# On three-dimensional Navier-Stokes equations with axi-symmetric vortex rings as initial vorticity

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# Dedication

To my grandmother, parents and wife

## Abstract

We consider the classical Cauchy problem for the 3d Navier-Stokes equation with the initial vorticity $\omega_0$ concentrated on a circle, or more generally, a linear combination of such data for circles with common axis of symmetry. We show that natural approximations of the problem obtained by smoothing the initial data satisfy good a-priori estimates which enable us to conclude that the original problem with the singular initial distribution of vorticity has a solution. We impose no restriction on the size of the initial data.

# Contents

# Chapter 1

# Introduction

In fluid dynamics, vortex flow is one of the fundamental types of fluid and gas motion. The most spectacular form, called concentrated vortices, is characterized by local circulation of fluid around a core [**AKO07**]. Among all vortical structures, vortex rings with closed-loop cores are perhaps the most familiar to our daily experience such as the well-known smoke rings of cigarettes and the vortex rings observed in the wakes of aircraft. Actually vortex rings are plentiful in turbulent flows of liquids and gases, but are rarely noticed unless the motion of the fluid is revealed by suspended particles. A striking feature of a vortex ring is that it moves in a direction that is perpendicular to the plane of the ring, in general changing its shape and propagation speed as it proceeds.

The fascinating vortex-ring structure has also been observed in many other physical systems. In [**MK96**, **MK97**], Malevanets and Kapral numerically observed stable links and knot structures in bistable chemical media (using particle simulation of the FitzHughNagumo model), including linked smoke-rings. In block copolymers, Pochan et al. [**P04**] produced a morphological phase of toroidal supramolecule assemblies using a triblock copolymer. In a quantum fluid, a vortex ring is formed by a loop of poloidal quantized flow pattern. It was detected in the superfluid helium by Rayfield and Reif [**RR64**] and more recently in Bose-Einstein condensates by Anderson et al. [**A01**].

It should be emphasized that the size, origin, and mechanism of quantized vortex rings are quite different from those in normal fluids since they exemplify superfluid properties (cf. [**B08**, **BD09**, **D91**]).

Here we are mainly concerning about the vortex rings in classical fluid mechanics. The simplicity of their generation and observation sparked interest of many researchers in mechanics and physics for more than a century [**A09**, **SL92**, **MGK11**]. It should be mentioned the celebrated work [**H1858**] of Helmholtz, which laid the theoretical foundations of the entire vortex dynamics and, in particular, of the theory of the motion of vortex rings in an ideal (inviscid) incompressible fluid. Following Helmholtz's work, many other interesting results on vortex rings (in ideal fluids) are obtained, such as Hill's spherical vortex, the existence of steady vortex rings of permanent shape (proved by [**F70**, **FB74**], see also [**Ni80**, **AS89**]), and the formula for the propagation speed $U$ of vortex rings (obtained by Kelvin in 1867),

$$U \approx \frac{\kappa}{4\pi R_0}\Big(\log\frac{8R_0}{\varepsilon} - \frac{1}{4}\Big) \qquad \text{for } \frac{R_0}{\varepsilon} >> 1, \tag{1.1}$$

where $\kappa$ is the strength of the ring, $\varepsilon$ the radius of the cross-section and $R_0$ the ring radius, see [**F70**, **S92**].

In this thesis, we consider vortex rings in viscous fluids. More precisely, we consider the classical Cauchy problem for the Navier-Stokes equation in $\mathbb{R}^3 \times (0,\infty)$:

$$\left.\begin{aligned} u_t + \operatorname{div}(u\otimes u) + \nabla p - \nu\Delta u &= 0 \\ \operatorname{div} u &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3\times(0,\infty)\,, \tag{1.2}$$

$$u(\,\cdot\,,0) = u_0 \qquad \text{in } \mathbb{R}^3\,. \tag{1.3}$$

We will consider the initial data $u_0$ with vorticity $\omega_0 = \operatorname{curl} u_0$ which is supported on a circle. In terms of the Geometric Measure Theory, $\omega_0$ is a *1-current* of strength $\kappa$ supported on a smooth circle $\gamma$. This means that for any smooth compactly supported test vector field (or, more precisely, 1-form)

$\varphi = (\varphi_1, \varphi_2, \varphi_3)$ we can write

$$\int_{\mathbb{R}^3} \omega_0 \cdot \varphi \, dx = \kappa \int_\gamma \varphi_i(x) \, dx_i \ , \tag{1.4}$$

where the last integral is the classical curve integral (summation over the repeated indices is understood). We will use the notation

$$\omega_0 = \kappa \delta_\gamma \tag{1.5}$$

in this situation. The initial velocity field is recovered from $\omega_0$ via the Biot-Savart law

$$u_0(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \wedge \omega_0(y)}{|x-y|^3} \, dy = -\frac{\kappa}{4\pi} \int_\gamma \frac{(x-y) \wedge dy}{|x-y|^3} \ . \tag{1.6}$$

We note that such $u_0$ has infinite kinetic energy:

$$\int_{\mathbb{R}^3} \frac{1}{2} |u_0|^2 \, dx = +\infty \ , \tag{1.7}$$

due to the contributions from the immediate neighborhood of $\gamma$. The initial datum of this form and its regularized variants are usually referred to as a *vortex ring*. Their study goes back to Kelvin. If $\gamma$ is the circle $(r_0 \cos\theta, r_0 \sin\theta, 0)$ (with $-\pi \le \theta < \pi$) and $\kappa > 0$, we expect from Kelvin's calculations and the regularization due to the viscosity that at time $t$ the ring $\kappa\delta_\gamma$ will "fatten" to thickness $\sim \sqrt{\nu t}$ and will be moving up along the $z-$axis at speed roughly

$$\frac{\kappa}{4\pi r_0} \log \frac{a}{\sqrt{\nu t}} \ , \tag{1.8}$$

where $a$ is a suitable reference length.

Our goal here is to establish the existence of such a solution, although we will not verify rigorously the detailed behavior suggested by Kelvin's calculations. Our estimates will be less precise. On the other hand, our method will be quite robust, and can handle not only one vortex ring, but also a finite or even continuous combination[1] of such as long as they have a common axis of

[1] with coefficients of the same sign

symmetry. The last condition is crucial, our method relies on the rotational symmetry of the situation.

It is instructive to compare our problem with the situation of parallel rectilinear vorticies. When the initial vorticity is supported on a line $l$,

$$\omega_0 = \kappa \delta_l \,, \tag{1.9}$$

the solution of the problem is given simply by the "heat extension" of the initial data. When $l$ is the $x_3-$ axis, one has the text-book solution

$$\omega(x,t) = (0, 0, \kappa \Gamma_2(x_1, x_2, \nu t)) \ , \tag{1.10}$$

where $\Gamma_2(x_1, x_2, \nu t) = \frac{1}{4\pi\nu t} e^{-\frac{x_1^2+x_2^2}{4\nu t}}$ is the 2d heat kernel. The non-linear term vanishes identically on these solutions. Uniqueness is a subtle problem. The uniqueness has been proved in the class of the solutions of the form

$$u = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0) \tag{1.11}$$

(2d Navier-Stokes solutions), see [**GW05**, **GGL05**], but uniqueness among the 3d solutions seems to be open.

When the line $l$ is replaced by a collection of parallel lines $l_i$ and

$$\omega_0 = \sum_i \kappa_i \delta_{l_i} \tag{1.12}$$

or possibly

$$\omega_0 = \int \kappa_i \delta_{l_i} \ d\mu(i) \ , \tag{1.13}$$

where $\mu$ is a probability measure, one no longer has explicit solutions. The existence problem becomes more difficult and was solved only in the 1980s in [**C86**, **GMO88**], see also [**BA94**, **K94**]. Uniqueness is again a subtle issue and is known only in the class (1.11) of 2d solutions, see [**GG05**].

Another class of existence results was obtained in [**GM89**] for small data, see also [**T92**]. In those papers the authors proved both existence and uniqueness (in suitable classes of functions) of the Cauchy problem (1.2), (1.3) for example in the case when the initial data $u_0$ is

$$\omega_0 = \kappa\delta_\gamma\,, \tag{1.14}$$

where $\gamma$ is a smooth closed curve and $\kappa$ is sufficiently small (with the notion of smallness depending on $\gamma$). These results are proved by perturbation theory, and also follow from later works based on perturbation theory, such as [**KT01**].

Our main result in this thesis is the following:

**Theorem 1.1.** *Let $\gamma$ be a circle, $\kappa \in \mathbb{R}$ and $\omega_0 = \kappa\delta_\gamma$. Then the Cauchy problem (1.2), (1.3) for the initial data $u_0$ given by $\omega_0$ has a global solution which is smooth for $t > 0$. The initial condition is satisfied in the following weak sense: for any $\varphi \in C_0^\infty\big(\mathbb{R}^3;\mathbb{R}^3\big)$,*

$$\lim_{t\to 0}\int_{\mathbb{R}^3} \omega(x,t)\cdot\varphi(x)\;dx = \int_{\mathbb{R}^3}\omega_0(x)\cdot\varphi(x)\;dx\;, \tag{1.15}$$

*where $\omega = \operatorname{curl} u$ is the vorticity field.*

**Remarks**

1. Our method can be used to show that the same results hold when $\omega_0 = \int \kappa(\gamma)\delta_\gamma\;d\mu(\gamma)$, where $\mu$ is a probability measure supported on the set of the circles with a given axis of symmetry, and $\kappa(\gamma) \geq 0$ is an integrable function with respect to $\mu$.

2. The sense in which the initial condition $u_0$ is assumed is somewhat weak, see (1.1). A more precise analysis than ours is needed to determine optimal convergence of $\omega(\,\cdot\,,t) \to \omega_0$ as $t \to 0_+$.

We now outline the main ideas involved in the proof. By using the following transformation

$$u(x,t) \mapsto \nu u(x,\nu t)\,,\quad p(x,t) \mapsto \nu^2 p(x,\nu t)\;, \tag{1.16}$$

we can change the first equation in (1.2) to

$$u_t + \operatorname{div}(u\otimes u) + \nabla p - \Delta u = 0\;. \tag{1.17}$$

Therefore, without loss of generality, we can assume $\nu = 1$. Let us work with the vorticity equation (obtained by taking the curl of the Navier-Stokes equations)

$$\omega_t + u\nabla\omega - \omega\nabla u = \Delta\omega \ , \tag{1.18}$$

which simplifies significantly for the axi-symmetric velocity fields with no swirl which we will be considering. The precise definition is as follows.

**Definition 1.2.** (Axi-symmetric vector field). *A vector field $u$ in $\mathbb{R}^3$ is axi-symmetric if there is a coordinate frame in which it can be written as*

$$u = u_r(r,z)e_r + u_\theta(r,z)e_\theta + u_z(r,z)e_z \ , \tag{1.19}$$

*where*

$$e_r = (x_1/r, x_2/r, 0), \quad e_\theta = (-x_2/r, x_1/r, 0), \quad e_z = (0,0,1) \tag{1.20}$$

*and $(r,\theta,z)$ are the usual cylindrical coordinates associated with the frame. The components $u_r$, $u_\theta$ and $u_z$ are independent of $\theta$. The component $u_\theta$ is referred to as the swirl component of the vector field $u$ (in the given frame). If $u_\theta$ vanishes, we say that $u$ has no swirl.*

It is easy to check that the curl of an axi-symmetric vector field $u = u_r e_r + u_z e_z$ with no swirl is of the form

$$\omega = \text{curl } u = (u_{r,z} - u_{z,r})\, e_\theta \ , \tag{1.21}$$

which has only the $e_\theta$ component, where $u_{r,z}$ denotes the partial derivative $\partial u_r/\partial z$, etc. We will seek the solution of (1.18) in the form $\omega = \omega_\theta(r,z,t)e_\theta$ and the velocity field in the form $u = u_r(r,z,t)e_r + u_z(r,z,t)e_z$. The vorticity equation (1.18) simplifies to

$$\left(\frac{\omega_\theta}{r}\right)_t + u\nabla\left(\frac{\omega_\theta}{r}\right) = \Delta\left(\frac{\omega_\theta}{r}\right) + \frac{2}{r}\left(\frac{\omega_\theta}{r}\right)_{,r} . \tag{1.22}$$

The right hand side of (1.22) can be interpreted as the Laplacian in $\mathbb{R}^5 = \{(y_1,\dots,y_4,z)\}$ on functions which depend only on $r = \sqrt{y_1^2 + \cdots + y_4^2}$ and $z$. Therefore the quantity $\frac{\omega_\theta}{r}$ satisfies a maximum principle, see Lemma 4.5.

There are three main ingredients of the proof:

1. Nash-type estimates for the quantity $\frac{\omega_\theta}{r}$ based on equation (1.22) and the div-free nature of the field $u$. These estimates give a good decay of $\left\|\frac{\omega_\theta(t)}{r}\right\|_{L^\infty_x(\mathbb{R}^3)}$ in terms of $t^{-\alpha}$ for suitable $\alpha > 0$, even when the initial condition for $\omega_\theta$ is a Dirac distribution, see (4.29).

2. The use of the conservation of the vorticity flux and momentum, which are respectively the quantities $\int \omega_\theta(r,z)\,dr\,dz$ and $\int r^2\omega_\theta(r,z)\,dr\,dz$.

3. Weighted inequalities for axi-symmetric fields with no swirl, such as

$$\|u\|_{L^\infty_x(\mathbb{R}^3)} \leq C\,\|r\omega_\theta\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega_\theta}{r}\right\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega_\theta}{r}\right\|^{\frac{1}{2}}_{L^\infty_x(\mathbb{R}^3)}. \tag{1.23}$$

Step 1 is achieved by applying of Nash's techniques [**N58**] for estimates of equations with div-free drift. In our case they cannot quite be used directly, due to the singular behavior of the coefficients of $\frac{2}{r}\left(\frac{\omega_\theta}{r}\right)_{,r}$ near the $z-$axis which give extra terms in the Nash-type estimates. Fortunately, the terms have a good sign, see the second term on line 6 in (4.26) in the proof of Lemma 4.9. Inequality (1.23) seems to be of independent interest, and it gives information about $u$ in terms of $\omega_\theta$, the quantity for which we have the most control.

Combining the results 1–3, we can then proceed along similar lines as [**GMO88**]. The uniqueness of the solutions from the above theorem seems to be a difficult open problem. We conjecture that it is possible to prove uniqueness in some natural classes of axi-symmetric solutions without swirl, but uniqueness in the class of all reasonable 3d vector fields may be much harder to prove and one might perhaps even have counter-examples. We plan to consider these topics in a future work.

- Chapter 2 introduce a subcritical theory for Navier-Stokes equations.

- In Chapter 3 prove some weighted inequalities for axi-symmetric vector fields with no swirl.

- Chapter 4 describe some a-priori estimates for natural approximate solutions and prove the main result: Theorem 1.1.

# Chapter 2

# A subcritical theory for Navier-Stokes equations

In this chapter, we introduce a subcritical theory for Navier-Stokes equations, which plays an indispensable role in proving the main result: Theorem 1.1. Although the contents of this section are classical, we can not locate a satisfactory version in the literature. We give a detailed exposition of this theory for future reference. Let us consider the Cauchy problem for the Navier-Stokes equation in $\mathbb{R}^n \times (0,T)$:

$$\left.\begin{array}{rcl} u_t + \operatorname{div}(u \otimes u) + \nabla p - \nu \Delta u & = & 0 \\ \operatorname{div} u & = & 0 \end{array}\right\} \qquad \text{in } \mathbb{R}^n \times (0,T)\,, \tag{2.1}$$

$$u(\,\cdot\,,0) = u_0 \qquad \text{in } \mathbb{R}^n\,. \tag{2.2}$$

The main result of this chapter is the following.

**Theorem 2.1.** *For any div-free initial data $u_0 \in L^p_x(\mathbb{R}^n)$ with $p \in (n,\infty)$, the Cauchy problem (2.1) and (2.2) has a unique solution $u \in C\left([0,T(u_0)), L^p_x(\mathbb{R}^n)\right)$, where $T(u_0) > 0$ is so-called maximum existence time and it is determined by $u_0$. This solution is smooth in $\mathbb{R}^n \times (0,T(u_0))$ and satisfies the Navier-Stokes equations in the classical sense. If $T(u_0) < \infty$, then $\lim_{t \to T(u_0)-} \|u(t)\|_{L^p_x(\mathbb{R}^n)} = \infty$.*

*Moreover for any nonnegative integers $k, h$ and any $0 < \tau < T < T(u_0)$, the following estimate holds*

$$\left\|\nabla_x^k \nabla_t^h u\right\|_{L_t^\infty L_x^p(\mathbb{R}^n \times [\tau, T])} \leq C \ , \tag{2.3}$$

*where $C$ depends only $n, p, k, h, \tau, T,$ and $\|u\|_{C\left([0,T], L_x^p(\mathbb{R}^n)\right)}$.*

## 2.1 Estimates for the heat extension

Let $\Gamma$ be the heat kernel in $\mathbb{R}^n$:

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \ . \tag{2.4}$$

Let $G$ be the Gaussian distribution in $n$-dimension:

$$G(z) = \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4}} \ . \tag{2.5}$$

Let $F$ be the heat extension of $f$:

$$\begin{aligned} F(x,t) = (\Gamma(\cdot,t) * f)(x) &= \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} f(y)\ dy \\ &= \int_{\mathbb{R}^n} G(z) f\left(x - \sqrt{t} z\right)\ dz \ . \end{aligned} \tag{2.6}$$

**Lemma 2.2.** *For any nonnegative integers $k$, $h$, $1 \leq p \leq q \leq \infty$ and $t > 0$, the following estimate holds*

$$\left\|\nabla_x^k \nabla_t^h F(t)\right\|_{L_x^q(\mathbb{R}^n)} \lesssim t^{-h-\frac{k}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L_x^p(\mathbb{R}^n)} \ . \tag{2.7}$$

**Proof** . It is easy to check that

$$\left|\nabla_x^k \nabla_t^h \Gamma(x,t)\right| \lesssim t^{-h-\frac{k+n}{2}} \exp\left(-C\frac{|x|^2}{t}\right) \tag{2.8}$$

for a suitable constant $C$ depending on $k$, $h$ and $n$. By (2.8) and Young's inequality,

$$\begin{aligned} &\left\|\nabla_x^k \nabla_t^h F(t)\right\|_{L_x^q(\mathbb{R}^n)} \\ &= \left\|\nabla_x^k \nabla_t^h \Gamma(\cdot,t) * f\right\|_{L_x^q(\mathbb{R}^n)} \lesssim \left\|\nabla_x^k \nabla_t^h \Gamma(\cdot,t)\right\|_{L_x^r(\mathbb{R}^n)} \|f\|_{L_x^p(\mathbb{R}^n)} \\ &\lesssim t^{-h-\frac{k}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L_x^p(\mathbb{R}^n)} \ . \end{aligned} \tag{2.9}$$

where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{p}$. □

**Remark 2.3.** *In view of (2.7), we have (in the case of $q=p$)*

$$\left\|t^{h+\frac{k}{2}}\nabla_x^k\nabla_t^h F(t)\right\|_{L_t^\infty L_x^p(\mathbb{R}^n\times(0,\infty))}\lesssim\|f\|_{L_x^p(\mathbb{R}^n)}\ . \tag{2.10}$$

We next prove $t^{h+\frac{k}{2}}\nabla_x^k\nabla_t^h F$ belongs to $C\left([0,\infty);L_x^p(\mathbb{R}^n)\right)$ if $f\in L_x^p(\mathbb{R}^n)$ for $1\le p<\infty$.

**Lemma 2.4.** *Assume $f\in L_x^p(\mathbb{R}^n)$ for $1\le p<\infty$. Let $F$ be the heat extension of $f$. Then for any nonnegative integers $k$ and $h$, $t^{h+\frac{k}{2}}\nabla_t^h\nabla_x^k F$ belongs to $C\left([0,\infty);L_x^p(\mathbb{R}^n)\right)$ and as $t\to 0+$,*

$$\begin{aligned}&F(\cdot,t)\to f\quad in\quad L_x^p(\mathbb{R}^n)\ ,\\&t^{h+\frac{k}{2}}\nabla_t^h\nabla_x^k F(\cdot,t)\to 0\quad in\quad L_x^p(\mathbb{R}^n)\quad if\quad h\neq 0\quad or\quad k\neq 0\ .\end{aligned} \tag{2.11}$$

**Proof** . By (2.6) and integration by parts, we obtain the spatial derivatives of $F$

$$\begin{aligned}\nabla_x^k F(x,t)&=\int_{\mathbb{R}^n}G(z)\nabla_x^k f\left(x-\sqrt{t}z\right)\,dz\\&=\frac{1}{\left(-\sqrt{t}\right)^k}\int_{\mathbb{R}^n}G(z)\left(-\sqrt{t}\right)^k\nabla_x^k f\left(x-\sqrt{t}z\right)\,dz\\&=\frac{1}{t^{\frac{k}{2}}}\int_{\mathbb{R}^n}\nabla_z^k G(z)f\left(x-\sqrt{t}z\right)\,dz\ .\end{aligned} \tag{2.12}$$

Since $F$ is the heat extension of $f$, the time derivative $\nabla_t$ of $F$ is the same as $\Delta_x$ of $F$ and hence by (2.12), we obtain

$$\nabla_t^h\nabla_x^k F(x,t)=\frac{1}{t^{h+\frac{k}{2}}}\int_{\mathbb{R}^n}\Delta_z^h\nabla_z^k G(z)f\left(x-\sqrt{t}z\right)\,dz\ . \tag{2.13}$$

Thus,

$$t^{h+\frac{k}{2}}\nabla_t^h\nabla_x^k F(x,t)=\int_{\mathbb{R}^n}\Delta_z^h\nabla_z^k G(z)f\left(x-\sqrt{t}z\right)\,dz\ . \tag{2.14}$$

By Minkowski's inequality, we have

$$\begin{aligned}&\left\|s^{h+\frac{k}{2}}\nabla_t^h\nabla_x^k F(\cdot,s)-t^{h+\frac{k}{2}}\nabla_t^h\nabla_x^k F(\cdot,t)\right\|_{L_x^p(\mathbb{R}^n)}\\&\le\int_{\mathbb{R}^n}\left|\Delta_z^h\nabla_z^k G(z)\right|\left\|f\left(\cdot-\sqrt{s}z\right)-f\left(\cdot-\sqrt{t}z\right)\right\|_{L_x^p(\mathbb{R}^n)}\,dz\ .\end{aligned} \tag{2.15}$$

