Homoclinic and Heteroclinic Orbits in Lagrangian Dynamical Systems

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Dedication

To my parents Zifan Yu and Xiaodan Hong.

Abstract

In this thesis, variational methods are used to study the existence of homoclinic and heteroclinic orbits in various contexts of Lagrangian dynamical systems:

1. Monotone twist maps, which can be presented as time-one maps of certain positive definite time-periodic Lagrangian systems whose configuration spaces are 1-dim tori.

2. Time-periodic Tonelli Lagrangian systems whose configuration spaces are finite dimensional closed (i.e., compact, boundaryless) and connected smooth Riemannian manifolds.

3. Time-independent Tonelli Lagrangian systems whose configuration spaces are a 2-dim tori.

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Chapter 1

Introduction

The study of area preserving maps is a classic topic in dynamical system. It goes back to Poincaré's work of celestial mechanics [1], in which he reduced the study of the restricted three body problem to the study of an area preserving map.

After Poincaré, it is Birkhoff who made essential progress in this topic. In [2], he proved that for a monotone twist map (see Definition 2.1.1), there exits a periodic orbit with rational rotation number for any rational number.

About 60 years later, such a result was extended to irrational rotation numbers by Mather [3]. Of course for irrational rotation numbers quasi-periodic orbits rather than periodic orbits were shown to exist.

Besides [3], monotone twist maps and related topics were studied by Mather in a series of papers [4], [5], [6], [7], [8], [9], [10].

At the same, Aubry and his collaborators applied ideas similar to Mather's to the study of the discrete Frenkel-Kontorova model in solid state physics.

In an excellent paper by Bangert [11], the author reduced the proofs of many results in the above two different fields to one common root by introducing a variational problem. Now people usually refer to this as **Aubry-Mather theory**.

Furthermore Bangert explained how the results of Morse [12] and Hedlund [13] of minimal geodesics on 2-dim torus can be obtained by Aubry-Mather theory.

As powerful as Aubry-Mather theory is, it can only be applied to Hamiltonian systems with two degrees or one and half degrees of freedom. Inspired by a result of Moser [14] that every monotone twist map can be viewed as the time-one map of a positive definite Lagrangian system, Mather started a program to generalize Aubry-Mather theory to Lagrangian systems with arbitrary degrees of freedom in a series of papers [15], [16], [17]. In these papers, the dynamics of the time periodic, positive definite and super-linear growth Lagrangian systems with the configuration spaces being closed (i.e., compact, boundaryless) and connected smooth Riemannian manifolds were studied.

The main feature of such systems is the existence of a series of families of invariant sets in the phase space satisfying different minimizing properties. Those invariant sets are called **Mather sets**, **Aubry sets** and **Mañé sets**, whose precise definitions will be given in Chapter 2.

Besides Mather, Mañé and his school have done a lot of important works in this area, see [18], [19], [20], [21], [22], [23] and [24]. Therefore in this paper we will refer to this generalization of Aubry-Mather theory as **Mather-Mañé theory**.

Another important result in this area is Fathi's work on weak KAM solutions of Hamilton-Jacobi equation [25].

The Mather-Mañé theory has been a very active area in the past two decades, because it is a powerful tool in the study of **"Arnold Diffusion"**, for example see works of Xia [26], Mather [27], Cheng and Yan [28], [29], Bernard [30], Bernard, Kaloshin and Zhang [31] and Kaloshin and Zhang [32].

This thesis is organized as follows:

1. In Chapter 2, we will give a brief introduction to Aubry-Mather theory and Mather-Mañé theory.

2. In Chapter 3, for a monotone twist map, under mild non-degeneracy conditions, we will prove the existence of infinitely many multi-bump homoclinic and heteroclinic orbits between two neighboring minimal periodic orbits with the same rational rotation number.

3. In Chapter 4, existence of heteroclinic orbits between different static classes of the Aubry sets of a time periodic, positive definite and super-linear growth Lagrangian system will be established under certain non-degeneracy conditions.

4. In Chapter 5, we will apply the basic technics from Mather-Mañé theory to the

study of a class of autonomous Lagrangian systems with the 2-dim torus as the configuration space.

Chapter 2

Aubry-Mather Theory and Mather-Mañé Theory

In this chapter we will give a brief introduction to **Aubry-Mather theory** and its generalization to Lagrangian systems with multiple degrees of freedom, which we will call as **Mather-Mañé theory** in this thesis.

2.1 Aubry-Mather theory

The **Aubry-Mather theory** is based on independent researches in three different fields:

1. Monotone twist maps in dynamical systems.
2. Minimal geodesics on a 2 dimensional torus in differential geometry.
3. The discrete Frenkel-Kontorova model in solid state physics.

In an excellent paper by Bangert [11], the author reduced the proofs of many results in the above three different fields to one common root by introducing a variational problem.

The study of dynamics of area preserving maps, which is a classical topic in dynamical system, goes back to the work of Poincaré and Birkhoff. These maps usually turn up as Poincaré maps of Hamiltonian systems with one and half or two degrees of freedom. Among the area preserving maps a special class called **monotone twist maps** has been studied intensively. For example see [3], [4], [7], [9], [10].

Definition 2.1.1. *An orientation preserving C^1-diffeomorphism $f : S^1 \times [a,b] \to S^1 \times [a,b]$ of an annulus is a* ***monotone twist map****, if it preserves the ends of the annulus and has a lift $\tilde{f} : \mathbb{R} \times [a,b] \to \mathbb{R} \times [a,b]; \tilde{f}(x_0, y_0) = (x_1, y_1)$ satisfying the following properties:*

1. *$\tilde{f}$ preserves area;*

2. *twist condition: $\frac{\partial x_1}{\partial y_0} > 0$.*

The reason that our study of the dynamics of monotone twist maps can be done through a variational problem is that a monotone twist map can be described by a C^2 **generating function** $h : \mathbb{R}^2 \to \mathbb{R}$, unique up to a constant, in the following way: $\tilde{f}(x_0, y_0) = (x_1, y_1)$ is equivalent to $y_0 = -\partial_1 h(x_0, x_1); y_1 = \partial_1 h(x_0, x_1)$.

To introduce the variational problem, we need to define a **variational principle**.

Definition 2.1.2. *A continuous function $h : \mathbb{R}^2 \to \mathbb{R}$ will be called a* ***variational principle*** *if it satisfies the following conditions:*

- $H_1 : h(\xi + 1, \xi' + 1) = h(\xi, \xi')$, *for all* $\xi, \xi' \in \mathbb{R}$;

- $H_2 : \lim_{|\zeta| \to \infty} h(\xi, \xi + \zeta) = \infty$, *uniformly in* ξ;

- $H_3 : h(\xi, \zeta') + h(\zeta, \xi') > h(\xi, \xi') + h(\zeta, \zeta')$, *if* $\xi < \zeta$ *and* $\xi' < \zeta'$;

- H_4 : *If* $(\xi, \eta, \zeta) \neq (\xi', \eta, \zeta')$ *both are minimal, i.e.,* $h(\xi, \eta) + h(\eta, \zeta) \leq h(\xi, \eta') + h(\eta', \zeta)$ *and* $h(\xi', \eta) + h(\eta, \zeta') \leq h(\xi', \eta') + h(\eta', \zeta')$ *for any* $\eta' \in \mathbb{R}$, *then* $(\xi - \xi')(\zeta - \zeta') < 0$.

- H_5 : *There exists a positive continuous function ρ on $\mathbb{R}^2$ such that*

$$h(\xi, \zeta') + h(\zeta, \xi') - h(\xi, \xi') - h(\zeta, \zeta') > \int_{\xi}^{\eta} \int_{\xi'}^{\eta'} \rho$$

  if $\xi < \zeta$ *and* $\xi' < \zeta'$;

- H_6 : *There is a* $\theta > 0$, *such that* $\xi \mapsto \theta\xi^2/2 - h(\xi, \xi')$ *is convex, for any* ξ', *and* $\xi' \mapsto \theta\xi'^2/2 - h(\xi, \xi')$ *is convex, for any* ξ.

Remark 2.1.1. *In Bangert's paper [11], only conditions H_1 to H_4 were required. Mather added conditions H_5 and H_6 in his papers [7], [9] and [10], these two conditions are necessary for our result. These conditions are not all independent, obviously H_5 implies H_3. H_5 and H_6 together imply H_4, see [7] for the details.*

It is not hard to see the generating function of a monotone twist map is a variational principle. Even though the generating function of a monotone twist map is C^2, we assume weaker regularity condition for two reasons: first, the variational principle associated with a smooth Riemannian metric on $\mathbb{T}^2$ is usually just Lipschitz continuous see [11] for details (although only conditions H_1 to H_4 are verified in [11], as Mather said in [7] it also satisfies conditions H_5 and H_6); second, we want our result to work for finite composition of monotone twist maps as well. The variational principle for such a composition is always a variational principle, but not necessarily C^2, see [7] and [10] for details.

We will work on a configuration space $\prod_{i=n_0}^{n_1} \mathbb{R}$, $-\infty \le n_0 < n_1 \le +\infty$ with the product topology. When $n_0 = -\infty$ and $n_1 = +\infty$, we will denote the configuration space by $\mathbb{R}^{\mathbb{Z}}$. By the **Aubry graph** $Au(x)$ of a configuration $x = \{x_i\}_{i=n_0}^{n_1}$, we mean the union of the line segments in $\mathbb{R}^2$ joining (i, x_i) and $(i+1, x_{i+1})$, for $n_0 \le i < n_1$. From now on, we will say two configurations intersect if their Aubry graphs intersect.

First we will extend the variational principle h to any finite configuration $\{x_i\}_{i=n_0}^{n_1}$ by

$$h(x_{n_0}, \dots, x_{n_1}) = \sum_{i=n_0}^{n_1-1} h(x_i, x_{i+1}).$$

Such a finite configuration is **minimal** w.r.t. h, if

$$h(x_{n_0}, \dots, x_{n_1}) \le h(x^*_{n_0}, \dots, x^*_{n_1}),$$

for any $\{x^*_i\}_{i=n_0}^{n_1}$ with $x^*_{n_0} = x_{n_0}$ and $x^*_{n_1} = x_{n_1}$. A configuration, finite or not, is minimal w.r.t. h, if any finite sub-configuration of it is minimal w.r.t. h. Similarly we say a configuration is **locally minimal** w.r.t. h, if all the above are true inside an open neighborhood of it in the configuration space. For the sake of convenience, we will omit h in all these notations when there is no confusion.

Definition 2.1.3. *We say* $x = \{x_i\}_{i=n_0}^{n_1}, -\infty \leq n_0 < n_1 \leq +\infty$ *is a* ***stationary configuration****, if*

$$\partial_2 h(x_{i-1}, x_i) + \partial_1 h(x_i, x_{i+1}) = 0, \quad \forall n_0 < i < n_1.$$

Remark 2.1.2. *Although* h *is usually just Lipschitz continuous, if* x *is a minimal configuration, both* $\partial_2 h(x_{i-1}, x_i)$ *and* $\partial_1 h(x_i, x_{i+1})$ *exist, for* $n_0 < i < n_1$ *as explained by Mather in [7]. The same argument also works, if* x *is locally minimal. Therefore every locally minimal configuration is a stationary configuration.*

It is not hard to see every stationary configuration corresponds to an orbit of the monotone twist map.

For any pair of integers $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, we define an operator $T_{(m,n)}$ on $\mathbb{R}^{\mathbb{Z}}$ by

$$(T_{(m,n)} x)_i = x_{i+n} - m.$$

If a configuration x satisfies $T_{(m,n)} x = x$, we say x is (m, n)-periodic.

For any two configurations x and x^*, we say x is ω-asymptotic to x^* or simply $x(+\infty) = x^*$, if $\lim_{i \to +\infty} |x_i - x_i^*| = 0$ and is α-asymptotic to x^* or simply $x(-\infty) = x^*$, if $\lim_{i \to -\infty} |x_i - x_i^*| = 0$. When x^* is a constant configuration, i.e., $x_i^* \equiv u \in \mathbb{R}, \forall i \in \mathbb{Z}$, we may also write $x(\pm\infty) = u$.

If a configuration x satisfies $x(-\infty) = y^1$ and $x(+\infty) = y^2$, we say x is a **heteroclinic configuration** between y^1 and y^2, if y^1 and y^2 are different, or a **homoclinic configuration**, if y^1 and y^2 are the same. Furthermore, if x is a stationary configuration, we will call it a **heteroclinic connection** or **homoclinic connection**.

We say a configuration $x = \{x_i\}_{i \in \mathbb{Z}}$ has a *rotation number* $\alpha \in \mathbb{R}$, if

$$\lim_{|i| \to +\infty} \frac{x_i}{i} \text{ exists and equal to } \alpha.$$

The following lemma lies in the core of the proofs of many propositions in Aubry-Mather theory and its proof can be found in [11] (unless specified all the proofs of the results listed in this section can be found in [11]).

Lemma 2.1.1. ***Aubry's Crossing Lemma***

1. *If* $\{x_i\}_{i=j}^k$ *and* $\{x_i^*\}_{i=j}^k$ *are two different minimal configurations and they do not have the same end points, then the Aubry graphs of them intersect at most once. Therefore if* $x \neq x^* \in \mathcal{M}$*, their Aubry graphs intersect at most at one point.*

2. *If $x \in \mathcal{M}$ is a minimal configuration, then x and $T_{(a,b)}x$ do not cross, for any $(a,b) \in \mathbb{Z}^2$ x.*

3. *If $x \neq x^* \in \mathcal{M}$ with $x(-\infty) = x^*$ or $x(+\infty) = x^*$, then x and x^* don't intersect.*

From now on we will use $\mathcal{M}$ to denote the set of all minimal configurations $x \in \mathbb{R}^{\mathbb{Z}}$.

Theorem 2.1.1. *Given a pair of integers (p, q) with $q \neq 0$ and relatively prime, there is at least one periodic minimal configuration with period (q, p) in $\mathcal{M}$.*

A very important property of the minimal configurations is that for every $x \in \mathcal{M}$ there exists an orientation preserving circle homeomorphism ϕ and a lift of it, $\tilde{\phi}$, such that $x_{i+1} = \tilde{\phi}(x_i)$. This implies the following theorem.

Theorem 2.1.2. *Given $\mathcal{M}$ the topology induced from $\mathbb{R}^{\mathbb{Z}}$, there exists a continuous map $\tilde{\alpha} : \mathcal{M} \to \mathbb{R}$ with the following properties:*

1. *For all $x \in \mathcal{M}, i \in \mathbb{Z}$ we have $|x_i - x_0 - i\tilde{\alpha}(x)| < 1$, in particular $\tilde{\alpha}(x) = \lim_{|i|\to\infty} x_i/i$.*

2. *if $x \in \mathcal{M}$ is periodic with period (p, q) then $\tilde{\alpha}(x) = p/q$.*

3. *$\tilde{\alpha}$ is invariant under T: $\tilde{\alpha}(T_{(p,q)}x) = \tilde{\alpha}(x)$ for all $(p, q) \in \mathbb{Z}^2$.*

By the above two theorems for every rational rotation number there is at least one minimal configuration from $\mathcal{M}$ with the given rotation number. The next theorem tells us that this is also true for each irrational rotation number.

Theorem 2.1.3. *For all $\alpha \in \mathbb{R}$ the set $\mathcal{M}_\alpha = \{x \in \mathcal{M} : \tilde{\alpha}(x) = \alpha\}$ is not empty.*

The remarkable thing about Aubry-Mather theory is that it gives us a very good understanding of the structure of $\mathcal{M}_\alpha$, $\forall \alpha \in \mathbb{R}$ and they are quite different depending on if α is rational or irrational. We will discuss them separately.

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{M}_\alpha$ is totally ordered. This means that for any $x \neq x^* \in \mathcal{M}_\alpha$, either $x < x^*$ or $x^* < x$. For two configurations $y^1, y^2 \in \mathcal{M}$, by $y^1 < y^2$, we mean $y^1_i < y^2_i$, for any $i \in \mathbb{Z}$.

The ordering structure implies $\mathcal{M}_\alpha$ can be embedded into a circle homeomorphism, i.e., there exists an orientation preserving circle homeomorphism ϕ and a lift $\tilde{\phi}$ of it with rotation number α, such that for each $x \in \mathcal{M}_\alpha$, we have $x_{i+1} = \tilde{\phi}(x_i)$.

There are two different cases for the structure of $\mathcal{M}_\alpha, \alpha \in \mathbb{R} \setminus \mathbb{Q}$:

1. If for any $a \in \mathbb{R}$, then a $x \in \mathcal{M}_\alpha$ with $x_0 = a$. This means $\mathcal{M}_\alpha$ foliate the entire configuration space $\mathbb{R}^{\mathbb{Z}}$. For the corresponding monotone twist map, there is an homotopically nontrivial invariant circle, i.e., it can not be continuously deformed to a point in the annulus with rotation number α. Conversely, if there is an homotopically nontrivial invariant circle in the annulus of the monotone twist map with rotation number α, then it can be shown that every orbit in the invariant circle has a corresponding minimal configuration $x \in \mathcal{M}_\alpha$ and $\mathcal{M}_\alpha$ is a foliation of $\mathbb{R}^{\mathbb{Z}}$.

2. If the projection of $\mathcal{M}_\alpha$ to a single $\mathbb{R}$ in $\mathbb{R}^{\mathbb{Z}}$ is not onto, then there is a unique recurrent invariant set on the circle for the orientation preserving circle homeomorphism ϕ related with $\mathcal{M}_\alpha$. Because of the closedness of $\mathcal{M}_\alpha$, the projection of $\mathcal{M}_\alpha$ must cover this invariant set and the invariant set in the annulus is a Cantor set or a Cantorus, as it is often called.

If $\alpha = p/q \in \mathbb{Q}$ with p, q relative primes, we set $\mathcal{M}_\alpha^{per}$ to be the set of all periodic minimal configurations with period (p, q), i.e.

$$\mathcal{M}_\alpha^{per} := \{x \in \mathcal{M}_\alpha : T_{(p,q)}x = x\}.$$

By Theorem 2.1.1, it is nonempty for any $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and **Aubry's crossing lemma** implies that it is totally ordered, i.e., for any $x \neq x^* \in \mathcal{M}_\alpha^{per}$, either $x < x^*$ or $x > x^*$.

Two different configurations $x^0 < x^1 \in \mathcal{M}_\alpha^{per}$ will be called a pair of ((p, q)-periodic) **neighboring minimal configurations** if there is no other $x \in \mathcal{M}_\alpha^{per}$ which lies in between them. This happens when $\mathcal{M}_\alpha^{per}$ does not foliate the whole configuration space $\mathbb{R}^{\mathbb{Z}}$. In other words there are gaps between elements of $\mathcal{M}_\alpha^{per}$.

Now we will study the complement of $\mathcal{M}_\alpha^{per}$ in $\mathcal{M}_\alpha$.

Theorem 2.1.4. *Given a pair of ((p, q)-periodic) neighboring minimal configurations $x^0 < x^1 \in \mathcal{M}_\alpha^{per}$, there is at least one minimal configuration $y^+ \in \mathcal{M}_\alpha$ with $y^+(-\infty) = x^0; y^+(+\infty) = x^1$ and one minimal configuration y^- with $y^-(-\infty) = x^1; y^-(+\infty) = x^0$.*

By the above theorem, it is natural to define the following nonempty sets

$$\mathcal{M}_\alpha^+(x^0, x^1) := \{x \in \mathcal{M}_\alpha : x(-\infty) = x^0 \text{ and } x(+\infty) = x^1\};$$

$$\mathcal{M}_\alpha^-(x^0, x^1) := \{x \in \mathcal{M}_\alpha : x(-\infty) = x^1 \text{ and } x(+\infty) = x^0\}.$$

Then we set $\mathcal{M}_\alpha^+$ resp. $\mathcal{M}_\alpha^-$ to be the union of the sets $\mathcal{M}_\alpha^+(x^0, x^1)$ resp. $\mathcal{M}_\alpha^-(x^0, x^1)$ extended over all pairs of ((p, q)-periodic) of neighboring minimal configurations.

Obviously $\mathcal{M}_\alpha^{per}, \mathcal{M}_\alpha^+, \mathcal{M}_\alpha^-$ are subset of $\mathcal{M}_\alpha$, and the next theorem tells us there is nothing else left in $\mathcal{M}_\alpha$.

Theorem 2.1.5. *For each $\alpha \in \mathbb{Q}$, $\mathcal{M}_\alpha = \mathcal{M}_\alpha^{per} \cup \mathcal{M}_\alpha^+ \cup \mathcal{M}_\alpha^-$.*

2.2 Mather-Mañé theory

Inspired by a result of Moser [14] that every monotone twist map can be viewed as the time-one map of a positive definite Lagrangian system, Mather started a program to generalize Aubry-Mather theory to Lagrangian systems with arbitrary degrees of freedom in a series of papers [15], [16], [17].

Given a closed (i.e., compact and boundaryless) and connected smooth Riemannian manifold M, Mather studied the dynamics of a special class of Lagrangians which are defined on $TM \times \mathbb{T}$, where TM is the tangent bundle of M and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Nowadays people usually refer to them as Tonelli Lagrangians.

Definition 2.2.1. *A Lagrangian $L \in C^2(TM \times \mathbb{R})$ is a* ***Tonelli Lagrangian****, if it satisfies the following conditions:*

1. ***Periodicity****: L is 1-periodic in time, i.e., $L(x, v, t) = L(x, v, t + 1)$ for all $(x, v, t) \in TM \times \mathbb{R}$;*

2. ***Positive definiteness****: $\partial^2 L/\partial v^2(x, v, t)$ is positive definite, as a quadratic form, for all $(x, v, t) \in TM \times \mathbb{R}$;*

3. ***Superlinear growth****: $\lim_{\|v\|_x \to +\infty} \frac{L(x,v,t)}{\|v\|_x} = +\infty$, in each fiber, i.e., for each $A \in \mathbb{R}$, there exists $B(A) \in \mathbb{R}$ such that $L(x, v, t) \geq A\|v\|_x - B(A)$, for all $(x, v, t) \in TM \times \mathbb{R}$.*

4. ***Completeness****: Every orbit of the Euler-Lagrange flow introduced by L is defined for all time.*

Remark 2.2.1. *$\|\cdot\|_x$ denotes the norm associated to the Riemannian metric on M. Since M is compact, condition (3) in the above definition is independent of the choice of Riemannian metrics on M. For any $x, y \in M$, we use $d(x, y)$ to denote the distance defined by the Riemannian metric.*

The main feature of a Tonelli Lagrangian system is the existence of a series of families of invariant objects satisfying various minimizing properties, which all are first introduced by Mather in [16].

Given a Tonelli Lagrangian $L : TM \times \mathbb{T} \to \mathbb{R}$, let η be a closed one form defined on M. By abusing of notation, we also use η to denote the function defined on $TM \times \mathbb{T}$ as

$$\eta(x, v, t) = \langle \eta(x)dx, v\frac{\partial}{\partial x}\rangle, \quad \forall (x, v, t) \in TM \times \mathbb{T}.$$

Notice that as a function η's value is independent of t.

Consider the new Lagrangian $L_\eta := L - \eta$, which is also Tonelli. As η is a closed one form, the Euler-Lagrange flows introduced by L and L_η are the same.

Our first family of invariant objects come from flow invariant measures. Let $\mathfrak{M}(L)$ be the space of Borel probability measures on $TM \times \mathbb{T}$ that are invariant under the Euler-Lagrange flow Φ_t^L introduced by L and have finite average action, i.e. $A_L(\mu) := \int_{TM\times\mathbb{T}} L d\mu < \infty$.

We assume that $\mathfrak{M}(L)$ is endowed with the *vague topology*, i.e., the weak* topology induced by the space $C_l^0(TM \times \mathbb{T}, \mathbb{R})$ of continuous functions $f : TM \times \mathbb{T} \to \mathbb{R}$ having at most linear growth

$$\sum_{(x,v,t)\in TM\times\mathbb{T}} \frac{|f(x, v, t)|}{1 + \|v\|} < +\infty.$$

Obviously $\mathfrak{M}(L) \subset (C_l^0)^*$.

Proposition 2.2.1. *Given a Tonelli Lagrangian L, for any closed one form η, $A_{L_\eta} : \mathfrak{M}(L) \to \mathbb{R}$ is lower semicontinuous with vague topology on $\mathfrak{M}(L)$.*

As a result we have the following corollary.

Corollary 2.2.2. *There exist at least one minimal measure $\mu \in \mathfrak{M}(L)$, i.e.,*

$$A_{L_\eta}(\mu) = \min\{A_{L_\eta}(\tilde{\mu}) : \tilde{\mu} \in \mathfrak{M}(L)\}.$$

A simple observation tells us the minimal measure only depends on the de Rham cohomology class $c := [\eta] \in H^1(M;\mathbb{R})$ as $\int_{TM\times\mathbb{T}} df\, d\mu = 0$ for any exact one form df and Φ_t^L invariant measure $\mu \in \mathfrak{M}(L)$.

Definition 2.2.2. *Let η_c be a closed one form with $c = [\eta_c] \in H^1(M;\mathbb{R})$, we call $\mu \in \mathfrak{M}(L)$ a **c-minimal measure** if*

$$A_{L_{\eta_c}}(\mu) = \min A_{L_{\eta_c}}(\tilde{\mu}) : \tilde{\mu} \in \mathfrak{M}(L).$$

For a given Tonelli Lagrangian L, we define a function $\alpha : H^1(M;\mathbb{R}) \to \mathbb{R}$ as following:

$$\alpha(c) = -\min\{A_{L_{\eta_c}}(\mu) : \mu \in \mathfrak{M}(L)\}, \quad \forall c = [\eta_c] \in H^1(M;\mathbb{R}).$$

As proved by Mather in [16]

Proposition 2.2.3. *α is a continuous convex function with superlinear growth.*

Let $\mathfrak{M}_c(L) \subset \mathfrak{M}(L)$ be the set of all c-minimal measures. Now we are ready to introduce the first important family of invariant sets: **Mather sets**.

Definition 2.2.3. *Given a Tonelli Lagrangian L and a cohomology class $c \in H^1(M;\mathbb{R})$,*

$$\widetilde{\mathcal{M}}_c(L) := \bigcup_{\mu\in\mathfrak{M}_c(L)} supp\, \mu \subset TM \times \mathbb{T},$$

*will be called the **Mather set of cohomology class** c. Let $\pi : TM \times \mathbb{T} \to M \times \mathbb{T}$ be the usual projection along each fiber of M, $\mathcal{M}_c(L) := \pi(\widetilde{\mathcal{M}}_c(L))$ will be called the **projected Mather set of cohomology class** c.*

From now on when there is no confusion we will omit the L in the notations of the above sets.

A very important proposition of Mather sets is the so-called *Mather graph theorem*, which was first proved by Mather in [16]

Theorem 2.2.1. *Let $\widetilde{\mathcal{M}}_c$ and $\mathcal{M}_c$ be defined as in Definition 2.2.3, then $\widetilde{\mathcal{M}}_c$ is compact and invariant under the Euler-Lagrange flow Φ_t^L and $\pi|_{\widetilde{\mathcal{M}}_c} : \widetilde{\mathcal{M}}_c \to \mathcal{M}_c$ is a bijective, Lipschitz continuous map.*

An alternative way to introduce those c-minimal measures is that instead of minimizing the action of a modified Lagrangian L, we can minimize the action of L while putting *constraints* on the invariant measures, namely fixing the *"rotation vector"* of the measures.

This approach is in some sense more similar to the Aubry-Mather theory we discussed in the previous section, because the *rotational vector* can be viewed as a generalization of the *rotation number* defined for minimal configurations.

Given an invariant measure $\mu \in \mathfrak{M}(L)$, for any closed one form η on M, because L is superlinear growth, $\int_{TM\times\mathbb{T}} \eta\, d\mu$ is well-defined and finite. As we mentioned before, $\int_{TM\times\mathbb{T}} df\, d\mu = 0$ for any exact one form df. Hence we can define a linear function from $H^1(M;\mathbb{R})$ to $\mathbb{R}$ by mapping each $c = [\eta_c] \in H^1(M;\mathbb{R})$ to $\int_{TM\times\mathbb{T}} \eta_c\, d\mu$.

Viewing $H_1(M;\mathbb{R})$ as the dual space of $H^1(M;\mathbb{R})$ with $\langle , \rangle$ denoting the canonical pairing between them, there is a unique $\rho(\mu) \in H_1(M;\mathbb{R})$, such that

$$\langle c, \rho(\mu)\rangle = \int_{TM\times\mathbb{T}} \eta_c d\mu, \quad \forall c \in H^1(M;\mathbb{R}).$$

As proved by Mather in [16], such a $\rho : \mathfrak{M}(L) \to H_1(M;\mathbb{R})$ is a continuous surjective function, i.e., for any $h \in H_1(M;\mathbb{R})$, there is a $\mu \in \mathfrak{M}(L)$ satisfying $\rho(\mu) = h$.

By Proposition 2.2.1, it is not hard to see for any $h \in H_1(M;\mathbb{R})$, there is at least one $\mu \in \mathfrak{M}(L)$ with $\rho(\mu) = h$ and

$$A_L(\mu) = \min\{A_L(\tilde{\mu}) : \tilde{\mu} \in \mathfrak{M}(L) \text{ and } \rho(\tilde{\mu}) = h\}.$$

We call such a μ an **h-minimal measure** or **minimal measure with rotation vector** h and $\mathfrak{M}^h(L)$ is used to denoted the set of all h-minimal measures.

Definition 2.2.4. *For a rotation vector or homology class $h \in H_1(M;\mathbb{R})$, we define the **Mather set with rotation vector** h as*

$$\widetilde{\mathcal{M}}^h(L) := \bigcup_{\mu\in\mathfrak{M}^h(L)} supp\, \mu \subset TM \times \mathbb{T},$$

and

$$\mathcal{M}^h(L) := \pi(\widetilde{\mathcal{M}}^h(L) \subset M \times \mathbb{T}$$

*the **projected Mather set with rotation vector** h.*

Again when there is no confusion, the Lagrangian L in the above notations will be omitted.

Like the α function, we can define a β function as

$$\beta : H_1(M;\mathbb{R}) \to \mathbb{R}; \quad \beta(h) = \min\{A_L(\mu) : \mu \in \mathfrak{M}(L) \text{ and } \rho(\mu) = h\}.$$

There is a nice relation between the α function and β function.

