Essays on Macroeconomics

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Dedication

I dedicate this dissertation to my parents.

Abstract

The dissertation includes two essays.

In the first essay I try to explain the slow recovery of the U.S. economy after the 2008-2009 recession. I develop a theory in which the slow recovery is due to continuing weak consumption demand arising from slowly resolved aggregate uncertainty. An exogenous belief shock creates uncertainty about firms' credit availability. This translates into income uncertainty at the household level because firms have to borrow from banks in order to hire workers. In response, household demand is low, which in turn causes firms to borrow little and produce little. This low level of economic activity impedes the resolution of uncertainty regarding credit availability. Therefore, the economy stagnates with high uncertainty, low demand, and low output, even though there is no shock to fundamentals. The resolution of uncertainty is not efficient because households do not fully internalize the effect of increasing their own demand on improving the quality of information. This leaves room for government intervention. Quantitatively my model generates a slow recovery, which is comparable to U.S. employment data. Model predictions are also consistent with U.S. county level data: counties with cheaper access to household credit have higher employment but all counties recover at the same rate.

In the second essay, which is co-authored with Yun Pei, we try to understand the impact of sovereign default on the lending countries. We develop a model in which banks in the lending countries can buy foreign sovereign bond, provide loan to domestic firms and borrow from domestic households. We find that a sovereign default in foreign countries can result in an output drop in the lending countries, because the default can worsen the balance sheet of banks. Therefore default risk in foreign countries can be transmitted to output risk in the lending countries. However, this does not mean that the governments in the lending countries should limit the purchase of foreign sovereign bond. A tax on the purchase of sovereign bond reduces the output fluctuation but also decreases the welfare of domestic bankers and households.

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Chapter 1

Slow Resolution of Uncertainty and Sluggish Recovery

1.1 Introduction

The recovery following the 2008-2009 recession has been disappointing. From July 2009 to April 2012, the civilian unemployment rate dropped from 9.5% to 8.1% – a reduction of only 1.4% over the three-year recovery period.[1] Given that the unemployment rate was 4.7% in November 2007, the month before the recession started, a natural question is: what is causing the slow recovery?

Recent empirical studies (e.g. [1]) show that weak household demand plays a big role in impeding the recovery from the 2008-2009 recession. In addition, the Survey of Professional Forecasts shows that there was a substantial increase in uncertainty about future income starting on the eve of recession, and that this uncertainty persists.

In this paper I develop a theory in which continuing weak household consumption arising from slowly resolved aggregate income uncertainty leads to the slow recovery. The resolution of uncertainty is slow because information is endogenous and the learning process is inefficient. In my model, output is driven by household demand and working capital constraints that force firms to borrow from banks to hire workers. Uncertainty

[1] According to the National Bureau of Economic Research, the 2008-2009 recession ended in June, 2009. The unemployment data is from FRED, Federal Reserve Economic Data, Federal Reserve Bank of St. Louis: Civilian Unemployment Rate (UNRATE); Bureau of Labor Statistics. The data is available at `http://research.stlouisfed.org/fred2/series/UNRATE`; accessed September, 2012.

about firms' credit availability leads to households' uncertainty about income. Income uncertainty suppresses household consumption demand. In response to low household demand, firms do not require much credit to hire workers. But modest credit demand means that households cannot infer whether firms will be able to get enough credit from banks to achieve a higher level of production and employment. Therefore, the economy stagnates with low demand and continued uncertainty, even if there is no realized deterioration in firms' credit availability. When households are financially constrained, the precautionary saving motive is more pronounced and the recovery is even slower. In my model, higher aggregate household demand leads to better information and faster resolution of uncertainty. However, individuals do not fully internalize the value of increasing their own demand on improving the quality of information, which in turn slows down the recovery and leaves room for government intervention.

The benchmark model includes a large number of identical households. Households make consumption decisions before they know the contemporaneous employment rate. As in [2], labor market matching is random and firms hire the exact number of workers to fulfill consumers' demand. The information friction in making consumption decisions and the labor market friction imply that output in my model is driven by household demand.

Firms face working capital constraint: they have to hire labor before production actually occurs. This working capital constraint requires firms to borrow from banks, and credit availability thus acts as the upper bound on the scale of production. Output is distributed to households in the form of labor income. Thus uncertainty regarding firms' credit availability leads to households' uncertainty about income.

In my model, a recession is triggered by a belief shock that increases the uncertainty about firms' credit availability. In response to increased uncertainty, households cut consumption demand. Because in my model output is determined by household demand, firms cut hiring as well as borrowing accordingly and a recession emerges. The speed of recovery depends on the rate at which uncertainty is resolved.

The resolution of uncertainty is slow for two reasons. First, the quality of information is endogenous. The uncertainty about firms' credit availability can be resolved if and only if firms borrow from banks. The amount firms want to borrow increases with household demand. A rapid increase in loans helps households adjust the lower bound of their beliefs about credit availability quickly. Therefore, observed demand and output today, which are endogenous, act as the information that households use to update beliefs in the future. When households choose low demand because of high initial uncertainty, the quality of information is low. Another reason the resolution of uncertainty is slow is because learning is inefficient. The quality of information is determined by aggregate household demand. However, each household does not internalize the effect of increasing its own demand on the quality of information. Therefore, the social value of increasing demand is larger than private value. This leaves room for government intervention. I show that a consumption subsidy schedule can improve the welfare of households and speed up the recovery.

When it is costly for households to get access to household credit, the resolution of uncertainty and recovery are slower. This is because the precautionary saving motive is stronger and household demand is weaker. Quantitatively my model generates a slow recovery, which is comparable to U.S aggregate employment data.

A natural question is whether my theory can also explain the recovery patterns at the county level. I extend the benchmark model to include counties that are heterogeneous in their access to household credit markets and basket of consumption goods. In response to uncertainty about aggregate credit availability, households living in counties with cheaper access to household credit can better insure against income uncertainty. Thus, they are willing to demand more goods. High local demand results in a large increase in local non-tradable sector employment and a tiny increase in tradable sector employment in all counties as in [1]. Thus, counties with better access to household credit markets have higher employment than other counties during the recovery. Meanwhile, the increase in demand relies on the evolution of beliefs about uncertainty. Because households in different counties share the same information and beliefs, all counties recover at the same speed. I show that these predictions are consistent with U.S. county level data.

The main contribution of this paper is to provide a novel explanation of the slow recovery from the angle of weak household demand and slowly resolved uncertainty.[2]

There is a long line of literature based on [5](for more recent examples, see [6]; [7];

[2] [3] and [4] also build models focusing on household side. But they focus on how tightened household credit constraint triggers the recession. In my paper, I focus on the slowly resolved aggregate uncertainty and its impact on slow recovery.

[8]) that tries to explain the gradual recovery after the bust. In these studies, firms' borrowing constraint is binding so that firms cannot hire more workers or increase investment. Thus, the recession and slow recovery come from the supply side. In my paper, firms' credit availability does not change. Firms are not willing to hire more workers because consumption demand is weak. Another line of literature (e.g. [9]) suggests that the slow recovery comes from some real permanent changes in economic fundamentals.[3] [12] builds a model in which the recession is triggered by a damage to capital stock. With sticky wages, the loss of capital has the same effect as a permanent shock to total factor productivity. However, in my paper, the slow recovery comes from slowly updated beliefs instead of a realized permanent change in fundamentals. Although some other studies (e.g., [13]; [2]) also focus on weak household demand in explaining the slow growth of output, they rely on households' pessimistic beliefs about other households' decisions. In their studies, aggregate states are determined by whether households can communicate with each other to convince others about their own decisions. My paper nests the self-fulfilling prophecy of these studies. However, the slow recovery result in my paper does not rely on the self-fulfilling prophecy. Instead, in my paper aggregate states are determined by households' beliefs about fundamentals (credit availability). In particular, I develop an algorithm that computes the equilibrium with the fastest recovery path among multiple equilibria arising from household coordination problems. I show that the recovery speed in this particular equilibrium is still slow.

Another contribution of this paper is to provide a model in which the resolution of uncertainty is slow because of endogenous information and inefficient learning. A consumption subsidy schedule can speed up the resolution of uncertainty and improve welfare. My paper is closely related to [14], [15], [16], [17], [18] and especially [19]. In [19], firms have to learn aggregate productivity and the quality of information is positively correlated with economic levels. Therefore, firms are more cautious in recession and recovery is slow. My paper is different from their paper along three dimensions. First, [19] focus on firm's behavior but I focus on household demand. Second, in [19], information flow is partly exogenous because higher productivity can directly increase output. However, in my model information flow is completely endogenous: the quality of information is uniquely determined by household demand. Third, my model does

[3] Other recent related studies include [10] and [11].

not rely on noise or private information, which are typically the key reasons why uncertainty cannot be resolved in other studies.[4] This paper is also related to studies that explore the macroeconomic implications of uncertainty shocks during the recession and recovery.[5] In those studies, uncertainty persists exogenously. However, in this paper I focus on the endogenously slow resolution of uncertainty and its macroeconomic implications.

The final contribution of this paper is to provide a theory that can explain the cross-county difference in recovery. The extension of the benchmark model predicts that counties with cheaper access to household credit have lower employment rates relative to other counties. In addition, all counties recover at the same speed. This is consistent with U.S. county level data.

1.2 Empirical Facts

This section documents some empirical facts: the economy experienced slow recovery after the most recent recession; uncertainty about future income increased during recession and stayed high.

1.2.1 The Slow Recovery

Figure 1.8[6] shows that U.S. real GDP per capita is below its long run trend after the most recent recession. Meanwhile, three years after the official end of the recession, the employment-population ratio does not show any sign of recovery. Both measures (GDP and employment) indicate that the recovery from the most recent recession is slow.

Figure 1.8[7] compares the recoveries of employment from the most recent recession and past ones. The dashed line plots the average recovery path from previous recessions

[4] The learning structure in this paper is also related to [20], in which slow learning is used to analyze the female labor participation.

[5] e.g. [21] and [22].

[6] Data source: The GDP data is from FRED, Federal Reserve Economic Data, Federal Reserve Bank of St. Louis: Real Gross Domestic Product; U.S. Department of Commerce: Bureau of Economic Analysis. The data is available at `http://research.stlouisfed.org/fred2/series/GDPC1?rid=53&soid=18`; accessed September, 2012. The employment data is from Bureau of Labor Statistics. The data is available at `http://data.bls.gov/timeseries/LNS12300000`.

[7] Data source: Federal Reserve Bank of Minneapolis; Bureau of Labor Statistics. The data is available at `http://www.minneapolisfed.org/publications_papers/studies/recession_perspective/`.

since 1948. Quarter 0 is the official end of the recession. As indicated by the slope of the lines, recovery from the most recent recession is slower than historical average. [8]

1.2.2 The Increase of Uncertainty About Future Income

I use the standard deviation of professional forecasts for the annual average rate of growth in real GDP over the next 10 years as a measure of uncertainty about future income.[9] Figure 1.3 shows that the standard deviation started to increase in 2008 and it has stayed high. This paper explains why uncertainty level stays high and how that impedes recovery.

1.3 A Simple Example: The Age of Exploration

In this section I use a simple example to illustrate the relationship between endogenous information and resolution of uncertainty.

Suppose the environment is sixteenth-century Spain. There are N captains. Each captain maximizes his own lifetime discounted utility from consumption of gold. Gold can be found only in the New World after paying the fixed cost of building a ship for carrying gold. The fixed cost is $F > 0$. After building a ship, the captain can sail to the New World and get $y^* \geq 0$ gold in every period. However, y^* is unknown initially. Although *The Travels of Marco Polo*, a book from the fourteenth century, states that the New World is full of gold ($y_{polo} > 0$ per year), it is not confirmed by captains. The exact value of y^* can be revealed only after at least one captain has been to the New World. I assume that in period 0 captains have identical initial beliefs Φ_0 with upper bound $y_{polo} > 0$ and lower bound $y_L \geq 0$: $\Phi_0(\cdot) \sim \text{Uniform}[y_L, y_{polo}]$.

[8] Some authors suggest that the recoveries from the most three recent recessions are *jobless* in the sense that employment recovers slower than other macroeconomic variables. e.g. [23] and [10]. Therefore I also document the facts about the recoveries in GDP and consumption. The slow patten of the recovery from the most recent recession is still valid.

[9] The data is from the Survey of Professional Forecasters, Federal Reserve Bank of Philadelphia. The data set contains forecasts for 32 economic variables such as GDP growth rates and inflation. The forecasts for the annual average rate of growth in real GDP over the next 10 years are made in the first quarter of each year.

Beliefs are updated by Bayes' rule in every period. Each captain's utility maximization problem is

$$Max \quad \mathbb{E}_{\Phi_0} \sum_t \beta^t (y_t - F\mathbb{I}_t) \tag{1.1}$$

$$\text{where } \mathbb{I}_t = \begin{cases} 1 & \text{if he builds a ship at } t, \\ 0 & \text{if he does not build a ship at } t. \end{cases} \tag{1.2}$$

$$y_t = \begin{cases} y^*, \quad \text{where } y^* \sim \Phi_t(\cdot) & \text{if he has paid for a ship up to } t, \\ 0 & \text{if he has not paid for a ship up to } t. \end{cases} \tag{1.3}$$

$$\Phi_t(\cdot) = \begin{cases} \Phi_0(\cdot) & \text{if nobody has been to the New World up to } t-1 \\ \{y^*\} & \text{if someone has been to the New World up to } t-1. \end{cases} \tag{1.4}$$

$$\Phi_0(\cdot) \sim \text{Uniform}[y_L, y_{polo}] \tag{1.5}$$

At an arbitrary time t, if a captain has built a ship, it is optimal for him to sail to the New World to get y^* since $y^* \geq y_L \geq 0$. If he has not built a ship but y^* has been revealed, he will choose to build a ship if and only if $\frac{y^*}{1-\beta} - F > 0$. i.e. the expected payoff from sailing is larger than the building cost.

If he does not own a ship and y^* has not been revealed, it is easy to show that the sufficient condition for him not to build a ship is

$$\mathbb{E}_{\Phi_t} \frac{y^*}{1-\beta} < F \tag{1.6}$$

Note that in (1.6), $\Phi_t = \Phi_0$ since nobody has been to the New World. Obviously, if (1.6) holds, a captain is not willing to build a ship at time 0. However, if nobody sails to the New World at time 0, the uncertainty about y^* persists and $\Phi_1(\cdot) = \Phi_0(\cdot)$. Therefore, each captain's problem at time 1 is exactly the same as the problem at time 0. Consequently, nobody builds a ship at time 1. The same argument goes through for every period and eventually output $y_t = 0$ for any $t \geq 0$.

Proposition 1. ***If*** $\mathbb{E}_{\Phi_0} \frac{y^*}{1-\beta} < F$***, captains never choose to sail to the New World and*** $y_t = 0$ ***for any*** $t \geq 0$***.***

In this example, sailing decisions today determine the quality of information tomorrow. If one captain chooses to sail, y^* is revealed at the end of the period (perfect information). If nobody chooses to sail, the uncertainty regarding y^* persists (no new information). Initial uncertainty regarding y^* makes captains cautiously choose not to sail. The cautious decisions arising from uncertainty, however, as endogenous information, impede the updating of beliefs. Therefore, the economy stagnates with low output (zero) and high uncertainty.

Now let's consider what would happen if there is a social planner Queen Isabella. Suppose at time 0, Queen Isabella orders one captain Columbus to build a ship to sail to the New World. Obviously, y^* is revealed at the end of period 0. If $\frac{y^*}{1-\beta} - F > 0$, it is optimal for Queen Isabella to ask every captain to sail to the New World from period 1.

If it is optimal for Queen Isabella to ask Columbus to build a ship at time 0 in order to improve total welfare, equilibrium allocation is obviously different from social optimal allocation. The possible failure of the First Welfare Theorem in this example comes from informational externality. If one captain pays F and sails to the New World, y^* is revealed and this information can be shared by everyone else from next period. However, for each individual, the private value of building the first ship does not include the value of information sharing that other captains can get. That is, the social value of building the first ship, which also includes the value of information sharing, might be larger than the private value of building the first ship. In particular, the social value of building the first ship at time 0 is $\mathbb{E}_{\Phi_0}\frac{y^*}{1-\beta} + (N-1)\beta\mathbb{E}_{\Phi_0}Max\{\frac{y^*}{1-\beta} - F, 0\}$. Therefore, a sufficient condition for Queen Isabella to optimally order one captain to build a ship at time 0 is

$$F < \mathbb{E}_{\Phi_0}\frac{y^*}{1-\beta} + (N-1)\beta\mathbb{E}_{\Phi_0}Max\{\frac{y^*}{1-\beta} - F, 0\}. \tag{1.7}$$

The left hand side of (1.7) is the social cost as well as the private cost of building a ship. The first term on the right hand side of (1.7) is the private value of building a ship. The second term on the right hand side of (1.7) is the value of information sharing brought by Columbus. As long as N is large enough so that (1.7) holds, it is optimal for Queen Isabella to order Columbus to sail at time 0. Output will be y^* if $F < \mathbb{E}_{\Phi_0}\frac{y^*}{1-\beta}$ and Ny^* otherwise for any periods after time 0. In other words, if (1.7) holds, there is room for government intervention to increase output and total welfare.

1.4 The Benchmark Model

In this section I use a general equilibrium setup to model the slow recovery. In domestic country, a continuum of homogenous competitive firms hire workers to produce. The product can be consumed and it is the numeraire in this model. The measure of households is one. Each household has one continuum of workers and shoppers. Members within each household share consumption evenly so that only total income and consumption within each household are of interest. All households can borrow from (and lend to) foreigners at international risk-free interest rate r. The risk-free bond is the only asset in this economy. I normalize the start of recession to time 0.

1.4.1 Household's Problem

Timing Friction

Let I_t denote the contemporaneous observables at time t. Let I^t denote the history up to time t. In this economy, $I_t = (D_t, n_t)$ where D_t and n_t are aggregate household demand and aggregate employment at time t, respectively.

As in [2], I assume that households face employment risk when they make their decisions due to timing friction. On the morning of time t, with information set I^{t-1}, households send out shoppers. Shoppers buy c_t units of goods for consumption. It is helpful to think of firms as restaurants and shoppers as customers. Thus, c_t is the size of meal order. Because c_t is determined when households are uncertain about current and future employment, it can be written as $c_t(I^{t-1})$. Furthermore, assume that shoppers should make the interest payment rb_{t-1} to foreign lenders right after shopping. Because b_{t-1} is determined at the end of time $t-1$ with information set I^{t-1}, it is written as $rb_{t-1}(I^{t-1})$. Thus, a representative shopper demands $d_t(I^{t-1}) = c_t(I^{t-1}) + rb_{t-1}(I^{t-1})$. The interest payment is directly shipped to foreign creditors and shoppers take c_t home. Workers look for jobs on the morning of time t. In the middle of the day, firms observe shoppers' demand and make hiring decisions accordingly. Contemporaneous information $I_t = (D_t, n_t)$ is available. At the end of time t, shoppers and workers of each household meet, they use income $w_t(I^t)n_t(I^t)$ to pay for demand $d_t(I^{t-1})$, adjust asset holdings, and consume.

