particular representation. Table 4.7 summarizes how Sara discussed the translations between or within representations.

Table 4.7

Sara's Case From Interview Questions 1-3

Sara	Instances of representational fluency	Pre	Post
The role of the mathematics teacher	Representations	P, M	L, R
	Translation within representations	P, M	L, R
	Translation between representations	T	T
What to teach	Representations	P, L	M, L
	Translation within representations	P, L	M, L
	Between representations	T	M & L
How to maximize	Representations	P, S	L
	Within representations	P, S	L
	Between representations	P & S, T	T

Note. M: Manipulatives, P: Pictorials, S: Symbols, L: Language, R: Realistic representations, and T: Translation among multiple representations (representations not identified).

Sara's beliefs about representations

The following sections elaborate on Sara's beliefs about how each representation helped and hindered her teaching of mathematics. The examples of using each representation are described. I also elaborated on her beliefs about the translation between and within multiple representations.

How representations help student learning. In the interview questions about each representation, Sara described what using each representation meant in her mathematics teaching. Sara said that manipulative representations could be thought of as visual representations; "that would be the way that any additional opportunities to see it in different ways" (Sara's pre-interview). She explained in her post-interview that these representations help deepen students' understanding. For pictorials, she maintained, in both pre- and post-interviews, that these representations enhance students' conceptual understanding and help connect mathematical concepts with students' thinking. For symbols, she said in her pre-interview that they are mathematical terms used to learn mathematics, and later she stated in her post-interview that symbols help with students' conceptual understanding. She also thought of the symbolic representations as another language representation in her post-interview. Sara said that both language and realistic representations help students connect something with mathematics, and these representations help build students' understanding in mathematics. She also elaborated that using language representations help students talk about mathematics using their own language. Moreover, she explained that realistic representations help students explore mathematical concepts, and provide students with the motivation to learn mathematics. Table 4.8 summarizes how Sara discussed the helpfulness of using each representation for student learning.

Table 4.8

Sara – Helpfulness of Using Representations From Interview Questions 5-9

Sara / Help	Manipulatives		Pictorial		Symbolic		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Deeper understanding		✓	✓	✓		✓	✓	✓	✓	
Visualization	✓									
Mathematical terms					✓					
Connection with students' thinking			✓	✓						
Connection with mathematics							✓			✓
Alternative representations						✓				
Paraphrasing								✓		
Exploration										✓
Motivation										✓

How representations hinder student learning. For manipulatives, she stated in her pre- and post-interview that students sometimes use manipulatives in improper ways. She also elaborated in her pre-interview that some students did not connect manipulatives with mathematical concepts that were supposed to be understood, so it could hinder students' learning in mathematics. Sara also said in both pre- and post-interviews that there was a hindrance when some students used incorrect pictorial representations to solve problems. For symbols, she said that if students did not understand the meaning of mathematical symbols, then it could hinder students' understanding in mathematics. For language and realistic representations, she believed that teachers should help students minimize having any hindrances while they use representations. She said in her post-interview that teachers need to help them when they have issues in using language. In order to use realistic representations without any hindrances, she said in her post-interview that teachers had students use contexts relating with their own lives. Table 4.9 summarizes how Sara discussed the hindrances of using each representation for student learning.

Table 4.9

Sara – Hindrances of Using Representations From Interview Questions 5-9

Sara / Hindrance	Manipulatives Pre	Manipulatives Post	Pictorial Pre	Pictorial Post	Symbolic Pre	Symbolic Post	Language Pre	Language Post	Realistic Pre	Realistic Post
Improper use	✓	✓								
No connection	✓									
Incorrect presentation			✓	✓						
May lack understanding					✓	✓				
No instruction when needed								✓		
Using irrelevant context										✓
No hindrance							✓		✓	

Sara's examples of using representations in her classrooms. Sara gave specific examples as to how she represented mathematical ideas in the five different modes from the interview questions about each representation. She gave examples of manipulatives, such as 16 tiles to find the largest perimeter (Sara's pre-interview), and counting blocks or algebra tiles to teach integer operations (Sara's post-interview). She also used how to teach integer operations using realistic representations in her pre-interview. The following excerpt illustrates Sara's descriptions involving how she used realistic representations in the classrooms with her students in the pre-interview:

> *In working with integers, giving examples of real-world situations with money – positive, negative, you owe, you earn – and temperature, that was when a lot of 'aha' moments were made…*(Sara's pre-interview).

For pictorials, Sara described connecting tables and graphs in linear algebra, and using pictures to explain adding fractions. She said, "We spend a lot of our time at the end of each class having opportunities to write down or verbally share something that they learned today" (Sara's pre-interview). She said that she thought students should be able to "explain how this mathematical concept relates to whatever and how to use it" (Sara's post-interview). For realistic representations, she said that she always tried to "throw back some sort of situation where you would use it in the future" (Sara's post-interview).

How translations among multiple representations help or hinder students' learning. Sara maintained in both the pre- and post-interviews that using multiple representations around the same concept helps students deepen their understanding in mathematics. She believed that using multiple representations was a way to represent students' own mathematical ideas. She said that there were not any hindrances in using translations among multiple representations. The following excerpt illustrates Sara's examples about using multiple representations in her classes, as she described in her post-interview:

> *I think it depends on their reasoning. Why does this table have a constant rate of change? Well, if I look at my graph, I can see that it increases at a constant rate. And if I look at the equation, I can see that it must have a constant slope. ...It's definitely going to help their learning and understanding*...(Sara's post-interview).

Table 4.10 summarizes how Sara discussed translations among multiple representations for student learning:

Table 4.10

Sara – Translations Among Multiple Representations

Sara	Help	Hindrance	Translation example	
	Deeper understanding	No hindrance	Within representations	Between representations
Pre	✓	✓		✓
Post	✓	✓		✓

In sum, Sara explained that the translations within and between modes of representations help students enhance their conceptual understanding in mathematics. She also expanded on the importance of correctly using representations, such as manipulatives and pictorials. She showed a big picture on how she taught integer operations using multiple representations in the separate interview questions. For example, she used algebra tiles to teach the operations of integers, and then she discussed “how students have done it” (Sara’s pre-interview). She gave examples of real-world situations with money and temperature to teach them. She has a scenario to translate within and between representations to help students deepen their understanding. Her belief changes will be described in the following section.

Sara’s belief changes

In her beliefs about teaching, Sara primarily elaborated on the translation within and between language representations in her post-interview, while she described the translation within and between pictorial representations in her pre-interview. For deciding what or what not to teach, she said that she had considered using manipulatives to improve students’ mathematical understanding in her post-interview, while she talked about the manipulatives as a visualization tool in the pre-interview. In the post-interview, she focused more on using realistic representations as the role of the mathematics teacher, and she also provided an example of using mathematical representations with realistic representations in her post-interview.

In the interview questions about each representation, Sara saw manipulatives as a good visual tool in her pre-interview, but she elaborated that manipulatives promoted students' deeper understanding in mathematics during her post-interview. She also argued that there were hindrances if students did not use them correctly. Thus, she believed that teachers should introduce manipulatives to students clearly so that they can use them in the right situations. The following illustrates an excerpt about using manipulatives from her post-interview.

> *It needs to be presented correctly. If they're not understanding how to use the counting blocks correctly, they're not going to make that connection and understand the purpose of using them…*(Sara's post-interview)

She also talked about a hindrance when students did not connect manipulatives with mathematics in her pre-interview, but she said that these representations could help students connect something with mathematics in her post-interview. I summarized her belief changes about using manipulatives in Figure 4.7.

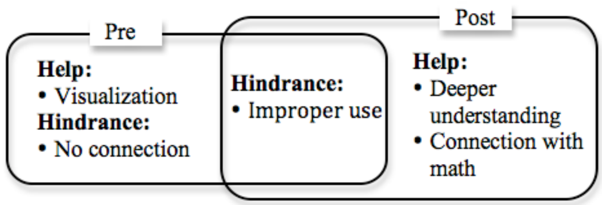


Figure 4.7. Sara's belief changes about manipulatives.

In her pre-interview, Sara explained that symbols were mathematical explanations; thus, they could hinder students' learning if they did not understand them. In her post-interview, she explained that using symbols helps students' understanding in mathematics, and it was a different way to describe mathematics. With respect to language representations, Sara stated in her pre-interview that language helps students connect something with mathematics so that they can enhance their mathematical understanding. In her post-interview, she said that language representations could make students explain mathematical concepts or their mathematical thinking using their own words. Also in her post-interview, she believed that when students use their own language, teachers need to help them by minimizing any hindrances in their learning. I summarized her belief changes about using symbolic and language representations in Figure 4.8.

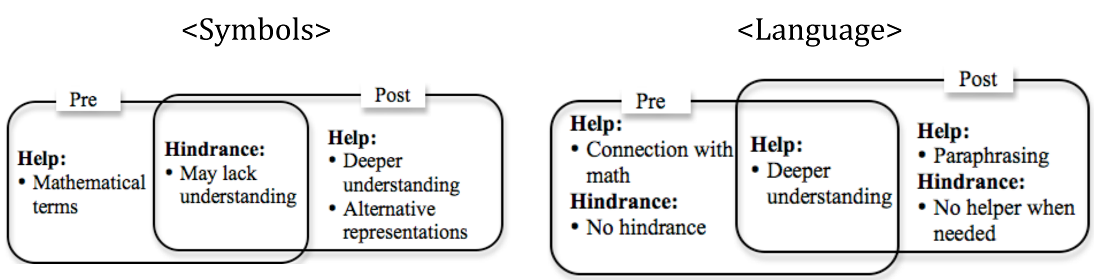


Figure 4.8. Sara's belief changes about symbols and language.

For realistic representations, Sara elaborated in her pre-interview that using real-world situations helps improve students' understanding. However, she stated in her post-interview that realistic representations can help student learning through connecting with mathematics, exploring real-world contexts, and giving students motivation. I summarized her belief changes about using realistic representations in Figure 4.9.

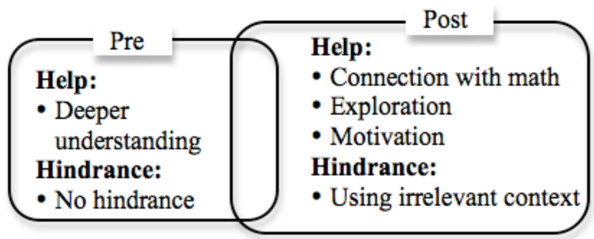


Figure 4.9. Sara's belief changes about realistic representations.

In sum, Sara' beliefs about using manipulatives was developed in her post-interview. She believed that manipulatives can help improve students' mathematical understanding and connections with mathematics, rather than just as a means of visualization. Sara also stated in her post-interview that mathematical symbols were a different means for mathematical talk, which deepened students understanding in mathematics.

Sara's practices

Observation 1. The first observation (October, 2011) was geometry classes, and Day1 was about embedded triangles. At the beginning of class, she reviewed two methods, such as the shadow and mirror methods to find out the missing values in embedded triangles. Sara gave out the problems using pictures and real-world situations, and she let the students discuss their methods they used to solve the problems with their neighbors. Then the students shared their methods with the whole class, and she talked about how they found the values using ratios and scale factors. When she used real-world problems, the questions were from the textbook.

Day 2 was about stretching and shrinking using proportions. She began class with homework that involved realistic, pictorial, language, and symbolic representations.

During the rest of the whole class, she set up 10 different problems on each table and 2-3 students solved the problem in a group. Each group moved around to solve the problems on the each table every 7 minutes. The problems included real-world situations, symbolic and pictorial representations. While students solved each problem in a group, she moved around and talked about the students' problem-solving strategies with the students in each group.

On Day 3, she reviewed a quarter of the fall semester for the next day's test. She showed a common mistake in setting up ratios for an embedded triangle problem. She discussed what the problems were in the ratios. She talked all about embedded triangles, ratios, scale factors in similar rectangles and triangles, and stretching and shrinking of polygons.

Using the Lesh translation model in the first observation. On Day 1, Sara used word problems with real-world contexts to teach about similar triangles, and she drew pictures of the embedded triangles that described the problems. She first talked about the corresponding sides from the triangles, and she set up two ratios to find the missing number from the problem. Then she used the scale factor to find a missing value in the equation that says that two ratios are equivalent. In a question, Sara discussed what each number in the ratios represented in the picture after writing out the ratios with the students. That is, she actively used the translation within and between pictorial, language, and symbolic representations. In addition, she used translating from realistic representations to symbolic and pictorial representations, and translating within and between language and realistic representations.

On Day 2, she assigned six groups with 2-3 students, and the students worked with ten problems in the worksheet, which included realistic, pictorial, symbolic, and language representations in a group. She was going around the groups, and she helped the students use spoken language. She used the students' pictures and equations that the students created using two ratios. That is, Sara used pictorial, symbolic, and language representations when the students needed them, but the problems on the worksheet were mostly solved by the students. Realistic representations were presented on the classroom worksheet.

On Day 3, she reviewed similar polygons. She mostly used language and symbolic representations. For example, she discussed the rules of (2x, 5y) or $(\frac{1}{3}x, \frac{1}{3}y)$ which could result in a similar figure. She also explained scale-factors to find missing numbers from two similar polygons. That is, Sara mostly used translating within and between symbolic, and language representations, and translating within and between pictorial and language representations. She also used translating from pictorial to symbolic representations. In addition, she used symbolic, pictorial, language, and realistic representations for checking homework. The diagrams in Figure 4.10 show the use of mathematical representations in the first classroom observation.

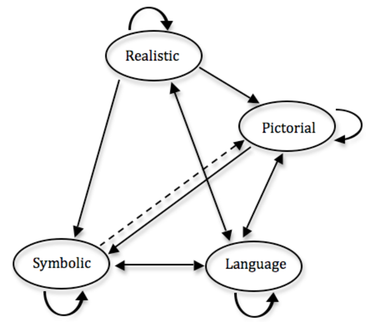
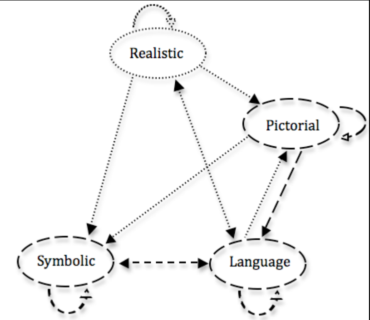
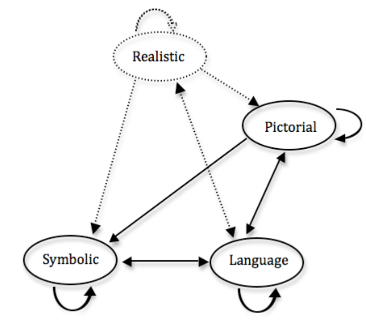
<table>
<tr><td>Day 1:
Embedded triangles</td><td>Day 2:
Stretching and Shrinking</td><td>Day 3:
Review</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td colspan="3">Note. Type of line:
Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class</td></tr>
</table>

Figure 4.10. The first classroom observation in Sara's geometry classes for three days.

Observation 2. The second observation (February, 2012) was algebra classes to teach linear equations. At the beginning of the Day 1 class, Sara showed a function machine that showed the input x at the top, the output y on the bottom, and a function in the middle of the body with the ordered pair (x, y). She asked to find y-values while adding random x values, and she made a table for (x, y) that the students found. She also talked about the patterns of the (x, y) in the tables. Then she gave warm-up questions to check whether the ordered pairs were in functions using a function machine. Then she talked about the steps of writing equations, so she discussed the rate of change, the y-intercept, and the y=mx+b form of equations. She also gave real-world examples, such as the rate of eating chicken wings. For solving linear equation problems, she used tables, graphs, equations, and real-world word problems.

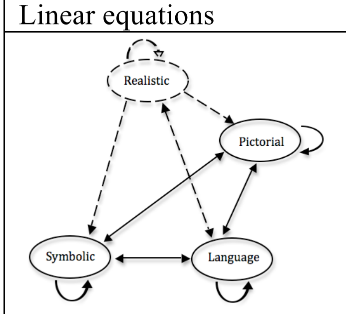
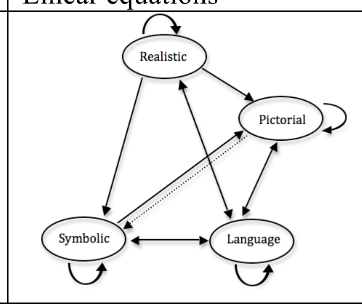
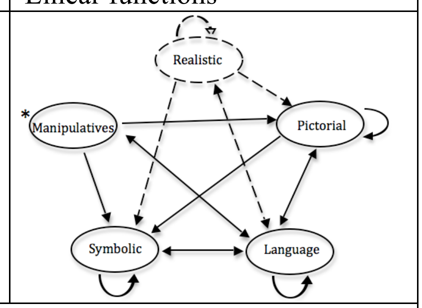
On Day 2, Sara talked about the problems in finding linear equations using function machines and tables for the warm-up. Then she talked about "what variable depends on the other variable," such as "miles depend on time." As a result, she wrote and talked about how linear equations showed in three different representations, such as graphs, tables, and equations. She used real-world word problems to find equations and variables using tables and graphs.

At the beginning of Day 3, she used a virtual manipulative called "the interaction function machine activity" on a website. In the activity, Sara had students input any numbers in the function machine and the virtual manipulative automatically created the table on the screen. Then based on the table, students inserted the rate of change and the y-intercept in the blank boxes of an equation (e.g., y=☐x + ☐) using the keypad. During the class, she showed two linear equation graphs that had time as an independent variable, distance as the dependent variable, and an intersection point. She talked about the graphs with students.

Using the Lesh translation model in the second observation. On Day 1, Sara first created a table for x and y values while students found the y-values (output) of linear equations by substituting any number for x-values (input) into a linear equation (e.g., y=2x+5), when reviewing the last class. In addition, for the warm-up, she asked several questions about whether or not an ordered pair (x, y) (e.g., (5, -2)) was on the line of a linear equation, $y = mx + b$ (e.g., $y = -2x + 12$). For the topic of the day, she explained the meaning of each symbol in a linear equation, $y = mx + b$, such as m means the rate of change ($\frac{\text{change in y}}{\text{change in x}}$), b is the y-intercept when x=0, and x and y are input and output. She also gave examples with real-world contexts to explain the rate of change (e.g., the rate of eating chicken nuggets). She also created tables and graphs to find the rates of change for linear equations, and she also used a graph to find a linear equation. That is, Sara actively used translating within and between pictorial, symbolic, and language representations. She also used realistic representations in the examples and problems for linear equations. Furthermore, she had her students solve problems to create tables and graphs to find linear equations in a handout, which she taught students during Day 1.

On Day 2, she explained how to find linear equations using tables and graphs for the warm-up. She used a real-word context (e.g., biking to win) to find a linear equation, and used a graph to prove if the equation was correct in checking homework. For the topic of the day, she explained what value depends on the other (e.g., y depends on the x) showing on the graph and the table (e.g., y (output) on the right depends on x (input) on the left). Then she explained an equation, such as distance per time (e.g., d=rt). She also emphasized that distance did not always depend on time. She used word problems with real-world contexts (e.g., running to win) to find linear equations and to create graphs for the equation. That is, she used translating within and between pictorial, symbolic, language, and realistic representations, but she only used translating from pictorial to symbolic representations in the warm-up and checking of homework.

On Day 3, she used a virtual manipulative representation from the online website, which was to find equations using a function machine. In the virtual manipulative, it created tables as she put x-values as inputs into a function match, and the students created linear equations based on the x and y values in the tables. For the topic of the day, she showed two linear lines in terms of x=time and y=distance with an intersection, and she talked about the intercept on the graphs. She talked about the rates of change, and the y-intercepts on the graphs. In addition, she also used the term the “coefficient of x” as the rate of change, and she found coefficients of x and y-intercepts from linear equations. She also explained them on the graphs. That is, she used translating between virtual manipulative, pictorial, symbolic, and language representations with little realistic representation. However, she used translating from virtual manipulative to pictorial and symbolic representations, and translating from realistic to pictorial and symbolic representations. She also used the translations within pictorial, symbolic, language, and realistic representations. During class, the students had time to solve the problems in order to find the coefficients of x and y-intercepts from linear equations. The diagrams in Figure 4.11 show the use of mathematical representations in the second classroom observation.

Day 1: Linear equations	Day 2: Linear equations	Day 3: Linear functions
		

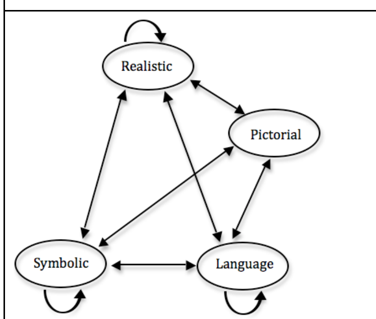
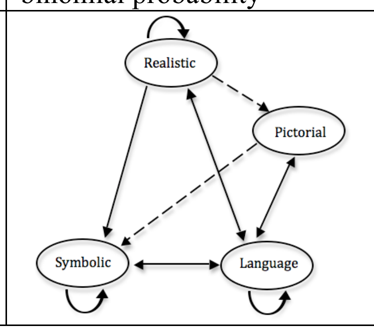
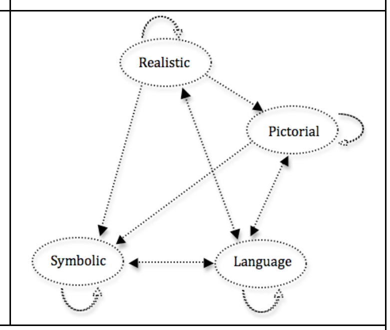
Note.
1. * means virtual manipulatives
2. Type of line:
   Strongly used in teaching (solid line)
   Weakly used in teaching (dashed line)
   Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students’ worksheets during class (dotted line)

Figure 4.11. The second classroom observation in Sara’s algebra classes for three days.

Observation 3. The third observation (May, 2012) was probability classes concerning binomial probability. At the beginning of Day 1, Sara used a real-world word problem to find Ben (a case in this study)'s five-digit-long favorite number. She also talked about the combination of an iPad password. For the rest of the class time, she discussed the definition of binomial probability, and then talked about examples of them. She gave different situations to solve problems of them. The problems involved realistic, pictorial, symbolic, and language representations. On Days 2 and 3, Sara reviewed the expected values and binomial probability using real-world word problems, tables, symbols, and language.

Using the Lesh translation model in the third observation. On Day 1, she gave the definition of binominal probabilities, and she asked her students to create two examples of the binomial probabilities. Most of the students' and her examples were from real-world contexts, such as having boys or girls or true or false quizzes. The students created a table for representing all different ways to have five children. She also explained that ½ is the probability of each child being a boy or a girl (50% chance). Then she asked different types of binomial probability questions, such as finding the probability that a family has exactly five girls, five kids with a least one boy, or five kids with at most one boy. Sara asked students to create the table to show each event in the binomial probability questions in front of the board individually, and she explained the problems using the table the students created. She also had students solve various binomial probability questions. Sara and her students translated between pictorial (tables), symbolic, language, and realistic representations, and the translation within symbolic (½ and 50% chances), language, and realistic representations in order to solve problems.

On Day 2, she used word problems with real-world contexts, such as selling 5 female ($750) or male ($200) puppies, having all three male, all 26 female hamsters, and so on. When she solved the problems, she used a table for the problem about having all three males, and she directly used the probability, $\frac{1}{2}$. On Day 3, she spent most of the time checking homework. She showed her answer keys for homework about binomial probabilities and expected values, which included tables, an area model, a tree diagram and symbols. She asked students and explained to them how to make each pictorial representation, and how to find the answers in symbolic types. The problems in the homework, which the students solved, were word problems with real-world contexts. The diagrams in Figure 4.12 show the use of mathematical representations in the third classroom observation.

Day 1: Binomial probability	Day 2: Review expected value and binomial probability	Day 3: Review binomial probability
		

Note. Type of line:
Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

Figure 4.12. The third classroom observation in Sara's Probability classes for three days.

In sum, Sara mostly used realistic and pictorial problems, and she talked about mathematical concepts and problem-solving strategies with the students through three period observations. In addition, she used a virtual manipulative representation to teach linear functions, and the virtual manipulative representation was the same as her teaching method by using pictorial representations, such as a function machine in the second observation. She created real-world word problems that were related to her students' school lives, as well as textbook word problems. She also gave chances for students to express their problem-solving processes during problem solving in her classes. She mainly used translating within and between pictorial, symbolic, and language representations, with some realistic representations. Moreover, she used a virtual manipulative representation in the second observation.

Nick's Case

Nick had 34 years of teaching experience in middle schools, and he was teaching high-achieving 7th-grade students using the McDougal Littell mathematics curriculum in School A. His class size was about 32 students, and the students sat in a row. He encouraged the students to solve problem in front of the whiteboard.

Nick's beliefs about teaching

In the following sections, I explore Nick's general beliefs about teaching mathematics, such as (1) what he believes the role of the mathematics teacher is; (2) how he decides what or what not to teach; and (3) how he maximizes student learning in the classroom. These beliefs are related to his beliefs about using multiple representations. Thus, this analysis helps investigate research questions about teachers' beliefs in using representations and their practice in this study.

The role of the mathematics teacher. Nick first thought of the role of the mathematics teacher as delivering the standards and curriculum for students' performance, and as motivating students in mathematics in both pre- and post-interviews. In the pre-interview, he said that teachers should develop "a positive relationship with his students", but in the post-interview, he said that choosing "problems that are related to the real-life" was the role of the teacher.

What or what not to teach. Nick said that he decided what or what not to teach within the seventh-grade level team in both the pre- and post-interviews. In the pre-interview, he said that he used pre-tests in order to make his decisions. In the post-interview, Nick elaborated that the decision was driven by the state standards, and he considered the standards that were related to real-life salutations.

How to maximize students learning. Nick emphasized verbalizing mathematically and keeping more students engaged in order to maximize student learning, in both the pre- and post-interviews. In the pre-interview, he said that he used key instruction strategies that are research-based. He also said that he tested students' prior knowledge and emphasized student collaboration. In the post-interview, Nick said that he tried to vary his teaching methods so that it would be more interesting. He emphasized that students should be able to write mathematically, as well as verbalize orally. I summarized Nick's beliefs about teaching mathematics in Table 4.11.

Table 4.11

Nick – Summary of His Beliefs About Teaching Mathematics

Nick	**1. The role of the mathematics teacher**
	• Deliver mathematics standards and curricula for students' performance (pre & post) • Motivate students in mathematics (pre & post) • Develop a positive relationship with students (pre) • Choose problems that are related to the real-word (post)
	2. What to teach
	• Work with colleagues to decide (pre & post) • Use pre-tests (pre) • Consider the curriculum and the standards (post) • Consider relevant to real-world situations (post)
	3. How to maximize
	• Verbalize mathematically (pre & post) • Create engagement (pre & post) • Use research-based classroom instruction (pre) • Test students' prior knowledge (pre) • Enable students to collaborate (pre) • Use different teaching methods (post) • Make mathematics interesting (post) • Enable students to write and speak mathematically (post)

Nick's beliefs about the Lesh translation model in the first three interview questions about teaching. The first three interview questions were about (1) what the role of the mathematics teacher is; (2) what or what not to teach; and (3) how to maximize student learning in the classroom. In this section, I described how Nick talked about the Lesh translation model in the three questions about teaching. Nick emphasized using various realistic representations, in both the questions about the role of the mathematics teacher and what to teach. Nick also described language representations as a way of maximizing students' learning, in both the pre- and post-interviews. He elaborated that language representations help students deepen their understanding in mathematics. He talked about translating between representations, such as pictures, real-world situations and functions. Table 4.12 summarizes how Nick discussed the translations between or within representations.

Table 4.12

Nick's Case From Interview Questions 1-3

Nick	Instances of representational fluency	Pre	Post
The role of the mathematics teacher	Representation		R
	Translation within representations		R
	Translation between representations		
What to teach	Representation		R
	Translation within representations		R
	Translation between representations		
How to maximize	Representation	L	L
	Translation within representations	L	L
	Translation between representations		T

Note. L: Language, R: Realistic representations, and T: Translation among multiple representations (representations not identified).

Nick's beliefs about representations

The following sections describe Nick's beliefs about how each representation that helped or hindered his teaching of mathematics. The examples of using each representation are described. I also elaborated on his beliefs about the translation between and within multiple representations.

How representations help student learning. Nick said that manipulatives deepen students' mathematical understanding, which could provide good concrete examples for students. He explained that manipulatives can help students visualize mathematics, and that they mostly help lower-achieving students learn mathematics. Nick said that pictorial representations were his key teaching model so that he always thought of different pictorial representations for his classes. Nick believed that symbols were mathematical language to communicate with others, and that it is necessary to learn in mathematics. On the other hand, he said that language representation is very important for students' futures in the workplace, because they should be able to describe their thinking about what they know. Nick elaborated that both realistic and language representations give students positive motivation to learn mathematics, as well as they help deepen students' understanding in mathematics. Table 4.13 summarizes how Nick discussed the helpfulness of using each representation for student learning.

Table 4.13

Nick – Helpfulness of Using Representations From Interview Questions 5-9

Nick / Help	Manipulatives		Pictorial		Symbolic		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Deeper understanding	✓	✓					✓	✓	✓	✓
Visualization		✓								
Concrete examples	✓									
Lower-ability students	✓									
A key teaching model			✓	✓						
Necessity of symbols					✓					
A means of communication						✓				
Future relevance							✓			
Motivation								✓	✓	✓

How representations hinder student learning. Nick described the hindrances of using manipulatives and symbols. For using manipulatives, he talked about management issues because of the class size in his pre-interview. The following excerpt illustrates Nick's description regarding hindrances of using manipulatives in his pre-interview.

The only thing I can think of as a hindrance is that we have time constraints to cover things and sometimes it's hard to do things and get creative when you have

35 students in a class and you have 55 minutes…(Nick's pre-interview).

In his post-interview, he also said that using manipulatives might "slow the pace down for those students who already understand the abstract" (Nick's post-interview). For the symbolic representation, he said that it could be a hindrance if students did not understand the mathematical meaning of symbols, in both the pre- and post-interviews. However, he said there were not any hindrances by using pictorials, language, and realistic representations. Table 4.14 summarizes how Nick discussed the hindrances of using each representation for student learning.