Proposition 2.2.4. *β is a convex, continuous and superlinear growth function. Furthermore α, β are convex conjugates to each other, i.e.,*

$$\beta(h) = \alpha^*(h) = \sup_{c \in H^1(M;\mathbb{R})} \{\langle h, c\rangle - \alpha(c)\}, \quad \forall h \in H_1(M;\mathbb{R});$$

$$\alpha(c) = \beta^*(c) = \sup_{h \in H_1(M;\mathbb{R})} \{\langle h, c\rangle - \beta(h)\}, \quad \forall c \in H^1(M;\mathbb{R}).$$

In order to introduce the remaining families of invariant objects, we are going to switch our focus from minimal measures to minimal curves with respect to the action of the Lagrangians.

Let $\mathcal{C}([a,b], M)$ be the set of all absolutely continuous curves defined on the interval $[a,b]$ with

$$\mathcal{C}_{[a,b]}(x,y) = \{\gamma \in \mathcal{C}([a,b], M) : \gamma(a) = x, \gamma(b) = y\}, \, \forall x, y \in M.$$

Theorem 2.2.2. *Given a Tonelli Lagrangian $L \in C^2(TM \times \mathbb{T}, \mathbb{R})$ and a closed one form η, for any $x, y \in M$ and $a < b \in \mathbb{R}$, there is at least one curve $\gamma \in \mathcal{C}_{[a,b]}(x,y)$ minimizing the action of L_η among all curves in $\mathcal{C}_{[a,b]}(x,y)$, i.e.,*

$$A_{L_\eta}(\gamma) := \int_a^b L_\eta(d\gamma(t), t)\, dt = \min\{\int_a^b L_\eta(d\tilde{\gamma(t)}, t)\, dt : \tilde{\gamma} \in \mathcal{C}_{[a,b]}(x,y)\}.$$

Furthermore such a γ is C^2 solution of the Euler-Lagrange equation of L.

Let η_c be a closed one form with $c = [\eta_c] \in H^1(M;\mathbb{R})$, we will define the following functions:

$$\begin{aligned} h_{\eta_c} &: M \times \mathbb{R} \times M \times \mathbb{R} \to \mathbb{R} \\ &((x,a),(y,b)) \mapsto \inf\{\int_a^b L_{\eta_c}(d\gamma(t), t) + \alpha(c)\, dt : \gamma \in \mathcal{C}_{[a,b]}(x,y)\}; \end{aligned}$$

$$
\begin{aligned}
h^\infty_{\eta_c} &: M \times \mathbb{T} \times M \times \mathbb{T} \to \mathbb{R} \\
&((x,\sigma),(y,\tau)) \mapsto \liminf_{T\in\mathbb{Z}^+, T\to+\infty} h_{\eta_c}((x,\sigma),(y,\tau+T));
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{\eta_c} &: M \times \mathbb{T} \times M \times \mathbb{T} \to \mathbb{R} \\
&((x,\sigma),(y,\tau)) \mapsto \inf_{T\in\mathbb{Z}^+} h_{\eta_c}((x,\sigma),(y,\tau+T));
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{d}_{\eta_c} &: M \times \mathbb{T} \times M \times \mathbb{T} \to \mathbb{R} \\
&((x,\sigma),(y,\tau)) \mapsto \Phi_{\eta_c}((x,\sigma),(y,\tau)) + \Phi_{\eta_c}((y,\tau),(x,\sigma)).
\end{aligned}
$$

Remark 2.2.2. *h_{η_c} is only defined on those points $((x,a),(y,b))$ with $a < b$ and the infimum in its definition is actually a minimum by Theorem 2.2.2.*

We summarize some important propositions of the above functions, whose proofs can be found in [33].

Proposition 2.2.5. *Let $h_{\eta_c}, h^\infty_{\eta_c}, \Phi_{\eta_c}$ and $\widetilde{d}_{\eta_c}$ be defined as above,*

1. *h_{η_c} is Lipschitz continuous and bounded on $\{((x,a),(y,b)) \in M \times \mathbb{R} \times M \times \mathbb{R} : a+1 \leq b\}$;*

2. *For any $\sigma, \tau \in \mathbb{T}$, $h^\infty_{\eta_c}((*,\sigma),(*,\tau)) : M \times M \to \mathbb{R}$ is Lipschitz continuous and satisfies the following triangle inequality:*

$$h^\infty_{\eta_c}((x,\sigma),(y,\tau)) + h^\infty_{\eta_c}((y,\tau),(z,\rho)) \geq h^\infty_{\eta_c}((x,\sigma),(z,\rho)),$$

   for any $(x,\sigma),(y,\tau),(z,\rho) \in M \times \mathbb{T}$.

3. *$\widetilde{d}_{\eta_c}$ is a non-negative function.*

The above functions will be used in the following definitions.

Definition 2.2.5. *Given an interval of time $I \subset \mathbb{R}$ and an absolutely continuous curve $\gamma \in \mathcal{C}(I, M)$, γ is a c-**minimal curve**, if*

$$A_{L_{\eta_c}+\alpha(c)}(\gamma|_{[a,b]}) = h_{\eta_c}((\gamma(a), a), (\gamma(b), b)) \quad \forall [a,b] \subset I;$$

*γ is a c-**semi-static curve**,*

$$A_{L_{\eta_c}+\alpha(c)}(\gamma|_{[a,b]}) = \Phi_{\eta_c}((x, a \text{ mod } 1), (y, b \text{ mod } 1)) \quad \forall [a,b] \subset I;$$

*γ is a c-**static curve**, if*

$$A_{L_{\eta_c}+\alpha(c)}(\gamma|_{[a,b]}) = -\Phi_{\eta_c}((y, b \text{ mod } 1), (x, a \text{ mod } 1)) \quad \forall [a,b] \subset I.$$

Remark 2.2.3. *1. By the definition of $\widetilde{d}_{\eta_c}$ and its non-negativity, it is easy to see*

$$-\Phi_{\eta_c}((y, b \text{ mod } 1), (x, a \text{ mod } 1)) \leq \Phi_{\eta_c}((x, a \text{ mod } 1), (y, b \text{ mod } 1)).$$

Therefore every c-static curve must be c-semi-static as well.

2. Obviously a c-semi-static curve is also c-minimal. In fact it satisfies an even stronger minimizing property, i.e., $A_{L_{\eta_c}+\alpha(c)}(\gamma|_{[a,b]}) \leq A_{L_{\eta_c}+\alpha(c)}(\xi)$ for any $\xi \in \mathcal{C}_{[a',b']}(\gamma(a), \gamma(b))$ with $a' - a \in \mathbb{Z}, b' - b \in \mathbb{Z}$ and $b' \geq a' + 1$.

A c-semi-static curve (or c-static) curve $\gamma \in \mathcal{C}(I, M)$ is called global c-semi-static (or c-static) when $I = \mathbb{R}$. Obviously such a curve is a solution of the Euler-Lagrange equation introduced by L. The corresponding orbit of it is called a global c-semi-static (or c-static) orbit.

Now we are ready to introduce the remaining two families of invariant sets.

Definition 2.2.6. *Given a closed one form η_c with $c = [\eta_c] \in H^1(M; \mathbb{R})$, we define the **Aubry set of cohomology class** c as*

$$\widetilde{\mathcal{A}}_c := \{(d\gamma(t), t \text{ mod } 1) : \gamma \text{ is global } c\text{-static } \} \subset TM \times \mathbb{T};$$

*the **Mañé set of cohomology class** c as*

$$\widetilde{\mathcal{N}}_c := \{(d\gamma(t), t \text{ mod } 1) : \gamma \text{ is global } c\text{-semi-static}\} \subset TM \times \mathbb{T}.$$

Obviously $\widetilde{\mathcal{A}}_c$ is a subset of $\widetilde{\mathcal{N}}_c$ and it is proved in [16] that

Theorem 2.2.3. *$\widetilde{\mathcal{M}}_c$ is a subset of $\widetilde{\mathcal{A}}_c$.*

As we mentioned a very important property of the Mather set $\widetilde{\mathcal{M}}_c$ is the *Mather's graph theorem*. In fact this theorem can be extended to an even larger set, namely the Aubry set $\widetilde{\mathcal{A}}_c$.

Theorem 2.2.4. *Let $\mathcal{A}_c := \pi(\widetilde{\mathcal{A}}_c) \subset M \times \mathbb{T}$ and called the* ***projected Aubry set of cohomology class*** *c, then $\pi|_{\widetilde{\mathcal{A}}_c} : \widetilde{\mathcal{A}}_c \to \mathcal{A}_c$ is a bijective, Lipschitz continuous map.*

Remark 2.2.4. *However such a graph theorem can not extended to the Mañé set $\widetilde{\mathcal{N}}_c$.*

Theorem 2.2.4 allows us to define an equivalence relation on $\widetilde{\mathcal{A}}_c$ through the function $\widetilde{d}_{\eta_c}$. We say $(x, v, \sigma), (y, w, \tau) \in \widetilde{\mathcal{A}}_c$ are equivalent iff

$$\widetilde{d}_{\eta_c}(\pi(x, v, \sigma), \pi(y, w, \tau)) = \widetilde{d}_{\eta_c}((x, \sigma), (y, \tau)) = 0.$$

By this equivalence relation, we break $\widetilde{\mathcal{A}}_c$ into classes which will be called static classes.

Furthermore for a c-semi-static curve $\gamma \in \mathcal{C}(I, M)$, if $I = [a, +\infty)$ (or $I = (-\infty, a]$) for some $a \in \mathbb{R}$, we say γ is a forward (or backward) c-semi-static curve and we set

$$\widetilde{\mathcal{N}}_c^+ := \{(d\gamma(t), t \ mod\ 1) : \gamma \text{ is forward } c - \text{semi-static } \};$$

$$\widetilde{\mathcal{N}}_c^- := \{(d\gamma(t), t \ mod\ 1) : \gamma \text{ is backward } c - \text{semi-static } \}.$$

While the number of c-semi-static curves is relatively small, forward (or backward) c-semi-static curves are unusually abundant, as indicated by the following proposition from [34].

Proposition 2.2.6. *Given a Tonelli Lagrangian L and a closed one form η_c, for each point $(x, \tau) \in M \times \mathbb{T}$, there is at least one forward c-semi-static curve started at (x, τ) and one backward c-semi-static curve ended at (x, τ).*

The next proposition tells us the asymptotic behaviors of these forward and backward c-semi-static orbits.

Proposition 2.2.7. *The ω-limit (or α-limit) set of a forward (or backward) c-semi-static orbits is contained in a unique static class of the Aubry set $\widetilde{\mathcal{A}}_c$.*

By Theorem 2.2.4, for each $(x, \tau) \in \widetilde{\mathcal{A}}_c$, there is a unique $(x, v, \tau) \in TM \times \mathbb{T}$ and a unique global c-static curve $\gamma_{(x,\tau)}$ with $(d\gamma_{(x,\tau)}(\tau)) = (x, v)$. Together with Proposition 2.2.6, we have the following proposition.

Proposition 2.2.8. *For each $(x, \tau) \in \widetilde{\mathcal{A}}_c$, there is a unique forward c-semi-static curve $\gamma^+ \in \mathcal{C}([\tau, +\infty), M)$ with $\gamma^+(\tau) = x$ and a unique backward c-semi-static curve $\gamma^- \in \mathcal{C}((-\infty, \tau], M)$ with $\gamma^-(\tau) = x$. Furthermore both γ^+, γ^- are c-static with*

$$\gamma^+(t) = \gamma_{(x,\tau)}(t), \;\; \textit{if } t \in [\tau, +\infty);$$

$$\gamma^-(t) = \gamma_{(x,\tau)}(t), \;\; \textit{if } t \in (-\infty, \tau],$$

where $\gamma_{(x,\tau)}$ is the unique global c-static curve passing x at time τ.

Chapter 3

Connecting orbits of Monotone Twist Maps

In this chapter, we will continue our study of Aubry-Mather theory. Given a variational principle h either corresponding to a finite composition of monotone twist maps or a smooth Riemannian metric on a 2-dim torus, let $x^0, x^1 \in \mathcal{M}_\alpha^{per}$, $\alpha \in \mathbb{Q}$, be a pair of periodic neighboring minimal configurations. By Theorem 2.1.4, there is at least one heteroclinic connection from x^0 to x^1 and one heteroclinic connection from x^1 to x^0.

What we will prove in this section is that if neither $\mathcal{M}_\alpha^+(x^0, x^1)$ nor $\mathcal{M}_\alpha^-(x^0, x^1)$ foliates the configuration space between x^0 and x^1, then there are infinitely many homoclinic and heteroclinic connections oscillating a prescribed number of times between x^0 and x^1 and asymptotic to x^0 or x^1 as we wish. To be precise, let's set $x^i = x^{i(mod 2)}$ for all $i \in \mathbb{Z}$, then we have

Theorem 3.0.5. *If*

$$I_\alpha^+(x^0, x^1) \neq (x_0^0, x_0^1) \text{ and } I_\alpha^-(x^0, x^1) \neq (x_0^0, x_0^1), \tag{gap}$$

where (x_0^0, x_0^1) *is an open interval and*

$$I_\alpha^+(x^0, x^1) := \{x_0 : x \in \mathcal{M}_\alpha^+(x^0, x^1)\};$$

$$I_\alpha^-(x^0, x^1) := \{x_0 : x \in \mathcal{M}_\alpha^-(x^0, x^1)\}.$$

Then for every $\hat{\delta} > 0$ small enough, there is an $m = m(\hat{\delta}) \in \mathbb{N}$ such that for every sequence of integers $q = \{q_i \in \mathbb{Z}\}_{i=-\infty}^{+\infty}$ with $q_{i+1} - q_i \geq 4m$ and for every $j, k \in \mathbb{Z}$ with $j < k$, there is a homoclinic or heteroclinic connection x satisfying

1. $x_i^0 < x_i < x_i^1$ *for all* $i \in \mathbb{Z}$;
2. $|x_{q_i - m} - x_{q_i - m}^i| \leq \hat{\delta}$ *and* $|x_{q_i + m} - x_{q_i + m}^{i+1}| \leq \hat{\delta}$ *for all* $i = j, \ldots, k$;
3. $x(-\infty) = x^j$ *and* $x(+\infty) = x^{k+1}$.

Remark 3.0.5. *The theorem immediately implies that the corresponding monotone twist map (resp. geodesic flow) has infinitely many homoclinic and heteroclinic orbits (resp. geodesics) between two periodic neighboring minimal orbits (geodesics), which shows chaotic dynamics.*

Given a monotone twist map $f : S^1 \times [a, b] \to S^1 \times [a, b]$ with an associated variational principle h, assume α_1 is the rotation number of $f|_{S^1 \times \{a\}}$ and α_2 is the rotation number of $f|_{S^1 \times \{b\}}$. If $\alpha \in (\alpha_1, \alpha_2) \cap \mathbb{Q}$ and (gap) is not true for any pair of periodic neighboring minimal configurations with rotation number α, $\mathcal{M}_\alpha$ must foliate the whole configuration space. Then f has an (homotopically) non-trivial invariant circle in the cylinder with rotation number α. If (gap) is true, it is not hard to see Theorem 3.0.5 implies f has positive topological entropy.

If f has zero topological entropy, then for every $\alpha \in (\alpha_1, \alpha_2) \cap \mathbb{Q}$, f must have an non-trivial invariant circle with rotation number α. Because the set of non-trivial invariant circles (and hence the set of rotation numbers which can occur) is closed, see [35] and [36], for every irrational $\alpha \in (\alpha_1, \alpha_2)$, f must have an non-trivial invariant circle as well. Hence we have the following result, which was proved by Angenent in [35] using a different method.

Corollary 3.0.9. *If the topological entropy of f vanishes, then f must have a non-trivial invariant circle of rotation number α, for any $\alpha \in (\alpha_1, \alpha_2)$.*

Similarly for a geodesic flow on $\mathbb{T}^2$ with an associated variational principle h, when (gap) is true, the geodesic flow must have positive topological entropy. If a geodesic flow on $\mathbb{T}^2$ has zero topological entropy, (gap) is not true for any pair of periodic neighboring minimal configurations with rational rotation number α, then minimal geodesics with rotation number α will foliate the whole torus.

Corollary 3.0.10. *If the topological entropy of a geodesic flow on $\mathbb{T}^2$ vanishes, then for every $\alpha \in \mathbb{Q}$, minimal geodesics with rotation number α is a foliation of $\mathbb{T}^2$.*

Related results about geodesic flows on $\mathbb{T}^2$ with vanishing topological entropy can be found in [37], [38].

Furthermore we would like to mention an interesting paper [39] by Bolotin and Rabinowitz, where a similar variational method was applied directly to the geodesics on $\mathbb{T}^2$ to show the existence of chaotic geodesics. Theorem 2.3 in [39] says that under certain geometric condition, there are infinitely many homoclinic and heteroclinic geodesics between two periodic neighboring minimal geodesics.

Although the geometric condition assumed in [39] is stronger than our (gap) condition, their result is also stronger than ours. Namely by our result the geodesics have to spend large enough time between every bump, while in [39] this is not necessary. There are other interesting homoclinic and heteroclinic geodesics in their paper beyond the reach of our result.

Our proof uses a variational method similar to those used in [40] and [41], where existence of multibump homoclinic and heteroclinic orbits of one-dimensional periodic forced pendulum was proved in the same spirit.

3.1 Asymptotic behaviors of configurations

In this section we will introduce a normalized function J, which is similarly to a normalized functional used in [40] and [41]. This normalized function will give us a convenient way to determine the asymptotic behaviors of the configurations.

Given an arbitrary variational principle $h : \mathbb{R}^2 \to \mathbb{R}$, following the notations in [42], we define a new function $\bar{h} : \mathbb{R} \to \mathbb{R}$ by $\bar{h}(\xi) = h(\xi, \xi)$.

By condition H_1 there is a finite $c := \min\{\bar{h}(\xi) : \xi \in \mathbb{R}\}$. If $\bar{h}(u) = c$, we will call u a *minimizer* of h. When there is no confusion u will also be used to represent the configuration $x = \{x_i\}_{i\in\mathbb{Z}}$ with $x_i \equiv u$ for all $i \in \mathbb{Z}$.

Two minimizers $u_0 < u_1$ will be called a pair of *neighboring minimizers*, if $\bar{h}(u) > \bar{h}(u_0)$, for all $u \in (u_0, u_1)$.

Proposition 3.1.1. *If $u \in \mathbb{R}$ is a minimizer of h, then as a configuration $u \in \mathcal{M}_0^{per}$, i.e., it is a minimal configuration.*

This is a well-know fact in Aubry-Mather theory, a proof can be found in [11] or [43]. Hence, for a given variational principle h, there is a one to one correspondence between $(0,1)$-periodic neighboring minimal configurations and neighboring minimizers of h.

For the remainder of this paper, we will fix an arbitrary variational principle h and an arbitrary pair of neighboring minimizers $u_0 < u_1$. Now several compact sets of configurations between u_0 and u_1 will be introduced.

Definition 3.1.1. *Let $U := \{u_0, u_1\}$, for any positive integer n, we define the following sets,*

$$X(n) := X(n;U) := \{x = \{x_i\}_{i=0}^{n} : x_i \in [u_0, u_1]\};$$

$$\hat{X}(n) := \hat{X}(n;U) := \{x = \{x_i\}_{i=0}^{n} : x \in X(n) \text{ and } x_0 = x_n\};$$

$$X := X(U) := \{x = \{x_i\}_{i=-\infty}^{+\infty} : x_i \in [u_0, u_1]\}.$$

Definition 3.1.2. *We define a normalized function J on X by*

$$J(x) = \sum_{i=-\infty}^{+\infty} a_i(x), \text{ where } a_i(x) = h(x_i, x_{i+1}) - c, \forall i \in \mathbb{Z},$$

Notice that even if c is the minimum of $\bar{h}$ on $\mathbb{R}$, if we choose $\xi \neq \eta$, it is possible that $h(\xi,\eta) - c < 0$, which means $J(x)$ may have no lower bound. However by the next several lemmas we will show that $\sum_{-n}^{n} a_i(x)$ has a finite lower bound independent of $n \in \mathbb{N}$ and $x \in X$, then we will prove that J is a well-defined function from X to $\mathbb{R} \cup \{+\infty\}$.

For simplicity, we define translation operators $T_k : X \to X$ by $T_k x = T_{(0,k)} x$ for any $k \in \mathbb{Z}$. It is easy to see J is invariant under T_k.

First we will show that for any configurations $x \in \hat{X}(n)$, $\sum_{i=0}^{n-1} a_i(x)$ is non-negative.

Lemma 3.1.2. *For any $x \in \hat{X}(n)$, we have*

$$\sum_{i=0}^{n-1} a_i(x) = h(x_0, \dots, x_n) - nc \geq 0,$$

the inequality is an equality iff $x_i \equiv u_0$ or $x_i \equiv u_1$ for all $0 \leq i \leq n$.

This lemma follows directly from a well-know result in Aubry-Mather theory, see Theorem 3.3 in [11], we omit the proof here.

The above result only gives us a lower bound for configurations in $\hat{X}(n)$, to find a lower bound for configuration in $X(n)$, we need the following definition and a better estimate for the lower bounds of configurations in $\hat{X}(n)$.

Definition 3.1.3. *For any $x \in X(n)$, we define*

$$d(x, U) := \max_{0 \le i \le n} \min_{j \in \{0,1\}} |x_i - u_j|.$$

Lemma 3.1.3. *For any $\delta > 0$, let*

$$\phi(\delta) := \inf_{n \in \mathbb{Z}^+} \inf\{\sum_{i=0}^{n-1} a_i(x) : x \in \hat{X}(n) \text{ and } d(x, U) \ge \delta\},$$

then ϕ is a monotonically increasing continuous function of δ satisfying $\phi(\delta) > 0$, if $\delta > 0$; $\phi(\delta) = 0$, if $\delta = 0$.

Moreover if $x \in \hat{X}(n)$ satisfies

$$\min_{j \in \{0,1\}} |x_i - u_j| \ge \delta \text{ for all } i = 1, \dots, n-1, \tag{3.1}$$

then,

$$\sum_{i=0}^{n-1} a_i(x) \ge n\phi(\delta). \tag{3.2}$$

Proof. First it is not hard to see $\phi(\delta) = 0$ if $\delta = 0$. For the rest of the proof, we let $\delta > 0$. If we fix an $n \in \mathbb{Z}^+$, by the compactness of $\{x \in \hat{X}(n) : d(x, U) \ge \delta\}$, we have

$$\phi(\delta, n) := \inf\{\sum_{i=0}^{n-1} a_i(x) : x \in \hat{X}(n) \text{ with } d(x, U) \ge \delta\} > 0.$$

Using induction on n, we will show that $\phi(\delta, n) \ge \phi(\delta, 1)$ for all $n \in \mathbb{N}$, then $\phi(\delta) = \phi(\delta, 1) > 0$.

Let us assume $\phi(\delta, k) \ge \phi(\delta, 1)$ for all $k = 1, \dots, n-1$, we will show that $\phi(\delta, n) \ge \phi(\delta, 1)$. Choose an arbitrary $x \in \hat{X}(n)$ with $d(x, U) \ge \delta$, if there is a $j = 1, \dots, n-1$, such that $x_j = x_0 = x_n$, then $\{x_i\}_{i=0}^j \in \hat{X}(j)$ and $\{x_i\}_{i=j}^n \in \hat{X}(n-j)$, and among $d(\{x_i\}_{i=0}^j, U) \ge \delta$ and $d(\{x_i\}_{i=j}^n, U) \ge \delta$, at least one is true. Therefore

$$\sum_{i=0}^{n-1} a_i(x) \ge \min\{\phi(\delta, j), \phi(\delta, n-j)\} \ge \phi(\delta, 1).$$

Now if $x_j \neq x_0$ for all $j = 1, \dots, n-1$, then there must be a $1 \leq k \leq n-1$, such that $(x_k - x_{k-1})(x_k - x_{k+1}) \geq 0$. By condition H_3,

$$h(x_{k-1}, x_k) + h(x_k, x_{k+1}) \geq h(x_{k-1}, x_{k+1}) + h(x_k, x_k).$$

We set $x^{n-1} = (x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ and $x^1 = (x_k, x_k)$, again $x^{n-1} \in \hat{X}(n-1), x^1 \in \hat{X}(1)$ and one of $d(x^{n-1}, U) \geq \delta, d(x^1, U) \geq \delta$ must be true. Therefore

$$\sum_{i=0}^{n-1} a_i(x) \geq h(x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n) - (n-1)c + h(x_k, x_k) - c \geq \phi(\delta, 1).$$

Since we choose x arbitrarily, $\phi(\delta, n) \geq \phi(\delta, 1)$. Hence $\phi(\delta) = \phi(\delta, 1)$. From the definition of $\phi(\delta, 1)$ it is not hard to see it is continuous and monotonically increasing with respect to δ. We finished the first part of the lemma.

For the second part, again by induction assume the statement is true for $k = 1, \dots, n-1$, if $x \in \hat{X}(n)$ satisfies (3.1), as above there are two cases.

Case 1: there is a $j = 1, \dots, n-1$, such that $x_j = x_0 = x_n$, then $\{x_i\}_{i=0}^{j} \in \hat{X}(j)$ satisfies (3.1) with n replaced by j, and $\{x_i\}_{i=j}^{n} \in \hat{X}(n-j)$ satisfies (3.1) with n replaced by $n-j$. By the induction assumption

$$\sum_{i=0}^{n-1} a_i(x) = \sum_{i=0}^{j-1} a_i(x) + \sum_{i=j}^{n-1} a_i(x) \geq j\phi(\delta) + (n-j)\phi(\delta) = n\phi(\delta).$$

Case 2: $x_j \neq x_0$ for all $j = 1, \dots, n-1$, then there is a $1 \leq k \leq n-1$ with $(x_k - x_{k-1})(x_k - x_{k+1}) \geq 0$. Now for x^{n-1} and x^1 defined as above, x^{n-1} satisfies (3.1) with n replaced by $n-1$. Hence

$$\sum_{i=0}^{n-1} a_i(x) \geq h(x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n) - (n-1)c + h(x_k, x_k) - c \geq n\phi(\delta).$$

□

Let $C := Lip(h)$ for the rest of this paper, i.e., $|h(\xi, \xi') - h(\zeta, \zeta')| \leq C(|\xi - \zeta| + |\xi' - \zeta'|)$, for any ξ, ξ', ζ, and ζ' in $\mathbb{R}$, with the aid of the above lemma, now we can show that $\sum_{i=0}^{n-1} a_i(x)$ has a uniform lower bound, for any $x \in X(n)$.

Lemma 3.1.4. *For any $\delta > 0$ and $n \in \mathbb{N}$, if $x \in X(n)$ satisfies $d(x, U) \geq \delta$, we have*

$$\sum_{i=0}^{n-1} a_i(x) \geq \phi(\delta) - C|x_n - x_0| \geq -C|x_n - x_0|,$$

where $\phi(\delta)$ is defined as in Lemma 3.1.3. Moreover there is a $B \in \mathbb{R}$, such that for any $x \in X(n)$ and $n \in \mathbb{N}$,

$$\sum_{i=0}^{n-1} a_i(x) = h(x_0, \dots, x_n) - nc \geq B.$$

Proof. First we will assume $d(x,U) = \min_{j\in\{0,1\}}\{|x_k - u_j|\}$ for a $k \neq 0$, then define a $\hat{x} \in \hat{X}(n)$ by

$$\hat{x}_i = \begin{cases} x_n & \text{if } i = 0 \\ x_i & \text{if } i = 1 \cdots n. \end{cases}$$

Obviously $\hat{x} \in \hat{X}(n)$ with $d(\hat{x}, U) \geq \delta$, so $\sum_{i=0}^{n-1} a_i(\hat{x}) \geq \phi(\delta)$. At the same time

$$|\sum_{i=0}^{n-1} a_i(x) - \sum_{i=0}^{n-1} a_i(\hat{x})| = |h(x_0, x_1) - h(x_n, x_1)| \leq C|x_n - x_0|.$$

Therefore,

$$\begin{aligned} \sum_{i=0}^{n-1} a_i(x) &\geq \sum_{i=0}^{n-1} a_i(\hat{x}) - C|x_n - x_0| \\ &\geq \phi(\delta) - C|x_n - x_0|. \end{aligned}$$

Now assuming $d(x,U) = \min\{|x_0 - u_0|, |x_0 - u_1|\}$, we define

$$\hat{x}_i = \begin{cases} x_i & \text{if } i = 0 \cdots n-1. \\ x_0 & \text{if } i = n \end{cases}$$

and repeat the previous argument. This finishes the first part of the lemma.

For any $n \in \mathbb{Z}^+$ and $x \in X(n)$, following what we just proved

$$\begin{aligned} \sum_{i=0}^{n-1} a_i(x) &\geq \phi(d(x,U)) - C|x_n - x_0| \\ &\geq \phi(0) - C|x_n - x_0| \\ &\geq -C|x_n - x_0| \geq -C(u_1 - u_0) =: B \end{aligned}$$

We are done. □

From Lemma 3.1.4, we know $\sum_{-n}^{n} a_i(x)$ has a finite lower bound B independent of n and x, however generally we can not expect B to be non-negative.

Proposition 3.1.5. *J is a well-defined function from X to $\mathbb{R} \cup \{+\infty\}$, i.e., $J(x)$ is either finite or diverges to infinity. Moreover, if $x \in X$ and $J(x) < +\infty$, then $x(\pm\infty) \in \{u_0, u_1\}$.*

A similar proposition as above was proved in [40], their idea works equally well in our case, we give the proof in Section 3.5 for the sake of completeness. This proposition gives us a convenient way to determine the asymptotic behaviors of a configuration, which makes it extremely useful in this paper.

As we said before, generally we can not except the lower bound of J to be non-negative. However if we restrict ourselves to a class of homoclinic configurations, it does have a non-negative lower bound, as shown by the following lemma.

Lemma 3.1.6. *If $x \in X$ with $x(+\infty) = x(-\infty) = u_0$ (or $x(+\infty) = x(-\infty) = u_1$) and $x_i \neq u_0$ (or $x_i \neq u_1$), for some $i \in \mathbb{Z}$ then $J(x) > 0$.*

Proof. First there is a $i_0 \in \mathbb{Z}$, such that $0 < \delta = |x_{i_0} - u_0| < \frac{u_1 - u_0}{2}$. For an large enough integer N, we have $|x_n - u_0| \leq \frac{\phi(\delta)}{2C}$ for all $|n| \geq N$. Therefore $|x_n - x_{-n}| \leq \frac{\phi(\delta)}{2C}$, for all $n > N$, by Lemma 3.1.4,

$$\sum_{i=-n}^{n-1} a_i(x) \geq \phi(\delta) - C|x_N - x_{-N}| \geq \frac{\phi(\delta)}{2} > 0.$$

Since this is true for any $n \geq N$, $J(x) > 0$.

□

In our proofs the multibump homoclinic and heteroclinic connections will be found as minimizers of J in some subsets of X, to make sure those minimizers are actually locally minimal configurations, we need to make sure each component of these minimizers does not equal to u_0 or u_1. Therefore the next two lemmas will be very useful.