Maximization Problem

At the beginning of time t, the representative household wakes up with debt $b_{t-1}(I^{t-1})$ and information set I^{t-1}. Shoppers go out shopping and decide demand (or equivalently, consumption, as $c_t = d_t - rb_{t-1}$). At the end of the day, after new information I_t is available, $b_t(I^t)$ is chosen to balance the budget. At any time k, the representative household's (or, the representative shopper's) problem can be written as:

$$Max_{\{d_t(I^{t-1})\}} \ \mathbb{E}\{\sum_{t\geq k} \beta^t[u(c_t(I^{t-1}))] \mid I^{k-1}\}, \tag{1.8}$$

Subject to

$$c_t(I^{t-1}) = d_t(I^{t-1}) - rb_{t-1}(I^{t-1}), \tag{1.9}$$

$$d_t(I^{t-1}) + b_{t-1}(I^{t-1}) = w_t n_t(I^t) + b_t(I^t), \tag{1.10}$$

$$b_t(I^t) \leq \bar{B}. \tag{1.11}$$

where $\bar{B}$ is large enough so that (1.11) is never binding in the equilibrium. Notice that the representative household chooses d_t without knowing current employment rate n_t. This is the uncertainty of the representative household's interest.

1.4.2 Production

Firms produce consumption goods subject to three kinds of frictions: demand driven output, credit constraints and labor market frictions. In equilibrium, as will be clear later, firms try to meet households' demand. However, credit constraints might limit firms' abilities to hire workers. Therefore, uncertainty about credit availability creates uncertainty about output and employment.

Technology and Demand Driven Output

The representative firm has a linear technology. However, as in the example of restaurants, only product ordered by households will be sold and product leftover will perish.

Formally, the representative firm's output is determined by the number of workers hired and household demand:

$$y_t = min\{n_t, D_t\} \tag{1.12}$$

Credit Constraint

The representative firm faces a working capital constraint[10] : it has to pay wage $w_t n_t$ to workers before output is produced. Thus, the representative firm has to borrow $w_t n_t$ from lenders if it wants to hire n_t workers at wage rate w_t, and pays back at the end of time t. Here I assume the net interest rate for this within period working capital loan is zero.[11] However, banks have a limited amount of credit ξ_t available. ξ_t stands for firms' credit availability at time t and it is unobservable. This reflects that in reality individuals do not know how much credit that banks such as Citi group and Bank of America can provide to firms. Formally, at any time t,

$$w_t n_t \leq \xi_t, \tag{1.13}$$

where n_t is the amount of labor input.

Thus, inequality (1.13) is a simplified version of the borrowing constraint in [26]. ξ_t limits the number of workers that firms can hire. Consequently, uncertainty about ξ_t creates uncertainty about employment and hence output.

Labor Market Frictions

As discussed in [12], without labor market frictions, it is hard to generate a slow output growth because households would like to increase labor supply during hard times. In this paper I use the random matching friction as in [2].

[10] The assumption of working capital constraint is also featured in [24] and [25]

[11] Notice that the intertemporal interest rate in household's problem is still $r > 0$. However, because firms' working capital loan is within period, I assume the net interest rate (for working capital loan), r_w, is zero. This assumption can be relaxed by assuming the within period interest rate r_w is endogenous and it will not change the equilibrium outcome. To see this, on one hand, with the assumption that lenders can always choose to hold cash in hand without any additional cost, it is easy to show that in equilibrium $r_w^* \geq 0$. On the other hand, As will be clear later, in equilibrium firms always make zero profit because wage cost= output. If $r_w^* > 0$, firms would choose to shut down and loan demand would be zero. Therefore, in equilibrium the within period working capital loan has interest rate $r_w^* = 0$. Firms' loan demand correspondence would be an interval from 0 to $w_t D_t$ instead of a point. The equilibrium quantity of loan is determined by ξ_t and firms' credit demand correspondence given $r_w^* = 0$. I thank Erzo Luttmer for pointing out this problem.

At time t, households send out workers with reservation wage w_t^r. Workers meet firms in a random fashion. Given that all workers have a reservation wage $w_t^r = 1$ (which will be verified later), the optimal response for a firm is to only hire workers with reservation wage weakly less than one: otherwise, the marginal cost of labor would be higher than marginal revenue. Given firms' strategy, households would not set the reservation wage $w_t^r > 1$. Households would not like to set $w_t^r < 1$ either because a lower reservation wage does not increase the chance of being employed due to the random meeting feature. Ex ante, the optimal strategy for households is to set $w_t^r = 1$. The first n_t workers who meet the representative firm are employed, and the rest $1 - n_t$ are unemployed. Consequently, the equilibrium wage $w_t = w_t^r = 1$.

1.4.3 Employment and Output

In each period t, each household decides its demand d_t. The representative firm faces aggregate demand D_t and hires workers to fulfill demand, which requires a loan $w_t n_t = w_t D_t$ from banks.

If household demand (as well as credit demand) is relatively small, i.e.

$$w_t D_t \leq \xi_t, \tag{1.14}$$

the representative firm hires $n_t = D_t$ workers to produce $y_t = D_t$ units of goods and sells them to consumers.

If household demand (as well as credit demand) is large enough, i.e.

$$w_t D_t > \xi_t, \tag{1.15}$$

the representative firm cannot hire enough workers to fulfill demand due to credit shortage and it hires $n_t = \frac{\xi_t}{w_t}$ workers and imports $IM_t = D_t - \frac{\xi_t}{w_t}$ from abroad.[12] Therefore, at the end of the day the representative firm still sells D_t units of output to shoppers. Given the information set I^{t-1}, the contemporaneous observables I_t are determined as

[12] Strictly speaking, the representative firm is indifferent with hiring zero or $\frac{\xi_t}{w_t}$ units of labor because the firm always get zero profit. To eliminate indeterminacy, I assume that the firm always try to hire as much labor as it can subject to demand and credit constraints. Alternatively, one can assume that the final consumption good is a CES composite of differentiable goods. Each firm is a monopolistic producer of a kind of differentiable good with markup infinitely close to 1. In this setup, the unique optimal strategy for each firm is to hire as much labor as it can subject to demand and credit constraints.

below:

$$I_{t,1} = D_t(I^{t-1}), \tag{1.16}$$

$$I_{t,2} = n_t = min\{\frac{\xi_t}{w_t}, D_t(I^{t-1})\}. \tag{1.17}$$

1.4.4 Foreign Demand

I must also specify foreign demand in this demand driven economy. In this model, both domestic firms and foreign firms offer a unit price so that households in the foreign country are indifferent between buying output from domestic and foreign firms. Because the purpose of this paper is to explore the connection between economic recovery and resolution of uncertainty, I close the foreign demand channel by assuming domestic firms face zero demand from foreigners. However, foreigners can still receive and consume debt repayment from domestic households.

1.4.5 Beliefs

Because in equilibrium $w_t = 1$, (1.13) can be simplified to

$$n_t \leq \xi_t \tag{1.18}$$

Households cannot observe firms' credit availability $\{\xi_k\}$ but they can form beliefs about the distribution of $\{\xi_k\}$ based on information available: $\{\xi_k\} \sim \Phi(\cdot \mid I^{t-1})$. In this economy, because households share the same information set at any given time, they always have the same beliefs. Assume that households are rational so that they fully understand firm's problem. Households update their beliefs using Bayes' rule.

1.4.6 Definition of Equilibrium

A competitive equilibrium in this economy is defined as:

a. The representative household's policy functions $\{d_t(I^{t-1})\}$, $\{c_t(I^{t-1})\}$, and $\{b_t(I^t)\}$;

b. The representative firm's policy functions $\{n_t(I^t)\}$, $\{y_t(I^t)\}$, and $\{IM_t(I^t)\}$;

c. The representative household's belief $\{\Phi(\cdot \mid I^{t-1})\}$;

d. Wage w_t; and

e. Aggregate demand functions $\{D_t(I^{t-1})\}$, such that, for any $t \geq 0$:

1. Given wage w_t, and aggregate demand functions $\{D_t(I^{t-1})\}$, household belief $\{\Phi(\cdot \mid I^{t-1})\}$, and the representative household's policy functions solve the household's problem in Section 1.4.1;

2. Given wage w_t and aggregate demand functions $\{D_t(I^{t-1})\}$, $\{n_t(I^t)\}$, $\{y_t(I^t)\}$ and $\{IM_t(I^t)\}$ solve the representative firm's problem;

3. Consistency: $D_t(I^{t-1}) = d_t(I^{t-1})$.

1.4.7 Multiplicity

There are multiple equilibria in this model due to self-fulfilling prophecy. To see this, to the extreme, suppose the true value of ξ_t is larger than 1 for any t (so that banks can always provide enough credit to firms) and households are fully aware of this. Thus, there is no uncertainty about firms' credit availability. On the morning of time t, if one believes that aggregate demand $\{D_k\}_{k\geq t}$ will be low (high), he or she can infer that current and future income will be low (high) due to low (high) aggregate demand. Consequently, everyone chooses a low (high) d_t. This in turn confirms the prediction of low (high) aggregate demand. To the extreme, suppose everyone believes that $D_k = \xi_{min}$ for any $k \geq 0$. It is optimal for each household to choose $d_k = \xi_{min} = D_k$ for any $k \geq 0$. It is trivial to verify that $D_k = d_k = \xi_{min}$ and $c_k = \xi_{min} - rb_{-1}$ contribute to an equilibrium allocation. In this equilibrium, the stagnation comes from households' failure to coordinate with others' decisions. If each household is convinced that other households will choose high demand so that aggregate demand and income will be high, it will also choose high demand. That is, the equilibrium depends on uncertainty about others' decisions and beliefs instead of uncertainty about fundamentals.[13]

In this paper, instead of focusing on the rationale of selecting an equilibrium in which households fail to form optimistic beliefs about others' decisions, I focus on the best equilibrium in which households' beliefs about others' decisions are as optimistic as possible. I first define and prove the existence of the best equilibrium. I then develop an algorithm which solves for the best equilibrium. The algorithm includes several main steps. First, I make an initial guess of equilibrium aggregate demand for each period and

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[13] In the context of recession and recovery, recent studies relying on the feature that agents are pessimistic about others' decisions include [2], [27] and [13].

state. Second, with the prior guess of aggregate demand, I can solve each individual's problem. Third, I use each individual's optimal policy functions to update the prior guess about aggregate demand and go back to the first step. Repeat this procedure until each individual's optimal demand is consistent with the aggregate demand, the iteration converges to the equilibrium allocation.

The technical difficulty is that frequently used algorithms based on Newton method do not work here due to multiplicity. Instead, an algorithm with *monotonic* property is required. To see this, denote $\{D_t^b\}$ as the aggregate demand path of the best equilibrium which we want to solve for. Denote $\{D_t^k\}$ as the aggregate demand path after the kth iteration. If $D_t^k < D_t^b$ for some t, $\{D_t^k\}$ will never converge to $\{D_t^b\}$. Therefore, I choose an optimistic initial guess about aggregate demand, which is definitely above the demand path of the best equilibrium. i.e. $\{D_t^0\} \geq \{D_t^b\}$. I then specify an iteration rule and show that iteration is *monotonic*: if we have two prior demand paths $\{D_t^k\}$ and $\{D_t'^k\}$ such that $\{D_t^k\} \leq \{D_t'^k\}$, then we must have $\{D_t^{k+1}\} \leq \{D_t'^{k+1}\}$. This guarantees that the iteration *monotonically* converges to the best equilibrium.

In the quantitative exercise I report the simulated recovery speed of the best equilibrium. As such the quantitative results in my model provide the upper bound of the recovery speed and quantitatively. As will be clear later, quantitatively the recovery speed of this best equilibrium is still slow, which is comparable with U.S employment data. In the rest of paper, to avoid confusion, the equilibrium should be considered as the best equilibrium.

Definition 1. Let $\{y_t\}_{t\geq 0}$ denote the path of output in an arbitrary equilibrium. Suppose there is one equilibrium with output path $\{y_t^b\}_{t\geq 0}$, which satisfies $y_t^b \geq y_t$ for any $t \geq 0$. Then the equilibrium with output path $\{y_t^b\}_{t\geq 0}$ is called the best equilibrium.

The next proposition provides the existence of the best equilibrium.

Proposition 2. ***If $u(\cdot)$ is twice differentiable, concave, increasing and $u'''(\cdot) > 0$, the benchmark model always has a best equilibrium.***

1.4.8 Pre-recession Steady State

Assume that before the recession (starting from $t = 0$), the economy is in a steady state with time invariant credit availability $\bar{\xi} \leq 1$. The underlying assumption is that the time horizon before the recession is long enough so that households are able to learn the value of $\bar{\xi}$. That is,

Assumption 1. *Before the recession,* $\bar{\xi}_t = \bar{\xi} \leq 1$ *and households are aware of the value of* $\bar{\xi}$.

Assumption 2. $\beta(1+r) = 1$ *and initial bond position* $b > 0$ *is given.*

It is trivial to show that in the pre-recession steady state, the aggregate and individual demand $D_t = d_t = \bar{\xi}$, debt $b_t = \bar{b}$, and $c_t = d_t - rb_t$. The representative firm chooses $n_t = \bar{\xi}$, $y_t = \bar{\xi}$ and $IM_t = 0$. Wage $w_t = 1$ and household's beliefs about firms' credit availability $\Phi(\cdot \mid I^{t-1}) = \{\bar{\xi}\}$. That is, aggregate demand and output exactly equal the maximum amount of credit that firms can get: $\bar{\xi}$.[14] The main reason is that there is no uncertainty so that households optimally choose high demand $\bar{\xi}$.

1.4.9 Recession and Recovery

I assume that at the end of $t = -1$, an unexpected belief shock creates uncertainty about firms' credit availability. One can think of it as the public panic about credit availability after banks made huge losses in the recession and regulatory authorities stepped up credit control. Because of the belief shock, households believe that $\bar{\xi}$ permanently shifts to $\bar{\xi}^*$ and $\bar{\xi}^* \sim \Phi(\cdot)$. Because it is a belief shock instead of a real shock, the true value of credit availability does not change, i.e. $\bar{\xi}^* = \bar{\xi}$. However, households are unaware of that. As long as the prior beliefs are somewhat pessimistic,[15] households cut demand and low demand causes a drop in output at time 0. Thus, the recession and slow recovery in my model are purely driven by households' beliefs about credit availability instead of a realized negative shock to fundamentals. This is the main difference between my paper and the literature in which recession and slow recovery are driven by real shocks.[16]

[14] Recall that $n_t \leq \frac{\xi_t}{w_t}$ and $w_t = 1$.

[15] In other words, households cannot exclude the possibility that $\bar{\xi}^*$ might be smaller than $\bar{\xi}$.

[16] [9] argues that the slow recovery is due to the slowdown in trend labor force growth after 2005. In [12] the recession starts from a damage of capital.

Assumption 3. *At $t = 0$ the economy is hit by an unexpected belief shock. Households believe that $\bar{\xi}$ permanently shifts to $\bar{\xi}^* \sim \Phi(\cdot)$. The true value of $\bar{\xi}^*$ is still $\bar{\xi}$ but it is unobservable to households.*

Updating Beliefs

Households observe $I_t = (D_t, n_t)$ and update their beliefs by Bayes' rule. Assume that $\{D_t\}$ is increasing in equilibrium, which will be verified later. Suppose at arbitrary time t, with prior belief $\Phi(\cdot|I^{t-1})$, one observes that aggregate demand D_t is larger than employment n_t. It must be true that $\bar{\xi}^* = n_t < D_t$ so that firms cannot obtain enough credit to hire D_t units of workers to fill up demand. However, if $D_t = n_t$, households can infer that firms are able to obtain enough credit from banks to hire D_t units of workers. Therefore, it must be true that $\bar{\xi}^* \geq D_t$. Formally, that is,

$$\Phi(\cdot|I^t) = \begin{cases} \Phi(\cdot|\bar{\xi}^* \geq D_t) & \text{if } n_t = D_t, \\ \{n_t\} & \text{if } n_t < D_t. \end{cases}$$

where $\Phi(\cdot)$ is the initial belief of households right after they are hit by the belief shock.

To simplify the analysis, I assume that the initial belief $\Phi(\cdot)$ follows an exponential distribution, although most qualitative results of this paper are valid without this assumption.

Assumption 4. *The initial prior belief $\Phi(\cdot)$ has the support $[\xi_{min}, +\infty)$, where $\xi_{min} < \bar{\xi} \leq 1$. In addition, it follows an exponential distribution on $[\xi_{min}, +\infty)$:*

$$\phi(\xi) = \begin{cases} f_\gamma(\xi - \xi_{min}) & \textit{if } \xi \in [\xi_{min}, +\infty), \\ 0 & \textit{otherwise.} \end{cases}$$

where $f_\gamma(\cdot)$ is the density function of exponential distribution with rate γ.

Policy Functions

Household demand depends on beliefs. Suppose at time $t \geq 0$, uncertainty has been resolved. That is, $\Phi(\cdot|I^{t-1})$ is a degenerate distribution with support $\{\xi\}$. It is trivial to show that $d_k = D_k = \xi$ and $c_k = \xi - rb_{t-1}$ for any $k \geq t$.

However, suppose at time $t \geq 0$, uncertainty has not been resolved. Let d_t^* and D_t^* denote individual and aggregate demand when uncertainty has not yet been resolved.[17]

The equilibrium recovery paths are fully characterized by $\{D_t^*\}$: output $y_t = D_t^*$ in every period until time t_1 such that $\{D_{t_1}^*\} > \bar{\xi}^*$. For any $t > t_1$, we have $D_t = y_t = \bar{\xi}^*$. Therefore, it is sufficient to focus on $\{D_t^*\}$, the path of aggregate demand when uncertainty has not been resolved.

Optimal Conditions

Given aggregate purchase $\{D_t\}$ and wage $w_t = 1$, each household's problem can be re-written as below:

$$Max_{\{d_t\}} \ \mathbb{E}\{\sum_{t\geq k} \beta^t u(d_t - rb_{t-1}) \mid I^{k-1}\} \tag{1.19}$$

$$\text{s.t.} \quad b_t = b_{t-1} + d_t - w_t n_t,$$

$$n_t = min\{\frac{\bar{\xi}^*}{w_t}, D_t\}, \quad \bar{\xi}^* \sim \Phi(\cdot|I^{t-1})$$

$$b_t \leq \bar{B}.$$

The Euler equation is:

$$\mathbb{E}\{u'(c_t) \mid I^{t-1}\} = \mathbb{E}\{u'(c_{t+1}) \mid I^{t-1}\}. \tag{1.20}$$

With the assumption that $\beta(1+r) = 1$, (1.20) is similar to standard Euler equations except the expectations are conditional upon I^{t-1} instead of I^t. Expand (1.20) when uncertainty has not yet been resolved, [18] and obtain the following[19]

$$\begin{aligned}\int u'(d_t^* - rb_{t-1})\mathrm{d}\Phi(\bar{\xi}^*|\bar{\xi}^* \geq D_{t-1}^*) = &\int_{\bar{\xi}^* < D_t^*} u'(\bar{\xi}^* - r(b_{t-1} + d_t^* - \bar{\xi}^*))\mathrm{d}\Phi(\bar{\xi}^*|\bar{\xi}^* \geq D_{t-1}^*) \\ &+ \int_{\bar{\xi}^* \geq D_t^*} u'(d_{t+1}^* - rb_{t-1})\mathrm{d}\Phi(\bar{\xi}^*|\bar{\xi}^* \geq D_{t-1}^*)\end{aligned} \tag{1.21}$$

The left hand side of (1.21) is simply the marginal utility at time t if the representative household chooses to demand d_t^*. Because uncertainty has not been resolved up

[17] That is, $d_t(I^{t-1}) = d_t^*$ and $D_t(I^{t-1}) = D_t^*$ if $\Phi(\cdot|I^{t-1})$ is non-degenerate.