Table 4.14

Nick – Hindrances of Using Representations From Interview Questions 5-9

Nick / Hindrance	Manipulatives		Pictorial		Symbolic		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Management issue	✓									
Time issue for curriculum	✓									
Excessive learning time		✓								
May lack understanding					✓	✓				
No hindrance			✓	✓			✓	✓	✓	✓

Nick's examples of using representations in his classrooms. Nick gave specific examples of how he represented mathematical ideas in the five different modes from the interview questions about each representation. For manipulatives, in the both pre- and post-interviews, he said he uses geo-boards to help students understand similar and congruent geometric shapes. He also said he uses algebra tiles to talk about multiplying binomials. Furthermore, he taught volume or surface area by using common household items, such as coffee cans. He explained that he uses physical objects that can be found in students' real lives. Also, his example of using a realistic representation was connected with using a pictorial representation. The following excerpt shows Nick's descriptions concerning examples about translating between pictorial and realistic representations in his pre-interview:

> *What I did use is an architect's drawing of a [land] lot, which had everything in the bearings. So it brought in angles and it had the fact that you needed to subtract out the home and the driveway…So I think what it did is it inspired students to learn more math to make sense out of the drawing and help them understand it…*(Nick's pre-interview).

Nick mentioned graphic calculators for a pictorial representation. He said that it was very useful for understanding graphs of functions in his pre-interview. In his post-interview, he said that Venn-diagrams are a key instructional strategy to talk about comparing and contrasting mathematical concepts (Nick's post-interview). Nick said that symbols are necessary to learn mathematics, so students should understand what the symbols mean. He talked about using translations among manipulatives, realistic, and pictorial in order to help students understand the symbols in his classes, in the both the pre- and post-interviews. The following excerpt illustrates Nick's description regarding using manipulative, pictorial or realistic representations in order to teach mathematical symbols in his pre-interview.

> *If they don't understand the symbol, I try to give them either a real-life example or a graph or a chart or a manipulative* (Nick's pre-interview).

The following excerpt shows Nick's descriptions about the relationship between symbols and language, and his example of using pictorial representations in order to teach mathematical symbols in his post-interview.

> *If students don't understand a linear inequality, you could graph that inequality with a line graph or shading. With two variables, graphing a line…So I would use a different representation to help them understand* (Nick's post-interview).

Nick thought of symbols as a language. He emphasized that language helps because they represent mathematical written words. He said that students should understand them. Thus, he tried to use another representation if students didn't understand them.

How translations among multiple representations help or hinder students' learning. Nick elaborated that using multiple representations around the same concept helps students fully understand mathematics. However, he thought that students might not need to see all different representations if students understand one method completely in both the pre- and post-interviews. As a result, he said that it was important for students to understand a method they could use efficiently around the same concepts. That is, he said that using multiple representations around some concept could be a hindrance if teachers have to spend a lot of time with multiple representations of something that most of them already understand. Therefore, he said, "If students don't understand one method, then teachers could move to another one" (Nick's post-interview). However, he said, "We talk about all those things (representations) and they have them make a decision" (Nick's post-interview). Table 4.15 summarizes how Nick discussed translations among multiple representations for student learning.

Table 4.15

Nick – Translations Among Multiple Representations

Nick	Help	Hindrance	Translation example	
	Deeper understanding	Time issue for curriculum	Within representations	Between representations
Pre	✓	✓		
Post	✓	✓		

In sum, Nick said that students could learn mathematics efficiently when they fully understand one method that would work best for them in a situation. Thus, he described his negative belief about using multiple representations to teach the same concept. However, he said that he considered the translation within language or realistic representations in his beliefs about teaching. He also elaborated on the translations between manipulatives and visual representations, and among symbolic, pictorial, and language representations. He also gave examples that he used translations among manipulatives, pictorial, symbolic, and realistic representations to improve students' mathematical understanding. The following section will describe the changes in his beliefs.

Nick's belief changes

Nick showed his different beliefs about using manipulatives, symbolic, and language representations between his pre- and post-interviews. For manipulatives, Nick said that manipulatives were helpful for students to enhance students' understanding, in both the pre- and post-interviews. However, he emphasized manipulatives as pictorial representations in his post-interview, while he described them as concrete examples, especially for lower-ability students in his pre-interview. He also elaborated that class size and class management could cause hindrances in using manipulatives in his pre-interview, but he said that it slowed the learning pace down for high-achieving students who already know about abstract mathematical concepts in his post-interview. I summarized Nick's belief changes about using manipulatives in Figure 4.13.

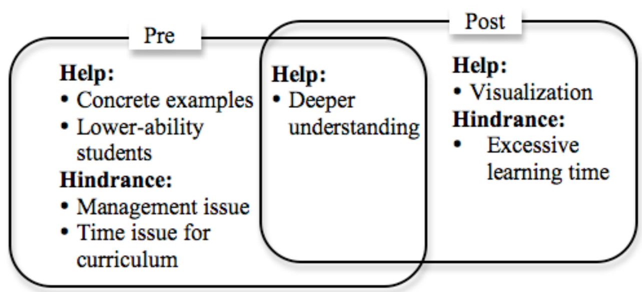


Figure 4.13. Nick's belief changes about Manipulatives.

For symbols, Nick argued that these were necessary to learn mathematics, so he believed that students should know them in his pre-interview. However, he said that symbols were a means to communicate with people, so students should understand them to learn mathematics in his post-interview. For language, he said that language was needed for students to succeed in the workplace for the future, so they should be able to communicate with mathematics in his pre-interview. In his post-interview, he elaborated that using language presentations gives students motivation, and improves their understanding in his post-interview. I summarized Nick's belief changes about using symbolic and language representations in Figure 4.14.

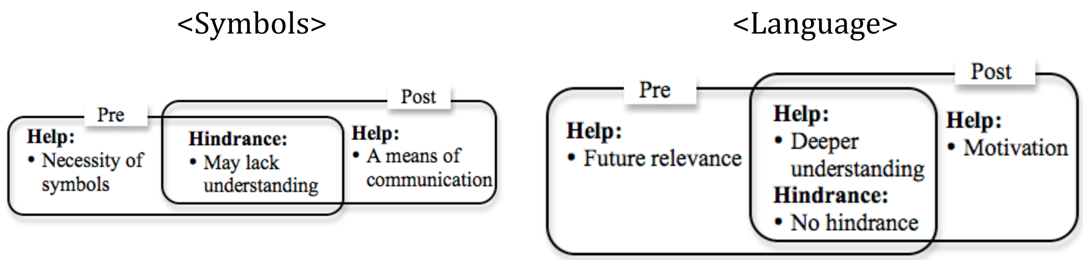


Figure 4.14. Nick's belief changes about symbols and language.

In sum, Nick' beliefs about using manipulative, symbolic, and language representations had changed. He believed that manipulatives could be used as a means of visualization for all students in his post-interview, rather than being used only for lower-achieving students to help them understand mathematical concepts. Nick said that mathematical symbols were a means of communication, rather than something needed to learn mathematics in his post-interview. He emphasized that realistic representations gave students motivation to learn mathematics in his post-interview, while he said that they were important for students' future careers in his pre-interview. The following section will describe how Nick used multiple representations in his classes.

Nick's practices

Observation 1. The first observation (October, 2011) was algebra classes for fractions. In the beginning of Day 1, he solved some problems from the previous day's homework. The problem included a real-world word problem and other problems to find the Greatest Common Factors. To solve the word problem, he drew a picture to describe the question, and used factor trees to find all prime factors of composite numbers. For the lesson of the day, Nick solved problems about simplifying fractions. Nick had students solve the problems on the board and explain how they solved them. He also solved a realistic word problem with symbolic and language representations.

Day 2 involved the Least Common Multiple (LCM). He gave the definition of LCM in written and spoken language at the beginning of the lesson. He solved problems, such as finding the LCM between 12 and 28, with the students using symbolic and language representations. Students solved problems on the board and explained them. On Day 3, he solved problems that involved pictures, symbols, and real-world situations in a review of the quiz from the previous day. In the lesson of the day, Nick explained several methods for comparing fractions and mixed numbers using LCM, and fraction circles as pictures. He solved the problems such as comparing $2\frac{7}{9}$, $2\frac{3}{15}$, and $\frac{11}{3}$. At the end of class, they solved a real-world question for comparing mixed numbers.

Using the Lesh Translation Model in the first observation. On Day 1, Nick used factor trees to find the Greatest Common Factors (GCF), and some real-world problems with pictures (e.g., finding the greatest length of a fence between two playgrounds) in checking homework. In the lesson of the day, he defined the meaning of the simplest form of a fraction, and the meaning of equivalent fractions in language representations. He gave two fractions, whether or not they were equivalent, and he also gave a fraction to find the equivalent fractions. Nick gave time to solve the problems in order to find the simplest forms of fractions and equivalent fractions, and he asked students for the answers they solved. Then he explained dividing a common factor or multiplying a number, both the numerator and denominator, to solve the problems. He also emphasized simplifying the variable expression using prime factorization. All problems were represented by symbolic representations. At the end of the class, he gave a real-world problem about the St. Wenceslas crown in the Czech Republic to find the simplest form of the fraction, $\frac{\text{emeralds}}{\text{total jewels}}$, and his students solved the problems. That is, he mostly used translation within and between symbolic and language representations. In addition, he used some realistic representations in the homework and the classroom work, and he also used some pictorial representations, such as factor trees, or pictures with little translation within them in the homework.

On Day 2, he defined the multiple of a number, a common multiple, and the least common multiple (LCM) in language representations. He gave various problems that were represented by symbolic representations to find the LCM of numbers and monomials by listing multiples and by using common factors. That is, he used translating within and between symbolic and language representations. On Day 3, he reviewed the quiz from the previous day, and the questions included pictorial, symbolic, language, and realistic representations. On the lesson of the day, by comparing fractions and mixed numbers, he shortly explained comparing two fractions, $\frac{3}{8}$ and $\frac{5}{8}$, using the picture of a fraction circle. He gave various problems to compare fractions or mixed numbers, and he made common denominators to solve the problems. Therefore, he focused on how to find a common denominator. At the end of the class, he used a real-world problem to compare two orangutans' heights, $3\frac{3}{4}$ feet and $3\frac{2}{5}$ feet. That is, he mostly used translating between symbolic and language representations, and little translating within language representations. In addition, he also used few translations within and between pictorial and realistic representations. The diagrams in Figure 4.15 show the use of mathematical representations in the first classroom observation.

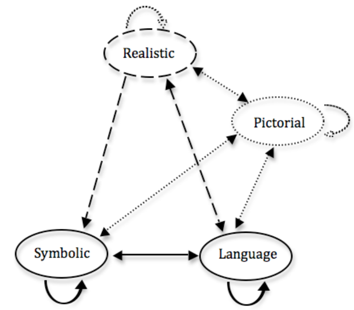
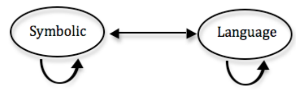
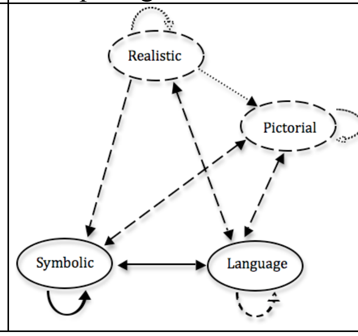
Day 1: Simplifying Fractions	Day 2: Least Common Multiple	Day 3: Comparing fractions
		
Note. Type of line: Strongly used in teaching Weakly used in teaching Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class		

Figure 4.15. The first classroom observation in Nick's algebra classes for three days.

Observation 2. The second observation (February, 2012) was about similarity and dilations in geometry. Nick checked the last homework questions. He had already written what (x, y) was reflected over the x-axis, the y-axis, 90 degrees clockwise or y=x on the board, and he explained them. For the day of the problems, he discussed 8% of 15 cents for delivery tips from a Sunday newspaper carton. In the warm-up questions, he solved questions about translating a given three points with the given conditions, such as across the x-axis. Using PowerPoint, Nick showed definitions of similar polygons, dilation, and distortion using language. He talked about wearing sunglasses to explain dilation. He showed an octagonal tin candy box he had brought, and he mentioned that the shape of the box was similar to the shape of stop signs. Nick explained a task to distort and enlarge a face that students had created. The students worked on the task for the rest of the class.

On Day 2, Nick talked about 3 by 5-inch notes for the next test, and showed how a student made a non-rectangle note. He cut three triangles on one side, and he moved them to the other side. He talked about the areas and the shape with the students. To teach similarity and dilation, he used the octagon tin candy box to explain the similarity with stop signs. He also showed a picture of two similar TV screens. Most questions students solved in class were used coordinate planes, ordered pairs, and geometric shapes. Some questions were related to real-world situations. For dilation, he drew a quadrilateral on graph paper, and he dilated the quadrilateral with a scale factor of 3. He also solved other problems about dilation. He used most questions from the textbook. He also used a ratio to find a missing value from two similar polygons. Day 3 was used to review Chapter 8 about polygons and transformation. The students solved mini-quizzes, and then they wrote their solutions on the board to explain in the whole class.

Using the Lesh translation model in the second observation. On Day 1, Nick checked homework about transformations such as translation, reflection, rotation, and dilation, which involved pictorial, symbolic, and realistic representations. For the lesson of Day 1, Nick explained the definitions of similar polygons, dilation, and distortion in written and spoken language representations. He showed an octagonal tin candy box to discuss its similarity with stop signs on the road. He also talked about dilations and contractions of pupils of the eyes, depending on light in order to explain dilations in mathematics. Then Nick explained hand-on tasks that would copy and trace a face using a handout involving a head, eyes, noses, ears and so on, and then enlarge and distort the original face. In the rest of the class, the students worked on the task using a grid sheet. That is, his lesson was mostly used translating within and between pictorial and language representations, and translating between symbolic and language representations. In addition, he used few manipulative and realistic representations.

On Day 2, Nick cut one side of a rectangle into two triangular pieces, and moved the pieces to another side in order to discuss the same area of two figures, a rectangle and a geometric shape that was not rectangular during the warm-up. He had students use polygons (i.e., geometric shapes) or real-world pictures (e.g., two TV screens that had different sizes or trees) to solve problems about similar polygons. He also used proportions (ratios) to find a value from two similar polygons. Furthermore, he dilated polygons with different scale factors on graph paper. Nick gave chances for students to solve problems that included pictorial, symbolic, and realistic representations, and to explain their solutions in language representations during class. That is, he translated within and between pictorial (e.g., two similar geometric shapes or real pictures), symbolic (e.g., a ratio between the similar figures), and language representations, and translated from realistic representations to pictorial ones. He also used few physical manipulatives to find a figure that had the same area as a rectangle during the warm-up.

On Day 3, the students solved mini-quizzes about the polygons and transformation, and the questions of the mini-quizzes involved pictorial, language, and symbolic representations. Nick had the students explain how they solved them, and he helped the students correctly explain them using appropriate mathematical terms (e.g., congruent, ratios, corresponding angles, similar, or symmetric). In addition, he also used pictures that the students drew to explain some problems. That is, he used translating within and between pictorial, symbolic, and language representations. The diagrams in

Figure 4.16 show the use of mathematical representations in the second classroom observation.

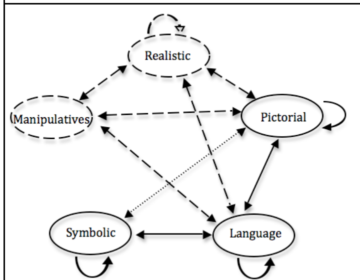
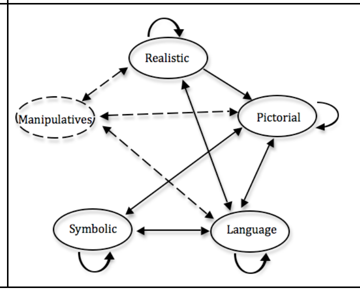
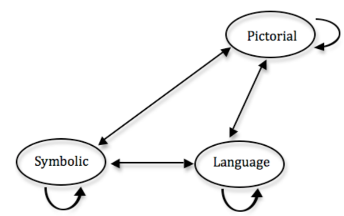
<table>
<tr><td>Day 1:
Similarity and dilations
Use the same PP as Holly's and Pam's</td><td>Day 2:
Similar polygons
Use the same PP as Pam's</td><td>Day 3:
Review</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td colspan="3">Note.
1. Manipulatives=physical manipulatives
2. Type of line:
Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class</td></tr>
</table>

Figure 4.16. The second classroom observation in Nick's geometry classes for three days.

Observation 3. The third class (May, 2012) was probability. On Day 1, Nick solved problems about combinations and permutations from the previous homework, which students asked him to solve. Most questions involved symbols and word problems, and some of them were realistic problems. The topic of the day was probability and odds, and Nick defined odds and probability (e.g., $\frac{\text{Winners}}{\text{Total}}$, $\frac{\text{Winners}}{\text{Losers}}$), and gave examples of them, such as using a spinner, letters (e.g., Mississippi), gambling, or the lottery. The students also individually solved some problems about the odds that were represented by symbolic, language, and realistic representations. Nick also brought a letter that his wife received. That was related to probability and odds that showed the chances of being a winner of prizes such as a TV or airplane tickets. As a result, he talked about the probability and odds in the letters with them.

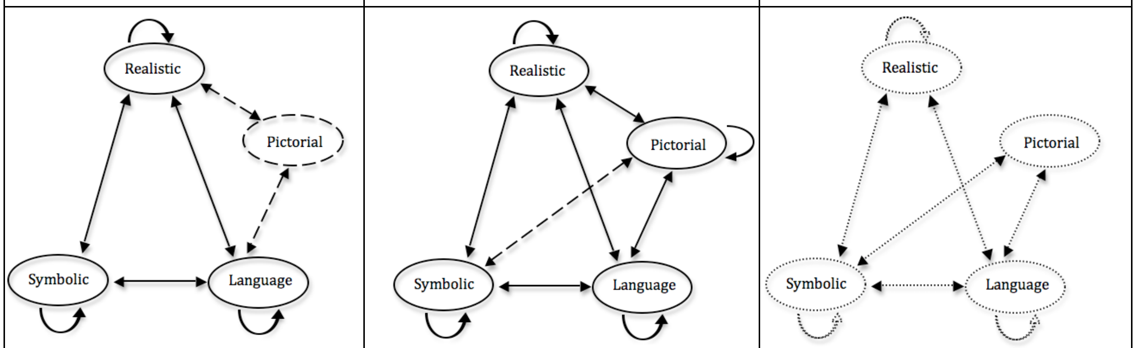
Day 2 covered independent and dependent events in probability. Nick started solving problems about probability and odds from the previous homework, when students asked questions about the processes of problem solving. The problems were presented by realistic, symbolic and language representations. On the day of the class, he first provided some questions to distinguish between independent and dependent events in written words, and he explained them. He showed the definitions of them from the textbook. He also gave examples that were presented by realistic (e.g., gumball), pictorial, symbolic, and language representations. The students individually solved the problems in realistic, symbolic, and language representations, and they discussed them in the whole class. Nick asked for the answers to problems that the students solved, and he also asked about mathematical reasoning within problem-solving processes. The PowerPoint slides were the same as Holly (a case in this study)'s and Pam (a case in this study)'s. On Day 3, he solved problems from the homework about combinations and permutations, and he reviewed Chapter 12 about data analysis and probability. For the review, he used slides that explained each of the topics such as plots, tree diagrams, permutations, combinations, probability and odds, and independent and dependent events. The slides showed realistic examples, pictures, definitions in words, and symbols, and they were same as Holly's and Pam's.

Using the Lesh translation model in the third observation. On Day 1, Nick checked homework combinations and permutations, and he used symbolic, language and realistic representations (e.g., flipping a coin, choosing good books). Nick defined the odds and probability in written and spoken language using terms of winners and losers (e.g., $\frac{\text{Winners}}{\text{Total}}$, $\frac{\text{Winners}}{\text{Losers}}$), and gave examples of them using pictorial, symbolic, language, and realistic representations (e.g., a spinner, gambling, or lottery, Prob (Heads) = $\frac{1}{2}$, and odds (heads) = 1:1). He talked about the Minnesota lottery to explain the odds in winning (1:4). He also brought a letter from home, which was related to probability and odds (e.g., chances to be a winner of prizes). During the class, he gave time for students to solve the problems about odds and probability. That is, he mostly used translating within and between symbolic, language, and realistic representations, with few pictorial ones.

On Day 2, Nick defined independent and dependent events using the terms of replacement or no replacement with pictures (e.g., different colored balls in bags), and he gave examples of each event (e.g., choosing a red ball in a bag with replacement or without replace involving pictures). He gave word problems with some real-world contexts (e.g., a bingo game and a coin flip) to find whether or not events were dependent or independent, and to solve the probabilities of dependent and independent events in symbolic and language representations (e.g., P(A and B), probability greater than one when rolling a die). That is, he used translating within and between symbolic, language, and realistic representations, with few pictorial representations. On Day 3, he checked problems from the homework, and he emphasized mathematical language (random, probability, or odds), and symbols to solve the problems. To review Chapter 12, he defined key terms or words in language representations, and showed a simple example through PowerPoint slides. All examples involved pictorial or symbolic representations.

The diagrams in Figure 4.17 show the use of mathematical representations in the third classroom observation.

Day 1: Probability and odds Use same PP with Pam's	Day 2: Independent and dependent Use same PP with Holly's and Pam's	Day 3: Review Use same PP with Holly's and Pam's



Note. Type of line:
Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

Figure 4.17. The third classroom observation in Nick's probability classes for three days.

In sum, Nick mostly used symbolic and pictorial representations in the first observation, and he talked more about realistic examples in the second and third observations. He also brought real-world examples from home, such as newspapers or letters to share with students in each class through the three-period observations.

Holly' Case

Holly had 12 years of teaching experience in middle schools, and she was teaching high-achieving seventh-grade students using McDougal Littell in School A. Her class size was about 37 students, and the students sat in a row. She encouraged her students to solve problem with worksheets.

Holly's beliefs about teaching

In the following sections, I explore Holly's general beliefs about teaching mathematics, such as (1) what she believes the role of the mathematics teacher; (2) how she decides what or what not to teach; and (3) how she maximizes student learning in the classroom. These beliefs are related to her beliefs about using multiple representations. Thus, this analysis helps investigate research questions about teachers' beliefs in using representations and their practice in this study.

The role of the mathematics teacher. Holly said, "The role of the mathematics teacher is to first assess where the students are" in her pre-interview. She also said that these assessments could be formal or a verbal discussion with students. It was to address "individual students' needs." In the post-interview, she said, "The math teacher is there to first give information to students and then to be a guide as they are practicing those new concepts to help them with their questions and their misunderstanding." She also elaborated that the mathematics teacher needs to understand how students read a problem, and how they understand what's important in that problem. She emphasized mathematics vocabulary, formulas and other things in her post-interview.

What or what not to teach. Holly said that she decided what or what not to teach based on state standards, the district's curriculum, and the textbook, and students' prior knowledge, in both the pre- and post-interviews. In the pre-interview, she said that she worked as a team teacher so that they made the decisions about their teaching topics and materials. She also said that she thought about what kinds of materials or activities she could use on a particular topic by herself. In the post-interview, she said that she considered activities from the textbook or on the Internet that were related to the concept in the standards and what other teachers have posted on websites in her pre-interview. She also explained that the activities could be games, using the graphing calculator, and team-building activities.

How to maximize student learning. Holly said, “Pretests make a difference” so that she obtained students’ individual ideas that came from their prior knowledge, in the both the pre- and post-interviews. She added that those tests she used usually came from the textbook. In the pre-interview, Holly said that she considered challenging assignments, and she gave opportunities to practice something new. She said that she also allowed “multiple strategies for a certain problem.” In the post-interview, Holly elaborated that she allowed students to differentiate practices. She used problems based on students’ levels. I summarized Holly’s beliefs about teaching mathematics in Table 4. 16.

Table 4.16

Holly – Summary of Her Beliefs About Teaching Mathematics

Holly	**1. The role of the mathematics teacher**
	• Assess students' prior knowledge (pre) • Address students' needs in class (pre) • Provide information and guide students' learning (post) • Understand students' problem-solving abilities (post)
	2. What to teach
	• Consider the curriculum and the standards (Pre & post) • Consider students' prior knowledge (pre & post) • Consider materials and activities (Pre & post) • Work with colleagues (pre)
	3. How to maximize
	• Differentiate lessons based on students' prior knowledge (pre & post) • Use challenging assignments (pre) • Practice new problems (pre) • Use multiple strategies to solve problems (pre) • Differentiate problems based on students' levels (post)

Holly's beliefs about the Lesh translation model in the first three questions about teaching. The first three interview questions were about (1) what the role of the mathematics teacher is; (2) what or what not to teach; and (3) how to maximize student learning in the classroom. In this section, I described how Holly talked about the Lesh translation model in the three questions about teaching. Holly said that she used translations within or between mathematical representations in her classes when she talked about the role of the mathematics teacher, what to teach, and how to maximize students' learning. Holly explained that students should be able to connect a language representation to symbols in a problem-solving process (Holly's post-interview). When she thought of lesson plans, she also considered various virtual manipulatives such as Internet activities, and the connection between virtual manipulatives and pictorial representations. She also said that visual representations such as graphs or charts are very helpful to talk about equations or data that they collected and organized. She showed her positive beliefs about translating between multiple representations. Table 4.17 summarizes how she discussed translations between or with representation.

Table 4.17

Holly's Case From Interview Questions 1-3

Holly	Instances of representational fluency	Pre	Post
The role of the mathematics teacher	Representation	L	L, S
	Translation within representations	L	L, S
	Translation between representations		L&S
What to teach	Representation	M	*M, P
	Translation within representations	M	*M, P
	Translation between representations		*M&P
How to maximize	Representation		P, S
	Translation within representations		P, S
	Translation between representations	T	T

Note. M: Physical manipulatives, *M: Virtual manipulatives, P: Pictorials, S: Symbols, L: Language, R: Realistic representations, and T: Translation among multiple representations (representations not identified).

Holly's beliefs about representations

The following sections describe Holly's beliefs about how each representation helped and hindered her teaching of mathematics. The examples of using each representation are described. I also elaborated on her beliefs about the translation between and within multiple representations.

How representations help students' learning. Holly said that manipulatives could be a visual representation in her pre-interview, to make students do something and get involved in learning, in both the pre- and post-interviews. She said in the post-interview that manipulatives also gave students motivation so that they would keep paying attention in the classroom. The following excerpt is Holly's description of how manipulatives help students' learning in Geometry class in her pre-interview. Holly illustrated how manipulatives help students' learning in her post-interview, such as the following excerpt:

> *It keeps their interest when they have something other than just paper in front of them. They pay better attention. It gets students with a different learning style, a kinesthetic learning style…*(Holly's post-interview).

For pictorials, Holly explained that it helps students see the relationships or patterns from the data. She also said that it could help students understand what problems were asking them through drawing pictures or making graphs, in both the pre- and post-interviews. She elaborated that symbols were critical to understanding mathematics, in both the pre- and post-interviews, and that this representation was a written language representation in her pre-interview. She also said that students could accelerate written explanations of mathematics if they used symbols, in her post-interview. For language representations, she said that they help students explain their mathematical thinking in her pre-interview, and connect something with mathematics, in both the pre- and post-interviews. She also elaborated that using language promotes student's deeper understanding, and that it also helps teachers understand students' misunderstanding in mathematical concepts from her post-interview. The following excerpt illustrates Holly's description involving how pictorials help students' learning in Geometry class during her pre-interview.

> *It is important for them to be able to explain what they were thinking so that*

…they can use what they were thinking in a previous problem to help solve the new problems…(Holly's pre-interview).

Holly said that realistic examples and problems gave students motivation to learn mathematics, and that those help students understand how mathematics is useful in their lives, in her both pre- and post-interviews. Thus, she said in her post-interview that it could be useful toward students' careers in the workplace. Table 4.18 summarizes how Holly discussed the helpfulness of using each representation for students' learning.

Table 4.18

Holly – Helpfulness of Using Representations From Interview Questions 5-9

Help \ Holly	Manipulatives		Pictorial		Symbolic		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Deeper understanding					✓	✓		✓		
Visualization	✓									
Active learning	✓	✓								
Motivation		✓							✓	✓
Maintaining attention		✓								
Tools to organize problems			✓	✓						
See patterns b/w data			✓	✓						
Alternative representation					✓					
Acceleration of mathematical processes						✓				
Connection with mathematics							✓	✓	✓	
Explanation of students' thinking							✓			
Knowing students' misunderstanding								✓		
Mathematical utility									✓	✓
Future relevance										✓

How representations hinder students' learning. Holly said that the materials in hands-on activities sometimes distracted students' learning, and that there was the time issue to cover all contents she had to teach. She also elaborated that students could lose some learning time that could have been used for practice with various methods if they spent time with manipulatives, in both the pre- and post-interviews. The following excerpt shows Holly's description regarding how manipulatives hindered students' learning in Geometry class during her pre-interview.

> *It sometimes hinders when the objects are a distraction. In some situations, they're playing with the object and not focusing on the mathematics of that…*(Holly's pre-interview).

Holly also illustrated her descriptions about how manipulatives hindered students' learning in her post-interview, such as the following excerpt:

> *When we're dealing with a short 50-minute period, we may be losing some learning time…*(Holly's post-interview).

For pictorials, she said that pictorial representations could make slow learning progress a hindrance if some students had difficulty with special skills, such as drawing pictures in her post-interview. For symbols, she elaborated in her post-interview that it could be a hindrance if students did not understand the meaning of mathematical symbols, and if they were spending much time on deciphering the symbols in her pre-interview. For language, she said that it could be a hindrance if there were language issues for some students, and if students were either not familiar with or were just slow at explanations or writing about it in her post-interview. For realistic representations, she said in her post-interview that they were sometimes difficult to use in class. The following excerpt shows Holly's description regarding how realistic representations hindered students' learning in Geometry class in her pre-interview:

> *But in the real world, using some of those numbers that you're going to deal with are going to make it more difficult than you'd want to start from…*(Holly's post-interview).

She also said that if teachers chose some parts of a whole real-world situation to make it simple, it could be an issue to give correct information, in both the pre- and post-interviews. Table 4.19 summarizes how Holly discussed the hindrances of using each representation for student learning.