Lemma 3.1.7. *For any $\delta \in (0, u_1 - u_0]$, if a configuration (x_0, x_1, x_2) with $x_i \in [u_0, u_1]$, for $i = 0, 1, 2$, satisfies*

1. *$x_0 \neq u_1$ or $x_2 \neq u_1$;*

2. *$x_1 \in [u_1 - \delta, u_1]$;*

3. *$h(x_0, x_1, x_2) \leq h(x_0, \xi, x_2)$ for any $\xi \in [u_1 - \delta, u_1]$.*

then $x_1 \neq u_1$.

The statement is still true if we replace every u_1 *by* u_0 *and every* $[u_1 - \delta, u_1]$ *by* $[u_0, u_0 + \delta]$.

Lemma 3.1.8. *If a finite configuration* $x = \{x_i\}_{i=n_0}^{n_1}$ *satisfies*

1. $x_i \in [u_0, u_1]$ *for all* $i = n_0, \ldots, n_1$;
2. $h(x_{n_0}, \cdots, x_{n_1}) \leq h(y_{n_0}, \cdots, y_{n_1})$, *for any* $\{y_i\}_{i=n_0}^{n_1}$ *with* $y_{n_0} = x_{n_0}, y_{n_1} = x_{n_1}$ *and* $y_i \in [u_0, u_1]$;

then x *is a minimal configuration.*

Furthermore if x *also satisfies* $x_{n_0} \notin \{u_0, u_1\}$ *or* $x_{n_1} \notin \{u_0, u_1\}$, *then* $x_i \notin \{u_0, u_1\}$ *for all* $i = n_0 + 1, \ldots, n_1 - 1$.

The proofs of the above two lemmas will be given in Section 3.6. We will finish this section by a comparison lemma, which will be used in several places of our proofs.

Definition 3.1.4. *For any* $j \in \{0, 1\}$ *and* $k \in \mathbb{Z}$, *we define the following operators,* $G_j^{\pm}(k) : X \to X$, *by*

$$(G_j^+(k)x)_i = \begin{cases} x_i & \textit{if } i \leq k, \\ u_j & \textit{if } i > k \end{cases}$$

and

$$(G_j^-(k)x)_i = \begin{cases} u_j & \textit{if } i < k, \\ x_i & \textit{if } i \geq k, \end{cases}$$

for any $x \in X$.

Lemma 3.1.9. *For any* $x \in X$,

$$|J(G_j^+(k)x) - \sum_{i=-\infty}^{k-1} a_i(x)| \leq C|u_j - x_k|;$$

$$|J(G_j^-(k)x) - \sum_{i=k}^{+\infty} a_i(x)| \leq C|u_j - x_k|.$$

It follows directly from Definition 3.1.4 and the Lipschitz continuity of h.

3.2 Minimal heteroclinic connections

In this section we will show the existence of minimal configurations that are heteroclinic from u_0 to u_1 and from u_1 to u_0. The result is not new. It is just an alternative proof of the fact that $\mathcal{M}_0^+(u_0, u_1)$ and $\mathcal{M}_0^-(u_0, u_1)$ are non-empty for a pair of ((0, 1)-periodic) neighboring minimal configurations. We include it here because it illuminates some of the ideas that will be needed in the next section.

The strategy here is to consider a class of configurations with the desired asymptotic behaviors. The existence of minimizers of J on such a class of configurations will be shown and we will prove such minimizers are minimal heteroclinic connections.

Definition 3.2.1.

$$X^0 := X^0(u_0, u_1) := \{x \in X : x(-\infty) = u_0 \textit{ and } x(+\infty) = u_1\};$$

$$c_0 := \inf\{J(x) : x \in X^0\}.$$

$$X^1 := X^1(u_0, u_1) := \{x \in X : x(-\infty) = u_1 \textit{ and } x(+\infty) = u_0\};$$

$$c_1 := \inf\{J(x) : x \in X^1\}.$$

Remark 3.2.1. *It is easy to see that c_0 and c_1 are finite.*

Definition 3.2.2.

$$\mathcal{M}^0 := \mathcal{M}^0(u_0, u_1) := \{x \in X^0 : J(x) = c_0\};$$

$$\mathcal{M}^1 := \mathcal{M}^1(u_0, u_1) := \{x \in X^1 : J(x) = c_1\}.$$

Theorem 3.2.1. *$\mathcal{M}^0$ (resp. $\mathcal{M}^1$) is a non-empty set. If $x \in \mathcal{M}^0$ (resp. $x \in \mathcal{M}^1$), then x is a minimal configuration, furthermore it is a heteroclinic connection from u_0 to u_1 (resp. from u_1 to u_0).*

Proof. Choose a minimizing sequence $x^n \in X^0$, i.e., $\lim_{n\to+\infty} J(x^n) = c_0$, and a $\delta \in (0, \frac{u_1-u_0}{2})$ small enough, since $J(x^n)$ are invariant under translation operators $T_k, \forall k \in \mathbb{Z}$, we may assume

$$\delta \le |x_1^n - u_0| \le \frac{u_1 - u_0}{2} \quad \text{and} \quad |x_i^n - u_0| \le \delta, \quad \forall i \le 0, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

By the compactness of X and lower semi-continuity of J, x^n (passing to a subsequence if necessary) converges to an $x \in X$ with $J(x) \leq c_0$. Then Proposition 3.1.5 tells us $x(\pm\infty) \in U$. Since every x^n satisfies (3.3), we must have $x(-\infty) = u_0$. To prove $x \in X^0$, we only need to show $x(+\infty) = u_1$.

Assume $x(+\infty) = u_0$, let $0 < \varepsilon \leq \phi(\delta)/4C$, then there is a N large enough such that $x_i \leq u_0 + \varepsilon/2$ for all $i \geq N$.

For n large enough, we have $x_N^n \leq u_0 + \varepsilon$. Because $x^n(-\infty) = u_0$, there are k_n small enough, such that $x_k^n \leq u_0 + \varepsilon$ for all $k \leq k_n$. Because $d((x_k^n, \dots, x_N^n), U) \geq |x_1^n - u_0| \geq \delta$, we have

$$\sum_{i=k}^{N-1} a_i(x^n) \geq \phi(\delta) - C|x_N^n - x_{k_n}^n| \geq \phi(\delta) - C\varepsilon,$$

As it is true for all $k \leq k_n$, we have

$$\sum_{i=-\infty}^{N-1} a_i(x^n) \geq \phi(\delta) - C\varepsilon.$$

Now define $\bar{x}^n = G_0^-(N)x^n$, obviously $\bar{x}^n \in X^0$, and from Lemma 3.1.9

$$\begin{aligned} J(\bar{x}^n) &\leq \sum_{i=N}^{+\infty} a_i(x^n) + C\varepsilon \\ &= J(x^n) - \sum_{i=-\infty}^{N-1} a_i(x^n) + C\varepsilon \\ &\leq J(x^n) - \phi(\delta) + C\varepsilon + C\varepsilon \\ &\leq J(x^n) - \phi(\delta)/2. \end{aligned}$$

Therefore $\liminf J(\bar{x}^n) \leq \liminf J(x^n) - \phi(\delta)/2 < c_0$, but this is absurd. So the assumption on $x(+\infty)$ is not true, and we have $x(+\infty) = u_1$. As a result, $x \in X^0$ and $J(x) \geq c_0$, so $J(x) = c_0$. Therefore $x \in \mathcal{M}^0$.

Although x is just a minimizer of J among configurations in X^0, Lemma 3.1.8 shows it is a minimal configuration. Furthermore Lemma 3.1.7 and Lemma 3.1.8 implies $x_i \notin U$ for any $i \in \mathbb{Z}$, so x is a heteroclinic connection from u_0 to u_1.

□

Theorem 3.2.1 tells us

$$\mathcal{M}^0 \subset \mathcal{M}_0^+(u_0, u_1) \text{ and } \mathcal{M}^1 \subset \mathcal{M}_0^-(u_0, u_1). \tag{3.4}$$

Right now we can not tell whether they are actually the same, even through we believe this should be the case.

Proposition 3.2.1. *If $x \in \mathcal{M}^0$ (resp. $x \in \mathcal{M}^1$), then x is strictly monotonically increasing (resp. decreasing), i.e., $x_i < x_{i+1}$ (resp. $x_i > x_{i+1}$) for all $i \in \mathbb{Z}$.*

Proof. First we will shown that $x_k = x_{k+1}$ will never happen for any $k \in \mathbb{Z}$, assume $x_k = x_{k+1}$ for some $k \in \mathbb{Z}$, from the proof of Theorem 3.2.1, $x_i \in (u_0, u_1)$ for any $i \in \mathbb{Z}$, so $a_i(x) = h(x_k, x_k) - c > 0$. Let $\bar{x} = (\dots, x_{k-1}, x_{k+1}, \dots)$, then $J(\bar{x}) = J(x) - (h(x_k, x_{k+1}) - c) < c_0$, a contradiction.

Next assume there is a $k \in \mathbb{Z}$ with $(x_k - x_{k-1})(x_k - x_{k+1}) > 0$, by condition H_3,

$$h(x_{k-1}, x_k) + h(x_k, x_{k+1}) > h(x_{k-1}, x_{k+1}) + h(x_k, x_k).$$

Let $\bar{x} = (\dots, x_{k-1}, x_{k+1}, \dots)$, then

$$c_0 = J(x) > J(\bar{x}) + h(x_k, x_k) - c \geq J(\bar{x}).$$

Obviously $\bar{x} \in X^0$ and we get a contradiction. Therefore x must be strictly monotonic and the asymptotic behaviors of x guarantee it is increasing. □

3.3 Multibump homoclinic and heteroclinic connections

In this section, we will prove Theorem 3.0.5, when $x^0 = u_0$ and $x^1 = u_1$, i.e., it is a pair of $(0, 1)$-periodic neighboring minimal configurations.

The proof is again based on a variational method. Minimizers of J on a class of configurations with desired asymptotic and oscillating behaviors will be found first. Then we will show that these minimizers are stationary configurations. A major difference between the result in this paper and the Aubry-Mather theory is that we will find locally minimal configurations rather than minimal configurations.

Definition 3.3.1. $I_j := \{x_0 : x \in \mathcal{M}^j\}$, *for* $j = 0, 1$.

If we replace x^0, x^1 and x_0^0, x_0^1 in Theorem 3.0.5 by u_0, u_1, then (3.4) tells us (gap) implies

$$I_0 \neq (u_0, u_1) \text{ and } I_1 \neq (u_0, u_1) \tag{$*$}$$

We assume $(*)$ is true for the rest of this section

Proposition 3.3.1. *For any* $\hat{\delta} > 0$*, there are* $\delta_i \in (0, \hat{\delta})$*,* $i = 0, \dots, 3$ *and* $e_0 = e_0(\delta_0, \delta_2), e_1 = e_1(\delta_1, \delta_3) > 0$ *such that,*

$$\inf\{J(x) : x \in X^0, x_0 = u_0 + \delta_0, \text{ or } x_0 = u_1 - \delta_2\} \geq c_0 + e_0$$

$$\inf\{J(x) : x \in X^1, x_0 = u_1 - \delta_1, \text{ or } x_0 = u_0 + \delta_3\} \geq c_1 + e_1$$

Proof. Details will be given on how to find δ_0, δ_2 and e_0, while the others are similar.

By $(*)$ there is a $u \in (u_0, u_1) \setminus I_0$. From the ordering structure of $\mathcal{M}^0$ inherited from $\mathcal{M}_0^+(u_0, u_1)$, there is a pair of minimal configurations $y < z \in \mathcal{M}^0$ with $u \in (y_0, z_0)$, such that no other element from $\mathcal{M}^0$ lies in between them.

By Proposition 3.2.1, it is not hard to see, there is an $n_0 \in \mathbb{Z}^-$ small enough such that

$$u_0 < y_{n_0} < z_{n_0} < u_0 + \hat{\delta}.$$

Choose a $\delta_0 \in (y_{n_0}, z_{n_0})$. Obviously $\delta_0 \in (u_0, u_1) \setminus I_0$. We set

$$Y_{n_0}^0 = \{x \in X^0 : x_{n_0} = u_0 + \delta_0\} \text{ and } b_0 = \inf\{J(x) : x \in Y_{n_0}^0\}.$$

Obviously b_0 is finite and $\delta_0 \in (u_0, u_1) \setminus I_0$ implies

$$b_0 > c_0. \tag{3.5}$$

Let $Y_0^0 = \{x \in X_0 : x_0 = u_0 + \delta_0\}$. Because $J(T_{n_0} x) = J(x)$,

$$\inf\{J(x) : x \in Y_0^0\} = b_0 > c_0.$$

Similarly we can find an $n_1 \in \mathbb{Z}^+$ large enough such that $u_1 - \hat{\delta} < y_{n_1} < z_{n_1} < u_1$. Then there is a $\delta_2 \in (y_{n_1}, z_{n_1})$, which satisfies

$$b_2 = \{J(x) : x \in X_0, x_0 = u_1 - \delta_2\} > c_0.$$

Let $e_0 = \min\{b_0 - c_0, b_2 - c_0\}$, we get the desired result. □

For the rest of this section, we choose a $0 < \bar{\delta} < (u_1 - u_0)/2$ and set

$$\varepsilon^* := \min\{\rho(\xi, \zeta) : \xi, \zeta \in [u_0, u_1]\}; \tag{3.6}$$

$$\bar{\varepsilon} := \min\{\phi(\bar{\delta}), \quad \varepsilon^*(u_1 - u_0 - 2\bar{\delta})^2\}. \tag{3.7}$$

Obviously both ε^* and $\bar{\varepsilon}$ are strictly positive.

From now on we require $\hat{\delta}$ in Proposition 3.3.1 satisfying

$$0 < \hat{\delta} < \bar{\delta} \text{ and } \hat{\delta} \leq \frac{1}{4C}\bar{\varepsilon}. \tag{3.8}$$

Definition 3.3.2. *For an $m \in \mathbb{N}$, we set*

$$Z_0 := \{x \in X : x_{-m} \leq u_0 + \delta_0, x_m \geq u_1 - \delta_2\};$$

$$Z_1 := \{x \in X : x_{-m} \geq u_1 - \delta_1, x_m \leq u_0 + \delta_3\}.$$

Obviously Z_0, Z_1 depends on the choice of m, however an m large enough will be chosen and fixed in all our results. The requirements will be specified later, right now we just say m is large enough, so that $\mathcal{M}^0 \cap Z_0$ and $\mathcal{M}^1 \cap Z_1$ are non-empty. We let $y^0 \in \mathcal{M}^0 \cap Z_0$ and $y^1 \in \mathcal{M}^1 \cap Z_1$ be two fixed minimal configurations for the rest of this section.

Remark 3.3.1. *For the sake of simplicity, from now on we make the following agreement that when we label c, e, u, y, Z and $G^{\pm}$ by an index, that index is considered mod 2.*

Definition 3.3.3. *Let $q = \{q_i\}_{i=-\infty}^{+\infty}$ be a bi-infinite sequence of integers, which satisfies $q_{i+1} - q_i \geq 4m$ for all $i \in \mathbb{Z}$. Furthermore, for any pair of integers $j \leq k$, we set*

$$Z(j,k) := \{x \in X : x(-\infty) = u_j, x(+\infty) = u_{k+1} \text{ and } T_{q_l}x \in Z_l \text{ for any } j \leq l \leq k\},$$

and,

$$c(j,k) := \inf\{J(x) : x \in Z(j,k)\}.$$

The following figure shows what the Aubry graph of an $x \in Z(0,2)$ should look like.

The $Z(j,k)$'s are the classes of configurations that have the desired asymptotic and oscillating behaviors. It is not hard to see every $c(j,k)$ is a finite number. We will show $c(j,k)$ is a minimum instead of infimum and configurations achieve this minimum will not bump up against the constraints we posted in Definition 3.3.3, therefore they are stationary configurations.

Proposition 3.3.2. *Let q be a bi-infinite sequence of integers defined as above, then for any pair of integers $j \leq k$, there is a $x^{jk} \in Z(j,k)$ with $J(x^{jk}) = c(j,k)$.*

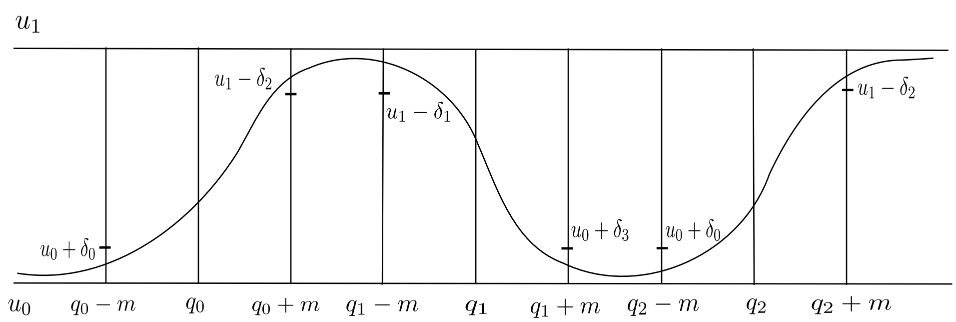

Figure 3.1: Monotone twist map 1

Proof. The proposition is trivial when $j = k$, since in this case $c(j,k) = c_j$ and $T_{-q_j}y^j \in Z(j,k)$ with $J(T_{-q_j}y^j) = J(y^j) = c_j$.

When $j < k$, we choose a minimizing sequence of configuration $\{x^n\}$ from $Z(j,k)$, such that $\lim_{n\to+\infty} J(x^n) = c(j,k)$. Similar to the argument in the proof of Theorem 3.2.1, we may assume x^n converges to a $x \in X$ then

$$J(x) \leq \liminf_{n\to+\infty} J(x^n) = c(j,k).$$

It is easy to see $T_{q_l}x \in Z_l$ for $j \leq l \leq k$, since $T_{q_l}x^n \in Z_l$ and x_i^n converges to x_i for all $i \in \mathbb{Z}$. To prove $x \in Z(j,k)$, we need to show that $x(-\infty) = u_j$ and $x(+\infty) = u_{k+1}$. Because $J(x)$ is finite. Proposition 3.1.5 implies $x(\pm\infty) \in U$. Therefore it is enough to show that $x(-\infty) \neq u_{j+1}$ and $x(+\infty) \neq u_k$. We will give the detailed proof for the case $x(+\infty) \neq u_k$ when k is even, the other cases can be proved similarly.

Assume $x(+\infty) = u_k = u_0$, then there is an N large enough ($N > q_k + m$), such that

$$x_N \leq u_0 + \delta_2/2,$$

where δ_2 is the same as defined in Proposition 3.3.1. Therefore when n is large enough,

$$x_N^n \leq u_0 + \delta_2.$$

On the other hand $x^n(+\infty) = u_{k+1} = u_1$, as $x^n \in Z(j,k)$, and therefore for each x^n there is a $p_n > N$ large enough such that $x_p^n \geq u_1 - \delta_2$ for all $p \geq p_n$.

Now for each $\{x_i^n\}_{i=q_k+m}^p$, there are two possibilities:

Case 1: there is a $j \in [q_k + m, p] \cap \mathbb{Z}$, such that $x_j^n \in [u_0 + \bar{\delta}, u_1 - \bar{\delta}]$. Then $d(\{x_i^n\}_{i=q_k+m}^p, U) \geq \bar{\delta}$, by Lemma 3.1.4

$$\sum_{i=q_k+m}^{p-1} a_i(x^n) \geq \phi(\bar{\delta}) - C|x_p^n - x_{q_k+m}^n| \geq \phi(\bar{\delta}) - C\delta_2, \tag{3.9}$$

the second inequality is because of $x_p^n, x_{q_k+m}^n \in [u_1 - \delta_2, u_1]$.

Case 2: For any $j \in [q_k + m, p] \cap \mathbb{Z}$, $x_j^n \notin [u_0 + \bar{\delta}, u_1 - \bar{\delta}]$. Because we have $x_{q_k+m}^n \in [u_1 - \delta_2, u_1], x_N^n \in [u_0, u_0 + \delta_2]$ and $x_p^n \in [u_1 - \delta_2, u_1]$, then there must be two integers $j_0 \leq j_1 \in [q_k + m + 1, p - 1] \cap \mathbb{Z}$ satisfying

$$x_{j_0}^n \in [u_1 - \bar{\delta}, u_1], x_{j_0+1}^n \in [u_0, u_0 + \bar{\delta}],$$

$$x_{j_1}^n \in [u_0, u_0 + \bar{\delta}], x_{j_1+1}^n \in [u_1 - \bar{\delta}, u_1].$$

By condition H_5,

$$\begin{aligned} h(x_{j_0}^n, x_{j_0+1}^n) + h(x_{j_1}^n, x_{j_1+1}^n) &\geq h(x_{j_1}^n, x_{j_0+1}^n) + h(x_{j_0}^n, x_{j_1+1}^n) + \int_{x_{j_1}^n}^{x_{j_0}^n} \int_{x_{j_0+1}^n}^{x^{j_1+1}} \rho \\ &. \geq h(x_{j_1}^n, x_{j_0+1}^n) + h(x_{j_0}^n, x_{j_1+1}^n) + \varepsilon^*(u_1 - u_0 - 2\bar{\delta})^2 \end{aligned} \tag{3.10}$$

At the same time

$$\begin{aligned} \sum_{i=q_k+m}^{p-1} a_i(x^n) &= h(x_{q_k+m}^n, \ldots, x_p^n) - (p - q_k - m)c \\ &= h(x_{q_k+m}^n, \ldots, x_{j_0}^n) + h(x_{j_0+1}^n, \ldots, x_{j_1}^n) + h(x_{j_1+1}^n, \ldots, x_p^n) \\ &\quad - (p - q_k - m)c + h(x_{j_0}^n, x_{j_0+1}^n) + h(x_{j_1}^n, x_{j_1+1}^n). \end{aligned} \tag{3.11}$$

Combine (3.10) and (3.11),

$$\begin{aligned} \sum_{i=q_k+m}^{p-1} a_i(x^n) \geq\ & h(x_{q_k+m}^n, \ldots, x_{j_0}^n) + h(x_{j_0}^n, x_{j_1+1}^n) + h(x_{j_1+1}^n, \ldots, x_p^n) \\ & + h(x_{j_1}^n, x_{j_0+1}^n) + h(x_{j_0+1}^n, \ldots, x_{j_1}^n) - (p - q_k - m)c \\ & + \varepsilon^*(u_1 - u_0 - 2\bar{\delta})^2 \end{aligned} \tag{3.12}$$

Following Lemma 3.1.3,

$$h(x_{j_1}^n, x_{j_0+1}^n) + h(x_{j_0+1}^n, \ldots, x_{j_1}^n) - (j_1 - j_0)c \geq \phi(0) \geq 0,$$

and Lemma 3.1.4,

$$h(x^n_{q_k+m},\dots,x^n_{j_0})+h(x^n_{j_0},x^n_{j_1+1})+h(x^n_{j_1+1},\dots,x^n_p)-(p-q_k-m-j_1+j_0)c$$
$$\geq \phi(0)-C\delta_2\geq -C\delta_2,$$

therefore,

$$\sum_{i=q_k+m}^{p-1} a_i(x^n)\geq \varepsilon^*(u_1-u_0-2\bar{\delta})^2-C\delta_2. \tag{3.13}$$

This finishes our discussion of the two possibilities of $\{x^n_i\}^p_{i=q_k+m}$.

Recall that $\bar{\varepsilon}:=\min\{\phi(\bar{\delta}),\varepsilon^*(u_1-u_0-2\bar{\delta})^2\}>0$, following (3.9) and (3.13) we get

$$\sum_{i=q_k+m}^{p-1} a_i(x^n)\geq \bar{\varepsilon}-C\delta_2.$$

Since this is true for all $p\geq p_n$,

$$\sum_{i=q_k+m}^{+\infty} a_i(x^n)\geq \bar{\varepsilon}-C\delta_2.$$

Now consider the sequence $\{G^+_1(q_k+m)x^n : n\in\mathbb{N}\}$, obviously it belongs to $Z(j,k)$, and by Lemma 3.1.9,

$$\begin{aligned}
J(G^+_1(q_k+m)x^n) &\leq J(x^n)-\sum_{i=q_k+m}^{+\infty} a_i(x^n)+C\delta_2\\
&\leq J(x^n)-\bar{\varepsilon}+C\delta_2+C\delta_2\\
&\leq J(x^n)-\bar{\varepsilon}+2C\hat{\delta}\\
&\leq J(x^n)-\bar{\varepsilon}/2,
\end{aligned} \tag{3.14}$$

the last inequality is because of the assumption on $\hat{\delta}$. Hence,

$$\liminf_{n\to+\infty} J(G^+_1(q_k+m)x^n)\leq \liminf_{n\to+\infty} J(x^n)-\bar{\varepsilon}/2\leq c(j,k)-\bar{\varepsilon}/2<c(j,k),$$

which is a contradiction to the definition of $c(j,k)$.

□

Use the same notation as in Proposition 3.3.2 in the next two lemmas.

Lemma 3.3.3. *Every $x^{jk} \in Z(j,k)$ with $J(x^{jk}) = c(j,k)$ satisfies $x^{jk} \notin \{u_0, u_1\}$ for all $i \in \mathbb{Z}$.*

Proof. This lemma is a immediate consequence of Lemma 3.1.7 and 3.1.8, once the readers notice that $x^{jk} \neq u_0$ and $x^{jk} \neq u_1$ as configurations. □

Lemma 3.3.4. *Every $x^{jk} \in Z(j,k)$ with $J(x^{jk}) = c(j,k)$ satisfies*

$$x_{i-1} < x_i, \quad \forall i \leq q_j - m, \ \textit{if } j(mod2) = 0;$$

$$x_{i-1} > x_i, \quad \forall i \leq q_j - m, \ \textit{if } j(mod2) = 1;$$

$$x_{i+1} > x_i, \quad \forall i \geq q_k + m, \ \textit{if } k(mod2) = 0;$$

$$x_{x+1} < x_i, \quad \forall i \geq q_k + m, \ \textit{if } k(mod2) = 1.$$

Proof. The proof is essentially the same as the proof of Proposition 3.2.1. □

To show that x^{jk}'s are stationary configurations, we need to specify the requirements on m. First, we choose a $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{\min\{e_1, e_2\}}{4C}, \tag{3.15}$$

for the rest of this section.

Definition 3.3.4. *We set m used in Definition 3.3.3 large enough to satisfy the following inequalities :*

$$m \geq \frac{2C\hat{\delta}}{\phi(\varepsilon)}, \tag{3.16}$$

$$y^0_{-2m} < u_0 + \delta_3 \quad \textit{and} \quad y^0_{2m} > u_1 - \delta_1, \tag{3.17}$$

$$y^1_{-2m} > u_1 - \delta_2 \quad \textit{and} \quad y^1_{2m} < u_0 + \delta_0. \tag{3.18}$$

We will fix an $m \in \mathbb{Z}^+$, which satisfies the conditions in Definition 3.3.4 from now on.

Lemma 3.3.5. *Let $q = \{q_i\}_{i=-\infty}^{+\infty}$ be a bi-infinite sequence of integers as defined in Definition 3.3.3, we have $c(j,k) < c(j,l) + c(l+1,k)$, for any $j \leq l < k$.*

Proof. We will give the detailed proof for the case l is odd, the other is similar.

Let $x \in Z(j,l)$ and $y \in Z(l+1,k)$ be two minimizers, i.e., $J(x) = c(j,l)$ and $J(y) = c(l+1,k)$, first we will show that

$$x_{q_{l+1}-m} < u_0 + \delta_0, \quad y_{q_l+m} < u_0 + \delta_3. \tag{3.19}$$

If $l = j$, simply let $x = T_{-q_l}y^1$, then by the assumption on m,

$$x_{q_{l+1}-m} < x_{q_{l+2m}} = y^1_{2m} < u_0 + \delta_0.$$

If $j < l < k$, let $\bar{x} = T_{-q_l}y^1$, then $\bar{x}_{q_{l-1}-m}$ is close to u_1, while $x_{q_{l-1}-m}$ is close to u_0, in a word,

$$\bar{x}_{q_{l-1}-m} > x_{q_{l-1}-m}. \tag{3.20}$$

At the same time, by (3.18),

$$\bar{x}_{q_{l+1}-m} < u_0 + \delta_0.$$

If the first inequalities in (3.19) is not true, then

$$x_{q_{l+1}-m} > \bar{x}_{q_{l+1}-m}. \tag{3.21}$$

Because of (3.20) and (3.21), we know $(x_{q_{l-1}-m}, \ldots, x_{q_{l+1}-m})$ and $(\bar{x}_{q_{l-1}-m}, \ldots, \bar{x}_{q_{l+1}-m})$ have at least one intersection. See the following figure where $Au(x)$ denoted by a solid curve and $Au(\bar{x})$ denoted by a dashed curve.

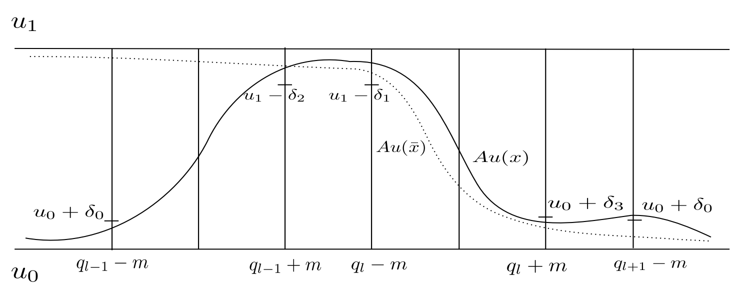

Figure 3.2: Monotone twist map 2

Let

$$x^+ = \{x_i^+\}_{i=-\infty}^{+\infty}, \quad \text{where } x_i^+ = \max\{x_i, \bar{x}_i\}, \forall i \in \mathbb{Z},$$

$$x^- = \{x_i^-\}_{i=-\infty}^{+\infty}, \quad \text{where } x_i^- = \min\{x_i, \bar{x}_i\}, \forall i \in \mathbb{Z},$$

then a simple application of condition H_5 gives

$$J(x^-) + J(x^+) \leq J(x) + J(\bar{x}).$$

It is easy to tell $x^- \in Z(j,l)$ and $T_{q_l} x^+ \in Z(l,l)$, therefore

$$J(x^-) \leq J(x^-) + J(x^+) - c_1 \leq J(x) + J(\bar{x}) - c_1 = c(j,l).$$

So if x doesn't satisfy the desired inequality we can replace it by x^-, in a similar way we can show that $y_{q_l+m} < u_0 + \delta_3$.