[18] As we discussed earlier, it is trivial to derive the policy functions if $\phi(\cdot|I^{t-1})$ follows a degenerate distribution.

[19] Assume $D_{-1}^* = n_{min}$.

to $t-1$, $\Phi(\cdot|I^{t-1}) = \Phi(\cdot|\bar{n}^* \geq D^*_{t-1})$. At t, if aggregate demand is larger than $\bar{\xi}^*$, uncertainty is resolved and households know that their permanent income will be $\bar{\xi}^*$. Thus, they demand $\bar{\xi}^*$ from $t+1$ and consumption at $t+1$ equals income $\bar{\xi}^*$ minus interest payment $r(b_{t-1} + d^*_t - \bar{\xi}^*)$. This is the first term on the right hand side of (1.21). If $\bar{\xi}^* \geq D^*_t$, uncertainty persists and households choose d^*_{t+1} at $t+1$. This is the second term on the right hand side.

In equilibrium, $d^*_t = D^*_t$ for any $t \geq 0$. Moreover, up to time t, if aggregate demand has never been larger than $\bar{\xi}^*$, household income always equals demand. Therefore, the debt position does not change from 0 to t. Replace d^*_t with D^*_t, b_{t-1} and b_t with b_{-1}, (1.21) is turned to

$$\begin{aligned}\int u'(D^*_t - rb_{-1})\mathrm{d}\Phi(\bar{\xi}^* \geq D^*_{t-1}) = &\int_{\bar{\xi}^* < D^*_t} u'(\bar{\xi}^* - r(b_{-1} + D^*_t - \bar{\xi}^*))\mathrm{d}\Phi(\bar{\xi}^*|\bar{\xi}^* \geq D^*_{t-1}) \\ &+ \int_{\bar{\xi}^* \geq D^*_t} u'(D^*_{t+1} - rb_{-1})\mathrm{d}\Phi(\bar{\xi}^*|\bar{\xi}^* \geq D^*_{t-1})\end{aligned} \tag{1.22}$$

Equation (1.22) characterizes the dynamics of D^*_t (with the condition $D^*_{-1} = \xi_{min}$). Note that in this demand driven economy, because I assume that there is no real shock (so that $\bar{\xi}^* = \bar{\xi}$), the recovery path of demand overlaps with the recovery paths of employment and output as long as they are less that $\bar{\xi}$. The aggregate demand (before uncertainty is resolved) is always weakly increasing in time. To see this, compare household's problems at time t and time $t+1$. In equilibrium, when households choose D^*_t, they understand that the lower bound of $\bar{\xi}^*$ is $max\{D^*_k, \xi_{min}\}_{-1 \leq k < t}$. However, when they choose D^*_{t+1}, in their beliefs, the lower bound of $\bar{\xi}^*$ is $max\{D^*_k, \xi_{min}\}_{-1 \leq k < t+1}$. Obviously, at time $t+1$ they have more optimistic beliefs. Thus, $D^*_{t+1} \geq D^*_t$.

The rate of exponential distribution γ defined in Assumption 4, is a measure of pessimism. A smaller γ implies that households assign larger probabilities on the right tail of the distribution of $\bar{\xi}^*$. That is, a smaller γ is equivalent to being more optimistic. Because households demand more when their beliefs are more optimistic, D^*_k is a decreasing function of γ. Therefore, recovery is slower if γ is larger.

Figure 1.4 illustrates the recovery of employment in the benchmark model when uncertainty has not been revealed.[20] The gradual growth of employment comes from the gradual growth of household demand. For a household, increasing consumption is risky. In equilibrium, $d_t^* = D_t^*$. If $\bar{\xi}^*$ is larger than D_t, it can be shown that $c_{t+1} \geq c_t$. However, if $\bar{\xi}^* < D_t$, it can be shown that $c_{t+1} = \bar{\xi}^* - r(b_{-1} + D_t^* - \bar{\xi}^*) < c_t$. Excessive consumption at time t might further decrease consumption at $t+1$. When $\bar{\xi}^*$ is small so that $c_{t+1} < c_t$, this is painful. Therefore, households are not willing to increase demand too fast even if they observe that $n_t = D_t$. The blue dashed line illustrates the evolution of the lower bound of the support of household beliefs $\Phi(\cdot|I^{t-1})$. Notice that the lower bound of the support of $\Phi(\cdot|I^{t-1})$ is exactly D_{t-1} as long as uncertainty has not been resolved. The two parallel lines in Figure 1.4 show that the growth of employment relies on the improvement of household beliefs and vice versa.

In this model, households use last period's output and demand as information to update their beliefs. However, last periods' demand and output are endogenously determined by the solution to household's optimization problem. When initial uncertainty is large, households endogenously choose low demand. Consequently, output and firms' credit demand are low. The low level of economic activity means that firms' credit availability cannot be revealed. Therefore, the feature of endogenous information impedes the resolution of uncertainty.

1.4.10 Benchmark Model With Household Credit Cost

In this section I extend the benchmark model by including household credit frictions. As will be clear later, it amplifies the effect of endogenous information and quantitatively it substantially slows down the recovery.

Assume that after the recession, households are financially constrained: they can increase debt only with costs. In particular, I assume that after the recession, at any t, if $b_t - b_{t-1} \geq 0$, a credit cost $\lambda(b_t - b_{t-1} - \epsilon)^+ \geq 0$ is incurred.[21] I assume that the credit cost is charged before consuming. That is, consumption at time t is $c_t = d_t - rb_{t-1} - \lambda(b_t - b_{t-1} - \epsilon)^+$ so that the debt position at the end of time t is still b_t after the credit cost has been charged. This assumption corresponds to the situation

[20] I will discuss the choice of parameters in later sections.

[21] The constant number $\epsilon > 0$ is infinitely close to zero. I need ϵ in f_t to ensure that household's objective function is differentiable around the equilibrium allocation. Because ϵ can be infinitely close to zero, except in this section and appendix, I will omit ϵ to simplify the notation.

that households find it hard to get more credit after the crisis.

As in the benchmark model without household credit costs, I still focus on the best equilibrium defined in Definition 1. The analogue of Proposition 2 still holds and it is proved in the Appendix.

Proposition 3. ***If $u(\cdot)$ is twice differentiable, concave, increasing and $u'''(\cdot) > 0$, the benchmark model with household credit cost always has the best equilibrium.***

The household's problem in the benchmark model with household credit cost can be written as below.

$$Max_{\{d_t\}} \ \mathbb{E}\{\sum_{t\geq k} \beta^t u(d_t - rb_{t-1} - f_t) \mid I^{k-1}\} \tag{1.23}$$

s.t.

$$f_t = \lambda(b_t - b_{t-1} - \epsilon)^+ \tag{1.24}$$

$$b_t = b_{t-1} + d_t - w_t n_t, \quad b_t \leq \bar{B}.$$

$$n_t = min\{\frac{\bar{\xi}^*}{w_t}, D_t\}, \quad \bar{\xi}^* \sim \Phi(\cdot|I^{t-1})$$

1.4.11 Optimal Conditions In the Benchmark Model With Household Credit Cost

The Euler equation is

$$\mathbb{E}\{u'(c_t) \mid I^{t-1}\} = \mathbb{E}\{u'(c_{t+1}) + \lambda u'(c_t)\mathbf{1}_{[d_t-\epsilon>n_t]} - \beta\lambda u'(c_{t+1})\mathbf{1}_{[d_{t+1}-\epsilon>n_{t+1}]} \mid I^{t-1}\}. \tag{1.25}$$

The intuition behind (1.25) is clear. Suppose at time $t-1$, given information I^{t-1}, the representative household decides to deviate from its optimal policy by increasing c_t to $c_t + \Delta$. Here Δ is positive and close to zero. The benefit of this deviation is simply the left hand side of (1.25) times Δ. The cost of it can be decomposed into three parts. First, households should consume $(1+r)\Delta$ less at time $t+1$ because of the budget constraint. This corresponds to the first component on the right hand side of (1.25). Second, an increase in consumption at time t leads to an increase in debt from b_t to $b_t + \Delta$. Consequently, the expected household credit cost from expanding debt at the end of time t increases by $\lambda\Delta$. This effect corresponds to the second component of the right hand side of (1.25), as credit cost is incurred only if $b_t > b_{t-1} + \epsilon$ (which is equivalent to $d_t - \epsilon > n_t$ from the budget constraint). The third component comes from the change in $d_{t+1} - n_{t+1}$. Consumption c_{t+1} decreases by $(1+r)\Delta$, and interest payment, rb_t, increases by $r\Delta$. Therefore the net change in demand d_{t+1} is $-\Delta$. This reduces the expected credit cost at $t+1$, which is the third term on the right hand side of (1.25).

Substitute the equilibrium condition $d_t = D_t$ into (1.25) we can characterize the recovery path as in the benchmark model.

1.4.12 The Role of Household Credit Cost

Other things equal, household credit costs decrease household demand. To the extreme, if λ is large enough, households do not want to take any risks. That is, they always set their demand exactly the same as the lower bound of their beliefs about $\bar{\xi}^*$. The economy stagnates at $D = \xi_{min}$.

Figure 1.5 illustrates the recoveries of employment in models with and without household credit costs. It clearly shows household credit costs significantly slow down the recovery. With credit costs, the precautionary saving motive is more pronounced, and hence household demand is further suppressed. Low demand leads to low employment and low output in this demand driven economy. Therefore, compared with the benchmark model without credit costs, household credit costs **amplify** the effect of endogenous information. In the extreme case in which λ goes to infinity, uncertainty is never resolved and the economy stagnates.

1.5 The Social Planner's Problem and Inefficient Learning

This section discusses the social planner's problem. The equilibrium presented in previous sections is not efficient due to the existence of two externalities. First, higher consumption demand weakly increases output. This is the traditional Keynesian multiplier. Second, higher consumption demand leads to higher borrowing and faster resolution of uncertainty. Because each households do not take into account the fact that by increasing consumption they can improve the quality of information, the learning process is inefficient. Consequently, the social planner increases efficiency in these two aspects. At the end of this section, I show that a consumption subsidy schedule can decentralize the social planner's allocation and speed up the recovery.

Let $V_t^s(b_{t-1}|\Phi_t(\cdot|I^{t-1}))$ denote the social planner's value function at time t if uncertainty *has not* been resolved up to $t-1$, where b_{t-1} is the debt position at the end of $t-1$. $\Phi_t(\cdot|I^{t-1})$ is the conditional beliefs at the beginning of time t. If demand $\{D_t\}$ is increasing over time before uncertainty is resolved (which is trivially true in the solution of the social planner's problem), $\Phi_t(\cdot|I^{t-1}) = \Phi_t(\cdot|\bar{\xi}^* \geq D_{t-1})$. Thus, D_{t-1} summarizes the beliefs about $\bar{\xi}^*$ and $V_t^s(b_{t-1}|\Phi_t(\cdot|I^{t-1}))$ can be written as $V_t^s(b_{t-1}|\bar{\xi}^* \geq D_{t-1})$.

Similarly, let $V_t^{s,r}(b_{t-1}, \bar{\xi}^*)$ denote the value function at time t if $\bar{\xi}^*$ has been revealed and the debt position inherited from last period is $(1+r)b_{t-1}$. The social planner's problem at time t can be written as:

$$V_t^s(b_{t-1}|\bar{\xi}^* \geq D_{t-1}) = Max_{\{D_t\}}\ \mathbb{E}\{u(c_t) + \beta \tilde{V}_{t+1}^s(b_t|D_t) \mid \bar{\xi}^* \geq D_{t-1}\} \tag{1.26}$$

$$\text{s.t.} \quad y_t = min\{D_t, \bar{\xi}^*\} \tag{1.27}$$

$$c_t = D_t - rb_{t-1} - \lambda(D_t - y_t)^+ \tag{1.28}$$

$$b_t = D_t + b_{t-1} - y_t \tag{1.29}$$

$$\tilde{V}_{t+1}^s(b_t|D_t) = \begin{cases} V_{t+1}^s(b_t|\bar{\xi}^* \geq D_t), & \text{if } \bar{\xi}^* \geq D_t \text{ so that } \bar{\xi}^* \text{ is not revealed,} \\ V_{t+1}^{s,r}(b_t, \bar{\xi}^*), & \text{if } \bar{\xi}^* < D_t \text{ so that } \bar{\xi}^* \text{ is revealed.} \end{cases} \tag{1.30}$$

The first order condition of the social planner's problem is:

$$\begin{aligned} &\mathbb{E}\{u'(c_t)(1 - \lambda \mathbb{I}_{\{D_t > y_t\}} + \lambda \frac{\partial y_t}{\partial D_t} \mathbb{I}_{\{D_t > y_t\}})|\bar{\xi}^* \geq D_{t-1}\} \\ &+ \beta \mathbb{E}\{\tilde{V}_{t+1}^{s'}(b_t|D_t)(1 - \frac{\partial y_t}{\partial D_t})|\bar{\xi}^* \geq D_{t-1}\} + \beta \left. \frac{\partial \mathbb{E}\{\tilde{V}_{t+1}^s(b_t|D)|\bar{\xi}^* \geq D_{t-1}\}}{\partial D} \right|_{D=D_t} = 0 \end{aligned} \tag{1.31}$$

Rearrange it gives[22]

$$
\begin{aligned}
\mathbb{E}u'(c_t) = &\ \mathbb{E}\{\lambda u'(c_t)\mathbb{I}_{\{D_t>y_t-\}} - \beta\tilde{V}^{s'}_{t+1}(b_t|D_t)\} \\
&+ \underbrace{\mathbb{E}\{[-\lambda u'(c_t)\mathbb{I}_{\{D_t>y_t\}} + \beta\tilde{V}^{s'}_{t+1}(b_t|D_t)]\frac{\partial y_t}{\partial D_t}\}}_{\text{externality I: } D_t\uparrow\Rightarrow y_t\uparrow} \\
&\underbrace{-\beta\left.\frac{\partial\mathbb{E}\{\tilde{V}^{s}_{t+1}(b_t|D)|\bar{\xi}^*\geq D_{t-1}\}}{\partial D}\right|_{D=D_t}}_{\text{externality II: } D_t\uparrow\Rightarrow \text{ more informative } I_t}
\end{aligned}
\tag{1.32}
$$

The left hand side of (1.32) is simply the expected marginal benefit of increasing demand at time t. The first line on the right hand side of (1.32) is the expected marginal cost of increasing demand in the equilibrium. The second and third lines on the right hand side of (1.32) represent the two externalities internalized by the social planner. First, an increase in demand D_t weakly increases income y_t. The increase in y_t decreases the expected household credit cost at time t and debt position b_t. This is characterized by the second line. Second, D_t summarizes the information that the social planner has at the beginning of time $t+1$: either $y_t < D_t$ so that the value of $\bar{\xi}^*$ is revealed, or $y_t \geq D_t$ so that the lower bound on the belief of $\bar{\xi}^*$ moves to D_t. Therefore, a larger D_t leads to more precise beliefs at time $t+1$, which increases the expected value function at time $t+1$. This is illustrated by the third line on the right hand side of (1.32). Both externalities give extra incentive to increase D_t.[23] Consequently, the social planner would always demand more than households in the equilibrium.

Figure 1.6 compares the recovery of employment in models with and without a social planner. In the model with a social planner, output drop is much smaller and the recovery only takes four quarters.

Compared with equilibrium allocation, the social planner's allocation results in faster recovery and larger welfare. A natural question is whether the social planner's allocation can be implemented as an equilibrium outcome. I use consumption subsidy as policy instruments. In particular, I assume that government subsidizes consumption at rate τ_t when uncertainty has not yet been revealed.[24] Government also charges lump sum

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[22] To simplify the notation, I use $\mathbb{E}(\cdot)$ to denote $\mathbb{E}(\cdot|\bar{\xi}^*\geq D_{t-1})$.

[23] It can be proved that the signs of the second and third lines of (1.32) are negative

[24] Obviously, the social planner's allocation is consistent with equilibrium allocation when uncertainty has been revealed.

tax in each period to balance its budget.

Each household's budget constraint at time t is

$$(1-\tau_t)(d_t - rb_{t-1}) + (1+r)b_{t-1} = b_t + w_t n_t + T_t \tag{1.33}$$

where $T_t = -\tau_t(D_t - rB_{t-1})$. The Euler equation of the representative household in this tax distorted economy is

$$\begin{aligned} &\frac{1}{1-\tau_t}\mathbb{E}_{t-1}u'(c_t) - \lambda\mathbb{E}_{t-1}u'(c_t)1_{[b_t > b_{t-1}]} \\ &= \mathbb{E}_{t-1}\frac{\beta R}{1-\tau_{t+1}}u'(c_{t+1}) - \lambda\beta\mathbb{E}_{t-1}u'(c_{t+1})1_{[b_{t+1} > b_t]} \end{aligned} \tag{1.34}$$

Intuitively, a regressive consumption subsidy schedule gives households extra incentive to increase consumption, which leads to a faster recovery. If the representative household increases expenditure by one unit at time t, consumption increases by $\frac{1}{1-\tau_t}$ at time t and decreases by $\frac{R}{1-\tau_{t+1}}$ at time $t+1$. The debt, however, changes by one unit at time t because only consumption is subsidized. Figure 1.7 shows the optimal subsidy rate that decentralizes the social planner's problem.

1.6 Model with Heterogeneous Counties

In this section I use the recovery patterns across U.S. counties to test my theory. I extend my benchmark model to include heterogeneous counties. Model predictions are consistent with U.S. county level data: counties with cheaper access to household credit have higher employment but all counties recover at the same rate.

1.6.1 Counties and Goods

Assume there are three counties: county m with size $S^m = 1$ and two ϵ-small counties u and c with size $S^u = S^c = \epsilon$. Here ϵ is infinitely close to zero so that county u and c have no impact on the aggregate economy. Each county consists of a continuum of households. Assume that labor is immobile across counties. Counties are heterogeneous in the basket of goods consumed and household credit costs. County m's credit cost is λ as in the benchmark model. County u and county c have zero and infinite credit cost, respectively.

I follow [1] to assume that there are two categories of goods: identical tradable goods and county-specific non-tradable goods. At time t, consumers in county $i(= u, m, c)$ consume tradable goods $c_t^{i,T}$ and non-tradable goods $c_t^{i,N}$. Firms have linear technology:

$$y_t^{i,g} = n_t^{i,g}. \tag{1.35}$$

where $i \in \{u, m, c\}$, $g \in \{T, N\}$ and $n_t^{i,g}$ stands for the number of workers hired in the representative firm in county i and sector g at time t. Firms in county i produce either tradable goods or county specific non-tradable goods. Assume further that foreigners can produce all kinds of goods. This assumption can help me clear the goods market as in the benchmark model. Tradable goods produced by each domestic county and foreigners are identical. As such I use tradable goods as the numeraire. In each period t, households send their workers with reservation wage w_t^r to look for jobs. Managers randomly meet workers. Assume that workers do not know the type of firms they are negotiating contracts with when they meet managers. This assumption implies that the reservation wage w_t^r does not depend on the type of firms workers work for. In equilibrium, as in the benchmark model, the reservation wage $w_t^r = 1$ because the price of tradable goods is normalized to one. Consequently, the price of non-tradable goods and the equilibrium wage rate also equal one.