Noting that $\left\| f\left(\cdot-\sqrt{s}z\right)-f\left(\cdot-\sqrt{t}z\right)\right\|_{L^p_x(\mathbb{R}^n)} \le 2\left\|f\right\|_{L^p_x(\mathbb{R}^n)}$, by Lebesgue's dominated convergence theorem and the mean-continuity property of $L^p$ integrals ($1 \le p < \infty$), we can conclude that the LHS of (2.15) goes to zero as $s \to t$, which implies $t^{h+\frac{k}{2}}\nabla_t^h\nabla_x^k F \in C\left([0,\infty)\,;L^p_x(\mathbb{R}^n)\right)$. In particular, we have

$$t^{h+\frac{k}{2}}\nabla_t^h\nabla_x^k F(\cdot,t) \to f\int_{\mathbb{R}^n}\Delta_z^h\nabla_z^k G(z)\,dz \quad \text{in} \quad L^p_x(\mathbb{R}^n) \quad \text{as} \quad t\to 0\ , \tag{2.16}$$

which is (2.11). $\square$

## 2.2 The mild solutions and bilinear form

By div-free condition, we rewrite Navier-Stokes equations (2.1) as

$$u_t+\nabla p-\Delta u=-\frac{\partial}{\partial x_k}\left(u_k u\right)\ . \tag{2.17}$$

We transform Navier-Stokes equations (2.1) into its integral form. Denoting the Helmholtz projection of vector fields on div-free fields by $P$, we have the well-known representation formula

$$u(x,t)=\int_{\mathbb{R}^n}\Gamma(x-y,t)u_0(y)\,dy-\int_0^t\int_{\mathbb{R}^n}\Gamma(x-y,t-s)P\left(\frac{\partial}{\partial x_k}\left(u_k u\right)\right)(y,s)\,dyds\ . \tag{2.18}$$

This can be written more concretely in terms of the Stokes kernel

$$K_{ij}(x,t)=\left(-\delta_{ij}\Delta+\frac{\partial^2}{\partial x_i\partial x_j}\right)\Phi(x,t)\ , \tag{2.19}$$

where the "generating function" $\Phi$ is defined in terms of the fundamental solution $G_\Delta$ of the Laplace operator and the heat kernel $\Gamma$ by

$$\Phi(x,t)=\int_{\mathbb{R}^n}\Gamma(x-y,t)G_\Delta(y)\,dy=\left(e^{t\Delta}G_\Delta\right)(x)\ , \tag{2.20}$$

where $e^{t\Delta}$ is the solution operator of the heat equation. Let

$$K_{ijk}(t)=\frac{\partial}{\partial x_k}K_{ij}(t)=\frac{\partial}{\partial x_k}\left(-\delta_{ij}\Delta+\frac{\partial^2}{\partial x_i\partial x_j}\right)\left(e^{t\Delta}G_\Delta\right)\ . \tag{2.21}$$

It is easy to see that

$$e^{\tau\Delta} K_{ijk}(t) = K_{ijk}(t+\tau) \ . \tag{2.22}$$

Using (2.21), we can rewrite (2.18) as

$$u_i(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t) u_{0i}(y)\, dy - \int_0^t \int_{\mathbb{R}^n} K_{ijk}(x-y,t-s) u_k(y,s) u_j(y,s)\, dyds \ , \tag{2.23}$$

We set

$$\begin{aligned} B(u,v)_i(x,t) &= -\int_0^t \int_{\mathbb{R}^n} K_{ijk}(x-y,t-s) u_k(y,s) v_j(y,s)\, dyds \\ &= -\int_0^t K_{ijk}(t-s) * \left(u_k(s) v_j(s)\right)(x)\, ds \ . \end{aligned} \tag{2.24}$$

We use $U$ to denote the heat extension of the initial data $u_0$. The equation (2.23) can be written as

$$u = U + B(u,u) \ . \tag{2.25}$$

By definition, a *mild solution* is a function $u$ satisfying (2.23) or (2.25). The kernel $K$ satisfies the following estimates:

$$\left|\nabla_x^l K_{ijk}(x,t)\right| \lesssim \frac{1}{\left(|x|^2+t\right)^{\frac{l+n+1}{2}}} \ . \tag{2.26}$$

For any $T>0$, $n<p<\infty$ and nonnegative integer $N$, let $X_{p,N,T}$ be the Banach space equipped with the norm

$$\|u\|_{X_{p,N,T}} = \sum_{l=0}^{N} \left\| t^{\frac{l}{2}} \nabla_x^l u \right\|_{C\left([0,T]; L_x^p(\mathbb{R}^n)\right)} \ , \tag{2.27}$$

where

$$\|w\|_{C\left([0,T]; L_x^p(\mathbb{R}^n)\right)} = \max_{0\le t\le T} \|w(t)\|_{L_x^p(\mathbb{R}^n)} \ . \tag{2.28}$$

The following lemma shows that $B : X_{p,N,T} \times X_{p,N,T} \to X_{p,N,T}$ is bounded.

**Lemma 2.5.** *Let $T>0$, $n<p<\infty$ and $N$ be a nonnegative integer. Then for any $u, v \in X_{p,N,T}$, the following estimate holds*

$$\|B(u,v)\|_{X_{p,N,T}} \le M_1 T^{\frac{1}{2}-\frac{n}{2p}} \|u\|_{X_{p,N,T}} \|v\|_{X_{p,N,T}} \ , \tag{2.29}$$

*where $M_1$ is independent of $u$, $v$, $T$ but depends on $n$, $p$ and $N$.*

**Proof** . Pick an arbitrary $i \in \{1, \cdots, n\}$. Fix $t \in (0, T]$. Fix $l \in \{0, \cdots, N\}$. We split (2.24) into two parts:

$$\begin{aligned} & B(u,v)_i(t) \\ & = -\int_0^{\frac{t}{2}} K_{ijk}(t-s) * (u_k(s)v_j(s))\ ds - \int_{\frac{t}{2}}^{t} K_{ijk}(t-s) * (u_k(s)v_j(s))\ ds \end{aligned} \tag{2.30}$$

Then

$$\begin{aligned} & t^{\frac{l}{2}} \nabla_x^l B(u,v)_i(t) \\ & = -t^{\frac{l}{2}} \int_0^{\frac{t}{2}} \nabla_x^l K_{ijk}(t-s) * (u_k(s)v_j(s))\ ds - t^{\frac{l}{2}} \int_{\frac{t}{2}}^{t} K_{ijk}(t-s) * \nabla_x^l (u_k(s)v_j(s))\ ds \\ & \equiv I_1 + I_2 \end{aligned} \tag{2.31}$$

By Minkowski's inequality, Young's inequality, Hölder's inequality and (2.26), we have

$$\begin{aligned} \|I_1\|_{L_x^p(\mathbb{R}^n)} & \lesssim t^{\frac{l}{2}} \int_0^{\frac{t}{2}} \left\| \nabla_x^l K_{ijk}(t-s) * (u_k(s)v_j(s)) \right\|_{L_x^p(\mathbb{R}^n)} ds \\ & \lesssim t^{\frac{l}{2}} \int_0^{\frac{t}{2}} \left\| \nabla_x^l K_{ijk}(t-s) \right\|_{L_x^{\frac{p}{p-1}}(\mathbb{R}^n)} \|u_k(s)v_j(s)\|_{L_x^{\frac{p}{2}}(\mathbb{R}^n)} ds \\ & \lesssim t^{\frac{l}{2}} \int_0^{\frac{t}{2}} (t-s)^{-\frac{l+1+n/p}{2}} \|u_k(s)\|_{L_x^p(\mathbb{R}^n)} \|v_j(s)\|_{L_x^p(\mathbb{R}^n)} ds \\ & \lesssim t^{\frac{1}{2}-\frac{n}{2p}} \|u\|_{X_{p,N,T}} \|v\|_{X_{p,N,T}} \lesssim T^{\frac{1}{2}-\frac{n}{2p}} \|u\|_{X_{p,N,T}} \|v\|_{X_{p,N,T}} \end{aligned} \tag{2.32}$$

since $n < p$ and $t \leq T$. We consider $I_2$. Observe that $\nabla_x^l (u_k(s)v_j(s))$ is a finite sum of some terms of the form $\nabla_x^{l_1} u_k(s) \nabla_x^{l_2} v_j(s)$ with $l_1 + l_2 = l$. Therefore, by the triangle inequality and Hölder's inequality, we obtain

$$s^{\frac{l}{2}} \left\| \nabla_x^l (u_k(s)v_j(s)) \right\|_{L_x^{\frac{p}{2}}(\mathbb{R}^n)} \lesssim \|u\|_{X_{p,N,T}} \|v\|_{X_{p,N,T}} \tag{2.33}$$

for every $\frac{t}{2} \leq s \leq t$ since $t \leq T$. By Minkowski's inequality, Young's inequality, (2.26) and (2.33), we have

$$\begin{aligned}
\|I_2\|_{L^p_x(\mathbb{R}^n)} &\lesssim t^{\frac{l}{2}} \int_{\frac{t}{2}}^{t} \left\| K_{ijk}(t-s) * \nabla_x^l \left(u_k(s) v_j(s)\right) \right\|_{L^p_x(\mathbb{R}^n)} ds \\
&\lesssim t^{\frac{l}{2}} \int_{\frac{t}{2}}^{t} \|K_{ijk}(t-s)\|_{L_x^{\frac{p}{p-1}}(\mathbb{R}^n)} \left\| \nabla_x^l \left(u_k(s) v_j(s)\right) \right\|_{L_x^{\frac{p}{2}}(\mathbb{R}^n)} ds \\
&\lesssim t^{\frac{l}{2}} \int_{\frac{t}{2}}^{t} \|K_{ijk}(t-s)\|_{L_x^{\frac{p}{p-1}}(\mathbb{R}^n)} s^{-\frac{l}{2}} \, ds \, \|u\|_{X_{p,N,T}} \|v\|_{X_{p,N,T}} \\
&\lesssim \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1+n/p}{2}} \left(\frac{t}{s}\right)^{\frac{l}{2}} ds \, \|u\|_{X_{p,N,T}} \|v\|_{X_{p,N,T}} \\
&\lesssim t^{\frac{1}{2}-\frac{n}{2p}} \|u\|_{X_{p,N,T}} \|v\|_{X_{p,N,T}} \lesssim T^{\frac{1}{2}-\frac{n}{2p}} \|u\|_{X_{p,N,T}} \|v\|_{X_{p,N,T}} \ .
\end{aligned} \tag{2.34}$$

In view of (2.31),(2.32) and (2.34), we get

$$\left\| t^{\frac{l}{2}} \nabla_x^l B(u,v)_i(t) \right\|_{L^p_x(\mathbb{R}^n)} \lesssim T^{\frac{1}{2}-\frac{n}{2p}} \|u\|_{X_{p,N,T}} \|v\|_{X_{p,N,T}} \ , \tag{2.35}$$

which implies (2.29) by the arbitrariness of $i$, $t$ and $l$. However we have one more work to do. We also need to show that $B$ is indeed a map from $X_{p,N,T} \times X_{p,N,T}$ into $X_{p,N,T}$. This amounts to prove that for any $u, v \in X_{p,N,T}$, any $i \in \{1, \cdots, n\}$ and any $l \in \{0, \cdots, N\}$, the mapping $t \mapsto t^{\frac{l}{2}} \nabla_x^l B(u,v)_i(t)$ belongs $C\left([0,T]\,; L^p_x(\mathbb{R}^n)\right)$. This can be done by using the ideas of obtaining (2.35) and of proving Lemma 2.4. This work is routine and we omit it. $\square$

## 2.3 Local well-posedness in $X_{p,N,T}$

For any nonnegative integer $N$, any $p \in (n, \infty)$ and any initial data $u_0 \in L^p_x(\mathbb{R}^n)$ (not necessarily small), we will use a fixed point argument to show the exitence and uniqueness of a mild solution in the space $X_{p,N,T}$ for some $T > 0$, namely there exits a unique function $u \in X_{p,N,T}$ satisfying the equation

$$u = U + B(u,u), \qquad \text{for } 0 < t \leq T \ , \tag{2.36}$$

where $U$ is the heat extension of $u_0$. By Lemma 2.4, $U \in X_{p,N,T}$ for any $T > 0$ and moreover by (2.10), we have

$$\|U\|_{X_{p,N,T}} \leq M_2 \|u_0\|_{L_x^p(\mathbb{R}^n)} \ , \tag{2.37}$$

where $M_2$ is independent of $T$ but depends on $n$, $p$ and $N$. In this section, we write $X_{p,N,T}$ simply as $X_T$. Let $M_1$ be the constant in (2.29). For each $T > 0$ such that

$$4M_1M_2 \|u_0\|_{L_x^p(\mathbb{R}^n)} T^{\frac{1}{2}-\frac{n}{2p}} < 1 \ , \tag{2.38}$$

we let $P_T$ be the quadratic polynomial:

$$P_T(R) = M_1 T^{\frac{1}{2}-\frac{n}{2p}} R^2 - R + M_2 \|u_0\|_{L_x^p(\mathbb{R}^n)} \ . \tag{2.39}$$

Under the condition (2.38), $P_T$ has two different roots,

$$R_T = \frac{1 + \sqrt{1 - 4M_1M_2 \|u_0\|_{L_x^p(\mathbb{R}^n)} T^{\frac{1}{2}-\frac{n}{2p}}}}{2M_1 T^{\frac{1}{2}-\frac{n}{2p}}} \ , \tag{2.40}$$

and

$$r_T = \frac{1 - \sqrt{1 - 4M_1M_2 \|u_0\|_{L_x^p(\mathbb{R}^n)} T^{\frac{1}{2}-\frac{n}{2p}}}}{2M_1 T^{\frac{1}{2}-\frac{n}{2p}}} \ . \tag{2.41}$$

It is easy to see that

$$r_T \leq 2M_2 \|u_0\|_{L_x^p(\mathbb{R}^n)} \ . \tag{2.42}$$

Note that the bound on RHD of (2.42) is independent of the choice of $T$. For $R \geq 0$, we use $B(R; X_T)$ to denote the open ball $\left\{v \in X_T,\ \|v\|_{X_T} < R\right\}$ and $\overline{B(R; X_T)}$ to denote the closed ball $\left\{v \in X_T,\ \|v\|_{X_T} \leq R\right\}$. The first conclusion is about the local existence of a solution to (2.36).

**Lemma 2.6.** *For each $T > 0$ satisfying (2.38), the equation (2.36) has a unique solution $u = u_T$ in the ball $B(R_T; X_T)$. Moreover, this solution $u$ lies in $\overline{B(r_T; X_T)}$ and satisfies the following estimates*

$$\left\|\nabla_x^N u(t)\right\|_{L_x^p(\mathbb{R}^n)} \leq 2M_2 \|u_0\|_{L_x^p(\mathbb{R}^n)} t^{-\frac{N}{2}} \tag{2.43}$$

*for $0 < t \leq T$.*

**Proof** . Let $S\colon X_T \to X_T$ be the operator defined by

$$S(v) = U + B(v, v) \tag{2.44}$$

We claim that if $r_T \le R \le R_T$, then $S$ maps the closed ball $\overline{B\left(R; X_T\right)}$ into itself. Indeed, it is easy to see that if $r_T \le R \le R_T$, the polynomial $P_T(R) \le 0$ and as a result, for every $v \in \overline{B\left(R; X_T\right)}$, we have, by Lemma 2.5 and (2.37),

$$\begin{aligned} \|S(v)\|_{X_T} &\le \|U\|_{X_T} + \|B(v,v)\|_{X_T} \le M_2 \|u_0\|_{L^p_x(\mathbb{R}^n)} + M_1 T^{\frac{1}{2}-\frac{n}{2p}} \|v\|^2_{X_T} \\ &\le M_2 \|u_0\|_{L^p_x(\mathbb{R}^n)} + M_1 T^{\frac{1}{2}-\frac{n}{2p}} R^2 \le R \ , \end{aligned} \tag{2.45}$$

which proves the claim.

Next we claim that if $R < \frac{1}{2M_1 T^{\frac{1}{2}-\frac{n}{2p}}} \equiv R_*$, then $S|_{\overline{B(R;X_T)}}$ is a contraction mapping. Indeed, for any $v_1, v_2 \in \overline{B\left(R; X_T\right)}$, by Lemma 2.5,

$$\begin{aligned} &\|S(v_1) - S(v_2)\|_{X_T} = \|B(v_1, v_1) - B(v_2, v_2)\|_{X_T} \\ &\le M_1 T^{\frac{1}{2}-\frac{n}{2p}} \left(\|v_1\|_{X_T} + \|v_2\|_{X_T}\right) \|v_1 - v_2\|_{X_T} \le 2M_1 R T^{\frac{1}{2}-\frac{n}{2p}} \|v_1 - v_2\|_{X_T} \ . \end{aligned} \tag{2.46}$$

Noting that $r_T < R_* < R_T$, by Banach fixed-point theorem, there is a unique a fixed point $u \in \overline{B\left(r_T; X_T\right)}$, that is,

$$u = S(u) = U + B(u, u) \ . \tag{2.47}$$

By the definition of $X_T = X_{p,N,T}$ and (2.42), we have

$$\left\| t^{\frac{N}{2}} \nabla_x^N u \right\|_{C\left([0,T]; L^p_x(\mathbb{R}^n)\right)} \le \|u\|_{X_T} \le r_T \le 2M_2 \|u_0\|_{L^p_x(\mathbb{R}^n)} \ , \tag{2.48}$$

which implies (2.43). We last claim that $S$ has no fixed point in $B\left(R_T; X_T\right) \setminus \overline{B\left(r_T; X_T\right)}$, which implies that $u$ is the unique solution in $B\left(R_T; X_T\right)$. Assume not, let $w \in B\left(R_T; X_T\right) \setminus \overline{B\left(r_T; X_T\right)}$ be a fixed point of $S$. Since $r_T < \|w\|_{X_T} < R_T$, $P_T\left(\|w\|_{X_T}\right) < 0$ and hence we have, by Lemma 2.5,

$$\begin{aligned} \|w\|_{X_T} &= \|S(w)\|_{X_T} \le \|U\|_{X_T} + \|B(w,w)\|_{X_T} \\ &\le M_2 \|u_0\|_{L^p_x(\mathbb{R}^n)} + M_1 T^{\frac{1}{2}-\frac{n}{2p}} \|w\|^2_{X_T} < \|w\|_{X_T} \ . \end{aligned} \tag{2.49}$$

We get a contradiction. ☐

Let $u(\cdot+\tau)(x,t)=u(x,t+\tau)$. The following result implies that the mild solutions has semigroup properties.

**Lemma 2.7.** *Let $u$ be a mild solution in $X_T$ with initial data $u_0$. Then, for every $0<\tau<T$, $u(\cdot+\tau)$ is a mild solution in $X_{T-\tau}$ with initial data $u(\tau)$, namely,*

$$u\left(\cdot+\tau\right)=U\left(u\left(\tau\right)\right)+B\left(u\left(\cdot+\tau\right),u\left(\cdot+\tau\right)\right)\ , \tag{2.50}$$

*where $U\left(u\left(\tau\right)\right)$ is the heat extension of $u(\tau)$.*

**Proof** . We first show $u(\cdot+\tau)\in X_{T-\tau}$. For each $l\in\{0,\cdots,N\}$ and any $0\le t,s\le T-\tau$, we have

$$\begin{aligned}
&\left\|s^{\frac{l}{2}}\nabla_x^l u(s+\tau)-t^{\frac{l}{2}}\nabla_x^l u(t+\tau)\right\|_{L^p_x(\mathbb{R}^n)}\\
=&\left\|\left(\frac{s}{s+\tau}\right)^{\frac{l}{2}}(s+\tau)^{\frac{l}{2}}\nabla_x^l u(s+\tau)-\left(\frac{t}{t+\tau}\right)^{\frac{l}{2}}(t+\tau)^{\frac{l}{2}}\nabla_x^l u(t+\tau)\right\|_{L^p_x(\mathbb{R}^n)}\\
\le&\left\|\left(\frac{s}{s+\tau}\right)^{\frac{l}{2}}(s+\tau)^{\frac{l}{2}}\nabla_x^l u(s+\tau)-\left(\frac{s}{s+\tau}\right)^{\frac{l}{2}}(t+\tau)^{\frac{l}{2}}\nabla_x^l u(t+\tau)\right\|_{L^p_x(\mathbb{R}^n)}\\
&+\left\|\left(\frac{s}{s+\tau}\right)^{\frac{l}{2}}(t+\tau)^{\frac{l}{2}}\nabla_x^l u(t+\tau)-\left(\frac{t}{t+\tau}\right)^{\frac{l}{2}}(t+\tau)^{\frac{l}{2}}\nabla_x^l u(t+\tau)\right\|_{L^p_x(\mathbb{R}^n)}\\
\le&\left\|(s+\tau)^{\frac{l}{2}}\nabla_x^l u(s+\tau)-(t+\tau)^{\frac{l}{2}}\nabla_x^l u(t+\tau)\right\|_{L^p_x(\mathbb{R}^n)}\\
&+\left|\left(\frac{s}{s+\tau}\right)^{\frac{l}{2}}-\left(\frac{t}{t+\tau}\right)^{\frac{l}{2}}\right|\left\|(t+\tau)^{\frac{l}{2}}\nabla_x^l u(t+\tau)\right\|_{L^p_x(\mathbb{R}^n)}\\
\le&\left\|(s+\tau)^{\frac{l}{2}}\nabla_x^l u(s+\tau)-(t+\tau)^{\frac{l}{2}}\nabla_x^l u(t+\tau)\right\|_{L^p_x(\mathbb{R}^n)}+\left|\left(\frac{s}{s+\tau}\right)^{\frac{l}{2}}-\left(\frac{t}{t+\tau}\right)^{\frac{l}{2}}\right|\|u\|_{X_T}\ ,
\end{aligned} \tag{2.51}$$

which implies that the mapping $t\mapsto t^{\frac{l}{2}}\nabla_x^l u(t+\tau)$ belongs to $C\left([0,T-\tau]\,;L^p_x(\mathbb{R}^n)\right)$.

$$\begin{aligned}
\|u(\cdot+\tau)\|_{X_{T-\tau}} &= \sum_{l=0}^{N} \left\| t^{\frac{l}{2}} \nabla_x^l u(t+\tau) \right\|_{C\left([0,T-\tau];L_x^p(\mathbb{R}^n)\right)} \\
&= \sum_{l=0}^{N} \left\| t^{\frac{l}{2}} (t+\tau)^{-\frac{l}{2}} (t+\tau)^{\frac{l}{2}} \nabla_x^l u(t+\tau) \right\|_{C\left([0,T-\tau];L_x^p(\mathbb{R}^n)\right)} \\
&\le \sum_{l=0}^{N} \left\| (t+\tau)^{\frac{l}{2}} \nabla_x^l u(t+\tau) \right\|_{C\left([0,T-\tau];L_x^p(\mathbb{R}^n)\right)} \le \|u\|_{X_T} \ .
\end{aligned} \tag{2.52}$$

Hence, $u(\cdot+\tau) \in X_{T-\tau}$. Next, we show that $u(\cdot+\tau)$ is a mild solution with initial data $u(\tau)$. Since $u$ is a mild solution with initial data $u_0$ in $X_T$, for any $0 < t \le T-\tau$ and any $i \in \{1, \cdots, n\}$, we have, by (2.23) and (2.22)

$$\begin{aligned}
&u_i(\cdot+\tau)(t) = u_i(t+\tau) \\
=&e^{(t+\tau)\Delta} u_{0i} - \int_0^{t+\tau} K_{ijk}(t+\tau-s) * (u_k(s)u_j(s))\ ds \\
=&e^{(t+\tau)\Delta} u_{0i} - \int_0^{\tau} K_{ijk}(t+\tau-s) * (u_k(s)u_j(s))\ ds - \int_\tau^{t+\tau} K_{ijk}(t+\tau-s) * (u_k(s)u_j(s))\ ds \\
=&e^{t\Delta} e^{\tau\Delta} u_{0i} - \int_0^{\tau} \left(e^{t\Delta} K_{ijk}(\tau-s)\right) * (u_k(s)u_j(s))\ ds - \int_0^{t} K_{ijk}(t-r) * (u_k(r+\tau)u_j(r+\tau))\ dr \\
=&e^{t\Delta} e^{\tau\Delta} u_{0i} - e^{t\Delta} \int_0^{\tau} K_{ijk}(\tau-s) * (u_k(s)u_j(s))\ ds - \int_0^{t} K_{ijk}(t-r) * (u_k(\cdot+\tau)(r)u_j(\cdot+\tau)(r))\ dr \\
=&e^{t\Delta} u_i(\tau) - \int_0^{t} K_{ijk}(t-r) * (u_k(\cdot+\tau)(r)u_j(\cdot+\tau)(r))\ dr
\end{aligned} \tag{2.53}$$

Hence $u(\cdot+\tau)$ is indeed a mild solution with initial data $u(\tau)$. □

The uniqueness statement in Lemma 2.6, together with the semiflow property in Lemma 2.7 can be used to obtain the following statement.