With x and y satisfy the desired inequalities, by Lemma 3.3.4 they must intersect at least once in between $q_l - m$ and $q_{l+1} + m$, as indicated in the following Figure with $Au(x)$ denoted by solid curve and $Au(y)$ denoted by dashed curve.

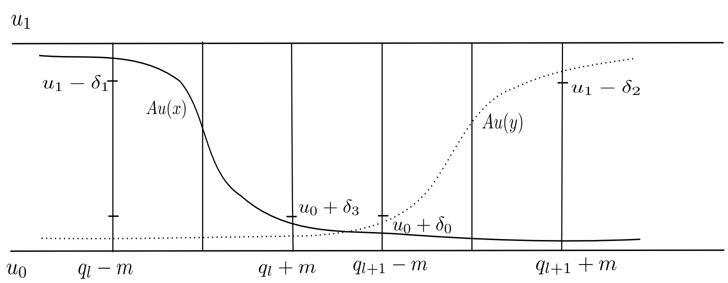

Figure 3.3: Monotone twist map 3

Let

$$z^+ = \{z_i^+\}_{i=-\infty}^{+\infty}, \quad \text{where } z_i^+ = \max\{x_i, y_i\}, \forall i \in \mathbb{Z},$$

$$z^- = \{z_i^-\}_{i=-\infty}^{+\infty}, \quad \text{where } z_i^- = \min\{x_i, y_i\}, \forall i \in \mathbb{Z},$$

it is not hard to see $z^+ \in Z(j,k)$ and $z^- \in X$ with $z^-(\pm\infty) = u_0$. Lemma 3.3.3 and Lemma 3.1.6 together imply $J(z^-) > 0$. Then

$$c(j,k) \leq J(z^+) < J(z^+) + J(z^-) \leq J(x) + J(y) = c(j,l) + c(l+1,k).$$

$\square$

Now we are ready to prove that those x^{jk}'s are stationary.

Theorem 3.3.1. *Under the assumption* $*$, *let* $\hat{\delta}, \varepsilon$ *satisfying conditions* (3.8), (3.15) *separately, and* q *be a bi-infinite sequence of integers with* $q_{i+1} - q_i \geq 4m, \forall i \in \mathbb{Z}$, *then* *for any pair of integers $j < k$, if $x^{jk} \in Z(j,k)$ satisfies $J(x^{jk}) = c(j,k)$, it is a stationary configuration, and therefore a homoclinic or heteroclinic connection from u_j to u_{k+1}.*

Proof. For the sake of simplicity, we set $x = x^{jk}$. By Lemma 3.3.3, $x_i \notin \{u_0, u_1\}$, so it is enough to show that for $l = j, \dots, k$,

$$x_{q_l - m} < u_0 + \delta_0, x_{q_l+m} > u_1 - \delta_2, \text{ for } l \text{ even}, \tag{3.22}$$

$$x_{q_l - m} > u_1 - \delta_1, x_{q_l+m} < u_0 + \delta_3, \text{ for } l \text{ odd}. \tag{3.23}$$

Let us define a finite configuration $x^* = \{x_i^*\}_{i=q_l+m}^{q_{l+1}-m}$, by

$$x_i^* = x_i, \text{ if } i = q_l + m, q_{l+1} - m$$
$$x_i^* = u_{l+1}, \text{ if } q_l + m < i < q_{l+1} - m$$

Because of the minimality of x,

$$\sum_{i=q_l+m}^{q_{l+1}-m-1} a_i(x) \le \sum_{i=q_l+m}^{q_{l+1}-m-1} a_i(x^*) \le 2C\hat{\delta}. \tag{3.24}$$

Let $I_l = [q_l + m, q_{l+1} - m] \cap \mathbb{Z}$, we claim that there is a $p_l \in I_l$, such that

$$|x_{p_l} - u_{l+1}| < \varepsilon, \text{ for all } l = j, \dots, k-1. \tag{3.25}$$

Before we prove the above claim, first we will show that $|x_i - u_{l+1}|$ is uniformly bounded by $\bar{\delta}$, i.e.,

$$|x_i - u_{l+1}| \le \bar{\delta}, \quad \forall i \in I_l, \quad \forall l = j, \dots, k-1. \tag{3.26}$$

We notice that there is no x_i satisfying $\min\{|x_i - u_l|, |x_i - u_{l+1}|\} \ge \bar{\delta}$, otherwise $d(\{x_i\}_{i \in I_l}, U) \ge \bar{\delta}$, then by Lemma 3.1.4

$$\begin{aligned} \sum_{i=q_l+m}^{q_{l+1}-m-1} a_i(x) &\ge \phi(\bar{\delta}) - C\hat{\delta} \\ &\ge \bar{\varepsilon} - C\hat{\delta} \\ &\ge 4C\hat{\delta} - C\hat{\delta} > 2C\hat{\delta}, \end{aligned}$$

the second and third inequalities follows from the requirements (3.7), (3.8) on $\bar{\varepsilon}$ and $\hat{\delta}$. This is a contradiction to (3.24).

Now we will show there is no x_i satisfying $|x_i - u_l| \leq \bar{\delta}$. If this is not true, because of $|u_{q_l+m} - u_{l+1}| \leq \hat{\delta}, |u_{q_{l+1}-m} - u_{l+1}| \leq \hat{\delta}$ and what we just showed, there must be two integers $j_0 < j_1 \in I_l$, such that

$$|x_{j_0} - u_{l+1}| \leq \bar{\delta} \text{ and } |x_{j_0+1} - u_l| \leq \bar{\delta},$$

$$|x_{j_1} - u_l| \leq \bar{\delta} \text{ and } |x_{j_1+1} - u_{l+1}| \leq \bar{\delta}.$$

By the same argument as in the proof of Proposition 3.3.2, we have

$$\begin{aligned}\sum_{i=q_l+m}^{q_{l+1}-m-1} a_i(x) &\geq \varepsilon^*(u_1 - u_0 - 2\bar{\delta})^2 - C\hat{\delta} \\ &\geq \bar{\varepsilon} - C\hat{\delta} \\ &\geq 4C\hat{\delta} - C\hat{\delta} > 2C\hat{\delta},\end{aligned}$$

this violates inequality (3.24).

Hence we showed (3.26) is true. Since $0 < \bar{\delta} < \frac{u_1 - u_0}{2}$, we have $\min\{|x_i - u_l|, |x_i - u_{l+1}|\} = |x_i - u_{l+1}|$, for any $i \in I_l$ and $l = j, \ldots, k$.

Now we are ready to prove our claim. For an arbitrary $l = j, \ldots, k-1$, we assume $|x_i - u_{l+1}| \geq \varepsilon$, for all $i \in I_l$ and set a finite configuration $\bar{x} = \{\bar{x}_i\}_{i=q_l+m-1}^{q_{l+1}-m+1}$ as

$$\begin{aligned}\bar{x}_i &= u_{l+1}, \text{ if } i = q_l + m - 1 \text{ or } q_{l+1} - m + 1; \\ \bar{x}_i &= x_i, \text{ if } i \in I_l.\end{aligned}$$

Because $\bar{x}$ belongs to $\hat{X}(q_{l+1} - q_l - 2m + 2; U)$, by Lemma 3.1.3,

$$\sum_{i=q_L+m-1}^{q_{l+1}-m+1} a_i(\bar{x}) \geq (q_{l+1} - q_l - 2m + 2)\phi(\varepsilon),$$

then together with (3.24),

$$\begin{aligned} 2C\hat{\delta} &\geq \sum_{i=q_l+m}^{q_{l+1}-m-} a_i(x) \\ &\geq \sum_{i=q_l+m-1}^{q_{l+1}-m} a_i(\bar{x}) - 2C\hat{\delta} \\ &\geq (q_{l+1} - q_l - 2m + 2)\phi(\varepsilon) - 2C\hat{\delta} \\ &> 2m\phi(\varepsilon) - 2C\hat{\delta}, \end{aligned}$$

so $m < 2C\hat{\delta}/\phi(\varepsilon)$, which violates the assumption (3.16) on m. Therefore we have shown, for any $l = j, \ldots, k-1$, there is at least one $p_l \in I_l$ satisfying (3.25).

Finally we are ready to prove the inequalities (3.22), (3.23). If $j < l < k$, let $x^+ = G_l^+(p_{l-1})x$, $x^- = G_{l+1}^-(p_l)x$ and $x' = G_l^-(p_{l-1}) \circ G_{l+1}^+(p_l)x$, it is not hard to see that $x^+ \in Z(j, l-1)$, $x^- \in Z(l+1, k)$ and $x' \in Z_l$.

Assume x violates one of the inequalities in (3.22) or (3.23), then x' also violates the same one. We give an illuminating graph for the case $l(mod2) = 0$ and $x_{q_l+m} = u_1 - \delta_2$, see the following figure, where $Au(x^+), Au(x^-)$ are denoted by solid curves and $Au(x')$ is denoted by dashed curve.

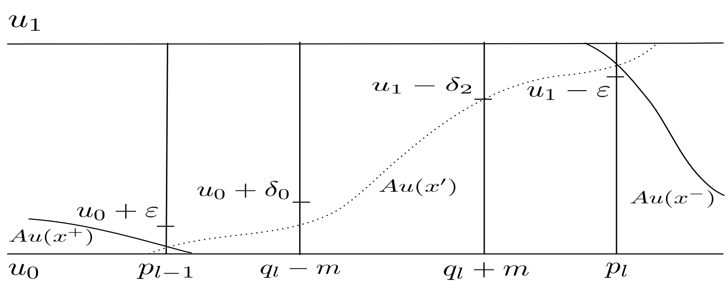

Figure 3.4: Monotone twist map 4

By Proposition 3.3.1, $J(x') \geq c_l + e_l$, then Lemma 3.1.9 and Lemma 3.3.5 imply

$$\begin{aligned} c_l + e_l \leq J(x') &\leq J(x) - J(x^+) - J(x^-) + 4C\varepsilon \\ &\leq c(j,k) - c(j,l-1) - c(l+1,k) + 4C\varepsilon \\ &< c(l,l) + 4C\varepsilon = c_l + 4C\varepsilon, \end{aligned}$$

which contradicts the requirement of ε in (3.15).

If $l = j$, we just choose p_{l-1} close enough to $-\infty$, such that $|x_{p_{l-1}} - u_l| < \varepsilon$, this is possible since for any $x \in Z(j,k)$, we have $x(-\infty) = u_j$. Now we just repeat the above argument with the only modification that $J(x^-) > 0$ because $x^- \in X$ and $x(\pm\infty) = u_j$. The proof for the case $l = k$ is similar. Therefore x satisfies the strict inequalities (3.22) and (3.23).

Therefore x is a locally minimal configuration. It means x is a homoclinic or heteroclinic connection from u_j to u_{k+1}. □

As a result we proved Theorem 3.0.5 when rotation number $\alpha = 0$ and $x^0 = u_0, x^1 = u_1$ as configurations.

In fact, Theorem 3.2.1 and 3.3.1 are still true if we assume $\{u_0, u_1\}$ is just a pair of local minimizers of $\bar{h}$ satisfies $\bar{h}(u) > \bar{h}(u_0) = \bar{h}(u_1)$ for all $u \in (u_0, u_1)$. Results about heteroclinic connection between local minimizers can be found in [42] and [44].

3.4 Generalization to Non-zero Rational Rotation Numbers

In this section we will generalize our result in the previous section to periodic neighboring minimal configurations with non-zero rational rotation numbers.

The basic idea is that given a variational principle h and a non-zero rational rotation number $\alpha = p/q \in \mathbb{Q}$. We can construct a new variational principle H, such that (p,q)-periodic minimal configurations of h are $(0,1)$-periodic minimal configurations of H. Then we will prove that every stationary configuration of H corresponds to a stationary configuration of h.

First a new operator called *conjunction* needs to be introduced before we can proceed. Given two variational principles h_1 and h_2, a conjunction of those two is a function $h_1 * h_2 : \mathbb{R}^2 \to \mathbb{R}$ with

$$h_1 * h_2(\xi, \xi') = \min_{\zeta \in \mathbb{R}} \{h_1(\xi, \zeta) + h_2(\zeta, \xi')\}.$$

Obviously $h_1 * h_2$ is well-defined and associative. In general, the conjunction of two C^∞ functions needs not to be C^1, however if h_1 and h_2 are variational principles, so is $h_1 * h_2$. All these were defined and proved by Mather in [7].

For any $\alpha = p/q \neq 0$ with p, q relatively prime and $q \in \mathbb{Z}^+$, we set $H(\xi, \xi') := h^{*q}(\xi, \xi' + p)$, where $h^{*q} = h * \cdots * h$ (q times) denotes the q-fold conjunction of h with itself. We choose an arbitrary pair of (p, q)-periodic neighboring minimal configurations, x^-, x^+, and fix it for the rest of this section. We define the following sets of configurations.

Definition 3.4.1.

$$X_\alpha := X_\alpha(x^-, x^+) := \{x = \{x_i\}_{i=-\infty}^{+\infty} : x_i \in [x_i^-, x_i^+], \forall i \in \mathbb{Z}\},$$

$$X_\alpha(q) := X_\alpha(q; x^-, x^+) := \{x = \{x_i\}_{i=0}^{q} : x_i \in [x_i^-, x_i^+], i = 0, \ldots, q\},$$

$$\hat{X}_\alpha(q) := \hat{X}_\alpha(q; x^-, x^+) := \{x \in X_\alpha(q) : x_q = x_0 + p\}.$$

Since $x^-, x^+ \in \mathcal{M}_\alpha^{per}$ is a pair of neighboring minimal configurations, it is not hard to see x_0^-, x_0^+ is a pair of neighboring minimizers of H. Otherwise there is a $\zeta \in (x_0^-, x_0^+)$ with

$$H(\zeta, \zeta) \leq H(x_0^-, x_0^-) = H(x_0^+, x_0^+),$$

then,

$$h^{*q}(\zeta, \zeta + p) \leq h^{*q}(x_0^-, x_0^- + p) = h^{*q}(x_0^+, x_0^+ + p),$$

violates the fact that x^-, x^+ is a pair of neighboring minimal configuration. Similarly it can be shown that every $\{x_i^-, x_i^+\}$, $i \in \mathbb{Z}$, is a pair of neighboring minimizers of H. Following Definition 3.1.2, we have a set of configurations $X(\{x_0^-, x_0^+\})$.

Definition 3.4.2. *For any configuration $y = \{y_i\}_{i=-\infty}^{+\infty} \in X(\{x_0^-, x_0^+\})$, we define a corresponding configuration $x = \{x_i\}_{i=-\infty}^{+\infty}$ as*

1. $x_{iq} = y_i + ip, \forall i \in \mathbb{Z}$;
2. $\{x_j\}_{j=iq}^{(i+1)q}$ *is a minimal configurations w.r.t. h, i.e.,*

$$h(x_{iq}, \cdots, x_{(i+1)q}) = H(x_{iq}, x_{(i+1)q}) = H(y_i, y_{i+1}), \ \forall i \in \mathbb{Z}.$$

Obviously $x \in X_\alpha(x^-, x^+)$.

Proposition 3.4.1. *Let $y \in X(\{x_0^-, x_0^+\})$ and $x \in X_\alpha(x^-, x^+)$ be defined as in Definition 3.4.2, if y is a stationary configuration of H, then x is a stationary configuration of h. Moreover if $y(\pm\infty) = x_0^\pm$, then $x(\pm\infty) = x^\pm$ correspondingly.*

It is not hard to see Theorem 3.0.5 follows immediately from Proposition 3.4.1 and Theorem 3.3.1. The rest of this section will be devoted to the proof of Proposition 3.4.1. The following definitions and technical lemmas will be needed.

Definition 3.4.3. *For any $x \in X_\alpha(q)$, we define*

$$d_\alpha(x) := d_\alpha(x, \{x^-, x^+\}) := \max_{0 \le i \le q} \min\{|x_i - x_i^-|, |x_i - x_i^+|\}.$$

We set $\lambda_i := \frac{1}{2}|x_i^+ - x_i^-|$ for $i = 0, \cdots, q$, and $\lambda := \min\{\lambda_i : i = 0, \cdots, q\}$.

Lemma 3.4.2. *We set $c_\alpha := h(x_0^-, \cdots, x_q^-) = h(x_0^+, \cdots, x_q^+)$. For any $0 \le \delta \le \lambda$, let*

$$\phi_\alpha(\delta) := \inf\{h(x_0, \cdots, x_q) - c_\alpha : x = \{x_i\}_{i=0}^q \in \hat{X}_\alpha(q), d_\alpha(x) \ge \delta\} \ge 0,$$

then ϕ_α is a monotonically increasing and continuous function of δ satisfying

$$\phi_\alpha(\delta) > 0, \text{ if } \delta > 0; \quad \phi_\alpha(\delta) = 0, \text{ if } \delta = 0.$$

The proof of this lemma is a simpler vision of the proof of Lemma 3.1.3, so we omit it here.

Lemma 3.4.3. *Let $C := Lip(h)$, then for any $0 \le \delta \le \lambda$, if $x = \{x_i\}_{i=0}^q \in X_\alpha(q)$ with $d_\alpha(x) \ge \delta$, we have*

$$h(x_0, \cdots, x_q) - c_\alpha \ge \phi_\alpha(\delta) - C|x_q - x_0 - p|.$$

Again the proof is similar to the proof of Lemma 3.1.4 and we omit it here.

Definition 3.4.4. *For $0 \le \varepsilon \le \frac{\phi_\alpha(\lambda)}{4C+1}$, we define*

$$\psi_\alpha(\varepsilon) := \inf\{0 \le \delta \le \lambda : \phi_\alpha(\delta) \ge (4C+1)\varepsilon\}.$$

By the continuity of ϕ_α,

$$\phi_\alpha(\psi_\alpha(\varepsilon)) \ge (4C+1)\varepsilon > 4C\varepsilon. \tag{3.27}$$

Because ϕ_α is a monotonically increasing continuous function, by simple calculation we see ψ_α is also a monotonically increasing continuous function w.r.t. ε and $\psi_\alpha(0) = 0$.

Let $y \in X(\{x_0^-, x_0^+\})$ and $x \in X_\alpha(x^-, x^+)$ be defined as in Definition 3.4.2, the key to the proof of Proposition 3.4.1 is to show that for any $i \in \mathbb{Z}$, $|x_j - x_j^\pm|, j = iq+1, \ldots, (i+1)q-1$ is controlled by $|x_i - x_i^\pm|$ and this can be done by the following two lemmas.

Lemma 3.4.4. *For $0 \le \varepsilon \le \frac{\phi_\alpha(\lambda)}{4C+1}$, if $x \in X_\alpha(q; x^-, x^+)$ is a minimal configuration of h, which satisfies $|x_i - x_i^+| \le \varepsilon, i = 0, q$ or $|x_i - x_i^-| \le \varepsilon, i = 0, q$, then $d_\alpha(x) < \psi_\alpha(\varepsilon)$ for $\psi_\alpha(\varepsilon)$ defined as above.*

Proof. We will give the detailed proof for the case $|x_i - x_i^+| \le \varepsilon, i = 0, q$, while the other is similar.

Assuming $d_\alpha(x) \ge \psi_\alpha(\varepsilon)$, then by Lemma 3.4.3,

$$h(x_0, \cdots, x_q) - c_\alpha \ge \phi_\alpha(\psi_\alpha(\varepsilon)) - C|x_q - x_0 - p| \ge \phi_\alpha(\psi_\alpha(\varepsilon)) - 2C\varepsilon. \tag{3.28}$$

The last inequality follows from

$$|x_q - (x_0 + p)| = |x_q - x_q^+ + x_0^+ + p - (x_0 + p)| \le |x_q - x_q^+| + |x_0^+ - x_0| \le 2\varepsilon.$$

Then the minimality of x and the Lipschitz continuity of h tell us

$$h(x_0, \cdots, x_q) \le h(x_0, x_1^+, \cdots, x_{q-1}^+, x_q) \le c_\alpha + 2C\varepsilon. \tag{3.29}$$

Combining (3.28) and (3.29), we have

$$\phi_\alpha(\psi_\alpha(\varepsilon)) \le 4C\varepsilon,$$

but this contradicts (3.27). Hence $d_\alpha(x) < \psi_\alpha(\varepsilon)$. □

Lemma 3.4.5. *There is a small enough $\varepsilon^* > 0$, such that if $x \in X_\alpha(q; x^-, x^+)$ is a minimal configuration and $|x_i - x_i^+| \le \varepsilon^*, i = 0, q$ (resp. $|x_i - x_i^-| \le \varepsilon^*, i = 0, q$), then $|x_i - x_i^+| < \lambda_i$, for all $i = 1, \cdots, q-1$ (resp. $|x_i - x_i^-| < \lambda_i$, for all $i = 1, \cdots, q-1$).*

Proof. We will only show the proof for the case $|x_i - x_i^+| \le \varepsilon^*, i = 0, q$. Assume the lemma is not true, then there is a sequence of positive numbers $\{\varepsilon_n\}_{n\in\mathbb{N}} \searrow 0$ and a sequence of minimal configurations $\{x^n\}_{n\in\mathbb{N}} \subset X_\alpha(q)$, such that,

$$|x_i^n - x_i^+| \le \varepsilon_n, i = 0, q, \quad \forall n \in \mathbb{N},$$

while there is a least on $i_n \in \{1, \ldots, q-1\}$ with

$$|x_{i_n}^n - x_{i_n}^+| \ge \lambda_{i_n} \quad \forall n \in \mathbb{N}.$$

Replacing $\{x^n\}$ by a subsequence if necessary, we may assume there is a fixed $j \in \{1, \ldots, q-1\}$, such that

$$|x_j^n - x_j^+| \geq \lambda_j, \quad \forall n \in \mathbb{N}. \tag{3.30}$$

Passing to a subsequence if necessary, x^n converges to an $x = \{x_i\}_{i=0}^q \in X_\alpha(q)$. Since every x^n is minimal, so is x.

Then $x_i = \lim_{n\to+\infty} x_i^n = x_i^+$, for $i = 0, q$. Because x is minimal, we must have $x_i = x_i^+$ for all $i = 0, \ldots, q$.

On the other hand, since $x_j = \lim_{n\to+\infty} x_j^n$, by (3.30), $|x_j - x_j^+| \geq \lambda_j$, which is a contradiction. So the assumption we made is incorrect and we are done.

□

Now we are ready to prove proposition 3.4.1

Proof. (**Proposition 3.4.1**) If $y \in X(\{x_0^-, x_0^+\})$ is a stationary configuration of H, i.e., locally minimal w.r.t. H, by the way $x \in X(x^-, x^+)$ is defined, it is not hard to see x is locally minimal w.r.t. h, so x is a stationary configuration of h.

By the monotonicity of ψ_α, for ε^* satisfying Lemma 3.4.5, we can find a $0 < \hat{\varepsilon} \leq \frac{\phi_\alpha(\lambda)}{4C+1}$, such that

$$0 < \psi_\alpha(\varepsilon) \leq \varepsilon^*, \text{ for } 0 < \varepsilon < \hat{\varepsilon}.$$

We will prove that $y(+\infty) = x_0^+$ implies $x(+\infty) = x^+$, the other cases are similar.

By the way x is defined, we have

$$\lim_{i\to+\infty} |x_{iq} - x_{iq}^+| = \lim_{i\to+\infty} |x_{iq} - (x_0^+ + ip)| = \lim_{i\to+\infty} |y_i - x_0^+| = 0.$$

Hence, for any $0 < \varepsilon < \hat{\varepsilon}$, there is a n_0 large enough, such that

$$|x_{iq} - x_{iq}^+| < \varepsilon, \quad i > n_0.$$

Because every $\{x_j\}_{j=iq}^{(i+1)q}$ is a minimal configuration of h, Lemma 3.4.4 tells us

$$d_\alpha(\{x_j\}_{j=iq}^{(i+1)q}) < \psi_\alpha(\varepsilon) \leq \varepsilon^*, \quad \forall i > n_0.$$

Then by Lemma 3.4.5

$$|x_j - x_j^+| < \lambda_{j(\text{mod} q)}, \text{ for } j = iq+1, \ldots, (i+1)q-1, \quad \forall i > n_0,$$

because of the periodicity of x^-, x^+, we have $\frac{|x_j^+ - x_j^-|}{2} = \lambda_{j(\text{mod}q)}$, for any $j \in \mathbb{Z}$. Therefore,

$$|x_j - x_j^+| < \psi_\alpha(\varepsilon), \text{ for } j = iq+1, \dots, (i+1)q-1, \quad \forall i > n_0.$$

Since $\psi(\varepsilon)$ goes to zero when ε goes to zero, we have $x(+\infty) = x^+$. □

3.5 Proof of Proposition 3.1.5

Before we can prove Proposition 3.1.5, one more lemma is needed

Lemma 3.5.1. *For all $x \in X$,*

$$\textit{if } \limsup_{n \to +\infty} \sum_{i=-n}^{n-1} a_i(x) = +\infty, \textit{ then } J(x) = +\infty,$$

in the sense that the series $J(x)$ is well defined and diverges to $+\infty$.

Proof. Assume the lemma is not true, then we have

$$\liminf_{n \to +\infty} \sum_{i=-n}^{n-1} a_i(x) = A_1 < +\infty.$$

Let $A_2 > A_1 + 1 - 2B$, where B is the constant defined in Lemma 3.1.4, there are two positive integers $n_0 < n_1$, such that $\sum_{i=-n_0}^{n_0-1} a_i(x) \geq A_2$ and $\sum_{i=-n_1}^{n_1-1} a_i(x) \leq A_1 + 1$.

Then

$$\sum_{i=-n_1}^{-n_0-1} a_i(x) + \sum_{i=n_0}^{n_1-1} a_i(x) = \sum_{i=-n_1}^{n_1-1} a_i(x) - \sum_{i=-n_0}^{n_0-1} a_i(x) \leq A_1 + 1 - A_2 < 2B,$$

which means at least one of the following is true,

$$\sum_{i=-n_0}^{n_0-1} a_i(x) < B \text{ or } \sum_{i=-n_1}^{n_1-1} a_i(x) < B.$$

This contradicts Lemma 3.1.4.

□

Now we are ready to prove Proposition 3.1.5.

Proof. (**Proposition 3.1.5**) We will prove the second part of the proposition first. For this it is enough to show that, for any $x \in X$, if x_i does not converges to u_0 or u_1 as $i \to \pm\infty$, then $J(x) = \infty$.

Details will be given for $i \to +\infty$, the other case is similar. Choose a proper $\delta > 0$, there is a sequence of positive integers $k_j \nearrow +\infty$ and $i_j \in [k_j, k_{j+1}) \cap \mathbb{Z}$ for all $j \in \mathbb{Z}^+$, such that $\lim_{j\to+\infty} |x_{k_j} - x_{k_{j+1}}| = 0$ and $\min\{|x_{i_j} - u_0|, |x_{i_j} - u_1|\} \geq \delta, \forall i \in \mathbb{Z}^+$. Replacing k_j by a subsequence if necessary, we may assume $|x_{k_{j+1}} - x_{k_j}| < \frac{\phi(\delta)}{2C}, \forall j \in \mathbb{Z}^+$. By Lemma 3.1.4,

$$\sum_{i=k_j}^{k_{j+1}-1} a_i(x) \geq \phi(\delta) - C|x_{k_{j+1}} - x_{k_j}| > \frac{\phi(\delta)}{2},$$

then for all $n \in \mathbb{Z}^+$,

$$\begin{aligned}\sum_{i=-k_n}^{k_n-1} a_i(x) &= B + \sum_{i=k_0}^{k_n-1} a_i(x) \\ &= B + \sum_{j=1}^{n-1} \sum_{i=k_j}^{k_{j+1}-1} a_i(x) \\ &= B + \frac{n\phi(\delta)}{2}.\end{aligned}$$

Therefore,

$$\limsup_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x) = +\infty,$$

and by Lemma 3.5.1, $J(x) = +\infty$. This finishes the second part of the proposition.

Now we will prove the first part of the proposition, it is enough to show that $J(x)$ is well-defined, i.e. it converges to a finite number or diverges to $+\infty$, if $x(\pm\infty) \in U$.

We will consider the case $x(-\infty) = u_0$ and $x(+\infty) = u_1$ (the other cases are similar), let

$$A_2 = \limsup_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x).$$

Let's assume $A_2 < +\infty$, since $J(x)$ diverges to $+\infty$ when $A_2 = +\infty$ by Lemma 3.5.1. Therefore we need to show that

$$A_1 := \liminf_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x) = A_2.$$

Assume this is not true, i.e. $A_1 < A_2$. Then there are two sequences of positive integers $l_j \nearrow +\infty$ and $n_j \nearrow +\infty$, satisfy $l_j + 1 < n_j < l_{j+1} - 1, \forall j \in \mathbb{Z}^+$ and

$$\lim_{j\to+\infty} \sum_{i=-l_j}^{l_j-1} a_i(x) = A_2 > \lim_{j\to+\infty} \sum_{i=-n_j}^{n_j-1} a_i(x) = A_1.$$

Hence for j large enough,

$$\sum_{i=-n_j}^{-l_j-1} a_i(x) + \sum_{i=l_j}^{n_j-1} a_i(x) = \sum_{i=-n_j}^{n_j-1} a_i(x) - \sum_{i=-l_j}^{l_j-1} a_i(x) < \frac{A_1 - A_2}{2} < 0. \tag{3.31}$$

On the other hand, by $x(-\infty) = u_0$, $|x_{-n_j} - x_{-l_j}| < \frac{|A_1 - A_2|}{4C}$ for j large enough, then following Lemma 3.1.4,

$$\sum_{i=-n_j}^{-l_j-1} a_i(x) \geq -C|x_{-n_j} - x_{-l_j}| > \frac{A_1 - A_2}{4}.$$

Similarly,

$$\sum_{i=l_j}^{n_j-1} a_i(x) \geq -C|x_{n_j} - x_{l_j}| > \frac{A_1 - A_2}{4}.$$

Hence for j large enough, we have

$$\sum_{i=-n_j}^{-l_j-1} a_i(x) + \sum_{i=l_j}^{n_j-1} a_i(x) > \frac{A_1 - A_2}{2},$$

which contradicts (3.31).

□

3.6 Proof of Lemma 3.1.7 and Lemma 3.1.8

Lemma 3.6.1. *(Aubry crossing lemma) Let* $x = \{x_i\}_{i=n_0}^{n_1}$ *and* $y = \{y_i\}_{i=n_0}^{n_1}$ *be two minimal configurations of a variational principle. Then* $Au(x) \cap Au(y)$ *contains at most two points. If it contains two points, then these are the endpoints of the two graphs, i.e.* $x_{n_0} = y_{n_0}$ *and* $x_{n_1} = y_{n_1}$.

This is a well-know result in Aubry-Mather theory, a proof can be found in [11] or [43] and it also works if x and y are locally minimal configurations.