1.6.2 Household Consumption and Income

As in [1], assume that consumers have Cobb-Douglas period utility function:

$$u(c_t^{i,N}, c_t^{i,T}) = (c_t^{i,N})^{\alpha}(c_t^{i,T})^{1-\alpha}. \tag{1.36}$$

where $c_t^{i,T}$ ($c_t^{i,N}$) stands for the tradable (non-tradable) consumption of the representative household living in county i at time t. Define total consumption $c_t^i = c_t^{i,N} + c_t^{i,T}$. Because all goods have unit price and preference is Cobb-Douglas,

$$c_t^{i,N} = \alpha c_t^i \quad \text{and} \quad c_t^{i,T} = (1-\alpha)c_t^i. \tag{1.37}$$

Therefore, c_t^i is sufficient to characterize the consumption bundle $(c_t^{i,T}, c_t^{i,N})$. Similarly, let $n_t^i = n_t^{i,T} + n_t^{i,N}$, $d_t^i = d_t^{i,T} + d_t^{i,N}$, $D_t^i = D_t^{i,T} + D_t^{i,N}$[25] . Assume that at time t,

[25] Here $D_t^{i,g}$ stands for the average demand of residents in county i

the interest payment, rb_{t-1} is paid in a composition of non-tradable goods and tradable goods with weight α.[26] Therefore, the demand of the representative consumer in county i is $d_t^i = c_t^i + rb_{t-1}^i$ where $b_t^i = b_t^{i,T} + b_t^{i,N}$. Define observables $I_t = \{I_t^m, I_t^u, I_t^c\} = \{(D_t^m, n_t^m), (D_t^u, n_t^u), (D_t^c, n_t^c)\}$, which includes the average demand per household[27] and employment rate in each county. As in the benchmark model, consumers decide c_t^i given information set I^{t-1}.

To determine household income in each county, it is necessary to figure out the size of demand faced by firms in different sectors and counties. Local non-tradable goods of county i are solely produced by non-tradable sector firms in county i. Tradable goods, however, can be produced in all counties. Assume that each household's demand for tradable goods is randomly distributed to tradable sector firms across the country. The number of firms in each county is proportional to county size. Therefore, if the representative household in county i increases local non-tradable goods demand, only employment in county i might increase in response. However, if the representative household in county i increases the demand of tradable goods, employment in all counties increases as long as firms can get enough credit from banks. The importance of the asymmetric effects of tradable and non-tradable goods demand are first proposed by [1].

In particular, suppose at time t, households in county i demand $S^i D_t^i$ units of (tradable and non-tradable) goods in total. The total demand for county i specific non-tradable goods, $\alpha S^i D^i$, is solely faced by firms in county i. The pool of tradable goods demand, $(1-\alpha)D_t$, however, is shared by firms in all three counties where $D_t = \sum_j S^j D_t^j$. Firms in county i face S^i share of it. Because county u and county c are ϵ-small, national average demand is determined by households in county m: $D_t = D_t^m$. Thus, I will drop the superscript m occasionally in the rest of this section. Let $M_t^{i,g}$ stand for the demand of sector g goods that each worker[28] in county i faces at time

[26] This is a technical assumption which can simplify our analysis. Alternatively, one can assume that the interest payment is in the form of tradable good. Since the relative magnitude of interest payment is small, it should not significantly change the quantitative result.

[27] Notice that in this model, because the sizes of county u and c are not one, I define D_t^i as the average demand per household instead of aggregate demand of that county at time t.

[28] Recall that each household consists of one unit of workers.

t,[29] then:

$$M^{i,N} = D_t^{i,N} = \alpha D_t^i \tag{1.38}$$

$$M^{i,T} = (1-\alpha) D_t. \tag{1.39}$$

Thus, average demand faced by each worker in county i, M_t^i,

$$M_t^i = M_t^{i,T} + M_t^{i,N} = \alpha D_t^i + (1-\alpha) D_t. \tag{1.40}$$

Banks can lend money to firms in any county. As in the benchmark model, employment (hence household income) depends on demand and whether firms can get a within period loan from banks. Assume all counties confront an aggregate credit availability $\bar{\xi}$. If $\bar{\xi} \geq D_t$, banks can provide enough credit for firms to hire labor. However, if $\bar{\xi} < D_t$, there is rationing of credit. Formally,

$$n_t^i = \underbrace{\alpha D_t^i + (1-\alpha) D_t}_{\text{demand faced by firms in county } i} - \underbrace{(D_t - \bar{\xi})^+}_{\text{aggregate credit shortage}} \tag{1.41}$$

Assume that if firms in county i cannot hire enough labor to fulfill demand due to low credit availability, they import goods from foreigners. Therefore, regardless of the value of $\bar{\xi}$, the representative firm in county i always supplies M_t^i units of goods to consumers (after divided by county size S^i). The amount of imports of county i representative firm is:

$$IM_t^i = IM_t^{i,T} + IM_t^{i,N} = M_t^i - n_t^i. \tag{1.42}$$

Notice that I do not specify the amount of non-tradable imports or tradable imports because only their sum is of interest.

1.6.3 Household's Problem

Let $\Phi(\cdot|I^{t-1})$ denote the household beliefs about $\bar{\xi}$ conditional on I^{t-1}. At the beginning of time k, households in all counties share the same information set I^{k-1} and have identical beliefs.

[29] That is, $M_t^{i,g} = \frac{TM_t^{i,g}}{S^i}$ where $TM_t^{i,g}$ is the total demand of sector g good that firms in county i face at time t.

Pre-recession (k < 0)

The representative household living in county i solves the following problem:

$$Max_{\{d_t^i\}} \ \mathbb{E}\{\sum_{t\geq k} \beta^t u(c_t^i) \mid I^{k-1}\} \tag{1.43}$$

s.t. for each t

$$c_t^i = d_t^i - rb_{t-1}^i, \tag{1.44}$$

$$d_t^i + b_{t-1}^i = w_t n_t^i + b_t^i, \qquad b_t^i \leq \bar{B}. \tag{1.45}$$

$$n_t^i = \alpha D_t^i + (1-\alpha)D_t - (D_t - \bar{\xi})^+, \qquad \bar{\xi} \sim \Phi(\cdot|I^{t-1}) \tag{1.46}$$

After the Recession $k \geq 0$

As in the benchmark model, households face extra credit cost when they increase debt. In particular, households living in county i can increase debt only with additional credit cost $\lambda^i(b_t^i - b_{t-1}^i)^+$. The county i representative household's problem is:

$$Max_{\{d_t^i\}} \ \mathbb{E}\{\sum_{t\geq k} \beta^t u(c_t^i) \mid I^{k-1}\} \tag{1.47}$$

subject to (1.45) (1.46) and

$$c_t^i = d_t^i - rb_{t-1}^i - \lambda^i(b_t^i - b_{t-1}^i)^+ \tag{1.48}$$

Note that the recession and credit costs are unexpected to households before the end of time -1.

1.6.4 Definition of Equilibrium

The competitive equilibrium is defined as:

a. The representative household's policy functions $\{d_t^i(I^{t-1})\}$, $\{c_t^i(I^{t-1})\}$ and $\{b_t^i(I^t)\}$,

b. The representative firm's policy functions $\{n_t^i(I^t)\}$, $\{y_t^i(I^t)\}$ and $\{IM_t^i(I^t)\}$,

c. The representative household's beliefs $\{\Phi(\cdot|I^{t-1})\}$

d. Wage $w_t^i = 1$,

e. Average household demand functions $\{D_t^i(I^{t-1})\}$, such that

for any t,

1. Given wage, average household demand functions $\{D_t^i(I^{t-1})\}$, and household beliefs, the representative household's policy functions solve households' utility maximization problem described in 1.6.3;

2. Given wage and per household demand functions $\{D_t^i(I^{t-1})\}$, the representative firm's policy functions $\{n_t^i(I^t)\}$, $\{y_t^i(I^t)\}$ and $\{IM_t^i(I^t)\}$ satisfy (1.38)-(1.42);

3. The updating of beliefs $\{\Phi(\cdot|I^{t-1})\}$ follows Bayes' rule;

4. The representative household's policy function is consistent with the average household demand function: $D_t^i(I^{t-1}) = d_t^i(I^{t-1})$.

1.6.5 Recession and Recovery

Households in the three counties have different levels of credit costs: $\lambda^m = \lambda$ as in the benchmark model, $\lambda^c = \infty$ and $\lambda^u = 0$. County m and c correspond to regions with more pre-crisis debt or sharper housing price drop, and thus households in those counties find it harder to increase debt after the recession. County u corresponds to regions with less pre-crisis debt or a modest housing price drop so that county u residents can still increase their debt without additional cost.

Assumption 5. *$\lambda^c = \infty$, $\lambda^m = \lambda$ and $\lambda^u = 0$.*

As in the benchmark model, a belief shock before time 0 triggers the recession. Households believe that credit availability permanently shifts to the new steady state $\bar{\xi}^*$ and $\bar{\xi}^* \sim \Phi(\cdot)$. As in the benchmark model, the true value of $\bar{\xi}^* = \bar{\xi}$ so the recession and slow recovery are purely driven by household beliefs instead of a negative real shock. Formally, At any time $t \geq 0$, if uncertainty has been resolved, policy functions are trivial: $d_k^i = D_k^i = n_k^i = \bar{\xi}^*$ for any $k \geq t$. Thus, I focus on the policy functions when uncertainty has not been resolved. As in the benchmark model, I use $\{d_t^{i*}\}$ and $\{D_t^{i*}\}$ to denote the demand when uncertainty has not been resolved. In addition, with exogenous initial belief $\Phi(\cdot|I^{-1}) = \Phi(\cdot)$, the equilibrium can be fully characterized by $\{D_t^{i*}\}$.

The Euler equations of households living in county $i = m, c$ are:

$$\mathbb{E}\{u'(c_t^i)|I^{t-1}\} = \mathbb{E}\{u'(c_{t+1}^i) + \lambda^i u'(c_t^i)\mathbf{1}_{[d_t^i > n_t^i]} - \beta\lambda^i u'(c_{t+1}^i)\mathbf{1}_{[d_{t+1}^i > n_{t+1}^i]} \mid I^{t-1}\} \quad (1.49)$$

As in the benchmarked model, substitute the consistency condition $d_t^i = D_t^i$ into the equation above, we can derieve the recovery paths. Notice that whether credit cost is incurred depends on whether the household income n_t^i is smaller than county i's demand D_t^{i*}. However, the updating of beliefs depend on the aggregate demand, if D_t^*.[30]

1.6.6 Some Qualitative Results

Some qualitative properties can be derived from the model with heterogeneous counties. Because the equilibrium allocation after uncertainty has been resolved is trivial, I only discuss the properties when uncertainty has not been revealed.

The aggregate dynamics are determined by county m because the other two counties are ϵ-small. Therefore, as in the benchmark model without heterogeneity, the recovery speed of aggregate output and employment depends on $\lambda^m = \lambda$. Households in county m have balanced budget in each period.

Households living in county u have zero credit cost. Thus, other things equal, the marginal cost for households to increase consumption is smaller in county u than in other counties. This can be seen by comparing (1.49) for different counties. Thus,

$$D_t^{u*} > D_t^* > D_t^{c*} \tag{1.50}$$

High demand of households living in county u leads to high employment in the non-tradable sector of county u because non-tradable goods specific to county u are produced locally. Each county has the same employment (adjusted by county size) in the tradable sector. This is because tradable goods are produced by all counties. Therefore, employment in county u is always higher than other counties during the recovery.

$$n_t^u > n_t > n_t^c \tag{1.51}$$

For households living in county u, both demand and employment are high. However, $D_t^{u*} > n_t^u$ because only part of their high demand (in particular, high demand for local non-tradable goods) leads to an exclusive increase of local employment. That is,

$$D_t^{u*} > n_t^u, \quad D_t^{c*} < n_t^c \tag{1.52}$$

[30] or equivalently, D_t^{m*} because the other two counties have nearly zero size.

Therefore, during the recovery, households living in county u (c) accumulate (decrease) debt.

$$b_t^u > b_{t-1}^u, \quad b_t^c < b_{t-1}^c \tag{1.53}$$

In this model, for households living in county u, their income levels are restricted by two factors: the aggregate credit availability $\bar{\xi}^*$ and the size of tradable goods demand from other counties, $(1-\alpha)D_t^*$. Given the assumption that there is no real shock to the value of credit availability, the increase of D_t^{u*} depends on the updating of beliefs about $\bar{\xi}^*$ and the growth of aggregate demand D_t^*. Because households in different counties share the same information and beliefs, the growth rates of demand in different counties are the same. That means all counties recover at the same rate.

1.7 Quantitative Analysis

1.7.1 Parameters

I use log preference.[31] I set the interest rate r to 0.01 and the time discount rate $\beta = \frac{1}{1+r}$. Based on [1], I set the share of local goods α to 64.5%.[32]

In my model, employment and output drop at time 0 and recovery starts at $t = 1$. A lower ξ_{min} (the lower bound of the initial belief) leads to a lower household demand and a lower employment at time $t = 0$. Thus, I choose $\xi_{min} = 89.9\%$ to match the trough of recession (in terms of employment).[33]

Because $\Phi(\cdot)$ follows an exponential distribution with rate γ defined in Assumption 4, higher γ means the distribution is less tilted to the right tail and households are more optimistic. More optimistic beliefs result in higher demand, faster resolution of uncertainty, and faster recoveries. Assume that during the recovery from the 1982 recession, households have zero credit cost.[34] Thus, the recovery speed after the 1982

[31] I also try CRRA preference with risk aversion 2 and I get similar quantitative results.

[32] In [1], the estimated ratio between non-tradable sector employment and tradable sector employment is about 0.55. Therefore I set $\alpha = 1 - \frac{0.55}{1+0.55} = 64.5\%$.

[33] Aggregate unemployment data in this section is from FRED, Federal Reserve Bank of St. Louis: *Civilian Unemployment Rate*; Bureau of Labor Statistics. The data is available at `http://research.stlouisfed.org/fred2/series/UNRATE`.

[34] Households were much less leveraged in early 1980s and there was very mild de-leverage after the recession. Thus, it is plausible to assume that in 1982 it was easier for households to get credit after the recession.

recession only depends on γ. When $\gamma = 171$, the model predicted recovery speed equals the recovery speed of employment after the 1981 recession.

Other things equal, if λ is larger, aggregate demand D_t^* is lower. Thus, $D_t^{u*} - D_t^*$ increases in λ. Because part of the cross county difference in demand is transmitted to the difference in employment, $n_t^{u*} - n_t$ also increases in λ. I use per capita debt data by year and county from Federal Reserve Bank of New York Consumer Credit Panel. County u in the model corresponds to counties which increase their unsecured debt the most after the recession in data. County c in the model corresponds to counties which reduce their unsecured debt most after the recession in data. I choose λ such that the model predicted difference in employment between county u and county m is exactly the same as that in data.[35]

1.7.2 Quantitative Results in the Aggregate Level

Figure 1.5 and Table 1.1 summarize the recovery of employment in the aggregate level. The average recovery speed of employment within nine quarters after the trough of the recession is 0.18% per quarter in data. The recovery speed predicted by my model with household credit cost is 0.25% per quarter. If household credit cost is eliminated, the recovery speed is 0.58%. This is essentially the average recovery speed after the 1981 recession because of my calibration strategy.

Table 1.1: Recovery of Employment (per quarter)

Data	Without Household Credit Cost	With Household Credit Cost
0.18%	0.58%	0.25%

[35] The county level data is from Federal Reserve Bank of New York, Consumer Credit Panel, statistics by county. The data is downloaded from `http://www.newyorkfed.org/householdcredit/historical-reports.html`. It should be noted that I only use credit card debt data because the debt in the model is unsecured debt. The county level employment data is from Bureau of Labor Statistics: Local Area Unemployment Statistics.

1.7.3 Quantitative Results in the County Level

My theory predicts that heterogeneity in household credit costs has *level effects* on the employment of different counties: counties with cheaper access to household credit markets have higher employment than other counties. However, heterogeneity in household credit costs has *no growth effects*: the growth rates of employment in all counties are the same. These predictions are illustrated in Figure 1.8. Figure 1.8 shows several important predictions. First, in the recession, employment in county c with infinite credit markets drops more than other county u with zero credit cost. Second, the difference in employment persists. Third, recovery speeds in both county u and county c are almost the same.

[1] group counties by the amount of pre-crisis debt. The underlying assumption is that counties with lower level of pre-crisis debt tend to have cheaper access to household credit markets after the recession. Thus, the counties with high (low) pre-crisis debt in their studies correspond to county c (county u) in my model, respectively. They find that counties with high (low) pre-crisis debt tend to suffer a larger (smaller) drop in employment. The difference in employment persists during the recovery and there is no clear difference in terms of recovery speed. These empirical findings are consistent with the three main predictions of my model.

Alternatively, I group counties by the change of debt during the recovery. The underlying assumption is that counties which have been able to increased debt more during the hard times after the recession tend to have better access to household credit markets. I document the employment levels of counties which have increased debt the most and the least during the recovery. The main results are illustrated in Figure 1.9.

From Figure 1.9 it is clear that the average employment in counties which increase debt most (the blue dashed line) during the recovery have a relatively higher trough. This confirms the first model prediction. The blue dashed line is always above the red solid line and the gap is persist and sizable, which confirms the second and third predictions.

1.8 Conclusion

In this paper I build a model in which the weak household demand arising from aggregate uncertainty leads to recession and slow recovery. The uncertainty is slowly resolved for two reasons. First, information is endogenous. Second, learning process is inefficient. At the aggregate level, the model generates a slow recovery, which is comparable to data. At the county level, my model can explain the cross county differences in levels and growth rates of employment.

Broadly speaking, this paper offers a mechanism by which a temporary belief shock can lead to a persistent output drop due to information frictions. It is worthwhile to explore other implications of this theory in future research. For example, given my theory, countries with larger information frictions are more likely to have persistent output drop. This is closely related to the studies by [28]. Moreover, it is well known in the quantitative sovereign default literature that a persistent output shock is necessary to generate defaults in equilibrium.[36] This paper suggests a positive relationship between default frequency and information frictions. In particular, developing countries default more often might be partially due to the more persistent output drop arising from larger information frictions in those countries.[37]

[36] See [29].
[37] See [30].

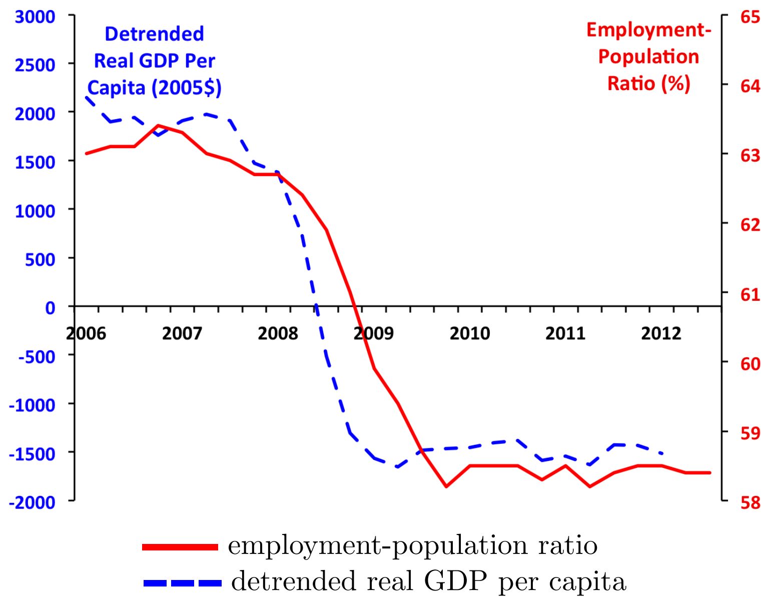


Figure 1.1: U.S. Employment-Population Ratio and Real GDP Per Capita

Data source: FRED, Federal Reserve Economic Data, Federal Reserve Bank of St. Louis: *Real Gross Domestic Product*; U.S. Department of Commerce: Bureau of Economic Analysis; Bureau of Labor Statistics.