**Proposition 2.8.** *For each $u_0 \in L_x^p(\mathbb{R}^n)$, the equation (2.36) has at most one solution in $X_T$ for any $T > 0$ (not necessarily small).*

**Proof** . Let $v_1$ and $v_2$ be two different solutions of (2.36) in $X_T$ with the same initial data $u_0$. By Lemma 2.4 and Lemma 2.5, we have

$$\|v_i(t) - u_0\|_{L_x^p(\mathbb{R}^n)} \to 0 \qquad \text{as} \quad t \to 0+ \ . \tag{2.54}$$

which implies

$$\|u_0\|_{L^p_x(\mathbb{R}^n)} \le \|v_i\|_{X_T} \qquad \text{for} \quad i = 1, 2 \ . \tag{2.55}$$

For any $0 < \tau < T$, it is obvious that $v_i$ $(i = 1, 2)$ are also solutions of (2.36) in $X_\tau$ and by Lemma 2.5, (2.36), (2.37) and (2.55), we have

$$\begin{aligned} \|v_i\|_{X_\tau} &\le \|U\|_{X_\tau} + \|B(v_i, v_i)\|_{X_\tau} \le M_2 \|u_0\|_{L^p_x(\mathbb{R}^n)} + M_1 \tau^{\frac{1}{2}-\frac{n}{2p}} \|v_i\|^2_{X_\tau} \\ &\le M_2 \|v_i\|_{X_T} + M_1 \tau^{\frac{1}{2}-\frac{n}{2p}} \|v_i\|^2_{X_T} \qquad \text{for} \quad i = 1, 2 \ . \end{aligned} \tag{2.56}$$

Since the root $R_\tau = \frac{1+\sqrt{1-4M_1M_2\|u_0\|_{L^p_x(\mathbb{R}^n)}\tau^{\frac{1}{2}-\frac{n}{2p}}}}{2M_1\tau^{\frac{1}{2}-\frac{n}{2p}}} \ge \frac{1}{2M_1\tau^{\frac{1}{2}-\frac{n}{2p}}} \to \infty$ as $\tau \to 0+$, we can find a $t_1 = t_1\left(\|v_1\|_{X_T}, \|v_2\|_{X_T}\right) > 0$ satisfying (for $i = 1, 2$)

$$\begin{aligned} &4M_1M_2 \|u_0\|_{L^p_x(\mathbb{R}^n)} t_1^{\frac{1}{2}-\frac{n}{2p}} \le 4M_1M_2 \max\left(\|v_1\|_{X_T}, \|v_2\|_{X_T}\right) t_1^{\frac{1}{2}-\frac{n}{2p}} < 1 \ , \\ &\text{and} \quad M_2 \|v_i\|_{X_T} + M_1 t_1^{\frac{1}{2}-\frac{n}{2p}} \|v_i\|^2_{X_T} < R_{t_1} \ , \end{aligned} \tag{2.57}$$

which, combining with (2.56), implies $v_i \in B\left(R_{t_1}; X_{t_1}\right)$ for $i = 1, 2$ (The first condition in (2.57) is just to make (2.38) true). By Lemma 2.6, $v_1 = v_2$ in $\mathbb{R}^n \times [0, t_1]$.

Next, we consider $v_1(\cdot + t_1)$ and $v_2(\cdot + t_1)$. By Lemma 2.7, they are both mild solutions in $X_{T-t_1}$ with the same initial data $v_1(t_1) = v_2(t_1)$. By (2.52), $\|v_i(\cdot + t_1)\|_{X_{T-t_1}} \le \|v_i\|_{X_T}$. Therefore, one can use the foregoing argument to show that $v_1(\cdot + t_1) = v_2(\cdot + t_1)$ in $\mathbb{R}^n \times [0, t_1]$. Hence $v_1 = v_2$ in $\mathbb{R}^n \times [t_1, 2t_1]$. It is not hard to see that one just need to continue this argument finite times to reach $T$. □

**Corollary 2.9.** *Let $u_1$ and $u_2$ be two solutions of (2.36) in $X_{T_1}$ and $X_{T_2}$, respectively, with the same initial data. Assume $T_1 \le T_2$. Then $u_1 = u_2$ in $\mathbb{R}^n \times [0, T_1]$.*

**Lemma 2.10.** *Let $0 < T_0 < T_1$. Let $u_1$ be a mild solution in $X_{T_1}$ with initial data $u_0$. Let $u_2$ be a mild solution in $X_{T_2}$ with initial data $u_1(T_0)$ for some $T_2 > T_1 - T_0$. Define*

$$u(x,t) = \begin{cases} u_1(x,t) & \text{in } \mathbb{R}^n \times [0, T_1] \\ u_2(x, t - T_0) & \text{in } \mathbb{R}^n \times [T_1, T_0 + T_2] \ . \end{cases} \tag{2.58}$$

*Then $u$ is a mild solution in $X_{T_0+T_2}$ with initial data $u_0$.*

**Proof** . By Lemma 2.7, $u_1(\cdot + T_0)$ is a mild solution in $X_{T_1-T_0}$ with initial data $u_1(T_0)$. Then by Corollary 2.9, $u_2 = u_1(\cdot + T_0)$ in $\mathbb{R}^n \times [0, T_1 - T_0]$. In particular, $u_2(x, T_1 - T_0) = u_1(x, T_1)$ and hence (2.58) is well-defined.

We first prove that the function $u$ defined in (2.58) is in $X_{T_0+T_2}$. Firstly, for every $l \in \{0, \cdots, N\}$, the mapping $t \mapsto t^{\frac{l}{2}} \nabla_x^l u(\cdot, t)$ is in $C\left([0,T]\,; L_x^p(\mathbb{R}^n)\right)$. We distinguish two cases:

**Case 1**: $0 \le t < T_1$ and $s \to t$, or, $t = T_1$ and $s \to T_1-$. Then

$$\left\| s^{\frac{l}{2}} \nabla_x^l u(\cdot, s) - t^{\frac{l}{2}} \nabla_x^l u(\cdot, t) \right\|_{L_x^p(\mathbb{R}^n)} = \left\| s^{\frac{l}{2}} \nabla_x^l u_1(\cdot, s) - t^{\frac{l}{2}} \nabla_x^l u_1(\cdot, t) \right\|_{L_x^p(\mathbb{R}^n)} \to 0 \tag{2.59}$$

since $u_1 \in X_{T_1}$.

**Case 2**: $T_1 < t \le T_0 + T_2$ and $s \to t$, or, $t = T_1$ and $s \to T_1+$. Then

$$\begin{aligned}
&\left\| s^{\frac{l}{2}} \nabla_x^l u(\cdot, s) - t^{\frac{l}{2}} \nabla_x^l u(\cdot, t) \right\|_{L_x^p(\mathbb{R}^n)} = \left\| s^{\frac{l}{2}} \nabla_x^l u_2(\cdot, s - T_0) - t^{\frac{l}{2}} \nabla_x^l u_2(\cdot, t - T_0) \right\|_{L_x^p(\mathbb{R}^n)} \\
&= \left\| \left( \frac{s}{s - T_0} \right)^{\frac{l}{2}} (s - T_0)^{\frac{l}{2}} \nabla_x^l u_2(\cdot, s - T_0) - \left( \frac{t}{t - T_0} \right)^{\frac{l}{2}} (t - T_0)^{\frac{l}{2}} \nabla_x^l u_2(\cdot, t - T_0) \right\|_{L_x^p(\mathbb{R}^n)} \\
&\le \left\| \left( \frac{s}{s - T_0} \right)^{\frac{l}{2}} (s - T_0)^{\frac{l}{2}} \nabla_x^l u_2(\cdot, s - T_0) - \left( \frac{t}{t - T_0} \right)^{\frac{l}{2}} (s - T_0)^{\frac{l}{2}} \nabla_x^l u_2(\cdot, s - T_0) \right\|_{L_x^p(\mathbb{R}^n)} \\
&\quad + \left\| \left( \frac{t}{t - T_0} \right)^{\frac{l}{2}} (s - T_0)^{\frac{l}{2}} \nabla_x^l u_2(\cdot, s - T_0) - \left( \frac{t}{t - T_0} \right)^{\frac{l}{2}} (t - T_0)^{\frac{l}{2}} \nabla_x^l u_2(\cdot, t - T_0) \right\|_{L_x^p(\mathbb{R}^n)} \\
&\le \left| \left( \frac{s}{s - T_0} \right)^{\frac{l}{2}} - \left( \frac{t}{t - T_0} \right)^{\frac{l}{2}} \right| \|u_2\|_{X_{T_2}} \\
&\quad + \left( \frac{t}{t - T_0} \right)^{\frac{l}{2}} \left\| (s - T_0)^{\frac{l}{2}} \nabla_x^l u_2(\cdot, s - T_0) - (t - T_0)^{\frac{l}{2}} \nabla_x^l u_2(\cdot, t - T_0) \right\|_{L_x^p(\mathbb{R}^n)} \to 0
\end{aligned} \tag{2.60}$$

since $u_2 \in X_{T_2}$. Secondly, we have

$$
\begin{aligned}
&\|u\|_{X_{T_0+T_2}} = \sum_{l=0}^{N} \left\| t^{\frac{l}{2}} \nabla_x^l u \right\|_{C\left([0,T_0+T_2];L_x^p(\mathbb{R}^n)\right)} \\
&= \sum_{l=0}^{N} \max_{0\le t\le T_0+T_2} \left\| t^{\frac{l}{2}} \nabla_x^l u(\cdot,t) \right\|_{L_x^p(\mathbb{R}^n)} \\
&= \sum_{l=0}^{N} \max\left( \max_{0\le t\le T_1} \left\| t^{\frac{l}{2}} \nabla_x^l u_1(\cdot,t) \right\|_{L_x^p(\mathbb{R}^n)} \ , \ \max_{T_1\le t\le T_0+T_2} \left\| t^{\frac{l}{2}} \nabla_x^l u_2(\cdot,t-T_0) \right\|_{L_x^p(\mathbb{R}^n)} \right) \\
&\le \sum_{l=0}^{N} \max_{0\le t\le T_1} \left\| t^{\frac{l}{2}} \nabla_x^l u_1(\cdot,t) \right\|_{L_x^p(\mathbb{R}^n)} + \sum_{l=0}^{N} \max_{T_1\le t\le T_0+T_2} \left\| t^{\frac{l}{2}} \nabla_x^l u_2(\cdot,t-T_0) \right\|_{L_x^p(\mathbb{R}^n)} \\
&= \|u_1\|_{X_{T_1}} + \sum_{l=0}^{N} \max_{T_1\le t\le T_0+T_2} \left( \frac{t}{t-T_0} \right)^{\frac{l}{2}} \left\| (t-T_0)^{\frac{l}{2}} \nabla_x^l u_2(\cdot,t-T_0) \right\|_{L_x^p(\mathbb{R}^n)} \\
&\le \|u_1\|_{X_{T_1}} + \left( \frac{T_0+T_2}{T_1-T_0} \right)^{\frac{N}{2}} \|u_2\|_{X_{T_2}} \quad .
\end{aligned}
\tag{2.61}
$$

Hence $u$ is indeed in $X_{T_0+T_2}$. Next we show that $u$ is a mild solution with initial data $u_0$. We distinguish two cases:

**Case 1**: $0 < t \le T_1$. Since $u_1$ is a mild solution in $X_{T_1}$ with initial data $u_0$, for any $i \in \{1, \cdots, n\}$, we have, by (2.23) and (2.58),

$$
\begin{aligned}
u_i(t) = u_{1i}(t) &= e^{t\Delta} u_{0i} - \int_0^t K_{ijk}(t-s) * (u_{1k}(s) u_{1j}(s)) \ ds \\
&= e^{t\Delta} u_{0i} - \int_0^t K_{ijk}(t-s) * (u_k(s) u_j(s)) \ ds \ ,
\end{aligned}
\tag{2.62}
$$

**Case 2**: $T_1 < t \le T_0 + T_2$. Since $u_2$ is a mild solution in $X_{T_2}$ with initial data

$u_1(T_0)$, for any $i \in \{1, \cdots, n\}$, we have, by (2.23) and (2.22)

$$
\begin{aligned}
u_i(t) = u_{2i}(t-T_0) &= e^{(t-T_0)\Delta} u_{1i}(T_0) - \int_0^{t-T_0} K_{ijk}(t-T_0-s) * (u_{2k}(s)u_{2j}(s))\, ds \\
=& e^{(t-T_0)\Delta} \left[ e^{T_0\Delta} u_{0i} - \int_0^{T_0} K_{ijk}(T_0-s) * (u_{1k}(s)u_{1j}(s))\, ds \right] \\
& - \int_{T_0}^{t} K_{ijk}(t-r) * (u_{2k}(r-T_0)u_{2j}(r-T_0))\, dr \\
=& e^{t\Delta} u_{0i} - \int_0^{T_0} K_{ijk}(t-s) * (u_{1k}(s)u_{1j}(s))\, ds - \int_{T_0}^{t} K_{ijk}(t-r) * (u_{2k}(r-T_0)u_{2j}(r-T_0))\, dr \\
=& e^{t\Delta} u_{0i} - \int_0^{T_0} K_{ijk}(t-s) * (u_{1k}(s)u_{1j}(s))\, ds - \int_{T_0}^{T_1} K_{ijk}(t-r) * (u_{2k}(r-T_0)u_{2j}(r-T_0))\, dr \\
& - \int_{T_1}^{t} K_{ijk}(t-r) * (u_{2k}(r-T_0)u_{2j}(r-T_0))\, dr \\
=& e^{t\Delta} u_{0i} - \int_0^{T_0} K_{ijk}(t-s) * (u_{1k}(s)u_{1j}(s))\, ds - \int_{T_0}^{T_1} K_{ijk}(t-r) * (u_{1k}(r)u_{1j}(r))\, dr \\
& - \int_{T_1}^{t} K_{ijk}(t-r) * (u_{2k}(r-T_0)u_{2j}(r-T_0))\, dr \\
=& e^{t\Delta} u_{0i} - \int_0^{T_1} K_{ijk}(t-s) * (u_{1k}(s)u_{1j}(s))\, ds - \int_{T_1}^{t} K_{ijk}(t-r) * (u_{2k}(r-T_0)u_{2j}(r-T_0))\, dr \\
=& e^{t\Delta} u_{0i} - \int_0^{t} K_{ijk}(t-s) * (u_k(s)u_j(s))\, ds
\end{aligned}
\tag{2.63}
$$

Hence $u$ is a mild solution with initial data $u_0$. □

We end this section by defining the maximum existence time of the mild solution in the space $X_T = X_{p,N,T}$. Let

$$
T_N(u_0) = \sup \{T > 0 : \text{ equation (2.36) has a solution in } X_{p,N,T} \text{ with initial data } u_0\} \ . \tag{2.64}
$$

By Lemma 2.6, $T_N(u_0) > 0$ for any $u_0 \in L^p_x(\mathbb{R}^n)$ with $p \in (n, \infty)$.

## 2.4 The mild solution in $X_{p,0,T} = C\left([0,T]\,; L^p_x(\mathbb{R}^n)\right)$ and proof of Theorem 2.1

By the definition of the maximum existence time $T_N(u_0)$ of the mild solution in $X_{p,N,T}$ and Proposition 2.8, it is clear that $T_{N_1}(u_0) \geq T_{N_2}(u_0)$ if $N_1 \leq N_2$. Actually, they are equal.

**Proposition 2.11.** *For every $u_0 \in L^p_x(\mathbb{R}^n)$ with $p \in (n,\infty)$ and every positive integer $N$, we have $T_0(u_0) = T_N(u_0)$.*

**Proof** . Fix an initial datat $u_0$. We write $T_0(u_0)$ and $T_N(u_0)$ simply as $T_0$ and $T_N$, respectively. If $T_N = \infty$, then $T_0 = \infty$ since $T_0 \geq T_N$. The assertion is true. We consider the case of $T_N < \infty$. We assume $T_0 > T_N$. We take a $T \in (T_N, T_0)$. By the definition of $T_0$, equation (2.36) has a solution $u$ in $X_{p,0,T} = C\left([0,T]\,; L^p_x(\mathbb{R}^n)\right)$ with initial data $u_0$. Take an $\varepsilon \in (0, T_N)$ such that

$$4M_1M_2 \left\|u\right\|_{X_{p,0,T}} \varepsilon^{\frac{1}{2}-\frac{n}{2p}} < 1 \ , \tag{2.65}$$

where $M_1$ and $M_2$ are the constants in (2.29) and (2.37), respectively, both of which depend only on $n$, $p$ and $N$. By the defintion of $T_N$, equation (2.36) has a solution $v$ in $X_{p,N,T_N-\frac{\varepsilon}{3}}$ with initial data $u_0$. Since $X_{p,N,T_N-\frac{\varepsilon}{3}} \subset X_{p,0,T_N-\frac{\varepsilon}{3}}$ and $T_N - \frac{\varepsilon}{3} < T_N < T$, by Corollay 2.9, $u = v$ in $\mathbb{R}^n \times \left[0, T_N - \frac{\varepsilon}{3}\right]$. In view of (2.65), we have

$$4M_1M_2 \left\|v\left(T_N - 2\varepsilon/3\right)\right\|_{L^p_x(\mathbb{R}^n)} \varepsilon^{\frac{1}{2}-\frac{n}{2p}} < 1 \ , \tag{2.66}$$

By (2.66) and Lemma 2.6, there exists a mild solution $w$ in $X_{p,N,\varepsilon}$ with the initial data $v\left(T_N - 2\varepsilon/3\right)$. Let

$$f(x,t) = \begin{cases} v(x,t) & \text{in } \mathbb{R}^n \times \left[0, T_N - \frac{\varepsilon}{3}\right] \ , \\ w\left(x, t - T_N + \frac{2\varepsilon}{3}\right) & \text{in } \mathbb{R}^n \times \left[T_N - \frac{\varepsilon}{3}, T_N + \frac{\varepsilon}{3}\right] \ . \end{cases} \tag{2.67}$$

By Lemma 2.10, $f$ is a mild solution in $X_{p,N,T_N+\frac{\varepsilon}{3}}$ with initial data $u_0$, which leads to a contraction, since $T_N$ is the maximum existence time of the mild solution with initial data $u_0$. $\square$

By Proposition 2.11, the maximum existence time $T_N(u_0)$ is independent of $N$. In the sequel, we use $T(u_0)$ to denote this time of $u_0$. We summarize our previous results as the following statement.

**Corollary 2.12.** *For any initial data $u_0 \in L^p_x(\mathbb{R}^n)$ with $p \in (n, \infty)$, there exists a unique mild solution $u \in C\left([0, T(u_0)), L^p_x(\mathbb{R}^n)\right)$ to equation (2.36). And this solution $u$ belongs to $X_{p,N,T}$ for any nonnegative integer $N$ and any $0 < T < T(u_0)$.*

The following statment says that if $T(u_0)$ is finite, then $\|u(t)\|_{L^p_x(\mathbb{R}^n)}$ will blow up as $t \to T(u_0)-$.