Lemma 3.6.2. *If* $(\xi, \eta, \zeta) \neq (\xi', \eta, \zeta')$ *are two locally minimal configurations of a variational principle* h*, then*

$$(\xi - \xi')(\zeta - \zeta') < 0.$$

This is merely a version of condition H_4 for locally minimal configurations. As we said before, Mather showed condition H_4 was implied by conditions H_5 and H_6. The same argument also works for locally minimal configurations

Proof. (**Lemma 3.1.7**) We claim $h(x_0, x_1, x_2) \leq h(x_0, \xi, x_2)$ for any $\xi \in (u_1, +\infty)$. Assume this is not true, then there is a $\eta \in (u_1, +\infty)$ such that

$$h(x_0, \eta, x_2) < h(x_0, x_1, x_2) \leq h(x_0, u_1, x_2). \tag{3.32}$$

By condition H_3,

$$\begin{aligned} h(x_0, u_1) + h(u_1, \eta) + h(\eta, u_1) + h(u_1, x_2) \\ < h(x_0, \eta) + h(\eta, x_2) + h(u_1, u_1) + h(u_1, u_1). \end{aligned}$$

Combine this with (3.32),

$$h(u_1, \eta) + h(\eta, u_1) < h(u_1, u_1) + h(u_1, u_1),$$

which is absurd. Now both (x_0, x_1, x_2) and (u_0, u_1, u_2) are locally minimal configurations with $(x_0 - u_1)(x_2 - u_1) \geq 0$, if $x_1 = u_1$ it will contradict Lemma 3.6.2. Hence $x_1 \neq u_1$. $\square$

Proof. (**Lemma 3.1.8**) First it is easy to see there must be a minimal configuration $z = \{z_i\}_{i=n_0}^{n_1}$ with $z_{n_0} = x_{n_0}$ and $z_{n_1} = x_{n_1}$.

If $z_i \in [u_0, u_1]$ for all $n_0 < i < n_1$, then

$$h(x_{n_0}, \cdots, x_{n_1}) = h(z_{n_0}, \cdots, z_{n_1}),$$

and x is a minimal configuration.

Assume there is an $n_0 < i_0 < n_1$, such that $z_{i_0} \notin [u_0, u_1]$, then $Au(z)$ has at least two intersections with $Au(u_0)$ or $Au(u_1)$. When the two intersections are the end point of $Au(z)$, a basic result of Aubry-Mather theory (see [11]) tells us $z_i = u_0$ or u_1 for all $n_0 \leq i \leq n_1$, so $z_i \in [u_0, u_1]$, for $n_0 < i < n_1$. When the two intersections are not the end points of $Au(z)$, it contradicts *Aubry crossing lemma*, because z, u_0 and u_1 are minimal configurations. This finishes the first part of the lemma.

For the second part, without loss of generality, we assume $x_{n_0} \notin \{u_0, u_1\}$. By Lemma 3.1.7, $x_{n_0+1} \notin \{u_0, u_1\}$. Repeating this process, we have $x_i \notin \{u_0, u_1\}$ for all $n_0 < i < n_1$. $\square$

Chapter 4

Heteroclinic Orbits of Time-periodic Tonelli Lagrangian Systems

In this chapter we will study heteroclinic orbits in time-periodic Tonelli Lagrangian systems.

Let M be a closed and connected smooth Riemannian manifold and $L \in C^2(TM \times \mathbb{T}, \mathbb{R})$, recall the results from Chapter 2, for any closed one form η_c with $[\eta_c] = c \in H^1(M;\mathbb{R})$, all the c-static orbits form an Aubry set of cohomology class c: $\widetilde{\mathcal{A}}_c \subset TM \times \mathbb{T}$.

Furthermore with the nonnegative function $\widetilde{d}_{\eta_c}$, we can define an equivalence relation on $\widetilde{\mathcal{A}}_c$ which breaks it into several c-static classes. Under the assumption that the number of c-static classes is finite, in this chapter we will prove that between any two different c-static classes there is a chain of heteroclinic orbits between them. When certain non-degeneracy condition hold, which will be specified later, we will show there is actually a heteroclinic orbit connecting them.

In this chapter all the proofs and results will be given for 0-static classes, as the Aubry set of cohomology class c of a Lagrangian L is the same as the Aubry set of cohomology class 0 of the modified Lagrangian $L - \eta_c$ with $[\eta_c] = c$.

To further simplify the notations, we will add the $\alpha(0)$ to L, where α is a function from $H^1(M;\mathbb{R})$ to $\mathbb{R}$ defined in Chapter 2. Such a Tonelli Lagrangian will be called critical. As a result, $\alpha(0) = 0$ for a critical Tonelli Lagrangian.

Remark 4.0.1. *In the rest of this chapter, α will be used to denote a point in $\mathbb{T}$ rather than a function as in chapter 2.*

For any compact interval $[a,b] \subset \mathbb{R}$, we denote $\mathcal{C}([a,b],M)$ by the set of all absolutely continuous curves defined on $[a,b]$. The action A_L of L on any $\gamma \in \mathcal{C}([a,b],M)$ is defined by

$$A_L(\gamma) := \int_a^b L(d\gamma(t),t)dt, \text{ where } d\gamma(t) = (\gamma(t),\dot{\gamma}(t)).$$

Given $x, y \in M$, we set $\mathcal{C}_{[a,b]}(x,y) := \{\gamma \in \mathcal{C}([a,b],M) : \gamma(a) = x, \gamma(b) = y\}$ and $\mathcal{C}_T(x,y) := \mathcal{C}_{[0,T]}(x,y)$ for any $T > 0$. The extremals of A_L in $\mathcal{C}_{[a,b]}(x,y)$ are solutions of the Euler-Lagrange equation which in local coordinates is given by

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t),\dot{\gamma}(t),t) = \frac{\partial L}{\partial x}(\gamma(t),\dot{\gamma}(t),t). \tag{EL}$$

For any $(x,a),(y,b) \in M \times \mathbb{R}$ with $a < b$, we define

$$h((x,a),(y,b)) = \min\{\int_a^b L(d\gamma(t),t)dt : \gamma \in \mathcal{C}_{[a,b]}(x,y)\}.$$

For any $(x,\alpha),(y,\beta) \in M \times \mathbb{T}$, we define

$$h^\infty((x,\alpha),(y,\beta)) = \liminf_{T\in\mathbb{Z}^+, T\to+\infty} h((x,\alpha),(y,\beta+T)),$$

$$\Phi((x,\alpha),(y,\beta)) = \inf_{T\in\mathbb{Z}^+} h((x,\alpha),(y,\beta+T)),$$

$$\widetilde{d}((x,\alpha),(y,\beta)) = \Phi((x,\alpha),(y,\beta)) + \Phi((y,\beta),(x,\alpha)).$$

Obviously the above functions are the corresponding functions $h_{\eta_c}, h^\infty_{\eta_c}, \Phi_{\eta_c}$ and $\widetilde{d}_\eta$ with $\eta_c = 0$. Hence by Proposition 2.2.5, we have

Proposition 4.0.3. *Let functions h, h^∞, Φ and $\widetilde{d}$ be defined as above.*

1. *The function $h : M \times \mathbb{R} \times M \times \mathbb{R} \to \mathbb{R}; ((x,a),(y,b)) \to h((x,a),(y,b))$ is Lipschitz continuous and bounded on $\{b - a \geq 1\}$.*

2. *For any $\alpha, \beta \in \mathbb{T}$, $h^\infty((*,\alpha),(*,\beta)) : M \times M \to \mathbb{R}$ is Lipschitz continuous and satisfies triangle inequality*

$$h^\infty((x,\alpha),(y,\beta)) + h^\infty((y,\beta),(z,\rho)) \geq h^\infty((x,\alpha),(z,\rho)),$$

for any $(x,\alpha),(y,\beta),(z,\rho) \in M \times \mathbb{T}$.

3. $\widetilde{d}$ is non-negative.

In this chapter all the notations and results from Chapter 2 will be about the 0 cohomology class. Hence all the sub-index in the notations will be simply omitted. However for the seek of completeness and to avoid unnecessary confusion, we will reintroduce them in the following.

Let $I \subset \mathbb{R}$ be an interval of time, a curve $\gamma \in \mathcal{C}(I, M)$ is called *semi-static* if

$$A_L(\gamma|_{[a,b]}) = \Phi((\gamma(a), a \ mod\ 1), (\gamma(b), b \ mod\ 1)), \quad \forall [a, b] \subset I.$$

When $I = \mathbb{R}$, we say γ is a global semi-static curve. When $I = [a, +\infty)$ (or $I = (-\infty, a]$) for some $a \in \mathbb{R}$, we say γ is a forward (or backward) semi-static curve. Obviously a semi-static curve is a solution of (EL).

If γ satisfies

$$A_L(\gamma|_{[a,b]}) = -\Phi((\gamma(b), b \ mod\ 1), (\gamma(a), a \ mod\ 1)), \quad \forall [a, b] \subset I,$$

we say it is *static*. When $I = \mathbb{R}$, we say γ is a global static curve. By 4.0.3 it is not hard to see a static curve must be semi-static.

We define the Mañé set $\widetilde{\mathcal{N}} \subset TM \times \mathbb{T}$ and Aubry set $\widetilde{\mathcal{A}} \subset TM \times \mathbb{T}$ as

$$\widetilde{\mathcal{N}} := \{(d\gamma(t), t \ mod\ 1) : \gamma \in \mathcal{C}(\mathbb{R}, M) \text{ is global semi-static.}\};$$

$$\widetilde{\mathcal{A}} := \{(d\gamma(t), t \ mod\ 1) : \gamma \in \mathcal{C}(\mathbb{R}, M) \text{ is global static.}\}.$$

Similarly we define

$$\widetilde{\mathcal{N}}^+ := \{(d\gamma(t), t \ mod\ 1) : \gamma \text{ is forward semi-static. }\};$$

$$\widetilde{\mathcal{N}}^- := \{(d\gamma(t), t \ mod\ 1) : \gamma \text{ is backward semi-static. }\}.$$

It is easy to see

$$\widetilde{\mathcal{A}} \subset \widetilde{\mathcal{N}} \subset \widetilde{\mathcal{N}}^{\pm}.$$

Let $\pi : TM \times \mathbb{T} \to M \times \mathbb{T}$ be the usual projection, we set $\mathcal{N} = \pi(\widetilde{\mathcal{N}})$ and $\mathcal{A} = \pi(\widetilde{\mathcal{A}})$. The famous *Mather's graph theorem* (see [16]) tells us $\pi|_{\widetilde{\mathcal{A}}} : \widetilde{\mathcal{A}} \to \mathcal{A}$ is bijective and its inverse $(\pi|_{\widetilde{\mathcal{A}}})^{-1} : \mathcal{A} \to \widetilde{\mathcal{A}}$ is Lipschitz.

Following Mather's graph theorem, for any $(x, \alpha) \in \mathcal{A} \subset M \times \mathbb{T}$, there is a unique global static curve, denoted by $\gamma_{(x,\alpha)}$, satisfying $\gamma_{(x,\alpha)}(\alpha) = x$. Furthermore using

$\widetilde{d}$ defined in Proposition 4.0.3, we can define an equivalence relation on $\widetilde{\mathcal{A}}$ by saying $(x, v, a), (y, w, b) \in \widetilde{\mathcal{A}}$ are equivalent iff $\widetilde{d}(\pi(x, v, a), \pi(y, w, b)) = \widetilde{d}((x, a), (y, b)) = 0$.

By this equivalence relation, we break $\widetilde{\mathcal{A}}$ into classes will be called *static classes.* Let $\widetilde{\mathbb{A}}$ be the set of static classes, through the entire paper $\widetilde{\Lambda}, \widetilde{\Omega}, \widetilde{\Gamma}$ and $\widetilde{\Delta}$ with or without super-index will be used to represent static classes.

Under the assumption that

$$\widetilde{\mathbb{A}} \text{ contains only finite elements,} \tag{$*_1$}$$

we will prove the following two theorems.

Remark 4.0.2. $(*_1)$ *is a generic condition in the sense of Mañé, see [45].*

Theorem 4.0.1. *For any two different static classes $\widetilde{\Lambda}^1, \widetilde{\Lambda}^2 \in \widetilde{\mathbb{A}}$, one of the following must be true:*

1. *There is a global semi-static curve γ with the α-limit set $\alpha(d\gamma) \subset \widetilde{\Lambda}^1$ and the ω-limit set $\omega(d\gamma) \subset \widetilde{\Lambda}^2$;*

2. *There is a finite set of static classes $\{\widetilde{\Omega}^1, \dots, \widetilde{\Omega}^n\} \subset \widetilde{\mathbb{A}} \setminus \{\widetilde{\Lambda}^1, \widetilde{\Lambda}^2\}$ and global semi-static curves $\gamma^i : i = 0, \dots, n$ with $\alpha(d\gamma^i) \subset \widetilde{\Omega}^i$ and $\omega(d\gamma^i) \subset \widetilde{\Omega}^{i+1}$ for $i = 0, \dots, n$, where $\widetilde{\Omega}^0 = \widetilde{\Lambda}^1$ and $\widetilde{\Omega}^{n+1} = \widetilde{\Lambda}^2$.*

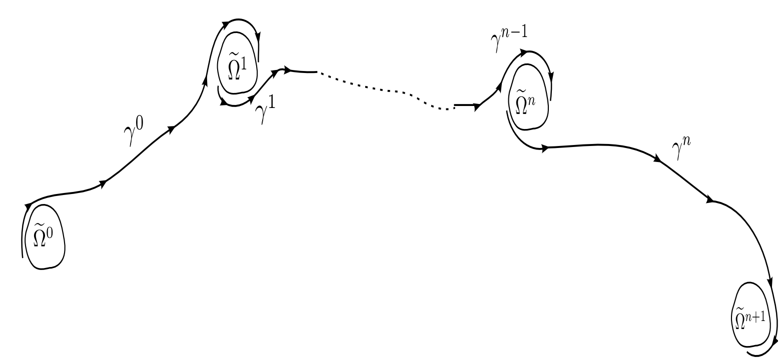

Figure 4.1: Static classes 1

Given an orbit $\{(\gamma(t), \dot{\gamma}(t), t \ mod\ 1) : t \in \mathbb{R}\}$ of the Euler-Lagrange flow, we use $\alpha(d\gamma)$ (or $\omega(d\gamma)$) to denote its α-limit set (or ω-limit set). Similarly we let $\alpha(\gamma)$ (or $\omega(\gamma)$) be the α-limit set (or ω-limit set) of $\{\gamma(t) : t \in \mathbb{Z}\}$ in M.

When L is time-independent the above theorem has been proved by Contreras and Paternain in [24].

Remark 4.0.3. *Through the entire paper, for any subset $U \subset M$ and $\delta > 0$, by $U(\delta)$ we mean $U(\delta) := \{x \in M : d(x, U) \leq \delta\}$.*

Under further assumption that

$$\mathcal{N}_0 \setminus \mathcal{A}_0(\delta) \text{ is totally isolated, for some } \delta > 0 \text{ small enough.} \tag{$*_2$}$$

Where $\mathcal{N}_0 := \mathcal{N} \cap M \times \{0\}$ and $\mathcal{A}_0 := \mathcal{A} \cap M \times \{0\}$.

We will prove that along a chain of heteroclinic orbits obtained in Theorem 4.0.1, there is a real heteroclinic orbit connecting the two given static classes.

Theorem 4.0.2. *If $(*_1), (*_2)$ are true, for any two different static classes $\widetilde{\Lambda}^1, \widetilde{\Lambda}^2 \in \widetilde{\mathbb{A}}$, there is a curve $\gamma \in C^2(\mathbb{R}, M)$, such that $(d\gamma(t), t \bmod 1)$ is an orbit of the Euler-Lagrange system introduced by L with $\alpha(d\gamma) \subset \widetilde{\Lambda}^1$ and $\omega(d\gamma) \subset \widetilde{\Lambda}^2$.*

Similar variational method has been used in [46] and [47], where the authors established the existence of homoclinic orbits to the Aubry set under various conditions different from ours.

4.1 Asymptotic behaviors of semi-static curves

By *Mather's graph theorem*, we can break $\mathcal{A}$ into a set of equivalent classes $\mathbb{A} := \{\Lambda = \pi(\widetilde{\Lambda}) : \widetilde{\Lambda} \in \widetilde{\mathbb{A}}\}$. By abusing of notation, they will also be called static classes and capital Greek letters Λ, Ω with or without super-index will be reserved to represent such static classes throughout this paper.

For all the sets $\widetilde{\mathcal{A}}, \widetilde{\mathcal{N}}, \widetilde{\Lambda}$ $(\mathcal{A}, \mathcal{N}, \Lambda)$, by putting a sub-index $t \in \mathbb{T}$ to them, we mean it is their intersection with $TM \times \{t\}$($M \times \{t\}$), for example $\widetilde{\mathcal{A}}_t = \widetilde{\mathcal{A}} \cap TM \times \{t\}$, $\mathcal{A}_t = \mathcal{A} \cap M \times \{t\}$.

For any $n \in \mathbb{N}$, we define $h^n : M \times M \to \mathbb{R}$ by

$$h^n(x, y) = h((x, 0), (y, n)),$$

and $h^\infty : M \times M \to \mathbb{R}$, $\Phi : M \times M \to \mathbb{R}$ by

$$h^\infty(x, y) = \liminf_{n \in \mathbb{N}, n \to +\infty} h^n(x, y);$$

$$\Phi(x, y) = \Phi((x, 0), (y, 0)).$$

In some cases, we need to shift the time parameterization of a given curve for which we introduce the following operator.

Definition 4.1.1. *Given $a < b \in \mathbb{R}$ for any $c \in \mathbb{R}$, we define an operator τ_c : $\mathcal{C}([a, b], M) \to \mathcal{C}([a + c, b + c], M)$ by*

$$\tau_c(\gamma)(t) = \gamma(t - c), \text{ for any } \gamma \in \mathcal{C}([a, b], M) \text{ and } t \in [a + c, b + c].$$

The variational study of Tonelli Lagrangian L depends on some standard results proved by Mather in [16].

Lemma 4.1.1. *Given a real number K and a compact interval $[a, b]$,*

$$\{\gamma \in \mathcal{C}([a, b], M) : A_L(\gamma) \leq K\}$$

is compact for the topology of uniform convergence.

Theorem 4.1.1. ***(Tonelli Theorem)*** *Given two points $x, y \in M$ and a compact interval $[a, b]$, there is a $\gamma \in \mathcal{C}_{[a,b]}(x, y)$ with $A_L(\gamma) = h((x, a), (y, b))$ and γ is a C^2 solution of (EL).*

If a curve $\gamma \in \mathcal{C}_{[a,b]}(x, y)$ satisfies $A_L(\gamma) = h((x, a), (y, b))$ we will call it a *minimizer*. Obviously a semi-static curve is a minimizer, but a minimizer is not necessarily a semi-static curve.

The next lemma is well-known to experts, however we can not locate a complete proof in the literature, therefore we give one at here.

Lemma 4.1.2. *Given any $p, q \in M$ and two sequences of positive integers $\{T_k^1\} \nearrow +\infty, \{T_k^2\} \nearrow +\infty$, if $\{\gamma_k \in \mathcal{C}_{[-T_k^1, T_k^2]}(p, q)\}$ is a sequence of minimizers, i.e., $A_L(\gamma_k) = h^{T_k^1 + T_k^2}(\gamma_k(-T_k^1), \gamma_k(T_k^2))$, for any $k \in \mathbb{N}$, satisfying*

$$\lim_{k \to +\infty} A_L(\gamma_k) = \liminf_{n \to +\infty} h^n(p, q) = h^\infty(p, q),$$

then there is a global semi-static curve γ, such that γ_k converges to γ uniformly on any compact interval along a subsequence.

Proof. By Proposition 1.1 and Lemma 4.1.1, it is not hard to see for any $T > 0$, along a subsequence γ_k converges uniformly to a $\gamma_T \in \mathcal{C}([-T,T], M)$ on $[-T,T]$. Apply this to a sequence of positive integers $\{T_n\} \nearrow +\infty$, then by a diagonal extraction, we can find a subsequence of γ_k, which we rename as γ_k, and a $\gamma \in \mathcal{C}(\mathbb{R}, M)$, such that γ_k converges uniformly to γ on any compact interval.

We claim for any $T \in \mathbb{Z}^+$, $A_L(\gamma|_{[-T,T]}) = \Phi(\gamma(-T), \gamma(T))$, so γ is a global semi-static curve.

Assume this is not true, then there is a $T \in \mathbb{Z}^+$, such that

$$\varepsilon = A_L(\gamma|_{[-T,T]}) - \Phi((\gamma(-T), \gamma(T)) > 0. \tag{4.1}$$

Since $\gamma_k|_{[-T,T]}$ converges uniformly to $\gamma|_{[-T,T]}$, the lower semi-continuity of A_L implies

$$A_L(\gamma_k|_{[-T,T]}) \geq A_L(\gamma|_{[-T,T]}) - \frac{\varepsilon}{4} \text{ for } k \text{ large enough.} \tag{4.2}$$

By the definition of Φ there is a $S \in \mathbb{Z}^+$, such that

$$h^S(\gamma(-T), \gamma(T)) \leq \Phi(\gamma(-T), \gamma(T)) + \frac{\varepsilon}{4}. \tag{4.3}$$

Because h^S is Lipschitz,

$$|h^S(\gamma_k(-T), \gamma_k(T)) - h^S(\gamma(-T), \gamma(T))| \leq C[d(\gamma_k(-T), \gamma(-T)) + d(\gamma_k(T), \gamma(T))].$$

As $\lim_{k\to+\infty} d(\gamma_k(\pm T), \gamma(\pm T)) = 0$, for k large enough we have

$$|h^S(\gamma_k(-T), \gamma_k(T)) - h^S(\gamma(-T), \gamma(T))| \leq \frac{\varepsilon}{4}. \tag{4.4}$$

By **Tonelli Theorem**, for any $k \in \mathbb{N}$, there are $\xi_k \in \mathcal{C}_{[0,S]}(\gamma_k(-T), \gamma_k(T))$ with

$$A_L(\xi_k) = h^S(\gamma_k(-T), \gamma_k(T)),$$

so by (4.4),

$$A_L(\xi_k) \leq h^S(\gamma(-T), \gamma(T)) + \frac{\varepsilon}{4}. \tag{4.5}$$

Combine this with (4.1), (4.2) and (4.3), we get

$$\begin{aligned} A_L(\xi_k) &\leq \Phi(\gamma(-T), \gamma(T)) + \frac{\varepsilon}{2} \\ &= A_L(\gamma|_{[-T,T]}) - \frac{\varepsilon}{2} \\ &\leq A_L(\gamma_k|_{[-T,T]}) - \frac{\varepsilon}{4}, \end{aligned}$$

for k large enough.

We define a new sequence of curves $\{\widetilde{\gamma}_k \in \mathcal{C}_{T_k^1+T_k^2-2T+S}(p,q)\}$ by

$$\widetilde{\gamma}_k(t) = \begin{cases} \gamma_k(t - T_k^1) & \text{if } t \in [0, T_k^1 - T] \\ \xi_k(t - T_k^1 + T) & \text{if } t \in [T_k^1 - T, T_k^1 - T + S] \\ \gamma_k(t - T_k^1 + 2T - S) & \text{if } t \in [T_k^1 - T + S, T_k^1 + T_k^2 - 2T + S]. \end{cases}$$

For all k large enough, $A_L(\widetilde{\gamma}_k) \leq A_L(\gamma_k) - \frac{\varepsilon}{4}$. Since $\{T_k^2 + T_k^1 - 2T + S\}$ goes to infinity as $k \to +\infty$, we have

$$\begin{aligned} h(p,q) &\leq \liminf_{k\to+\infty} h^{T_k^2+T_k^1-2T+S}(p,q) \leq \liminf_{k\to+\infty} A_L(\widetilde{\gamma}_k) \\ &\leq \lim_{k\to+\infty} A_L(\gamma_k) - \frac{\varepsilon}{4} = h(p,q) - \frac{\varepsilon}{4}, \end{aligned}$$

which is absurd and we proved our claim. $\square$

Remark 4.1.1. *In the previous lemma, if we assume $\{T_k^1\}$ (resp. $\{T_k^2\}$) is bounded, by the same arguments it is not hard to see there is a forward semi-static curve (resp. backward semi-static curve) γ, such that, γ_k converges uniformly to γ on any compact interval in the domain of γ along a subsequence.*

In this paper a stronger version of Lemma 4.1.2 will be needed.

Lemma 4.1.3. *Given any $p, q \in M$ and a sequence of positive integers $\{T_k\} \nearrow +\infty$, let $\{\gamma_k \in \mathcal{C}_{T_k}(p,q)\}$ be a sequence of minimizers, i.e., $A_L(\gamma_k) = h((p,0),(q,T_k))$, satisfies $\lim_{k\to+\infty} A_L(\gamma_k) = h^\infty(p,q)$.*

If $\{a_k\} \nearrow +\infty, \{b_k\} \nearrow +\infty$ are two sequence of positive integers satisfying

1. *$0 \leq a_k \leq b_k \leq T_k$, for all $k \in \mathbb{N}$;*
2. *$b_k - a_k \to +\infty$ as $k \to +\infty$;*
3. *there are $x, y \in M$, such that along a subsequences*

$$d(\gamma_k(a_k), x) \to 0 \text{ and } d(\gamma_k(b_k), y) \to 0 \text{ as } k \to +\infty,$$

then along a subsequence, $\lim_{k\to+\infty} A_L(\gamma_k|_{[a_k,b_k]}) = h^\infty(x,y)$.

Proof. Noticing that $\{A_L(\gamma_k|_{[a_k,b_k]})\}$ has a finite upper bound, passing to a subsequence if necessary, we can say

$$\lim_{k\to+\infty} h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) = \lim_{k\to+\infty} A_L(\gamma_k|_{[a_k,b_k]}) = B < +\infty.$$

We claim $B = h^\infty(x,y)$ and the lemma follows immediately from this claim.

By Lipschitz continuity of h^n,

$$|h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) - h^{b_k-a_k}(x,y)| \le C(d(\gamma_k(a_k),x) + d(\gamma_k(b_k),y)),$$

by passing to a proper subsequence, we can say $d(\gamma_k(a_k),x) + d(\gamma_k(b_k),y)) \to 0$ as $k \to +\infty$, then

$$\lim_{k\to+\infty} h^{b_k-a_k}(x,y) = \lim_{k\to+\infty} h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)).$$

Hence,

$$B = \lim_{k\to+\infty} h^{b_k-a_k}(x,y) \ge \liminf_{n\to+\infty} h^n(x,y) = h^\infty(x,y).$$

Assume $\varepsilon = B - h^\infty(x,y) > 0$, then there are non-negative integers $a'_k < b'_k$ with $b'_k - a'_k \to +\infty$ as $k \to +\infty$ satisfying

$$\lim_{k\to+\infty} h^{b'_k-a'_k}(x,y) = h^\infty(x,y) = B - \varepsilon.$$

Then for k large enough, we have

$$h^{b'_k-a'_k}(x,y) \le B - \frac{3}{4}\varepsilon. \tag{4.6}$$

On the other hand, $B = \lim_{k\to+\infty} h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k))$, so for k large enough

$$h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) \ge B - \frac{\varepsilon}{4}. \tag{4.7}$$

Combine (4.6) and (4.7), we get

$$h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) \ge h^{b'_k-a'_k}(x,y) + \frac{\varepsilon}{2}. \tag{4.8}$$

Again by the Lipschitz continuity of h^n and $d(\gamma_k(a_k),x) + d(\gamma_k(b_k),y)) \to 0$ as $k \to +\infty$, for k large enough we have

$$h^{b'_k-a'_k}(\gamma_k(a_k),\gamma_k(b_k)) \le h^{b'_k-a'_k}(x,y) + \frac{\varepsilon}{4}. \tag{4.9}$$

(4.8) and (4.9) imply

$$h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) \geq h^{b'_k-a'_k}(\gamma_k(a_k),\gamma_k(b_k)) + \frac{\varepsilon}{4}.$$

Now following the same argument as in the proof of Lemma 4.1.2, we can construct a new sequence $\{\gamma_k^* \in \mathcal{C}_{T_k-(b_k-a_k)+(b'_k-a'_k)}(p,q)\}$ with

$$h^\infty(p,q) \leq \liminf_{k\to+\infty} A_L(\gamma_k^*) \leq \liminf_{k\to+\infty} A_L(\gamma_k) - \frac{\varepsilon}{4} = h^\infty(p,q) - \frac{\varepsilon}{4},$$

which is a contradiction.

□

In next lemma we will strengthen the triangle inequality of h^∞ given in Proposition 1.1.

Lemma 4.1.4. *Given a $\Lambda \in \mathbb{A}$ and a compact set $U \subset M$ with $\Lambda_0 \cap U = \emptyset$, then we can find an $\varepsilon > 0$ small enough and a $\delta = \delta(\varepsilon) > 0$, such that*

$$h^\infty(x,y) + h^\infty(y,z) \geq h^\infty(x,z) + \varepsilon, \quad \forall x,z \in \Lambda_0(\delta), \quad \forall y \in U.$$

Proof. Because $\Lambda_0 \cap U = \emptyset$ and U is compact, there is an $\varepsilon' > 0$, such that

$$\widetilde{d}(p,y) = h^\infty(p,y) + h^\infty(y,p) \geq \varepsilon' > 0, \quad \forall p \in \Lambda_0, \quad \forall y \in U.$$

Then for any $q \in \Lambda_0$, as h^∞ satisfies triangle inequality,

$$h^\infty(p,y) + h^\infty(y,q) + h^\infty(q,p) \geq h^\infty(p,y) + h^\infty(y,p) \geq \varepsilon',$$

so

$$h^\infty(p,y) + h^\infty(y,q) \geq -h^\infty(q,p) + \varepsilon' = h^\infty(p,q) + \varepsilon', \tag{4.10}$$

the last equality is because of $h^\infty(p,q) + h^\infty(q,p) = \widetilde{d}(p,q) = 0$, as $p,q \in \Lambda_0$ are in the same static class.