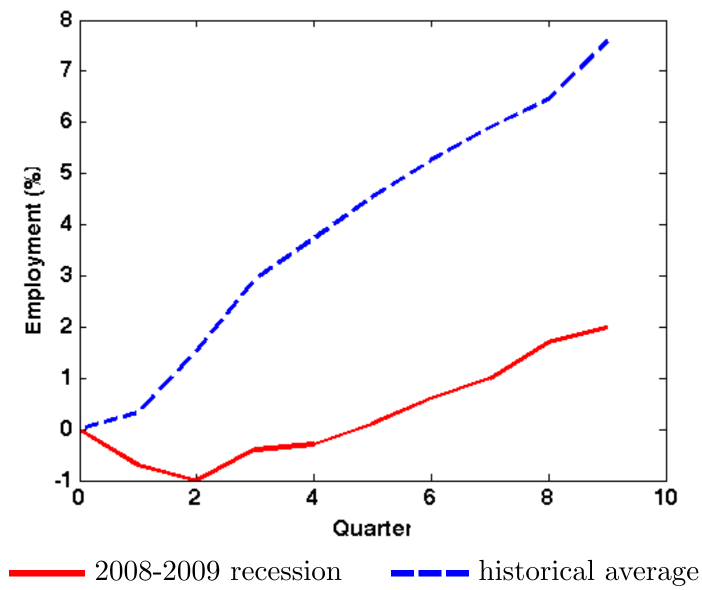


Figure 1.2: Change in U.S. Employment: Recoveries

The horizontal axis shows the number of quarters from the start of recovery. The vertical axis shows the cumulative change of U.S. employment from the start of recovery.
Data source: Federal Reserve Bank of Minneapolis: *The Recession and Recovery in Perspective*; Bureau of Labor Statistics.

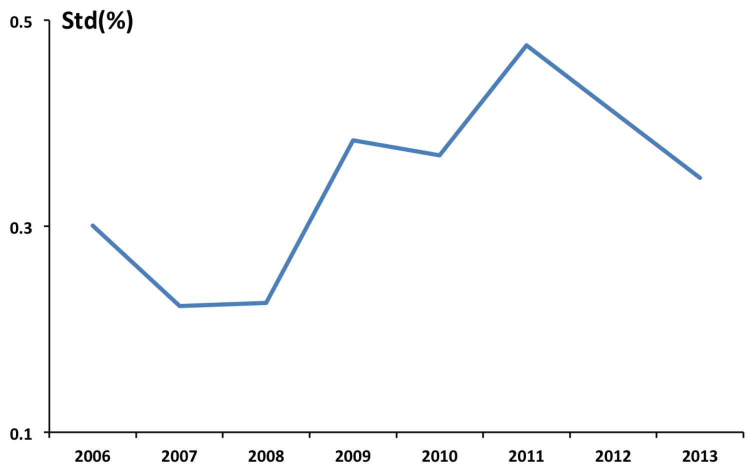
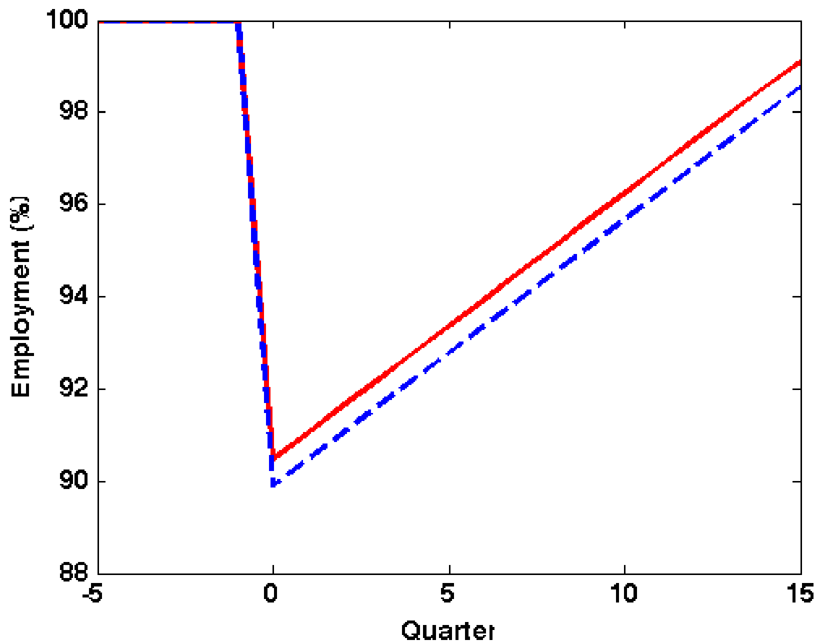


Figure 1.3: Standard Deviation of Professional Forecasts for the Annual Average Rate of Growth in Real GDP Over the Next 10 Years

Data source: Survey of Professional Forecasters, Research Department, Federal Reserve Bank of Philadelphia.



employment   lower bound of the support of $\Phi(\cdot|I^{t-1})$

Figure 1.4: The Recovery of Employment in the Benchmark Model

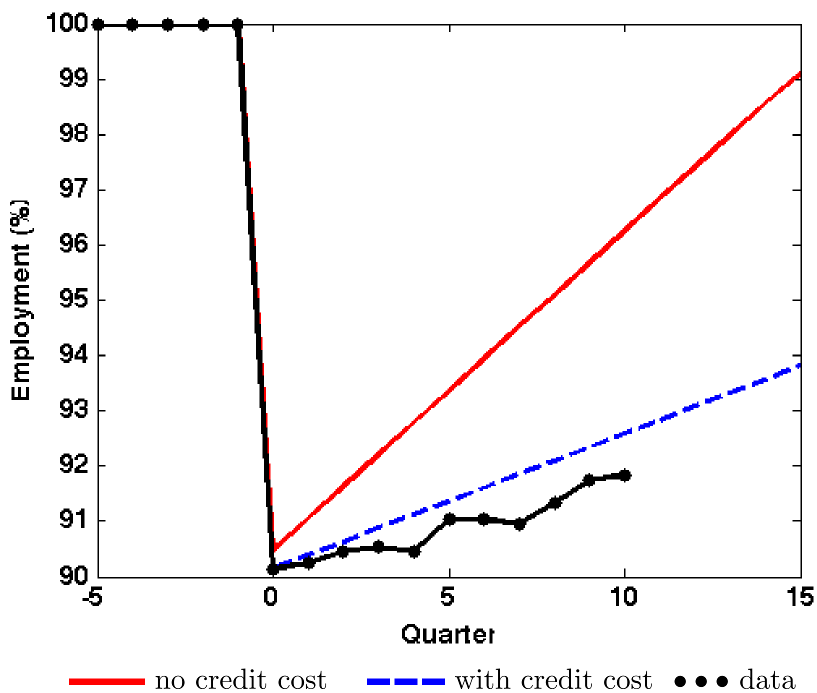


Figure 1.5: The Recovery of Employment: Benchmark Model with Household Credit Costs

Data source: FRED, Federal Reserve Bank of St. Louis: *Civilian Unemployment Rate*; Bureau of Labor Statistics.

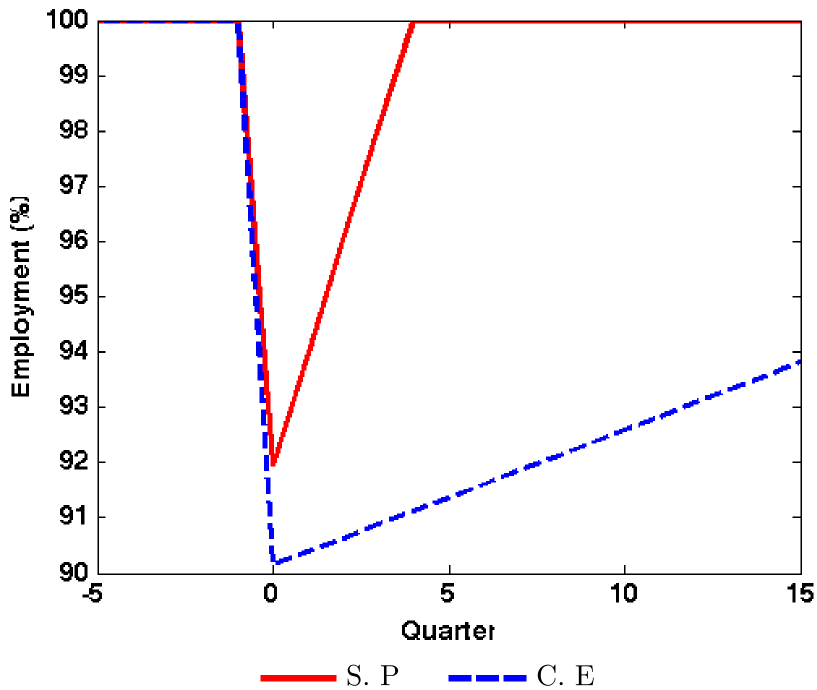


Figure 1.6: The Recovery of Employment: Social Planner and Competitive Equilibrium

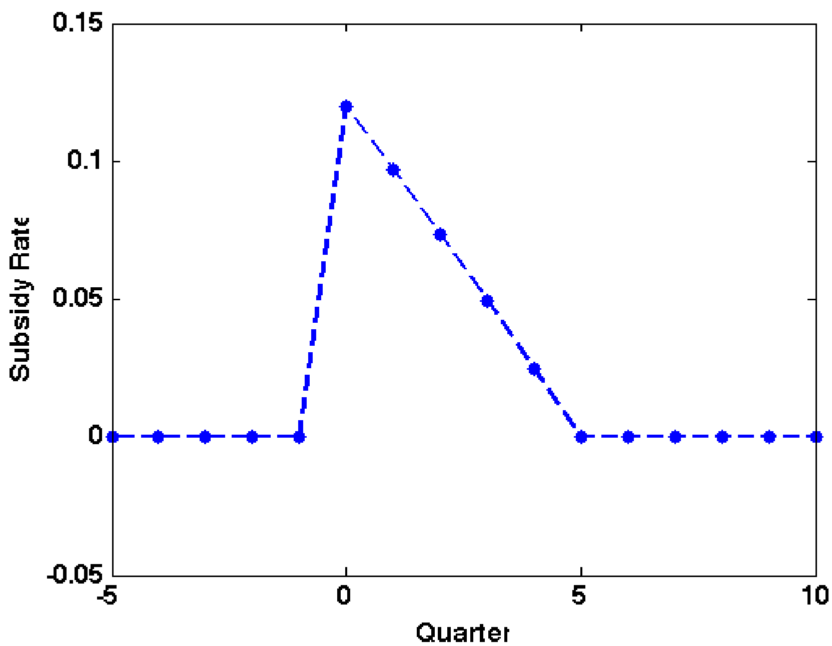


Figure 1.7: Optimal consumption subsidy rate when the uncertainty has not been resolved

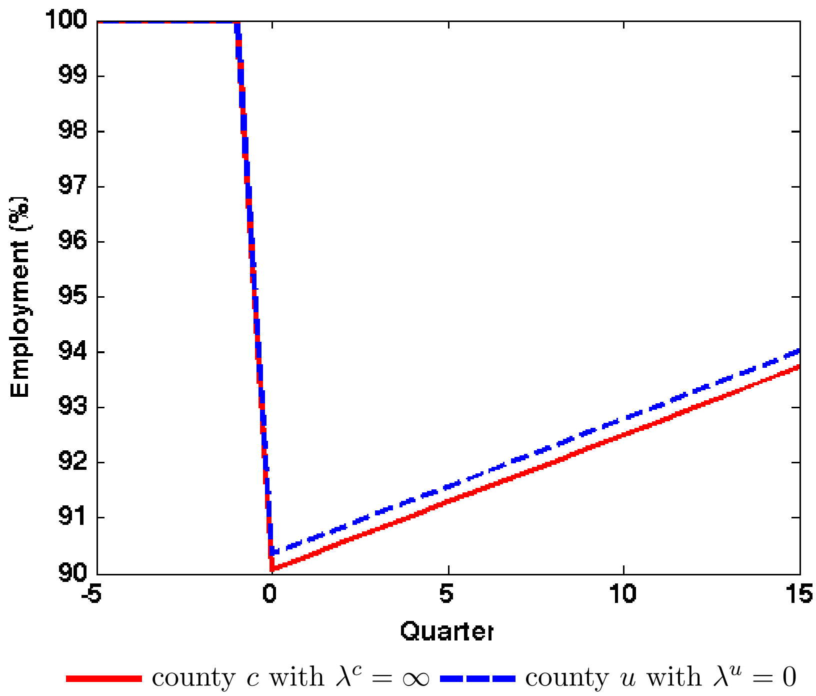


Figure 1.8: Employment in County u and County c: Model Predictions

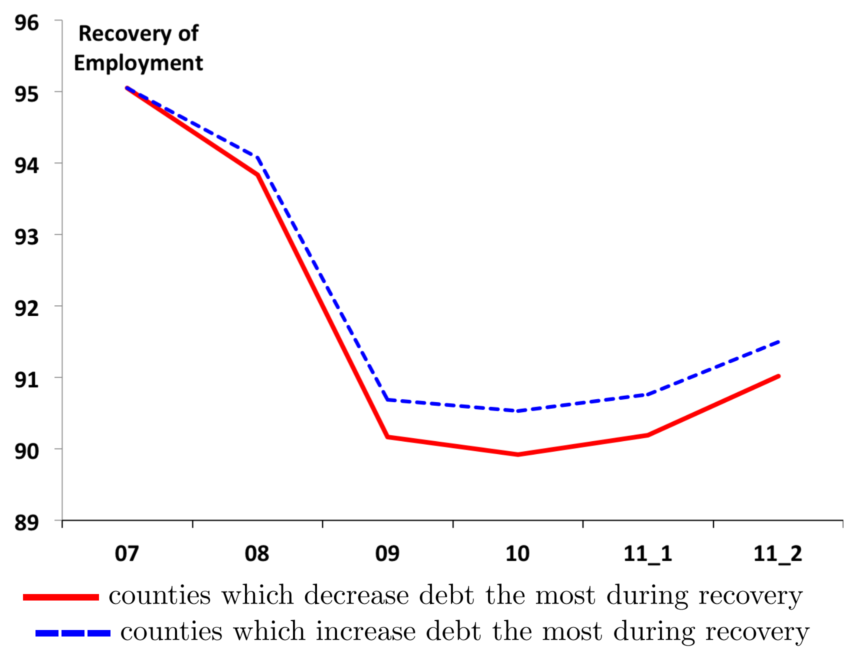


Figure 1.9: Employment Across Counties: Data

Data source: Federal Reserve Bank of New York, Consumer Credit Panel, statistics by county; Bureau of Labor Statistics: Local Area Unemployment Statistics. X-axis represents year and Y-axis represents one minus unemployment (unit: percent). The last two points on the X-axis denote the first and second half of 2011.

Chapter 2

The Impact of Sovereign Default on Lending Countries

2.1 Introduction

[1] The recent European debt crisis creates concerns on the risk exposure of those lending countries that hold a large amount of foreign sovereign debt. The possible default of Greece (or Portugal, Spain and Italy) may worsen the balance sheet of banks in lending countries (e.g., Germany). The deterioration of banks' balance sheet in lending countries may be transmitted to the real economy eventually.

In this paper we build a dynamic model with sovereign bond and banking sectors to analyze the impact of sovereign default on lending countries. We also evaluate whether it is optimal for the governments in lending countries to regulate banks' purchase of foreign sovereign bond. We find that a default in borrowing countries can cause an output drop in lending countries. But it does not necessarily mean that banks' purchase of sovereign bond should be regulated.

In our model, the economy includes two groups of countries: borrowing countries and lending countries. Borrowing countries provide supply of risky sovereign bond, which can be bought by banks in lending countries. Banks can also provide loans to domestic firms to facilitate production. We find that a sovereign default in borrowing

[1] This chapter is co-authored with Yun Pei.

countries always causes a deterioration of banks' balance sheet in the lending countries. In response, banks cut their debt and supply of loans to domestic firms to rebalance their portfolio. Consequently, the lending countries will experience an output drop. This implies that the default risk of borrowing countries results in the output fluctuations in lending countries.

We also conduct policy experiment to see if it is optimal for lending countries' governments to charge a tax on banks' purchase of sovereign bond. We find that an increase in tax will reduce banks' holdings of sovereign bond and therefore output fluctuations. However, bankers' welfare will drop due to higher cost (lower interest rate) of holding sovereign bond. Consequently banks have less wealth and they are less willing to provide loans to domestic firms. Households' welfare will also drop due to lower labor demand. This implies that although the purchase of sovereign bond creates exposure to borrowing countries' default risk, it may not be optimal for governments to regulate the purchase of foreign sovereign bond.

Our paper is closely related to the sovereign default literature.[2] The main difference is that we focus on the effect of default risk on lending countries instead of borrowing countries. We model default as an exogenous event and we try to understand how default risk could cause output drops in lending countries through the wealth loss of banks.[3]

Our paper is also related to the literature on over borrowing problem. Previous studies[4] have discussed the pecuniary externality in international borrowing and lending and its implications on the borrowing countries' policy. In our model, the main externality is that the choice of banks in lending countries affects aggregate output. However, each bank does not fully internalize this effect.

Our paper is also closely related to the literature about the effect of sovereign defaults on real economy through the banking sectors. For example, [38] and [39] discuss the connection between banking crisis and the default of domestic government. In this paper, we focus on the transmission of crisis from borrowing countries to lending countries via banks.

The remainder of the paper is organized as follows. Section 2.2 describes the model

[2] For example, [31], [29] and [25].
[3] This is also related to [32], [33] and[34].
[4] For example, [35], [36] and [37]

environment. Section 2.3 defines and characterizes the equilibrium. Section 2.4 illustrates the connection between sovereign defaults and output fluctuations. Section 2.5 describes the policy experiment in which government imposes a tax on buying sovereign bond. Section 2.6 provides a revision of the model with exogenous and constant debt limit. Section 2.7 concludes.

2.2 Model

We describe the model environment in this section. Time is discrete and infinite. There are a continuum of identical borrowing countries and a continuum of identical lending countries. In each lending country, there are continuum of households, firms and bankers, which are all homogeneous.

2.2.1 Borrowing Countries

The governments in borrowing countries can sell sovereign bond to the banks in lending countries. To focus on the effects of default on lending countries' banking sector and output, we model borrowing countries in a partial equilibrium fashion. The aggregate supply of sovereign bonds is characterized by a price function $q(B')$ such that $q'(B') > 0$, where B' is the amount of bond issued by the sovereign countries. This implies that the interest rate on sovereign bond is decreasing in the amount of debt borrowed. We also assume an exogenous constant default risk $d' = \{0, 1\}$ in each period. Because all borrowing countries are identical, when $d' = 1$ all borrowing countries default.