**Proposition 2.13.** *Let $u_0 \in L^p_x(\mathbb{R}^n)$ with $p \in (n, \infty)$. Let $u$ be the solution in Corollary 2.12 with $u_0$ as initial data. If $T(u_0) < \infty$, then*

$$\lim_{t \to T(u_0)-} \|u(t)\|_{L^p_x(\mathbb{R}^n)} = +\infty \tag{2.68}$$

**Proof** . Assume not. Then there exists a strictly increasing sequence $t_n \uparrow T(u_0)$ with

$$\sup_{n \in \mathbb{N}} \|u(t_n)\|_{L^p_x(\mathbb{R}^n)} \le M_* \tag{2.69}$$

for some $0 < M_* < \infty$. Take a $0 < T < T(u_0)$ such that

$$4 M_1 M_2 M_* T^{\frac{1}{2} - \frac{n}{2p}} < 1 \ , \tag{2.70}$$

where $M_1$ and $M_2$ are the constants in (2.29) and (2.37), respectively, both of which depend only on $n$ and $p$ (in this case $N = 0$). Take a $t_n > T(u_0) - T$. By Corollary 2.12, $u \in X_{p,0,t_{n+1}}$. By (2.69) and (2.70), we have

$$4 M_1 M_2 \|u(t_n)\|_{L^p_x(\mathbb{R}^n)} T^{\frac{1}{2} - \frac{n}{2p}} < 1 \ . \tag{2.71}$$

By Lemma 2.6, there exits a mild solution $v \in X_{p,0,T}$ with $u(t_n)$ as initial data. Let

$$w(x,t) = \begin{cases} u(x,t) & \text{in } \mathbb{R}^n \times [0, t_{n+1}] \ , \\ v\left(x, t - t_n\right) & \text{in } \mathbb{R}^n \times [t_{n+1}, t_n + T] \ . \end{cases} \tag{2.72}$$

By Lemma 2.10, $w$ is a mild solution in $X_{p,0,t_n+T}$ with initial data $u_0$, which leads to a contraction since $T(u_0) < t_n + T$. $\square$

We now prove Theorem 2.1. All assertions in Theorem 2.1 have been covered by Corollary 2.12 and Proposition 2.13 except the estimate (2.3). We prove the estimate (2.3). Let $k$ be a nonnegative integer and let $0 < \tau < T < T(u_0)$. We first consider the spatial derivatives $\nabla_x^k u$. We will use the local-in-times estimates (2.43) and the semigroup property (Lemma 2.7) of the mild solution to get estimates over the time interval $[\tau, T]$. Let

$$M(T) = \|u\|_{C\left([0,T], L_x^p(\mathbb{R}^n)\right)} \ . \tag{2.73}$$

Note $M(T) < \infty$ since $T < T(u_0)$. Fix a $T_*$ such that

$$4M_1 M_2 M(T) T_*^{\frac{1}{2} - \frac{n}{2p}} < 1 \ , \tag{2.74}$$

where $M_1$ and $M_2$ are the constants in (2.29) and (2.37), respectively. Both $M_1$ and $M_2$ depend only $n$, $p$ and $k$ (in this case $N = k$). Let

$$\sigma = \frac{1}{2} \min(\tau, T_*) \ . \tag{2.75}$$

For each $s \in [\tau, T]$, by (2.73), (2.74) and (2.75), we have

$$4M_1 M_2 \|u(s-\sigma)\|_{L_x^p(\mathbb{R}^n)} T_*^{\frac{1}{2} - \frac{n}{2p}} < 1 \ . \tag{2.76}$$

By Lemma 2.6 and (2.76), there exists a mild solution $v$ in $X_{p,k,T_*}$ with $u(s-\sigma)$ as initial data. Note that we must have $T_* + s - \sigma < T(u_0)$. By (2.43) and (2.73), we have

$$\left\|\nabla_x^k v(t)\right\|_{L_x^p(\mathbb{R}^n)} \leq 2M_2 \|u(s-\sigma)\|_{L_x^p(\mathbb{R}^n)} t^{-\frac{k}{2}} \leq 2M_2 M(T) t^{-\frac{k}{2}} \tag{2.77}$$

for $0 < t \leq T_*$. In particular, since $0 < \sigma < T_*$, we have

$$\left\|\nabla_x^k v(\sigma)\right\|_{L_x^p(\mathbb{R}^n)} \leq 2M_2 M(T) \sigma^{-\frac{k}{2}} \ . \tag{2.78}$$

On the other hand, by Lemma 2.7 and Corollary 2.12, $u\Big(\cdot+(s-\sigma)\Big)$ is a mild solution in $X_{p,k,T_*}$ with initial data $u(s-\sigma)$. By Corollary 2.9, we have $u\Big(\cdot+(s-\sigma)\Big)=v$ in $\mathbb{R}^n\times[0,T_*]$ and hence by (2.78)

$$\left\|\nabla_x^k u(s)\right\|_{L_x^p(\mathbb{R}^n)}=\left\|\nabla_x^k v(\sigma)\right\|_{L_x^p(\mathbb{R}^n)}\leq 2M_2M(T)\sigma^{-\frac{k}{2}}\ . \tag{2.79}$$

By the arbitrariness of $s$, we thus obtain the following estimate

$$\left\|\nabla_x^k u\right\|_{L_t^\infty L_x^p(\mathbb{R}^n\times[\tau,T])}\leq 2M_2M(T)\sigma^{-\frac{k}{2}}\ , \tag{2.80}$$

which implies (2.3) in the case of $h=0$. In view of (2.18), we have

$$\partial_t u=\Delta u+\left(\partial_t-\Delta\right)u=\Delta u-P\left(\frac{\partial}{\partial x_k}\left(u_k u\right)\right)\ , \tag{2.81}$$

where $P$ the orthogonal projection onto the subspace of divergence-free vector fields, which is a singular integral operator of convolution type. It is easy to see that (2.3) can obtained from (2.80) and (2.81) by induction. Theorem 2.1 is thus proved.

# Chapter 3

# Weighted inequalities

In this chapter, we present some weighted inequalities. We will have a-priori bounds on three quantities related to the vorticity: $\|r\omega\|_{L^1_x(\mathbb{R}^3)}$, $\left\|\frac{\omega}{r}\right\|_{L^1_x(\mathbb{R}^3)}$, $\left\|\frac{\omega}{r}\right\|_{L^\infty_x(\mathbb{R}^3)}$, and our aim is to obtain further estimates on the velocity $u$ from these bounds.

**Proposition 3.1.** *Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be such that $\|rf\|_{L^1_x(\mathbb{R}^3)}$, $\left\|\frac{f}{r}\right\|_{L^1_x(\mathbb{R}^3)}$ and $\left\|\frac{f}{r}\right\|_{L^\infty_x(\mathbb{R}^3)}$ are finite, where $r = \sqrt{x_1^2 + x_2^2}$. Then for every $1 \le p \le 2$, $f \in L^p_x(\mathbb{R}^3)$ and*

$$\|f\|_{L^p_x(\mathbb{R}^3)} \le \|rf\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{\frac{1}{p}-\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{1-\frac{1}{p}}_{L^\infty_x(\mathbb{R}^3)} .$$

**Proof** . We first prove the two cases of $p = 1$ and $p = 2$ and then use interpolation to prove the other cases. We can write

$$\begin{aligned}
&\int_{\mathbb{R}^3} |f|\, dx = \int_{\mathbb{R}^3} r^{\frac{1}{2}}\, |f|^{\frac{1}{2}}\, \frac{|f|^{\frac{1}{2}}}{r^{\frac{1}{2}}} dx \\
&\le \left(\int_{\mathbb{R}^3} r\, |f|\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \frac{|f|}{r} dx\right)^{\frac{1}{2}} = \|rf\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} ,
\end{aligned}$$

which proves the case $p = 1$.

Next we consider

$$\left(\int_{\mathbb{R}^3} |f|^2\, dx\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} r\,|f|\,\frac{|f|}{r} dx\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{\mathbb{R}^3} r\,|f| \left\|\frac{f}{r}\right\|_{L^\infty_x(\mathbb{R}^3)} dx\right)^{\frac{1}{2}} = \|rf\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{\frac{1}{2}}_{L^\infty_x(\mathbb{R}^3)} ,$$

which proves the case $p = 2$.

Let $1 < p < 2$. We have

$$\|f\|_{L^p_x(\mathbb{R}^3)} \leq \|f\|^{\frac{2}{p}-1}_{L^1_x(\mathbb{R}^3)} \|f\|^{2-\frac{2}{p}}_{L^2_x(\mathbb{R}^3)}$$
$$\leq \left(\|rf\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)}\right)^{\frac{2}{p}-1} \left(\|rf\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{\frac{1}{2}}_{L^\infty_x(\mathbb{R}^3)}\right)^{2-\frac{2}{p}}$$
$$= \|rf\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{\frac{1}{p}-\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{f}{r}\right\|^{1-\frac{1}{p}}_{L^\infty_x(\mathbb{R}^3)} .$$

□

**Remark 3.2.** *Under the assumption of Proposition 3.1, one can not control $\|f\|_{L^p_x(\mathbb{R}^3)}$ for $p > 2$. It is not hard to exhibit counterexamples.*

**Corollary 3.3.** *Assume that $\omega$ is a vector field on $\mathbb{R}^3$ such that*

$$\|r\omega\|_{L^1_x(\mathbb{R}^3)} < \infty,\ \left\|\frac{\omega}{r}\right\|_{L^1_x(\mathbb{R}^3)} < \infty,\ \left\|\frac{\omega}{r}\right\|_{L^\infty_x(\mathbb{R}^3)} < \infty\ . \tag{3.1}$$

*Let $u$ be the vector field constructed from $\omega$ via the Biot-Savart Law,*

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y)\ dy\ . \tag{3.2}$$

*Then for any $\frac{3}{2} < q \leq 6$, $u \in L^q_x(\mathbb{R}^3)$ and*

$$\|u\|_{L^q_x(\mathbb{R}^3)} \lesssim \|r\omega\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega}{r}\right\|^{\frac{1}{q}-\frac{1}{6}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega}{r}\right\|^{\frac{2}{3}-\frac{1}{q}}_{L^\infty_x(\mathbb{R}^3)}\ . \tag{3.3}$$

**Proof** . By Proposition 3.1 and (3.1), for any $1 \leq p \leq 2$, we have

$$\|\omega\|_{L^p_x(\mathbb{R}^3)} \leq \|r\omega\|^{\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega}{r}\right\|^{\frac{1}{p}-\frac{1}{2}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega}{r}\right\|^{1-\frac{1}{p}}_{L^\infty_x(\mathbb{R}^3)}\ . \tag{3.4}$$

Then by the classical Hardy-Littlewood-Sobolev inequality (see for instance [**S93**, **T86**]), one can get

$$\|u\|_{L^q_x(\mathbb{R}^3)} \lesssim \|\omega\|_{L^p_x(\mathbb{R}^3)}\,, \quad \text{for } \ p \in (1,3) \ \text{ and } \ \frac{1}{q} = \frac{1}{p} - \frac{1}{3}\,,$$

which, combining with (3.4), implies (3.3). □

**Remark 3.4.** *By interpolation, the a-priori bounds (3.1) imply*

$$\left\|\frac{\omega}{r}\right\|_{L^p_x(\mathbb{R}^3)} < \infty, \qquad \text{for all } 1 < p < \infty\ .$$

*What can we say about the full gradient $\nabla u$ from the above bounds and (3.1)? This question is related to the theory of singular integral operators with weights. Here we will only consider this question for vector fields which are axi-symmetric.*

It is natural to ask whether we can control other $L^q_x(\mathbb{R}^3)$ norms of $u$ except $\frac{3}{2} < q \le 6$ under the assumptions of Corollary 3.3. The inequality (3.5) below indicates what can be expected in this situation. We prove this inequality as a warm-up for the proof of our main inequality (1.23).

**Proposition 3.5.** *Assume $f = f(x_1, x_2, z) = f\big(\sqrt{x_1^2 + x_2^2}, z\big)\colon \mathbb{R}^3 \to \mathbb{R}$ is smooth and vanishes at infinity. Assume in addition that $\|r\nabla f\|_{L^1_x(\mathbb{R}^3)}$, $\left\|\frac{\nabla f}{r}\right\|_{L^1_x(\mathbb{R}^3)}$ and $\left\|\frac{\nabla f}{r}\right\|_{L^\infty_x(\mathbb{R}^3)}$ are finite. Then we have*

$$\|f\|_{L^\infty_x(\mathbb{R}^3)} \lesssim \|r\nabla f\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\nabla f}{r}\right\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\nabla f}{r}\right\|^{\frac{1}{2}}_{L^\infty_x(\mathbb{R}^3)}\,. \tag{3.5}$$

**Proof** . Assume $|f(r,z)|$ achieve its supremum at $(r_0, z_0)$, that is,

$$\|f\|_{L^\infty_x(\mathbb{R}^3)} = |f(r_0, z_0)|\ .$$

By the boundedness of $\frac{\nabla f}{r}$, $\nabla f$ must vanish at $r = 0$ (the $z$-axis). In particular, $\nabla_z f = 0$ along the $z$-axis. Thus, $f(0,z) \equiv 0$ by the assumption that $f$ vanishes at infinity. Therefore, without loss of generality, we can assume $r_0 > 0$. By the fundamental theorem of calculus and the Hölder's inequality

$$
\begin{aligned}
&\|f\|_{L^\infty_x(\mathbb{R}^3)} = |f(r_0,z_0)| = \left|f(r_0,z_0)^2\right|^{\frac{1}{2}} = \left|\int_{z_0}^{\infty} \partial_z f(r_0,z)^2 \; dz\right|^{\frac{1}{2}} \\
&\lesssim \left(\int_{z_0}^{\infty} |f(r_0,z)|\,|\partial_z f(r_0,z)| \; dz\right)^{\frac{1}{2}} \\
&= \left(\int_{z_0}^{\infty} \left|\int_{r_0}^{\infty} \partial_r f(r,z) \; dr\right| |\partial_z f(r_0,z)| \; dz\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{z_0}^{\infty} \int_{r_0}^{\infty} |\partial_r f(r,z)| \; dr\, |\partial_z f(r_0,z)| \; dz\right)^{\frac{1}{2}} \\
&= \left(\int_{z_0}^{\infty} \int_{r_0}^{\infty} \left(r\,|\partial_r f(r,z)|^{\frac{1}{2}}\,|\partial_r f(r,z)|^{\frac{1}{2}}\,\frac{1}{r}\right) dr\; |\partial_z f(r_0,z)| \; dz\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{z_0}^{\infty} \int_{r_0}^{\infty} \left(r\,|\partial_r f(r,z)|^{\frac{1}{2}}\,|\partial_r f(r,z)|^{\frac{1}{2}}\right) dr\; \frac{|\partial_z f(r_0,z)|}{r_0} \; dz\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{z_0}^{\infty} \int_{r_0}^{\infty} r\,|\partial_r f(r,z)|^{\frac{1}{2}}\,|\partial_r f(r,z)|^{\frac{1}{2}} \; dr\, dz\right)^{\frac{1}{2}} \left(\sup_z \frac{|\partial_z f(r_0,z)|}{r_0}\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{-\infty}^{\infty} \int_{0}^{\infty} r^2\,|\partial_r f(r,z)| \; dr\, dz\right)^{\frac{1}{4}} \left(\int_{-\infty}^{\infty} \int_{0}^{\infty} |\partial_r f(r,z)| \; dr\, dz\right)^{\frac{1}{4}} \left(\sup_{\mathbb{R}^3} \frac{|\partial_z f(r,z)|}{r}\right)^{\frac{1}{2}} \\
&\lesssim \|r\nabla f\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\nabla f}{r}\right\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\nabla f}{r}\right\|^{\frac{1}{2}}_{L^\infty_x(\mathbb{R}^3)} .
\end{aligned}
$$

□

In light of (3.5), one might ask whether the following inequality is true.

$$
\|u\|_{L^\infty_x(\mathbb{R}^3)} \lesssim \|r\omega\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega}{r}\right\|^{\frac{1}{4}}_{L^1_x(\mathbb{R}^3)} \left\|\frac{\omega}{r}\right\|^{\frac{1}{2}}_{L^\infty_x(\mathbb{R}^3)} . \tag{3.6}
$$

We do not know whether (3.6) is true for general vector fields, but we will show that it turns out to be true for the class of axi-symmetric vector fields with no swirl, which is enough for our purposes here. We will use the *axi-symmetric Biot-Savart Law*. To introduce it, we start from the so-called axi-symmetric stream function.

In cylindrical coordinates, the class of axi-symmetric vector fields with no swirl is in the form $u = u_r(r,z)e_r + u_z(r,z)e_z$, see Definition 1.2, and the divergence-free condition $\mathrm{div} u = 0$ turns out to be

$$(ru_r)_{,r} + (ru_z)_{,z} = 0 \ ,$$

which means that

$$ru_r = -\psi_{,z} \, , \qquad ru_z = \psi_{,r}$$

for a suitable function $\psi = \psi(r,z)$, called the axi-symmetric stream function, similar to the 2d situation. Hence

$$u_r = -\frac{1}{r}\psi_{,z}, \quad u_z = \frac{1}{r}\psi_{,r} \ . \tag{3.7}$$

It is easy to check that the curl of an axi-symmetric field $u$ with no swirl is in the form

$$\mathrm{curl} u = \omega_\theta e_\theta$$

with $\omega_\theta = u_{r,z} - u_{z,r}$. Therefore, we obtain

$$L\psi \colon = -\frac{1}{r}\psi_{,rr} + \frac{1}{r^2}\psi_{,r} - \frac{1}{r}\psi_{,zz} = \omega_\theta \ .$$

The inverse operator $L^{-1}$ is given by

$$\psi(\bar{r},\bar{z}) = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\bar{r}r}{4\pi}\int_0^{2\pi} \frac{\cos\varphi \; d\varphi}{\left[r^2 + \bar{r}^2 - 2\bar{r}r\cos\varphi + (z-\bar{z})^2\right]^{\frac{1}{2}}}\,\omega_\theta(r,z)\,dr\,dz \, . \tag{3.8}$$

For the axi-symmetric stream function and the derivation of (3.8), we refer the readers to [**S11**]. For completeness, we give a derivation of (3.8). We recall a simple formula. For a vector field $A = A_\theta e_\theta$, we have

$$\mathrm{curl}\, A = -A_{\theta,z}e_r + \frac{1}{r}\left(rA_\theta\right)_{,r} e_z \ . \tag{3.9}$$

For a div-free, axi-symmetric vector field $u = u_r(r,z)e_r + u_z(r,z)e_z$ with no swirl, we look for a vector potential $A$ such that

$$\mathrm{curl}\, A = u, \quad \mathrm{div}\, A = 0 \tag{3.10}$$

and in view of (3.7) and (3.9), one can take

$$A = \frac{\psi}{r} e_\theta \tag{3.11}$$

By (3.10), we have

$$\omega_\theta e_\theta = \omega = \operatorname{curl} u = \operatorname{curl}\operatorname{curl} A = -\Delta A + \nabla(\operatorname{div} A) = -\Delta A \tag{3.12}$$

Thus,

$$A(x) = (-\Delta)^{-1}\omega = \int_{\mathbb{R}^3} \frac{1}{4\pi |x-y|}\omega(y)\,dy \tag{3.13}$$

Letting $\left(\bar{r}, \bar{\theta}, \bar{z}\right)$ and $(r, \theta, z)$ be the cylindrical coordinates of $x$ and $y$, respectively, by elementary trigonometry, we can write

$$|x-y| = \sqrt{r^2 + \bar{r}^2 - 2r\bar{r}\cos\left(\theta - \bar{\theta}\right) + (z-\bar{z})^2} \tag{3.14}$$

By (3.11), (3.13) and (3.14), we obtain

$$\begin{aligned}
&\frac{\psi\left(\bar{r}, \bar{z}\right)}{\bar{r}}\left(-\sin\bar{\theta}, \cos\bar{\theta}, 0\right) \\
&= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{r\omega_\theta(r,z)\left(-\sin\theta, \cos\theta, 0\right)\ d\theta drdz}{4\pi\sqrt{r^2+\bar{r}^2-2r\bar{r}\cos\left(\theta-\bar{\theta}\right)+(z-\bar{z})^2}} \\
&= \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\bar{\theta}}^{2\pi-\bar{\theta}} \frac{r\omega_\theta(r,z)\left(-\sin\left(\varphi+\bar{\theta}\right), \cos\left(\varphi+\bar{\theta}\right), 0\right)\ d\varphi drdz}{4\pi\sqrt{r^2+\bar{r}^2-2r\bar{r}\cos\varphi+(z-\bar{z})^2}} \\
&= \left(-\cos\bar{\theta}, -\sin\bar{\theta}, 0\right)\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\bar{\theta}}^{2\pi-\bar{\theta}} \frac{r\omega_\theta(r,z)\sin\varphi\ d\varphi drdz}{4\pi\sqrt{r^2+\bar{r}^2-2r\bar{r}\cos\varphi+(z-\bar{z})^2}} \\
&\quad + \left(-\sin\bar{\theta}, \cos\bar{\theta}, 0\right)\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\bar{\theta}}^{2\pi-\bar{\theta}} \frac{r\omega_\theta(r,z)\cos\varphi\ d\varphi drdz}{4\pi\sqrt{r^2+\bar{r}^2-2r\bar{r}\cos\varphi+(z-\bar{z})^2}} \\
&= \left(-\cos\bar{\theta}, -\sin\bar{\theta}, 0\right)\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\pi}^{\pi} \frac{r\omega_\theta(r,z)\sin\varphi\ d\varphi drdz}{4\pi\sqrt{r^2+\bar{r}^2-2r\bar{r}\cos\varphi+(z-\bar{z})^2}} \\
&\quad + \left(-\sin\bar{\theta}, \cos\bar{\theta}, 0\right)\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{r\omega_\theta(r,z)\cos\varphi\ d\varphi drdz}{4\pi\sqrt{r^2+\bar{r}^2-2r\bar{r}\cos\varphi+(z-\bar{z})^2}} \\
&= \left(-\sin\bar{\theta}, \cos\bar{\theta}, 0\right)\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{r\omega_\theta(r,z)\cos\varphi\ d\varphi drdz}{4\pi\sqrt{r^2+\bar{r}^2-2r\bar{r}\cos\varphi+(z-\bar{z})^2}}
\end{aligned} \tag{3.15}$$

Therefore,

$$\frac{\psi\left(\bar{r},\bar{z}\right)}{\bar{r}}=\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{2\pi}\frac{r\omega_{\theta}(r,z)\cos\varphi\;d\varphi drdz}{4\pi\sqrt{r^{2}+\bar{r}^{2}-2r\bar{r}\cos\varphi+(z-\bar{z})^{2}}}\tag{3.16}$$

which implies (3.8).