Let $\varepsilon = \frac{\varepsilon'}{4}$ and $\delta = \frac{\varepsilon}{8C}$, then for any $x,z \in \Lambda_0(\delta)$, there are $p',q' \in \Lambda_0$ with $d(x,p') \leq \delta$ and $d(z,q') \leq \delta$. By the Lipschitz continuity of h^∞, we have

$$h^\infty(x,y) \geq h^\infty(p',y) - C\delta,$$

$$h^\infty(y,z) \geq h^\infty(y,q') - C\delta,$$

$$h^\infty(x,z) \le h(p',q') + 2C\delta,$$

combine these with (4.10), we get

$$\begin{aligned} h^\infty(x,y) + h^\infty(y,z) &\ge h^\infty(p',y) + h^\infty(y,q') - 2C\delta \ge \ h(p',q') + \varepsilon' - 2C\delta \\ &\ge h^\infty(x,z) - 2C\delta + 2\varepsilon - 2C\delta = h^\infty(x,z) + 2\varepsilon - 4C\delta \\ &= h^\infty(x,z) + \frac{3}{2}\varepsilon \end{aligned}$$

□

Given an arbitrary $(x,\alpha) \in M \times \mathbb{T}$, by Fathi's weak KAM theory [25], there are at least one forward semi-static curve $\gamma^+ \in C^2([a,+\infty), M)$ with $\gamma^+(a) = x$, $a \ mod\ 1 = \alpha$ and one backward semi-static curve $\gamma^- \in C^2((-\infty, a], M)$ with $\gamma^-(a) = x$, $a \ mod\ 1 = \alpha$.

One of the most important feature of these forward (or backward) semi-static curves is that the must asymptotic to a unique static class of $\widetilde{\mathcal{A}}$.

Lemma 4.1.5. *If γ is a forward (or backward) semi-static curve, then there is a unique static class $\widetilde{\Lambda} \in \widetilde{\mathbb{A}}$, such that $\omega(d\gamma)$ (or $\alpha(d\gamma)$) $\subset \widetilde{\Lambda}$.*

Generally speaking there may be more than one forward (or backward) semi-static curves starting from (or ending at) a given point $(x,\alpha) \in M \times \mathbb{T}$, however if (x,α) belongs to $\mathcal{A}$, then they are unique.

Lemma 4.1.6. *If $(x,\alpha) \in \mathcal{A}$, then there is a unique $v \in T_xM$, such that $(x,v,\alpha) \in \widetilde{\mathcal{A}} \subset \widetilde{\mathcal{N}}^\pm$.*

As a consequence there is a unique forward semi-static curve $\gamma^+ \in \mathcal{C}([\alpha,+\infty), M)$ with $\gamma^+(\alpha) = x$ and a unique backward semi-static curve $\gamma^- \in \mathcal{C}((-\infty,\alpha], M)$ with $\gamma^-(\alpha) = x$, and γ^+, γ^- must be static, with

$$\gamma^+(t) = \gamma_{(x,\alpha)}(t), \ \ \textit{if}\ t \in [\alpha, +\infty);$$

$$\gamma^-(t) = \gamma_{(x,\alpha)}(t), \ \ \textit{if}\ t \in (-\infty, \alpha],$$

where $\gamma_{(x,\alpha)}$ is the unique global static curve passing (x,α).

The proofs of the above two lemmas can be found in [20], [24], [25], which we will not repeat here.

4.2 Heteroclinic chains

We will give the proof of Theorem 4.0.1 in this section. By assumption $(*_1)$, we can fix a $\delta^* > 0$ small enough, such that for any two different static classes $\Lambda^1, \Lambda^2 \in \mathbb{A}$, $\Lambda^1_0(\delta^*) \cap \Lambda^2_0(\delta^*) = \emptyset$. In order to distinguish the two different cases in Theorem 4.0.1, we introduce the following definition

Definition 4.2.1. *Given a $\delta > 0$ and a curve $\gamma \in \mathcal{C}([a,b], M)$, for any two static classes $\Lambda^1, \Lambda^2 \in \mathbb{A}$, we defined the following set*

$$K(\gamma, \delta, \Lambda^1, \Lambda^2) := \{\Lambda \in \mathbb{A} \setminus \{\Lambda^1, \Lambda^2\} : \min_{t \in [a,b] \cap \mathbb{Z}} d(\gamma(t), \Lambda_0) \le \delta\}.$$

A similar idea was used by Maxwell [48] and Rabinowitz [49] on a special class of Hamiltonian systems including periodic forced multiple pendulum with time reversibility assumption.

For the remainder of this section, we fix two arbitrary static classes $\Lambda^1 \neq \Lambda^2 \in \mathbb{A}$, and two points $p \in \Lambda^1_0, q \in \Lambda^2_0$. By **Tonelli Theorem**, there is a sequence of minimizers $\{\gamma_k \in \mathcal{C}_{[-T^1_k, T^2_k]}(p,q)\}$ satisfying

1. $\{T^1_k \in \mathbb{Z}^+\} \nearrow +\infty, \{T^2_k \in \mathbb{Z}^+\} \nearrow +\infty$ as $k \to +\infty$;
2. $\lim_{k \to +\infty} A_L(\gamma_k) = \lim_{k \to +\infty} h^{T^2_k + T^1_k}(p,q) = h^\infty(p,q)$.

Let's consider the collection of sets $K(\gamma_k, \delta, \Lambda^1, \Lambda^2)$, for all $k \in \mathbb{N}$ and $\delta > 0$, there are two possibilities:

Case 1: There is a $\delta_0 > 0$ small enough, such that for each $m \in \mathbb{N}$, there is a $k > m$ with $K(\gamma_k, \delta_0, \Lambda^1, \Lambda^2) = \emptyset$;

Case 2: For each $\delta > 0$, there is a $m \in \mathbb{N}$, such that $K(\gamma_k, \delta, \Lambda^1, \Lambda^2) \neq \emptyset, \quad \forall k > m$.

First we shall assume case 1 hold, then by passing $\{\gamma_k\}$ to a subsequence, we may assume $K(\gamma_k, \delta_0, \Lambda^1, \Lambda^2) = \emptyset$ for all $k \in \mathbb{N}$.

Proposition 4.2.1. *If* Case 1 *hold, there is a global semi-static curve γ satisfying*

$$\alpha(d\gamma) \subset \widetilde{\Lambda}^1 \text{ and } \omega(d\gamma) \subset \widetilde{\Lambda}^2.$$

Proof. For each $k \in \mathbb{N}$, we set $S_k := \min\{t \in [-T^1_k, T^2_k] \cap \mathbb{Z} : d(\gamma_k(t), \Lambda^1_0) > \delta^*\}$, it is not hard to see, for each $k \in \mathbb{N}$, S_k is a well-defined integer and $d(\gamma_k(t), \Lambda^1_0) \le \delta^*,$ for all $t \in [-T^1_k, S_k) \cap \mathbb{Z}$.

Lemma 4.2.2. *Both $\{S_k + T_k^1\}$ and $\{T_k^2 - S_k\}$ goes to infinity as k goes to infinity.*

Proof. (**Lemma 4.2.2**) We will only give the detailed proof for $S_k + T_k^1 \to +\infty$, while $T_k^2 - S_k \to +\infty$ can be proven similarly.

Suppose $\{S_k + T_k^1\}$ is bounded, then passing $\{\gamma_k\}$ to a subsequence, we can say $S_k + T_k^1 \equiv T \in \mathbb{Z}^+$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, we define $\gamma_k^* \in \mathcal{C}([0, T_k^1 + T_k^2], M)$ by $\gamma_k^* = \tau_{T_k^1}(\gamma_k)$, where $\tau_{T_k^1}$ is defined as in Definition 4.1.1, i.e., we shift the time parameterization on γ_k forward by T_k^1.

Replace γ_k^* by a proper subsequence, by Remark 4.1.1, there is a forward semi-static curve $\gamma^* \in \mathcal{C}([0, +\infty), M)$, such that γ_k^* converges to γ^* on any compact interval of $[0, +\infty)$. Since

$$d(\gamma^*(T), \Lambda_0^1) = \lim_{k \to +\infty} d(\gamma_k^*(T), \Lambda_0^1) = \lim_{k \to +\infty} d(\gamma_k(S_k), \Lambda_0^1) \geq \delta^*,$$

$\gamma^*(T) \notin \Gamma_0^1$. At the same time $\gamma^*(0) = \lim \gamma_k^*(0) = p \in \Lambda_0^1$, then γ^* can not be a static curve.

Because $p \in \Lambda_0^1 \subset \mathbb{A}_0$, there must be a global static curve ξ with $\xi(0) = p$. As a result we have two different forward semi-static curves starting from p and this is a contradiction to Lemma 4.1.6. □

By Lemma 4.2.2, without loss of generality, we may assume $S_k \equiv 0$ for all $k \in \mathbb{N}$. Passing γ_k to a subsequence, by Lemma 4.1.2, there is a global semi-static curve γ, such that γ_k converges uniformly to γ on any compact interval.

Since $d(\gamma_k(t), \Lambda_0^1) \leq \delta^*$, for all $t \in [-T_k^1, 0) \cap \mathbb{Z}$ and $k \in \mathbb{N}$, we have $d(\gamma(t), \Lambda_0^1) \leq \delta^*$, for all $t \in (-\infty, 0) \cap \mathbb{Z}$.

By Lemma 4.1.5, there is a unique $\widetilde{\Lambda} \in \widetilde{\mathbb{A}}$, such that $\alpha(d\gamma) \subset \widetilde{\Lambda}$, hence we must have $\widetilde{\Lambda} = \widetilde{\Lambda}^1$ and $\alpha(d\gamma) \subset \widetilde{\Lambda}^1$.

Because *Case 1* is true, either $\omega(d\gamma) \subset \widetilde{\Lambda}^2$ or $\omega(d\gamma) \subset \widetilde{\Lambda}^1$ must be true. Assume $\omega(d\gamma) \subset \widetilde{\Lambda}^1$, then there is a sequence of positive integers $\{T_j\} \nearrow +\infty$ with $\lim_{j \to +\infty} d(\gamma(T_j), x) = 0$ for some $x \in \Lambda_0^1$.

On the other hand we can find a subsequence $\{\gamma_{k_j} \in \mathcal{C}_{[-T_{k_j}^1, T_{k_j}^2]}(p, q)\}$ of $\{\gamma_k\}$ with $T_j \leq T_{k_j}^2$ for all $j \in \mathbb{N}$ and $d(\gamma_{k_j}(T_j), \gamma(T_j))$ approaches to 0 as $j \to +\infty$. Therefore $d(\gamma_{k_j}(T_j), x)$ approaches to 0 as $j \to +\infty$.

Replacing $\{\gamma_{k_j}\}$ by a proper subsequence, by Lemma 4.1.3

$$h(p,x) = \lim_{j\to+\infty} A_L(\gamma_{k_j}|_{[-T^1_{k_j},T_j]}). \tag{4.11}$$

Since $\{T^1_{k_j}\}$ and $\{T_j\}$ goes to infinity, as $j \to +\infty$, both sequences $\{\gamma_{k_j}|_{[-T^1_{k_j},0]}\}$ and $\{\gamma_{k_j}|_{[0,T_j]}\}$ satisfy conditions of Lemma 4.1.3, passing $\{\gamma_{k_j}\}$ to a proper subsequence, we have

$$h^\infty(p,\gamma(0)) = \lim_{j\to+\infty} A_L(\gamma_{k_j}|_{[-T^1_{k_j},0]}); \quad h^\infty(\gamma(0),x) = \lim_{j\to+\infty} A_L(\gamma_{k_j}|_{[0,T_j]}). \tag{4.12}$$

By (4.11) and (4.12), $h^\infty(p,\gamma(0)) + h^\infty(\gamma(0),x) = h^\infty(p,x)$.

However it is easy to see $d(\gamma(0),\Lambda^1_0) \geq \delta^*$, at the same time $p,x \in \Lambda^1_0$, by Lemma 4.1.4, $h^\infty(p,\gamma(0)) + h^\infty(\gamma(0),x) > h^\infty(p,x)$, which is a contradiction and we finished our proof. □

Now we assume *Case 2* is true, because of $(*_1)$, by passing $\{\gamma_k\}$ to a subsequence, we can say that for a sequence of positive real numbers $\{\delta_k \in (0,\delta^*)\} \searrow 0$, there is a finite set of static classes $\{\Omega^1,\dots,\Omega^n\} \subset \mathbb{A} \setminus \{\Lambda^1,\Lambda^2\}$ satisfying $K(\gamma_k,\delta_k,\Lambda^1,\Lambda^2\} \equiv \{\Omega^1,\dots,\Omega^n\}$ for all $k \in \mathbb{N}$.

We set $\Omega^0 = \Lambda^1$ and $\Omega^{n+1} = \Lambda^2$.

Proposition 4.2.3. *If* Case 2 *is true, there is a chain of global semi-static curves $\gamma^i : i = 0,\dots,n$ satisfying $\alpha(d\gamma^i) \subset \widetilde{\Omega}^i$ and $\omega(d\gamma^i) \subset \widetilde{\Omega}^{i+1}$ for $i = 0,\dots,n$.*

Proof. By Definition 4.2.1, for every $i = 1,\dots,n$ and $k \in \mathbb{N}$, we can find an $S^i_k \in (-T^1_k, T^2_k) \cap \mathbb{Z}$ with $d(\gamma_k(S^i_k),\Omega^i_0) \leq \delta_k$ and we set $S^0_k = -T^1_k$ and $S^{n+1}_k = T^2_k$ for all $k \in \mathbb{N}$.

Without loss of generality, we can assume $S^i_k \leq S^{i+1}_k$ for all $i = 0,\dots,n$ and $k \in \mathbb{N}$. Furthermore by passing $\{\gamma_k\}$ to a proper subsequence , we can say there are $x_i \in \Omega^i_0$ for $i = 0,\dots,n+1$ with

$$\lim_{k\to+\infty} d(\gamma_k(S^i_k), x_i) = 0,$$

where $x_0 = p$ and $x_{n+1} = q$.

Lemma 4.2.4. *For every $i = 0,\dots,n$, $\{S^{i+1}_k - S^i_k\} \to +\infty$ as $k \to +\infty$.*

Proof. (**Lemma 4.2.4**) It can be proven similarly as we did in the proof of Lemma 4.2.2 and we will not repeat it here. □

Lemma 4.2.5. *There is a $\widetilde{\delta} \in (0, \delta^*)$ small enough, such that for every k large enough*

$$K(\gamma_k|_{[S_k^i, S_k^{i+1}]}, \widetilde{\delta}, \Omega^i, \Omega^{i+1}) = \emptyset, \forall i = 0, \ldots, n.$$

Proof. (**Lemma 4.2.5**) If not, without loss of generality we can assume there are $0 \leq i_0 \neq i_1 \leq n$ and a sequence of positive numbers $\{\widetilde{\delta}_k \in (0, \delta^*)\} \searrow 0$ satisfying

$$\Omega^{i_1} \in K(\gamma_k|_{[S_k^{i_0}, S_k^{i_0+1}]}, \widetilde{\delta}_k, \Omega^{i_0}, \Omega^{i_0+1})$$

for all k large enough.

By the definition of K, we must have $i_1 \neq i_0+1$. Let's assume $i_1 > i_0+1$ (the case of $i_1 < i_0$ can be proven similarly), then there is a sequence of integers $\{S_k \in (S_k^{i_0}, S_k^{i_0+1})\}$, such that $d(\gamma_k(S_k), \Omega_0^{i_1}) \leq \widetilde{\delta}_k$.

First we will show that $\{S_k^{i_0+1} - S_k\}$ approaches positive infinity as k goes to $+\infty$. If this is not true, replacing $\{\gamma_k\}$ by a proper subsequence, we may assume $S_k^{i_0+1} - S_k \equiv S \in \mathbb{R}$ for all $k \in \mathbb{N}$.

We define a new sequence of minimizers $\{\gamma_k^* = \tau_{-S_k}(\gamma_k)\}$, then

$$d(\gamma_k^*(0), \Omega_0^{i_1}) = d(\tau_{-S_k}(\gamma_k)(0), \Omega_0^{i_1}) = d(\gamma_k(S_k), \Omega_0^{i_1}) \leq \widetilde{\delta}_k.$$

Since $\widetilde{\delta}_k \to 0$ as $k \to +\infty$, we have $\lim_{k\to+\infty} \gamma_k^*(0) = x$ for some $x \in \Omega_0^{i_1}$, when passing $\{\gamma^*\}$ to a proper subsequence.

Again passing $\{\gamma_k^*\}$ to a proper subsequence, by Lemma 4.1.2, there is a global semi-static curve γ^* such that γ_k^* converges to γ^* on any compact interval. Hence

$$\gamma^*(0) = \lim_{k\to+\infty} \gamma_k^*(0) = x \in \Omega_0^{i_1}.$$

On the other hand

$$\begin{aligned}\gamma^*(S) &= \lim_{k\to+\infty} \gamma_k^*(S) = \lim_{k\to+\infty} \gamma_k(S + S_k) \\ &= \lim_{k\to+\infty} \gamma_k(S_k^{i_0+1}) = x_{i_0+1} \in \Omega_0^{i_0+1}.\end{aligned}$$

As $\Omega_0^{i_0+1} \cap \Omega_0^{i_1} = \emptyset$, γ^* can not be a static curve. Since $(x, 0) \in \Omega_{i_1} \subset \mathcal{A}$, there is a global static curve $\gamma_{(x,0)}$ with $\gamma_{(x,0)}(0) = x$. Obviously $\gamma_{(x,0)} \neq \gamma^*$ and we have two different forward semi-static curves starting from x, which is a contradiction to Lemma 4.1.6. Therefore $\{S_k^{i_0+1} - S_k\} \to +\infty$ as $k \to +\infty$.

At the same time by Lemma 4.2.4, $\{S_k^{i_1} - S_k^{i_0+1}\} \to +\infty$ as $k \to +\infty$. Noticing that $\{\gamma_k|_{[S_k, S_k^{i_1}]}\}$ is a sequence of minimizers with

$$\lim_{k\to+\infty} \gamma_k(S_k) = x, \lim_{k\to+\infty} \gamma(S_k^{i_0+1}) = x_{i_0+1} \text{ and } \lim_{k\to+\infty} \gamma_k(S_k^{i_1}) = x_{i_1},$$

by Lemma 4.1.3, passing $\{\gamma_k\}$ to a proper subsequence, we have

$$\begin{aligned} h^\infty(x, x_{i_1}) &= \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k, S_k^{i_1}]}) \\ &= \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k, S_k^{i_0+1}]}) + \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k^{i_0+1}, S_k^{k_1}]}) \\ &= h^\infty(x, x_{i_0+1}) + h^\infty(x_{i_0+1}, x_{i_1}). \end{aligned}$$

However $x, x_{i_1} \in \Omega_0^{i_1}$ and $x_{i_0+1} \in \Omega_0^{i_0+1}$, by Lemma 4.1.4,

$$h^\infty(x, x_{i_1}) < h^\infty(x, x_{i_0+1}) + h^\infty(x_{i_0+1}, x_{i_1}),$$

which is absurd, so we are done. $\square$

Now we resume our proof of Proposition 4.2.3.

For each $i = 0, \dots, n$, by Lemma 4.2.4 and Lemma 4.2.5, we showed that $\{\gamma|_{[S_k^i, S_k^{i+1}]}\}$ is a sequence of minimizers satisfying the following:

1. $S_k^{i+1} - S_k^i \to +\infty$ as $k \to +\infty$;
2. $\lim_{k\to+\infty} d(\gamma_k(S_k^i), x_i) = 0$ and $\lim_{k\to+\infty} d(\gamma_k(S_k^{i+1}), x_{i+1}) = 0$;
3. $K(\gamma_k|_{[S_k^i, S_k^{i+1}]}, \widetilde{\delta}, \Omega^i, \Omega^{i+1}) = \emptyset$, for all k large enough.

Let $\{\gamma_k^i = \tau_{c_k}(\gamma|_{[S_k^i, S_k^{i+1}]})\}$, where $c_k = -\frac{S_k^i + S_k^{i+1}}{2}$, then following the same argument as in the proof of Proposition 4.2.1, we can show that along a subsequence $\{\gamma_k^i\}$ converges uniformly on any compact interval to a global semi-static curve γ^i satisfying $\alpha(d\gamma^i) \subset \widetilde{\Omega}^i$ and $\omega(d\gamma^i) \subset \widetilde{\Omega}^{i+1}$.

Hence $\{\gamma^i : i = 0, \dots, n\}$ form a chain of heteroclinic orbits as we wanted and $\{x_i : i = 0, \dots, n\}$ satisfies $x_i \in \omega(\gamma^{i-1}) \cap \alpha(\gamma^i) \cap \Omega_0^i$ for $i = 1, \dots, n$, and $x_0 \in \alpha(\gamma^0) \cap \Omega_0^0$, $x_{n+1} \in \omega(\gamma^n) \cap \Omega_0^{n+1}$.

Furthermore we have

$$h^\infty(x_i, x_{i+1}) = \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k^i, S_k^{i+1}]}), \text{ for all } i = 0, \dots, n, \tag{4.13}$$

Hence,

$$h^{\infty}(p,q)=\lim_{k\to+\infty}A_L(\gamma_k|_{[S_k^0,S_k^{n+1}]})=\sum_{i=0}^{n}\lim_{k\to+\infty}A_L(\gamma_k|_{[S_k^i,S_k^{i+1}]})=\sum_{i=0}^{n}h^{\infty}(x_i,x_{i+1}). \tag{4.14}$$

□

Obviously Theorem 4.0.1 follows directly from Proposition 4.2.1 and Proposition 4.2.3.

4.3 Heteroclinic orbits

This section will be devoted to the proof of Theorem 4.0.2. In the previous section we proved that for any two different static classes, there is a chain of heteroclinic orbits connecting them. Under the assumption $(*_2)$, we will show that there is a real heteroclinic orbit connecting those two static classes along such a chain of heteroclinic orbits.

Such a heteroclinic orbits will be found as a constraint minimizer of the action of Lagrangian L. Using the minimizing properties of static curves and semi-static curves, we will show that the constraint minimizers we find will not bump up to the boundary conditions we posted and it is a real orbit.

First we will introduce some technical lemmas.

Lemma 4.3.1. *There is a $\hat{\delta}\in(0,\delta^*)$ small enough, such that if $p,q\in\Lambda_0(\hat{\delta})$, for some $\Lambda\in\mathbb{A}$, then*

$$h^{\infty}(p,x)+h^{\infty}(x,q)=h^{\infty}(p,q),\quad \forall x\in\Lambda_0, \tag{4.15}$$

and

$$h^{\infty}(p,x)+h^{\infty}(x,q)>h^{\infty}(p,q),\ \text{if } x\in\mathcal{A}_0\setminus\Lambda_0. \tag{4.16}$$

Proof. Choose an arbitrary $p,q\in\Lambda_0(\delta^*)$, first we will show that

$$h^{\infty}(p,x)+h^{\infty}(x,q)=h^{\infty}(p,x')+h^{\infty}(x',q),\ \text{if } x,x'\in\Lambda_0.$$

By the triangle inequality of h^{∞},

$$\begin{aligned}h^{\infty}(p,x)+h^{\infty}(x,q)&\le h^{\infty}(p,x')+h^{\infty}(x',x)+h^{\infty}(x,x')+h^{\infty}(x',q)\\&=h^{\infty}(p,x')+h^{\infty}(x',q),\end{aligned}$$

the last equality is due to the fact that x, x' are contained in the same static class, which means

$$h^\infty(x,x') + h^\infty(x',x) = \widetilde{d}(x,x') = 0.$$

Therefore $h^\infty(p,x) + h^\infty(x,q) \le h^\infty(p,x') + h^\infty(x',q)$, the other direction of the inequality can be proven similarly.

First we will prove (4.15) under the assumption that both p, q are contained in Λ_0.

For any $x \in \Lambda_0$, by triangle inequality

$$h^\infty(p,x) + h^\infty(x,q) \ge h^\infty(p,q).$$

On the other hand, since q, x are contained in the same static class Γ_0, $\widetilde{d}(q,x) = h^\infty(q,x) + h^\infty(x,q) = 0$, then

$$h^\infty(p,q) - h^\infty(x,q) = h^\infty(p,q) + h^\infty(q,x) \ge h^\infty(p,x),$$

so

$$h^\infty(p,q) \ge h^\infty(p,x) + h^\infty(x,q).$$

As a result, we proved equality (4.15) under the assumption $p, q \in \Gamma_0$.

Now we will drop our previous assumption. Without loss of generality, let's say $q \notin \Lambda_0$. Then there is a sequence of minimizers $\{\gamma_k \in \mathcal{C}_{[0,T_k]}(p,q)\}$ with $\{T_k \in \mathbb{Z}^+\} \nearrow +\infty$, and a forward semi-static curve $\gamma \in \mathcal{C}([0,+\infty), M)$, such that γ_k converges uniformly to γ on any compact sub-interval of $[0,+\infty)$.

Then $\omega(\gamma) \subset \Omega$ for some $\Omega \in \mathbb{A}$. We can find a sequence of positive integers $\{S_k\} \nearrow +\infty$ and a $x \in \Omega_0$, such that $\lim_{k\to+\infty} d(\gamma(S_k), x) = 0$, then

$$\lim_{k\to+\infty} d(\gamma_k(S_k), x) = 0$$

.

We claim $T_k - S_k \to +\infty$ as $k \to +\infty$, if this is not true, passing to a subsequence, we may assume $T_k - S_k \equiv T \in \mathbb{Z}^+$. Let $\xi_k \in \mathcal{C}_{[-T_k,0]}(p,q); \xi_k = \tau_{-T_k}(\gamma_k)$ be a new sequence of minimizers.

Then passing to a subsequence ξ_k converges uniformly to a backward semi-static curve ξ on any compact sub-interval of $(-\infty, 0]$.

Noticing that $\xi_k(-T) = \xi_k(S_k - T_k) = \gamma_k(S_k)$ approaches to x as k goes to $+\infty$, so $\xi(-T) = \lim_{k\to+\infty} \xi_k(-T) = x \in \Omega_0$. Then by Lemma 4.1.6, ξ must be part of a global static curve and $(\xi(t), 0) \in \mathcal{A}_0$ for any $t \in (-\infty, 0] \cap \mathbb{Z}$. However $\xi(0) = \lim \xi_k(0) = \lim \gamma_k(T_k) = q \notin \mathcal{A}_0$, which is a contradiction.

As we just showed, when k goes to $+\infty$, $\{T_k - S_k\}$ approaches to $+\infty$, $\{S_k\}$ approaches to $+\infty$ and $\gamma_k(S_k)$ approaches to x, by Lemma 4.1.3, passing $\{\gamma_k\}$ to a proper subsequence, we have

$$\begin{aligned} h^\infty(p, q) &= \lim_{k \to +\infty} A_L(\gamma_k|_{[0,T_k]}) \\ &= \lim_{k \to +\infty} A_L(\gamma_k|_{[0,S_k]}) + \lim_{k \to +\infty} A_L(\gamma_k|_{[S_k,T_k]}) \\ &= h^\infty(p, x) + h^\infty(x, q). \end{aligned}$$

Now we will show that for a small enough $\hat{\delta} > 0$, we must have $\Omega = \Lambda$, then $x \in \Lambda_0$ and (4.15) follows immediately from that.

Assume $\Omega \neq \Lambda$, then $x \in \mathcal{A}_0 \setminus \Lambda_0$, which is a compact set without any intersection with Λ_0, then by Lemma 4.1.4, we can find an $\hat{\delta} \in (0, \delta^*)$, such that for any $p, q \in \Lambda_0(\hat{\delta})$ and $y \in \mathcal{A}_0 \setminus \Lambda_0$, we have

$$h^\infty(p, y) + h^\infty(y, q) > h^\infty(p, q).$$

This is a contradiction to to what we just proved. Therefore we must have $\Omega = \Gamma$.

Finally (4.16) follows directly from Lemma 4.1.4. $\square$

In section 2, we have mentioned that for any $(x, \alpha) \in M \times \mathbb{T}$, there is at least one forward semi-static curve γ^+ (or backward semi-static curve γ^-) starting from (or ending at) (x, α) with its ω-limit set (or α-limit set) contained in a unique static class. Generally we can not determine which static class it will approach to, however by the above lemma we can determine where it asymptotic to in some special case.

Lemma 4.3.2. *Let $\hat{\delta}$ be defined as in Lemma 4.3.1, if $p \in \Lambda_0(\hat{\delta})$, $q \in \Lambda_0$ for some $\Lambda \in \mathbb{A}$ and $\{\gamma_k \in C_{T_k}(p, q)\}$ is a sequence of minimizers with $\lim_{k\to+\infty} A_L(\gamma_k) = h^\infty(p, q)$ and $\{T_k\} \nearrow +\infty$, then there is a forward semi-static curve $\gamma^+ \in C^2([0, +\infty), M)$ satisfying the following conditions*

1. *$\gamma^+(0) = p$;*

2. *$\{\gamma_k\}$ converges uniformly to γ^+ along a subsequence on any compact interval;*

3. $\omega(d\gamma^+) \subset \widetilde{\Lambda}$.

Proof. By Remark 4.1.1, there is a forward semi-static curve γ satisfying conditions $(1), (2)$, we claim it also satisfies condition (3). If not, by Lemma 4.1.5, there is another static class $\Omega \neq \Lambda$, such that $\omega(d\gamma) \subset \widetilde{\Omega}$.

Then there is a $x \in \Omega_0$ and a sequence of integers $\{S_k \in (0, T_k)\} \nearrow +\infty$, such that

$$\lim_{k \to +\infty} d(\gamma(S_k), x) = 0.$$

Hence passing $\{\gamma_k\}$ to a subsequence, we have

$$\lim_{k \to +\infty} d(\gamma_k(S_k), x) = 0.$$

Similar to the proof of Lemma 4.3.1, we have $\{T_k - S_k\} \to +\infty$ as $k \to +\infty$.

Since $\{T_k - S_k\} \to +\infty$, $\{S_k\} \to +\infty$ and $\lim_{k \to +\infty} d(\gamma_k(S_k), x) = 0$ as $k \to +\infty$, Lemma 4.1.3 tells us that along a subsequence of $\{\gamma_k\}$,

$$h^\infty(p, x) = \lim_{k \to +\infty} A_L(\gamma_k|_{[0, S_k]});$$

$$h^\infty(x, q) = \lim_{k \to +\infty} A_L(\gamma_k|_{[S_k, T_k]}).$$

Therefore

$$h^\infty(p, q) = \lim_{k \to +\infty} A_L(\gamma_k|_{[0, T_k]}) = h^\infty(p, x) + h^\infty(x, q). \tag{4.17}$$

If $x \in \Omega_0 \nearrow \Gamma_0$, Lemma 4.3.1 tells us

$$h^\infty(p, q) < h^\infty(p, x) + h^\infty(x, q),$$

which is a contradiction to (4.17). As a result we proved our claim and γ is the forward semi-static curve γ^+ we are looking for. $\square$

Remark 4.3.1. *It is not hard to see that similar argument can to be used to show the existence of a backward semi-static curve* γ^- *with* $\gamma^-(0) = p$ *and* $\alpha(d\gamma^-) \subset \widetilde{\Lambda}$.