Table 4.19

Holly – Hindrances of Using Representations From Interview Questions 5-9

Holly / Hindrance	Manipulatives		Pictorial		Symbolic		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Excessive learning time	✓	✓			✓					
Distracted by the material	✓	✓								
Slow progress for some				✓						
May lack understanding						✓				
Language issue								✓		
Slow explanations inefficient								✓		
Sampling issue									✓	✓
Messy data/ numbers										✓
Calculation										✓
No hindrance			✓				✓			

Holly's examples of using representations in her classrooms. Holly gave specific examples how she represented mathematical ideas in the five different modes from the interview questions about each representation. She said that she did not use manipulatives much in her classroom, but she would imagine that it might help "students who need to visualize the geometric shape or example" (Holly's pre-interview). In her post-interview, she said that manipulatives introduced students to a different learning style such as "a kinesthetic learning style." She also said that manipulatives were physical objects that were used as teaching tools to engage students in learning mathematics. The following excerpt illustrates Holly's descriptions involving how she used manipulative representations in the classroom with her students:

> *I had cut out trapezoids that were flat and I animated them. And then I passed out rulers as well as so that they could measure their own dimensions. So instead of just having the picture in the book…needing to collect some data from actual objects. It's just different ways to engage them…*(Holly's post-interview).

For pictorial representations, Holly said that tables and linear graphs were helpful to understand the patterns of data in her post-interview. She also mentioned geometry diagrams "to see the relationship of where that data belonged in the problem and how it related to the other numbers" (Holly's pre-interview). In the symbols, she emphasized the importance of understanding the meaning of x in a linear equation. Holly said that students needed language representations to connect one problem to other problems.

How translations among multiple representations help or hinder students' learning. Holly said that using multiple representations across the same concept help students understand the concept deeply, and improve problem-solving skills in both the pre- and post-interviews. She said that it was important to have multiple ways to solve problems, and to see how different representations were showing the same thing. However, she said in her pre-interview that the representations might be a hindrance if students tried to use all of the representations step-by-step any time, even though they did not need to do so. The following excerpt illustrates Holly's descriptions concerning examples about how she used language representations in the classroom with her students:

> *You're expecting them to work through the equation and then make the graph from the table. That's going to take a longer period of time. I think it would probably be more beneficial to do a few problems from multiple representations than to do depend on only one way...* (Holly's post-interview).

Table 4.20 summarizes how Holly discussed translations among multiple representations for student learning:

Table 4.20

Holly – Translations Among Multiple Representations

Holly	Help		Hindrance		Translation example	
	Deeper understanding	Problem-solving skills	May use inefficient	No hindrance	Within representations	Between representations
Pre	✓	✓	✓			✓
Post	✓	✓		✓		✓

In sum, Holly believed that using translations within language or symbols helps deepen students' understanding. She also elaborated that using multiple representations across the same concept helps students' understanding. She emphasized the translation between manipulative and pictorial representations. I will describe the changes of her beliefs in the following section.

Holly's belief changes

Holly elaborated that it helps students to visualize some mathematical concepts in her pre-interview. However, she stated that manipulatives help students pay attention in class when they used them, and that using manipulatives increases students' motivation in learning mathematics from her post-interview. For pictorials, she said that there were not any hindrances in her pre-interview. However, she said in her post-interview that some students had difficulty drawing pictures and making graphs, and those students could experience slow progress while using some pictorial representations. I summarized Holly's belief changes about using manipulatives and pictorial representations in Figure 4.18.

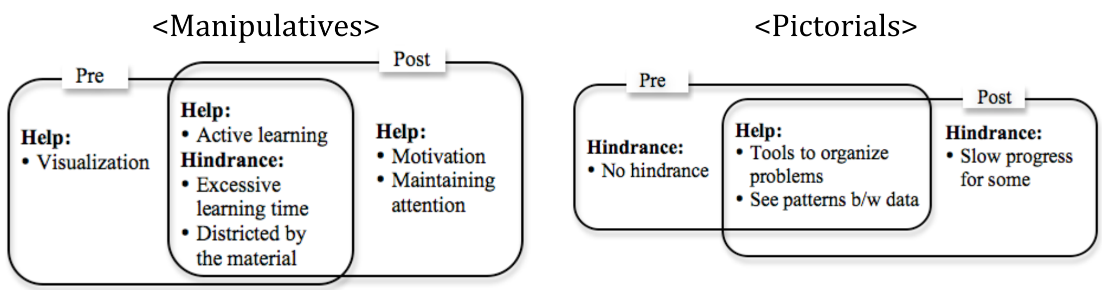


Figure 4.18. Holly's belief changes about manipulatives and pictorials.

For symbols, Holly elaborated that symbols were a way to explain mathematics, so students sometimes needed more time to understand them than she supposed in her pre-interview. However, in her post-interview, she said that if students understood the meaning of mathematical symbols, they could learn and explain mathematics efficiently. Holly said that language helps students solve problems through explaining their own thinking about mathematics in her pre-interview. In her post-interview, she explained that using language helps students develop their mathematical understanding, and that teachers could figure out students' misunderstanding while students explained how they solved problems. Therefore, she said that language helps teachers create effective lessons for students' conceptual understanding. However, she also talked about the hindrances of using language for some students, such as students' language issues and students' difficulties in explaining their understanding in the post-interview. I summarized Holly's belief changes about using symbolic and language representations in Figure 4.19.

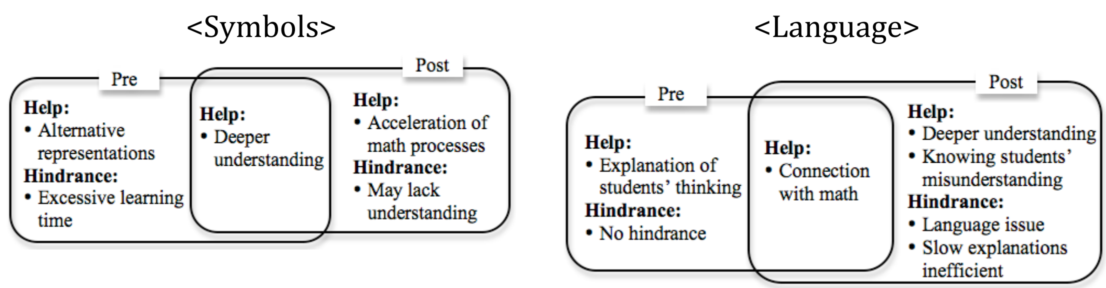


Figure 4.19. Holly's belief changes about symbols and language.

Holly said that students put more effort into learning mathematics when they see that there is a connection to the real world in her pre-interview. In her post-interview, she emphasized the usefulness toward students' careers about using realistic representations. In her post-interview, she also said that there was the hindrance of using realistic representations when students work with messy real-world data or numbers to solve problems. She said that using real numbers makes it difficult for students to calculate them. I summarized Holly's belief changes about using realistic representations in Figure 4.20.

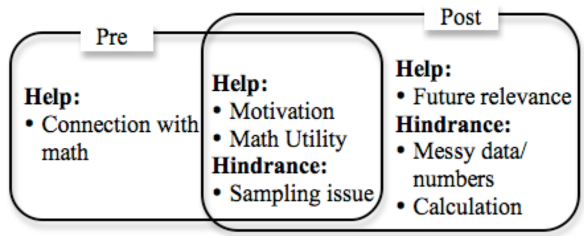


Figure 4.20. Holly's belief changes about realistic representations.

In sum, Holly showed her belief changes in all of the representations. In particular, she showed more positive beliefs about using manipulatives and language in her post-interview, For examples, she said that manipulatives could help students pay attention and give them motivation rather than be a means of visualization. For language, she said that using language helps teachers better understand students' misunderstanding, so they can help the students deepen their understanding in mathematics.

Holly's practices

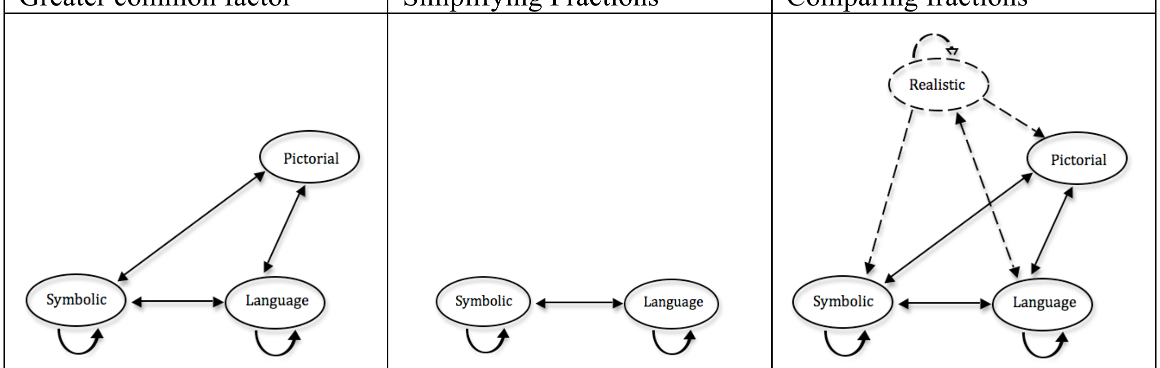
Observation 1. The first observation was algebra classes about fractions. On Day 1, Holly taught greater common factors (GCF). She gave the definition of GCF in verbal language. While solving problems about GCF between two numbers, she used writing all lists of factors and factor trees. She also defined the relatively prime numbers, and she asked students, "What do they look like?" She also gave examples of the relatively prime numbers, and compared them with composite numbers. Finally, she solved problems to find the GCF between two monomials, such as 6x and 8x. Thus, she used all problems that were presented by symbols, and she used pictorial representations, such as factor trees, and language representations to define GCF, the relatively prime, and composite numbers, and she encouraged students solve the problems. In addition, she also used some real-world problems to compare fractions or mixed numbers.

On Day 2, Holly taught simplifying fractions using GCF. At the beginning of the class, she solved the problems from homework, such as finding GCF among $12x^2y^2$, $42x^3y^2$, and $36x^2y^3$. For the topic of the day, first, she defined the simplest forms of fractions in language. She gave students problems to simplify fractions (e.g., $\frac{22}{116}$) and a variable expression (e.g., $\frac{14x}{7xy}$). When she showed the problems on PowerPoint, she first asked what the common factors were. All questions that were represented by symbols came from the textbook. She allowed students to use a calculator to practice the problems. On Day 3, Holly used real-world word problems for comparing fractions. She also used factor trees to find GCF and LCM. Holly told them that there were many methods to compare fractions, but they would use common denominators following the textbook's suggestions during the class day.

Using the Lesh translation model in the first observation. On Day 1, Holly used both written and verbal language using PowerPoint in order to explain mathematical concepts and to solve problems. To teach GCF, she first listed all factors and then made factor trees (prime factors). She explained how to find the GCF of numbers (e.g., 4 and 9) and monomials (e.g., 6x and 8x) using lists of all factors or factor trees. For the rest of the time, she gave students a worksheet to solve the similar types of problems she had explained. That is, she used translating among pictorial, symbolic, and language representations, and translating within symbolic and language representations.

On Day 2, Holly defined the simplest forms of fractions in language. She found equivalent fractions using the simplest forms. She also practiced simplifying variable expresses (e.g., $\frac{14x}{7xy}$). That is, she used translating within and between symbolic and language representations. On Day 3, she used word problems with real-world contexts to compare two fractions using a common denominator. She explained how to compare fractions and mixed numbers using common denominators. She also used some problems to find numbers that satisfied some conditions (e.g., the GCF of two numbers whose sum is 120 and whose difference is 4). That is, she used translating within and between symbolic and language representations with some realistic representations. For the rest of the time on both Day 2 and Day 3, she gave students a worksheet to solve similar types of problems she had explained. The diagrams in Figure 4.21 show the use of mathematical representations in the first classroom observation.

Day 1: Greater common factor	Day 2: Simplifying Fractions	Day 3: Comparing fractions



Note. Type of line:
Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

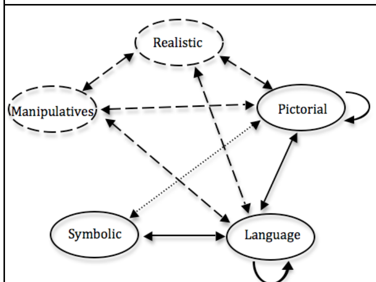
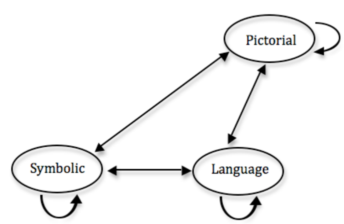
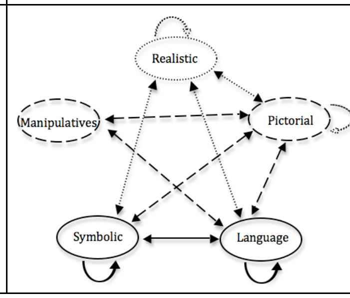
Figure 4.21. The first classroom observation in Holly's algebra class in three days.

Observation 2. The second observation was geometry classes. On Day 1, Holly taught similarity and dilations. In the warm-up, she gave questions to find the missing sides from two equal quadrilaterals in pictures. Then she solved questions from the homework, which were about translations in given directions. For the topic of the day, she defined similar polygons, dilation, and distortion in language. Then she explained a task students would do for the rest of class by themselves. The task was to trace a face in the handout, and to enlarge and distort the face. At first she demonstrated how to draw on a grid sheet, and she gave all directions about how to use a coordinate plane in order to double the area of the original face in language. She also showed a sample from the previous year.

Day 2 was about similar polygons. For the warm-up, Holly used two similar polygons to find missing angles, and she talked about the sum of interior angles in a polygon using $(n-2)\bullet 180°$. For the topic of the day, she first defined the similar polygon the same as the ones on the Day 1 lesson, such as "the same shape, but not always the same size," and then she gave them another way to define similar polygons using alternative interpretations, such as "corresponding angles are congruent, and corresponding side lengths are in proportion." She provided students with questions as to whether or not two triangles were similar. Then she asked the students to find missing values if the triangles were similar. Holly also gave an alternative definition of dilation using $(x, y) \rightarrow (kx, ky)$. She gave an example with k=2, and drew the polygons. On Day 3, she reviewed the transformations such as translation, reflection, rotation, and dilation. She also used some real-world problems from the textbook, which were about the height of a flagpole or shadow. She mainly discussed transformation of triangles using given directions, such as moving right 3 and down 4, and rotating 90 degrees clockwise.

Using the Lesh translation model in the second observation. On Day 1, Holly asked students to find the length of the sides from congruent polygons in the warm-up. She also used different pictures for showing various transformations in the given directions for checking the homework. All problems in the homework were presented with pictures and symbols. In the lesson for the day, she defined similar polygons, dilation, and distortion in both written and spoken language. She gave directions about how to copy and trace a face using the handout involving a head, eyes, noses, ears, and so on. Then she explained how to enlarge and distort the original face. For the rest of the class, the students worked on the task using a grid sheet. That is, she used translating within and between pictorial and language representations, and translating between symbolic and language representations. In addition, she used few manipulative and realistic representations.

On Day 2, she defined similar polygons in written and spoken language and symbols. Then she solved problems involving two polygons to prove whether or not they were similar. She used side proportions (ratios) to solve the problems. She discussed dilations to explain similar polygons using pictorial, symbolic, and written and spoken language representations. She also gave time for students to solve problems about similar polygons. Holly used translations within and between pictorial, symbolic, and language representations. On Day 3, she used real-world problems (e.g., a flag pole and the shadow) about similar polygons in checking homework, and she used pictorial and symbolic representations to solve the problems. Other problems in the homework involved pictorial, symbolic, and language representations. When Holly discussed the transformations of triangles with her students, all of them used translations within and between symbolic (e.g., $(x, y) \rightarrow (kx, ky)$) and language representations. In addition, when she solved a problem rotating 90 degrees clockwise, she manipulated rotating paper drawing a triangle, and drew the transformation of the original triangle. Holly used translations within and between symbolic and language representations, and she used few manipulative representations and pictorial ones. In addition, she used realistic representations while checking homework. The diagrams in Figure 4.22 show the use of mathematical representations in the second classroom observation.

Day 1: Similarity and dilations Use the same PP with Nick's and Pam's	Day 2: Similar polygons and dilations	Day 3: Transformation
		

Note.
1. Manipulatives=physical manipulatives
2. Type of line:
   - Strongly used in teaching (solid line)
   - Weakly used in teaching (dashed line)
   - Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class (dotted line)

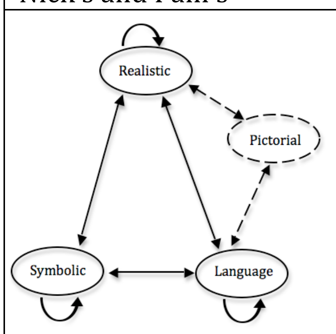
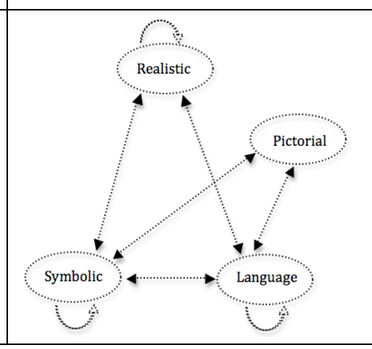
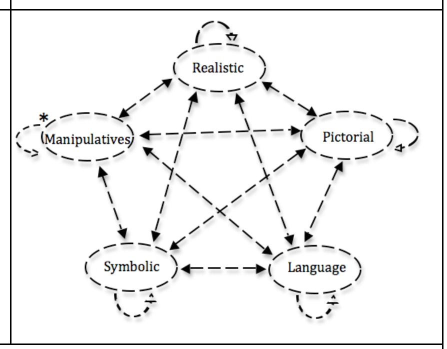
Figure 4.22. The second classroom observation in Holly's geometry class for three days.

Observation 3. The third observation was probability classes. On Day 1, Holly taught independent and dependent events, and she defined some concepts such as compound, independent, and dependent events in language, and she also gave the algebraic definitions from the textbook such as P(A)P(B)=P(A and B). She used realistic examples such as tossing a coin, a bingo game, and choosing a lunch bag. All examples were shown in PowerPoint with pictures.

Day 2 was spent on reviewing the data analysis and probability in Chapter 12. In the review, she used the slides that explained each of topics such as plots, tree diagrams, permutations, combinations, probability and odds, and independent and dependent events. The slides showed the realistic examples, pictures, definitions in words, and symbols from the textbook. On Day 3, after checking the homework, Holly and her students went to the Media Center. The students solved problems that were provided by the McDougal Littell (textbook) website. She solved problems and activities that were related to pictorial, symbol, realistic, and language representations.

Using the Lesh translation model in the third observation. On Day 1, Holly defined independent and dependent events with examples of each event (e.g., choosing gumballs in a bag with replacement or without replacement involving pictures). She also gave students word problems with some real-world contexts to find out if events were dependent or independent. Then she defined probabilities of dependent and independent events in symbolic and language representations. She also solved problems to find out the probabilities of each event with her students, which had real-world situations such as a bingo game and a coin flip. In addition, she had students practice problems solving P(A and B) in symbolic representations. That is, she used translating within and between symbolic, language, and realistic representations, with few pictorial representations.

On Day 2, she reviewed Chapter 12, which involved eight sections about data analysis and probability. For each section, she defined key terms or words in language representations, and showed a simple example through PowerPoint slides. All examples involved pictorial or symbolic representations. On Day 3, she checked homework about probabilities and odds, and she solved a real-world problem using symbolic and language representations. For the rest of the class, the students worked with virtual manipulatives about probabilities (e.g., choosing clothes). The virtual manipulatives involved pictorial, symbolic, and realistic representations. The diagrams in Figure 4.23 show the use of mathematical representations in the third classroom observation.

Day 1: Independent and dependent Use the same PP as Nick's and Pam's	Day 2: Review Use the same PP as Nick's and Pam's	Day 3: Check Homework and Go to Media Center
		

Note.
1. * means virtual manipulatives
2. Type of line:

Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

Figure 4.23. The third classroom observation in Holly's probability class for three days.

In sum, Holly used symbolic and pictorial representations for algebra classes, and she used pictorial, language, and symbolic representations for geometry and many realistic examples for the probability classes from the textbook. She also used physical manipulatives in the second observation, and virtual manipulatives in the third observations.

Mary 's Case

Mary had 21 years of teaching experience in middle schools, and she was teaching low-achieving 7th-grade students using McDougal Littell in School B. However, she mostly taught eighth grade students before this study. Her class size was about 30 students, and the students sat in a row. She used students' work to share with others.

Mary's beliefs about teaching

In the following sections, I explore Mary's general beliefs about teaching mathematics, such as (1) what she believes the role of the mathematics teacher; (2) how she decides what or what not to teach; and (3) how she maximizes student learning in the classroom. These beliefs are related to her beliefs about using multiple representations. Thus, this analysis helps investigate research questions about teachers' beliefs in using representations and their practice in this study.

The role of the mathematics teacher. Mary said that the role of the mathematics teacher is to facilitate and guide students through the state and the district standards that were set for the course or grade level, in the both the pre- and post-interviews. In the pre-interview, Mary said that she, as a mathematics teacher, encouraged students to ask questions and let them use more than one way to solve problems. She also said that she let them express how they set it up. In the post-interview, the role of the teacher is to figure out the most effective ways to get information to the students. She elaborated that she, as a mathematics teacher, guides students with problems and lets them work with a partner or within a group. Furthermore, she said that she gives students as much individual help as needed.

What or what not to teach. Mary said that every decision is made by the district standards and they use a textbook to make their own course sequencing in her pre-interview. She also talked with Pam, who had more recent teacher prep courses than she had, so that she followed many of Pam's ideas. However, she did not consider many activities because of class time issues or class size. In her post-interview, she said that she worked within a mathematics team to decide what types of problems they wanted to put into their PowerPoint. She also considered more manipulatives if students were more willing to work in groups.

How to maximize students learning. Mary said that she often used guided practices in problem solving in order to maximize student learning, in both the pre- and post-interviews. In the pre-interview, she said that she used formative and summative assessments. She also said that she was showing how other kids would do it. In the post-interview, Mary elaborated that she checked their assignments from the day before. She said that it was good to connect "what they're doing one day in class to the next day." She said that students needed to hear and see "how it is related to what we did the day before," and "why we are doing this today." She said that teachers should know their students' prior knowledge to enhance student learning. Table 4.21 summarizes Mary's beliefs about teaching mathematics.

Table 4.21

Mary – Summary of Her Beliefs About Teaching Mathematics

Mary	**1. The role of the mathematics teacher**
	• Guide students through standards (pre& post) • Encourage students to ask questions (pre) • Have students solve problems in different ways (pre) • Have students explain problem-solving processes (pre) • Know best teaching strategies for students (post) • Guide students with problems (post) • Let students work with a partner or within a group (post) • Help students individually (post)
	2. What to teach
	• Based on standards and the textbook they used (pre) • Work with a colleague who had recent teacher prep courses (pre) • Do not consider many activities (pre) • Work within a mathematics team (post) • Consider more manipulatives (post)
	3. How to maximize
	• Guide practices (pre & post) • Use assessments (pre) • Show students' work (pre) • Connect a lesson to the next one (post) • Let students know the purposes of the lesson (post) • Know students' prior knowledge (post)

Mary's beliefs about the Lesh translation model in the first three questions about teaching. The first three interview questions were about (1) what the role of the mathematics teacher is; (2) what or what not to teach; and (3) how to maximize student learning in the classroom. In this section, I described how Mary talked about the Lesh translation model in the three questions about teaching. Mary emphasized language representations as a key teaching method to enhance students' understanding in mathematics. Therefore, she talked about translating within language representations in her pre-interview. Mary believed that she could not represent mathematics without symbols, so she emphasized translation within symbols in her pre-interview. In addition, she described the importance of translating within representations as her key teaching way in her post-interview. Table 4.22 summarizes how Mary discussed the translations between or within representations.

Table 4.22

Mary's Case From Interview Questions 1-3

Mary	Instances of representational fluency	Pre	Post
The role of the mathematics teacher	Representation	L	
	Translation within representations	L	
	Translation between representations	T	T
What to teach	Representation	WL, S	M, R, L
	Translation within representations	WL, S	M, R, L
	Translation between representations		T
How to maximize	Representation		
	Translation within representations		
	Translation between representations		T

Note. M: Manipulatives, S: Symbols, WL: Written language, L: Language, R: Realistic representations, and T: Translation among multiple representations (representations not identified).

Mary's beliefs about representations

The following sections describe Mary's beliefs about how each representation helped and hindered her teaching mathematics. The examples of using each representation are described. I also elaborated on her beliefs about the translation between and within multiple representations.

How representations help student learning. In the interview questions about each representation, Mary said that manipulatives could be helpful to lower-achieving students. She said that these representations could help students' visualization, and that the provided students with active learning in her post-interview. She elaborated that using pictorials was her key teaching method, in her both pre- and post-interviews, and she encouraged students to connect mathematical concepts with their thinking by using pictorials. She also said that explaining problem-solving processes by using pictorial representations helps students connect with their thinking mathematically in her post-interview. For symbols, she said that those were necessary to learn about mathematics, in both the pre- and post-interviews, and they served as a means for communication in mathematics in her post-interview. Thus, Mary said that these representations help students acquire mathematical thinking, and support students' problem-solving skills in her post-interview. The following excerpt illustrates Mary's description about symbolic and pictorial representations in her post-interview:

> *Symbols and pictures are kind of a way to communicate. Not only to keep your own thoughts clear, but to explain to somebody else why this number is going there, where you got this number from. I think it supports good thinking. And it supports the problem-solving process…*(Mary's post-interview)

For language, she said that this was helpful for students to understand mathematical vocabulary in her pre-interview. She stated that these representations help students build mathematical understanding, and use their own language to explain mathematical concepts and problem-solving processes in her post-interview. She elaborated that realistic representations were useful to connect with mathematics, in both the pre- and post-interviews, and students could see the usefulness of mathematics in her post-interview. Table 4.23 summarizes how Mary discussed the helpfulness of using each representation for student learning.

Table 4.23

Mary – Helpfulness of Using Representations From Interview Questions 5-9

Help \ Mary	Manipulatives		Pictorial		Symbolic		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Deeper understanding								✓		
Visualization		✓								
Active learning		✓								
Lower-ability students	✓									
A means of communication				✓		✓				
Connection with students' thinking				✓						
A key teaching model			✓	✓						
Necessity of symbols					✓	✓				
Vocabulary							✓			
Paraphrasing								✓		
Connection with mathematics									✓	✓
Mathematical utility										✓

How representations hinder student learning. For manipulatives, Mary said that manipulatives were difficult to implement with about 30 students in her pre-interview. For symbols, she said that if students did not understand the meaning of mathematical symbols, then there would be hindrances. In the language representations, she emphasized language issues for some students to appropriately explain mathematical concepts or their thinking in her post-interview. For the realistic representations, she said that there would be a hindrance if the realistic examples that were given were not relevant to students' lives. The following excerpt illustrates Mary's description about realistic representations in her pre-interview:

How real world is that? Another one we laugh about is the profile picture of a *daffodil, and it talks about the angle measures and about the congruency-taking place. Who ever takes a daffodil and is measuring the angles?* …(Mary's pre-interview).

Table 4.24 summarizes how Mary discussed the hindrances of using each representation for student learning.

Table 4.24

Mary – Hindrances of Using Representations From Interview Questions 5-9

Mary / Hindrance	Manipulatives		Pictorial		Symbol		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Management issue	✓									
May lack understanding					✓					
Language issue								✓		
Using irrelevant context									✓	
No hindrance		✓	✓	✓		✓	✓			✓

Mary's examples of using representations in his classrooms. For manipulatives, Mary said that she preferred to use online manipulatives rather than use real hands-on activities such as using algebra tiles (Mary's pre-interview) so that she could prevent classroom management issues from happening. However, in her post-interview, she gave examples of using algebra tiles to describe the benefits of using manipulatives. In another interview question, Mary provided examples with the translation within pictorial representations, and the translation between pictorial and symbolic representations. The following excerpt illustrates Mary's descriptions involving how she used pictorial and symbolic representations in the classroom with her students:

When we get to our linear algebra and the kids are given the equation and they *have to graph it, or they're given a graph and they have to write an equation for it. The tables, the graphs, and the equations…having them see all three at the same time is good…*(Mary's pre-interview).

How translations among multiple representations help or hinder student learning. Mary said that using multiple representations across the same concept promotes students' deeper understanding, because they could see the concept many different ways. But she said that students need plenty of time to practice all representations to be familiar with them. She also gave examples of connecting between equations, tables, and graphs to solve algebra problems. Table 4.25 summarizes how Mary discussed translations among multiple representations for student learning:

Table 4.25

Mary – Translations Among Multiple Representations

Mary	Help		Hindrance		Translation example	
	Deeper understanding	Multiple interpretations	Lack of practice time	No hindrance	Within representations	Between representations
Pre	✓	✓		✓	✓	✓
Post	✓		✓		✓	✓

In sum, Mary believed that translating within language representations and translating between language and realistic representations were critical for a student's conceptual understanding in mathematics. The translation between pictorial and symbolic representations was also helpful to understand mathematical concepts. She believed that pictorial representations easily translated with language representations. Her belief changes will be described in the following section.

Mary's Belief changes

Mary showed her belief changes about using each representation between the pre- and post-interviews. For manipulatives, she said that when she decided what to teach, she did not consider any hands-on activities in her pre-interview, but she thought about using more manipulatives in her post-interview. She also said that using manipulatives helps lower-achievement students, and that it could be a hindrance because of class management issues in her pre-interview. However, she said that it helps students visualize mathematics and make students do mathematics in her post-interview. The following excerpt shows Mary's descriptions concerning manipulatives in her pre-interview:

> *When I taught 8th grade, I would use the algebra tiles for a day and two. But the classroom I was in didn't have carpet, and all those tiles falling on the floor all day long, and I got to the point where the kids would prefer to just draw them quickly instead of using the tiles…*(Mary's pre-interview).

However, she emphasized the helpfulness of using manipulatives instead of the hindrances in her post-interview. The following excerpt illustrates her descriptions about using them in her post-interview:

> *I like the algebra tiles. Kids are doing more of the algebra* (Mary's post-interview).