We can express (3.8) somewhat more explicitly as

$$\begin{aligned}\psi(\bar{r},\bar{z})&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\sqrt{\bar{r}r}}{2\pi}\int_{0}^{\pi}\frac{\cos\varphi\;d\varphi}{\left[2(1-\cos\varphi)+\frac{(r-\bar{r})^{2}+(z-\bar{z})^{2}}{\bar{r}r}\right]^{\frac{1}{2}}}\,\omega_{\theta}(r,z)\,dr\,dz\\&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\sqrt{\bar{r}r}}{2\pi}F\left(\frac{(r-\bar{r})^{2}+(z-\bar{z})^{2}}{\bar{r}r}\right)\omega_{\theta}(r,z)\,dr\,dz\,,\end{aligned}\tag{3.17}$$

where the function $F:(0,\infty)\to\mathbb{R}$ is defined by

$$F(s)\colon=\int_{0}^{\pi}\frac{\cos\varphi\;d\varphi}{\left[2(1-\cos\varphi)+s\right]^{\frac{1}{2}}}\;.\tag{3.18}$$

Let

$$G(\bar{r},\bar{z},r,z)=\frac{\sqrt{\bar{r}r}}{2\pi}F\left(\frac{(r-\bar{r})^{2}+(z-\bar{z})^{2}}{\bar{r}r}\right)\;.\tag{3.19}$$

Then

$$\psi(\bar{r},\bar{z})=\int_{-\infty}^{\infty}\int_{0}^{\infty}G(\bar{r},\bar{z},r,z)\,\omega_{\theta}(r,z)\,dr\,dz\;.$$

By (3.7) and (3.19), we get

$$\begin{aligned}u_{r}(\bar{r},\bar{z})&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\left[-\frac{1}{\bar{r}}\frac{\partial G}{\partial\bar{z}}(\bar{r},\bar{z},r,z)\right]\omega_{\theta}(r,z)\,dr\,dz\\&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{z-\bar{z}}{\pi\bar{r}^{\frac{3}{2}}\sqrt{r}}F'\left(\frac{(r-\bar{r})^{2}+(z-\bar{z})^{2}}{\bar{r}r}\right)\omega_{\theta}(r,z)\,dr\,dz\,,\end{aligned}\tag{3.20}$$

$$\begin{aligned}u_{z}(\bar{r},\bar{z})&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\left[\frac{1}{\bar{r}}\frac{\partial G}{\partial\bar{r}}(\bar{r},\bar{z},r,z)\right]\omega_{\theta}(r,z)\,dr\,dz\\&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\mathscr{Z}(\bar{r},\bar{z},r,z)\,\omega_{\theta}(r,z)\,dr\,dz\;,\end{aligned}\tag{3.21}$$

where

$$\mathscr{L}(\bar r,\bar z,r,z)=\frac{1}{\bar r}\frac{\partial G}{\partial \bar r}(\bar r,\bar z,r,z)\ .$$

The formulae (3.20) and (3.21), representing the relations between $u_r, u_z$ and $\omega_\theta$, represent the *axi-symmetric Biot-Savart Law.* We calculate the kernel $\mathscr{L}$. Let $d^2=(r-\bar r)^2+(z-\bar z)^2$. Let $\xi=\xi(\bar r,\bar z,r,z)=\frac{d}{\sqrt{\bar r r}}$. Then by (3.19), we have

$$G(\bar r,\bar z,r,z)=\frac{d}{2\pi\xi}F(\xi^2)=\frac{d}{2\pi}H(\xi)\ ,$$

where $H(t)=\frac{F(t^2)}{t}$. Direct calculation shows that

$$H'(t)=2F'(t^2)-\frac{F(t^2)}{t^2}\ ,\quad \frac{\partial\xi}{\partial\bar r}=\xi\Big(\frac{\bar r-r}{d^2}-\frac{1}{2\bar r}\Big)\ , \tag{3.22}$$

$$\begin{aligned}\mathscr{L}=\frac{1}{\bar r}\frac{\partial G}{\partial\bar r}&=\frac{1}{2\pi}\frac{\bar r-r}{\bar r^{\frac32}r^{\frac12}}\Big[\frac{H(\xi)}{\xi}+H'(\xi)\Big]-\frac{1}{4\pi}\xi^2H'(\xi)\frac{\sqrt r}{\bar r^{\frac32}}\\&=\frac{1}{\pi}\frac{\bar r-r}{\bar r^{\frac32}r^{\frac12}}F'(\xi^2)+\frac{1}{4\pi}\Big[F(\xi^2)-2\xi^2F'(\xi^2)\Big]\frac{\sqrt r}{\bar r^{\frac32}}\ .\end{aligned} \tag{3.23}$$

In the sequel, we are mainly interested in $\mathscr{L}$ at $(\bar r,\bar z)=(1,0)$. We write it down explicitly:

$$\begin{aligned}\mathscr{L}(1,0,r,z)=&\frac{1-r}{\pi r^{\frac12}}F'\left(\frac{(r-1)^2+z^2}{r}\right)\\&+\frac{\sqrt r}{4\pi}\left[F\left(\frac{(r-1)^2+z^2}{r}\right)-2\frac{(r-1)^2+z^2}{r}F'\left(\frac{(r-1)^2+z^2}{r}\right)\right]\ .\end{aligned} \tag{3.24}$$

At the first glance, comparing with the usual Biot-Savart Law (3.2), the axi-symmetric Biot-Savart Law (3.20) and (3.21) look more complicated and have no advantages. But (3.20) and (3.21) indeed capture some features of axi-symmetric fields with no swirl. Although the function $F$ in (3.18) cannot be expressed in terms of elementary functions, it has nice asymptotic properties near $s=0$ and $s=\infty$. By (3.18), it is obvious that

$$|F(s)|\lesssim\Big(\frac{1}{s}\Big)^{\frac12}. \tag{3.25}$$

However, $F$ actually has a slower blow-up at $s = 0$ and a faster decay at $s = \infty$ than (3.25) as: $|F(s)| \lesssim \log \frac{1}{s}$ near $s = 0$ and $|F(s)| \lesssim \left(\frac{1}{s}\right)^{\frac{3}{2}}$ near $s = \infty$. We will use the following simple properties of $F$.

**Lemma 3.6.** *For every non-negative integer $k$, the $k$th-derivative of $F$ satisfies*

$$\left|F^{(k)}(s)\right| \lesssim_k \frac{1}{s^{k+\frac{1}{2}}}\,, \tag{3.26}$$

*for all $s \in (0, \infty)$.*

**Proof** . By (3.18),

$$|F(s)| \lesssim \int_0^\pi \frac{d\varphi}{s^{\frac{1}{2}}} \lesssim \frac{1}{s^{\frac{1}{2}}}\,.$$

Hence (3.26) is true for the case of $k = 0$. The first derivative of $F$ is

$$F'(s) = -\frac{1}{2}\int_0^\pi \frac{\cos\varphi\ d\varphi}{\big[2(1-\cos\varphi)+s\big]^{\frac{3}{2}}}\,.$$

Therefore,

$$|F'(s)| \lesssim \int_0^\pi \frac{d\varphi}{s^{\frac{3}{2}}} \lesssim \frac{1}{s^{\frac{3}{2}}}\,.$$

Hence the case of $k = 1$ is also true. The remaining cases can be proved similarly. □

**Lemma 3.7.** *There exists an absolute constant $0 < \varepsilon_0 < 1$ such that for all $s \in (0, \varepsilon_0)$, the $k$th-derivative of $F$ satisfies*

$$\begin{aligned} |F(s)| &\lesssim \log\frac{1}{s} \lesssim_\tau \frac{1}{s^\tau}\,, \quad \text{for every } \tau > 0, \text{ if } k = 0\,, \\ \left|F^{(k)}(s)\right| &\lesssim_k \frac{1}{s^k}\,, \qquad\qquad\qquad\qquad \text{if } 0 < k \in \mathbb{N}\,. \end{aligned} \tag{3.27}$$

**Proof** . $F(s)$ has the following expansion near $s = 0$.

$$F(s) = \left(\log\frac{1}{s}\right)(a_0 + a_1 s + a_2 s^2 + \cdots) + (b_0 + b_1 s + b_2 s^2 + \cdots)\,, \tag{3.28}$$

with $a_0 = \frac{1}{2}$ and $b_0 = \log 8 - 2$. Hence

$$F(s) = \frac{1}{2}\log\frac{1}{s} + \log 8 - 2 + O\Big(s\log\frac{1}{s}\Big), \qquad s \to 0_+\,. \tag{3.29}$$

The estimates (3.27) follows easily from the above expansion. We now derive (3.28) by doing the direct calculations with the integral (3.18).

$$
\begin{aligned}
F(s) &= \int_0^{\pi} \frac{\cos\varphi \; d\varphi}{\left[2(1-\cos\varphi)+s\right]^{\frac{1}{2}}} = \int_0^{\pi} \frac{1-2\sin^2\frac{\varphi}{2}}{\left[4\sin^2\frac{\varphi}{2}+s\right]^{\frac{1}{2}}}\, d\varphi = \int_0^{\frac{\pi}{2}} \frac{1-2\sin^2\varphi}{\left[\sin^2\varphi+\frac{s}{4}\right]^{\frac{1}{2}}}\, d\varphi \\
&= \int_0^{\frac{\pi}{2}} \frac{1-2\sin^2\varphi}{\left[\sin^2\varphi+\sigma^2\right]^{\frac{1}{2}}}\, d\varphi, \qquad \sigma^2 = \frac{s}{4}\,,
\end{aligned}
\tag{3.30}
$$

or

$$
F(s) = \int_0^{\frac{\pi}{2}} \frac{1+2\sigma^2}{\left[\sin^2\varphi+\sigma^2\right]^{\frac{1}{2}}}\, d\varphi - 2\int_0^{\frac{\pi}{2}} \sqrt{\sin^2\varphi+\sigma^2}\, d\varphi, \qquad \sigma^2 = \frac{s}{4}. \tag{3.31}
$$

The leading term as $s \to 0$ is

$$
\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\left[\sin^2\varphi+\sigma^2\right]^{\frac{1}{2}}}\ . \tag{3.32}
$$

We write

$$
\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\left[\sin^2\varphi+\sigma^2\right]^{\frac{1}{2}}} = \int_0^{\frac{\pi}{2}} \frac{\cos\varphi\; d\varphi}{\left[\sin^2\varphi+\sigma^2\right]^{\frac{1}{2}}} + \int_0^{\frac{\pi}{2}} \frac{(1-\cos\varphi)\; d\varphi}{\left[\sin^2\varphi+\sigma^2\right]^{\frac{1}{2}}}\ . \tag{3.33}
$$

The first integral can be written as

$$
\begin{aligned}
\int_0^{1} \frac{dt}{\sqrt{t^2+\sigma^2}} &= \int_0^{\frac{1}{\sigma}} \frac{dt}{\sqrt{t^2+1}} = \log\frac{1}{\sigma} + \log\left(1+\sqrt{1+\sigma^2}\right) \\
&= \log\frac{1}{\sigma} + \log 2 + O\left(\sigma^2\right), \qquad \sigma \to 0\,.
\end{aligned}
\tag{3.34}
$$

The second integral in (3.33) can be approximated as

$$
\int_0^{\frac{\pi}{2}} \frac{1-\cos\varphi}{\sin\varphi} d\varphi + O\left(\sigma^2 \log\frac{1}{\sigma}\right) = \log 2 + O\left(\sigma^2\log\frac{1}{\sigma}\right), \qquad \sigma \to 0\,. \tag{3.35}
$$

For the second integral in (3.31) we can write

$$
2\int_0^{\frac{\pi}{2}} \sqrt{\sin^2\varphi+\sigma^2}\; d\varphi = 2 + O\left(\sigma^2\log\frac{1}{\sigma}\right)\ . \tag{3.36}
$$

We thus obtain (3.29). Proceeding in a similar way, one can show

$$F(s) = \log \frac{8}{\sqrt{s}} \left(1 + \frac{3}{16}s - \frac{15}{1024}s^2 + \cdots\right) - 2 - \frac{1}{16}s + \frac{31}{2048}s^2 + \cdots \quad (3.37)$$

$\square$

**Lemma 3.8.** *There exists an absolute constant $N_0 > 1$ such that for every non-negative integer $k$, the $k$th-derivative of $F$ satisfies*

$$\left|F^{(k)}(s)\right| \lesssim_k \frac{1}{s^{k+\frac{3}{2}}} \quad (3.38)$$

*for all $s \in (N_0, \infty)$.*

**Proof** . $F(s)$ has such an expansion near $s = \infty$,

$$F(s) = \frac{\pi}{2}\frac{1}{s^{\frac{3}{2}}} - \frac{3\pi}{2}\frac{1}{s^{\frac{5}{2}}} + O(s^{-\frac{7}{2}})$$

Then (3.38) follows easily from this expansion. $\square$

The estimates in Lemma 3.7 and Lemma 3.8 are local. But those restrictions can be easily removed with the aid of Lemma 3.6. As a consequence of Lemma 3.6, Lemma 3.7 and Lemma 3.8, we have

**Corollary 3.9.** *For every non-negative integer $k$, the $k$th-derivative of $F$ satisfies*

$$|F(s)| \lesssim_\tau \ \min\left(\left(\frac{1}{s}\right)^\tau, \left(\frac{1}{s}\right)^{\frac{1}{2}}, \left(\frac{1}{s}\right)^{\frac{3}{2}}\right), \quad \text{for every } 0 < \tau < \frac{1}{2}, \text{ if } k = 0,$$

$$\left|F^{(k)}(s)\right| \lesssim_k \ \min\left(\left(\frac{1}{s}\right)^k, \left(\frac{1}{s}\right)^{k+\frac{1}{2}}, \left(\frac{1}{s}\right)^{k+\frac{3}{2}}\right), \qquad \text{if } 0 < k \in \mathbb{N},$$

*for all $s \in (0, \infty)$.*

With the aid of Corollary 3.9, controlling $L_x^\infty(\mathbb{R}^3)$ of $u$ via the a-priori bounds (3.1) becomes tractable. We need the following technical lemma.

**Lemma 3.10.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $\|f\|_{L^1(\mathbb{R}^2)} < \infty$ and $\|f\|_{L^\infty(\mathbb{R}^2)} < \infty$. Let $K : \mathbb{R}^2 \to \mathbb{R}$ be such that $|K(x)| \leq \frac{C}{|x-x_0|}$ for some positive constant $C$, some point $x_0 \in \mathbb{R}^2$ and for all $x \in \mathbb{R}^2$. Then*

$$\left| \int_{\mathbb{R}^2} K(x) f(x)\, dx \right| \leq 2\sqrt{2\pi} C \, \|f\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} \|f\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} .$$

**Proof** . For any $\rho > 0$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^2} K(x) f(x)\, dx \right| &\leq \int_{|x-x_0|\leq \rho} \frac{C}{|x-x_0|} |f(x)|\, dx + \int_{|x-x_0|>\rho} \frac{C}{|x-x_0|} |f(x)|\, dx \\ &\leq 2\pi C \rho \|f\|_{L^\infty(\mathbb{R}^2)} + \frac{C}{\rho} \|f\|_{L^1(\mathbb{R}^2)} \ . \end{aligned}$$

After minimizing the last term, we can get the desired result. □

Since an axi-symmetric vector field $u$ with no swirl is of the form $u = u_r(r,z)e_r + u_z(r,z)e_z$, to estimate the $L^\infty_x(\mathbb{R}^3)$ norm of $u$, it is enough to estimate the $L^\infty$ norms of $u_r$ and $u_z$ over the $rz$-plane $\Omega := \{r \geq 0, z \in \mathbb{R}\}$. We will use the following simple identities.

$$\|r\omega\|_{L^1_x(\mathbb{R}^3)} = 2\pi \left\|r^2 \omega_\theta\right\|_{L^1(\Omega)} , \ \left\|\frac{\omega}{r}\right\|_{L^1_x(\mathbb{R}^3)} = 2\pi \|\omega_\theta\|_{L^1(\Omega)} , \ \left\|\frac{\omega}{r}\right\|_{L^\infty_x(\mathbb{R}^3)} = \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} .$$

We first estimate the $r$-component $u_r$.

**Proposition 3.11.** *Let $u_r$ be given by the formula (3.20) with $\omega_\theta$ satisfying*

$$\left\|r^2\omega_\theta\right\|_{L^1(\Omega)} < \infty, \ \|\omega_\theta\|_{L^1(\Omega)} < \infty, \ \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} < \infty \ .$$

*Then*

$$\|u_r\|_{L^\infty(\Omega)} \leq \ C_1 \left\|r^2\omega_\theta\right\|_{L^1(\Omega)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)}^{\frac{1}{2}} \ , \tag{3.39}$$

*where $C_1$ is an absolute constant.*

**Proof** . The estimate (3.39) is invariant under the scaling and the translation in the $z$ variable

$$u_r(r,z) \mapsto u_r(\lambda r, \lambda z + z_0), \quad \omega_\theta(r,z) \mapsto \lambda \omega_\theta(\lambda r, \lambda z + z_0)$$

for every $\lambda > 0$ and every $z_0 \in \mathbb{R}$, and therefore it is enough to prove

$$|u_r(1,0)| \lesssim \left\|r^2\omega_\theta\right\|_{L^1(\Omega)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)}^{\frac{1}{2}} . \tag{3.40}$$

By (3.20)

$$u_r(1,0) = \int_{-\infty}^{\infty}\int_0^\infty \frac{z}{\pi\sqrt{r}} F'\left(\frac{(r-1)^2+z^2}{r}\right)\omega_\theta(r,z)\,dr\,dz . \tag{3.41}$$

We split the right hand side of (3.41) into two parts. One is on the region

$$I_1 = \left\{\frac{1}{2} \le r \le 2, -1 \le z \le 1\right\}$$

and the other on the complement $I_2 = \Omega \setminus I_1$.

On $I_1$, by Corollary 3.9 (using $|F'(s)| \lesssim \frac{1}{s}$), the kernel of (3.41) can be estimated as

$$\left|\frac{z}{\pi\sqrt{r}} F'\left(\frac{(r-1)^2+z^2}{r}\right)\right| \lesssim \frac{|z|}{\sqrt{r}}\frac{r}{(r-1)^2+z^2} \lesssim \frac{1}{\sqrt{(r-1)^2+z^2}} = \frac{1}{|(r,z)-(1,0)|} .$$

Therefore, by Lemma 3.10 and the fact that $r \sim 1$ on $I_1$, we obtain

$$\begin{aligned} &\left|\iint_{I_1} \frac{z}{\pi\sqrt{r}} F'\left(\frac{(r-1)^2+z^2}{r}\right) \omega_\theta(r,z)\,dr\,dz\right| \\ =&\left|\iint \frac{z}{\pi\sqrt{r}} F'\left(\frac{(r-1)^2+z^2}{r}\right) \omega_\theta(r,z)\,\chi_{I_1}\,dr\,dz\right| \\ \lesssim& \|\omega_\theta\|_{L^1(I_1)}^{\frac{1}{2}} \|\omega_\theta\|_{L^\infty(I_1)}^{\frac{1}{2}} \lesssim \left\|r^2\omega_\theta\right\|_{L^1(I_1)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(I_1)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(I_1)}^{\frac{1}{2}} , \end{aligned} \tag{3.42}$$

where $\chi_{I_1}$ is the characteristic function of $I_1$.

On $I_2$, by Corollary 3.9, (using $|F'(s)| \lesssim \left(\frac{1}{s}\right)^{\frac{5}{2}}$), the kernel of (3.41) can be estimated as

$$\left|\frac{z}{\pi\sqrt{r}} F'\left(\frac{(r-1)^2+z^2}{r}\right)\right| \lesssim \frac{|z|}{\sqrt{r}}\left(\frac{r}{(r-1)^2+z^2}\right)^{\frac{5}{2}} \lesssim \frac{1}{(r-1)^2+z^2} ,$$

which is square-integrable on $I_2$. Therefore, noting that $|\omega_\theta| = r^{\frac{1}{2}}\,|\omega_\theta|^{\frac{1}{4}}\,|\omega_\theta|^{\frac{1}{4}}\,\frac{|\omega_\theta|^{\frac{1}{2}}}{r^{\frac{1}{2}}}$, by Hölder's inequality, we obtain

$$\left|\iint_{I_2} \frac{z}{\pi\sqrt{r}} F'\left(\frac{(r-1)^2+z^2}{r}\right) \omega_\theta(r,z)\,dr\,dz\right| \lesssim \left\|r^2\omega_\theta\right\|_{L^1(I_2)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(I_2)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(I_2)}^{\frac{1}{2}} . \tag{3.43}$$

Clearly, (3.41), (3.42) and (3.43) imply (3.40). The proposition is proved. $\square$

To estimate $u_z$, we need the following technical lemma.

**Lemma 3.12.** *Assume that $\omega_\theta$ is a function on $\Omega$ satisfying*

$$\left\|r^2\omega_\theta\right\|_{L^1(\Omega)} < \infty,\ \ \|\omega_\theta\|_{L^1(\Omega)} < \infty,\ \ \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} < \infty\ .$$

*Then*

$$\int_{z=-\infty}^{\infty}\int_{r=2}^{\infty} |\omega_\theta(r,z)|\,\frac{r^2}{[(r-1)^2+z^2]^{\frac{3}{2}}}\,dr\,dz \lesssim \left\|r^2\omega_\theta\right\|_{L^1(\Omega)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)}^{\frac{1}{2}}\ . \tag{3.44}$$

We remark that the integral domain $\Omega$ of the right hand side of (3.44) can be replaced by $\{r \geq 2\}$, where $\{r \geq 2\}$ is shorthand for the set $\{r \geq 2, z \in \mathbb{R}\}$. But (3.44) is enough for our purpose.

**Proof** . We can't use the Hölder's inequality directly to get (3.44) because on the region $\{r \geq |z|\}$, the weight $\frac{r^2}{[(r-1)^2+z^2]^{\frac{3}{2}}} \sim \frac{1}{[(r-1)^2+z^2]^{\frac{1}{2}}}$, which is not square-integrable on that region. We introduce some notations. Let $d^2 = r^2 + z^2$ and $f(r,z) = \frac{\omega_\theta(r,z)}{r}$. To prove (3.44), it is enough to show

$$\int_{z=-\infty}^{\infty}\int_{r=2}^{\infty} |f|\,\frac{r^3}{d^3}\,dr\,dz \lesssim \left\|r^3 f\right\|_{L^1(\Omega)}^{\frac{1}{4}} \|rf\|_{L^1(\Omega)}^{\frac{1}{4}} \|f\|_{L^\infty(\Omega)}^{\frac{1}{2}}\,. \tag{3.45}$$

By Cauchy-Schwartz inequality, we have

$$\left\|r^2 f\right\|_{L^1(\Omega)} \leq \left\|r^3 f\right\|_{L^1(\Omega)}^{\frac{1}{2}} \|rf\|_{L^1(\Omega)}^{\frac{1}{2}}\,.$$

Therefore, to prove (3.45), it is enough to prove

$$\int_{z=-\infty}^{\infty}\int_{r=2}^{\infty} |f|\,\frac{r^3}{d^3}\,dr\,dz \lesssim \left\|r^2 f\right\|_{L^1(\{r\geq 2\})}^{\frac{1}{2}} \|f\|_{L^\infty(\{r\geq 2\})}^{\frac{1}{2}}\,, \tag{3.46}$$

since $\{r \geq 2\} \subset \Omega$. We may assume that $f$ is a function supported in $\{r \geq 2\}$ and vanishing elsewhere in $\Omega$, otherwise, we can just replace $f$ by $f\chi_{\{r\geq 2\}}$. Under this assumption, it is enough to prove

$$\left\|f\frac{r^3}{d^3}\right\|_{L^1(\Omega)} \lesssim \left\|r^2 f\right\|_{L^1(\Omega)}^{\frac{1}{2}} \|f\|_{L^\infty(\Omega)}^{\frac{1}{2}}\,. \tag{3.47}$$

For $\lambda > 0$, let $f_\lambda(r,z) = \lambda^2 f(\lambda r, \lambda z)$. Clearly, $f_\lambda$ is supported on $\left\{r \geq \frac{2}{\lambda}\right\}$. It is easy to check that for every $\lambda > 0$, we have

$$\left\| f_\lambda \frac{r^3}{d^3} \right\|_{L^1(\Omega)} = \left\| f \frac{r^3}{d^3} \right\|_{L^1(\Omega)}, \|f_\lambda\|_{L^\infty(\Omega)} = \lambda^2 \|f\|_{L^\infty(\Omega)}, \left\|r^2 f_\lambda\right\|_{L^1(\Omega)} = \lambda^{-2} \left\|r^2 f\right\|_{L^1(\Omega)} .$$

We find $\lambda_0 > 0$ so that $\|f_{\lambda_0}\|_{L^\infty(\Omega)} = \|r^2 f_{\lambda_0}\|_{L^1(\Omega)}$. By calculation,

$$\lambda_0 = \left( \frac{\|r^2 f\|_{L^1(\Omega)}}{\|f\|_{L^\infty(\Omega)}} \right)^{\frac{1}{4}} .$$

To prove (3.47), it is enough to prove

$$\left\| f_{\lambda_0} \frac{r^3}{d^3} \right\|_{L^1(\Omega)} \lesssim \left\|r^2 f_{\lambda_0}\right\|_{L^1(\Omega)} + \|f_{\lambda_0}\|_{L^\infty(\Omega)} . \tag{3.48}$$

We distinguish two cases $0 < \lambda_0 \leq 1$ and $\lambda_0 > 1$.