Lemma 4.3.3. *Given a global semi-static curve* γ*, if* $p \in \alpha(\gamma) \cap \mathcal{A}_0$ *and* $q \in \omega(\gamma) \cap \mathcal{A}_0$*, then*

$$h^\infty(p, q) = h^\infty(p, \gamma(S)) + A_L(\gamma|_{[S,T]}) + h^\infty(\gamma(T), q), \text{ for any } S \leq T \in \mathbb{Z}.$$

Proof. For arbitrary $p \in \alpha(\gamma) \cap \mathcal{A}_0$ and $q \in \omega(\gamma) \cap \mathcal{A}_0$, there are two sequences of integers $\{S_k\} \searrow -\infty$ and $\{T_k\} \nearrow +\infty$, such that $\lim_{k\to+\infty} \gamma(S_k) = p$ and $\lim_{k\to+\infty} \gamma(T_k) = q$.

Following Lemma 4.1.3, without loss of generality, we can say

$$h^\infty(p,q) = \lim_{k\to+\infty} A_L(\gamma|_{[S_k,T_k]}),$$

$$h^\infty(p,\gamma(S)) = \lim_{k\to-\infty} A_L(\gamma|_{[S_k,S]}),$$

$$h^\infty(\gamma(T),q) = \lim_{k\to+\infty} A_L(\gamma|_{[T,T_k]}).$$

Therefore

$$\begin{aligned} h^\infty(p,q) &= \lim_{k\to+\infty} [A_L(\gamma|_{[S_k,S]}) + A_L(\gamma|_{[S,T]}) + A_L(\gamma|_{[T,T_k]})] \\ &= \lim_{k\to+\infty} A_L(\gamma|_{[S_k,S]}) + A_L(\gamma|_{[S,T]}) + \lim_{k\to+\infty} A_L(\gamma|_{[T,T_k]}) \\ &= h^\infty(p,\gamma(S)) + A_L(\gamma|_{[S,T]}) + h^\infty(\gamma(T),q). \end{aligned}$$

$\square$

Lemma 4.3.4. *Given a global semi-static curve γ, if there are two compact sets $U, V \subset M$ satisfying*

1. $\gamma(S) \in U,\ \gamma(T) \in V$ *for some* $S \leq T \in \mathbb{Z}$;
2. $\partial U \cap \mathcal{N}_0 = \emptyset, \quad \partial V \cap \mathcal{N}_0 = \emptyset$, *where $\partial U, \partial V$ are the boundaries of U, V.*

Then for any $p \in \alpha(\gamma) \cap \mathcal{A}_0$, $q \in \omega(\gamma) \cap \mathcal{A}_0$ and $(x,y) \in U \times V.$, we have

$$\Delta(p,x,y,q;\gamma) := h^\infty(p,x) + h^{T-S}(x,y) + h^\infty(y,q) - h^\infty(p,q) \geq 0.$$

Furthermore, if $x \in U \setminus \mathcal{N}_0$ or $y \in V \setminus \mathcal{N}_0$, then

$$\Delta(p,x,y,q;\gamma) > 0$$

and there is a $\varepsilon > 0$ such that

$$\min\{\Delta(p,x,y,q;\gamma) : (x,y) \in U \times V \setminus Int(U) \times Int(V)\} \geq \varepsilon,$$

where $Int(U), Int(V)$ are the interiors of U, V.

Proof. Assume there are $x \in U$, $y \in V$ with $\Delta(p,x,y,q;\gamma) < 0$, then

$$h^{\infty}(p,x) + h^{T-S}(x,y) + h^{\infty}(y,q) < h^{\infty}(p,q). \tag{4.18}$$

By Remark 4.1.1, we can find two sequences of minimizers

$$\{\xi_k \in \mathcal{C}_{[0,S_k]}(p,x)\} \text{ with } \{S_k \in \mathbb{Z}\} \nearrow +\infty;$$

$$\{\zeta_k \in \mathcal{C}_{[0,T_k]}(y,q)\} \text{ with } \{T_k \in \mathbb{Z}\} \nearrow +\infty,$$

satisfying

$$h^{\infty}(p,x) = \lim_{k\to+\infty} A_L(\xi_k); \quad h^{\infty}(y,q) = \lim_{k\to+\infty} A_L(\zeta_k).$$

Let $\eta \in \mathcal{C}_{[S,T]}(x,y)$ be a minimizer, i.e., $A_L(\eta) = h^{T-S}(x,y)$. We define a new sequence of curves $\{\gamma_k \in \mathcal{C}_{[-S_k,T-S+T_k]}(p,q)\}$ by

$$\gamma_k(t) = \begin{cases} \xi_k(t+S_k), & \text{if } t \in [-S_k, 0]; \\ \eta(t+S), & \text{if } t \in [0, T-S]; \\ \zeta_k(t-T+S), & \text{if } t \in [T-S, T-S+T_k]. \end{cases}$$

Obviously $S_k + T_k + T - S \to +\infty$ as $k \to +\infty$, and

$$\lim_{k\to+\infty} A_L(\gamma_k) = h^{\infty}(p,x) + h^{T-S}(x,y) + h^{\infty}(y,q).$$

However,

$$\begin{aligned} h^{\infty}(p,q) &= \liminf_{n\to+\infty} h^n(p,q) \le \liminf_{k\to+\infty} h^{S_k+T_k+T-S}(p,q) \\ &\le \lim_{k\to+\infty} A_L(\gamma_k) = h^{\infty}(p,x) + h^{T-S}(x,y) + h^{\infty}(y,q) \\ &< h^{\infty}(p,q), \end{aligned}$$

where the last inequality follows from (4.18). However this is absurd and we proved the first part of the lemma.

For the second part, without loss of generality we can assume there are $x \in U \setminus \mathcal{N}_0$ and $y \in V$ with $\Delta(p,x,y,q;\gamma) = 0$. Let $\{\gamma_k\}$ be defined as above then

$$\lim_{k\to+\infty} A_L(\gamma_k) = h^{\infty}(p,x) + h^{T-S}(x,y) + h^{\infty}(y,q) = h^{\infty}(p,q). \tag{4.19}$$

Although γ_k is not necessary a minimizer, by (4.19), it is not hard to see, for any $T' \in \mathbb{Z}^+$, $\{A_L(\gamma_k|_{[-T',T']})\}$ has a finite upper bound, and the argument in the proof of Lemma 4.1.2 will still hold. Therefore along a subsequence $\{\gamma_k\}$ converges uniformly to a global semi-static curve γ^* on any compact interval. Which means

$$\gamma^*(0) = \lim_{k\to+\infty} \gamma_k(0) = \eta(S) = x \in \mathcal{N}_0,$$

and this is a contradiction to our assumption.

Hence $\Delta(p, x, y, q; \gamma) > 0$, if $x \in U \setminus \mathcal{N}_0$ or $y \in V \setminus \mathcal{N}_0$. By the Lipschitz continuity of h^n and h^∞, there is a $\varepsilon > 0$ such that

$$\min\{\Delta(p, x, y, q; \gamma) : (x, y) \in U \times V \setminus Int(U) \times Int(V)\} \geq \varepsilon > 0.$$

□

Now we are ready to prove Theorem 4.0.2.

Proof. (**Theorem 4.0.2**) We will follow the notations from the previous section, let $p \in \Lambda_0^1,\ q \in \Lambda_0^2$ and $\{\gamma_k \in \mathcal{C}_{[-T_k^1, T_k^2]}(p, q)\}$ is a sequence of minimizers satisfying

$$h^\infty(p, q) = \lim_{k\to+\infty} A_L(\gamma_k).$$

Again there are two different cases as we discussed in the previous section.

If *Case 1* is true, nothing needs to be done here.

If *Case 2* is true, by the proof of Proposition 4.2.3, we have a set of finite static classes $\{\Omega^1, \ldots, \Omega^n\} \subset \mathbb{A} \setminus \{\Lambda^1, \Lambda^2\}$, a chain of global semi-static curves $\{\gamma^i : i = 1, \ldots, n\}$ and $\{x_i \in \Omega_0^i : i = 0, 1, \ldots, n+1\}$, where $x_0 = p, x_{n+1} = q$, satisfying

1. $\alpha(d\gamma^i) \subset \widetilde{\Omega}^i$ and $\omega(d\gamma^i) \subset \widetilde{\Omega}^{i+1}$, for $i = 0, \ldots, n$, where $\Omega^0 = \Lambda^1$ and $\Omega^{n+1} = \Lambda^2$;

2. $x_i \in \omega(\gamma^{i-1}) \cap \alpha(\gamma^i) \cap \Omega_0^i$, for $i = 1, \ldots, n$, and $x_0 = p \in \alpha(\gamma^0) \cap \Omega_0^0$, $x_{n+1} \in \omega(\gamma^n) \cap \Omega_0^{n+1}$.

3. $h^\infty(x_0, x_{n+1}) = \sum_{i=0}^n h^\infty(x_i, x_{i+1})$.

Let $\hat{\delta} > 0$ be defined as in Lemma 4.3.1, for each γ^i, we can choose $S^i < T^i \in \mathbb{Z}$ satisfying

$$\gamma^i(S^i) \in Int(\Omega_0^i(\hat{\delta})) \text{ and } \gamma^i(T^i) \in Int(\Omega_0^{i+1}(\hat{\delta})).$$

Then by assumption $(*_2)$, for each $i = 0, \dots, n$ there are compact sets $U^i, V^i \subset M$ satisfying

$$\gamma^i(S^i) \in Int(U^i), \quad U^i \subset \Omega_0^i(\hat{\delta}), \quad \partial U^i \cap \mathcal{N}_0 = \emptyset;$$

$$\gamma^i(T^i) \in Int(V^i), \quad V^i \subset \Omega_0^{i+1}(\hat{\delta}), \quad \partial V^i \cap \mathcal{N}_0 = \emptyset.$$

For each $(Y, Z) := \{(y_0, z_0), \dots, (y_n, z_n)\} \in \prod_{i=0}^n U^i \times V^i$, we define a function J by

$$J(Y, Z) := h^\infty(x_0, y_0) + \sum_{i=0}^{n} h^{T^i - S^i}(y_i, z_i) + \sum_{i=0}^{n-1} h^\infty(z_i, y_{i+1}) + h^\infty(z_n, x_{n+1}). \quad (4.20)$$

For each $i = 0, \dots, n-1$, $z_i \in V^i \subset \Omega_0^{i+1}(\hat{\delta})$ and $y_{i+1} \in U^{i+1} \subset \Omega_0^{i+1}(\hat{\delta})$, by Lemma 4.3.1,

$$h^\infty(z_i, y_{i+1}) = h^\infty(z_i, x_{i+1}) + h^\infty(x_{i+1}, y_{i+1}).$$

Hence the function J can be rewritten as

$$J(Y, Z) = \sum_{i=0}^{n} [h^\infty(x_i, y_i) + h^{T^i - S^i}(y_i, z_i) + h^\infty(z_i, x_{i+1})]. \quad (4.21)$$

For any $T \in \mathbb{Z}^+$, we define $J_T : \prod_{i=0}^n U^i \times V^i \to \mathbb{R}$, by

$$J_T(Y, Z) := h^\infty(x_0, y_0) + \sum_{i=0}^{n} h^{T^i - S^i}(y_i, z_i) + \sum_{i=0}^{n-1} h^T(z_i, y_{i+1}) + h^\infty(z_n, x_{n+1}).$$

Let

$$c_T := \inf\{J_T(Y, Z) : (Y, Z) = \{(y_0, z_0), \dots, (y_n, z_n)\} \in \prod_{i=0}^{n} U^i \times V^i\},$$

by the compactness of U^i, V^i and Lipschitz continuity of h^n, h^∞, it is easy to see the above infimum is in fact a minimum.

Lemma 4.3.5. *For T large enough, if $(Y', Z') = \{(y'_0, z'_0), \dots, (y'_n, z'_n)\} \in \prod_{i=0}^n U^i \times V^i$ satisfies $J_T(Y', Z') = c_T$, then*

$$(Y', Z') = \{(y'_0, z'_0), \dots, (y'_n, z'_n)\} \in \prod_{i=0}^{n} Int(U^i) \times Int(V^i).$$

The proof of the above lemma will be postponed.

By Lemma 4.3.5, for T large enough, there is a

$$(Y', Z') = \{(y'_0, z'_0), \dots, (y'_n, z'_n)\} \in \prod_{i=0}^{n} Int(U^i) \times Int(V^i)$$

with $J_T(Y', Z') = c_T$.

Then for each $i = 0, \dots, n$, there are minimizers $\zeta_i \in \mathcal{C}_{[0,T^i-S^i]}(y'_i, z'_i)$ with $A_L(\zeta_i) = h^{T^i-S^i}(y'_i, z'_i)$ and for each $i = 0, \dots, n-1$, there are minimizers $\eta_i \in \mathcal{C}_{[0,T]}(z'_i, y'_{i+1})$ such that $A_L(\eta_i) = h^T(z'_{i+1}, y'_{i+1})$.

By Lemma 4.3.2, there is a forward semi-static $\gamma^+ \in \mathcal{C}([0, +\infty), M)$ with $\gamma^+(0) = z'_n$ and $\omega(d\gamma^+) \subset \widetilde{\Omega}^{n+1}$, and a backward semi-static curve $\gamma^- \in \mathcal{C}((-\infty, 0], M)$ with $\gamma^-(0) = y'_0$ and $\alpha(d\gamma^-) \subset \widetilde{\Omega}^0$.

Gluing these curves together by the following order

$$\gamma^- * \zeta_0 * \eta_0 * \cdots * \eta_{n-1} * \zeta_n * \gamma^+,$$

we get a new curve $\gamma \in \mathcal{C}(\mathbb{R}, M)$ with $\alpha(d\gamma) \subset \widetilde{\Lambda}^1$ and $\omega(d\gamma) \subset \widetilde{\Lambda}^2$.

By the standard variational argument, it is not hard to see γ is a classical solution of (EL) and we are done.

□

We still need to give the proof of Lemma 4.3.5.

Proof. (**Lemma 4.3.5**) Let's fix an arbitrary

$$(Y, Z) = \{(y_0, z_0), \dots, (y_n, z_n)\} \in \prod_{i=0}^{n} U^i \times V^i \setminus \prod_{i=0}^{n} Int(U^i) \times Int(V^i),$$

for the rest of the proof. It is enough to show that there is a $T^* \in \mathbb{Z}^+$ large enough, such that if $T \geq T^*$, we have

$$J_T(Y, Z) > J_T(\{\gamma^i(S^i), \gamma^i(T^i)\}),$$

where

$$\{\gamma^i(S^i), \gamma^i(T^i)\} := \{(\gamma^0(S^0), \gamma^0(T^0)), \dots, (\gamma^n(S^n), \gamma^n(T^n))\},$$

and obviously

$$\{\gamma^i(S^i), \gamma^i(T^i)\} \in \prod_{i=0}^{n} Int(U^i) \times Int(V^i).$$

By (4.21),

$$J(Y, Z) - J(\{\gamma^i(S^i), \gamma^i(T^i)\}) = \sum_{i=0}^{n} \{h^\infty(x_i, y_i) + h^{T^i - S^i}(y_i, z_i) + h^\infty(z_i, x_{i+1}) \\ - [h^\infty(x_i, \gamma^i(S^i)) + h^{T^i - S^i}(\gamma^i(S^i), \gamma^i(T^i)) + h^\infty(\gamma^i(T^i), x_{i+1})]\}.$$

According to Lemma 4.3.3,

$$h^\infty(x_i, \gamma^i(S^i)) + h^{T^i - S^i}(\gamma^i(S^i), \gamma^i(T^i)) + h^\infty(\gamma^i(T^i), x_{i+1}) = h^\infty(x_i, x_{i+1}),$$

for each $i = 0, \ldots, n$. Therefore

$$J(Y, Z) - J(\{\gamma^i(S^i), \gamma^i(T^i)\}) = \sum_{0}^{n} \Delta(x_i, y_i, z_i, x_{i+1}; \gamma^i).$$

Following Lemma 4.3.4, we can find a small enough $\varepsilon > 0$, such that

$$J(Y, Z) - J(\{\gamma^i(S^i), \gamma^i(T^i)\}) \geq \varepsilon. \tag{4.22}$$

By the compactness of U^i, V^i and Lipschitz continuity of h^n, h^∞, there is a $T' \in \mathbb{Z}^+$ large enough, such that for all integers $T \geq T'$, we have

$$\sum_{i=0}^{n-1} |h^\infty(z_i, y_{i+1}) - h^T(z_i, y_{i+1})| \leq \frac{\varepsilon}{4},$$

then by the definitions of J and J_T,

$$J_T(Y, Z) \geq J(Y, Z) - \frac{\varepsilon}{4}. \tag{4.23}$$

Similarly there is a $T'' \in \mathbb{Z}^+$ large enough, such that for all integers $T \geq T''$,

$$\sum_{i=0}^{n} [|h^\infty(x_i, \gamma^i(S^i)) - h^T(x_i, \gamma^i(S^i))| + |h^\infty(\gamma^i(T^i), x_{i+1}) - h^T(\gamma^i(T^i), x_{i+1})|] \leq \frac{\varepsilon}{4},$$

therefore

$$\sum_{i=0}^{n} [h^T(x_i, \gamma^i(S^i)) + h^{T^i - S^i}(\gamma^i(S^i), \gamma^i(T^i) + h^T(\gamma^i(T^i), x_{i+1})] \\ \leq J(\{\gamma^i(S^i), \gamma^i(T^i)\}) + \frac{\varepsilon}{4}. \tag{4.24}$$

Without loss of generality, we may say T'' is large enough, such that for all integers $T \geq T''$,

$$h^{\infty}(x_0, \gamma^0(S^0)) + h^{\infty}(\gamma^n(T^n), x_{n+1}) \leq h^T(x_0, \gamma^0(S^0)) + h^T(\gamma^n(S^n), x_{n+1}) + \frac{\varepsilon}{4}.$$

Combine the above inequality with (4.24), it is easy to see

$$J_{2T}(\{\gamma^i(S^i), \gamma^i(T^i)\}) \leq J(\{\gamma^i(S^i), \gamma^i(T^i)\}) + \frac{\varepsilon}{2}, \tag{4.25}$$

once the reader noticing that

$$h^{2T}(\gamma^i(T^i), \gamma^{i+1}(S^{i+1})) \leq h^T(\gamma^i(T^i), x_{i+1}) + h^T(x_{i+1}, \gamma^{i+1}(S^{i+1})),$$

for all $i = 0, \ldots, n$.

Let $T^* = \max\{T', 2T''\}$, for all integers $T \geq T^*$, following (4.22), (4.23) and (4.25), we have

$$\begin{aligned} J_T(Y, Z) &\geq J(\{\gamma^i(S^i), \gamma^i(T^i)\}) + \frac{3}{4}\varepsilon \\ &\geq J_T(\{\gamma^i(S^i), \gamma^i(T^i)\}) + \frac{\varepsilon}{4}, \end{aligned}$$

and we are done for the proof of the lemma. $\square$

Chapter 5

Ray and Heteroclinic Orbits of Autonomous Lagrangian systems

In this chapter we will apply the basic results of Mather-Mañé theory to a class of autonomous Lagrangian systems with 2 degrees of freedom. This will generalize certain results obtained by Rabinowitz in [50], [51].

In [50], Rabinowitz considered a Hamiltonian system of double pendulum type

$$\ddot{q} + U'(q) = 0, \tag{HS}$$

with an associated Lagrangian $L(x, v) = \frac{1}{2}|v|^2 - U(x)$ satisfying

$$U \in C^2(\mathbb{R}^2, \mathbb{R}) \text{ and is } 1-\text{periodic in } x_1, x_2, \text{ the components of } x; \tag{U1}$$

$$U(0) = 0 > U(x), x \in \mathbb{R}^2 \setminus \mathbb{Z}^2. \tag{U2}$$

The system can also be considered defined on the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then every $\tau \in \mathbb{Z}^2 \setminus \{0\}$ represents a non-trivial homotopy class of $\mathbb{T}^2$. Let

$$\Gamma_\tau = \{q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^2) : \text{ there is a } T = T(q) > 0 \text{ s.t. } q(t+T) = q(t) + \tau, \forall t \in \mathbb{R}\}$$

and

$$c_\tau = \inf\{A_L(q) : q \in \Gamma_\tau\},$$

where

$$A_L(q) := \int_0^{T(q)} L(dq(t))dt, \quad dq(t) = (q(t), \dot{q}(t)). \tag{5.1}$$

Taking $\tau = (\tau_1, \tau_2)$ with τ_1, τ_2 relatively prime, it was proved in [50] that if

$$c_\tau > \inf\{\int_{-\infty}^{+\infty} L(dq(t))dt : q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^2) \text{ with } q(-\infty) = 0, q(+\infty) = \tau\}, \tag{5.2}$$

then

$$\mathcal{M}_\tau = \{q \in \Gamma_\tau : A_L(q) = c_\tau\}$$

is a non-empty set with every $q_\tau \in \mathcal{M}_\tau$ being a zero energy solution of (HS) satisfying $q_\tau(\mathbb{R}) \cap \mathbb{Z}^2 = \emptyset$.

Let's consider the component of $\mathbb{R}^2 \setminus \bigcup_{q\in\mathcal{M}_\tau} q(\mathbb{R})$ which contains 0. Its boundary consists two elements v^-, v^+ from $\mathcal{M}_\tau$. After an appropriate normalization of the time parameters on v^- and v^+, it was shown in [50] that there are four zero energy heteroclinic solutions $Q^\pm, P^\pm$ of (HS) satisfying:

- $Q^+(t) \to 0$ as $t \to -\infty$ and $Q^+(t) - v^+(t) \to 0$ as $t \to +\infty$;
- $Q^-(t) \to 0$ as $t \to -\infty$ and $Q^-(t) - v^-(t) \to 0$ as $t \to +\infty$;
- $P^+(t) \to 0$ as $t \to +\infty$ and $P^+(t) - v^+(t) \to 0$ as $t \to -\infty$;
- $P^-(t) \to 0$ as $t \to +\infty$ and $P^-(t) - v^-(t) \to 0$ as $t \to -\infty$

In a later paper [51], under a condition stronger than (5.2), Rabinowitz showed there are additional zero energy heteroclinic solutions $\bar{Q}^+, \bar{Q}^-$ of (HS) satisfying

- $\bar{Q}^+(t) - v^-(t) \to 0$ as $t \to -\infty$ and $\bar{Q}^+(t) - v^+(t) \to 0$ as $t \to +\infty$;
- $\bar{Q}^-(t) - v^+(t) \to 0$ as $t \to -\infty$ and $\bar{Q}^-(t) - v^-(t) \to 0$ as $t \to +\infty$.

As mentioned by Rabinowitz the proofs apply equally well to a Hamiltonian system with the associated Lagrangian of the form

$$L(x, v) = \frac{1}{2}\langle A(x)v, v\rangle - U(x),$$

where $A(x)$ satisfies the following conditions

$$A(x) \text{ is a positive definite quadratic form in } v \text{ for all } x \in \mathbb{R}^2; \tag{A1}$$

$$A(x) \text{ is 1-periodic in } x_1, x_2, \tag{A2}$$

and U satisfies conditions (U1) and (U2). We will call such a Lagrangian *Mechanical Lagrangian.*

In this chapter we will generalize the above results to a larger class of autonomous Hamiltonian systems with 2 degrees of freedom called **Tonelli Hamiltonian.**

A Hamiltonian is Tonelli, if the Legendre transformation of it is a Lagrangian $L \in C^2(\mathbb{R}^4, \mathbb{R})$ satisfying the following conditions:

1. L is 1-periodic in x_1, x_2;
2. $\frac{\partial^2 L}{\partial v^2}(x, v)$ is positive definite, as a quadratic form, for all $(x, v) \in \mathbb{R}^4$;
3. $\lim_{|v| \to +\infty} \frac{L(x,v)}{|v|} = +\infty$, for all $x \in \mathbb{R}^2$.

Recall the definition of Tonelli Lagrangians in Chapter 2, it is not hard to see such an L is the lift of a time-independent Tonelli Lagrangian defined on $T\mathbb{T}^2$ to its universal cover $\mathbb{R}^2 \times \mathbb{R}^2$. In this chapter we will also call such Lagrangians Tonelli Lagrangians.

Given an arbitrary Tonelli Lagrangian L, we will consider its associated Euler-Lagrangian equation

$$\frac{d}{dt}\frac{\partial L}{\partial v}(q(t), \dot{q}(t)) = \frac{\partial L}{\partial x}(q(t), \dot{q}(t)). \tag{LS}$$

Because L is time independent, a solution $q(t)$ of (LS) will have constant energy, i.e., $\frac{\partial L}{\partial v}(q(t), \dot{q}(t))\dot{q}(t) - L(q(t), \dot{q}(t)) \equiv \mathit{constant}$, see [52].

Every Tonelli Lagrangian L has a Mañé's Critical Value $c(L)$, which will be defined later. For any $k > c(L)$ and $\tau \in \mathbb{Z}^2 \setminus \{0\}$, we set

$$c_\tau^k := \inf\{A_{L+k}(q) : q \in \Gamma_\tau\},$$

where $A_{L+k}(q)$ is as defined in (5.1) with L replaced by $L + k$. It was proved in [21] Theorem 27 that

$$\mathcal{M}_\tau^k = \{q \in \Gamma_\tau : A_{L+k}(q) = c_\tau^k\}$$

is not empty and every element in $\mathcal{M}_\tau^k$ is a k energy solution of (LS).

The reason that no assumption like (5.2) is needed for the above result is that those solutions have energies strictly greater than $c(L)$, while $c(L) = 0$ for a Mechanical Lagrangian L considered in [50], see remark 5.1.2.

In Proposition 5.2.3, we will show that $\mathcal{M}_\tau^k$ is an ordered set, i.e., for any $u, v \in \mathcal{M}_\tau^k$, either $u(\mathbb{R}) = v(\mathbb{R})$ or $u(\mathbb{R}) \cap v(\mathbb{R}) = \emptyset$. When $u(\mathbb{R}) \cap v(\mathbb{R}) = \emptyset$ and there is no curve from $\mathcal{M}_\tau^k$ with its image lying between the images of u and v, we say u, v is a pair of (τ-periodic) *neighboring minimal curves*.

Given a Tonelli Lagrangian L, for any $\tau \in \mathbb{Z}^2 \setminus \{0\}$ and $k > c(L)$, if $v^- \neq v^+ \in \mathcal{M}_\tau^k$ is a pair of *neighboring minimal curves* and $\mathcal{R}$ is the closed region between $v^-(\mathbb{R})$ and $v^+(\mathbb{R})$ in $\mathbb{R}^2$, our main results are the following two theorems.

Theorem 5.0.1. *For any $p_0 \in int(\mathcal{R})$, where $int(\mathcal{R})$ is the interior of $\mathcal{R}$, there exist $Q^\pm \in C^2([0, +\infty), \mathcal{R})$ and $P^\pm \in C^2((-\infty, 0], \mathcal{R})$, which are k energy solutions of (LS) satisfying*

- *$Q^\pm(0) = p_0$, $Q^+(+\infty) = v^+$ and $Q^-(+\infty) = v^-$;*
- *$P^\pm(0) = p_0$, $P^+(-\infty) = v^+$ and $P^-(-\infty) = v^-$.*

Remark 5.0.2. *Given a $u \in \mathcal{M}_\tau^k$, for any curve q, by $q(\pm\infty) = u$, we mean $q(t) - u(t - \sigma(q)) \to 0$ as $t \to \pm\infty$, for some $\sigma(q) \in \mathbb{R}$.*

We call these *Ray solutions* because Bangert proved a similar result about geodesic rays on $\mathbb{T}^2$, see [53].

Theorem 5.0.2. *There exist $\bar{Q}^\pm \in C^2(\mathbb{R}, \mathcal{R})$, which are k energy solutions of (LS) satisfying*

- *$\bar{Q}^+(-\infty) = v^-$ and $\bar{Q}^+(+\infty) = v^+$;*
- *$\bar{Q}^-(-\infty) = v^+$ and $\bar{Q}^-(+\infty) = v^-$.*

Because the energies of these solutions are strictly greater than $c(L)$, our results does not imply the results in [50] and [51].

There are possibly other approaches to prove the above theorems.

First for any $u \in \mathcal{M}_\tau^k$, the invariant measure supported on $\{\pi(du(t)) : t \in \mathbb{R}\}$, where $\pi : T\mathbb{R}^2 \to T\mathbb{T}^2$, is a minimizing measure for $L + w_u$, where w_u is a closed one form on $\mathbb{T}^2$, see [16] for a precise definition of minimizing measure. Now given a pair of neighboring minimal curves v^-, v^+. If the corresponding closed one forms w_{v^-} and w_{v^+}

are the same, then Theorem 5.0.2 will be a direct corollary of results in [24]. However we are not able to prove it at this moment.

Second for $k > c(L)$, the dynamics of the Lagrangian flow on k-energy surface is merely a reparameterization of the geodesic flow of a Finsler metric, see [21]. If the Finsler metric is symmetric then our results will follow from the classic Aubry-Mather theory, see [11] and [53]. However we don't know if the same approach will still hold if the Finsler metric is non-symmetric.

Besides the above, the main reason that we are following Rabinowitz's approach is that it will allow us to prove the existence of multibump homoclinic and heteroclinic solutions between two neighboring minimal curves (which will be done in a different paper by the author), as Rabinowitz did for the Mechanical Lagrangian systems in [54] and [55]. Such kind of results will beyond the reach of the above two approaches we mentioned.

Our main motivation of this paper is its application to the following two examples.

The first example is a *Magnetic Lagrangian*, $L \in C^2(\mathbb{R}^4, \mathbb{R})$, with the form

$$L(x,v) = L_0(x,v) - \langle \theta(x), v\rangle := \frac{1}{2}\langle A(x)v, v\rangle - U(x) - \langle \theta(x), v\rangle,$$

where $\theta(x) \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ satisfies

$$\theta(x) \text{ is a 1-form in } v \text{ for every } x \in \mathbb{R}^2; \tag{$\theta1$}$$

$$\theta(x) \text{ is 1-periodic in } x_1, x_2, \tag{$\theta2$}$$

and L_0 is a Mechanical Lagrangian. If $\theta(x)$, which can also be seen as a 1-form on $\mathbb{T}^2$, is not closed, then the Euler-Lagrange flow of L is different from the Euler-Lagrange flow of L_0, and it models the motion of a charged particle with unit mass and unit charge under the effect of potential energy $U(x)$ and a magnetic field induced by $\theta(x)$.

While there have been many results about existence of periodic orbits in Magnetic Lagrangian systems, see [56] and the references there, up to our knowledge, there are not many results on the existence of ray and heteroclinic orbits in such systems.