I summarized Mary's belief changes about using manipulatives in Figure 4.24.

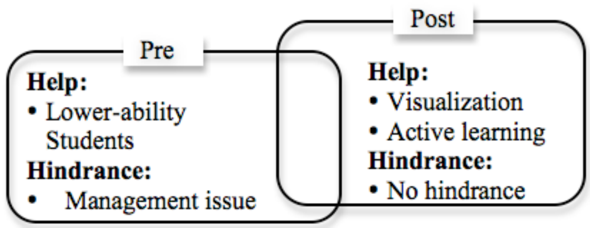


Figure 4.24. Mary's belief changes about Manipulatives.

Mary emphasized pictorial representations as a key teaching model, in both the pre- and post-interviews. She gave more detailed descriptions about the helpfulness of using pictorials in her post-interview rather than in her pre-interview. For example, in her post-interview, she said that pictorial representations could be a good means of mathematical description to communicate with others. She stated that it helps students see mathematical relationships so that they could make connections with their thinking. For symbols, she said that she could not think about mathematics without symbols, so it would be a hindrance if students did not understand them in her pre-interview. However, in her post-interview, she said that symbols were a means of communication, as well as necessary things to learn in mathematics during her post-interview. I summarized Mary's belief changes about using pictorials and symbols in Figure 4.25.

<Pictorials> <Symbols>

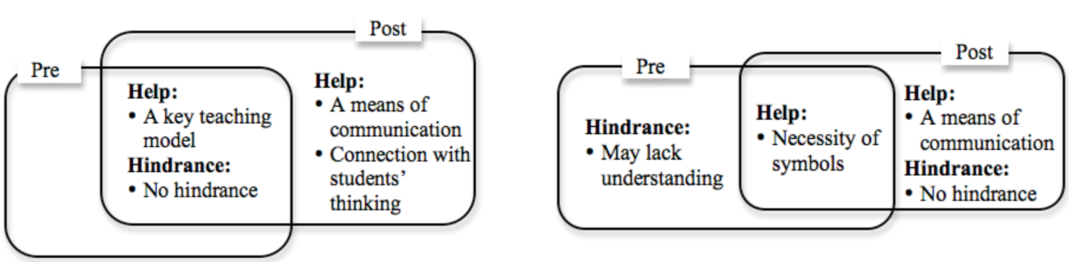


Figure 4.25. Mary's belief changes about pictorials and symbols.

For language, Mary emphasized the mathematical vocabulary in her pre-interview. In her post-interview, she explained that it helps students explain mathematical concepts with their own language, so they could deepen their understanding in mathematics. She also added language issues as a hindrance to use in her post-interview. For realistic representations, she said that if the real-world situations were not relevant to students' lives, it could be a hindrance in her pre-interview. However, in the post-interview, she said that it helps students know how to use mathematics in the real world. I summarized Mary's belief changes about using language and realistic representations in Figure 4.26.

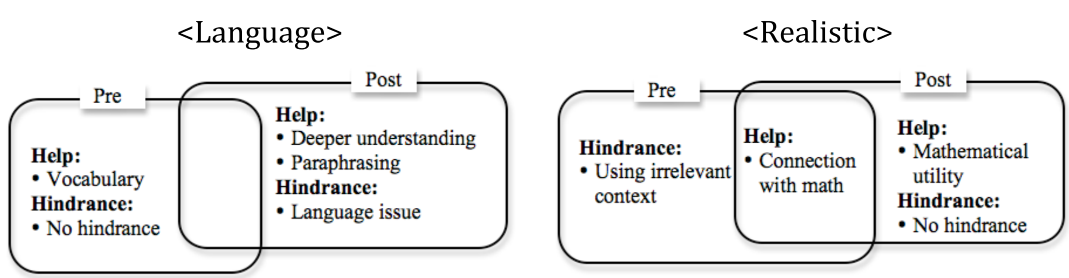


Figure 4.26. Mary's belief changes about language and realistic representations.

In sum, Mary stated that manipulatives help students' visualization and active learning in her post-interview, while she elaborated that there was a hindrance because of the management issues in her pre-interview. Mary showed negative beliefs about using manipulatives in her pre-interview, but she said that she wanted to use more manipulatives in her post-interview. She said in her pre-interview that language representation helped her understand mathematical vocabularies, but she said in her post-interview that language representation promotes students' deeper understanding. Mary also emphasized the use of translating among pictorial, symbolic, and language representations in her post-interview. She elaborated that she could promote students' deeper understanding when using the language representations in her post-interview.

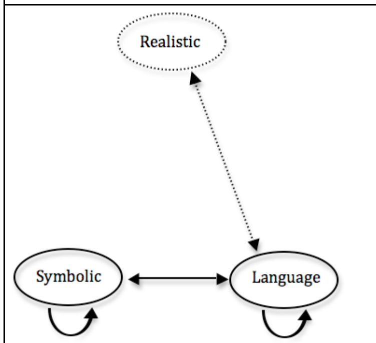
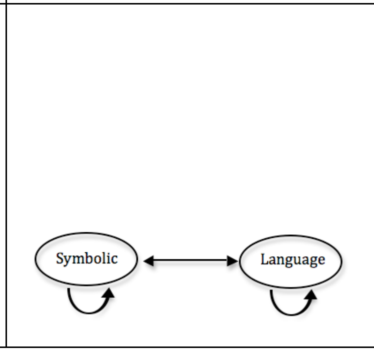
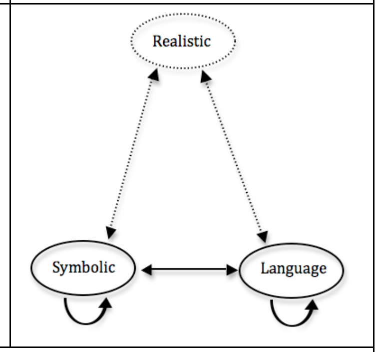
Mary's practices

Observation 1. The first observation was algebra classes about fractions. On Day 1 of the first observation, Mary taught comparing fractions using common denominators. She gave three warm-up questions using symbols and real-world contexts, such as bus schedules to compare two fractions. In the day's lesson, she explained a step on how to compare fractions using common denominators. She asked students how they solved the problems such as, "What are we going to multiply to make 36?" However, she mostly asked answers from problems. She allowed students to use calculators in some questions. She used some real-world contexts in word problems that came from the textbook.

On Day 2, Mary taught students ordering mixed and improper fractions. She most used symbolic (e.g., $\frac{19}{6}$, $3\frac{5}{12}$) and language representations to solve problems. She also explained her strategies and textbook's methods to solve ordering problems. At the beginning of the lesson, she gave the definitions of mixed numbers and improper fractions in language. She also used a word problem, but it did not involve real-world contexts. On Day 3, she reviewed fractions, decimals, and mixed numbers. She solved problems to change improper fractions to mixed numbers, and to change fractions to decimals and decimals to fractions. In addition, she taught her students how to use a calculator to change the numbers of fractions. She also used quarters (25¢) to explain $\frac{1}{4}$ in checking the homework.

Using the Lesh Translation Model in the first observation. On Day 1, she checked problems from the homework. Some of the problems included using the same numerator to compare fractions, and she explained that the more pieces a pizza is divided into, the smaller the pieces will be. In the lesson of Day 1, Mary had students compare fractions (e.g., $\frac{4}{15}$, $\frac{7}{12}$, $\frac{4}{5}$, and $\frac{23}{30}$) using a common denominator (e.g., 60 for the denominators 15, 12, 5 and 30). She talked about how to make the same denominator from fractions. The questions she provided students during class were represented by symbolic representations. Thus, Mary used translating within and between symbolic and language representations, with few realistic representations.

On Day 2, she solved problems to change mixed numbers to improper fractions before comparing mixed numbers and improper fractions. She used all symbolic representations for the problems to compare the numbers using a common denominator. On Day 3, she gave students the problems involving changing number forms, such as fractions, decimals, and mixed numbers, between each other. She asked students what they found, but inquired little as to how they solved their problems. She translated within and between symbols (e.g., $2\frac{2}{5} = 2.4$) and language (e.g., changing denominators to 10 or 100 in order to change to decimals, or dividing numerators by the denominators) in problem solving. She also used quarters to explain one fourth in checking the homework. That is, she used little realistic representation in checking the homework. The diagrams in Figure 4.27 show the use of mathematical representations in the first classroom observation.

Day 1: Comparing fractions	Day 2: Ordering mixed numbers and improper fractions	Day 3: Review decimals, fractions, and mixed numbers
		

Note. Type of line:

Strongly used in teaching

Weakly used in teaching

Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

Figure 4.27. The first classroom observation in Mary's algebra class for three days.

Observation 2. The second observation was algebra classes about the unit rate and slope. On Day 1, Mary reviewed unit rates and taught the slopes of linear lines. In the warm-up, she talked about what a unit rate is and what a calculator shows when students type the ratio in the word problem such as \$ 26.01 for 9 cups of cocoa. She explained how to use a calculator for this problem. She also talked about how to convert meters to centimeters, showing a ruler when she checked the homework. She explained the definition of slope as rise to run (slope $=\frac{\text{rise}}{\text{run}}$) and steepness, and she talked about skiing and riding a snowboard. In the problem-solving time, Mary asked to find the slopes from linear line graphs, and she also asked students some questions in different problems, such as, "How many units are we going to right/ going up or down?" and, "an increasing line or decreasing line?" She used the textbook information to show different methods to draw a graph with her teaching method to draw them. She also used word problems with real-world contexts, such as lava flow from the textbook. After finding the slopes from the graph, she asked students to draw a linear line, given the conditions such as a "slope of $\frac{1}{2}$ passes through (2,5)." When she gave time to solve problems using dot paper, she talked about the y=4 graph that did not have a slope. She asked why the slope of y=4 is not $\frac{4}{0}$, and she had students type $\frac{4}{0}$ into a calculator. Thus, she explained that it made an error, and it was not a number.

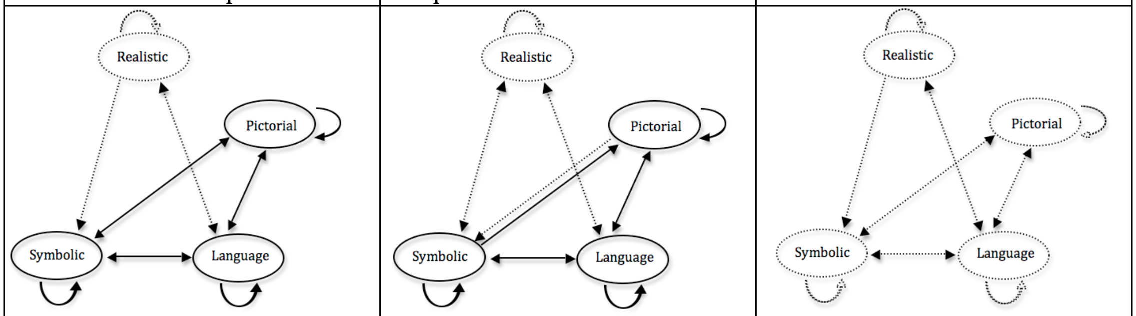
On Day 2, Mary explained how to find slopes from graphs. In the warm-up, she gave four different graphs, and then found the slopes. In checking the homework, she also solved some real-world problems, such as the speed of canoeing, or the changing speed of a river. She allowed students to use a calculator. Then she asked students if they could see the rise to run from 1.2, which showed up on a calculator. She showed how to use a calculator to change 1.2 into an improper fraction in order to see $\frac{\text{rise}}{\text{run}}$. The practice time was the same as the first day of the second observation. Students drew the graphs on the line that passed through two points, and then found the slopes. On Day 3, she reviewed chapter 8 about ratios and proportions. She corrected homework using students' work. There were some word problems with real-world contexts, such as Alabama and Colorado's gasoline tax rate, which were from the textbook. For the rest of the time, students worked on problems about ratios and proportions with their partners.

Using the Lesh translation model in the second observation. On Day 1, Mary provided students with problems involving real-world contexts to find unit rates (e.g., $\frac{\$534}{\text{12 lift tickets}}=\frac{\$44.50}{\text{1 lift ticket}}$) for the warm-up. She defined a slope as a ratio of the rise to run between any two points ($=\frac{\text{rise}}{\text{run}}$) in language. Then she solved problems to find slopes from the graphs (e.g., decreasing lines, increasing lines, and horizontal lines) with her students. She also had students solve problems about drawing graphs based on given conditions, such as two points (e.g., (3,4) and (6,3)) or one point and a slope (e.g., a slope of $\frac{1}{2}$; passes through (2, 5)). That is, she used translating within and between pictorial, symbolic, and language representations, with little realistic representations in the warm-up.

On Day 2, she explained slopes from linear graphs in the warm-up. In checking the homework, she used some real-world contexts to give the meaning of slopes that the students solved. The problems of homework also involved real-world problems. In the lesson of the day, she provided students with problems involving drawing linear graphs with the given conditions. It was the same as Day 1. That is, she used translating within and between pictorial (e.g., finding graphs from the given linear equations), symbolic (e.g., linear equations), and language representations, with few realistic representations in checking the homework. However, she only used translating pictorial to symbolic representations (e.g., finding linear equations from the given graphs) for the warm-up.

On Day 3, she solved the same types of problems as the last day in the warm-up and checking the homework, so she used symbolic, pictorial, language, and realistic representations. For the rest of the time, the students solved problems to review chapter 8 about ratios and proportions. The problems involved symbolic, pictorial, language, and realistic representations. The diagrams in Figure 4.28 show the use of mathematical representations in the second classroom observation.

Day 1: Unit rate and slope	Day 2: Slope	Day 3: Review



Note. Type of line:
Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

Figure 4.28. The second classroom observation in Mary's algebra class for three days.

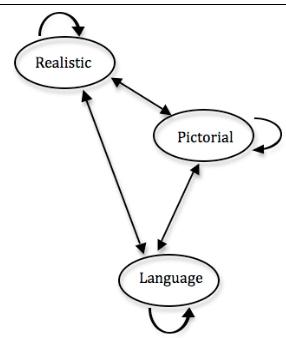
Observation 3. The third observation was geometry classes about 3D shapes and triangular prisms. On Day 1, she defined prisms, pyramids, cylinders, cones, and spheres. She also classified solids, and compared them between prisms and pyramids. Then she defined and compared faces, edges, and vertices for the solids. She showed a step-by-step strategy to sketch the solids, and she also provided real pictures that could be a geometric solid shape such as the shape of cheese for a triangular prism. Students sketched the solids by themselves. Pictorial and language representations were mostly used.

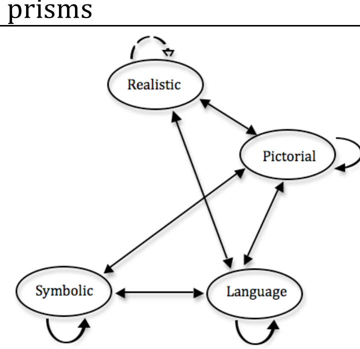
The Day 2 lesson was about surface areas of rectangular prisms. Mary explained nets of rectangular prisms using a tissue box and wrapping paper. She also used different colors on a net to explain how many faces were the same in a rectangular prism. She emphasized the formula of the surface area (e.g., S=2lw+2lh+2wh) in class. On Day 3, she taught the surface areas of triangular prisms. She used the same teaching strategies as the last class. At the beginning of the class, she showed different nets of triangular prisms, and she explained several different ways to find the surface area using nets of the triangular prisms. She explained the formula (S=2B+Ph, where B is the area of a base and P is the perimeter of a base) to find surface areas of triangular prisms and rectangular ones.

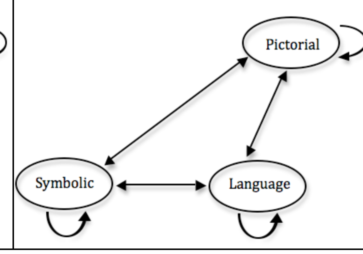
Using the Lesh translation model in the third observation. On Day 1, Mary defined and classified 3-dimensional shapes, such as prisms, pyramids, cylinders, cones, and spheres, using language, pictorial, and realistic representations. She explained their characteristics with written and spoken language, and showed pictures of them. Then she defined and compared faces, edges, and vertices for the solids using language and pictures. She also explained how to sketch solids, such as a rectangular prism and a pentagonal pyramid, and students practiced sketching solids such as triangular prisms, square prisms, and rectangular pyramids during the rest of the class. That is, she used translating within and between pictorial, language, and realistic representations.

On Day 2, Mary explained the surface areas of rectangular prisms using realistic representations (e.g., a tissue box and wrapping paper). Then she talked about the number of faces in a rectangular prism, and showed a net of the rectangular prism using a pictorial representation. She explained a formula to find the surface area of a rectangular prism (e.g., S=2lw+2lh+2wh) with pictorial and language representations. She had students solve problems to find surface areas of the rectangular prisms, which were represented by pictorial representations. On Day 3, she showed different nets of triangular prisms, and she explained different methods from the last class to find the surface area using pictorial representations (e.g., pictures of nets or solids). She explained a formula (e.g., S=2B+Ph) with pictorial representations to finding areas of triangular and rectangular prisms. She gave time for students to solve the types of problems she explained. That is, Mary used translating within and between pictorial, symbolic, and language representations. The diagrams in Figure 4.29 show the use of mathematical representations in the third classroom observation.

Day 1: 3D shapes	Day 2: Surface area of rectangular prisms	Day 3: Surface area of triangular prisms







Note. Type of line:
Strongly used in teaching
Weakly used in teaching
Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

Figure 4.29. The third classroom observation in Mary's geometry class for three days.

In sum, Mary used symbolic and language representations in the first observation. She used more pictorial and realistic representations in the second observation. In the third observation, she used pictorial, language, symbolic, and some real-world contexts. She connected pictorial representations with symbolic representations. She used real-world contexts from the textbook.

Pam's Case

Pam had six years of teaching experience in middle schools, and she was teaching high-achieving 7^{th}-grade students, using the McDougal Littell curriculum in School B.

Her class size was about 33 students, and the students sat in a row. She used three or four word problems at the beginning of class for a warm-up.

Pam's beliefs about teaching

In the following sections, I explore Pam's general beliefs about teaching mathematics, such as (1) what she believes the role of the mathematics teacher; (2) how she decides what or what not to teach; and (3) how she maximizes student learning in the classroom. These beliefs are related to her beliefs about using multiple representations. Thus, this analysis helps investigate research questions about teachers' beliefs in using representations and their practice in this study.

The role of the mathematics teacher. In the pre-interview, Pam said that the role of the mathematics teacher is to understand students' prior knowledge and to meet the state and district standards. The teachers need "not necessarily just have one single method to do things, …but understanding how to solve problems and how to apply it to new situations." She said that teachers needed to have "a ton of real-life examples." In the post-interview, Pam said that teachers should "help students understand math" and "how to apply their thinking to new situations, how to go deeper with their thinking, and how to explain how they thought about it." She also said that teachers had to encourage students to be involved in "real-world applications that might involve several different concepts. Moreover, she said that the role of the teacher is to help students figure out what they already know.

What or what not to teach. Pam said in the both pre- and post-interviews that what or what not to teach was decided based on state and district standards and the textbook, and a 7th-grade team had discussed what they were going to cover. In the pre-interview, she added that she considered "what I think is important," and what/how she could add other supplements. In the post-interview, Pam said that she considered "what students have done, what they are going to have to be able to do, and how we bridge the gap." She also considered more of the other materials, such as hands-on things than the curriculum would provide per se. However, she said that there were a lot of behavioral issues because students did not stay focused.

How to maximize student learning. In the pre-interview, Pam said that she had students talk to each other, to articulate their thinking through teaching each other or triggering other problem solving. She also differentiated homework based on students' ability based on pre-tests. She used various word problems in order to have students apply mathematical concepts. In the post-interview, Pam said that she gave good examples and then tried to get them to share their thinking and talking about multiple ways. She said that she guided students to draw connections between different methods, and she encouraged them to develop good questioning skills and to connect their prior knowledge with new situations. I summarized Pam's beliefs about teaching mathematics in Table 4.26.

Table 4.26

Pam – Summary of Her Beliefs About Teaching Mathematics

Pam	1. The role of the mathematics teacher
	• Apply mathematics in real-world situations (pre & post) • Understand students' prior knowledge (pre) • Meet the standards (pre) • Understand students' problem-solving strategies (pre) • Use real-life examples (pre) • Help students explain their thinking (post) • Help students know their level of understanding mathematics (post)
	2. What to teach
	• Based on standards and the textbook they used (pre & post) • Work within a mathematics team (pre & post) • Based on what she thinks is important (pre) • Consider supplements (pre) • Consider the sequence of the lesson (post) • Think about students' gaps • Consider more manipulatives (post)
	3. How to maximize
	• Have students talk with one another (pre) • Have peer teaching (pre) • Use various problem-solving methods (pre) • Differentiate homework based on pre-tests (pre) • Use work problems (pre) • Give good examples (post) • Let students share their thinking and work in multiple ways (post) • Have students connect between different methods (post) • Encourage students to have good questioning skills (post) • Have students connect their prior knowledge with new situations (post)

Pam's beliefs about the Lesh translation model in the first three questions about teaching. The first three interview questions were about (1) what the role of the mathematics teacher is; (2) what or what not to teach; and (3) how to maximize student learning in the classroom. In this section, I described how Pam talked about the Lesh translation model in the three questions about teaching. Pam believed that using realistic, language, and realistic representations was critical to understanding mathematics and applying it. She mostly emphasized translation within language or realistic representations. She said that students should be able to explain "what they are doing or why it works" (Pam's post-interview), so that she got them "to share their thinking and talking about multiple ways" (Pam's post-interview). She prepared "a ton of real-life examples (Pam's pre-interview) to help students understand "how to apply it to new situations" (Pam's pre-interview). Table 4.27 summarizes how Pam discussed the translations between or within representations.

Table 4.27

Pam's Case From Interview Questions 1-3

Pam	Instances of representational fluency	Pre	Post
The role of the mathematics teacher	Representation	R	R, L
	Translation within representations	R	R, L
	Translation between representations	T	R&L
What to teach	Representation		M, L, R
	Translation within representations		M, L, R
	Translation between representations	T	T
How to maximize	Representation	L, R	L, R
	Translation within representations	L, R	L, R
	Translation between representations		T

Note. M: Manipulatives, L: Language, R: Realistic representations, and T: Translation among multiple representations (representations not identified).

Pam's beliefs about representations

The following sections describe Pam's beliefs about how each representation helped and hindered her teaching mathematics. The examples of using each representation were described. I also elaborated on her beliefs about the translation between and within multiple representations.

How representations help student learning. In the interview questions about each representation, Pam said that manipulatives could help to see visuals in both the pre- and post-interviews, and students could be encouraged to talk about what they see in different objects in her pre-interview. She said that these representations help students develop mathematical understanding in her pre-interview. She maintained that these also help students motivate student learning, so it could make a class more active. Pam explained that pictorial representations promote students' deeper understanding in mathematics, and help them to better see mathematical relationships in the both the pre- and post-interviews. She said that these representations help students connect mathematical concepts with their thinking in her pre-interview. For symbols, she elaborated that these representations were a means of communications in mathematics, in both the pre- and post-interviews. Pam emphasized that language representations help students explain their own thinking (Pam's pre- and post-interviews) so that they could deepen their understanding (Pam's pre-interview). She said that realistic representations could improve student's motivation to learn, in both the pre- and post-interviews. She said that students could see the usefulness of mathematics through real world examples (Pam's pre-interview). She said that these representations were helpful in connecting with mathematics, so it created effective learning in mathematics (Pam's post-interview). Table 4.28 summarizes how Pam discussed the helpfulness of using each representation for student learning.

Table 4.28

Pam – Helpfulness of Using Representations From Interview Questions 5-9

Pam / Help	Manipulatives		Pictorial		Symbolic		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Deeper understanding	✓		✓	✓				✓		
Visualization	✓	✓								
Motivation		✓							✓	✓
Active learning		✓								
Connection with students' thinking			✓							
See patterns b/w data			✓	✓						
A means of communication	✓				✓	✓				
Explanation of students' thinking							✓	✓		
Connection with mathematics										✓
Mathematical utility									✓	
Effective learning										✓

How representations hinder student learning. Pam described the hindrances in manipulatives and symbols. For manipulatives, she said that 7th-grade students should graduate from needing manipulatives to being able to think about mathematics in their head. The following excerpt illustrates Pam's description about manipulative representations in her post-interview:

> *I think it depends on what we're talking about. Sometimes I think manipulatives are good. I think especially at younger ages, it's important. But by the time you get to high school, not that they're bad and you can't use them…*(Pam's post-interview).

For symbols, Pam said that it could be a hindrance if students do not understand the meaning of symbols: "A lot of kids don't understand what [symbols] mean, so they just get more confused" (Pam's pre-interview). Table 4.29 summarizes how Pam discussed the hindrances of using each representation for student learning.

Table 4.29

Pam – Hindrances of Using Representations From Interview Questions 5-9

Pam / Hindrance	Manipulatives		Pictorial		Symbol		Language		Realistic	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Depends on grades		✓								
May lack understanding					✓	✓				
No hindrance	✓		✓	✓			✓	✓	✓	✓

Pam's examples of using representations in her classrooms. Pam gave examples of how she represented mathematical ideas in some different modes from the interview questions about each representation. For manipulatives, she used unit cubes to create rectangular prisms that have the same area. The following excerpt illustrates Pam's descriptions involving how she used manipulative representations in the classroom with her students:

> *When we talk about that all the difference was that you could create rectangular prisms that have the same area, but have different surface areas. So, looking at all the possibilities. So, I'll give them a set of unit cubes and they physically create all the different rectangular prisms that would have that same area* (Pam's pre-interview).

She also elaborated on the translation within pictorial representations using examples of pictures, tables, and graphs in her pre-interview. She also said that drawing pictures was a great way to understand mathematical situations. For symbol representations, she talked about the equal sign as a mathematical symbol that students had difficulty understanding in terms of what it means in an equation in her pre-interview. She said that writing mathematical symbols is similar to writing English sentences such as, “We capitalize at the beginning of a sentence and add punctuation at the end” (Pam’s post-interview).

How translations among multiple representations help or hinder student learning. Pam said that using multiple representations across the same concept promotes students’ deeper understanding (Pam’s pre- and post-interview), and that “they could think about mathematics in many different ways” (Pam’s post-interview). However, students might “be overwhelmed by the multiple representations” (Pam’s post-interview), and they might not connect them with the same concept. Table 4.30 summarizes how Pam discussed translations among multiple representations for student learning:

Table 4.30

Pam – Translations Among Multiple Representations

Pam	Help		Hindrance		Translation example	
	Deeper understanding	Multiple interpretations	Overwhelmed	No hindrance	Within representations	Between representations
Pre	✓			✓		✓
Post	✓	✓	✓			✓

In sum, Pam believed that using mathematical representations such as manipulatives, pictorial, and language representations improved students’ conceptual understanding. She said that manipulatives could be a way of visualizing mathematics. She also elaborated on the translation with language or realistic representations, and the connection between different methods.

Pam's belief changes

Pam showed belief changes in manipulatives, pictorial, language and realistic representations. For manipulatives, Pam said in her pre-interview that these representations help students deepen their understanding, and students could have chances to talk about mathematical concepts through seeing different objects. However, in her post-interview, she emphasized that 7th-grade students should graduate from using manipulatives, even though she thought that with manipulatives, students would be more motivated about learning mathematics and would more actively work with mathematics. I summarized her belief changes about using manipulatives in Figure 4.30.

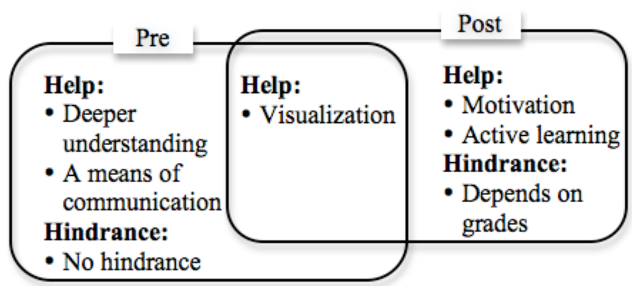


Figure 4.30. Pam's belief changes about manipulatives.

For pictorials, Pam elaborated that these representations help students understand mathematics deeply and that they see patterns between data in both the pre- and post-interviews. In her pre-interview, she added that these could make students connect mathematical concepts with their thinking when they see something, such as graphs and tables. For language, she emphasized to a greater degree students' deeper understanding while they were explaining their thinking in her post- versus her pre-interview. I summarized her belief changes about using pictorials and language in Figure 4.31.

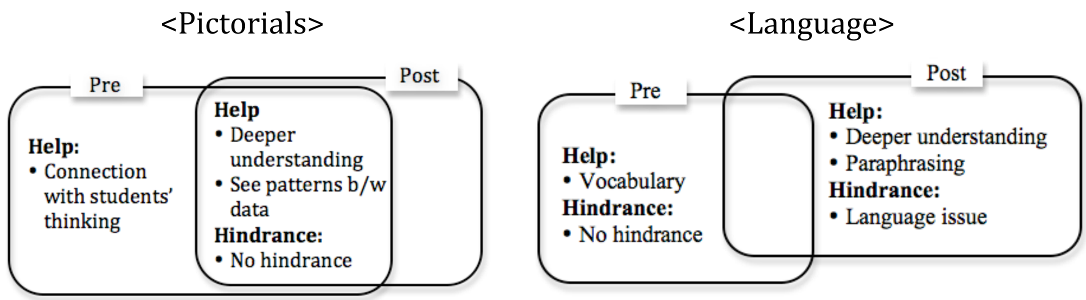


Figure 4.31. Pam's belief changes about pictorials and language

For realistic representations, Pam stated that these help students understand where they were going to really use mathematics in her pre-interview. However, she emphasized that real-world situations help students connect something with mathematics, so it could create effective learning in her post-interview. I summarized Pam's belief changes about using realistic representations in Figure 4.32.

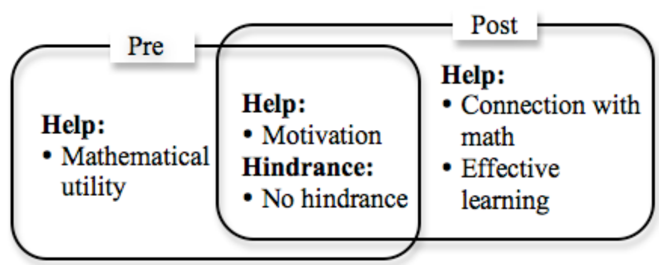


Figure 4.32. Pam's belief changes about realistic representations.