We assume that after default, borrowing countries are excluded from bond market for one period. They can re-enter the market to sell sovereign from the next period.

2.2.2 Lending Countries: Households

Households in lending countries choose consumption, labor supply and savings to maximize their discounted utility $\sum_{t=0}^{\infty} \delta^t u^h(c_t^h, n_t)$, where $0 < \delta < 1$ is their discount factor, c_t^h is consumption and n_t is labor supply in period t, and $u^h(\cdot, \cdot)$ is the period utility function. They have access to risk free bond issued by bankers. Their period budget constraint is $c_t^h + q_t^m m_{t+1} = \omega_t n_t + m_t + \Pi_t$, where w_t is the wage rate, m_t is the current asset, m_{t+1} is choice of next period's risk free asset holding, q_t^m is the price of risk free asset and Π_t is the profit from firms. Households take prices and profit as given when they make consumption, labor and asset trading decisions. Their maximization problem is thus

$$\max_{c_t, n_t, m_{t+1}} \sum_{t=0}^{\infty} \delta^t u^h(c_t^h, n_t) \tag{2.1}$$

$$\text{s.t.} \qquad c_t^h + q_t^m m_{t+1} = \omega_t n_t + m_t + \Pi_t \tag{2.2}$$

2.2.3 Lending Countries: Firms

Firms produce output using a Cobb-Douglas technology. The representative firm hires labor n_t from households and rents capital from bankers. Output is realized at time $t+1$. For simplicity we assume capital stock is time-invariant, $\bar{k}$. The productivity A is constant. Output is therefore

$$y_{t+1} = A\bar{k}^{\alpha} n_t^{1-\alpha}$$

where capital share $0 < \alpha < 1$.

We assume that firms face working capital constraints. Firms have to make wage payment before production takes place. To hire labor, firms need to take loans from banks in order to finance wage payment. This working capital loan is crucial to our mechanism, and will be explained in more details. For simplicity, we assume that the capital rental cost, however, can be paid to the capital owners (banks) after production takes place. Therefore, firms' problem is

$$\Pi_{t+1} = \max_{\bar{k}, n_t, l_{t+1}^d} A\bar{k}^{\alpha} n_t^{1-\alpha} - r_t^k \bar{k} - l_{t+1}^d \tag{2.3}$$

$$\text{s.t.} \qquad \omega_t n_t = q_t^w l_{t+1}^d \tag{2.4}$$

where r_t^k is the rental rate of capital, $\frac{1}{q_t^w}$ is the gross interest rate on working capital loan and l_{t+1}^d is the amount of loan that has to be paid back at time $t+1$. Firms' problem can be simplified into

$$\Pi_{t+1} = \max_{\bar{k}, n_t} A\bar{k}^{\alpha} n_t^{1-\alpha} - r_t^k \bar{k} - \frac{\omega_t n_t}{q_t^w}$$

2.2.4 Lending Countries: Bankers

Bankers own banks. They have access to three markets. First, they can buy risky sovereign bond from borrowing countries. Second, they can trade risk free asset with domestic households. Third, they can offer working capital loans to domestic firms.

At the beginning of time t, bankers' wealth consists of several parts. First, bankers get the working capital loan (lent to firms at time $t-1$) payment l_t^s from firms. Second, they pay risk free debt m_t to households. Third, they get sovereign bond payment $(1-d_t)b_t$. Fourth, we let bankers own capital stock $\bar{k}$, so they also receive capital income $r_{t-1}^k\bar{k}$. Note the rental rate is determined at time $t-1$ due to the delay in output realization. Last, we also assume that bankers have endowment X each period. Therefore, a banker's wealth at the beginning of time t is $w_t = l_t^s - m_t + b_t + r_{t-1}^k\bar{k} + X$ if default does not happen at time t. However, if borrowing countries default, i.e. $d_t = 1$, bankers suffer wealth loss b_t. Thus a banker's wealth is $w_t = l_t^s - m_t + r_{t-1}^k\bar{k} + X$ if default happens.

Given the wealth at the beginning of the period, bankers choose consumption c_t, next period's risk free debt position m_{t+1}, the amount of newly issued working capital loans $q_t^w l_{t+1}^s$, and next period's sovereign bond holding b_{t+1}. Bankers confront a collateral constraint, $m_{t+1} \leq \theta b_{t+1}$. That is, the amount of debt m_{t+1} that each banker has cannot be more than a fraction θ of his or her holding of sovereign bond b_{t+1}.

We set up the representative banker's problem as a value function with state variables $(w, b; W)$, where w and b are his wealth and bond holding and W is the aggregate wealth of all bankers in all countries. Having w in the value function reduces the number of state variables and simplifies computation, because we do not need to keep track of banker's holding of debts m and loans l^s. A banker's value function is

$$V(w, b; W) = (1-d)V^{nd}(w; W) + dV^d(w-b; W) \tag{2.5}$$

Each individual banker's problem can be written as below:

If borrowers do not default today,

$$\begin{aligned} V^{nd}(w;W) &= \max_{c,m',l^{s'},b'} u(c) + \beta EV(w',b';W') \\ \text{s.t.} \quad & c + q(B')b' = w + q^m m' - q^w l^{s'} \\ & w' = l^{s'} - m' + b' + r^k \bar{k} + X \\ & m' \leq \theta b' \\ & B' = \Gamma_B(W) \\ & W' = \Gamma_W(W) \end{aligned}$$

where β is bankers' discount factor and we assume $0 < \beta < \delta < 1$.

Individual bankers take prices, aggregate wealth and aggregate bond holding as given. The first equation is an individual banker's budget constraint. With wealth w, a banker borrows $q^m m'$ from household, lends $q^w l^{s'}$ to firms, purchases bond b' at unit price $q(B')$ and consumes c. Bond price depends on the aggregate amount of bond issued, so bankers take this price $q(B')$ as given and does not think his choice of b' affects bond price. The second equation is the evolution of the individual banker's wealth. The third equation is the collateral constraint. The last two equations are the evolution of aggregate bond and aggregate wealth.

If borrowing countries default on their sovereign bond,

$$\begin{aligned} V^{d}(w;W) &= \max_{c,m',l^{s'}} u(c) + \beta EV(w',0;W') \\ \text{s.t.} \quad & c = w + q^m m' - q^w l^{s'} \\ & w' = l^{s'} - m' + r^k \bar{k} + X \\ & m' \leq 0 \\ & W' = \Gamma_W(W) \\ & B' = 0 \end{aligned}$$

Notice that here the w and W in $V^d(w;W)$ denote the individual and aggregate wealth after default.

2.3 Recursive Competitive Equilibrium

In this section, we define and characterize the recursive competitive equilibrium in this economy. For the representative bank in each lending country, the current state consists of its own wealth w, the aggregate wealth of all lending countries W and default shock d. Similarly, for the representative household and firm in each country, the current state can be also summarized by (w, W, d).

2.3.1 Definition of Equilibrium

Let $i \in \{d, nd\}$. The recursive competitive equilibrium is a list of

1. individual bankers' value functions $\{V^i(w, W)\}$
2. individual bankers' policy functions $\{c_i(w, W), b'_i(w, W), l_i^{s'}(w, W), m_i^{d'}(w, W)\}$
3. firms' policy functions $\{n_i^d(w, W), l_i^d(w, W)\}$
4. households' policy functions $\{m_i^{s'}(w, W), n_i^s(w, W), c_i^h(w, W)\}$
5. wage, return to capital, risk free rate and the price of working capital loan $\{\omega_i(w, W), r_i^k(w, W), q^m, q_i^w(w, W)\}$

such that

1. given bond price $q(B')$, default risk d', wage, return to capital, risk free rate and the price of working capital loan $\{\omega_i(w, W), r_i^k(w, W), q^m, q_i^w(w, W)\}$, individual bankers' policy functions $\{c_i(w, W), b'_i(w, W), l_i^{s'}(w, W), m_i^{d'}(w, W)\}$ solve bankers' maximization problem.
2. given wage, return to capital and the price of working capital loan, firms' policy functions $\{n_i^d(w, W), l_i^d(w, W)\}$ solve firms' problem.
3. given wage and risk free rate, households' policy functions solve households' problem.

4. markets are clear

$$\begin{aligned} n_i^d &= n_i^s \\ l_i^d &= l_i^s \\ m_i^{d'} &= m_i^{s'} \\ c_i + c_i^h + q(B')b_i' &= A\bar{k}^\alpha n_{-1}^{1-\alpha} + (1-d)b + X \end{aligned}$$

5. consistency conditions for aggregate bond and aggregate wealth are satisfied

$$\begin{aligned} B_i' &= b_i'(W,W) \\ W_i' &= w_i'(W,W) \\ &= b_i'(W,W) - m_i^{d'}(W,W) + l_i^{s'}(W,W) + r_i^k(W,W)\bar{k} + X \end{aligned}$$

2.3.2 Characterization of Equilibrium

The optimality conditions for household's labor supply and saving decisions are respectively

$$-\frac{u_{nt}^h}{u_{ct}^h} = \omega_t \tag{2.6}$$

$$q_t^m u_{ct}^h = \delta u_{ct+1}^h \tag{2.7}$$

For computational tractability, we define utility function as $u^h(c,n) = c - \psi\frac{n^{1+\kappa}}{1+\kappa}$, where $\psi > 0$ and $\kappa > 0$. Therefore optimality conditions are simplified into

$$\psi n_t^\kappa = \omega_t \tag{2.8}$$

$$q_t^m = \delta \tag{2.9}$$

The preference is quasi-linear in consumption. It helps us get rid of the income effect on labor supply and to pin down the saving's return rate with the discount factor, i.e., $\delta\frac{1}{q_t^m} = 1$.

Firms take prices r_t^k and q_t^w as given and the optimal demand for capital and labor are characterized by

$$F_{kt} = \alpha A\bar{k}^{\alpha-1} n_t^{1-\alpha} = r_t^k \tag{2.10}$$

$$F_{nt} = (1-\alpha)A\bar{k}^\alpha n_t^{-\alpha} = \frac{\omega_t}{q_t^w} \tag{2.11}$$

Because firms are competitive and they have constant return to scale production technology, profit is always zero in equilibrium.

Given the equilibrium gross interest rate of working capital loan q_t^w, we can use households' first order conditions and the labor marketing clearing condition to derive wage ω_t and labor n_t

$$
\begin{aligned}
n_t &= \left(\frac{(1-\alpha)q_t^w}{\psi}\right)^{\frac{1}{\alpha+\kappa}} A\bar{k}^{\frac{\alpha}{\alpha+\kappa}} \\
w_t &= \psi\left(\frac{(1-\alpha)q_t^w}{\psi}\right)^{\frac{\kappa}{\alpha+\kappa}} A\bar{k}^{\frac{\alpha\kappa}{\alpha+\kappa}}
\end{aligned}
$$

Once wage and labor are determined, we can pin down all the variables on the production side. Working capital loan is needed to finance the wage payment, so the demand for working capital is

$$
l_t^{d'} = \frac{\omega_t n_t}{q_t^w} = \psi\left(\frac{1-\alpha}{\psi}\right)^{\frac{1+\kappa}{\alpha+\kappa}} A\bar{k}^{\frac{\alpha(1+\kappa)}{\alpha+\kappa}} (q_t^w)^{\frac{1-\alpha}{\alpha+\kappa}}
$$

Because the production is constant return to scale, capital income is α fraction of the output while labor income is $1-\alpha$ fraction of the output. Therefore, output is

$$
y_{t+1} = \frac{1}{1-\alpha}\frac{\omega_t n_t}{q_t^w} = \left(\frac{1-\alpha}{\psi}\right)^{\frac{1-\alpha}{\alpha+\kappa}} A\bar{k}^{\frac{\alpha(1+\kappa)}{\alpha+\kappa}} (q_t^w)^{\frac{1-\alpha}{\alpha+\kappa}}
$$

and capital income is

$$
r_t^k\bar{k} = \frac{\alpha}{1-\alpha}\frac{\omega_t n_t}{q_t^w} = \alpha\left(\frac{1-\alpha}{\psi}\right)^{\frac{1-\alpha}{\alpha+\kappa}} A\bar{k}^{\frac{\alpha(1+\kappa)}{\alpha+\kappa}} (q_t^w)^{\frac{1-\alpha}{\alpha+\kappa}}
$$

Wage, labor, working capital loan, capital income and output are all functions of q_t^w, price of loan from bankers. This is how the real sector and the financial sector are tied together, through the price of reallocating funds. When the borrowing countries default on the debt, the collateral value of bankers' assets decreases and bankers have to rebalance its portfolio in order to satisfy the collateral constraint. This causes an increase in q_t^w because bankers are now less willing to lend due to the tightened collateral constraint. This change in q_t^w in turn causes fluctuations in labor, wage and output, even though firms do not directly face any other shocks.

To characterize banker's optimality conditions, let λ_t and μ_t be the Lagrange multipliers on the budget constraint and the collateral constraint respectively. First-order conditions on b_{t+1}, m_{t+1} and l^s_{t+1} are

$$
\begin{aligned}
q_t(B_{t+1})u'(c_t) &= \beta E(1-d_{t+1})u'(c_{t+1}) + \theta\mu_t && (2.12)\\
q^m u'(c_t) &= q^m \mu_t + \beta E u'(c_{t+1}) && (2.13)\\
q^w_t u'(c_t) &= \beta E u'(c_{t+1}) && (2.14)
\end{aligned}
$$

Equation (2.12) says that the expected marginal value of one more unit of sovereign bond includes two parts: the expected value of payoff when default does not happen and the collateral value of sovereign bond, $\theta\mu_t$. Thus, bankers hold sovereign bonds not only because they want to save but also because they could use it as collateral in order to borrow more from households. Equation (2.13) indicates that the marginal cost of one more unit of debt includes two parts: the decrease of consumption tomorrow, $\beta E u'(c_{t+1})$, and a further tightening of the collateral constraint, $q^m\mu_t$. If the collateral constraint does not bind, i.e., $\mu_t = 0$, then the interest on borrowing from households and the interests on lending to firms are equalized, $q^w = q^m$. If the collateral constraint binds, i.e., $\mu_t > 0$, then $q^m > q^w_t$ or the interest rate on lending is higher than the interest rate on borrowing. There is now a profit opportunity to be made by borrowing more from households at q^m and lend more to firms at q^w_t. But due to the collateral constraint, this is limited.

2.4 Default and Output

To illustrate the effects of default on lending country's output, we do a numerical experiments in this section. The parameters used in the model are listed in Table 2.1. We also set the price schedule of sovereign bond to be a linear function of B' around $B' = 0.3$ with the slope $q'(B') = 0.1$. When $B' = 0.3$, the bond price offered by borrowers is exactly the same as the expected discounted payoff $\beta(1 - P_d)$.

We start the economy at the steady state. We simulate a path where borrowers default in period 0. We are interested in understanding how the production sector is affected due to this default risk.

Table 2.1: Parameter Values

capital share	α	0.32
banker's discount factor	β	0.9
household's discount factor	δ	0.99
collateral ratio	θ	0.5
default probability	P_d	0.1
labor weight in utility	ψ	1
inverse labor elasticity	κ	0.45
capital stock	$\bar{k}$	1
banker's endowment	X	0.35
productivity	A	0.60

Figure (2.1) shows the output path following the default. X-axis is the time period while y-axis is the output level. The economy is at the steady state in period -1 and the borrowers default in period 0. Output drop occurs in period 1 due to our assumption that output is realized with a one period delay. We can see there is a significant drop in output. Following the default, although the borrowers never default again, it takes a few periods for output to slowly grows back to the pre-default level.

In this economy, there is no TFP shock and the capital stock is time-invariant, so the decline in output is fully due to the decline in labor input. Figure (2.2) shows the labor path following the default. The labor path is similar to the output path. It decreases after default and slowly recovers back to the pre-default level in a few periods.

Firms and households do not own any sovereign bonds, so why are they affected by default? When borrowers default, there are two effects on the lending countries. The direct effect is that bankers lose their bond holdings, and the indirect effect is the drop in output. As described above, firms have to borrow funds from banks to finance wage payment. Bankers, on the other hand, borrow funds from households. Bankers use their own assets and the borrowing from households to meet firms' demand for working capital loan. However, bankers have a concave utility function. When borrowers default, bankers lose part of their wealth and become poorer, thus their marginal utility of consuming today increases. To meet their increased desire to consume, bankers would like to borrow more from household and/or lend less to firms. However, the amount of borrowing is limited by the amount of collateral they hold.[5] When default happens, bankers have a wealth loss, and they are forced to borrow less from households. Therefore, the only channel for bankers to smooth consumption is to provide less working capital loans. Less loan supply translates into higher borrowing cost (lower q^w) for firms. Faced with higher borrowing costs, firms cut down their demand for labor, as shown in Figure (2.2). With less labor inputs, output decreases (see Figure (2.1)) despite there is no change in productivity. This is how the default risk propagates to the real economy.

2.5 Government Regulation

In the previous section, we have shown that default risk of borrowing countries alone is able to generate output fluctuations in lending countries. It is trivial to see that if bankers in the lending countries are not allowed to hold any sovereign bond, the shocks in borrowing countries are not going to be transmitted to the lending countries at all. This gives rise to the question: should government in the lending countries regulate the bankers' purchase of sovereign bond?

Government intervention can improve the welfare only if the competitive equilibrium is not efficient. In our model, there are some externalities that the bankers cannot fully internalize, which may justify the need of government intervention. In particular, bankers' choice of sovereign bond holding affects the extent of wealth loss and de-leverage in default episodes, and hence affects working capital loan provision and output. That is, bankers' choice determines the aggregate output risk. However, when bankers decide how much bond to hold, they only consider the private cost of default risk, which is the loss of bond holdings. This at least creates two externalities.

First, the bankers do not take into account of the welfare loss of households in default episodes. To the extreme, if bankers do not hold any sovereign bond, the economy will stay in a steady state without any output fluctuation. Households would not suffer any income loss when default happens. If the government cares about both bankers and households, it can potentially improve households' welfare by limiting bankers' bond holdings.

[5] This is because $m' \leq \theta b'$. In default episode $b' = 0$.

Second, even if the government only cares about bankers' welfare, it may still be a Pareto improvement for the government to limit the bankers' bond holdings. The reason is that bankers' bond holdings affect aggregate income risk. However, each banker takes the aggregate income risk as given. He or she does not internalize the fact that by buying one more unit of sovereign bond, there will be one more dollar aggregate wealth loss tomorrow when default happens and hence the output drop will be deeper. Specifically, each banker does not realize that tomorrow's capital income $r_k'\bar{k}$ is a function of today's choice of B', especially when default happens.

In order to evaluate whether government should regulate the purchase of sovereign bond, we do the following numerical experiment. Specifically, the government in one country can charge a tax τ on buying bond. This affects the price faced by bankers in his country. To these bankers, price is now $q(B')+\tau$. This does not affect the price faced by the borrowers or bankers in other countries. To them, price is still $q(B')$. Bankers still take aggregate wealth W and aggregate bond B as given. It is worth noting that the government in this setup does not have the power to affect aggregate bond or the price function. It should be thought of as the government in a small open economy who can affect the residents in his country but not other international agents.