**Case 1.** $0 < \lambda_0 \leq 1$.

By definition, $f_{\lambda_0}$ is supported on $\left\{r \geq \frac{2}{\lambda_0}\right\}$, which lies in $\{r \geq 1\}$. On the support of $f_{\lambda_0}$, it is clear that $\frac{r^3}{d^3} \leq 1 \leq r^2$ and hence (3.48) is true.

**Case 2.** $\lambda_0 > 1$.

In this case, we have

$$\begin{aligned} \left\| f_{\lambda_0} \frac{r^3}{d^3} \right\|_{L^1(\Omega)} &\lesssim \int_{-\infty}^{\infty} \int_2^{\infty} |f_{\lambda_0}| \, dr \, dz + \|f_{\lambda_0}\|_{L^\infty(\Omega)} \int_{-\infty}^{\infty} \int_{\frac{2}{\lambda_0}}^{2} \frac{r^3}{d^3} \, dr \, dz \\ &\lesssim \left\|r^2 f_{\lambda_0}\right\|_{L^1(\Omega)} + \|f_{\lambda_0}\|_{L^\infty(\Omega)} . \end{aligned}$$

Therefore (3.48) is true. The lemma is proved. $\square$

We now estimate the $z$-component $u_z$. The work for $u_z$ is similar to that for $u_r$ in Proposition 3.11 but some part have to be treated differently.

**Proposition 3.13.** *Let $u_z$ be given by the formula (3.21) with $\omega_\theta$ satisfying*

$$\left\|r^2 \omega_\theta\right\|_{L^1(\Omega)} < \infty, \ \|\omega_\theta\|_{L^1(\Omega)} < \infty, \ \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\Omega)} < \infty .$$

*Then*

$$\|u_z\|_{L^\infty(\Omega)} \leq \ C_2 \left\|r^2 \omega_\theta\right\|_{L^1(\Omega)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{4}} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\Omega)}^{\frac{1}{2}} , \tag{3.49}$$

*where $C_2$ is an absolute constant.*

**Proof** . Since the estimate (3.49) is invariant under the scaling and the translation in the $z$ variable, it is enough to prove

$$|u_z(1,0)| \lesssim \left\|r^2\omega_\theta\right\|_{L^1(\Omega)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)}^{\frac{1}{2}} . \tag{3.50}$$

By (3.21),

$$u_z(1,0) = \int_{-\infty}^{\infty}\int_0^{\infty} \mathscr{L}(1,0,r,z)\,\omega_\theta(r,z)\,dr\,dz \ , \tag{3.51}$$

where $\mathscr{L}(1,0,r,z)$ is given by (3.24) as

$$\begin{aligned}\mathscr{L}(1,0,r,z) =&\frac{1-r}{\pi r^{\frac{1}{2}}}F'\left(\frac{(r-1)^2+z^2}{r}\right)\\ &+\frac{\sqrt{r}}{4\pi}\left[F\left(\frac{(r-1)^2+z^2}{r}\right)-2\frac{(r-1)^2+z^2}{r}F'\left(\frac{(r-1)^2+z^2}{r}\right)\right]\\ :=&\mathscr{L}_1(r,z)+\mathscr{L}_2(r,z) \ .\end{aligned} \tag{3.52}$$

We split the right hand side of (3.51) into two parts. One is on the region

$$I_1 = \left\{\frac{1}{2} \le r \le 2, -1 \le z \le 1\right\}$$

and the other on the complement $I_2 = \Omega \setminus I_1$.

On $I_1$, by Corollary 3.9, $\mathscr{L}_1$ can be estimated as (using $|F'(s)| \lesssim \frac{1}{s}$)

$$|\mathscr{L}_1(r,z)| \lesssim \frac{|1-r|}{r^{\frac{1}{2}}}\frac{r}{(r-1)^2+z^2} \lesssim \frac{1}{|(r,z)-(1,0)|}$$

and $\mathscr{L}_2$ can be estimates as (using $|F(s)| \lesssim \left(\frac{1}{s}\right)^{\frac{1}{2}}$ and $|F'(s)| \lesssim \left(\frac{1}{s}\right)^{\frac{3}{2}}$)

$$|\mathscr{L}_2(r,z)| \lesssim \sqrt{r}\left[\left(\frac{r}{(r-1)^2+z^2}\right)^{\frac{1}{2}} + \frac{(r-1)^2+z^2}{r}\left(\frac{r}{(r-1)^2+z^2}\right)^{\frac{3}{2}}\right] \lesssim \frac{1}{|(r,z)-(1,0)|} \ .$$

Therefore, by Lemma 3.10 and the fact that $r \sim 1$ on $I_1$, we obtain

$$\begin{aligned}&\left|\iint_{I_1} \mathscr{L}(1,0,r,z)\,\omega_\theta(r,z)\,dr\,dz\right|\\ &\lesssim \|\omega_\theta\|_{L^1(I_1)}^{\frac{1}{2}} \|\omega_\theta\|_{L^\infty(I_1)}^{\frac{1}{2}} \lesssim \left\|r^2\omega_\theta\right\|_{L^1(I_1)}^{\frac{1}{4}} \|\omega_\theta\|_{L^1(I_1)}^{\frac{1}{4}} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(I_1)}^{\frac{1}{2}} .\end{aligned} \tag{3.53}$$

On $I_2$, by Corollary 3.9, $\mathscr{Z}_1$ can be estimated as (using $|F'(s)| \lesssim \left(\frac{1}{s}\right)^{\frac{5}{2}}$)

$$|\mathscr{Z}_1(r,z)| \lesssim \frac{|1-r|}{r^{\frac{1}{2}}} \left(\frac{r}{(r-1)^2+z^2}\right)^{\frac{5}{2}} \lesssim \frac{1}{(r-1)^2+z^2} \, ,$$

which is square-integrable on $I_2$. Therefore, by Hölder's inequality, we obtain

$$\left| \iint_{I_2} \mathscr{Z}_1(r,z)\, \omega_\theta(r,z)\, dr\, dz \right| \lesssim \left\| r^2 \omega_\theta \right\|_{L^1(I_2)}^{\frac{1}{4}} \left\| \omega_\theta \right\|_{L^1(I_2)}^{\frac{1}{4}} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(I_2)}^{\frac{1}{2}} . \tag{3.54}$$

Unfortunately, the foregoing argument of $\mathscr{Z}_1$ does not work for $\mathscr{Z}_2$ because $\mathscr{Z}_2$ is not square-integrable on the region $I_2$. By Corollary 3.9, the best estimate for $\mathscr{Z}_2$ on $I_2$ is (using $|F(s)| \lesssim \left(\frac{1}{s}\right)^{\frac{3}{2}}$ and $|F'(s)| \lesssim \left(\frac{1}{s}\right)^{\frac{5}{2}}$)

$$|\mathscr{Z}_2(r,z)| \lesssim \sqrt{r} \left[ \left(\frac{r}{(r-1)^2+z^2}\right)^{\frac{3}{2}} + \frac{(r-1)^2+z^2}{r} \left(\frac{r}{(r-1)^2+z^2}\right)^{\frac{5}{2}} \right] \sim \frac{r^2}{[(r-1)^2+z^2]^{\frac{3}{2}}} \, . \tag{3.55}$$

To overcome this difficulty, we split the region $I_2$ into two parts, "good" part $I_{21} := I_2 \cap \{r \le 2\}$ and "bad" part $I_{22} := I_2 \cap \{r > 2\} = \{r > 2\}$. By (3.55), $\mathscr{Z}_2$ is clearly square-integrable on $I_{21}$ and therefore by Hölder's inequality, we obtain

$$\left| \iint_{I_{21}} \mathscr{Z}_2(r,z)\, \omega_\theta(r,z)\, dr\, dz \right| \lesssim \left\| r^2 \omega_\theta \right\|_{L^1(I_{21})}^{\frac{1}{4}} \left\| \omega_\theta \right\|_{L^1(I_{21})}^{\frac{1}{4}} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(I_{21})}^{\frac{1}{2}} . \tag{3.56}$$

On the "bad" part $I_{22}$, by Lemma 3.12 and (3.55), we have

$$\begin{aligned} \left| \iint_{I_{22}} \mathscr{Z}_2(r,z)\, \omega_\theta(r,z)\, dr\, dz \right| &\lesssim \int_{z=-\infty}^{\infty} \int_{r=2}^{\infty} |\omega_\theta(r,z)| \frac{r^2}{[(r-1)^2+z^2]^{\frac{3}{2}}}\, dr\, dz \\ &\lesssim \left\| r^2 \omega_\theta \right\|_{L^1(\Omega)}^{\frac{1}{4}} \left\| \omega_\theta \right\|_{L^1(\Omega)}^{\frac{1}{4}} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\Omega)}^{\frac{1}{2}} . \end{aligned} \tag{3.57}$$

Clearly, (3.51), (3.52), (3.53), (3.54), (3.56) and (3.57) imply (3.50). The proposition is proved. □

The following proposition concerns the decay as $|x| \to \infty$.

**Proposition 3.14.** *Let $u = u_r e_r + u_z e_z$ with $u_r$ given by (3.20) and $u_z$ given by (3.21) and with $\omega_\theta$ satisfying*

$$\left\|r^2\omega_\theta\right\|_{L^1(\Omega)} < \infty,\ \left\|\omega_\theta\right\|_{L^1(\Omega)} < \infty,\ \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} < \infty\ .$$

*Then for every $\varepsilon > 0$, there exists a $R > 0$ such that for every $x \in \mathbb{R}^3$ with $|x| > R$, we have*

$$|u(x)| \leq \frac{\left\|r^2\omega_\theta\right\|_{L^1(\Omega)}^{\frac{1}{2}} \left\|\omega_\theta\right\|_{L^1(\Omega)}^{\frac{1}{2}}}{2(|x|-R)^2} + \frac{\varepsilon}{2}\ .$$

*In particular, we have*

$$\lim_{|x|\to\infty} |u(x)| = 0\ .$$

**Proof** . We can assume

$$\left\|r^2\omega_\theta\right\|_{L^1(\Omega)} > 0,\ \left\|\omega_\theta\right\|_{L^1(\Omega)} > 0,\ \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} > 0\ ,$$

otherwise, $u \equiv 0$ and the assertions are obviously true. For any $\varepsilon > 0$, we can find a $R > 0$ so that $\omega_1 : = \omega_\theta \chi_{\{r^2+z^2 \geq R^2\}}$ satisfies

$$\left\|\omega_1\right\|_{L^1(\Omega)} < \frac{\varepsilon^4}{16(C_1^2+C_2^2)^2 \left\|r^2\omega_\theta\right\|_{L^1(\Omega)} \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)}^2}\ ,$$

where $C_1$ and $C_2$ are the constants from Proposition 3.11 and 3.13. Let $\omega_2 = \omega_\theta - \omega_1$. Let $u_1$ and $u_2$ be the vector fields constructed from $\omega_1$ and $\omega_2$ via (3.20) and (3.21), respectively. Clearly, $u = u_1 + u_2$. By Proposition 3.11 and 3.13, we have

$$\left\|u_1\right\|_{L_x^\infty(\mathbb{R}^3)} \leq \sqrt{C_1^2+C_2^2}\left\|r^2\omega_1\right\|_{L^1(\Omega)}^{\frac{1}{4}} \left\|\omega_1\right\|_{L^1(\Omega)}^{\frac{1}{4}} \left\|\frac{\omega_1}{r}\right\|_{L^\infty(\Omega)}^{\frac{1}{2}} \leq \frac{\varepsilon}{2}\ . \tag{3.58}$$

We can also express $u_2$ in terms of $\omega_2$ via the Biot-Savart Law in Cartesian coordinates

$$u_2(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega_2\, e_\theta\, dy\ .$$

Since $\omega_2$ is supported in the ball $B_R(0)$, for any $|x| > R$, we have

$$|u_2(x)| \le \frac{1}{4\pi}\frac{\|\omega_2\|_{L^1_x(\mathbb{R}^3)}}{(|x|-R)^2} = \frac{1}{2}\frac{\|r\omega_2\|_{L^1(\Omega)}}{(|x|-R)^2} \le \frac{\|r^2\omega_\theta\|^{\frac{1}{2}}_{L^1(\Omega)}\|\omega_\theta\|^{\frac{1}{2}}_{L^1(\Omega)}}{2(|x|-R)^2} \ . \tag{3.59}$$

Clearly, (3.58) and (3.59) imply the first assertion. The second assertion follows immediately from the first one. □

**Remark 3.15.** *In the statement of Proposition 3.14, the $R$ depends not only on the norms*

$$\left\|r^2\omega_\theta\right\|_{L^1(\Omega)}, \ \|\omega_\theta\|_{L^1(\Omega)}, \ \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} \tag{3.60}$$

*but also on the distribution of $\omega_\theta$. For example, let $\omega_\theta(r,z) = \chi_{\{1\le r\le 2, |z|\le 1\}}$. Let $\omega_\theta^{z_0}(r,z) = \omega_\theta(r, z-z_0)$. Let $u^{z_0} = u_r^{z_0}e_r + u_z^{z_0}e_z$ be the vector field constructed from $\omega_\theta^{z_0}$ via (3.20) and (3.21). Obviously, we have*

$$\left\|r^2\omega_\theta^{z_0}\right\|_{L^1(\Omega)} = \left\|r^2\omega_\theta\right\|_{L^1(\Omega)}, \ \|\omega_\theta^{z_0}\|_{L^1(\Omega)} = \|\omega_\theta\|_{L^1(\Omega)}, \ \left\|\frac{\omega_\theta^{z_0}}{r}\right\|_{L^\infty(\Omega)} = \left\|\frac{\omega_\theta}{r}\right\|_{L^\infty(\Omega)} ,$$

$$u_r^{z_0}(r,z) = u_r(r, z-z_0), \quad u_z^{z_0}(r,z) = u_z(r, z-z_0) \ ,$$

*but $u$ and $\{u^{z_0}\}_{z_0\in\mathbb{R}}$ do not have a uniform decay since the profile of $u^{z_0}$ is just the translation of that of $u$ by $z_0$ in the $z$-direction. Nevertheless, they have the uniform decay rate in the $r$-direction. Actually, we can prove the following result that for any $0 < \varepsilon < \frac{1}{2}$ and any $x \in \mathbb{R}^3$ with $r = \sqrt{x_1^2 + x_2^2} \ge 1$,*

$$|u(x)| \le \frac{C}{r^{\frac{1}{2}-\varepsilon}} \ , \tag{3.61}$$

*where the constant $C$ depends only on $\varepsilon$ and the size of the norms in (3.60). But it is not clear whether (3.61) is optimal.*

# Chapter 4

# A-priori estimates and proof of the main result

## 4.1 A-priori estimates

In this chapter, we present the a-priori estimates for natural approximate solutions obtained by regularizing the initial data, and prove the main result of this thesis: the existence of globally smooth solutions for vortex rings, before which, we introduce the notations used. The superscript "$(\varepsilon)$" indicates the quantity (scalar or vector or tensor-valued) is induced by regularized initial data. Sometimes we use a function $f = f(r, z)$ defined on $[0, \infty) \times \mathbb{R}$ as a function defined on $\mathbb{R}^3$ in the following way:

$$f(x_1, x_2, z) = f\left(\sqrt{x_1^2 + x_2^2}, z\right), \quad \text{for} \quad (x_1, x_2, z) \in \mathbb{R}^3 \ .$$

Let us get back to our problem. The initial vorticity is

$$\omega_0 = \kappa \delta_\gamma \ , \tag{4.1}$$

where $\kappa \in \mathbb{R}$ and $\gamma$ is a circle. Without loss of generality, we assume that $\gamma$ is $(r_0 \cos\theta, r_0 \sin\theta, z_0)$ for some $r_0 > 0$, $z_0 \in \mathbb{R}$ and $-\pi \leq \theta < \pi$. Then (4.1) is equivalent to, in the sense of distribution,

$$\omega_0 = \kappa \delta_{r_0,z_0} e_\theta \ , \tag{4.2}$$

where $\delta_{r_0,z_0}$ is the Dirac mass at $(r_0, z_0)$ in the $rz$-plane. We will search a solution in the class of axi-symmetric velocity fields with no swirl, which have the form

$$u = u_r(r,z,t)e_r + u_z(r,z,t)e_z \ . \tag{4.3}$$

The related vorticity fields have the form

$$\omega = \omega_\theta(r,z,t)e_\theta \tag{4.4}$$

with $\omega_\theta = u_{r,z} - u_{z,r}$. Note that a solution of the form (4.4) is formally compatible to the initial condition (4.2). The equation for $\omega_\theta$ is

$$\partial_t \omega_\theta + u_r \omega_{\theta,r} + u_z \omega_{\theta,z} - \frac{u_r}{r}\omega_\theta = \omega_{\theta,rr} + \frac{1}{r}\omega_{\theta,r} - \frac{1}{r^2}\omega_\theta + \omega_{\theta,zz} \ , \tag{4.5}$$

which can also be written as:

$$\partial_t \omega_\theta + u \cdot \nabla \omega_\theta - \frac{u_r}{r}\omega_\theta = \Delta \omega_\theta - \frac{1}{r^2}\omega_\theta \ , \tag{4.6}$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$ is the scalar Laplacian in $\mathbb{R}^3$, expressed in the cylindrical coordinates. $u \cdot \nabla \omega_\theta = u_1 \omega_{\theta,1} + u_2 \omega_{\theta,2} + u_z \omega_{\theta,z}$ is equal to $u_r \omega_{\theta,r} + u_z \omega_{\theta,z}$. In terms of $\omega_\theta$, the initial condition (4.2) can be formulated as:

$$\omega_\theta(r,z,0) = \kappa \delta_{r_0,z_0} \ . \tag{4.7}$$

But we will not use either (4.5) or (4.6) in our method because these two equations have a vortex-stretching term $-\frac{u_r}{r}\omega_\theta$. It is easier to work with the quantity $\eta = \omega_\theta / r$, which satisfies

$$\eta_t + u_r \eta_{,r} + u_z \eta_{,z} = \eta_{,rr} + \frac{3}{r}\eta_{,r} + \eta_{,zz} \ , \tag{4.8}$$

or

$$\eta_t + u \cdot \nabla \eta = \Delta \eta + \frac{2}{r}\eta_{,r} \ . \tag{4.9}$$

**Remark 4.1.** *For a smooth vector field $u$, the apparent singularity of $\eta = \omega_\theta / r$ is only an artifact of the coordinate choice. The quantity $\eta$ is actually a smooth function, even across the $z$-axis, as long as $u$ is smooth, see [***LW09***].*

### 4.1.1 Regularized initial data

In terms of $\eta$, the initia data (4.7) reads:

$$\eta_0(r,z) := \eta(r,z,0) = \frac{\omega_\theta(r,z,0)}{r} = \frac{\kappa \delta_{r_0,z_0}}{r} = \frac{\kappa}{r_0}\delta_{r_0,z_0} \ . \tag{4.10}$$

The last equality of (4.10) holds in the sense of distribution. If we take an arbitrary test function $\psi = \psi(r,z)$, then

$$\left(\frac{\kappa\delta_{r_0,z_0}}{r}, \psi\right) = \left(\kappa\delta_{r_0,z_0}, \frac{\psi}{r}\right) = \kappa\frac{\psi(r_0,z_0)}{r_0} = \left(\frac{\kappa}{r_0}\delta_{r_0,z_0}, \psi\right) \ .$$

Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be the standard mollifier such that $\phi \in C_0^\infty(B_1(0))$, $\phi \geq 0$ and $\int_{\mathbb{R}^2} \phi(y)\, dy = 1$. And Let $\phi^{(\varepsilon)}(y_1, y_2) := \varepsilon^{-2}\phi(\frac{y_1}{\varepsilon}, \frac{y_2}{\varepsilon})$. Here and in the sequel, we assume $0 < \varepsilon < \frac{r_0}{2}$. We define $\eta_0^{(\varepsilon)}$ by

$$\eta_0^{(\varepsilon)}(r,z) := \big(\phi^{(\varepsilon)} * \eta_0\big)(r,z) = \frac{\kappa}{r_0}\varepsilon^{-2}\phi\Big(\frac{r-r_0}{\varepsilon}, \frac{z-z_0}{\varepsilon}\Big) \ . \tag{4.11}$$

Clearly, for every $0 < \varepsilon < \frac{r_0}{2}$, $\eta_0^{(\varepsilon)}$ has a compact support which stays away from the $z$-axis at least $\frac{r_0}{2}$. It is easy to check

$$\begin{aligned} &\pi\,|\kappa| \leq \left\|\eta_0^{(\varepsilon)}\right\|_{L^1_x} \leq 3\pi\,|\kappa| \ , \\ &\frac{\pi}{4}\,|\kappa|\, r_0^2 \leq \frac{2\pi\,|\kappa|}{r_0}(r_0-\varepsilon)^3 \leq \left\|r^2\eta_0^{(\varepsilon)}\right\|_{L^1_x} \leq \frac{2\pi\,|\kappa|}{r_0}(r_0+\varepsilon)^3 \leq \frac{27\pi}{4}\,|\kappa|\, r_0^2 \ . \end{aligned} \tag{4.12}$$

**Remark 4.2.** *Note that $\left\|\eta_0^{(\varepsilon)}\right\|_{L^1_x} \sim |\kappa|$ and $\left\|r^2\eta_0^{(\varepsilon)}\right\|_{L^1_x} \sim |\kappa|\, r_0^2$. The bounds for $\left\|\eta_0^{(\varepsilon)}\right\|_{L^1_x}$ depends only on the strength $|\kappa|$ of the ring $\kappa\delta_{r_0,z_0}e_\theta$ but the bounds for $\left\|r^2\eta_0^{(\varepsilon)}\right\|_{L^1_x}$ depends on both the strength and $r_0$. Nevertheless, they are both independent of $\varepsilon$ and will serve the a-priori bounds. The inequalities in (4.12) are dimensionally consistent.*

Corresponding to $\eta_0^{(\varepsilon)}$, the initial vorticity field $\omega_0^{(\varepsilon)}$ and velocity field $u_0^{(\varepsilon)}$ are

$$\omega_0^{(\varepsilon)} := r\,\eta_0^{(\varepsilon)}\,e_\theta \quad \text{and} \quad u_0^{(\varepsilon)}(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}\times\omega_0^{(\varepsilon)}(y)\,dy\ , \tag{4.13}$$

respectively and $\omega_0^{(\varepsilon)}$ has compact support.

### 4.1.2 Approximate solutions for regularized initial data

Obviously the velocity $u_0^{(\varepsilon)}$ in (4.13) is axi-symmetric and swirl-free. And for each $\varepsilon$, $u_0^{(\varepsilon)} \in H_x^k(\mathbb{R}^3)$ for any $k \geq 0$ and satisfies

$$\operatorname{div} u_0^{(\varepsilon)} = 0, \qquad \operatorname{curl} u_0^{(\varepsilon)} = \omega_0^{(\varepsilon)}\ . \tag{4.14}$$

**Remark 4.3.** *We don't have a uniform bound for $H_x^k(\mathbb{R}^3)$ norms of $u_0^{(\varepsilon)}$, not even for the $L_x^2(\mathbb{R}^3)$ norms of $u_0^{(\varepsilon)}$.*

We use the result of [**L68**, **UY68**, **LMNP99**] to get a global-in-time smooth solution for initial data $u_0^{(\varepsilon)}$. Their result is stated as follows, where we use the version of [**LMNP99**].