In sum, Pam said that using multiple representations across the same mathematical concept helps students interpret mathematical concepts in different ways in her post-interview. She also elaborated that language, manipulative, and realistic representations help students' understanding, active learning, and effective learning in her post-interview. However, she emphasized an understanding of abstract mathematical concepts rather than using concrete models for seventh-grade students in her post-interview.

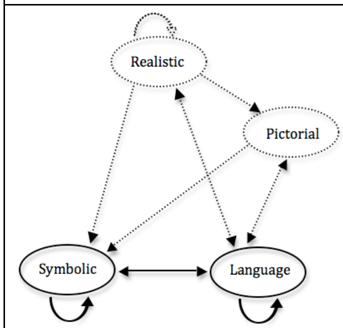
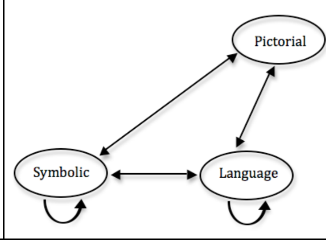
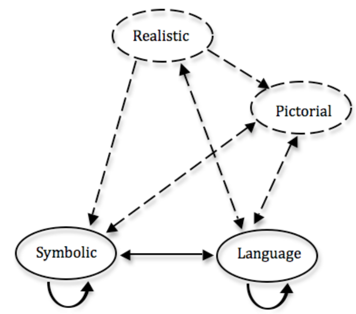
Pam's practices

Observation 1. The first observation was algebra classes for fractions. In the warm-up, Pam used a word problem with a real-world context, such as decorating picture frames, to use the Greatest Common Factor (GCF) and common factors. She also used factor trees to find prime factors. For the lesson of the day, she defined the simplest forms of fractions as the numerator and denominator being relatively prime. She solved problems about simplifying fractions. For examples, she gave fractions (e.g., $\frac{30}{45}$), and asked some questions such as, "What does it mean in the simplest fraction?" and, "How do you know that $\frac{2}{3}$ is in the simplest form?" She gave students time to talk with their neighboring peers, and they presented their thoughts in the whole class. Using students' answers, she asked the following questions, such as, "How do you know if two fractions are equivalent?" "Do you use another method?" and, "How did you get the %?" She also used a variable expression (e.g., $\frac{24a^3b^2}{18a^2b^3c}$) to find the simplest form. She was very focused on language to encourage talking with peers and in the whole class, as well as symbols to provide problems about the simplest forms.

Day 2 was about the Least Common Multiple (LCM). She talked about the differences between factors and multiples. She gave three methods to find LCM, such as 1) making a list; 2) prime factorization; and 3) using GCF. She gave students problems about finding LCM, and asked them how to find the answers with these methods. She also used factor trees to solve them. On Day 3, she solved the problems about comparing two fractions and mixed numbers using common denominators. She solved problems to compare them, such as comparing $\frac{7}{10}$ and $\frac{3}{4}$, or finding the greater distance in kayak racing. She quickly drew fraction circles to compare fractions. She also gave students a challenging problem, such as "$\frac{4}{x}+\frac{1}{2}$. How do you find a common denominator?"

Using the Lesh translation model in the first observation. On Day 1, Pam gave students a real-world problem to find GCF, and she explained factor trees to solve them for the warm-up. For the topic of the day, she used symbolic (e.g., $\frac{30}{45}$ or $\frac{24a^3b^2}{18a^2b^3c}$) and language representations to solve problems about simplifying fractions. She used factors or GCF to solve the simplest forms of fractions, and she defined the simplest forms in language representations. That is, she used translating within and between symbolic and language representations, with few pictorial or realistic representations in the warm-up and in checking homework.

On Day 2, she distinguished between factors and multiples in language, and she gave examples in symbolic representations. Then she talked about three methods to find LCM, such as making a list, using prime factorization, and using GCF. She also used factor trees to find factors. She solved problems to find LCM (e.g., LCM for 16, 20, and 28, or LCM for $6x^2y$ and $9x^4z$). She had students solve the problems to find the LCM during the class. Pam emphasized using symbolic, pictorial (e.g., factor trees) and language representations. However, there was no translation with pictorials. On Day 3, she compared fractions and mixed numbers using common denominators. She solved problems with a real-world problem (e.g., finding the greater distance in kayak racing). When she solved the real-world problem, she drew fraction circles to compare two fractions. That is, she used language and symbolic representations with few pictorial or realistic representations. The diagrams in Figure 4.33 show the use of mathematical representations in the first classroom observation.

Day 1: Simplifying Fractions	Day 2: Least Common Multiple	Day 3: Comparing fractions and mixed numbers
		

Note. Type of line:

Strongly used in teaching

Weakly used in teaching

Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

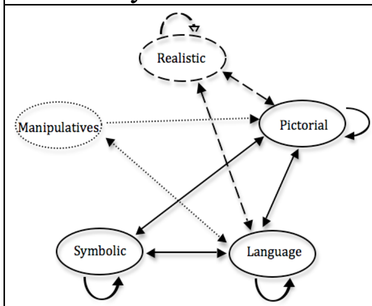
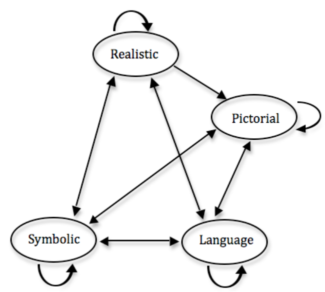
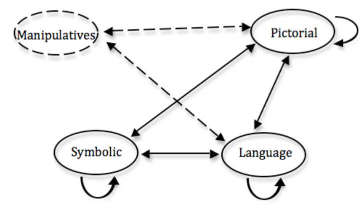
Figure 4.33. The first classroom observation in Pam's algebra class for three days.

Observation 2. The second observation was geometry classes. In the warm-up, Pam asked about reflecting across the x-axis, translating 8 units left and 6 units down, and rotating 90-degrees counter-clockwise at the given three points on graph paper. She demonstrated rotating her paper physically by 90-degrees counter-clockwise, and she explained by using the symbol (x, y) → (-y, x). Then she talked about the homework that students had questions about. Those questions came from the textbook. For the topic of the day, she explained that the pupils of the eyes change in size with the brightness of light. Then she explained the terms of similar polygons, dilation, and distortion in language. For the rest of the time, she talked about a project to create a face, dilate the face, and distort the face, and she showed some samples about how to do it.

Day 2 involved similarity and dilation. For the warm-up, Pam used two similar triangles to find missing angles. She also had students discuss how they could know if two triangles were congruent. For the topic of the day, she defined similar polygons using language, and she gave examples about similar polygons. She also solved problems about dilation using scale factors (e.g., scale factor > 1 or scale factor <1). These problems were selected from the textbook. She also asked students how to make polygons larger or smaller than their original size. On Day 3, she reviewed for the next day's test, such as angle pairs, finding missing angles, classifying polygons, congruent polygons, and translations on graphs. She handed out worksheets to solve problems in groups. Then she solved some problems that were asked by students in the whole class. Pam emphasized students' communication with peers or in the whole class.

Using the Lesh translation model in the second observation. On Day 1, Pam drew a triangle using three points, and transformed them based on the given conditions in the warm-up. She also physically demonstrated the rotations by using her paper. As a result, she used translating between pictorial, symbolic (e.g., (x, y) → (-y, x)), and language (e.g., translating a figure based on 90-degrees counterclockwise) representations, with little manipulative representation. For the lesson for the day, she defined similar polygons, dilation, and distortion in written and spoken language using some realistic representations (e.g., dilation of pupils' eyes). For the rest of class, she explained tasks to enlarge and distort a face. Thus, she demonstrated how to copy and trace a face and how to enlarge and distort the face by showing examples that she did. She gave time for students to do the tasks. That is, she used translating within and between pictorial and language representations, and translating between symbolic and language representations with few realistic ones.

On Day 2, Pam explained similarity using symbolic and language representations. She solved problems to find out whether or not two polygons or real-world pictures (e.g., a TV screen) were similar, or problems to find missing numbers using similar polygons. She used proportions (ratios) to find the missing numbers. For the rest of the time, she talked about creating similar figures by using dilation. As a result, she used scale factors to draw bigger or smaller figures than the original ones. She also gave chances for students to solve the problems and to create similar figures. That is, she used translating within and between symbolic, pictorial, and language representations, and translations between symbolic and realistic representations. She also used translating from realistic representations to pictorial ones. On Day 3, she solved problems about transformations, and she used translations within and between symbolic and language representations. She also drew triangles to solve a transformation problem and physically demonstrated rotating it. She also gave time for students to solve the problems she had solved, and to discuss how the students found the answers with peers after solving the problems individually. That is, she translated within and between pictorial, symbolic, and language representations, with few manipulative ones. The diagrams in Figure 4.34 show the use of mathematical representations in the second classroom observation.

Day 1: Similarity and dilations Use the same PP as Nick's and Holly's	Day 2: Similarity and dilations/ similar polygons	Day 3: Review
		

Note.

1. Manipulatives=physical manipulatives
2. Type of line:

Strongly used in teaching

Weakly used in teaching

Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class

Figure 4.34. The second classroom observation in Pam's geometry class in three days.

Observation 3. The third classroom observation was probability. On Day 1, Pam taught about probability and odds. When collecting the homework, she asked how to create their own word problems using the mathematical concepts they learned, and she said she would use some problems the students created on their test. In the warm-up, she used a visual brainstorm card game to solve mathematical word problems. Students shared their thoughts with peers and within a whole group setting. On the day of the topic, probability and odds, Pam talked about odds using real-world examples such as gambling, and pictorial representations such as the picture of a spinner, and gave its definition. She also defined the complement of an event in language and symbols. She selected many word problems with real-world contexts, such as free throws and dogs from the textbook.

On Day 2, she taught about independent and dependent events in probability. In the warm-up, she gave four word problems in real-world contexts. The problems about permutations and combinations were solved by using symbols (e.g., ${}_nC_r$ and ${}_nP_r$) and language. On the day of the class, she defined independent and dependent events and she also showed the definitions of them in words and in algebraic terms from the textbook. Then she gave the problems using real-world situations, language, symbols, and pictures. Day 3 was used to review Chapter 12 about data analysis and probability. For the review, she used slides that explained each of the topics, such as plots, tree diagrams, permutations, combinations, probability and odds, and independent and dependent events. The slides showed realistic examples, pictures, definitions in words, and symbols.

Using the Lesh translation model in the third observation. On Day 1, Pam defined probabilities and odds in language, and she solved problems about them using language and symbolic (e.g., P(Event A)=1-P(Event B)) and realistic (e.g., free throws in basket- ball games) representations. She also showed a spinner that was divided into four equal parts and colored blue and red to solve problems about probabilities. In addition, she also talked about complementary events to explain probabilities. During the class, she gave time for students to solve problems about probabilities and odds, and to explain what they solved. That is, she used translating within and between symbolic, language, and realistic representations, with few pictorial ones.

On Day 2, Pam talked about independence and dependence in reporting taxes, and then she defined independent and dependent events in mathematics. She also gave examples about choosing gumballs using the terms of replacement or no replacement with pictures. She explained how to find probabilities of independent and dependent events, and she emphasized that the probability of the second event could change, based on the first event. She gave students problems involving real-world contexts (e.g., tossing coins or bingo games) to find out whether or not these events were dependent or independent, and in order to solve the probabilities of dependent and independent events in symbolic and language representations. She asked students how they solved their problems. That is, she used translating within and between symbolic, language, and realistic representations, with few pictorial representations. On Day 3, there were warm-up real-world questions about permutations and combinations using symbolic, language, and realistic representations. She reviewed Chapter 12 about data analyses and probability. She defined key terms or words in language representations, and all examples for each term involved pictorial or symbolic representations. The diagrams in Figure 4.35 show the use of mathematical representations in the third classroom observation.

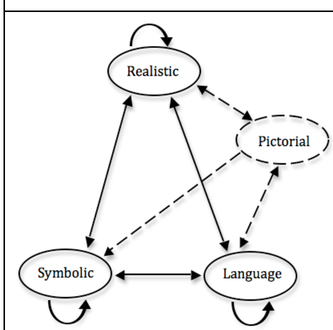
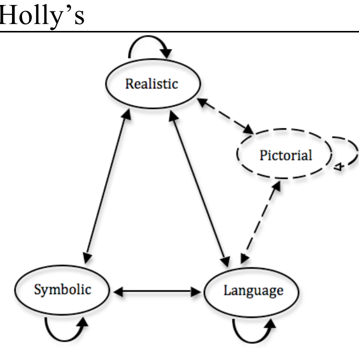
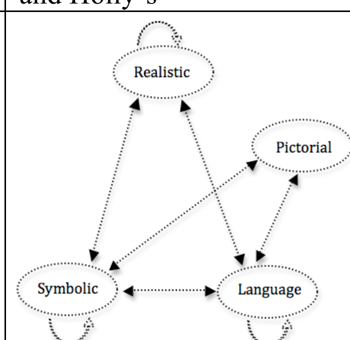
Day 1: Probability and odds	Day 2: Independent and dependent Use the same PP as Nick's and Holly's	Day 3: Review Use the same PP as Nick's and Holly's
		
Note. Type of line: Strongly used in teaching Weakly used in teaching Used only for reviewing the last class, checking homework, or solving warm-up problems, or used in students' worksheets during class		

Figure 4.35. The third classroom observation in Pam's probability class for three days

In sum, Pam had students talk about how to find answers, what kinds of strategies they used, and how to define mathematical terminology with their peers. Thus, she used many language representations. She also used symbolic representations to connect language that students used in all three observations. In the second observation, she often used pictorial representations. For the third observation, she used many word problems with real-world contexts.

Summary of Chapter 4

Chapter 4 presented the six teacher cases for this study. In the following section, a brief summary of each case will be described. In the beliefs about teaching, Ben emphasized the importance of real-word situation use and mathematical communication. He also said that teachers needed to make students work in many different ways. In the questions about each representation, Ben said that manipulatives representations are visualization tools, and pictorial representations help students successfully explain their own thinking using language representations. He also explained that language helps teachers understand students' ability levels, so that it helps teachers create lessons to maximize their own students' learning. Ben said that he uses a variety of real-world examples to improve students' conceptual understanding. For symbols, he said that students needed to understand the meaning of symbols for deeper understanding. Ben explained that the translation within and between multiple representations helps students build their mathematical understanding.

There were noticeable changes in Ben's beliefs about using manipulatives and language. He elaborated in the pre-interview that he tried to use manipulative as much as he could, but he just emphasized visualization through manipulatives in his pre-interview. For language, he mostly mentioned negative beliefs about them in his pre-interview, but said only positive beliefs during his post-interview. In the classroom observations, he used few manipulative representations using copy paper to teach perpendicular lines and right angles in the second observation. He used the translation within and between language representations through the three observations. In particular, in the third observation, he connected geometric concepts with algebraic concepts using language and symbolic representations. He mostly used translating within and between realistic language, and symbolic representations through the entire observation.

Sara believed that creating effective learning environments was important to help students develop their mathematical understanding. Thus, she said that using multiple representations was a good teaching strategy to create a learning environment. Sara elaborated that manipulative representations help students visualize mathematical concepts, and pictorial representations help students connect the concepts with students' thinking. She said that mathematical symbols were another language for mathematics, and language and realistic representations help students connect their thinking and knowledge with mathematics. Sara believed that using multiple representations was a way to represent student's own mathematical ideas. However, those representations could be a hindrance if students used them improperly or without any other help when they needed it.

Sara showed her belief changes on manipulative, symbols, language, and realistic representations. She said in her post-interview that students could develop their mathematical understanding using mathematical representations such as manipulatives and symbols, while she emphasized in her post-interview visualization for manipulatives and mathematical terms for symbols. In addition, she explained in her post-interview that symbols are another form of representation to communicate with others. Sara emphasized to a greater extent that teachers should help students properly use representations such as language and realistic representations in her post-interview. In the classroom observations, Sara mostly used realistic and pictorial problems, and she encouraged students to use language representations in problem solving through three period observations. She also used virtual manipulatives in the second observation. When she used real-world situations, she used the scenarios that were related to her students' lives. She also used language to encourage students to explain their mathematical reasoning rather than to explain their answers within all three observations.

Nick expanded upon the importance of using realistic and language representations to improve students' learning in his belief about teaching mathematics. For each representation question, he explained that manipulatives help students by giving them concrete models and seeing something in mathematics. Thus, he said that manipulatives could help students build mathematical understanding. However, he thought that they could also hinder some students who already understood abstract concepts in mathematics. Nick said that he often used pictorial representations as his teaching strategies. Nick believed that symbols were a communication tool in mathematics, so students should know symbols in order to learn mathematics. However, if students did not understand the meaning of symbols, it could hinder their learning. He said that language representation was very important for student's future career to communicate with others. Nick specified that both realistic and language representations gave students positive motivation to learn mathematics, as well as it could help deepen students' understanding in mathematics. He said that there were hindrances to overcome in mathematics curriculum when using multiple representations, even though they might improve some students' conceptual understanding.

Nick showed his belief changes about using manipulatives, symbolic, and language representations between his pre- and post-interviews. For manipulatives, Nick explained in his post-interview that manipulatives visualize mathematics while giving concrete examples, especially for lower-ability students (said in his pre-interview). He also expanded upon management issues as a hindrance of using manipulatives in his pre-interview, but he said that it slowed the learning pace down for high-achieving students in his post-interview. For symbols, Nick emphasized the necessity of symbols in his pre-interview, but he described symbols as a means of communication in his post-interview. In terms of language, he stated in his pre-interview that it was needed for students' future careers in the workplace, but he said in his post-interview that this presentation was important in order to give students motivation to learn mathematics. In the classroom observation, Nick used pictorial and symbolic representations in the first observation, and he talked about realistic examples in the second and third observations. He brought real-world examples such as newspapers, letters, and a tin candy box from home to share with students in each class through the three period observations.

In beliefs about teaching mathematics, Holly said that students should work in translating between a language and symbols in the problem-solving process. When she considered lesson plans, she was thinking about various virtual manipulatives such as Internet activities, and visual representations to teach abstract concepts. Holly elaborated that using multiple representations helps students build mathematical understanding and enhances problem-solving abilities. However, she said that students should understand how each representation could effectively be used in different problems. Sometimes, she said, it was not necessary to use all of the representations, step-by-step, every time. Holly expanded upon the impacts of each mathematical representation. For manipulatives, she said that these representations were easily connected with pictorial representations. Yet, she mentioned, there were some issues to cover all of the standards if they spent time on working with manipulatives. For pictorials, Holly explained that they help students understand mathematical relationships from data, and improve problem-solving skills, but she thought that students should fully understand how to create pictorial representations in mathematics problems. She said that mathematics could represent all mathematical symbols, so students should understand the meanings of the symbols. Holly said that students should explain their mathematical thinking through mathematical language, and should connect it with mathematics during their explanations. Thus, using language improves student's mathematical understanding, and teachers can also understand students' learning progress while observing their language use. However, she said, teachers always consider some students' language issues and their inefficient language use. She mentioned that realistic representations give students motivation to learn mathematics, and help them understand the usefulness of mathematics in their lives. However, she cautioned, real-world numbers are messy, so it sometimes makes it more difficult for students more to use them.

Holly showed her belief changes about each representation between the pre- and post-interviews. For manipulatives, she focused on helping students visualize mathematics in her pre-interview, but she expanded upon students' inspiration in learning mathematics through using them in her post-interview. For pictorials, she said in her post-interview that she considered them as a hindrance to a greater extent for some lower-ability students who were not able to create correct pictures within problem-solving processes. For symbols, Holly said in her post-interview that students needed to spend extra time in order to understand symbols as an alternative representation. However, she expanded upon the importance of understanding the meaning of symbols so as to accelerate students' learning in mathematics in her post-interview. Holly emphasized to a greater degree in her post-interview that using language helps students develop their conceptual understanding in mathematics, so teachers should help students overcome some of their language issues while students use mathematical language. Holly described to a greater extent in her post-interview the usefulness of realistic representations for students' future careers. In the classroom observations, Holly primarily used symbolic, language and pictorial representations. In the third observation, she used many realistic examples and problems from the textbook. She used both written and spoken language throughout the three observation periods while solving problems. She also used online activities in the third observation, which provided the curriculum website.

Mary said that teachers should give students opportunities to work with different problem-solving strategies. Thus, she considered using manipulatives to a greater degree when she decided lessons. She explained that using multiple representations helps students see mathematical concepts in many different ways, so that students could improve their mathematical understanding. Mary said that language representations were her key teaching strategies to develop student's mathematical understanding. Mary believed that she could not think of teaching mathematics without symbols, so students could fully understand the meaning of mathematical symbols. In addition, through using mathematical symbols and their own language, students could communicate with others about mathematics. These strategies enabled students to develop their mathematical understanding. She said that realistic representation helps students connect with mathematics and understand the usefulness of mathematics in their lives.

Mary showed significant belief changes about manipulatives. In her pre-interview, she did not consider many activities because of class time issues and class size. However, in her post-interview, she wanted to use manipulatives as much as she could. She also said in her pre-interview that using manipulatives helps lower-achievement students, but she said that these representations created active learning for all students in her post-interview. In her post-interview, Mary connected manipulatives with pictorial representations, and she stated that pictorials help students to use language representations. She emphasized that symbols were a language representation, and that language representation helps build mathematical understanding in her post-interview. In the classroom observation, Mary mostly used language and symbolic representations through the three observation periods. In the third observation, she spent most of the time connecting pictorial representations with symbolic representations. She also used more realistic word problems in the third observation, but most questions were selected from the textbook.

Pam said that she emphasized using multiple strategies to solve problems, and that using multiple representations helps improve students' mathematical understanding. She said that teachers should help students apply mathematics to real-world situations, and that realistic representations give students the motivation to learn mathematics. Pam elaborated that teachers considered many hands-on activities in their lessons. She explained that manipulative representations can connect with pictorial representations, and that they help students communicate mathematical concepts with others. However, she said that middle-school students needed to think about mathematics through abstract rather than concrete models. Pam explained that pictorial representations help students see mathematical patterns from data, so that students can develop their mathematical understanding. Pam elaborated that students could use symbols to communicate in mathematics with others, and that they could explain their mathematical thinking with language.

Pam showed belief changes about using manipulatives, pictorial, language, and realistic representations. She stated that the benefits of using manipulatives were giving motivation and making an active learning environment in her post-interview rather than building understanding and making better communication in her pre-interview. She also said that manipulatives were not necessary to seventh-grade students in her post-interview. Pam emphasized to a greater degree that language helps students deepen their understanding, and that using realistic representations helps make effective learning in her post-interview. In the classroom observation, Pam emphasized using language representations such as explain problem-solving processes and mathematical reasoning through the three observation periods. She also used multiple methods to solve problems. In the second and third observations, she used more translating within and between realistic representations that came from the textbook.

In the following chapter, I will analyze data across all of the cases in order to identify similarities and differences in the teachers' beliefs, belief changes, and practices.

Chapter 5. Cross Case Analysis

This chapter will report the results of an analysis across all six participants and will be divided into each representation with four main sections. The first section will describe the teachers' beliefs about each mathematical representation, such as manipulatives (physical and virtual ones), pictorial, symbolic, language, and realistic ones, described in the Lesh translation model. In the second section, I explore how the teachers' beliefs about teaching mathematics had changed. In the third section, I illustrate how teachers' beliefs about Lesh translation model were reflected in their classroom practices. Finally, I will make connections between teachers' beliefs and practices based on the Lesh translation model.

Teachers' Beliefs About Using Mathematical Representations

Overall, the teachers said that it was important for students to use each mathematical representation such as manipulatives (physical and virtual ones), pictorial, symbolic, language, and realistic representations. The teachers elaborated that translations within or between mathematical representations were helpful in enhancing students' mathematical conceptual understanding. Their responses indicated that their beliefs were aligned with the teaching and learning aspects in the Lesh translation model.

Manipulatives. In this section, I explore the six teachers' beliefs and practices regarding the use of manipulatives (physical and virtual ones) described in the Lesh translation model. I also describe the connection between their beliefs and practices regarding the use of manipulatives.

What teachers believe about manipulatives. All six of the teachers stated in the pre- or post-interviews that manipulatives helped students easily make connections with the visuals. They explained how students could better visualize mathematical concepts with manipulatives, such as algebra tiles or chips. However, Pam added that seventh-grade students need to go beyond hands-on concrete models. She also said in both the pre- and post-interviews that they should be able to think about mathematics concepts more abstractly than concretely. Most teachers, except for Holly also mentioned that manipulatives could help students develop their conceptual understanding. However, Nick and Holly said that manipulatives could slow down the learning pace of students who already understand abstract concepts. Holly, Mary, and Pam believed that these manipulatives could create an active learning environment. Sara, Holly, Mary, and Pam said in their post-interviews that they considered using physical or virtual manipulatives when deciding upon their lessons in their post-interviews. In addition, Mary said that it was easier to manage her classes when using virtual manipulatives than physical ones. Nick and Mary said in their pre-interviews that they were helpful for lower-achieving students to improve their mathematical understanding. Holly and Mary stated that using manipulatives could maintain students' interests and motivate them to do more mathematics. Nick and Mary said that there were management issues involved in handling all students, which was a hindrance in using manipulatives. All six of them talked about the hindrances of using physical manipulatives (e.g., algebra tiles), such as excessive learning time or management issues. Furthermore, Holly and Mary seemed to believe that virtual manipulatives could reduce those hindrances in the classroom. Table

5.1 summarizes how all six of the teachers pointed out the helpfulness and hindrances of using manipulatives in their teaching.

Table 5.1

Teachers' Beliefs About Manipulatives

		School A				School B							
	Manipulatives	Ben		Sara		Nick		Holly		Mary		Pam	
		Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Help	Deeper understanding		✓		✓	✓	✓					✓	
	Visualization	✓	✓	✓			✓	✓			✓	✓	✓
	Concrete examples					✓							
	Connection with math				✓								
	Active learning							✓	✓		✓		✓
	Motivation								✓				✓
	Maintaining attention								✓				
	Lower-ability students					✓				✓			
	A means of communication											✓	
Hindrance	Over-generalization	✓											
	Improper use			✓	✓								
	No connection			✓									
	Excessive learning time						✓	✓	✓				
	Management issue					✓				✓			
	Time issue for curriculum					✓							
	Distracted by the material							✓	✓				
	Depends on grades												✓
	No hindrance		✓								✓	✓	

In sum, the teachers believed that using various manipulatives helped students' conceptual understanding in mathematics, and that it was easy to translate between manipulatives and pictorial representations. However, some of them considered particular issues of using physical manipulatives, such as excessive learning time or management difficulty. These issues seemed to make the teachers hesitate to use physical manipulatives in the classrooms.

How teachers' beliefs about using manipulatives were changed. Most teachers (Ben, Sara, Holly, and Mary) spoke more about positive beliefs of using manipulatives in the post-interviews rather than negative aspects in the pre-interviews. In particular, Mary showed significant belief changes between her pre- and post-interviews. She had a negative belief about using physical manipulatives in her pre-interview, but she showed positive beliefs in her post-interview. For example, in her pre-interview, Mary said that she "used it for something, but very little." She also stated that she would use virtual manipulatives that students seem more interested in if some students needed them. She said that using virtual manipulatives could reduce some hindrances of using physical ones. However, in her post-interview, she said, "I like the algebra tiles. Kids are doing more of the algebra." She stated that she wanted to use as many manipulatives as she could in her post-interview.

Ben, Sara, and Holly mostly believed that manipulatives helped students visualize mathematics in the pre-interviews. However, they emphasized to a greater degree that manipulatives contributed to students' learning in the post-interviews. Ben and Sara also highlighted that manipulatives helped deepen students' understanding in the post-interviews. Ben said in his post-interview that he tried "to use as much hands-on materials as possible." Holly elaborated that using manipulatives helped create a learning-centered environment, such as motivating students to learn mathematics, and maintaining their attention in her post-interview. Pam also emphasized that manipulatives helped teachers build an effective learning environment, such as encouraging active learning and improving student motivation to learn in her post-interview. On the other hand, Nick and Mary said that manipulatives helped only low-achieving students in the pre-interviews. However, he stated that they underscored visualizing mathematical concepts using manipulatives in his post-interview. I summarized all six of the teachers' belief changes about the helpfulness of using manipulatives in Figure 5.1.

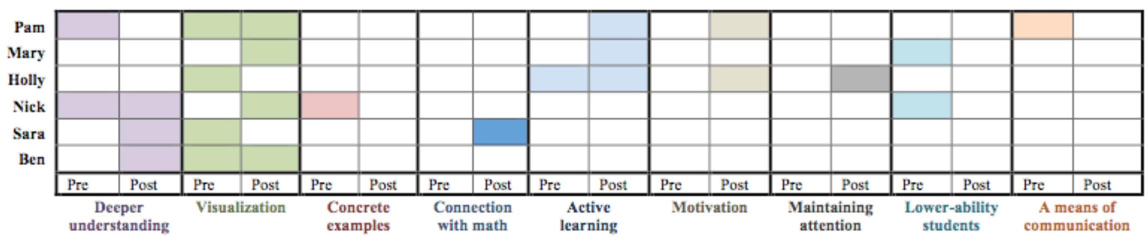


Figure 5.1. Summary of the teachers' beliefs about the helpfulness of using manipulatives.

Most teachers, except for Holly showed belief changes about the hindrances of using manipulatives between their pre- and post-interviews. Ben and Mary did not describe any hindrances in their post-interviews, even though they expanded upon the overgeneralization and management issues for the hindrances of using manipulatives in their pre-interviews. However, Pam stated that she did not want the seventh-grade students to use manipulatives any more during her post-interview, while she said that there were not any hindrances in her pre-interview. I summarized all six of the teachers' belief changes about the hindrances of using manipulatives in Figure 5.2.