If borrowers do not default, the banker's problem with tax rate τ can be written as

$$\begin{aligned}
V^{nd}(w;W) \quad &= \quad \max_{c,m',l^{s'},b'} u(c) + \beta E V(w',b';W') \\
\text{s.t.} \qquad &\qquad c + [q(B')+\tau]b' = w + q^m m' - q^w l^{s'} \\
&\qquad w' = l^{s'} - m' + b' + r^k\bar{k} + X \\
&\qquad m' \leq \theta b' \\
&\qquad B' = \Gamma_B(W) \\
&\qquad W' = \Gamma_W(W)
\end{aligned}$$

and if borrowers default,

$$\begin{aligned}
V^d(w;W) &= \max_{c,m',l^{s'}} u(c) + \beta E V(w', 0; W') \\
\text{s.t.} \quad & c = w + q^m m' - q^w l^{s'} \\
& w' = l^{s'} - m' + r^k \bar{k} + X \\
& m' \leq 0 \\
& W' = \Gamma_W(W)
\end{aligned}$$

The Euler equation with respect to b' is

$$(q(B') + \tau)u'(c) = \beta E(1 - d')u'(c') + \theta\mu \tag{2.15}$$

where μ is still the multiplier on the collateral constraint.

Figure (2.3) illustrates the change of bankers' and workers' welfare in response to different tax rates of buying sovereign bonds. Both workers and bankers will have welfare loss if the government charges higher tax on the purchase of sovereign bonds.

For the bankers, although the increase of tax rate could decrease the fluctuation of capital income, it increases the cost of buying sovereign bond. In equilibrium, the increase of the cost of buying bonds (first order) dominates the benefit from the decrease of capital income risk (second order). Therefore bankers have lower welfare when the tax rate is higher.

For the workers, the first order effect of increasing tax rate is the change of wage income. Figure (2.4) shows that if the tax rate moves from 0 to 1.6%, workers' mean wage will drop by about 0.85%. The main reason is that an increase of tax will increase bankers' cost of buying sovereign bonds, hence it has a negative effect on the wealth of bankers. When bankers have lower wealth, they decrease the supply of working capital loan, which in turn lowers the demand of labor.

Figure (2.5) shows the second order effect of increasing tax for the workers: it reduces the fluctuation of labor input. Because workers' utility function is concave, the decrease of labor input fluctuation should have a positive effect on workers' welfare. However, in equilibrium the first order effect (the change of mean wage income) dominates and workers will experience welfare loss if the government increases the tax.

2.6 Constant Debt Limit

In Section 2.4 we show that a default can result in an output drop in the lending countries. One concern is that when default happens, bankers' decrease of debt limit[6] and wealth take place simultaneously. One might argue that a drop of output might come from an exogenous decrease of debt limit after default instead of wealth loss of bankers. Therefore, in this section we assume that all bankers have an exogenous and constant debt limit. That is, a default does not change the debt limit of a banker. We show that a default could still result in an output drop, which comes from the wealth loss of bankers.

In particular, we assume that each banker's borrowing limit is $\bar{M} = 0.15$. We also assume that borrowing countries can immediately re-enter the sovereign bond market after default.[7] Figure 2.6 shows even if bankers' debt limit is exogenous, a default still result in an output loss. This is exactly because the wealth loss causes bankers to consume less and provide less working capital loans.

2.7 Conclusion

In this paper, we propose a mechanism where sovereign default in one country can have adverse effect on the output of another country. Banks in the lending countries purchase risky sovereign bond provided by the borrowing countries. These banks also provide working capital loans to domestic firms. A default by borrowing countries imply a negative shock to banks' balance sheet. Since bankers are risk averse, they reduce their supply of loans to domestic firms in order to smooth consumption. Faced with a higher borrowing cost from banks, firms have to cut their demand for labor. Hence, output decreases. This explains how the default risk in a borrowing country can propagate to the production sector of a lending country. It is perhaps one reason why the European Union is very concerned with the defaults and potential defaults in PIIGS countries.

[6] In our numerical example, in the default episode bankers' debt limit is zero. This is because $b' = 0$ and sovereign bond is the only asset that can be used as collateral.

[7] If we still assume that it takes one period for borrowers to enter the market, ideally we should have another asset A so that bankers can invest more on asset A when they cannot buy sovereign bond. Then we can explore the impact of wealth loss on the portfolio choice and output. Here because we simply assume that bankers can always buy sovereign bonds (even in default episode) to serve this purpose.

On the normative side, we conduct a policy experiment to see if it is optimal for the government in the lending countries to regulate the purchase of foreign sovereign bond. We allow the government to impose a tax on bond price. We find that although a higher tax reduces the volatility of the output, bankers and households' welfare are nevertheless decreased because bankers' wealth and households' labor income are reduced. This negative effect of lowered income dominates the positive effect of decreased fluctuations.

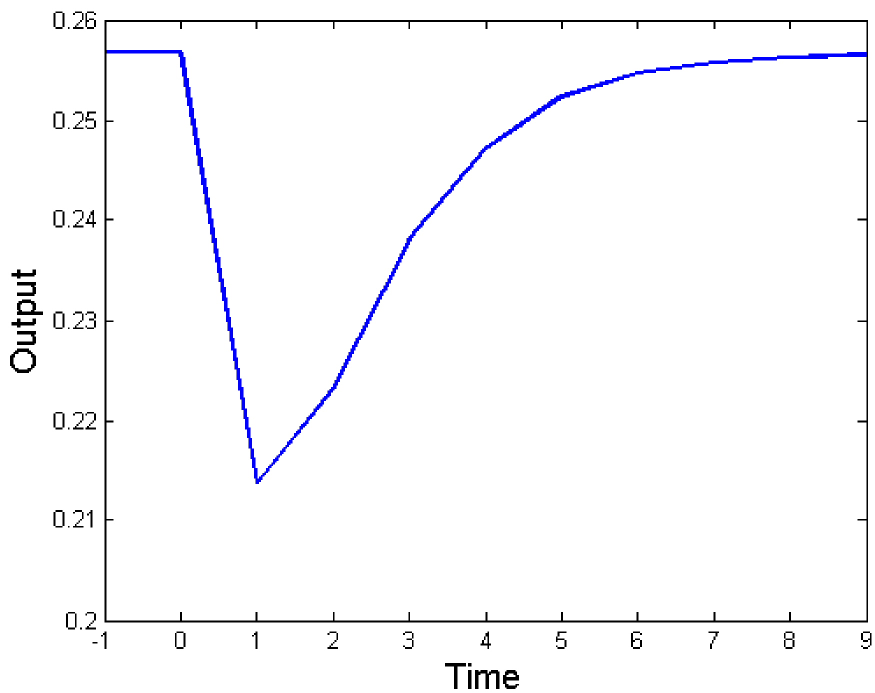


Figure 2.1: Output after Default

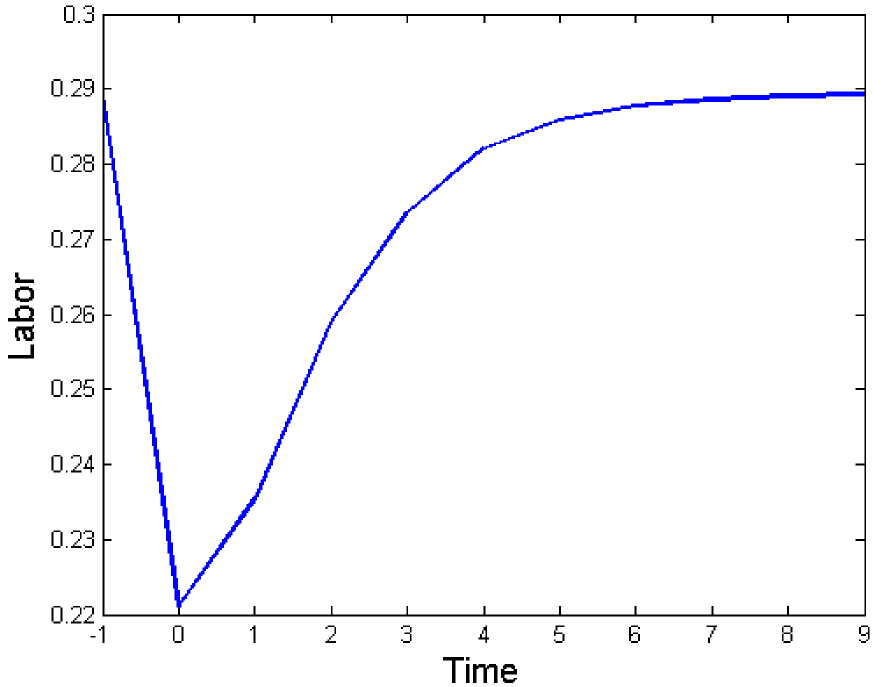


Figure 2.2: Labor after Default

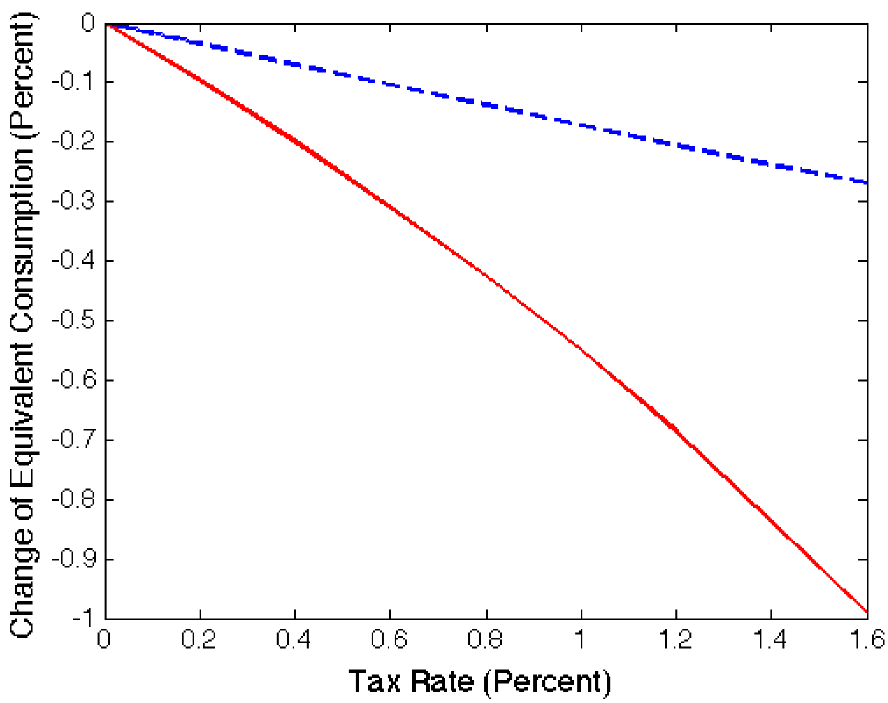


Figure 2.3: Change of Welfare Under Different Tax Rates
Bankers Workers

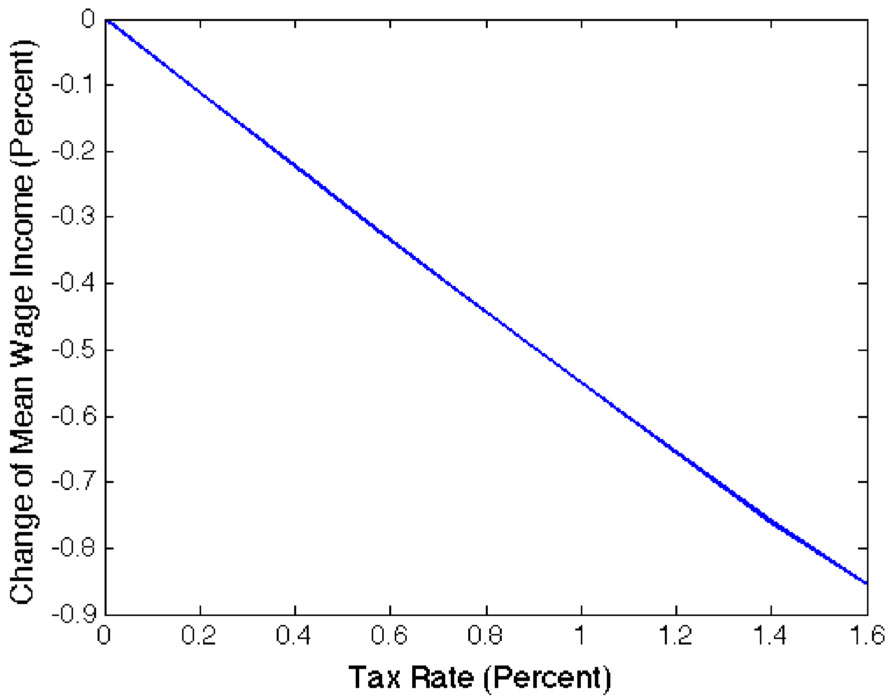


Figure 2.4: Change of Mean Wage Income Under Different Tax Rates

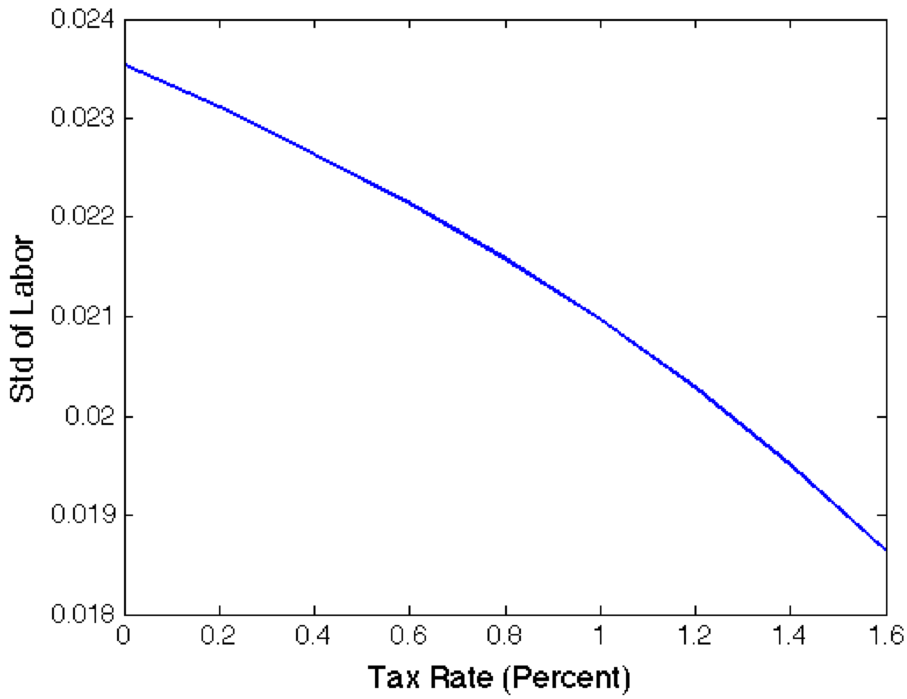


Figure 2.5: Standard Deviation of Labor Input Under Different Tax Rates

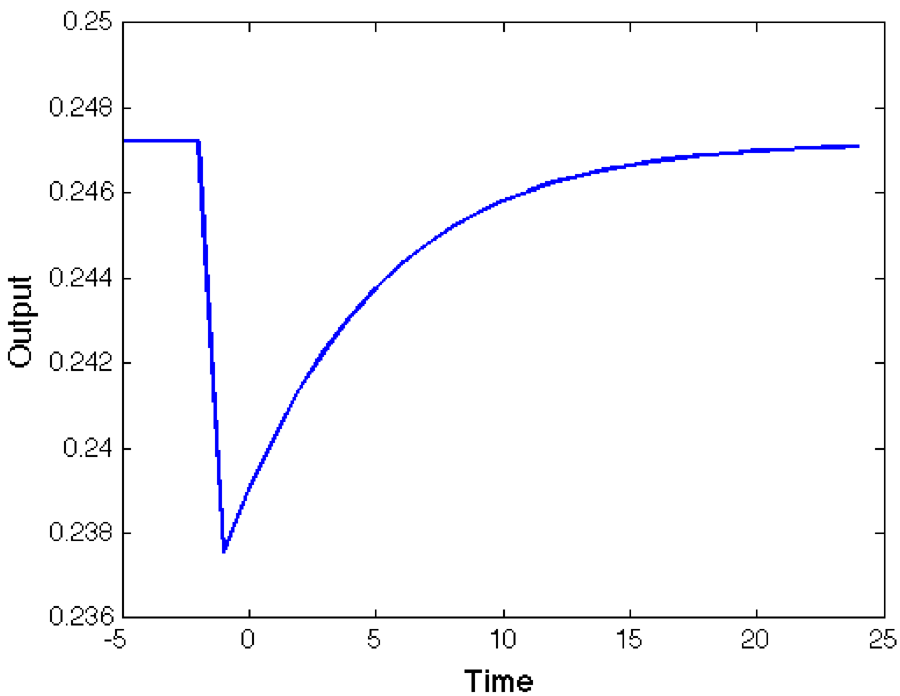


Figure 2.6: Output after Default (with exogenous debt limit)

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Appendix A

The Best Equilibrium

I prove the existence of the best equilibria in the benchmark model with household credit cost (Proposition 3). I will also show how I select the best equilibrium among the set of equilibria.

Denote $\bar{n}^* = min\{1, \bar{\xi}^*\}$ as the maximum level of employment. I assume that the credit cost $f_t = (b_t - b_{t-1} - \epsilon)^+$ where $\epsilon > 0$ is an arbitrarily small positive constant. This makes the household's problem differentiable around equilibrium so that I can use first order conditions to characterize equilibrium.

Define $\Omega_1 = \{\{X_t\}_{t\geq 0} | 1 \geq X_t \geq n_{min}\}$. For a given sequence $\{X_t\}_{t\geq 0} \in \Omega_1$, we construct $\{X'_t\}_{t\geq 0}$ as below: let $X_{-1} = n_{min}$, for any $t \geq 0$, given X_{t-1}, X_{t+1}, we find X'_t that solves[1] :

$$\begin{aligned}
&\int u'(X'_t - rb_{-1} - \lambda(X'_t - \bar{n}^* - \epsilon)^+)\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* \\
&- \lambda \int_{\bar{n}^* < X'_t - \epsilon} u'(X'_t - rb_{-1} - \lambda(X'_t - \bar{n}^* - \epsilon)^+)\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* \\
= &\int_{\bar{n}^* < X'_t} u'(\bar{n}^* - r(b_{-1} + X'_t - \bar{n}^*))\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* \\
&+ \int_{\bar{n}^* \geq X'_t} u'(X_{t+1} - rb_{-1} - \lambda(X_{t+1} - \bar{n}^* - \epsilon)^+)\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* \\
&- \lambda\beta \int_{\bar{n}^* \in [X'_t, X_{t+1} - \epsilon)} u'(X_{t+1} - rb_{-1} - \lambda(X_{t+1} - \bar{n}^* - \epsilon)^+)\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^*.
\end{aligned} \tag{A.1}$$

[1] We will prove the existence of X'_t later.

Where b_{-1} denotes the amount of debt at time $t = -1$. The last term of the RHS of (A.1) is zero if $X_{t+1} - \epsilon < X'_t$. The distribution of $\bar{n}^*$ follows time-varying density function $\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})$. Define $\psi(\{X_t\}) = \{X'_t\}$ and $\psi_k(\{X_t\}) = X'_k$.