**Theorem 4.4.** *Let $T \in (0,\infty)$ be arbitrary, and let $u_0 \in H_x^2(\mathbb{R}^3)$, $\operatorname{div} u_0 = 0$ be axi-symmetric and swirl-free. Then there exists a unique axi-symmetric and swirl-free solution $u$ to the Cauchy problem (1.2), (1.3) satisfying*

$$u \in L_t^\infty H_x^2\left(\mathbb{R}^3\times(0,T)\right)\cap L_t^2 H_x^3\left(\mathbb{R}^3\times(0,T)\right), \quad u_t \in L_t^2 H_x^1\left(\mathbb{R}^3\times(0,T)\right) \tag{4.15}$$

*Moreover, this solution is smooth.*

Since the initial data $u_0^{(\varepsilon)}$ satisfies the assumption of Theorem 4.4, there exists a unique global-in-time smooth solution $u^{(\varepsilon)}$ for 3d Navier-Stokes equations satisfying the initial condition

$$u^{(\varepsilon)}(0) = u_0^{(\varepsilon)}\ . \tag{4.16}$$

And moreover $u^{(\varepsilon)}$ is axi-symmetric with no swirl, that is, in cylindrical coordinates,

$$u^{(\varepsilon)} = u_r^{(\varepsilon)}(r,z,t)e_r + u_z^{(\varepsilon)}(r,z,t)e_z \ .$$

We shall show that a subsequence of $\left\{u^{(\varepsilon)}\right\}_{0<\varepsilon<\frac{r_0}{2}}$ converges to a smooth solution with the ring $\kappa\delta_{r_0,z_0}e_\theta$ as initial vorticity. Corresponding to $u^{(\varepsilon)}$, the vorticity field $\omega^{(\varepsilon)}$ and the scalar quantity $\eta^{(\varepsilon)}$ are

$$\omega^{(\varepsilon)} = \text{curl } u^{(\varepsilon)} = \left(u_{r,z}^{(\varepsilon)} - u_{z,r}^{(\varepsilon)}\right)e_\theta \quad \text{and} \quad \eta^{(\varepsilon)} = \frac{u_{r,z}^{(\varepsilon)} - u_{z,r}^{(\varepsilon)}}{r} \ , \tag{4.17}$$

respectively. As a result of (4.13), (4.14), (4.16) and (4.17), $\omega^{(\varepsilon)}$ and $\eta^{(\varepsilon)}$ satisfy the initial data in (4.13)

$$\omega^{(\varepsilon)}(0) = \omega_0^{(\varepsilon)}, \ \eta^{(\varepsilon)}(0) = \eta_0^{(\varepsilon)} \ . \tag{4.18}$$

By (4.9) and Remark 4.1, $\eta^{(\varepsilon)}$ is a smooth solution of the following equation:

$$\eta_t^{(\varepsilon)} + u^{(\varepsilon)} \cdot \nabla\eta^{(\varepsilon)} = \Delta\eta^{(\varepsilon)} + \frac{2}{r}\eta_{,r}^{(\varepsilon)}, \quad \text{in } \ \mathbb{R}^3 \times (0,\infty) \ . \tag{4.19}$$

### 4.1.3 A-priori estimates for approximate solutions

The following lemma says that $\eta^{(\varepsilon)}$ enjoys the strong maximum priciple, which is crucial for our arguments of obtaining the a-priori estimates.

**Lemma 4.5.** *If $\kappa > 0$ (or, $< 0$), then $\eta^{(\varepsilon)}(r,z,t) > 0$ (or, $< 0$) for any $r \geq 0$, $z \in \mathbb{R}$ and $t > 0$.*

**Proof** . We just prove the case of $\kappa > 0$. The case of $\kappa < 0$ can be proved similarly. We can not apply the maximum principle directly to (4.19) since the coefficient of $\frac{2}{r}\eta_{,r}^{(\varepsilon)}$ is singular. Recalling that the Laplacian of a radially symmetric function $v(r)$ defined on $\mathbb{R}^n$ is $\Delta v = v''(r) + \frac{n-1}{r}v'(r)$, the right hand side of (4.19) can be appropriately interpreted as the Laplacian in $\mathbb{R}^5$ and we can recast (4.19) in $\mathbb{R}^5 \times (0,\infty)$. To this end, we introduce some notations.

Define

$$\hat{\eta}^{(\varepsilon)}(x_1,x_2,x_3,x_4,z,t){:=}\eta^{(\varepsilon)}\Big(\sqrt{x_1^2+x_2^2+x_3^2+x_4^2},z,t\Big)\ ,$$

$$\hat{u}^{(\varepsilon)}(x_1,x_2,x_3,x_4,z,t)$$

$${:=}u_r^{(\varepsilon)}\Big(\sqrt{x_1^2+x_2^2+x_3^2+x_4^2},z,t\Big)\hat{e}_r+u_z^{(\varepsilon)}\Big(\sqrt{x_1^2+x_2^2+x_3^2+x_4^2},z,t\Big)\hat{e}_z\ ,$$

where

$$r=\sqrt{x_1^2+x_2^2+x_3^2+x_4^2},\quad \hat{e}_r=\Big(\frac{x_1}{r},\frac{x_2}{r},\frac{x_3}{r},\frac{x_4}{r},0\Big),\quad \hat{e}_z=\Big(0,0,0,0,1\Big)\ .$$

Then by (4.11), (4.18) and (4.19), we have

$$\begin{cases} \hat{\eta}_t^{(\varepsilon)}+\hat{u}^{(\varepsilon)}\cdot\nabla_5\hat{\eta}^{(\varepsilon)}=\Delta_5\hat{\eta}^{(\varepsilon)}, & \text{in } \mathbb{R}^5\times(0,\infty)\ , \\ \hat{\eta}^{(\varepsilon)}(0)\geq 0,\ \text{and } \not\equiv 0 & \text{in } \mathbb{R}^5\ , \end{cases}$$

where,

$$\nabla_5=\Big(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4},\frac{\partial}{\partial z}\Big),\quad \Delta_5=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\partial^2}{\partial x_4^2}+\frac{\partial^2}{\partial z^2}\ .$$

By strong maximum principle, we get

$$\hat{\eta}^{(\varepsilon)}>0,\quad \text{in } \mathbb{R}^5\times(0,\infty)\ ,$$

which implies

$$\eta^{(\varepsilon)}>0\ .$$

Thus the lemma is proved. □

One of the important a-priori estimates is the conservation of momentum.

**Lemma 4.6.** (Conservation of momentum). *For all $t\geq 0$, we have*

$$\big\|r\omega^{(\varepsilon)}(t)\big\|_{L^1_x}=\big\|r\omega^{(\varepsilon)}(0)\big\|_{L^1_x}\leq\frac{27\pi}{4}\,|\kappa|\,r_0^2\ . \tag{4.20}$$

**Proof** . By $\omega^{(\varepsilon)} = r\eta^{(\varepsilon)} e_\theta$, (4.20) is identical to

$$\left\| r^2 \eta^{(\varepsilon)}(t) \right\|_{L^1_x} = \left\| r^2 \eta^{(\varepsilon)}(0) \right\|_{L^1_x} \leq \frac{27\pi}{4} |\kappa| \, r_0^2 \ . \tag{4.21}$$

The "inequality" part of (4.21) follows from (4.12) and (4.18). It remains to prove the "equality" part, which is actually the conservation of momentum.

Since the initial vorticity field $\omega_0^{(\varepsilon)}$ in (4.13) is smooth and compactly supported, the vorticity field $\omega^{(\varepsilon)}$ remains Schwartz (smooth and having fast decay in all spatial derivatives) for all the time. Therefore the momentum can be defined by using the vorticity as

$$\frac{1}{2} \int_{\mathbb{R}^3} \left( x \times \omega^{(\varepsilon)}(x,t) \right) \, dx \ \ ,$$

and moreover, the momentum conserved globally in time, that is

$$\frac{1}{2} \int_{\mathbb{R}^3} \left( x \times \omega^{(\varepsilon)}(x,t) \right) \, dx = \frac{1}{2} \int_{\mathbb{R}^3} \left( x \times \omega^{(\varepsilon)}(x,0) \right) \, dx \, , \qquad \text{for all t} > 0 \, , \tag{4.22}$$

which can be checked by the vorticity equations (1.18), integration by parts and Schwartz property of the vorticity field $\omega^{(\varepsilon)}$.

By $\omega^{(\varepsilon)} = r\eta^{(\varepsilon)} e_\theta$,

$$\begin{aligned} & x \times \omega^{(\varepsilon)} = x \times r\eta^{(\varepsilon)} e_\theta \\ =& (x_1, x_2, x_3) \times \left( -x_2 \eta^{(\varepsilon)}, x_1 \eta^{(\varepsilon)}, 0 \right) \\ =& \left( -x_1 x_3 \eta^{(\varepsilon)}, -x_2 x_3 \eta^{(\varepsilon)}, r^2 \eta^{(\varepsilon)} \right) \ . \end{aligned}$$

Noting that the first two components are odd in $x_1$ and $x_2$, respectively, we thus have

$$\int_{\mathbb{R}^3} \left( x \times \omega^{(\varepsilon)}(x,t) \right) dx = \left( 0 \, , 0 \, , \int_{\mathbb{R}^3} r^2 \eta^{(\varepsilon)}(x,t) \, dx \right) \ , \tag{4.23}$$

which, combining with (4.22), implies

$$\int_{\mathbb{R}^3} r^2 \eta^{(\varepsilon)}(x,t) \, dx = \int_{\mathbb{R}^3} r^2 \eta^{(\varepsilon)}(x,0) \, dx \, , \qquad \text{for all t} > 0 \ .$$

Finally by Lemma 4.5, $\eta^{(\varepsilon)}(x,t)$ is nonnegative if $\kappa > 0$ (or, nonpositive if $\kappa < 0$) for all points $(x,t) \in \mathbb{R}^3 \times [0,\infty)$ and therefore we can get

$$\int_{\mathbb{R}^3} \left| r^2 \eta^{(\varepsilon)}(x,t) \right| dx = \int_{\mathbb{R}^3} \left| r^2 \eta^{(\varepsilon)}(x,0) \right| dx \ .$$

We get (4.21) and the lemma is proved. □

**Remark 4.7.** *The lemma says* $\left\| r\omega^{(\varepsilon)}(t) \right\|_{L^1_x} \lesssim |\kappa| \, r_0^2$. *(4.23) implies the total momentum of the fluid flow is in the* $z$*-direction. This is due to the special structure of axi-symmetric velocities with no swirl.*

The following lemma claims that the $L^1_x$ norms of $\frac{\omega^{(\varepsilon)}}{r}$ are uniformly bounded from above, which thus gives us the second a-priori estimate.

**Lemma 4.8.** *For all* $t \geq 0$*, we have,*

$$\left\| \frac{\omega^{(\varepsilon)}(t)}{r} \right\|_{L^1_x} \leq 3\pi \, |\kappa| \ .$$

**Proof** . By $\omega^{(\varepsilon)} = r\eta^{(\varepsilon)} e_\theta$, it suffices to prove

$$\left\| \eta^{(\varepsilon)}(t) \right\|_{L^1_x} \leq 3\pi \, |\kappa| \, , \quad \text{for all } t \geq 0 \ .$$

We just prove the case of $\kappa > 0$. The case of $\kappa < 0$ can be proved similarly. By Lemma 4.5, $\eta^{(\varepsilon)} \geq 0$, Direct calculation shows that

$$\begin{aligned} &\frac{d}{dt} \left\| \eta^{(\varepsilon)}(t) \right\|_{L^1_x(\mathbb{R}^3)} = \frac{d}{dt} \int_{\mathbb{R}^3} \eta^{(\varepsilon)} \Big( x_1, x_2, z, t \Big) \, dx_1 \, dx_2 \, dz \\ =& \int_{\mathbb{R}^3} \left( \Delta \eta^{(\varepsilon)} - u^{(\varepsilon)} \cdot \nabla \eta^{(\varepsilon)} + \frac{2}{r} \eta^{(\varepsilon)}_{,r} \right) dx_1 \, dx_2 \, dz = \int_{\mathbb{R}^3} \frac{2}{r} \eta^{(\varepsilon)}_{,r} \, dx_1 \, dx_2 \, dz \\ =& 4\pi \int_{-\infty}^{\infty} \int_0^{\infty} \eta^{(\varepsilon)}_{,r}(r,z,t) \, dr \, dz = -4\pi \int_{-\infty}^{\infty} \eta^{(\varepsilon)}(0,z,t) \, dz \leq 0 \ . \end{aligned}$$

Thus $\left\| \eta^{(\varepsilon)}(t) \right\|_{L^1_x}$ is decreasing in time. Combining this with (4.12), we get

$$\left\| \eta^{(\varepsilon)}(t) \right\|_{L^1_x} \leq \left\| \eta^{(\varepsilon)}(0) \right\|_{L^1_x} = \left\| \eta_0^{(\varepsilon)} \right\|_{L^1_x} \leq 3\pi \, |\kappa| \ .$$

The lemma is proved. □

By Nash's method, we will now get uniform estimates of the $L^p_x$ norms of $\frac{\omega^{(\varepsilon)}}{r}$, for all $1 \le p \le \infty$, which also serve as our a-priori estimates. This generalization has been further generalized in [**FS86**]. The key point in the proof below is that the drift term $\frac{2}{r}\eta^{(\varepsilon)}_{,r}$ has a good sign.

**Lemma 4.9.** *For every $1 \le p \le \infty$, we have,*

$$\left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L^p_x} \le C_p t^{-\frac{3}{2}(1-\frac{1}{p})}, \quad t \in (0,\infty) \ , \tag{4.24}$$

*where the constants $C_p$ are independent of $\varepsilon$.*

**Proof** . Note that (4.24) is valid for $p = 1$ with $C_1 = 3\pi\,|\kappa|$ by Lemma 4.8. Again by $\omega^{(\varepsilon)} = r\eta^{(\varepsilon)}e_\theta$, it suffices to prove

$$\left\|\eta^{(\varepsilon)}(t)\right\|_{L^p_x} \le C_p t^{-\frac{3}{2}(1-\frac{1}{p})}, \quad t \in (0,\infty) \ . \tag{4.25}$$

Under the spirit of the energy method, for $p = 2^n$ with nonnegative integers $n$, we define

$$E^{(\varepsilon)}_p(t) := \left\|\eta^{(\varepsilon)}(t)\right\|^p_{L^p_x} = \int_{\mathbb{R}^3} \left|\eta^{(\varepsilon)}(x,t)\right|^p \, dx \ .$$

For $p=2^n$ with $n\geq 1$, direct calculation yields that

$$\begin{aligned}
-\frac{dE_p^{(\varepsilon)}}{dt} &= -\frac{d}{dt}\int_{\mathbb{R}^3}\left|\eta^{(\varepsilon)}\right|^p\,dx = -\frac{d}{dt}\int_{\mathbb{R}^3}\left(\eta^{(\varepsilon)}\right)^p dx = -\int_{\mathbb{R}^3} p\left(\eta^{(\varepsilon)}\right)^{p-1}\eta_t^{(\varepsilon)}\,dx\\
&= -\int_{\mathbb{R}^3} p\left(\eta^{(\varepsilon)}\right)^{p-1}\left(\Delta\eta^{(\varepsilon)}+\frac{2}{r}\eta_{,r}^{(\varepsilon)}-u^{(\varepsilon)}\nabla\eta^{(\varepsilon)}\right)dx\\
&= -\int_{\mathbb{R}^3}\left\{p\left[\eta^{(\varepsilon)}\right]^{p-1}\Delta\eta^{(\varepsilon)}+\frac{2}{r}\left[(\eta^{(\varepsilon)})^p\right]_{,r}-u^{(\varepsilon)}\nabla\left[(\eta^{(\varepsilon)})^p\right]\right\}dx\\
&= \int_{\mathbb{R}^3} p(p-1)\left[\eta^{(\varepsilon)}\right]^{p-2}\left|\nabla\eta^{(\varepsilon)}\right|^2\,dx - 4\pi\int_{-\infty}^{\infty}\int_0^{\infty}\left[(\eta^{(\varepsilon)})^p\right]_{,r}dr\,dz\\
&= \int_{\mathbb{R}^3} p(p-1)\left|\left[\eta^{(\varepsilon)}\right]^{\frac{p-2}{2}}\nabla\eta^{(\varepsilon)}\right|^2\,dx - 4\pi\int_{-\infty}^{\infty}\left[(\eta^{(\varepsilon)})^p\right]_{r=0}^{r=\infty}dz\\
&= \int_{\mathbb{R}^3} p(p-1)\left|\frac{2}{p}\nabla\left[(\eta^{(\varepsilon)})^{\frac{p}{2}}\right]\right|^2 dx + 4\pi\int_{-\infty}^{\infty}\left[(\eta^{(\varepsilon)})^p\right]_{r=0}dz\\
&\geq \frac{4(p-1)}{p}\int_{\mathbb{R}^3}\left|\nabla\left[(\eta^{(\varepsilon)})^{\frac{p}{2}}\right]\right|^2 dx\ .
\end{aligned} \tag{4.26}$$

Recall the Nash's inequality [**N58**, P936]

$$\int_{\mathbb{R}^3}|\nabla u|^2\,dx \geq M\left(\int_{\mathbb{R}^3}|u|\right)^{-\frac{4}{3}}\left(\int_{\mathbb{R}^3}|u|^2\right)^{\frac{5}{3}}\ . \tag{4.27}$$

For $p=2^n$ with $n\geq 1$, by Nash's inequality, we get the following iteration scheme from (4.26),

$$\begin{aligned}
-\frac{dE_p^{(\varepsilon)}}{dt} &\geq \frac{4(p-1)}{p}\int_{\mathbb{R}^3}\left|\nabla\left[(\eta^{(\varepsilon)})^{\frac{p}{2}}\right]\right|^2 dx\\
&\geq \frac{4(p-1)}{p}M\left(\int_{\mathbb{R}^3}\left|(\eta^{(\varepsilon)})^{\frac{p}{2}}\right|\right)^{-\frac{4}{3}}\left(\int_{\mathbb{R}^3}\left|(\eta^{(\varepsilon)})^{\frac{p}{2}}\right|^2\right)^{\frac{5}{3}}\\
&= \frac{4(p-1)}{p}M\left(E_{p/2}^{(\varepsilon)}\right)^{-\frac{4}{3}}\left(E_p^{(\varepsilon)}\right)^{\frac{5}{3}}\ .
\end{aligned} \tag{4.28}$$

We first prove (4.25) for $p=2^n$ with nonnegative integers $n$ by induction. Assume (4.25) is valid for $q=2^k$ with $k\geq 0$. Let $p=2^{k+1}$. By (4.28), we have,

$$-\frac{dE_p^{(\varepsilon)}}{dt} \geq \frac{4(p-1)}{p}M\left(E_q^{(\varepsilon)}\right)^{-\frac{4}{3}}\left(E_p^{(\varepsilon)}\right)^{\frac{5}{3}} \geq \frac{4(p-1)}{p}M\left(C_q^q t^{-\frac{3}{2}(q-1)}\right)^{-\frac{4}{3}}\left(E_p^{(\varepsilon)}\right)^{\frac{5}{3}}\ ,$$

$\Longrightarrow$

$$\frac{3}{2}\Big[(E_p^{(\varepsilon)})^{-\frac{2}{3}}\Big]_t = -\frac{\frac{dE_p^{(\varepsilon)}}{dt}}{\left(E_p^{(\varepsilon)}\right)^{\frac{5}{3}}} \geq \frac{4(p-1)}{p} M C_q^{-\frac{4q}{3}} t^{2(q-1)} = \frac{4(p-1)}{p} M C_q^{-\frac{2p}{3}} t^{p-2} \ ,$$

$\Longrightarrow$

$$(E_p^{(\varepsilon)})^{-\frac{2}{3}}(t) \geq (E_p^{(\varepsilon)})^{-\frac{2}{3}}(t) - (E_p^{(\varepsilon)})^{-\frac{2}{3}}(0) \geq \frac{8(p-1)}{3p} M C_q^{-\frac{2p}{3}} \int_0^t s^{p-2} ds = \frac{8M}{3p} C_q^{-\frac{2p}{3}} t^{p-1} \ ,$$

$\Longrightarrow$

$$\left\|\eta^{(\varepsilon)}(t)\right\|_{L_x^p} = E_p^{(\varepsilon)}(t)^{\frac{1}{p}} \leq \left(\frac{3p}{8M}\right)^{\frac{3}{2p}} C_q t^{-\frac{3}{2}(1-\frac{1}{p})} \ .$$

Hence (4.25) is valid for $p = 2^{k+1}$ with $C_p = \left(\frac{3p}{8M}\right)^{\frac{3}{2p}} C_q$. In fact, $C_p$ is uniformly bounded from above.

$$C_p = \left(\frac{3}{8M}\right)^{\frac{3}{2^{k+2}}} 2^{\frac{3(k+1)}{2^{k+2}}} C_{2^k} \leq \left(\frac{3}{8M}\right)^{\sum \frac{3}{2^{i+2}}} 2^{\sum \frac{3(i+1)}{2^{i+2}}} C_1 =: C_\infty \ .$$

$\Longrightarrow$

$$\left\|\eta^{(\varepsilon)}(t)\right\|_{L_x^\infty} \leq C_\infty t^{-\frac{3}{2}} \ .$$

For other $p$, we can prove (4.25) by interpolation. Therefore the lemma is proved. $\square$

**Remark 4.10.** *From the proof of Lemma 4.9, we see the constants $C_p$ in (4.24) linearly depends on $C_1 = 3\pi\,|\kappa|$. In particular,*

$$\begin{aligned} C_\infty &= \left(\frac{3}{8M}\right)^{\sum \frac{3}{2^{i+2}}} 2^{\sum \frac{3(i+1)}{2^{i+2}}} C_1 \lesssim |\kappa| \ , \\ \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L_x^\infty} &\leq C_\infty t^{-\frac{3}{2}} \lesssim |\kappa|\, t^{-\frac{3}{2}} \ , \end{aligned} \tag{4.29}$$

*which gives us the third a-priori estimate, where $M$ is the absolute constant in Nash's inequality (4.27).*

**Remark 4.11.** *If the fluid is inviscid, then $\eta^{(\varepsilon)}$ satisfies*

$$\eta_t + u^{(\varepsilon)} \cdot \nabla \eta = 0, \quad \text{in} \ \ \mathbb{R}^3 \times (0, \infty) \ . \tag{4.30}$$

*Since $\eta^{(\varepsilon)}$ is conserved along particle trajectories, $\eta^{(\varepsilon)}$ keeps its sign in later time. We still have the uniform estimates of the $L^1_x$ norms:*

$$\left\|\eta^{(\varepsilon)}(t)\right\|_{L^1_x} = \left\|\eta^{(\varepsilon)}(0)\right\|_{L^1_x} = \left\|\eta_0^{(\varepsilon)}\right\|_{L^1_x} \le 3\pi\,|\kappa| \ .$$

*However, the argument in Lemma 4.9 yields: for any $1 < p \le \infty$,*

$$\left\|\eta^{(\varepsilon)}(t)\right\|_{L^p_x} = \left\|\eta^{(\varepsilon)}(0)\right\|_{L^p_x} = \left\|\eta_0^{(\varepsilon)}\right\|_{L^p_x} \ ,$$

*which will blow up as $\varepsilon$ goes to 0. Therefore we lose uniform controls of the $L^p_x$ norms in the inviscid case.*

We now use the weighted inequalities of the previous section and the three a-priori estimates from Lemma 4.6, Lemma 4.8 and Remark 4.10 to get further estimates on vorticity, the gradient of velocity, velocity and pressure.