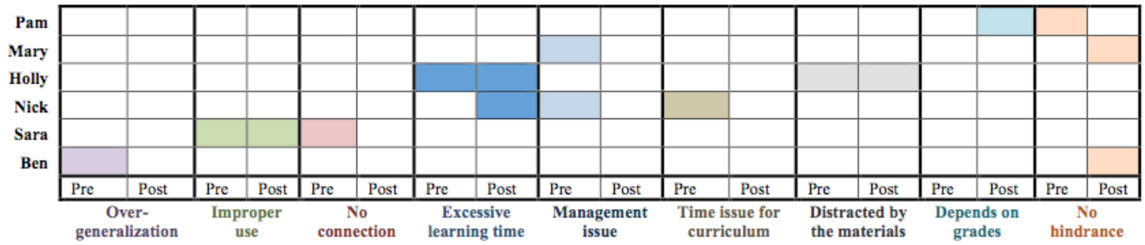


Figure 5.2. Summary of the teachers' beliefs about the hindrances of using manipulatives.

In sum, all six of the teachers believed that using manipulatives helped students learn mathematics, in both the pre- and post-interviews. They elaborated to a greater degree on positive rather than negative aspects during their post-interviews. However, most teachers (Sara, Nick, Holly, and Pam) also spoke about the hindrances of using them, such as excessive learning time, in the post-interviews. In addition, Pam believed that seventh-grade students should learn mathematics without manipulatives.

How teachers used manipulatives in their classrooms. All six of the teachers used few physical or virtual manipulative representations during the entire observations. For example, all six of them did not use any manipulatives in the first observation. . In the second observation, Ben, Nick, Holly, and Pam weakly used physical manipulatives, but there were no translations within them. They translated between manipulative and other representations, such as pictorial and symbolic ones, but Ben translated only from manipulative to pictorial ones. Ben, Nick, and Holly also used translating between manipulative and realistic representations. On the other hand, Sara used virtual manipulatives without translating within them. However, she actively translated between virtual manipulative and language representations, and translating virtual manipulatives to pictorial and symbolic ones. She did not translate between virtual manipulatives and realistic representations. Yet, Mary did not use any manipulatives in the second observation. In the third observation, most teachers except for Holly did not use any manipulatives, but Holly used virtual manipulative representations. The teachers seemed to want to focus on abstract concepts in their teaching without using manipulatives.

How teachers connect their beliefs about manipulatives with their practices. All six of the teachers elaborated on the different reasons as to why manipulatives significantly helped or hindered students in learning mathematics between the pre- and post-interviews. In particular, all six of them claimed that manipulatives helped students visualize mathematics in their pre- or post-interviews. The teachers had more positive beliefs toward using manipulatives in the classroom during their post-interviews compared to their pre-interviews. In the observations, Nick, Mary, and Pam used physical manipulatives during the second observation. While using the manipulatives, they enabled their students to see how a figure was enlarged and distorted in a coordinate plane. As a result, they used translations between manipulative and other representations, such as language and pictorial ones. Holly also stated that manipulative representations helped motivate students to learn mathematics and maintain their attention in her post-interview, and the representations helped create active learning in both the pre- and post-interviews. She also used manipulatives in the second observation. However, she elaborated that students could be distracted by some materials in both her pre- and post-interviews, and she emphasized to a greater extent translating within and between virtual manipulatives and pictorial representations during her post-interview rather than translating within physical manipulatives in her pre-interview. In her classroom practices, she used virtual manipulatives in the third observation. These helped students actively do various things that were provided by a website, and the students were also motivated to do them in her class. Sara also used virtual manipulatives in the second observation. The virtual manipulatives involved linear equations using a function machine. She also used the function machine as a pictorial representation to explain linear equations in her classes. Thus, she emphasized that students had connected the method to what they had already learned with a virtual manipulative. In Sara's beliefs about using manipulatives, she believed in the importance of connecting manipulatives with mathematical concepts directly in her pre- and post-interviews. It seems that these beliefs were reflected in her practices. Mary showed significant belief changes between her pre- and post-interviews, and she said that she wanted to use manipulatives. However, she did not use any of them throughout the three observations. Therefore, the teachers used few manipulatives throughout all of the three observations, even though they conveyed positive beliefs about using manipulatives. It seemed that the teachers needed to have more experiences and understanding of how to use them without dealing with hindrances. Moreover, it appeared as though teachers needed to know what kinds of representations were useful in understanding the abstract mathematical concepts that seventh-grade students should learn in order to use them in the classroom.

Pictorials. In this section, I explore the six teachers' beliefs and practices regarding the use of pictorials described in the Lesh translation model. I also describe the connection between their beliefs and practices regarding the use of pictorials.

What teachers believe about manipulatives. Ben, Sara, and Pam said in both the pre- and post-interviews that pictorials were the best way to promote students' deep mathematical understanding. Ben, Sara, Holly, and Mary stated that they often used translating within pictorial representations, such as tables and graphs in their classes. Ben, Sara, and Mary also elaborated that they connected pictorials with symbolic representations (e.g., equations or scale factors). Nick said that using graphing calculators was very helpful in using pictorial representations, and in connecting with symbolic representations. Holly and Pam also said that pictorials helped students see "the relationships between the pieces of data" (Holly's pre-interview). However, Ben added that there could be a hindrance, such as the following: "If the students are seeing too much data, they might be a little bit slower of a learner" (Ben's pre-interview). Sara and Pam mentioned that using pictorials was one of the best means of connecting mathematical concepts with students' thinking. Ben and Mary also said that pictorials were useful when they talked about mathematical concepts with students. For example, Nick said that Venn diagrams were great to talk about when comparing and contrasting many mathematical concepts. Mary said that these representations were "a good written description of something" (Mary's post-interview). Nick and Mary elaborated that pictorial representations were "one of the key instruction strategies" (Nick's post-interview) because any pictorials could help teach mathematics. Most teachers, except for Sara said that there were no hindrances in using pictorial representations, in either their pre- or post-interviews. However, Sara stated in both the pre- and post-interviews, that there could be hindrances if the pictorials were not presented correctly, and if they caused students to have incorrect interpretations when using them. Table 5.2 summarizes how all six of the teachers thought about the helpfulness and hindrances of using pictorial representations in their teaching.

Table 5.2

Teachers' Beliefs About Pictorials

		School A				School B							
	Pictorials	Ben		Sara		Nick		Holly		Mary		Pam	
		Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Help	Deeper understanding	✓	✓	✓	✓							✓	✓
	A means of communication		✓								✓		
	Connection with students' thinking			✓	✓						✓	✓	
	A key teaching model					✓	✓			✓	✓		
	Tools to organize problems							✓	✓				
	See patterns b/w data							✓	✓			✓	✓
Hindrance	No connection	✓											
	Too much data	✓											
	Incorrect interpretation			✓	✓								
	Slow progress for some	✓							✓				
	No hindrance		✓			✓	✓	✓		✓	✓	✓	✓

In sum, the teachers believed that translating within pictorial representations was helpful in teaching and learning mathematics. They also said that pictorial representations were easy to connect with symbols and language so that they facilitated students' learning in mathematics. Thus, it was a good teaching model for them. However, if teachers or students improperly used them, hindrances would result. They seemed to want to effectively or properly use pictorial representations in order to make connections with mathematical concepts and to enable students to express the concepts in using them.

How teachers' beliefs about using pictorials were changed. The teachers showed small belief changes about using pictorial representations between the pre- and post-interviews, compared with the other representations. For example, Ben and Mary said only during the post-interviews that pictorial representations could be used as a good means of communication in order to discuss mathematical concepts or problem-solving processes. Mary (only pre-interview) and Pam (only post-interview) elaborated that pictorials helped students connect mathematical concepts with their thinking. However, other beliefs about them were the same between the pre- and post-interviews. I summarized all six of the teachers' belief changes about the helpfulness of using pictorials in Figure 5.3.

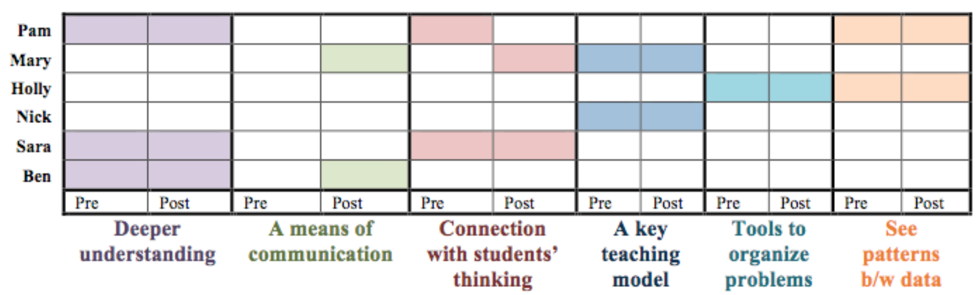


Figure 5.3. Summary of the teachers' beliefs about the helpfulness of using pictorials.

Ben showed significant changes about the hindrances of using pictorials between the pre- and post-interviews. He stated in his pre-interview that pictorial representations could hinder students' learning if there were no connections, if too much data were used in class, and if it created slow progress for some of the students. However, he said in his post-interview that these representations did not hinder students' learning at all. On the other hand, Holly said that there were not any hindrances in using pictorials in her pre-interview, but she stated that there was very slow progress for some students if they used too many different types of pictorial representations to solve a problem during her post-interview. I summarized all six of the teachers' belief changes about the hindrances of using pictorials in Figure 5.4.

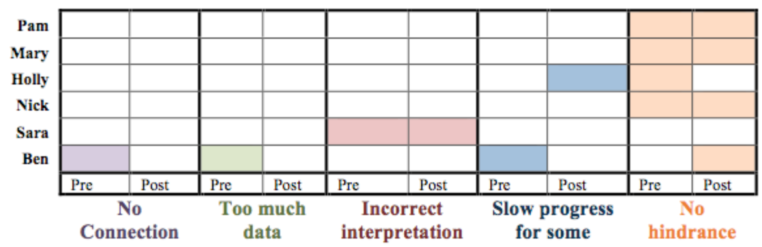


Figure 5.4. Summary of the teachers' beliefs about the hindrances of using pictorials.

In sum, there were small belief changes about using pictorial representations, such as a means of communication (Ben & Mary-Only post), connection with students' thinking (Pam-Only pre; Mary-Only post), and slow progress for some students (Ben-Only pre; Holly-Only Post). Most teachers, except for Sara and Holly did not talk about the hindrances of using pictorials in the post-interviews. In particular, Ben talked about many hindrances of using them in his pre-interview, but he did not mention any hindrances in his post-interview. That is, they believed that pictorial representations could be a good teaching strategy to use in the classroom without hindering student learning.

How teachers used pictorials in their classrooms. The teachers mostly used pictorial representations during the three observations. For example, in the first observation, most teachers except Mary used pictorial representations. Sara strongly used translating within pictorial representations. Ben, Holly, and Pam did not use translating within pictorial representations, but they actively used one pictorial representation, such as an area model or factor tree, while solving problems. In the second observation, all six of the teachers actively used translation within pictorials, and translation between pictorial and other representations, such as language and symbolic ones. Ben translated from pictorial to symbolic representation, but not vice-versa. In addition, Sara and Nick strongly used translating from realistic to pictorial representations, while Nick weakly used translating from pictorial to realistic representations. In the third observation, Sara, Nick, and Mary actively used pictorial representations, and other teachers weakly used them. Nick and Mary actively used translating within pictorials, but others used translating within them weakly or in checking homework. On the other hand, Ben and Pam translated from pictorial to symbolic representations. However, Ben did not translate in the opposite direction. In addition, some teachers used translating from realistic to pictorial representations, and then translating from the pictorial to symbolic representations, while solving problems.

How teachers connect their beliefs about pictorials with their practices. All six of the teachers mostly showed similar beliefs about using pictorial representations, such as building deeper understanding, using them as a key teaching model, or seeing patterns from data, between the pre- and post-interviews. They all said that this representation was very helpful in learning mathematics, and Nick, Mary, and Pam did not mention any hindrances in either the pre- or post-interviews. In their classroom practices, most teachers actively translated within pictorial representations and translated between pictorial and other representations, especially symbolic and language ones throughout the entire observations. Furthermore, Ben and Mary emphasized that pictorial representations were a means of communication in the post-interviews, but not in their pre-interviews. In their classroom practices, Ben actively used translations from pictorial to symbolic representations and language representations in the first and second observations, while he weakly used the same translations in the third observation. In addition, he weakly used translating from symbolic to pictorial representations in the first observation. Mary actively used within and among them in the second and the third observations, while she did not use any pictorial representations in the first observation. Therefore, teachers' positive beliefs about using pictorials were reflected in their practices in the classrooms. Many teachers used pictorials as an instructional strategy in order to find solutions that were represented by symbols from real-world problems. That is, they used pictorials as a good tool to solve real-world problems.

Symbols. In this section, I explore the six teachers' beliefs and practices regarding the use of symbols described in the Lesh translation model. I also describe the connection between their beliefs and practices regarding the use of symbols.

What teachers believe about symbols. Ben, Sara, and Nick elaborated that students' understanding of symbols was strongly tied with their mathematical understanding. Nick, Mary, and Pam stated that mathematical symbols were needed to communicate mathematical concepts, and Sara said that these were alternative representations and mathematical terms to express mathematics. Nick and Mary also said that mathematics could not be taught without using symbols. Mary mentioned that these symbols could help accelerate problem-solving processes if they fully understood the meaning of mathematical symbols. However, all six of the teachers said in the pre- or post-interviews that there could be a hindrance if students did not understand mathematical symbols. Holly also said that some students needed more time to understand mathematical symbols. Table 5.3 summarizes how all six of the teachers discussed the helpfulness and hindrances of using symbolic representations in their teaching.

Table 5.3

Teachers' Beliefs About Symbols

		School A				School B							
	Symbols	Ben		Sara		Nick		Holly		Mary		Pam	
		Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Help	Deeper understanding	✓	✓		✓			✓	✓				
	Mathematical terms			✓									
	Alternative representations				✓			✓					
	Necessity of symbols					✓				✓	✓		
	A means of communication						✓				✓	✓	✓
	Acceleration of math processes								✓				
Hindrance	May lack understanding	✓		✓	✓	✓	✓		✓	✓		✓	✓
	Excessive learning time							✓					
	No hindrance		✓								✓		

In sum, all six of the teachers elaborated on the importance of understanding the meaning of mathematical symbols in using them without any hindrances. They believed that symbolic representations were necessary to learn mathematics, and a communication means to teach mathematics. That is, they believed that using symbolic representations helped teachers teach mathematics, and students could learn mathematics by fully understanding these symbols.

How teachers' beliefs about using symbols were changed. Most teachers, except for Pam changed their beliefs about using symbols. Sara believed that symbols were only mathematical terms to learn mathematics in her pre interview. However, she said that symbol representations helped students build their mathematical understanding and communication skills in mathematics in her post-interview. Nick and Mary stated in their pre-interviews that students could not learn mathematics without symbols, so it was necessary for students. However, they said in the post-interviews that symbols were a means of communication to learn mathematics. Thus, they connected symbols with language representations. Holly elaborated in her pre-interview that symbols were alternative representations in order to explain mathematics, but she said in her post-interview that symbols helped students save time to solve problems through writing with symbols instead of using all words. I summarized all six of the teachers' belief changes about the helpfulness of using symbols in Figure 5.5.

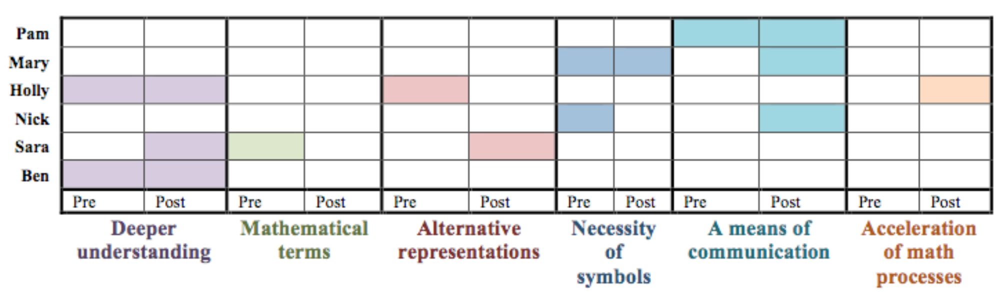


Figure 5.5. Summary of the teachers' beliefs about the helpfulness of using symbols.

All six of the teachers said during their pre- or post-interviews that there were hindrances if students did not understand the meaning of mathematical symbols. However, Ben and Mary said that there were not any hindrances of using mathematical symbols in their post-interviews. Holly said that some students would require excessive learning time to understand their meanings, so it could be a hindrance in her pre-interview. I summarized all six of the teachers' belief changes about the hindrances of using symbols in Figure 5.6.

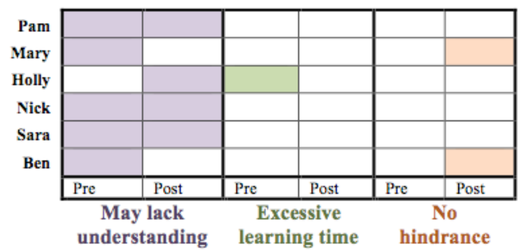


Figure 5.6. Summary of the teachers' beliefs about the hindrances of using symbols.

In sum, Sara, Nick, Holly, and Mary changed their beliefs about the helpfulness of using symbolic representations between the pre- and post-interviews, even though they claimed that using symbols was helpful for students to learn mathematics, in both the pre- and post-interviews. In particular, Nick and Mary elaborated on translations between symbols and language to learn mathematics in only the post-interviews. Sara emphasized that using symbols helped deepen students' understanding in her post-interview, while she said that symbols were merely mathematical terms in her post-interview.

How teachers used symbols in their classrooms. In the entire observations, all six of the teachers actively used symbolic representations. For example, throughout the all three observation, all six of them strongly used translations within and between symbolic and language representations. In the first observation, Ben, Sara, Holly, and Pam actively used translations between symbolic and other representations, such as pictorial or language ones. In addition, Ben and Sara primarily translated from pictorial to symbolic representations, but they weakly used translations from symbolic to pictorial ones. On the other hand, most teachers except for Mary used translations from realistic to symbolic representations while solving problems, but not vice-versa. Mary used them only in checking homework. In the second observation, most teachers except for Ben actively used translation within symbols and translation between symbolic and other representations, such as pictorial or language representations. However, Ben used translating from pictorial to symbolic representations, and he weakly used translating within symbolic ones. Most teachers except for Nick also used translations between symbolic and realistic representations. Sara and Pam strongly used them. Sara translated from realistic to symbolic ones, but not vice-versa. In addition, Sara used translations from virtual manipulative to symbolic representations, but not vice-versa. In the third observation, Sara, Nick, and Pam strongly used symbolic representations. Sara and Nick strongly used translating within symbolic representations and translating between symbolic and realistic representations. The other teachers weakly used translations between them. Ben and Pam weakly translated from pictorial to symbolic ones. In addition, Holly used translations between symbolic and virtual manipulative ones.

How teachers connect their beliefs about symbols with their practices. All six of the teachers talked about a hindrance of using symbols when students lacked an understanding of mathematical symbols in their pre- or post-interviews. Nick, Mary, and Pam emphasized that symbols were fully connected with language representations. In their practices, all of them actively used translating within mathematical symbols, and they also actively used translations between symbolic and language representations throughout all of the three observations. They spent plenty of time defining mathematical symbols in written and spoken language representations. In addition, all six of them used translations within and between symbolic and pictorial representations during the second and third observations. However, in the first observation, Ben and Sara strongly used translating from pictorial to symbolic representations, while they weakly used translating from symbolic to pictorial ones. That is, teachers believed that students needed plenty of time to fully understand mathematical symbols. They also believed that connecting with other representations, such as pictorial or language ones, was helpful for students to effectively understand the symbols. The teachers also used translations between pictorial and other representations, such as pictorial or language ones, in the classroom during the entire observations.

Language. In this section, I explore the six teachers' beliefs and practices regarding the use of language described in the Lesh translation model. I also describe the connection between their beliefs and practices regarding the use of language.

What teachers believe about language. All six of the teachers said in the pre- or post interviews that written or verbal language was one of the best teaching strategies to strengthen students' understanding. Ben, Sara, and Holly said that language helped students connect mathematical concepts with their thinking, so they could "clarify their own thinking" (Sara's pre-interview). Holly elaborated that teachers also figured out students' misunderstanding while students explained their thinking. Sara and Mary mentioned that teachers could help students explain mathematical concepts with their own words through using language representations. Mary said in her pre-interview that language representations could be considered as mathematical vocabulary to explain mathematical concepts. Nick elaborated on the usefulness of using language representations for students' future careers in his pre-interview. He also said in his post-interview that these representations provided students with motivation to learn mathematics. However, Ben said, "Some students got lost on mathematical concepts during their explanation of what the point of the problems was" (Ben's pre-interview). Sara stated that some students needed a teacher's good guidance to explain their mathematical thinking. If such guidance were not provided, it could be a hindrance (Sara's post-interview). Nevertheless all six of the teachers said that there were not any hindrances in using language representations in their mathematics teaching in the pre- or post-interviews. Table 5.4 summarizes how all six of the teachers discussed the helpfulness and hindrances of using language representations in their teaching.

Table 5.4

Teachers' Beliefs About Language

	Language	School A				School B							
		Ben		Sara		Nick		Holly		Mary		Pam	
		Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Help	Deeper understanding	✓	✓	✓	✓	✓	✓		✓		✓		✓
	Connection with math		✓	✓				✓	✓				
	Paraphrasing				✓						✓		
	Future relevance					✓							
	Motivation						✓						
	Explanation of students' thinking							✓				✓	✓
	Knowing students' mis-understanding								✓				
	Vocabulary									✓			
Hindrance	May become lost in math concepts	✓											
	If no Instruction when needed				✓								
	Language issue								✓		✓		
	Slow explanations inefficient								✓				
	No hindrance		✓	✓		✓	✓	✓		✓		✓	✓

In sum, all six of the teachers believed that using language representations helped students build mathematical conceptual understanding. It helped students explain concepts mathematically, and connect these mathematical concepts with their thinking. Thus, it could be the best teaching model for mathematics teachers. They believed that there were no hindrances, as long as the teachers helped some students who needed extra guidance.

How teachers' beliefs about using language were changed. Ben, Sara, and Nick said that language representations helped students enhance their mathematical conceptual understanding, in both their pre- and post-interviews, and all six of them said the same thing in the post-interviews. In addition, Mary and Sara emphasized that this representation helped students use their own language to explain mathematical concepts in the post-interviews. However, Mary said in her pre-interview that language representations were a vocabulary to learn mathematics, and Sara said in her pre-interview that this helped students connect certain things with mathematics. On the other hand, Nick said in his pre-interview that language fluency would help students work in the workplace, but in his post-interview, he said that this representation motivated to learn mathematics. Holly also elaborated that language representations helped students explain their thinking, and helped teachers know students' misunderstandings in her post-interview. I summarized all six of the teachers' belief changes about the helpfulness of using language in Figure 5.7.

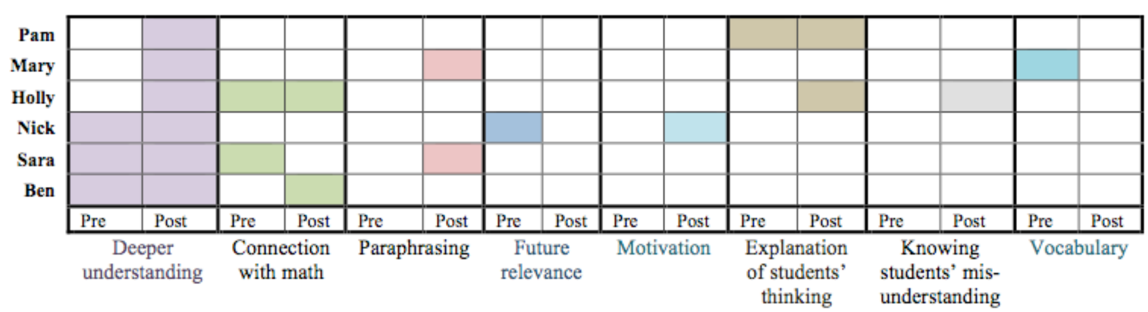


Figure 5.7. Summary of the teachers' beliefs about the helpfulness of using language.

Sara, Holly, and Mary talked about some hindrances of using language in the post-interviews, even though they did not mention any hindrances in the pre-interviews. For example, Sara said in her post-interview that using language could be a hindrance if there is no helper when students do not have the proper language to express themselves. Holly and Mary expanded upon the language issues for some students to explain concepts in the post-interviews. Holly also said in her post-interview that some students explained mathematical concepts inefficiently, so that it could be a hindrance for them. However, Ben said in his pre-interview that there could be a hindrance if students became lost in mathematical concepts when they were trying to explain them, but in his post-interview, he claimed that there were no hindrances at all. I summarized all six of the teachers' belief changes about the hindrances of using language in Figure 5.8.

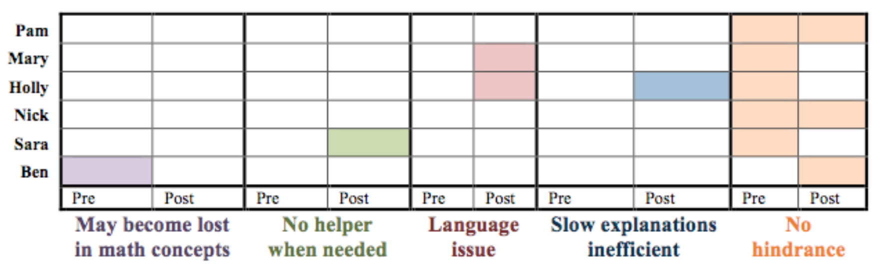


Figure 5.8. Summary of the teachers' beliefs about the hindrances of using language.

In sum, all six of the teachers described that using language representations helped students enhance their understanding of mathematics in the post-interviews. It seems that the teachers believed that using language representations would be the biggest influence to students' learning. However, Sara, Holly, and Mary began to talk about some hindrances of using language, such as the possibility of no instruction being available when students needed assistance, language issues, and inefficient explanations in their post-interviews.

How teachers used language in their classrooms. Throughout the entire observations, all six of the teachers strongly used translations within language representations. They all actively used translation between language and symbolic representations during all of the three observations. Furthermore, they also translated between language and other representations, such as manipulative, pictorial, or realistic ones. For example, in the first observation, Ben, Sara, Holly, and Pam actively used translations among language and other representations, such as pictorial or realistic ones. However, Ben, Holly, and Mary used them weakly or only in checking homework. In the second observation, all six of them actively used translations within and between language and pictorial representations. Sara, Nick, and Mary also strongly used translations between language and realistic representations. In addition, Ben, Nick, Holly, and Pam weakly translated between language and physical manipulative representations, and Sara strongly translated between language and virtual manipulative ones. In the third observation, Ben, Sara, Holly, and Pam strongly used translations between language and pictorial representations, and Nick weakly used them. All six of them used translations between language and realistic representations, and Sara especially used them strongly.

How teachers connect their beliefs about language with their practices. All six of the teachers described that language representations helped students build mathematical conceptual understanding, and there were no hindrances in the pre- or post-interviews. They all also emphasized translations within language in their beliefs about teaching and learning mathematics, and they showed little belief changes between the pre- and post-interviews. In their classroom practices, they actively used language representations, and mostly used written and spoken language at the same time throughout all of the three observations. All of them also connected these types of language with other mathematical representations, such as manipulative, symbolic, pictorial, or realistic ones, in all of the three observations. It seemed that the teachers used language representations in order to facilitate other representations.

Realistic representations. In this section, I explore the six teachers' beliefs and practices regarding the use of realistic representations described in the Lesh translation model. I also describe the connection between their beliefs and practices regarding the use of realistic representations.

What teachers believe about realistic representations. Ben, Sara, and Nick said that using realistic representations was very useful in their teaching because these representations helped students enhance their conceptual understanding in mathematics. Ben, Sara, Mary, and Pam stated that using realistic examples and problems could help students connect mathematics with their own lives, and Sara, Nick, Holly, and Pam elaborated that it could give students motivation to learn mathematics. Holly, Mary, and Pam said that it helped students understand the usefulness of mathematics in their future. In addition, Pam said that using realistic representations created effective learning environments. However, Sara said that if students could not understand the contexts, then it could be a hindrance. Holly said that there could be hindrances if the real data were too messy to use in the classroom, and there also could be a sampling issue when she modified data easily to use in the classroom. However, most teachers except for Holly said that there were not any hindrances in using realistic representations in their teaching, in the pre- or post-interviews. Table 5.5 summarizes how all six of the teachers discussed the helpfulness and hindrances of using realistic representations in their teaching.

Table 5.5

Teachers' Beliefs About Realistic Representations

	Realistic	School A				School B							
		Ben		Sara		Nick		Holly		Mary		Pam	
		Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Help	Deeper understanding	✓	✓	✓		✓	✓						
	Connection with math	✓	✓		✓			✓		✓	✓	✓	
	Exploration				✓								
	Motivation				✓	✓	✓	✓	✓				✓
	Math utility							✓	✓		✓		✓
	Effective learning											✓	
	Future relevance								✓				
Hindrance	Using irrelevant context			✓						✓			
	Sampling issue							✓	✓				
	Messy data/ numbers								✓				
	Calculation								✓				
	No hindrance	✓	✓		✓	✓	✓				✓	✓	✓

In sum, all six of the teachers said that using realistic representations facilitated students' learning mathematics. For example, these representations helped students develop their mathematical understanding, and helped connect some phenomena around their lives with mathematics. Using realistic representations also gave students motivation to learn mathematics, and it made them understand mathematical utility. Some teachers (Sara, Holly, and Mary) described several hindrances in the pre- or post-interviews, such as when they used irrelevant contexts and complicated data or numbers in class. It seemed that the teachers believed students should have experience with real contexts used in the classroom so that their understanding would be enhanced.

How teachers' beliefs about using realistic representations were changed. Sara, Holly, Mary, and Pam showed small belief changes about using realistic representations between their pre- and post-interviews. Sara said that real-world examples helped students' mathematical understanding only in her pre-interview, but she described many other things, such as connecting real-world situations with mathematics, exploring mathematical problems, and motivating students to learn mathematics in her post-interview. Holly emphasized in her post-interview that using this type of representation (i.e., solving real-world problems) was useful toward students' careers, while she said in her pre-interview that such a representation helped connect real-world situations with mathematics. Mary and Pam also elaborated in their pre-interview that they could connect real-world situations with mathematics, so it was very helpful for students. However, they stated in the post-interviews that mathematics teachers could teach the usefulness of mathematics in the world through solving real-world problems, and Pam also said that such problem solving gave students the motivation to learn mathematics in her post-interview. I summarized all six of the teachers' belief changes about the helpfulness of using realistic representations in Figure 5.9.