Lemma 1. *For any $\{X_k\} \in \Omega_1$, we have $X'_t \leq X_{t+1}$. The inequality is strict if $X_{t+1} = 1$ and $X_{t-1} < 1$.*

Proof of Lemma 1:

Case I: if $X_{t-1} > X_{t+1}$. In this case the last term of the RHS is zero. Suppose $X'_t > X_{t-1}$, it is trivial to prove that the first term of the LHS is strictly smaller than the RHS since $u'(\cdot)$ is strictly decreasing. Therefore we have LHS<RHS since the second term of RHS is non-positive. Therefore it cannot be true that $X'_t > X_{t-1}$ in this case. When $X'_t \leq X_{t-1}$, only the first term of the LHS and the second term of RHS are non-zero. It is easy to see that the only X'_t that can equate (A.1) is $X'_t = X_{t+1}$.

Case II: if $X_{t-1} \leq X_{t+1}$. Suppose $X'_t > X_{t+1}$. The last term of the RHS is zero. It is easy to see that the first term of the LHS is strictly smaller the RHS. Since the second term of the LHS is non-positive, we have LHS<RHS. Therefore it must be true that $X'_t \leq X_{t+1}$.

If $X_{t+1} = 1$ and $X_{t-1} < 1$, from the proof above we have $X'_t \leq X_{t+1} = 1$. Suppose $X'_t = 1$. The last term of the RHS is zero. Since $X_{t-1} < 1$ and $\phi(\cdot|\bar{n}^* \geq X_{t-1})$ is continuous on $[X_{t-1}, 1)$, it is easy to show that the first term of the LHS is strictly smaller than the RHS. Therefore we must have $X'_t < 1$.

Lemma 2. *The left hand side (LHS) of (A.1) is strictly decreasing in X'_t as long as $\lambda < 1$.*

Proof of Lemma 2:

$$\begin{aligned}\frac{\partial LHS}{\partial X'_t} &= -\lambda u'(X'_t - rb_{-1})\phi(X'_t - \epsilon|\bar{n}^* \geq X_{t-1}) \\ &+ (1-\lambda)^2 \int_{\bar{n}^* < X'_t - \epsilon} u''(X'_t - rb_{-1} - \lambda(X'_t - \bar{n}^* - \epsilon)^+)\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* \\ &+ (1-\lambda) \int_{\bar{n}^* \geq X'_t - \epsilon} u''(X'_t - rb_{-1} - \lambda(X'_t - \bar{n}^* - \epsilon)^+)\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* < 0.\end{aligned}$$

The last inequality comes from $u' > 0$ and $u'' < 0$.

Lemma 3. *The right hand side (RHS) of (A.1) is increasing in X'_t on $[n_{min}, X_{t+1}] \cap [n_{min}, 1)$.*

Proof of Lemma 3: for $X'_t \in [n_{min}, X_{t+1}] \cap [n_{min}, 1)$, we have

$$\begin{aligned}\frac{\partial RHS}{\partial X'_t} &= -r \int_{\bar{n}^* < X'_t} u''(\bar{n}^* - r(b_{-1} + X'_t - \bar{n}^*))\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* \\ &+ \lambda\beta u'(X_{t+1} - rb_{-1} - \lambda(X_{t+1} - X'_t - \epsilon)^+)\phi(X'_t|\bar{n}^* \geq X_{t-1}) \\ &+ u'(X'_t - rb_{-1})\phi(X'_t|\bar{n}^* \geq X_{t-1}) \\ &- u'(X_{t+1} - rb_{-1} - \lambda(X_{t+1} - X'_t - \epsilon)^+)\phi(X'_t|\bar{n}^* \geq X_{t-1}) \geq 0\end{aligned}$$

The inequality is strict if $X'_t \geq X_{t-1}$ or $X'_t < X_{t+1}$.

Lemma 4. *The right hand side (RHS) of (A.1) is strictly decreasing in X_{t+1} as long as $\lambda < 1$.*

Proof of Lemma 4:

When $X_{t+1} \geq X'_t + \epsilon$, we have

$$\begin{aligned}\frac{\partial RHS}{\partial X_{t+1}} = &(1-\lambda\beta)(1-\lambda)\int_{\bar{n}^*\in[X'_t, X_{t+1}-\epsilon)} u''(X_{t+1} - rb_{-1} - \lambda(X_{t+1} - \epsilon - \bar{n}^*))\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* \\ &- \lambda\beta u'(X_{t+1} - rb_{-1})\phi(X_{t+1} - \epsilon|\bar{n}^* \geq X_{t-1}) \\ &+ \int_{\bar{n}^*\geq X'_t} u''(X_{t+1} - rb_{-1})\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* < 0\end{aligned}$$

When $X_{t+1} \leq X'_t + \epsilon$, we have

$$\frac{\partial RHS}{\partial X_{t+1}} = \int_{\bar{n}^*\geq X'_t} u''(X_{t+1} - rb_{-1})\phi(\bar{n}^*|\bar{n}^* \geq X_{t-1})\mathrm{d}\bar{n}^* < 0$$

Proposition 4. *$\psi(\cdot)$ is a mapping from Ω_1 to Ω_1.*

Proof of Proposition 4:

Case I: When $X_{t-1} < X_{t+1}$. By the proof of Lemma 1 it is easy to show that LHS<RHS when $X'_t = X_{t+1}$. Suppose $X'_t < X_{t-1}$. Then the second term of the LHS is zero. The first term of the RHS is also zero. It is easy to show that the first term of the LHS is strictly smaller than the second term of the RHS since $u'(\cdot)$ is strictly decreasing. Because the third term of the RHS is non-positive, we have LHS>RHS. Because both LHS and RHS are continuous in X'_t, there must be at least one X'_t to equate both sides. By Lemma 1, $X'_t \leq X_{t+1}$ and $X'_t < 1$. It is also easy to see that $X'_t \geq X_{t-1}$. By Lemma 2 and 3, X'_t is unique and $X'_t \in [X_{t-1}, X_{t+1}]$.

Case II: When $X_{t-1} \geq X_{t+1}$. By the proof of Lemma 1, it is easy to show that $X'_t = X_{t+1}$ is the only X'_t that can equate both sides.

Remark Equation (A.1) essentially comes from the Euler equation. To see this, simply let $X'_t = X_t = D^*_t$, $X_{t-1} = D^*_{t-1}$ and $X_{t+1} = D^*_{t+1}$. Thus finding an equilibrium path $\{D^*_k\}$ is equivalent to find a fixed point of $\psi(\cdot)$.

The next proposition says that $\psi(\cdot) : \Omega_1 \rightarrow \Omega_1$ is monotomic.

Proposition 5. *For any $\{X_t\}_{t\geq 0} \in \Omega_1, \{Y_t\}_{t\geq 0} \in \Omega_1$ and $n_{min} \leq X_t \leq 1$, $n_{min} \leq Y_t \leq 1$, if $X_t \geq Y_t$ for any $t \geq 0$, we have $\psi(\{X_t\}) \geq \psi(\{Y_t\})$.*

To prove it, it is helpful to have two lemmas first.

Lemma 5. *Suppose* $\{X_t\}_{t\geq 0} \in \Omega_1$ *and* $\{Y_t\}_{t\geq 0} \in \Omega_1$. $X_{-1} = Y_{-1} = n_{min}$. *In addition, suppose* $X_{k-1} = Y_{k-1}$ *and* $X_{k+1} \geq Y_{k+1}$ *for some* $k \geq 1$, *then* $X'_k \geq Y'_k$ *where* $X'_k = \psi_k(\{X_t\})$ *and* $Y'_k = \psi_k(\{Y_t\})$.

Proof of Lemma 5:

We prove it by contradiction. Suppose $X'_k < Y'_k$. To simplify notation, let $RHS(Z'_k, Z_{k+1})$ and $LHS(Z'_k, Z_{k+1})$ denote the value of the RHS and LHS of (A.1) when $X'_k = Z'_k$ and $X_{k+1} = Z_{k+1}$. By definition, we have $LHS(X'_k, X_{k+1}) = RHS(X'_k, X_{k+1})$ and $LHS(Y'_k, Y_{k+1}) = RHS(Y'_k, Y_{k+1})$. Since $X'_k < Y'_k$ and $X_{k+1} \geq Y_{k+1}$, by Lemma 3 and 4, we have $RHS(X'_k, X_{k+1}) \geq RHS(X'_k, Y_{k+1}) > RHS(Y'_k, Y_{k+1})$[2] . The LHS is irrelevant to X_{t+1}. By Lemma 2, we have $LHS(X'_k, X_{k+1}) < LHS(Y'_k, X_{k+1}) = LHS(Y'_k, Y_{k+1})$. Therefore we get a contradiction and it cannot be true that $X'_k < Y'_k$.

Lemma 6. *Suppose* $\{X_t\}_{t\geq 0} \in \Omega_1$ *and* $\{Y_t\}_{t\geq 0} \in \Omega_1$. $X_{-1} = Y_{-1} = n_{min}$. *In addition, suppose* $X_{k-1} \geq Y_{k-1}$ *and* $X_{k+1} = Y_{k+1}$ *for some* $k \geq 1$, *then* $X'_k \geq Y'_k$ *where* $X'_k = \psi_k(\{X_t\})$ *and* $Y'_k = \psi_k(\{Y_t\})$.

Proof of Lemma 6:

Case I: $X_{k-1} \geq X_{k+1}$, by the proof of Lemma 1, it is easy to show that $X'_k = X_{k+1}$. We have also shown that $Y'_k \leq Y_{k+1}$. Thus $X'_k \geq Y'_k$ since $X_{k+1} = Y_{k+1}$.

Case II: $X_{k-1} < X_{k+1}$. We prove it by contradiction. Suppose $X'_k < Y'_k$.

Since $X_{k-1} \geq Y_{k-1}$ and $\phi(n)$ follows a truncated exponential distribution when $n < 1$, we have $\phi(\bar{n}^*|\bar{n}^* \geq X_{k-1})$ first order stochastically dominates (FSD) $\phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1})$. In addition, if $X'_k < X_{k+1} - \epsilon$, for any $n \in [X'_k, X_{k+1} - \epsilon)$, we have $\phi(n|n \geq X_{k-1}) > \phi(n|n \geq Y_{k-1})$ since we have proved that $X'_k \geq X_{k-1}$. Therefore we have

$$\begin{aligned}
& -\lambda\beta \int_{\bar{n}^* \in [X'_k, X_{k+1}-\epsilon)} u'(X_{k+1} - rb_{-1} - \lambda(X_{k+1} - \bar{n}^* - \epsilon)^+)\phi(\bar{n}^*|\bar{n}^* \geq X_{k-1})\mathrm{d}\bar{n}^* \\
\leq & -\lambda\beta \int_{\bar{n}^* \in [X'_k, X_{k+1}-\epsilon)} u'(X_{k+1} - rb_{-1} - \lambda(X_{k+1} - \bar{n}^* - \epsilon)^+)\phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1})\mathrm{d}\bar{n}^* \\
\leq & -\lambda\beta \int_{\bar{n}^* \in [Y'_k, Y_{k+1}-\epsilon)} u'(X_{k+1} - rb_{-1} - \lambda(X_{k+1} - \bar{n}^* - \epsilon)^+)\phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1})\mathrm{d}\bar{n}^*
\end{aligned} \tag{A.2}$$

The inequality is strict if $X'_k < X_{k+1} - \epsilon$.

Next, consider the second term of the LHS. Define a function

$$v_1(\bar{n}^*) = \begin{cases} u'(X'_k - rb_{-1} - \lambda(X'_k - \bar{n}^* - \epsilon)^+) & \text{if } \bar{n}^* < X'_k - \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then $v_1(\cdot)$ is decreasing in $\bar{n}^*$. Since $\phi(\bar{n}^*|\bar{n}^* \geq X_{k-1})$ first order stochastically dominates (FSD)

[2] We have proved that $Y'_k < Y_{k+1}$ in Proposition 4.

$\phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1})$, By [40] we have[3] .

$$\lambda \int_{\bar{n}^* < X'_k - \epsilon} v_1(\bar{n}^*) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq X_{k-1}) \leq \lambda \int_{\bar{n}^* < X'_k - \epsilon} v_1(\bar{n}^*) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1}) \tag{A.3}$$

This implies

$$\begin{aligned} &\lambda \int_{\bar{n}^* < X'_k - \epsilon} u'(X'_k - rb_{-1} - \lambda(X'_k - \bar{n}^* - \epsilon)^+) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq X_{k-1}) \\ \leq &\lambda \int_{\bar{n}^* < X'_k - \epsilon} u'(X'_k - rb_{-1} - \lambda(X'_k - \bar{n}^* - \epsilon)^+) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1}) \end{aligned} \tag{A.4}$$

Similarly, define function $v_2(\cdot)$ as

$$v_2(\bar{n}^*) = \begin{cases} u'(\bar{n}^* - r(b_{-1} + X'_k - \bar{n}^*)) - u'(X'_k - rb_{-1} - \lambda(X'_k - \bar{n}^* - \epsilon)^+) & \text{if } \bar{n}^* < X'_k, \\ u'(X_{k+1} - rb_{-1} - \lambda(X_{k+1} - \bar{n}^* - \epsilon)^+) - u'(X'_k - rb_{-1} - \lambda(X'_k - \bar{n}^* - \epsilon)^+) & \text{otherwise.} \end{cases}$$

It is easy to show that $v_2(\cdot)$ is decreasing in $\bar{n}^*$ as long as $u'''(\cdot) > 0$(which is true for CRRA preferences). Similar with (A.3), we have

$$\int_{\bar{n}^* < X'_k - \epsilon} v_2(\bar{n}^*) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq X_{k-1}) \leq \int_{\bar{n}^* < X'_k - \epsilon} v_2(\bar{n}^*) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1}) \tag{A.5}$$

This implies

$$\begin{aligned} &\int_{\bar{n}^* < X'_k} u'(\bar{n}^* - r(b_{-1} + X'_k - \bar{n}^*)) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq X_{k-1}) \\ &+ \int_{\bar{n}^* \geq X'_k} u'(X_{k+1} - rb_{-1} - \lambda(X_{k+1} - \bar{n}^* - \epsilon)^+) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq X_{k-1}) \\ &- \int u'(X'_k - rb_{-1} - \lambda(X'_k - \bar{n}^* - \epsilon)^+) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq X_{k-1}) \\ \leq &\int_{\bar{n}^* < X'_k} u'(\bar{n}^* - r(b_{-1} + X'_k - \bar{n}^*)) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1}) \\ &+ \int_{\bar{n}^* \geq X'_k} u'(X_{k+1} - rb_{-1} - \lambda(X_{k+1} - \bar{n}^* - \epsilon)^+) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1}) \\ &- \int u'(X'_k - rb_{-1} - \lambda(X'_k - \bar{n}^* - \epsilon)^+) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1}) \end{aligned} \tag{A.6}$$

Define $RHS(Z_1, Z_2)$ as the value of the RHS when $X'_k = Z_1$ and $X_{k-1} = Z_2$. Define $LHS(Z_1, Z_2)$ as the value of the LHS when $X'_k = Z_1$ and $X_{k-1} = Z_2$. From (A.2), (A.4) and (A.6), we have

$$RHS(X'_k, X_{t-1}) - LHS(X'_k, X_{t-1}) \leq RHS(X'_k, Y_{t-1}) - LHS(X'_k, Y_{t-1}) \tag{A.7}$$

[3] Strictly speaking (A.3) holds only when $v_1(\cdot)$ is continuous. Here $v_1(\cdot)$ is not continuous at $X'_k - \epsilon$. However, we can always find a sequence of continuous decreasing functions $v_1^j(\cdot) \to v_1(\cdot)$. For each j, we have $v_1^j(n) = v_1(n)$ for $n \notin (X'_k - \epsilon - \delta^j, X'_k - \epsilon + \delta^j)$ where $0 < \delta < 1$ and $\lambda \int_{\bar{n}^* < X'_k - \epsilon} v_1^j(\bar{n}^*) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq X_{k-1})\bar{n}^* \leq \lambda \int_{\bar{n}^* < X'_k - \epsilon} v_1^j(\bar{n}^*) \mathrm{d}\Phi(\bar{n}^*|\bar{n}^* \geq Y_{k-1})\bar{n}^*$. Since $\phi(\cdot)$ has zero measure at $X'_k - \epsilon$, take $j \to \infty$, (A.3) holds.

We have proved that RHS is strictly increasing in the first argument and the LHS is strictly decreasing in the first argument. Since $Y_k' > X_k'$, we have

$$RHS(X_k', Y_{t-1}) - LHS(X_k', Y_{t-1}) < RHS(Y_k', Y_{t-1}) - LHS(Y_k', Y_{t-1}) \tag{A.8}$$

In sum, we have $0 = RHS(X_k', X_{t-1}) - LHS(X_k', X_{t-1}) < RHS(Y_k', Y_{t-1}) - LHS(Y_k', Y_{t-1}) = 0$, which is a contradiction. Thus it must be true that $X_k' \geq Y_k'$.

Proof of Proposition 5:

We prove it by contradiction. Suppose there is a $k > 0$, such that $Y_k' > X_k'$ where $Y_k' = \psi_k(\{Y_t\})$ and $X_k' = \psi_k(\{X_t\})$. Construct a sequence $\{Z_t\}$, such that: $Z_t = Y_t$ for any $t \leq k-1$ and $Z_t = X_t$ for any $t \geq k$. Obviously $\{Z_t\} \in \Omega_1$. Let $Z_k' = \psi_k(\{Z_t\})$. By Lemma 6, $Z_k' \geq Y_k'$. By Lemma 5, $Z_k' \leq X_k'$. Therefore we have $X_k' \geq Y_k'$.

Proof of Proposition 3:

Define a sequence $\{X_t^0\}_{t\geq 0}$ such that $X_t^0 = 1$ for any $t \geq 0$. Obviously $\{X_t^0\}_{t\geq 0} \in \Omega_1$. Recursively define $\{X_t^j\} = \psi(\{X_t^{j-1}\})$ for $j = 1, 2, \ldots$. Using Proposition 4 we have $\{X_t^j\} \in \Omega_1$. Since $X_t^0 = 1$ for any t, obviously $\{X_t^0\} \geq \{X_t^1\}$. Using Proposition 5 we have $\psi(\{X_t^0\}) \geq \psi(\{X_t^1\})$, i.e. $\{X_t^1\} \geq \{X_t^2\}$. Repeat the iteration, we have $\{X_t^j\} \geq \{X_t^{j+1}\}$ for any $j \geq 0$. Since $\{X_t^j\} \in \Omega_1$, it has lower bound n_{min}. Therefore there is a sequence $\{X_t^*\}_{t\geq 0}$ such that $\{X_t^j\}_{t\geq 0} \to \{D_t^{b*}\}_{t\geq 0}$. Obviously $\{D_t^{b*}\}_{t\geq 0}$ is a fixed point of $\psi(\cdot)$ so it is an equilibrium aggregate demand path.

Suppose we have another arbitrary equilibrium aggregate demand path $\{D_t^*\}_{t\geq 0}$. Then $\{D_t^*\}_{t\geq 0} \in \Omega_1$ and $\psi(\{D_t^*\}) = \{D_t^*\}$. Since $\{X_t^0\}_{t\geq 0} \geq \{D_t^*\}_{t\geq 0}$, by Proposition 5, we have $\psi(\{D_t^0\}) \geq \psi(\{D_t^*\})$. Take $\psi(\cdot)$ on both sides and using Proposition 5 repeatedly, we have $\{D_t^{b*}\}_{t\geq 0} \geq \{D_t^*\}_{t\geq 0}$.