**Lemma 4.12.** *For $0 < t < \infty$, we have the following estimates:*
i) *for any $1 \le p \le 2$*

$$\left\|\omega^{(\varepsilon)}(t)\right\|_{L^p_x} \lesssim |\kappa|\, r_0 t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}, \tag{4.31}$$

ii) *for any $1 < p \le 2$*

$$\left\|\nabla u^{(\varepsilon)}(t)\right\|_{L^p_x} \lesssim |\kappa|\, r_0 t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}, \tag{4.32}$$

iii) *for any $\frac{3}{2} < q \le 6$*

$$\left\|u^{(\varepsilon)}(t)\right\|_{L^q_x} \lesssim |\kappa|\, r_0 t^{-\left(1-\frac{3}{2q}\right)}, \tag{4.33}$$

iv) *for any $1 < q \le 3$*

$$\left\|p^{(\varepsilon)}(t)\right\|_{L^q_x} \lesssim |\kappa|^2\, r_0^2 t^{-\left(2-\frac{3}{2q}\right)}, \tag{4.34}$$

v)

$$\left\|u^{(\varepsilon)}(t)\right\|_{L^\infty_x} \lesssim |\kappa|\, r_0^{\frac{1}{2}} t^{-\frac{3}{4}} \ . \tag{4.35}$$

**Proof** .

i). By Proposition 3.1, for any $1 \le p \le 2$, we have

$$\left\|\omega^{(\varepsilon)}(t)\right\|_{L^p_x} \le \left\|r\omega^{(\varepsilon)}(t)\right\|^{\frac{1}{2}}_{L^1_x} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|^{\frac{1}{p}-\frac{1}{2}}_{L^1_x} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|^{1-\frac{1}{p}}_{L^\infty_x} . \tag{4.36}$$

Then (4.31) is an easy consequence of (4.36), Lemma 4.6, Lemma 4.8 and (4.29) in Remark 4.10.

ii) By $\operatorname{div} u^{(\varepsilon)}{=}0$, $\operatorname{curl} u^{(\varepsilon)}{=}\omega^{(\varepsilon)} = (\omega_1^{(\varepsilon)}, \omega_2^{(\varepsilon)}, \omega_3^{(\varepsilon)})$ and Fourier transform, one can get

$$\nabla u^{(\varepsilon)} = \begin{bmatrix} R_1R_2\omega_3^{(\varepsilon)} - R_1R_3\omega_2^{(\varepsilon)} & R_2R_2\omega_3^{(\varepsilon)} - R_2R_3\omega_2^{(\varepsilon)} & R_2R_3\omega_3^{(\varepsilon)} - R_3R_3\omega_2^{(\varepsilon)} \\ R_1R_3\omega_1^{(\varepsilon)} - R_1R_1\omega_3^{(\varepsilon)} & R_2R_3\omega_1^{(\varepsilon)} - R_1R_2\omega_3^{(\varepsilon)} & R_3R_3\omega_1^{(\varepsilon)} - R_1R_3\omega_3^{(\varepsilon)} \\ R_1R_1\omega_2^{(\varepsilon)} - R_1R_2\omega_1^{(\varepsilon)} & R_1R_2\omega_2^{(\varepsilon)} - R_2R_2\omega_1^{(\varepsilon)} & R_1R_3\omega_2^{(\varepsilon)} - R_2R_3\omega_1^{(\varepsilon)} \end{bmatrix} ,$$

where $R_j$, $j = 1, 2, 3$ are the classical Riesz transformations, which are well-defined and continuous on $L^p_x(\mathbb{R}^3)$ for all $1 < p < \infty$, see for instance [**S93**, **T86**]. Therefore

$$\left\|\nabla u^{(\varepsilon)}(t)\right\|_{L^p_x} \lesssim \left\|\omega^{(\varepsilon)}(t)\right\|_{L^p_x} ,$$

which, combining with (4.31), implies (4.32).

iii) By Corollary 3.3, for any $\frac{3}{2} < q \le 6$,

$$\left\|u^{(\varepsilon)}(t)\right\|_{L^q_x} \lesssim \left\|r\omega^{(\varepsilon)}(t)\right\|^{\frac{1}{2}}_{L^1_x} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|^{\frac{1}{q}-\frac{1}{6}}_{L^1_x} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|^{\frac{2}{3}-\frac{1}{q}}_{L^\infty_x} . \tag{4.37}$$

Then (4.33) is an easy consequence of (4.37), Lemma 4.6, Lemma 4.8 and (4.29).

iv) Recall the pressure $p^{(\varepsilon)}$ and the velocity $u^{(\varepsilon)} = \left(u_1^{(\varepsilon)}, u_2^{(\varepsilon)}, u_3^{(\varepsilon)}\right)$ satisfy the following equation (which can be easily obtained from Navier-Stokes equations and divergence-free condition $\operatorname{div} u^{(\varepsilon)}{=}0$):

$$\Delta p^{(\varepsilon)} = -\partial_j\partial_k(u_j^{(\varepsilon)}u_k^{(\varepsilon)}) . \tag{4.38}$$

Then by (4.33), we can use the Riesz transformation $R_j$ to solve (4.38) to get

$$p^{(\varepsilon)} = R_jR_k(u_j^{(\varepsilon)}u_k^{(\varepsilon)}) .$$

Hence

$$\left\|p^{(\varepsilon)}(t)\right\|_{L_x^q} \lesssim \left\|u^{(\varepsilon)}(t)\right\|_{L_x^{2q}}^2 ,$$

which, combining with (4.33), implies (4.34).

v) By Proposition 3.11 and 3.13,

$$\left\|u^{(\varepsilon)}(t)\right\|_{L_x^\infty} \lesssim \left\|r\omega^{(\varepsilon)}(t)\right\|_{L_x^1}^{\frac{1}{4}} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L_x^1}^{\frac{1}{4}} \left\|\frac{\omega^{(\varepsilon)}(t)}{r}\right\|_{L_x^\infty}^{\frac{1}{2}} . \tag{4.39}$$

Then (4.35) is an easy consequence of (4.39), Lemma 4.6, Lemma 4.8 and (4.29). □

By Lemma 4.12 and the subcritical theory of Navier-Stokes equations, we can control the spatial and time derivatives of the velocity and pressure of any order pointwise.

**Lemma 4.13.** *For any $k, h \geq 0$ and for any $0 < s < T$, we have the following pointwise estimate*

$$\left\|\nabla_x^k \nabla_t^h u^{(\varepsilon)}\right\|_{C^0_{x,t}(\mathbb{R}^3 \times [s,T])} \leq C, \qquad \left\|\nabla_x^k \nabla_t^h p^{(\varepsilon)}\right\|_{C^0_{x,t}(\mathbb{R}^3 \times [s,T])} \leq C ,$$

*where $C$ is independent of $\varepsilon$ and depends only on $k, h, s, T, |\kappa|, r_0$.*

**Proof** . This lemma is a consequence of the subcritical well-posedness theory of Navier-Stokes equations. Fix $0 < s < T$. By (4.33), we have the following subcritical estimate

$$\left\|u^{(\varepsilon)}(t)\right\|_{L_x^6} \lesssim |\kappa|\, r_0 t^{-\frac{3}{4}} , \tag{4.40}$$

since $L_x^6(\mathbb{R}^3)$ is a subcritical space for Navier-Stokes equations with respect to the scaling

$$u(x,t) \longmapsto \lambda u(\lambda x, \lambda^2 t),\ p(x,t) \longmapsto \lambda^2 p(\lambda x, \lambda^2 t) .$$

By Theorem 2.1, there exists a local-in-time unique solution $v^{(\varepsilon)}$ for Navier-Stokes equations with $u^{(\varepsilon)}\left(\frac{s}{2}\right)$ as initial velocity in the space $C\left(\left[\frac{s}{2}, T^*\right), L_x^6(\mathbb{R}^3)\right)$ for some $\frac{s}{2} < T^* \leq \infty$. $v^{(\varepsilon)}$ coincides with $u^{(\varepsilon)}$ on the time interval $[\frac{s}{2}, T^*)$ by weak-strong uniqueness. The decay property (4.40) implies $T^* = \infty$. Hence $u^{(\varepsilon)} = v^{(\varepsilon)}$ for all $t \in [\frac{s}{2}, \infty)$. Again by Theorem 2.1, $u^{(\varepsilon)}$ satisfies

$$\left\| \nabla_x^k \nabla_t^h u^{(\varepsilon)} \right\|_{L_t^\infty L_x^6(\mathbb{R}^3 \times [s,T])} \leq C \ , \tag{4.41}$$

where $C$ depends only $k, h, s, T, \left\| u^{(\varepsilon)} \left( \frac{s}{2} \right) \right\|_{L_x^6}$. Then by Sobolev embedding, we prove the first estimate. The second estimate is a consequence of (4.41) and (4.38). □

The estimate (4.33) in Lemma 4.12 imply the set $\left\{ u^{(\varepsilon)} \right\}_{0<\varepsilon<\frac{r_0}{2}}$ has weak compactness in Lebesgue spaces. To show the strong convergence of $\left\{ u^{(\varepsilon)} \right\}_{0<\varepsilon<\frac{r_0}{2}}$, we need to establish certain uniform weak continuity of $u^{(\varepsilon)}$ as functions of time $t$. To this end, we use the standard method, the Aubin-Lions Theorem, see [**C86**, **L98**, **T77**]. We state the theorem in a general setting, see Chapter III of [**T77**].

Let $X_0, X, X_1$ be three Banach spaces such that

$$X_0 \subset X \subset X_1 \, , \tag{4.42}$$

where the injections are continuous, $X_i$ is reflexive, $i = 0, 1$, and the injection $X_0 \to X$ is compact. Let $T > 0$ be a fixed finite number, and let $\alpha_0$, $\alpha_1$ be two finite numbers such that $\alpha_i > 1$, $i = 0, 1$. We consider the space

$$\mathscr{Y} = \left\{ v : v \in L^{\alpha_0} \left( 0, T; X_0 \right), \frac{dv}{dt} \in L^{\alpha_1} \left( 0, T; X_1 \right) \right\} \ . \tag{4.43}$$

The space $\mathscr{Y}$ is provided with the norm

$$\| v \|_{\mathscr{Y}} = \| v \|_{L^{\alpha_0}(0,T;X_0)} + \left\| \frac{dv}{dt} \right\|_{L^{\alpha_1}(0,T;X_1)} \, , \tag{4.44}$$

which makes it a Banach space. It is evident that

$$\mathscr{Y} \subset L^{\alpha_0} \left( 0, T; X \right) \ , \tag{4.45}$$

with a continuous injection. Actually this injection is compact.

**Theorem 4.14.** (Aubin-Lions) *Under the assumptions (4.42) to (4.44) the injection of* $\mathscr{Y}$ *into* $L^{\alpha_0}(0,T;X)$ *is compact.*

**Proof** . See [**T77**]. $\square$

Let $H_x^{-2}(\mathbb{R}^3)$ be the dual space of $H_x^2(\mathbb{R}^3)$.

**Lemma 4.15.** *Let* $0<T<\infty$. *Then we have*

$$\left\|\frac{\partial u^{(\varepsilon)}}{\partial t}\right\|_{L_t^{\frac{5}{4}}\left(0,T;H_x^{-2}(\mathbb{R}^3)\right)} \le C\ , \tag{4.46}$$

*where the constant* $C$ *is independent of* $\varepsilon$ *and depends on* $T$.

**Proof** . Let $\phi \in H_x^2(\mathbb{R}^3)$. By Navier-Stokes equations and Lemma 4.12, we have

$$\begin{aligned}
&\left|\left(\frac{\partial u^{(\varepsilon)}}{\partial t},\phi\right)\right| \\
=&\left|\left(-\mathrm{div}(u^{(\varepsilon)}\otimes u^{(\varepsilon)})-\nabla p^{(\varepsilon)}+\Delta u^{(\varepsilon)},\phi\right)\right| \\
\le&\left|(u^{(\varepsilon)}\otimes u^{(\varepsilon)},\nabla\phi)\right|+\left|(p^{(\varepsilon)},\mathrm{div}\phi)\right|+\left|(u^{(\varepsilon)},\Delta\phi)\right| \\
\lesssim&\left\|u^{(\varepsilon)}(t)\right\|_{L_x^{p_1}}\left\|u^{(\varepsilon)}(t)\right\|_{L_x^{p_2}}\|\nabla\phi\|_{L_x^{p_3}}+\left\|p^{(\varepsilon)}(t)\right\|_{L_x^{q_1}}\|\nabla\phi\|_{L_x^{q_2}}+\left\|u^{(\varepsilon)}(t)\right\|_{L_x^2}\|\Delta\phi\|_{L_x^2} \\
\lesssim&|\kappa|^2 r_0^2 t^{-2+\frac{3}{2p_1}+\frac{3}{2p_2}}\|\nabla\phi\|_{L_x^{p_3}}+|\kappa|^2 r_0^2 t^{-2+\frac{3}{2q_1}}\|\nabla\phi\|_{L_x^{q_2}}+|\kappa| r_0 t^{-\frac{1}{4}}\|\phi\|_{H_x^2}\ ,
\end{aligned}$$

where

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1,\ \frac{3}{2}<p_1,p_2\le 6,\ \frac{1}{q_1}+\frac{1}{q_2}=1,\ 1<q_1\le 3\ .$$

One can take, for example,

$$p_1=p_2=\frac{12}{5},\ p_3=6,\ q_1=\frac{6}{5},\ q_2=6\ .$$

Then by Sobolev embedding, we have for $0<t\le T$,

$$\begin{aligned}
&\left|\left(\frac{\partial u^{(\varepsilon)}}{\partial t},\phi\right)\right| \\
\lesssim&|\kappa|^2 r_0^2 t^{-\frac{3}{4}}\|\nabla\phi\|_{L_x^6}+|\kappa|^2 r_0^2 t^{-\frac{3}{4}}\|\nabla\phi\|_{L_x^6}+|\kappa| r_0 t^{-\frac{1}{4}}\|\phi\|_{H_x^2} \\
\lesssim&\left(|\kappa|^2 r_0^2 t^{-\frac{3}{4}}+|\kappa| r_0 t^{-\frac{1}{4}}\right)\|\phi\|_{H_x^2}\ .
\end{aligned}$$

Hence

$$\left\|\frac{\partial u^{(\varepsilon)}}{\partial t}(t)\right\|_{H_x^{-2}} \lesssim |\kappa|^2 r_0^2 t^{-\frac{3}{4}} + |\kappa| r_0 t^{-\frac{1}{4}} \ .$$

Finally integrating with respect to time from $(0,T)$ yields the desired result. $\square$

**Lemma 4.16.** *For any* $0 < T < \infty$, $\left\{u^{(\varepsilon)}\right\}_{0<\varepsilon<\frac{r_0}{2}}$ *is precompact in* $L_t^{\frac{8}{5}}\left(0,T;L^2_{x,\mathrm{loc}}(\mathbb{R}^3)\right)$.

**Proof** . By Lemma 4.12, one has

$$\left\|u^{(\varepsilon)}(t)\right\|_{L_x^{\frac{8}{5}}} \lesssim |\kappa| r_0 t^{-\frac{1}{16}}, \qquad \left\|\nabla u^{(\varepsilon)}(t)\right\|_{L_x^{\frac{8}{5}}} \lesssim |\kappa| r_0 t^{-\frac{9}{16}} \ ,$$

which implies

$$u^{(\varepsilon)} \in L_t^{\frac{8}{5}}\left(0,T;W_x^{1,\frac{8}{5}}(\mathbb{R}^3)\right) \ . \tag{4.47}$$

Then (4.47), Lemma 4.15 and Theorem 4.14 imply the desired result, where we use $X_0 = W_x^{1,\frac{8}{5}}(\mathbb{R}^3)$, $X = L^2_{x,\mathrm{loc}}(\mathbb{R}^3)$, $X_1 = H_x^{-2}(\mathbb{R}^3)$ and $\alpha_0 = \frac{8}{5}$, $\alpha_1 = \frac{5}{4}$. $\square$

## 4.2 Proof of Theorem 1.1

(4.33) implies

$$\left\|u^{(\varepsilon)}(t)\right\|_{L_x^2} \lesssim |\kappa| r_0 t^{-\frac{1}{4}} \ , \tag{4.48}$$

which in turn implies

$$\left\{u^{(\varepsilon)}\right\} \text{ is a bounded set in } L_t^{\frac{8}{3}}L_x^2\big(\mathbb{R}^3\times(0,T)\big), \qquad \text{for any } 0<T<\infty \ . \tag{4.49}$$

Arzela-Ascoli's Theorem, Lemma 4.13, Lemma 4.16 and (4.49) allow us to extract a subsequece of $\left\{u^{(\varepsilon)},p^{(\varepsilon)}\right\}$, still denoted as $\left\{u^{(\varepsilon)},p^{(\varepsilon)}\right\}$ such that for a smooth vector field $u$ and a smooth scalar function $p$, for any nonnegative integers $k,h$ and for any $0<T<\infty$, we have

$$\begin{aligned} u^{(\varepsilon)} &\to u \quad \text{in } L_t^{\frac{8}{5}}\left(0,T;L^2_{x,\mathrm{loc}}(\mathbb{R}^3)\right) \ , \\ u^{(\varepsilon)} &\rightharpoonup u \quad \text{in } L_t^{\frac{8}{3}}L_x^2\big(\mathbb{R}^3\times(0,T)\big) \ , \end{aligned} \tag{4.50}$$

and

$$\begin{aligned} \nabla_x^k \nabla_t^h u^{(\varepsilon)} &\rightrightarrows \nabla_x^k \nabla_t^h u \quad \text{locally in } \mathbb{R}^3 \times (0,\infty) \ , \\ \nabla_x^k \nabla_t^h p^{(\varepsilon)} &\rightrightarrows \nabla_x^k \nabla_t^h p \quad \text{locally in } \mathbb{R}^3 \times (0,\infty) \ , \end{aligned} \tag{4.51}$$

which imply the limit $\big(u,p\big)$ is a global-in-time smooth solution of Navier-Stokes equations in $\mathbb{R}^3 \times (0,\infty)$ and $u$ is axi-symmetric with no swirl.

We prove the initial condition (1.15). Take a $\varphi \in C_0^\infty\big(\mathbb{R}^3;\mathbb{R}^3\big)$ with $B_R(0)$ as its support. By Navier-Stokes equations, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \Big\{ \big(u^{(\varepsilon)} \otimes u^{(\varepsilon)}\big) \cdot \nabla \mathrm{curl} \varphi + u^{(\varepsilon)} \cdot \Delta \mathrm{curl} \varphi \Big\} \, dx \, dt \\ &= \int_{\mathbb{R}^3} \omega^{(\varepsilon)}(x,T) \cdot \varphi(x) \, dx - \int_{\mathbb{R}^3} \omega_0^{(\varepsilon)}(x) \cdot \varphi(x) \, dx \ . \end{aligned} \tag{4.52}$$

We claim that we are able to pass to the limit in (4.52) to get

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \Big\{ \big(u \otimes u\big) \cdot \nabla \mathrm{curl} \varphi + u \cdot \Delta \mathrm{curl} \varphi \Big\} \, dx \, dt \\ &= \int_{\mathbb{R}^3} \omega(x,T) \cdot \varphi(x) \ dx - \int_{\mathbb{R}^3} \kappa \delta_{r_0,z_0} e_\theta \cdot \varphi \ dx \ . \end{aligned} \tag{4.53}$$

To this end, it suffices to check the nonlinear term in (4.52) and (4.53). By (4.49) and (4.50), we have

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{R}^3} \big(u^{(\varepsilon)} \otimes u^{(\varepsilon)}\big) \cdot \nabla \mathrm{curl} \varphi \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} \big(u \otimes u\big) \cdot \nabla \mathrm{curl} \varphi \, dx \, dt \right| \\ &\lesssim \big\| u^{(\varepsilon)} - u \big\|_{L_t^{\frac{8}{5}} L_x^2 \left(B_R(0) \times (0,T)\right)} \left( \big\| u^{(\varepsilon)} \big\|_{L_t^{\frac{8}{3}} L_x^2 \left(\mathbb{R}^3 \times (0,T)\right)} + \| u \|_{L_t^{\frac{8}{3}} L_x^2 \left(\mathbb{R}^3 \times (0,T)\right)} \right) \| \nabla \mathrm{curl} \varphi \|_{L_x^\infty} \ , \end{aligned}$$

which goes to 0 as $\varepsilon \to 0$. Thus (4.53) is obtained.

Fatou's lemma and (4.48) imply

$$\| u(t) \|_{L_x^2} \lesssim |\kappa| \, r_0 t^{-\frac{1}{4}} \ . \tag{4.54}$$

Hence in view of (4.53)

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \omega(x,T)\cdot\varphi(x)\,dx - \int_{\mathbb{R}^3} \kappa\delta_{r_0,z_0} e_\theta \cdot \varphi\,dx\right| \\
=&\left|\int_0^T \int_{\mathbb{R}^3} \Big\{ \big(u\otimes u\big)\cdot \nabla \mathrm{curl}\varphi + u\cdot \Delta \mathrm{curl}\varphi \Big\}\,dx\,dt\right| \\
\lesssim&\int_0^T |\kappa|^2\, r_0^2 t^{-\frac{1}{2}}\,dt + \int_0^T |\kappa|\, r_0 t^{-\frac{1}{4}}\,dt \lesssim |\kappa|^2\, r_0^2 T^{\frac{1}{2}} + |\kappa|\, r_0 T^{\frac{3}{4}},
\end{aligned} \tag{4.55}$$

which implies (1.15). Theorem 1.1 is thus proved.

**Remark 4.17.** *Theorem 1.1 is also true if we replace the initial condition by finite many vortex rings*

$$\omega(\cdot,0) = \sum_{i=1}^{n} \kappa_i \delta_{r_i,z_i} e_\theta\ , \tag{4.56}$$

*where all* $\kappa_i > 0$ *(or, all* $\kappa_i < 0$*), or more generally, by*

$$\omega(\cdot,0) = \mu e_\theta\ , \tag{4.57}$$

*where* $\mu$ *is a positive or negative finite measure with a compact support in the* $rz$*-plane. Without any modification, the preceding proof for single vortex ring also works for the cases of (4.56) and (4.57).*

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