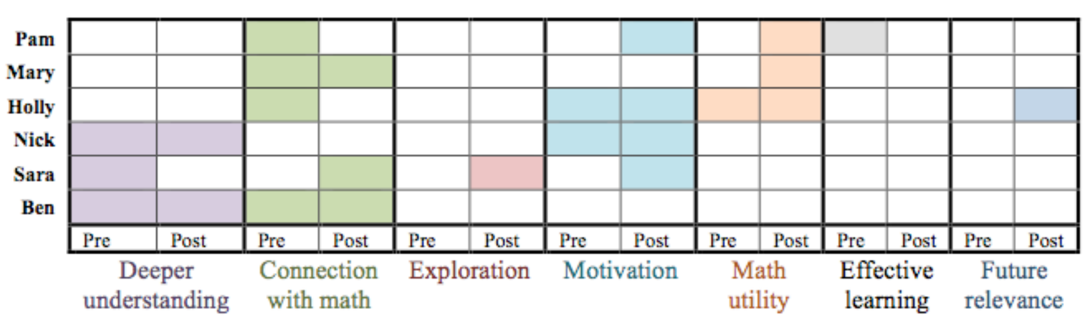


Figure 5.9. Summary of the teachers' beliefs about the helpfulness of using realistic representations.

Sara and Mary said in the pre-interviews that there was the possibility of a hindrance if students used them with irrelevant contexts, but in the post-interview, they did not mention any hindrances. Holly stated in her post interview that if teachers used messy data or complicated real numbers, it could hinder students' learning in mathematics. She also said that real numbers made it difficult for students to calculate problems in her post-interview. I summarized all six of the teachers' belief changes about the hindrances of using realistic representations in Figure 5.10.

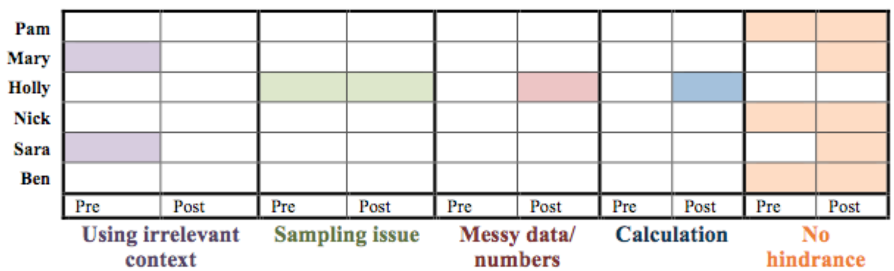


Figure 5.10. Summary of the teachers' beliefs about the hindrances of using realistic representations.

In sum, the teachers (Sara, Holly, Mary, and Pam) slightly changed their beliefs about using realistic representations between the pre- and post-interviews. Sara explained that realistic representations helped deepen students' understanding in her pre-interview, but she described in detail how and why she used realistic representation, such as its connection with mathematics, exploration, and motivation, in her post-interview. Mary and Pam emphasized mathematical utility when they used realistic representations in their post-interviews, while they pointed out the connection between real contexts and mathematics in their pre-interviews. Most teachers, except for Holly said that there were not any hindrances of using realistic representations in the post-interviews. In other words, the teachers thought that students should learn mathematics with realistic representations so as to understand the rationales of learning mathematics and to promote student learning in schools.

How teachers used realistic representations in their classrooms. All six of the teachers mostly used realistic representations in solving problems, and they sometimes used them to explain mathematical concepts. In the first observation, when most teachers, except for Mary used realistic representations, they translated from realistic to symbolic representations, but not vice-versa. However, they used translations within and between language and realistic representations; Mary only used them when checking homework. In the second observation, Sara, Nick, and Pam actively used realistic representations, and Pam strongly used translating between realistic and other representations, such as pictorial, symbolic, or language ones. Sara and Nick actively translated within and between language and realistic representations. Sara also translated from realistic to symbolic ones, while the other teachers did not translate between them. They strongly used translating from realistic to pictorial representations, while Nick weakly used translating from pictorial to realistic representations. In addition, Ben, Nick, and Holly weakly used translating between realistic and physical manipulative representations. In the third observation, most teachers, except for Ben strongly used translating within and among symbolic, language, and realistic representations, but Mary did not translate between symbolic and realistic representations. Sara, Nick, and Mary used translating within and between realistic and other representations, such as pictorial or language ones, but Ben translated only from realistic to pictorial ones while solving problems. Most teachers, except for Sara used the realistic representations that were provided in their textbooks.

How teachers connect their beliefs about realistic representations with their practices. The teachers elaborated on the importance of using realistic representations to learn mathematics in mainly four categories, such as deeper understanding, connection with mathematics, motivation, and mathematical utility. Most teachers, except for Holly said that there were no hindrances in the pre- or post-interviews. However, Holly described some hindrances of using them, such as sampling issues, messy data or numbers to use in class, and the difficulty in calculating them. In their classroom practices, the use of realistic representations was limited and mostly depended on the textbooks, even though some teachers used one or two real-world questions they created based on their own students' lives during the entire observations. Sara strongly used translations within and among realistic and other representations (e.g., pictorial, symbolic, and language) in all of the three observations. However, she used translating only from realistic to symbolic and pictorial representations, but not in the opposite direction from the first to the second observation. Ben, Holly, and Pam also used translating only from realistic to symbolic and pictorial representations, but not in the opposite directions in the first observation, and Ben also did the same translations during the third observation. That is, the teachers mostly used realistic representations in solving problems. It seemed that the teachers tried to build a deeper understanding, generate motivation, or connect mathematical concepts through practicing realistic problems.

Translation within or between mathematical representations. The following section will describe the six teachers' beliefs and practices regarding the use of translation within or between mathematical representations described in the Lesh translation model. I also describe the connection between their beliefs and practices regarding the use of translation within or between mathematical representations.

Teachers' beliefs about translations within or between representations. All six of the teachers said in both the pre- and post-interviews that translations within or between modes of mathematical representations were helpful for students to understand mathematical concepts or to enhance problem-solving strategies. Ben, Mary, and Pam said that seeing mathematical concepts and problems in multiple ways was very important to learn mathematics. They said in their pre- or post-interviews that they used translations within or between mathematical representations as a teaching strategy to show them in multiple ways. Most teachers, except for Nick said in the pre- or post interviews that there were not any hindrances of translations within or between mathematical representations. However, Nick said in both the pre- and post-interviews that there would be a hindrance if students spent a lot of time with multiple representations. Pam also said in her post-interview that there would be a hindrance because students "might be overwhelmed by the multiple representations." Table 5.6 summarizes how all six of the teachers discussed the helpfulness and hindrances of using translations within and between representations in their teaching.

Table 5.6

Teachers' Beliefs About Translating Within or Between Representations

	Translation	School A				School B							
		Ben		Sara		Nick		Holly		Mary		Pam	
		Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Help	Deeper understanding	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	Connection	✓											
	Multiple interpretations	✓	✓							✓			✓
	Problem-solving skills							✓	✓				
Hindrance	Time issue for curriculum					✓	✓						
	Overwhelmed												✓
	Lack of practice time										✓		
	May use inefficient							✓					
	No hindrance	✓	✓	✓	✓				✓	✓		✓	

While answering the interview question about using multiple representations, most teachers, except for Nick elaborated on their own teaching strategies on how they used translations within or between them in the pre- or post-interviews. Ben described how he used translations within or between representations, such as translating within pictorial representations (e.g., tables and graphs), between verbal or written language and pictorial representations in his pre-interview. He also talked about translation between language and pictorial representations or between symbolic and pictorial representations in his post-interview. The following excerpt illustrates Ben's descriptions about using translations within and between representations in the pre-interviews:

> *If a student could look at graphs, he could also prove it in a table. He could also prove it in written form and he could also prove it with the equations itself.*

…Sometimes when we do assessments, I will expect to say, 'Prove it with a graph.' And the next question is, 'Prove it with a table. What would this equation be of this relationship? Create a table that shows this relationship. Create a graph that shows that relationship.' So the kids see it in multiple ways… (Ben's pre-interview)

Sara also elaborated on her teaching strategies as to how she used multiple representations, in both her pre- and post-interviews. In her pre-interview, she explained it with using cards games, number lines, chips and visual, real examples in her classes. In her post-interview, she described the teaching strategies for teaching a linear algebra class in order to explain how to use multiple representations. The following excerpt illustrates Sara's descriptions about using translations within and between representations in the post-interview:

So I think the more representations that you can make, it's definitely going to help their learning and understanding. Some kids really struggle with the table stuff. And by just looking at it as a graph, it's really going to make that connection for them... giving them multiple representations gives them that other stepping stone to understand it. It helps to scaffold their understanding to that main idea that they need to know (Sara's post-interview).

Holly described her examples of using translations between mathematical representations, which helped students' learning in her class. She described her linear algebra class to give an example of using multiple representations. The following excerpt illustrates Holly's descriptions about using translations between representations in the pre-interview:

> *It may offer different ways to finish the problem. Approaching a problem one way, starting with an equation, for example, may not lead to an answer they can come up with. Maybe we haven't gotten to the level of solving required for the equation that they've built. So going to a different representation, a table or something, could actually allow them to compete the problem…(Holly's pre-interview)*

Mary talked about the example of translation within symbols with the mixed number, the decimal, and the improper fractions in her pre-interview. She said, "If they understand the connection between them, it's going to help their number sense." Mary mentioned the connection between tables, graphs, and equations in order to explain the hindrance of using multiple representations in her post-interview. The following excerpt illustrates Mary's descriptions about using translations within and between representations in the post-interviews:

> *How does it hinder them? I think some students are more comfortable with a table. If they have a table, they can plot the points and make the graph. If they just have the equation, I think some of them struggle to go from the equation to the graph because they haven't memorized that set of rules or concepts that are taking place. So having all of them and practicing with all of them is only going to help them…(Mary's post-interview).*

Pam gave an example with linear equations in her pre-interview. However, she did not give any examples in her post-interview. The following excerpt illustrates Pam's descriptions about using translations within or between representations in the pre-interview:

> *So if they understand the symbols with the equation and what those mean, they can connect that to the table, and then they can connect that with the actual graph. And, if possible, if they can connect that to a real-life situation, then they're going to understand it a lot better …(Pam's pre-interview).*

As shown above, Ben, Sara, Holly, Mary, and Pam described their teaching examples of translating within pictorial representations, such as graphs or tables to teach algebra in their class. Ben and Sara expanded upon translation between multiple representations, including pictorial representations. Holly, Mary, and Pam said that they used connecting pictorials with symbols (e.g., equations). Pam emphasized translating among pictorials, symbols, and realistic representations, as well.

The following section will provide the teachers' descriptions of using multiple representations in teaching mathematics, based on the teachers' responses in the first three interview questions about their beliefs of teaching mathematics; what the role of the mathematics teacher is, what or what not to teach, and how to maximize learning.

Teachers' descriptions of using mathematical representations in the first three interview questions. All six of the teachers talked about translating within or between mathematical representations when they answered the three interview questions about what teachers' roles are, what to teach or how to maximize learning. All six of them elaborated on translations within language representations to enhance students' mathematical learning as one of the main roles of mathematics teachers in the pre- or post-interviews. Sara and Holly also talked about using translations between manipulatives and symbols, respectively, as a role of mathematics teachers. Ben, Sara, Nick, and Pam said that the role of the mathematics teacher was to use various real-world examples or problems for their students. Pam especially emphasized the importance of translating within realistic representations in all of the first three interview questions. Sara, Holly, Mary, and Pam explained that they thought about various physical manipulatives or virtual ones when they decided what or what not to teach. Sara elaborated on the significance of translating within pictorial representations to teach mathematics in all of the three interview questions. Holly also said that she considered using various pictorial representations when she thought about what to teach and how to maximize learning in her post-interview.

Sara, Mary, and Pam described in the pre- or post-interviews that they were thinking about translating among multiple representations in the first three interview questions. Ben also believed that using various representations was the role of the mathematics teacher in his post-interview, and Nick said in his post-interview that teachers could help students maximize their learning if students worked in different ways. Holly said in her post-interview that connecting between language and symbols was one of the mathematics teachers' roles, and she considered translating between virtual manipulatives and pictorials when deciding upon her lessons. Sara elaborated that connecting between manipulatives and language was important in deciding what to teach in her post-interview, and connecting between pictorials and symbols helped to maximize students' learning in her pre-interview. Table 5.7 summarizes how all six of the teachers discussed translations between or within representations.

Table 5.7

Mathematical Representations in the Teachers' Beliefs About Teaching

Teachers	Instances of representational fluency	General Belief Questions for Teaching (Coded in representations, not considered other beliefs about teaching)					
		The role of the math teacher		What to teach or not		How to maximize learning	
		Pre	Post	Pre	Post	Pre	Post
All	Representation	M, P, L, R	L, S, R	M, P, L, WL, S	M, *M, P, L, R	P, S, L, R	P, S, L, R
Ben	Translation within representations	R	L			L	L
	Translation between representations		T				
Sara	Translation within representations	P, M	L, R	P, L	M, L	P, S	L
	Translation between representations	T	T	T	M & L	P & S, T	T
Nick	Translation within representations		R		R	L	L
	Translation between representations						T
Holly	Translation within representations	L	L, S	M	*M, P		P, S
	Translation between representations		L&S		*M&P	T	T
Mary	Translation within representations	L		WL, S	M, R, L		
	Translation between representations	T	T		T		T
Pam	Translation within representations	R	R, L		M, L, R	L, R	L, R
	Translation between representations	T	R&L	T	T		T

Note. M: Physical manipulatives, *M: Virtual manipulatives, P: Pictorials, S: Symbols, L: Language, WL: Written language, R: Realistic representations, and T: Translation among multiple representations (representations not identified).

In sum, all six of the teachers believed that using translating within each mathematical representation or between multiple representations was a key means to teach mathematics as mathematics teachers, to decide on teaching strategies, or to maximize students' learning. The teachers especially maintained that they considered translations within language representations, and translations between multiple representations to teach mathematics effectively. However, most teachers, except for Sara and Mary did not identify representations when they talked about translations between representations.

Belief changes about translating within and among representations. Ben, Holly, Mary, and Pam slightly changed their beliefs about translations between the pre- and post-interviews. All six of the teachers said that using translations within and among multiple representations helped students build their mathematical conceptual understanding, in both the pre- and post-interviews. Ben said in his pre-interview that these translations were intended for students to make connections between mathematical representations in class, such as writing a linear equation and then explaining why it is linear; thus, they believed it would enhance their understanding in mathematics. Holly stated in her pre-interview that students could waste time learning mathematics if they tried all representations, but she said in her-post-interview that there were not any hindrances. Mary said that students needed to interpret mathematical concepts in different representations such as tables, graphs, and equations in her pre-interview. Pam also said that the more students interpreted mathematical concepts in different ways, the more they would understand them in her post-interview. On the other hand, both Mary and Pam said in their pre-interviews that there were not any hindrances. However, Mary mentioned in her post-interview that there would be a hindrance if students did not have enough time to practice using all representations, and Pam said in her post-interview that some students might be overwhelmed by multiple representations. I summarized all six of the teachers' belief changes about using translations among multiple representations in Figure 5.11.

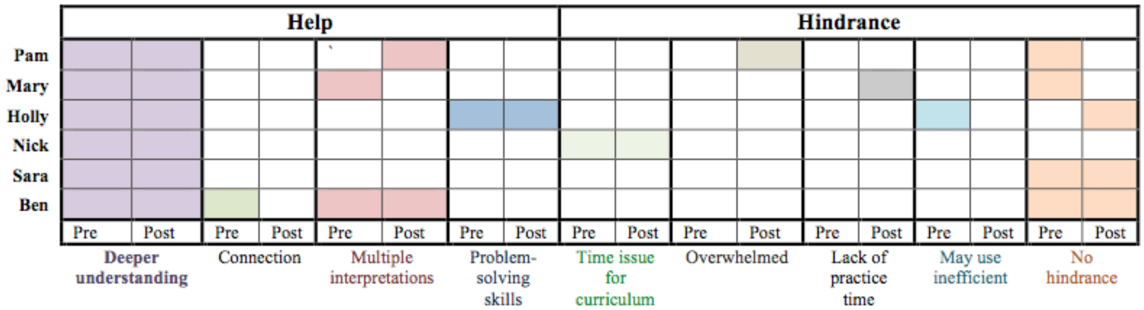
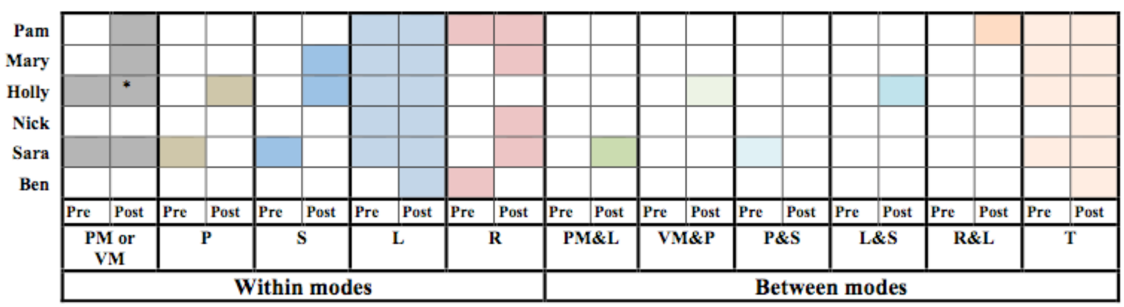


Figure 5.11. Summary of the teachers' beliefs about the helpfulness and hindrances of using translations among multiple representations.

The following paragraphs summarize what kinds of translations the teachers described, while they responded to the first three interview questions, such as the role of the mathematics teacher, deciding what or what not to teach, and maximizing student learning. Most teachers, except for Ben said that using translation within language representations was helpful for teaching mathematics in both their pre- and post-interviews, and Ben said the same thing in his post-interview. Sara, Nick, and Mary elaborated that connecting realistic representations with mathematics was a useful teaching means for students' effective mathematics learning in their post-interviews, and Ben elaborated the same sentiment in his pre-interview. Holly elaborated on translation within physical manipulatives in her pre-interview, but on translation within virtual manipulatives in her post-interview to teach mathematics effectively. Mary and Pam also described that they considered using many manipulatives to teach mathematics in the classroom during their post-interviews. Holly and Mary said that using translation within symbols was important to understand mathematics in the post-interviews, but Sara said the same idea in her pre-interview.

All six of the teachers also said that translations among multiple representations helped students better understand mathematics in the post-interviews, even though Ben and Nick did not say so in their pre-interviews. Sara elaborated on the effectiveness of translating between pictorials and symbols in her pre-interview, along with in her post-interview translation between manipulatives and language in her post-interview to teach mathematics. Holly also elaborated on the usefulness of translating between virtual manipulatives and pictorials, and between language and symbols in her post-interview. Figure 5.12 summarizes how all six of the teachers changed their beliefs about using translations between or within representations in the first three interview questions.



Note. PM: Physical Manipulatives, VM: Virtual manipulatives, P: Pictorials, S: Symbols, L: Language, R: Realistic representations, T: Translation among multiple representations, and * : Virtual manipulatives

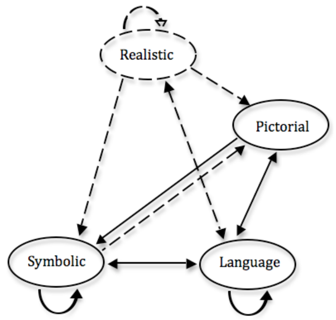
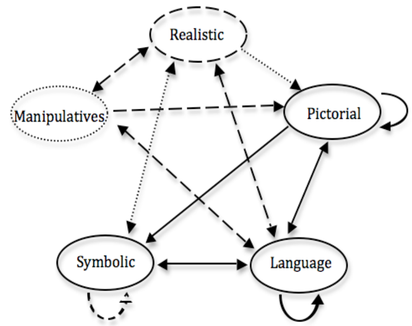
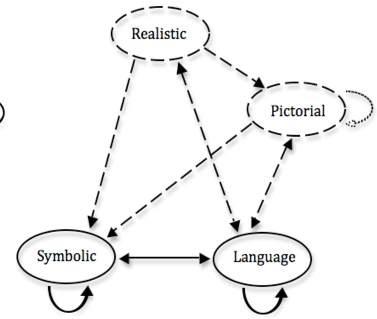
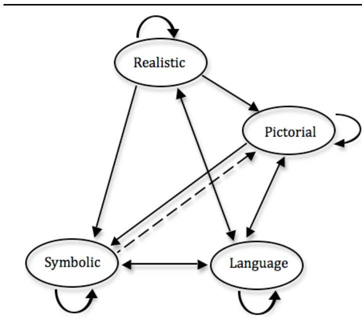
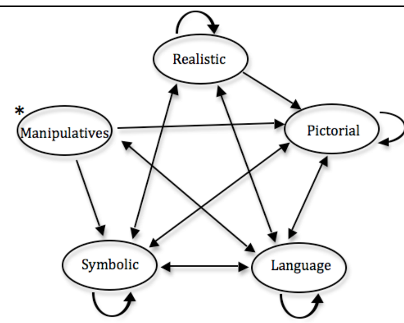
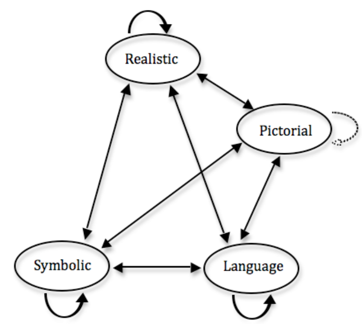
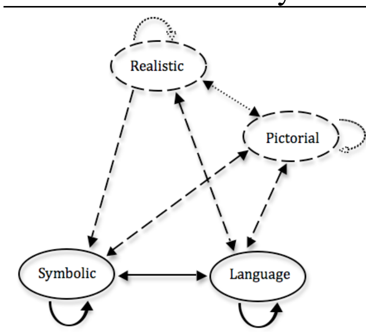
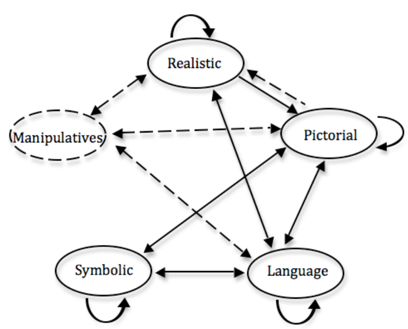
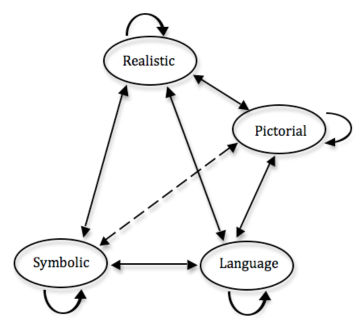
Figure 5.12. Summary of how all six of the teachers changed their beliefs about using translations within or between representations in the first three interview questions.

In sum, the teachers had positive beliefs about using translations within and between multiple representations to teach mathematics in both the pre- and post-interviews. All six of the teachers considered using them more, such as translating within manipulatives, symbolic, or realistic representations, and translating among multiple representations in their post-interviews versus in their pre-interviews. They also claimed that translation with language was important to teach mathematics in both their pre- and post-interviews.

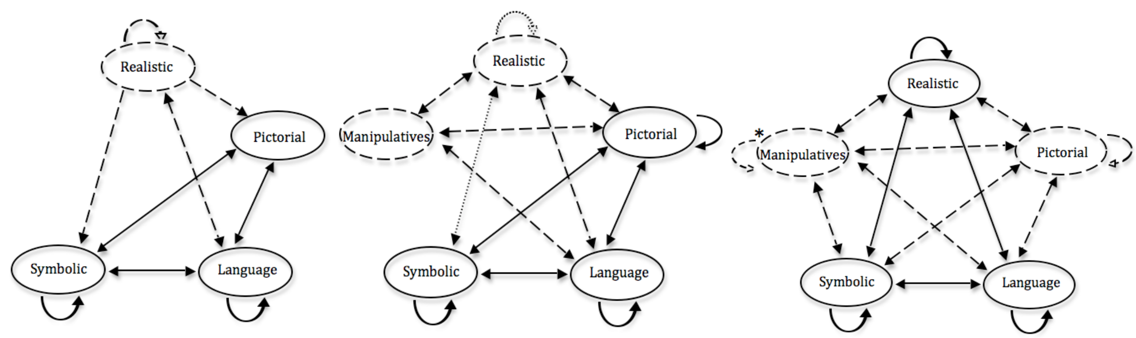
Translations between representations and their practices in the classroom. All six of the teachers elaborated that using translations among multiple representations helped students build their mathematical understanding, in both the pre- and post-interviews. Most teachers, except for Nick said that there were no hindrances to using them in their pre- or post-interviews. However, Nick said that there could be time issues for covering the curriculum they needed to teach if they used multiple representations every time. In their classroom practices, all six of the teachers mostly used translations between symbolic and language representations throughout all of the three observations. They all used translating within or between pictorial, symbolic, language, and realistic representations in the second and third observations. In particular, Sara actively used translations among pictorial, symbolic, language, and realistic representations in all of the three observations. For manipulatives, Nick, Holly, and Pam translated among physical manipulatives and other representations in the second observation, and Sara and Holly used translations among virtual manipulatives and other representations in the second and third observations, respectively. Table 5.8 contains a summary of using translations within and between mathematical representations from the data of the classroom observations.

Table 5.8

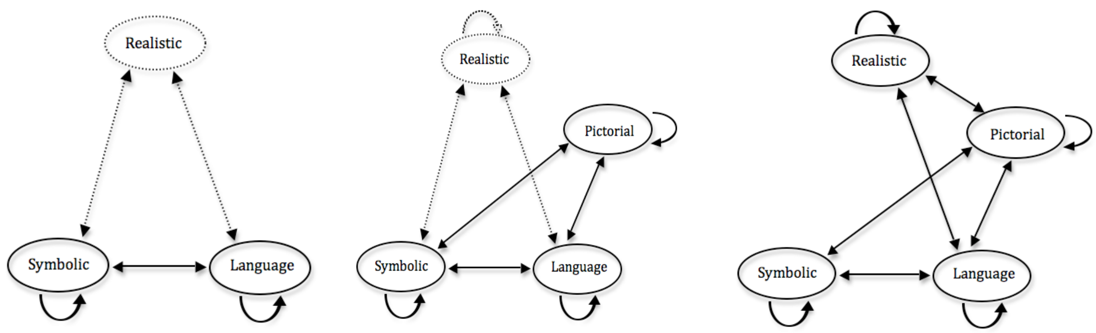
Summary of the classroom observations

First observation	Second observation	Third observation
		
Ben-Algebra	Ben-Geometry	Ben-Algebra
		
Sara-Geometry	Sara-Algebra	Sara-Probability
		
Nick-Algebra	Nick-Geometry	Nick-Probability

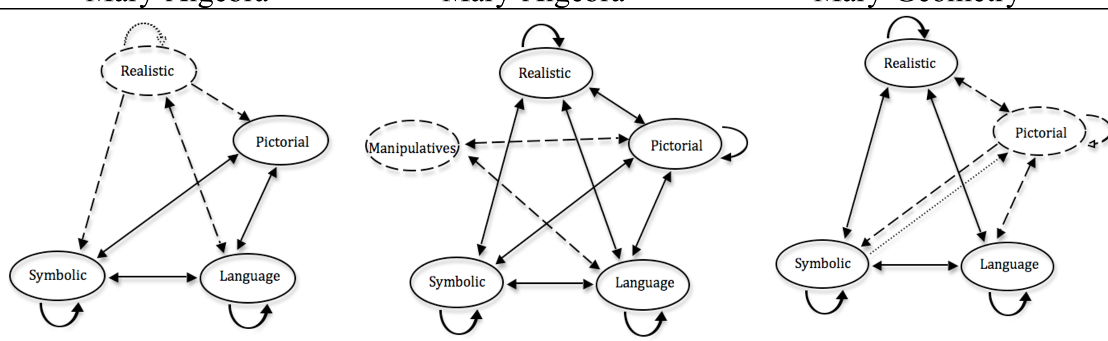
(continued)



Holly-Algebra	Holly-Geometry	Holly-Probability



Mary-Algebra	Mary-Algebra	Mary-Geometry



Pam-Algebra	Pam-Geometry	Pam-Probability

Note.
1. * means virtual manipulatives
2. Type of line:

One or more days teacher strongly use
One or more days teacher weakly use
One or more days used for reviewing the last class checking homework, solving warm-up problems or used in students' worksheets during class

In sum, all six of the teachers believed that using translations within and among multiple representations help students build deeper understanding. They also translated within and among pictorial, symbolic, and language representations in the classroom. In particular, they emphasized translations within language representations, and they actively used them in the classroom. On the other hand, in the post-interviews, more teachers thought of using physical or virtual manipulatives, but they used little them in the classroom. It seemed that the teachers needed to have a better understand of the effectiveness of using physical or virtual manipulatives for seventh-grade students so that they could implement them more in their classes.

In the following chapter, the results of this study that were related to using multiple representations in learning and teaching mathematics, will be discussed. The implications of the findings of the present study and the suggestions for future research about using multiple representations in mathematics will also be given in Chapter 6.

Chapter 6. Discussion, Implications, and Future Research

The purposes of this research were to examine mathematics teachers' beliefs about using mathematical representations that were described in the Lesh translation model and their belief changes toward using them after participating in a yearlong professional development program designed after the Lesh translation model, based on a theory of multiple representations. Furthermore, it aimed to reveal how their beliefs about using multiple representations, such as manipulatives (physical and virtual manipulatives), pictorial, symbolic, language, and realistic ones, that were described in the Lesh translation model were reflected in mathematics classes. The discussion section was presented with respect to each variable in this study.

This chapter is organized into three parts. First, a brief summary and discussion of the results are provided. Second, implications from the study are discussed. Fourth, recommendations for future research are discussed.

Summary and Discussion

Multiple representation studies in mathematics education have emphasized that students should learn mathematics in different ways. Through using them, they can build conceptual mathematical understanding, and can enhance their problem-solving abilities, as well as improve their mathematical performance (Brenner et al., 1997; Cramer, 2003; Lesh & Doerr, 2003). Concrete representations help in constructing a more abstract representation (Dreyfus, 1991; Markovits & Vachon, 1990; Pape & Tchoshanov, 2001). On the other hand, using multiple representations is important to teachers. By using them, teachers can create effective or active learning environments to help students build mathematical knowledge, and to help them think, communicate, and record mathematically. Teachers also use multiple representations to navigate students' ideas, understanding, or misunderstandings. NCTM standards (2000) have also suggested using multiple representations. These standards suggest that teachers and curricula provide students with opportunities to show what they know in multiple ways. Furthermore, students should be able to relate representations of concepts or procedures to one another, describe the results using multiple representations, such as graphs, tables, or other representations, recognize equivalent representations of the same concept, solve problems in multiple ways, and translate from the concrete to the abstract. Thus, students learn better through interactive experiences with multiple representations than through traditional teaching, where they are passive and receptive. However, students continue to learn mathematics through the traditional instructional approach.

In this study, I explored teachers' beliefs and their belief changes after having experience with multiple representations that were described in the Lesh translation model through a professional development program. Knowing teachers' beliefs is important in understanding teachers' practices and knowledge (Handal & Herrington, 2003; Wilkins & Ma, 2003). Belief studies have found that teachers' beliefs influence instructional practices and effective learning environments, which can improve students' achievement (Handal & Herrington, 2003; Thompson, 1992; Wilkins & Ma, 2003). If teachers maintain beliefs that are compatible with an instructional approach, such as the theory of multiple representations, then the implemented approach is more likely to be successful in their classrooms (Handal & Herrington, 2003). However, teachers' beliefs are not always consistent with the way they teach because of their lack of experience and knowledge.

In the following sections, I summarize and discuss my research questions about six teachers' beliefs about using multiple representations that are described in the Lesh translation model, and how they changed their beliefs and practices through gaining experience with multiple representations in a yearlong professional development program.

Research question 1.

How do the beliefs about teaching mathematics of teachers participating in a yearlong professional development program align with the teaching and learning aspects in the Lesh translation model?

Based on the interviews with six of the mathematics teachers at the beginning and at the end of the training, the teachers showed evidence of holding beliefs that aligned with the teaching and learning aspects of the Lesh translation model. For example, the teachers believed that students should learn mathematics in different ways, and should work with mathematical representations to develop mathematical conceptual understanding. All six of them said that translating among multiple representations helps student learning in mathematics. Thus, these results suggest that the teachers' beliefs about using multiple representations are aligned with the teaching and learning aspects of the Lesh translation model as a theory of multiple representations. Furthermore, in the first three interview questions about teaching, the teachers demonstrated via their beliefs that using physical or virtual manipulatives, language, and realistic representations is a powerful means of enhancing mathematics learning and teaching. All six of them stated that they were considering using translations among multiple representations in teaching mathematics in their pre- or post-interviews. The teachers considered using mathematical representations, especially physical or virtual manipulatives, language, and realistic representations, in preparing their lessons for maximizing students' learning. Thus, it seems reasonable to assume that the teachers may use multiple representations in the classroom because previous belief research has argued that teachers' positive beliefs about instructional strategies influence their classroom practices (Crawley & Koballa, 1992; Roehrig & Kruse, 2005; Speer, 2005).

The results of the rest of the seven interview questions may help clarify how the teachers believe about each representation, such as manipulatives, pictorial, symbolic, language, and realistic representations, as well as translating among multiple representations. All six of the teachers described how each representation, such as manipulatives, pictorial, symbolic, language, and realistic representations, helped or hindered students in learning mathematics. In this study, all six of them believed that manipulatives including physical and virtual ones were helpful in connection with pictorial representations during their pre- or post-interviews. Some studies have also indicated that manipulatives help visualize mathematical concepts (Bouck Flanagan, 2009; Lamberty & Kolodner, 2002). That is, the teachers' beliefs in this study were consistent with those of previous studies. All six of them also said that translation within symbolic and language representations helped deepen students' understanding in learning mathematics during their pre- or post-interviews. Many studies have also found that using language helps students develop their own mathematical understanding (Daniels & Anghileri, 1995; Topping et al., 2003; Yackel, Cobb, & Wood, 1991). In addition, Ben and Sara said that each representation helped students enhance their mathematical understanding in their pre- or post-interviews. I summarized the teachers' beliefs about the helpfulness of using mathematical representations, which were demonstrated by two or more teachers during their pre- or post-interviews, such as the following:

(1) Using manipulatives helped students (a) build deeper understanding (Ben, Sara, Nick, and Pam); (b) visualize mathematics concepts (all); (c) create active learning (Holly, Mary, and Pam); (d) be more motivated to learn mathematics (Holly and Pam); and (e) understand mathematics more effectively, particularly those who are lower achieving (Nick and Mary);

(2) Using pictorials helped teachers (a) build students' deeper understanding (Ben, Sara, and Pam); (b) communicate in mathematics (Ben and Mary); (c) connect mathematics with their students' thinking (Sara, Mary, and Pam); (d) create better instructional models (Nick and Mary); and (e) make students see patterns from the data (Holly and Pam);

(3) Using symbols helped students (a) build a deeper understanding (Ben, Sara, and Holly); (b) learn mathematics as alternative representations (Nick and Mary); and (c) communicate in mathematics (Nick, Mary, and Pam);

(4) Using language helped students (a) build a deeper understanding (all); (b) connect their thinking with mathematics (Ben, Sara, and Holly); (c) paraphrase mathematical

concepts (Sara and Mary); and (d) explain their own thinking (Holly and Pam);

(5) Using realistic representations helped students (a) build a deeper understanding (Ben, Sara, and Nick); (b) connect their thinking with mathematics (Ben, Sara, Holly, Mary, and Nick); (c) be motivated to learn mathematics (Sara, Nick, Holly, and Pam); and (d) understand mathematical utility (Holly, Mary, and Pam).

Therefore, these results regarding the helpfulness of using the five representations are consistent with the arguments of previous multiple representation studies, including the Lesh translation model (Cramer, 2003; Keller, 1993; Suh & Moyer, 2007). For example, the teachers believed that students could build conceptual mathematical understanding through using representations. They also believed that using representations, such as language and realistic representations, helps students connect with mathematics. That is, it helps them think and communicate their ideas mathematically. Previous studies have also found that using pictorial, language, and realistic representations helps students build their own understanding in mathematics (Topping et al., 2003; Yackel et al., 1991) and facilitates the development of mathematical concepts (Keller, 1993; Suh & Moyer, 2007). However, few studies have identified teachers' beliefs regarding the helpfulness of each representation described in the theories of multiple representations, such as manipulatives, pictorial, symbolic, language, and realistic representations. As shown above, the results of this study indicate an impact on the teachers' beliefs about the helpfulness of each representation described in the Lesh translation model as a theory of multiple representations.

On the other hand, some teachers expressed doubts about students being able to use manipulatives effectively. Nick and Holly said that students needed excessive learning time if they worked with manipulatives in the classroom. In particular, Pam claimed that seventh-grade students should be able to understand mathematics without manipulatives in her post-interview. Moreover, all six of the teachers stated that hindrances could result if students did not understand the meaning of mathematical symbols. Thus, it took time for students to be able to understand them. I summarized the beliefs about the hindrances of using mathematical representations, which were expressed by at least two or more teachers in their pre- or post-interviews, such as the following:

(1) Using manipulatives might lead to (a) excessive learning time (Nick and Holly); and (b) classroom management issues (Nick and Mary);
(2) Using pictorials might lead to slow progress for some students (Ben and Holly);
(3) Using symbols might lead to hindrances if there was a lack of understanding (all);
(4) Using language in teaching mathematics might lead to hindrances if there were language issues (Holly and Mary); and
(5) Using realistic representations might lead to hindrances when using irrelevant contexts (Sara and Mary).

There are a few studies about the hindrances of using each representation, such as pictorial, symbolic, and language representations, even though there are some studies about the hindrances of using manipulatives (physical and virtual ones) and realistic representations (Boaler, 1993; Midenhall et al., 2008; Uttal & DeLoache, 1997). The results shown above clarified teacher beliefs about the hindrances of each representation described in the Lesh translation model as a theory of multiple representations. The results may help teachers understand how to use multiple representations effectively while reducing the hindrances indicated in this study.

In sum, all six of the teachers said that using translation between multiple representations helped students enhance their mathematical conceptual understanding in both their pre- and post-interviews. Ben, Mary, and Pam stated that it was helpful for students to understand a mathematical concept with multiple interpretations. Therefore, the teachers' beliefs appeared to be aligned with the teaching and learning aspects in the Lesh translation model. Furthermore, some teachers distinguished between physical manipulatives and virtual ones. It seems that the teachers believed that virtual manipulatives were more effective to use in a large classroom than physical ones. Thus, they seemed to want to try various online activities, such as virtual manipulatives. Recent studies have also made efforts to determine the effectiveness of using virtual manipulatives (Moyer et al., 2002; Suh 2005). On the other hand, the teachers thought of some hindrances in using representations. I felt that their beliefs about hindrances might prevent the teachers from using representations in the classroom. Previous research has also argued that teachers' negative beliefs prevent them from implementing instructional models (Lloyd, 2003). Thus, teachers' belief changes toward new instructional models are required if they have negative beliefs about instructional models (Crawley & Koballa, 1992; Roehrig & Kruse, 2005; Speer, 2005).

In the following section, I summarize and discuss the teachers' belief changes after participating in a yearlong professional development program that was designed after the Lesh translation model as a theoretical framework.

Research question 2.

How do the math teachers' beliefs about teaching mathematics change through participating in a yearlong professional development program that is designed by using the Lesh translation model as a theoretical foundation?

All six of the teachers participated in a five-day teacher-training program during one academic year. This training in a professional development program was created with core features that were most important in influencing teachers' learning, such as a content focus, active learning, fostering coherence, duration, and collective participants (Bransford et al., 2000; Cohen & Hill, 2000; Desimone, 2009; Desimone, 2011; Garet et al., 2001; Loucks-Horsley & Matsumoto; 1999). Furthermore, the training was designed based on the Lesh translation model as a theoretical framework. The teachers who participated in the training had experience in using mathematical representations, and they discussed the impacts of using them in the classroom. Previous studies have found that teachers' beliefs could be changed, based on their experience and knowledge (Burton, 2003). The results from the data indicate that the teachers showed more belief adherence related to the Lesh translation model in their post-interviews, even though they continuously expressed some hindrances of using representations to teach mathematics. In the interview questions about teaching and learning mathematics (in the first three questions), the teachers emphasized the importance of mathematical connections within and among representations. In particular, all six of the teachers stated that translation within language and among multiple representations were important for mathematics teachers to decide upon the teaching topics and to enhance students' learning in their post-interviews. On the other hand, the teachers felt in their post-interviews in response to the first three interview questions that they would like to use more manipulatives in their teaching because it was a good means of engaging students, creating an interactive classroom, and making it easier for students to connect mathematical concepts with pictorial representations. They said that the use of manipulatives was a powerful way to ensure that all students, including middle-school students, can understand mathematical ideas in the post-interviews. Holly spoke about translation between virtual manipulatives and pictorials in her post-interview. However, only Sara talked about translations between manipulatives and language in her pre-interview. On the other hand, most teachers continuously elaborated on the hindrances of using manipulatives from the pre- to post-interviews. It seemed that they wanted students to master concrete models from elementary school, even though they believed that middle school students still needed manipulatives to promote deeper mathematical understanding. The results from the rest of the interview questions about using representations showed evidence of teachers' belief changes after gaining experience in using multiple representations through the professional development program.

In the interview questions about using representations, all six of the teachers showed belief changes about using each representation, such as manipulatives, pictorial, symbolic, language, and realistic representations between the pre- and post-interviews. For manipulatives, the teachers demonstrated that they held more positive beliefs about using them in the class during their post-interviews than during their pre-interviews. In particular, Mary said in her pre-interview that manipulatives would help low-achieving students, even though she did not use manipulatives in her class. However, in the post-interview, she elaborated on the positive benefits of using them, such as visualization and active learning, and she wanted to use them in her class. On the other hand, Holly emphasized virtual manipulatives more than physical ones in her post-interview. For language, Ben and Mary showed slight belief changes about using languages between their pre- and post-interviews. For example, they emphasized translation between pictorial and language representations in their post-interviews. However, the other teachers did not show any belief changes about the benefit of using them between their pre- and post-interviews. For symbols, most teachers, except for Pam demonstrated belief changes between their pre- and post-interviews. For example, Sara said in her pre-interview that symbols were mathematical terms, but in her post-interview, she said that these were alternative representations used in order to develop a deep understanding of mathematics. Nick believed in the pre-interview that symbols were necessary to learn mathematics, but he said that they were important to communicate with others about mathematics in his post-interview. For language, all six of the teachers emphasized that using mathematical language helped students build their mathematical conceptual understanding in their post-interviews, but only Ben, Sara, and Nick said so in their pre-interviews. There were also belief changes for realistic representations. For example, Sara and Pam highlighted motivations, and Mary and Pam emphasized mathematical utility in their post-interview, but not in their pre-interview. However, there was a slight belief change in terms of using translation among multiple representations, such as multiple interpretations between their pre- and post-interviews. All six of them emphasized that translations among them helped students develop a deep understanding of mathematics in their both pre- and post-interviews. Therefore, the teachers had more positive versus negative beliefs about using each representation after having experienced the multiple representations from the professional development program in this study. That is, they talked more about the helpfulness versus hindrances of using representations in the post-interviews. On the other hand, these results of teacher belief changes could be evidence of the argument from previous studies, in that teachers' beliefs can be changed by their experience and knowledge (Burton, 2003). In addition, this study found that a professional development program designed based on core features, such as content focus, active learning, fostering coherence, appropriate duration, collective participants, data driven by students, and research-based models could be a means to improve teachers' beliefs, which can influence their instructional practices.

The following section will summarize and discuss the use of multiple representations, based on the results of the classroom observations.

Research question 3.

How are teacher beliefs about the Lesh translation model reflected in their classroom practices?

Teacher beliefs shape the way in which a teaching method is implemented (Lumpe et al., 2000; Pedersen & Liu, 2003). Moreover, when teachers have negative beliefs about an instructional approach, the approach is generally not effective for their students (Handal & Herrington, 2003). However, it is not well known how belief changes about teaching and learning impact teachers' practices in the classroom. The results from all of the data showed how teachers' beliefs are reflected in their classroom practices.

The complete Lesh translation model is provided in Figure 2.1, which explains what fully using multiple representations would look like. In this study, manipulatives included both physical and virtual manipulatives to fully explore the teachers' classroom practices. Recent studies have identified the effectiveness of using virtual manipulatives, as well as using physical ones in the classroom (Suh, 2005; Suh et al., 2005). The results of the teachers' beliefs in this study also show evidence that studies about representations need to consider both physical and virtual manipulatives in mathematical representations.

In the results from the data about the teachers' beliefs, the teachers believed that using mathematical representations was critical to teach and learn mathematics. All six of the teachers elaborated that translation among multiple representations helped students enhance their mathematical conceptual understanding. In the results of the classroom observational data, they used translations within and among pictorial, symbolic, language, and realistic representations. However, some teachers used translating from realistic to symbolic and language representations while solving problems, but not when translating in the opposite direction. In addition, some of them used more translating from pictorial to symbolic representation than translating from symbolic to pictorial ones. On the other hand, all six of the teachers used few manipulatives in their classes throughout all of the three observations. However, the teachers developed their beliefs about using manipulatives throughout the three observations. That is, they developed positive beliefs about using manipulatives more in their post-interviews than in their pre-interviews. In the results of the classroom observations, some teachers weakly used physical manipulative representations during the second observations, but they did not use them in the first or third observations. In particular, Mary said that she wanted to use manipulatives in her class during the post-interview, but using manipulatives was not observed in any of her three observations. In addition, Holly believed that virtual manipulatives could be helpful in learning mathematics while reducing hindrances associated with using manipulatives. Holly used physical manipulatives in the second observation, but she used virtual ones in third observation. It seemed that the teachers wanted to use manipulatives, but they still thought about the hindrances of using them in their classes. It seemed that little use of manipulatives led to little use of translations within manipulatives and between them and other representations. It appeared as though the teachers needed more time or experience to use manipulatives in their classes, even though they had changed their beliefs toward implementing them. In other words, it seems that the teachers need more time to fully reflect on their beliefs about the Lesh translation model in their classroom practices. Previous research about beliefs and practice has also argued that teachers' belief changes precede changes in their practices (Czerniak & Lumpe, 1996). The teachers slowly tried to implement new teaching models after they started to develop positive beliefs about them. They also did not spend much time when implementing these new models. It seems that the teachers wanted to try to revise them for their own students in order to reduce any hindrances.

In the results of the teachers' beliefs, the teachers emphasized the importance of translations within and among pictorial, symbolic, and language representations. In the results of the classroom observations, they also actively used translations within and among them. Furthermore, the teachers believed that translation between pictorial and language representations was very helpful in teaching mathematics. All six of them actively used translations between the representations. All six of them also emphasized students' understanding of the meaning of mathematical symbols to learn mathematics effectively. The results of the classroom observations showed that they used translating within and between written and spoken language representations so as to enable the students to understand the meaning of mathematical symbols. Furthermore, the teachers believed that using language representations helped students explain their own thinking, and made them use their own words in learning mathematics However, when the teachers asked questions while solving mathematics problems, they usually wanted to know what answers the students had, but inquired little about the students' mathematical reasoning. It seemed that the teachers needed to improve their questioning skills in their classes. These questioning skills can help students build their mathematical understanding that the teachers described in their beliefs about the helpfulness of using language representations, so that their teaching via language representations can better influence student learning. Previous research has also argued that teachers need to build questioning skills to improve students' understanding in mathematics (Borko & Livingston, 1989; Burns, 1985). Thus, teachers need to practice instructional skills so that their beliefs about new instructional models fully reflect their practices.

On the other hand, in the results of the teachers' beliefs, the teachers expressed various advantages about using realistic representations, and some teachers talked about the hindrances of using them. In their practices, they used some realistic representations in their classes, and they also translated within and between realistic and other representations. However, they mostly depended on their textbooks to use real-world problems, and they used translation within them one or two times during the observations. The real-world problems in the textbook would not always be related to students' experience and their lives. Research has argued that using realistic problems sometimes fails to encourage student learning when it is not related to students' experience or their lives (Boaler, 1993). Thus, it seemed that the teachers needed to use various contexts in their problems to relate with their own students' experience and their lives in their practices. Indeed, Sara believed that using irrelevant contexts might be a hindrance in using realistic representations in her pre-interview, but she did not explain any hindrances in her post-interview. In the classroom observations, Sara actively used translating within and between realistic representations, and some real-world problems she used were related to the students' school lives. That is, she tried to use realistic representations with relevant contexts. It seems that teachers can use various real-world mathematics problems that are related with their own students' lives if they believe that they need to use those real-world problems.

In sum, teacher beliefs about the Lesh translation model appeared to be mostly reflected in their classroom practices. However, some beliefs about using representations, such as manipulatives and language, were not fully reflected in their teaching. Such a practice might need more time to be implemented in classes. They also might need to develop some instructional skills, such as questioning skills, so that their beliefs fully reflect their practices. On the other hand, the development of teacher beliefs about using realistic representations, such as using relevant realistic representations with their own students' experience, is imperative in order to implement the representations effectively. In addition, teachers' beliefs toward new instructional models may be developed by acquiring experience though well-designed professional development programs.

Limitations

Since this study was a case study of six seventh-grade mathematics teachers from two schools, the generalizability is limited. The results may not be the same with different teachers from different schools. In addition, the researcher observed different mathematics topics, such as algebra, geometry, and probability during the academic year when the data were collected. Moreover, the professional development program was focused on algebra, especially rational numbers. For the classroom observations, this study only focused on teachers' instruction, not students' work or their learning during the classes. Thus, this study is limited to investigating students' individual work during classes, which may have been related to the teachers' instruction for this study.

However, the study found that the teacher beliefs were aligned with the theory of multiple representations from the Lesh translation model. The rich description involving the teachers' beliefs in this study shows the benefits of using translations within and among representations in the classroom. Furthermore, the beliefs that the teachers developed through having experience with multiple representations helped the teachers align their practices more with the theory of multiple representations.

Implications

The Principles and Standards for School Mathematics (NCTM, 2000) emphasize the role of representation in learning mathematics. Research has shown a critical impact on using representations and interacting within and among the representations for students' mathematical conceptual understanding (Cramer, 2003; Goldin & Shteingold, 2001; Perry & Atkins, 2002). On the other hand, researchers argue that greater understanding of teachers' beliefs is essential to the improvement of instructional practices (Lumpe et al., 1998). Research has found that changes in teachers' beliefs precede changes in their teaching practices in implementing a particular innovation that does not initially conform to their prior beliefs (Richards et al., 2001). Teachers' experience and knowledge help them improve their beliefs (Burton, 2003), and professional development programs are a means of building experience and knowledge. Therefore, it is important for educators to understand teacher beliefs about using mathematical representations when developing instructional materials, activities, and instructions that can be used by teachers in the classroom.

This study classified the teachers' beliefs about using physical and virtual manipulatives, pictorial, symbolic, language, and realistic representations. These classifications are likely to support the idea of what educators need to consider in developing instructional tools, and using representations to teach mathematical concepts. If educators create classroom instructions and activities that reflect teachers' beliefs about using physical and virtual manipulatives, pictorial, symbolic, language, and realistic representations in a particular mathematical context, teachers will be willing to implement them in the classroom. Teachers' beliefs lead their teaching practices in implementing a particular teaching and learning model (Richards et al., 2001). On the other hand, there is a lack of research on manipulative use in middle-school classrooms, and there would seem to be a need to develop effective physical and virtual manipulatives for middle-school students. The teachers had doubts in using physical manipulative representations in the classroom because of some perceived hindrances of using them, such as management and time issues, and thinking of graduating from concrete models for middle-school students. However, they also believed that physical manipulatives had benefits for middle-school students, if implemented efficiently. Furthermore, the teachers also believed that virtual manipulatives instead of physical ones could be more helpful for students to teach mathematics without hindrances. Researchers have also argued that virtual representations are more accessible for learning mathematics, in that they immediately connect between concrete models and symbolic representations for middle-school students (Clements, 1999; Meira, 1998). However, other researchers have indicated that physical manipulatives and virtual ones have different functions in teaching and learning, and that both types should be used in middle schools (Suh & Moyer-Packenham, 2007; Takahashi, 2002). Thus, educators might need to create two types of activities, physical and virtual manipulatives, and make connections between them to teach a particular mathematical concept for middle-school students. On the other hand, teachers might need to understand how they can help students understand mathematical symbols to avoid the hindrances of learning mathematics. The question about how teachers can help students understand symbols for effective teaching and learning mathematics needs to be further addressed in research.

For language representations, most teachers asked what answers the students had rather than how and why they found their answers. However, the teachers believed that they should ask effective questions to improve students' mathematical reasoning. Therefore, this result suggests that teachers need to improve their questioning skills in order to develop students' reasoning. For realistic representations, the teachers believed that this representation helped students' learning in mathematics. In the classroom observations, the teachers also tried to use them in class. However, most teachers depended on examples and problems in the textbooks, so they seemed not to effectively use them for their own students. The examples and problems were not relevant to the students' lives. The teachers' use of textbooks was not only for realistic representation, but also for other representations, such as pictorials and symbols. Researchers have noted that textbooks significantly affect teachers' instruction and involve various mathematical representations. However, these representations are isolated, and the connections among them are sometimes not valued (Brenner et al., 1995; Reys, Reys, & Chavez, 2004). There still exist the need for guidelines in terms of how districts choose their textbooks, which can be effectively used and easily adapted by teachers in the classroom.

This study showed the benefits of using a professional development program for changing teachers' beliefs into aligning with the Lesh translation model as a multiple representation theory. Researchers have also found that professional development programs, including core features such as content focus, active learning, fostering coherence, duration, and collective participants, help teachers improve their beliefs and knowledge in order to enhance student learning (Desimone, 2009; Garet et al., 2001). The professional development program used in this study included core features, as well as involving data driven by students and research-based models. Thus, the program also provided an experience and discussion for the teachers to see the importance of developing conceptual understanding using mathematical representations. After the experience in the professional development program, the teachers had their belief changes about teaching and learning mathematics, which were more in line with the Lesh translation model. The results seems to indicate that the experience of the research-based models in teaching and learning, which provides teachers with an understanding of how to build students' conceptual understanding, as well as the core features presented by research, is important for their belief changes to improve their teaching models and to become in line with research-based learning and teaching models. However, we as mathematics educators still need to develop various teacher professional development programs to help teachers improve their beliefs toward effective teaching theories and models.

Through the classroom observations, this study has found how teachers' beliefs affect their classroom practices. Research has also noted that instructional practices are dependent on what they believe effective teaching entails (Speer, 2005). In addition, research has identified that changes in teachers' beliefs precede changes in their teaching practices in implementing a particular innovation that does not initially conform to their prior beliefs (Richards et al., 2001). In the classroom observations, the teachers used translations within and among pictorial, symbolic, language, and realistic representations more in the second and third observations than in the first one. The results of the teachers' beliefs also showed that they believed those representations were very helpful in helping students improve their mathematical understanding, in both the pre- and post-interviews. These findings tend to point toward the relationship between teacher beliefs and their classroom practices in the previous research. However, in the case of manipulatives, the teachers showed belief changes toward using manipulatives. They stated that they wanted to use many manipulatives. In the classroom observations, however, there were few uses of manipulatives in the classrooms. They also used few translations within and between manipulatives and other representations. It seemed that they needed more time to change their classroom practices, even though the teachers changed their beliefs about teaching and learning. Thus, educators need to conduct more research to help teachers efficiently change their practices in the classroom after changing their beliefs.

Future Research

There are several areas of future research that can develop this study. Future research could explore the relative relationship between different types of teacher professional development programs and teacher changes in their beliefs and instructional practices. Additional next steps could include exploring the links between teacher professional development programs, teacher belief changes, and student achievement. Gaining a better understanding of the relative impacts of these programs on raising student achievement will provide useful information to students, school leaders, and policymakers.

Future research could also focus on a longitudinal study to examine the interactions between beliefs and practices, and a study on the impacts of professional development programs on teachers' belief changes for all grades of teachers. Furthermore, research could explore more effective activities involving mathematical representations, physical or virtual manipulatives for K-16 grade students, and a study on how these activities help teachers develop their teaching models, and also how they help students develop their conceptual understanding. For example, a future study could explore the development of teachers' beliefs before, after, and one year after professional development programs. This longitudinal study could provide a valuable understanding of teachers' beliefs and their changes toward the purpose of professional development programs. A longitudinal study could also look at how teachers' beliefs that were changed by such professional development programs affect the development of their teaching models in their lessons during one academic year after these programs. Such studies can shed light on the interaction among teachers' belief changes, professional development programs, and their classroom practice changes.

Studies of mathematical representations are still an interesting area for future research in all grades. Furthermore, research could focus on how students believe that using mathematical representations help or hinder them in building their mathematical understanding during middle and high school, as well as during elementary school. In this study, we have already identified teachers' beliefs about using representations. Furthermore, future research could focus on a comparison between students' and teachers' beliefs regarding their use. In addition, we could conduct more research about how students' experiences of using multiple representations in elementary education affect their achievement in middle-school education as a longitudinal study. We could also investigate how elementary teachers' experiences of using multiple representations affect their teaching strategies in different subjects.

Conclusion

This study clarified teachers' beliefs about the helpfulness and hindrances of using each representation, such as manipulatives, pictorial, symbolic, language, and realistic representations, which was described in the Lesh translation model. This study found that teacher beliefs about hindrances could prevent them from using representations in the classroom. Teachers should have positive beliefs toward new instructional models so that these models can be successfully implemented in the classroom. This study suggests that a well-designed professional development program can help teachers change their beliefs. Furthermore, this study found that changed teachers' beliefs most likely lead to changes in their practices in the classroom. However, changing teachers' practices requires more time after their beliefs are changed. In addition, this study found that teachers sometimes needed to improve their instructional skills in order to reflect their beliefs in their practices.

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Appendices

Appendix A. Interview Protocol

Teacher Beliefs Interview*

1. What is the role of the mathematics teacher?
2. How do you decide what to teach or what not to teach?
3. How do you maximize student learning in your classroom?
4. How do you use mathematical representations in your teaching?
5. How does student use of manipulatives such as concrete models (counting blocks, geometric shapes, or algebraic tiles) help or hinder their learning of math?
6. How does student use of pictures, tables, or graphs help or hinder their learning of math?
7. How does student use of symbols help or hinder their learning of math?
8. How does student explanations of mathematical ideas (either orally or in writing) help or hinder their learning of math?
9. How does student connection of mathematics to real world situations help or hinder their learning of math?
10. How does student use of multiple representations around the same concept help or hinder their learning of math?

Appendix B. Observation Protocol

The classroom observation protocol was adapted from The Oregon Teacher Observation Protocol (OTOP; Morrell, Wainwright, & Flick, 2004).

I. Background Information

Teacher Name		School				
Subject observed		Grade level				
Observation is (circle or bold one)	In-field	Out- of-field				
Date		Start time		End time		
Traditional/Block		Meet 5 days or 3-4 days				
Observer		Observation# (circle or bold one)	1	2	3	4
Number of students in class:						
Brief description of students in class:						

II. Contextual Background and Activities

A. Objective for lesson (ask teacher before observing):

B. How does the lesson fit in the current context of instruction (e.g. connection to previous or other lessons)?

C. Classroom setting: (space, seating arrangements, etc. Include a diagram, if possible).

D. Any relevant details about the time, day, students, or teacher that you think are important? (i.e.: teacher bad day, day before spring break, pep rally previous hour, etc.)

III. Detail log/transcript of the classroom observation (indicate time when the activity changes)

Time: change topics, teaching strategies, or activities	Observation Notes (Note: *=Critical Incident)	Check lists the teacher uses (Note: M*=Virtual manipulatives)					
		R	M/ M*	P	S	L	T
Start time –							
End time:							