The second example is a Hamiltonian system $H \in C^2(\mathbb{R}^4, \mathbb{R})$ with

$$H(x,y) = \frac{1}{2}\langle A(x)v, v\rangle + U(x) + \langle \theta(x), y\rangle,$$

where $A(x), U(x), \theta(x)$ satisfy conditions (A1, A2), (U1, U2) and (θ1, θ2) correspondingly. Here $\theta(x)$ represents the *Coriolis Effect* in the system. Simple calculation shows that H has a corresponding Lagrangian

$$L(x,v) = \frac{1}{2}\langle v - \theta(x), A^{-1}(x)(v - \theta(x))\rangle - U(x).$$

It is not hard to see L is a Tonelli Lagrangian, but not necessarily a Mechanical Lagrangian.

Before we move on, we want to say a little more about Tonelli Lagrangian systems, since this has been a very active area in the past two decades and variational methods have been proved extremely useful. Homoclinic and heteroclinic orbits in the same spirits of this paper have been found in such systems by various authors. We refer the interested readers to [17], [34], [57], [24], [33], [46] and the references there.

5.1 Time-independent Tonelli Lagrangian systems

Since we will only study time-independent Tonelli Lagrangians in this chapter and in many cases we will work in the universal cover of a closed Remannian manifold, the related results from Mather-Mañé theory will be repeated here in a slightly modified form.

Let M be a connected compact smooth Riemannian manifold without boundary, and $\pi : \widetilde{M} \to M$ the universal covering of M.

Definition 5.1.1. *(Tonelli Lagrangian) A C^2 function $\mathbb{L} : TM \to \mathbb{R}$ is a Tonelli Lagrangian, if it satisfies the following conditions,*

1. *$\mathbb{L}$ is strictly convex in the fibers, i.e., $\partial^2\mathbb{L}/\partial v^2(x,v)$ is positive definite, as a quadratic form, for all $(x,v) \in TM$.*
2. *$\mathbb{L}$ is superlinear, $\lim_{\|v\|_x \to +\infty} \frac{\mathbb{L}(x,v)}{\|v\|_x} = +\infty$, in each fiber, i.e., for each $A \in \mathbb{R}$, there exists $B(A) \in \mathbb{R}$ such that*

$$\mathbb{L}(x,v) \geq A\|v\|_x - B(A), \quad \forall (x,v) \in TM.$$

Remark 5.1.1. *$\|\cdot\|_x$ denotes the norm associated to a Riemannian metric on M and its pull back on $\widetilde{M}$ as well. Since M is compact, condition (2) in the above definition is* *independent of which Riemannian metric is chosen on M. For any $x, y \in M$ or $\widetilde{M}$, we use $d(x, y)$ to denote the distance between x and y defined by the Riemannian metric.*

Given a Tonelli Lagrangian $\mathbb{L}$, let $L := \mathbb{L} \circ d\pi : T\widetilde{M} \to \mathbb{R}$ be a lift of $\mathbb{L}$ to $T\widetilde{M}$, obviously L also satisfies the conditions in Definition 5.1.1, and we will call L a Tonelli Lagrangian as well. The action of L on an absolutely continuous curve $\gamma : [a, b] \to \widetilde{M}$ will be defined as

$$A_L(\gamma) = \int_a^b L(d\gamma(t))dt, \text{ where } d\gamma(t) := (\gamma(t), \dot{\gamma}(t)).$$

All the curves we encounter in this paper are absolutely continuous and belong to $W^{1,2}_{loc}$. For any $I \subset \mathbb{R}$, let $\mathcal{C}(I, \widetilde{M})$ be the set of all absolutely continuous curves defined on I and belonging to $W^{1,2}_{loc}(I, \widetilde{M})$. For any $x, y \in \widetilde{M}$ and $a < b$, we set

$$\mathcal{C}_{[a,b]}(x, y) := \{\gamma \in \mathcal{C}([a, b], \widetilde{M}) : \gamma(a) = x, \gamma(b) = y\},$$

and for any $T > 0$,

$$\mathcal{C}_T(x, y) := \{\gamma \in \mathcal{C}_{[a,b]}(x, y) : a = 0, b = T\}.$$

If $\gamma \in \mathcal{C}_{[a,b]}(x, y)$ is a critical point of $A_L : \mathcal{C}_{[a,b]}(x, y) \to \mathbb{R}$, then it is a solution of (LS), see [52].

The variational study of a Tonelli Lagrangian L relies on the following standard results, whose proofs can be found in [16] and [25].

Lemma 5.1.1. *If $L : T\widetilde{M} \to \mathbb{R}$ is a Tonelli Lagrangian, K is a finite real number and $[a, b]$ is a compact interval, then $\{\gamma \in \mathcal{C}([a, b], \widetilde{M}) : A_L(\gamma) \leq K\}$ is a compact set for the topology of uniform convergence.*

Theorem 5.1.1. *Given a Tonelli Lagrangian $L : T\widetilde{M} \to \mathbb{R}$, a compact interval $[a, b]$ and $x, y \in \widetilde{M}$, $\inf\{A_L(\gamma) : \gamma \in \mathcal{C}_{[a,b]}(x, y)\}$ is a finite number and can be achieved at a $\xi \in \mathcal{C}_{[a,b]}(x, y)$, which is a C^2 solution of (LS).*

Obviously the above two results are still true if we replace L by $L + k$ for any $k \in \mathbb{R}$.

Definition 5.1.2. *For a Tonelli Lagrangian $L : T\widetilde{M} \to \mathbb{R}$, we define its **Mañé's Critical Value**, $c(L)$, as*

$$c(L) := \inf\{k \in \mathbb{R} : A_{L+k}(\gamma) \geq 0, \textit{ for any closed absolutely continuous curve } \gamma\}$$

By the superlinearity of L, it is not hard to see $c(L)$ is well-defined and finite.

Remark 5.1.2. *Mañé's Critical Value can also be defined for* $\mathbb{L}$ *and* $c(\mathbb{L}) \geq c(L)$. *A proof of this can be found in [22]. By the definition, it is easy to see for a Mechanical Lagrangian we considered in the previous section, its Mañé's Critical Value is 0.*

Definition 5.1.3. *For any* $k \in \mathbb{R}$, *we define the* action potential $\Phi_k : \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ *by*

$$\Phi_k(x, y) = \inf\{A_{L+k}(\gamma) : \gamma \in \cup_{T>0}\mathcal{C}_T(x, y)\}.$$

Proposition 5.1.2. *Given a Tonelli Lagrangian* L *with Mañé's Critical Value* $c(L)$, *we have*

1. *if* $k < c(L)$, *then* $\Phi_k(x, y) = -\infty$, *for all* x, y *in* $\widetilde{M}$;
2. *if* $k \geq c(L)$, *then* $\Phi_k(x, y) > -\infty$ *for all* x, y *in* $\widetilde{M}$ *and* Φ_k *is a Lipschitz function;*
3. *if* $k \geq c(L)$, *then*
$$\Phi_k(x, z) \leq \Phi_k(x, y) + \Phi_k(y, z)$$
for all x, y *and* z *in* $\widetilde{M}$ *and*
$$\Phi_k(x, y) + \Phi_k(y, x) \geq 0,$$
for all x, y *in* $\widetilde{M}$;
4. *if* $k > c(L)$, *then*
$$\Phi_k(x, y) + \Phi_k(y, x) > 0,$$
for any $x \neq y$ *in* $\widetilde{M}$.

We omit the proof here and refer the interested readers to [22], where the proposition is proven for $\mathbb{L}$, however the proof also works when L is a lift of $\mathbb{L}$.

For any $k \geq c(L)$, if a curve $\gamma \in \mathcal{C}_T(x, y)$ satisfies $A_{L+k}(\gamma) = \Phi_k(x, y)$ then

$$A_{L+k}(\gamma) \leq A_{L+k}(\xi), \quad \forall \xi \in \mathcal{C}_S(x, y) \text{ and } S > 0,$$

and we say γ is a *a free-time minimizer* of $L + k$. An absolutely continuous curve $\gamma \in \mathcal{C}(\mathbb{R}, \widetilde{M})$ will be called *a free-time minimizer* of $L + k$, when $A_{L+k}(\gamma|_{[a,b]}) =$

$\Phi_k(\gamma(a), \gamma(b))$ for any $[a, b] \subset \mathbb{R}$. We will simply call it a free-time minimizer, when there is no confusion.

Obviously the free-time minimizers depend on the choice of k, and we have the following proposition

Proposition 5.1.3. *If γ is a free-time minimizer of $L+k$, then it is a solution of (LS) with energy k.*

Again a proof can be found in [22].

Remark 5.1.3. *However for some of the solutions we will find in this paper, it can only be shown that they are* locally free-time minimizers *of $L+k$. A curve γ is called a locally free-time minimizer, if for any small enough interval $[a, b]$ in the domain of γ, we can find a small open neighborhood U of $\gamma([a, b])$, such that for any $\xi \in \mathcal{C}_s(\gamma(a), \gamma(b)), \forall S > 0$ with $\xi([a, b]) \subset U$, we have $A_{L+k}(\gamma|_{[a,b]}) \leq A_{L+k}(\xi)$. It is not hard to see a local free-time minimizer γ is also a solution of (LS) with energy k.*

The idea of the proof of the next proposition comes from [22]

Proposition 5.1.4. *Given a Tonelli Lagrangian $\mathbb{L} : TM \to \mathbb{R}$, let $L : T\widetilde{M} \to \mathbb{R}$ be a lift of $\mathbb{L}$, for any $x, y \in \widetilde{M}$ and $T > 0$, we set*

$$\Phi_k(x, y; T) := \inf\{A_{L+k}(\gamma) : \gamma \in \mathcal{C}_T(x, y)\},$$

then

1. *for $k \geq c(L), \lim_{T \to 0^+} \Phi_k(x, y; T) = +\infty$, $\forall x \neq y \in \widetilde{M}$;*
2. *for $k > c(L)$, $\lim_{T \to +\infty} \Phi_k(x, y; T) = +\infty$, $\forall x, y \in \widetilde{M}$. Moreover the convergence is uniform if $x \in U, y \in V$, where U, V are two compact subsets of $\widetilde{M}$.*

Remark 5.1.4. *By Theorem 5.1.1, for any $x, y \in \widetilde{M}$ and $T > 0$, there is a $\gamma \in \mathcal{C}_T(x, y)$ such that $\Phi_k(x, y; T) = A_{L+k}(\gamma)$. Therefore $\Phi_k(x, y; T)$ is finite and well-defined.*

Proof. By the superlinearity of L, for any $A > 0$, $\exists B = B(A)$ with $L(x, v) \geq A\|v\| - B$,

$\forall(x,v) \in T\widetilde{M}$, therefore

$$\begin{aligned}\Phi_k(x,y;T) &\geq \inf\{\int_0^T A\|\dot{\gamma}(t)\| - B + kdt : \gamma \in \mathcal{C}_T(x,y)\} \\ &= \inf\{A\int_0^T \|\dot{\gamma}(t)\|dt : \gamma \in \mathcal{C}_T(x,y)\} + (k-B)T \\ &\geq Ad(x,y) + (k-B)T,\end{aligned} \tag{5.3}$$

where $d(x,y)$ is the Riemannian distance, which is positive if $x \neq y$. Hence,

$$\liminf_{T\to 0+} \Phi(x,y;T) \geq Ad(x,y).$$

Since we can choose A arbitrarily large, we have $\lim_{T\to 0^+} \Phi_k(x,y;T) = +\infty$. This proves the first part of the proposition.

For any $x, y \in \widetilde{M}$ and $T > 0$,

$$\begin{aligned}\Phi_k(x,y;T) &= \inf\{A_{L+k}(\gamma) : \gamma \in \mathcal{C}_T(x,y)\} \\ &= \inf\{A_{L+c(L)}(\gamma) : \gamma \in \mathcal{C}_T(x,y)\} + (k-c(L))T \\ &\geq \Phi_{c(L)}(x,y) + (k-c(L))T.\end{aligned} \tag{5.4}$$

Because $k > c(L)$ and $\Phi_{c(L)}(x,y)$ is finite, $\Phi_k(x,y;T)$ goes to infinity as T goes to $+\infty$.

After fixing a $(x_0,y_0) \in U \times V$, we have two finite numbers

$$D_0 := \max\{d(x,x_0) : x \in U\};$$

$$D_1 := \max\{d(y,y_0) : y \in V\},$$

since U and V are compact. Let $Lip(\Phi_{c(L)})$ be the Lipschitz constant of $\Phi_{c(L)}$, we have

$$\begin{aligned}\Phi_k(x,y;T) &= \Phi_{c(L)}(x,y;T) + (k-c(L))T \\ &\geq \Phi_{c(L)}(x,y) + (k-c(L))T \\ &\geq \Phi_{c(L)}(x_0,y_0) - Lip(\Phi_{c(L)})(d(x,x_0) + d(y,y_0)) + (k-c(L))T \\ &\geq \Phi_{c(L)}(x_0,y_0) - Lip(\Phi_{c(L)})(D_0 + D_1) + (k-c(L))T.\end{aligned}$$

Because $\Phi_{c(L)}(x_0,y_0) - Lip(\Phi_{c(L)})(D_0 + D_1)$ is finite and $k - c(L) > 0$, $\Phi_k(x,y;T)$ goes to $+\infty$ uniformly as T goes to $+\infty$. $\square$

The above proposition has the following two obvious corollaries.

Corollary 5.1.5. *For any $x \neq y \in \widetilde{M}$, if $k > c(L)$, then there is a $T_0 > 0$ and $\gamma \in \mathcal{C}_{T_0}(x, y)$, such that $A_{L+k}(\gamma) = \Phi_k(x, y)$.*

Proof. By proposition 5.1.4,

$$\Phi_k(x, y; T) \to +\infty \text{ as } T \to +\infty;$$

$$\Phi_k(x, y; T) \to +\infty \text{ as } T \to 0^+.$$

Hence there is a $T_0 \in (0, +\infty)$ such that

$$\Phi_k(x, y; T_0) = \inf_{T \in (0,+\infty)} \Phi_k(x, y; T) = \Phi_k(x, y).$$

By *Tonelli's Theorem*, there is a $\gamma \in C_{T_0}(x, y)$ with $A_{L+k}(\gamma) = \Phi_k(x, y; T_0)$. Then

$$A_{L+k}(\gamma) = \Phi(x, y),$$

and we are done. $\square$

Corollary 5.1.6. *For any two compact subsets U and V of $\widetilde{M}$ with empty intersection, if $k > c(L)$, then for any positive $E > \min\{\Phi_k(x, y); x \in U, y \in V\} > -\infty$,*

1. *there is a $T_0 > 0$, such that for any $(x, y) \in U \times V$ and $\gamma \in \mathcal{C}_T(x, y)$, $A_{L+k}(\gamma) > E$, if $T \in (0, T_0)$;*

2. *there is a $T_1 > 0$, such that for any $(x, y) \in U \times V$ and $\gamma \in \mathcal{C}_T(x, y)$, $A_{L+k}(\gamma) > E$, if $T \in (T_1, +\infty)$.*

This corollary follows directly from (5.3) and (5.4).

We conclude this section with the well-known *Mather's crossing lemma*, which will be useful in the next section.

Lemma 5.1.7. *Given a Tonelli Lagrangian $\mathbb{L} : TM \to \mathbb{R}$, if $K > 0$, then there exist $\varepsilon, \delta, \zeta, C > 0$, such that if $\gamma, \xi : [-\varepsilon, \varepsilon] \to M$ are solutions of (LS) satisfying*

$$\max\{\|\dot{\gamma}(0)\|_{\gamma(0)}, \|\dot{\xi}(0)\|_{\xi(0)}\} \leq K,\ d(\gamma(0), \xi(0)) \leq \delta,$$

and

$$d((\gamma(0),\dot\gamma(0)),(\xi(0),\dot\xi(0))) \geq Cd(\gamma(0),\xi(0)),$$

then there exist C^1 curves $\alpha,\beta : [-\varepsilon,\varepsilon] \to M$ such that $\alpha(-\varepsilon) = \gamma(-\varepsilon), \alpha(\varepsilon) = \xi(\varepsilon)$, and $\beta(-\varepsilon) = \xi(-\varepsilon), \beta(\varepsilon) = \gamma(\varepsilon)$, satisfying

$$A_{\mathbb{L}}(\gamma) + A_{\mathbb{L}}(\xi) - A_{\mathbb{L}}(\alpha) - A_{\mathbb{L}}(\beta) \geq d((\gamma(0),\dot\gamma(0)),(\xi(0),\dot\xi(0)))^2.$$

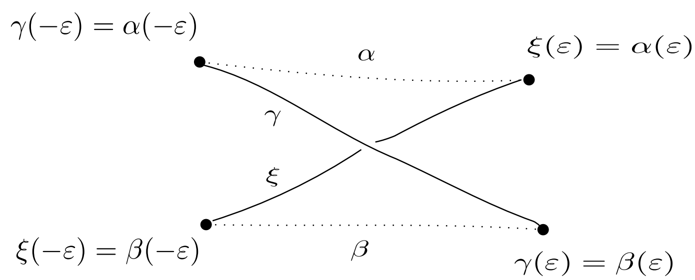

Figure 5.1: Tonelli Lagrangian 1

The proof can be found in [16], where it was proved for the more general case that $\mathbb{L}$ is time periodic. Since it is a local property, it is not hard to see it is still true if we replace $\mathbb{L} : TM \to \mathbb{R}$ by a lift $L : T\widetilde{M} \to \mathbb{R}$.

Even though the lemma does not require γ,ξ crossing each other. We call it Mather's crossing lemma because this is a generalization of an old theorem of Morse about geodesics on a two dimensional surface, where two geodesics are required to cross each other.

5.2 Periodic Free-Time minimizers

In the remainder of this paper we set $M = \mathbb{T}^2$, then $\widetilde{M} = \mathbb{R}^2$, and we simply choose the Euclidean metric on $\mathbb{R}^2$ and $\mathbb{T}^2$. Given an arbitrary Tonelli Lagrangian $\mathbb{L} : T\mathbb{T}^2 \to \mathbb{R}$ and a lift of $\mathbb{L}$, $L : T\mathbb{R}^2 \to \mathbb{R}$, if $\tau = (\tau_1,\tau_2) \in \mathbb{Z}^2 \setminus \{0\}$, let

$$\Gamma_\tau := \{\gamma \in \mathcal{C}(\mathbb{R},\mathbb{R}^2) : \gamma(t+T) = \gamma(t) + \tau, \forall t \in \mathbb{R} \text{ for some } T = T(\gamma) > 0\}.$$

When $\gamma \in \Gamma_\tau$, we say it is a τ-periodic curve with period $T(\gamma)$. To simplify notation, for $\gamma \in \Gamma_\tau$, we set

$$A_{L+k}(\gamma) := A_{L+k}(\gamma|_{[0,T(\gamma)]}).$$

Definition 5.2.1. *For any $k \in \mathbb{R}$ and $\tau \in \mathbb{Z}^2 \setminus \{0\}$, let*

$$c_\tau^k := \inf\{A_{L+k}(\gamma) : \gamma \in \Gamma_\tau\},$$

$$\mathcal{M}_\tau^k := \{\gamma \in \Gamma_\tau : A_{L+k}(\gamma) = c_\tau^k\}.$$

The following theorem can be found in [21].

Theorem 5.2.1. *For any $k > c(L)$ and $\tau \in \mathbb{Z}^2 \setminus \{0\}$, $\mathcal{M}_\tau^k$ is a non-empty set. Moreover every element in $\mathcal{M}_\tau^k$ is a solution of (LS) with energy k.*

If $u \in \mathcal{M}_\tau^k$, it is not hard to see $u|_{[nT(u),(n+1)T(u)]}$ is a free-time minimizer, for any $n \in \mathbb{Z}$. Our next proposition will show that in fact $u|_{\mathbb{R}}$ is a free-time minimizer.

Proposition 5.2.1. *For any $\tau \in \mathbb{Z}^2 \setminus \{0\}$ and $k > c(L)$, if $v \in \mathcal{M}_\tau^k$, then it is a free-time minimizer of $L + k$ and a solution of (LS) with energy k.*

The proposition will follow directly from the next lemma.

Lemma 5.2.2. *If $k > c(L)$ and $\tau = (\tau_1, \tau_2) \in \mathbb{Z}^2 \setminus \{0\}$ with τ_1, τ_2 relatively prime, then $\mathcal{M}_\tau^k = \mathcal{M}_{n\tau}^k$ for any $n \in \mathbb{Z}^+$.*

Proof of Lemma 5.2.2. First we choose another homotopy class $\sigma \in \mathbb{Z}^2 \setminus \{0\}$, such that τ and σ form a basis of $\mathbb{Z}^2$. By Theorem 5.2.1, $\mathcal{M}_\sigma^k$ is a non-empty set. Choose a $w \in \mathcal{M}_\sigma^k$, for any $i \in \mathbb{Z}$, we define $w_i \in \mathcal{C}(\mathbb{R}, \mathbb{R}^2)$ by $w_i(t) = w(t) + i\tau, \forall t \in \mathbb{R}$. If two different points $x, y \in w_i(\mathbb{R})$ satisfy $x = w_i(a), y = w_i(b)$ with $a < b \in \mathbb{R}$, we will say y is above x.

The detailed proof of the lemma will be given for $n = 2$, while the other cases can be proven similarly. For an arbitrary $u \in \mathcal{M}_{2\tau}^k$ with period $S = S(u)$, it must intersect every w_i. Without loss of generality, let's say $u(0) \in w_0(\mathbb{R}), u(S) \in w_2(\mathbb{R})$ and $u(S_0) \in w_1(\mathbb{R})$ for some $S_0 \in (0, S)$. We claim $u(S_0) - \tau = u(0)$.

Assuming this is not true, we define $u^* \in \mathcal{C}(\mathbb{R}, \mathbb{R}^2)$ by $u^*(t) := u(t + S_0) - \tau$ for all $t \in \mathbb{R}$, obviously u^* is 2τ-periodic with period S as well. By the assumption $u(0) \neq u^*(0) = u(S_0) - \tau$, let's say $u^*(0) = u(S_0) - \tau$ is above $u(0)$ on $w_0(\mathbb{R})$. Then $u^*(S)$ is also above $u(S)$ on $w_2(\mathbb{R})$. On the other hand $u(S_0)$ is above $u^*(S - S_0)$ on $w_2(\mathbb{R})$, because $u^*(S - S_0) = u(0) + \tau$. Hence $u((0, S_0))$ and $u^*((0, S - S_0))$ must intersect at least once and so do $u((S_0, S))$ and $u^*((S - S_0, S))$. Assume $u(a) = u^*(b)$ for $a \in$

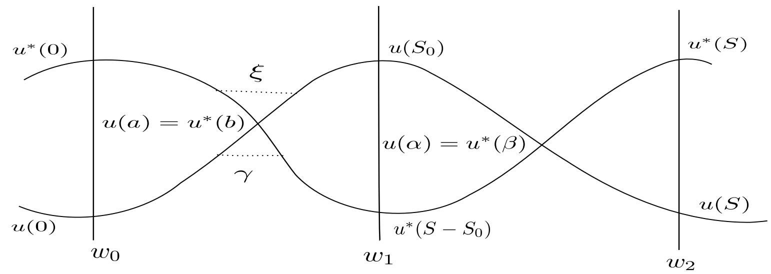

Figure 5.2: Tonelli Lagrangian 2

$(0, S_0), b \in (0, S - S_0)$ and $u(\alpha) = u^*(\beta)$ for $\alpha \in (S - S_0, S), \beta \in (S_0, S)$, as been showed in the following graph.

Because $u \in \mathcal{M}_\tau^k$ and u^* is a shift of u, both $u|_{[0,S]}$ and $u^*|_{[0,S]}$ are free-time minimizers of $L + k$ and solutions of (LS).

Then $\dot{u}(a)$ and $\dot{u^*}(b)$ exist and $\|\dot{u}(a)\|, \|\dot{u^*}(b)\|$ are bounded by some finite number. At the same time $d((u(a), \dot{u}(a)), (u^*(b), \dot{u^*}(b))) > d(u(a), u^*(b)) = 0$ because of the uniqueness of the solution of (LS). Although a, b may not equal to each other, we can shift the time parametrization on u and u^* by a constant as we want.

Therefore we can apply *Mather's crossing lemma* to $L + k$, to get two C^1 curves $\gamma : [b - \varepsilon, b + \varepsilon] \to \mathbb{R}^2, \xi : [a - \varepsilon, a + \varepsilon] \to \mathbb{R}^2$ with $\gamma(b - \varepsilon) = u(a - \varepsilon), \gamma(b + \varepsilon) = u^*(b + \varepsilon)$ and $\xi(a - \varepsilon) = u^*(b - \varepsilon), \xi(a + \varepsilon) = u(a + \varepsilon)$, for a $\varepsilon > 0$ small enough. They satisfy

$$A_{L+k}(u|_{[a-\varepsilon,a+\varepsilon]}) + A_{L+k}(u^*|_{[b-\varepsilon,b+\varepsilon]}) > A_{L+k}(\gamma|_{[b-\varepsilon,b+\varepsilon]}) + A_{L+k}(\xi|_{[a-\varepsilon,a+\varepsilon]}).$$

Therefore,

$$\begin{aligned} A_{L+k}(u|_{[a-\varepsilon,\alpha]}) + A_{L+k}(u^*|_{[b-\varepsilon,\beta]}) &> A_{L+k}(\xi|_{[a-\varepsilon,a+\varepsilon]}) + A_{L+k}(u|_{[a+\varepsilon,\alpha]}) \\ &+ A_{L+k}(\gamma|_{[b-\varepsilon,b+\varepsilon]}) + A_{L+k}(u^*|_{[b+\varepsilon,\beta]}). \end{aligned}$$

Then at least one of the following is true,

$$A_{L+k}(\xi|_{[a-\varepsilon,a+\varepsilon]}) + A_{L+K}(u|_{[a+\varepsilon,\alpha]}) < A_{L+k}(u^*|_{[b-\varepsilon,\beta]}),$$

$$A_{L+k}(\gamma|_{[b-\varepsilon,b+\varepsilon]}) + A_{L+k}(u^*|_{[b+\varepsilon,\beta]}) < A_{L+k}(u|_{[a-\varepsilon,\alpha]}).$$

But this is a contradiction to the fact that $u|_{[0,S]}$ and $u^*|_{[0,S]}$ are free-time minimizers of $L + k$.

Hence $u(S_0) = u(0)+\tau$, which means $[\pi(u|_{[0,S_0]})] = \tau$ and $[\pi(u|_{[S_0,S]})] = \tau$. It follows that $A_{L+k}(u|_{[0,S^*]}) \geq c_\tau$ and $A_{L+k}(u|_{[S^*,S]}) \geq c_\tau$. Therefore $c_{2\tau} \geq 2c_\tau$, on the other hand it is a simple observation that $2c_\tau \geq c_{2\tau}$, so $c_{2\tau} = 2c_\tau$. And the lemma follows immediately from this. $\square$

Proof of Proposition 5.2.1. As we can see if $v \in \mathcal{M}^k_\tau$ is τ-periodic with period T, then $v|_{[0,T]}$ is a free-time minimizer. By Lemma 5.2.2, $v \in \mathcal{M}^k_{n\tau}$ and it is a $n\tau$-periodic curve with period nT, for any $n \in \mathbb{Z}^+$. Therefore, $v|_{[0,nT]}$'s are free-time minimizers, for all $n \in \mathbb{Z}^+$. Since we can shift the time parameter on v, we are done. $\square$

Remark 5.2.1. *We can not generalize Proposition 5.2.1 to Tonelli Lagrangians defined on a manifold with dimension greater than* 2*, because of the examples given by Hedlund [13] and Bangert [58].*

Proposition 5.2.3. *For any $k > c(L)$ and $\tau \in \mathbb{Z}^2 \setminus \{0\}$, $\mathcal{M}^k_\tau$ is an ordered set, i.e., if $u, v \in \mathcal{M}^k_\tau$, either $u(\mathbb{R}) = v(\mathbb{R})$ or $u(\mathbb{R}) \cap v(\mathbb{R}) = \emptyset$.*

Proof. If $u(\mathbb{R}) \neq v(\mathbb{R})$, we claim $u(\mathbb{R}) \cap v(\mathbb{R}) = \emptyset$, otherwise there are $a, b \in \mathbb{R}$ with $u(a) = v(b)$.

Notice that $\dot{u}(a) \neq \dot{v}(b)$. Because u, v both are solutions of (LS), if $\dot{u}(a) = \dot{v}(b)$, then the uniqueness of solutions of (LS) tells us

$$u(t) = v(t + c), \forall t \in \mathbb{R}, \text{ for some } c \in \mathbb{R}.$$

Then $u(\mathbb{R}) = v(\mathbb{R})$ contradicts to our assumption.

By the periodicity of u and v, $u(a + T(u)) = v(b + T(v))$, because $u(a) = v(b)$ and $\dot{u}(a) \neq \dot{v}(b)$, using Mather's crossing lemma and similar arguments as in the proof of Lemma 5.2.2, we can find an $\varepsilon > 0$, such that $u|_{[a-\varepsilon,a+T(u)]}$ and $v|_{[b-\varepsilon,b+T(v)]}$ can not both be free-time minimizers, which is a contradiction to Proposition 5.2.1.

$\square$

5.3 Ray orbits

In the remainder of this paper, some arbitrary $k > c(L)$ and $\tau \in \mathbb{Z}^2 \setminus \{0\}$ will be chosen and fixed, and we assume $v^- \neq v^+ \in \mathcal{M}^k_\tau$ is a pair of neighboring minimal curves.

For any $p \in \mathbb{T}^2$, let

$$\Gamma_{\tau_p} := \{q \in \Gamma_\tau : p \in \pi(q(\mathbb{R}))\},$$

and

$$c^k_{\tau_p} := \inf\{A_{L+k}(q) : q \in \Gamma_{\tau_q}\}.$$

obviously $c^k_{\tau_p} \geq c^k_\tau$.

Let $\mathcal{R}$ be the closed region between $v^-(\mathbb{R})$ and $v^+(\mathbb{R})$ in $\mathbb{R}^2$, then

Proposition 5.3.1. *For any $p \in \mathbb{T}^2$, if $\pi^{-1}(p) \subset int(\mathcal{R})$, then $c^k_{\tau_p} > c^k_\tau$.*

The proposition follows from the definition of neighboring minimal curves.

The rest of this section will be devoted to the proof of Theorem 5.0.1. The detailed proof will be given for the existence of Q^+, while the proof of the others is similar. Our proof follows the variational methods used by Rabinowitz in [50], [51], [54], [55] and Rabinowitz and Stredulinsky in [59], [60], where in the last two papers certain nonlinear elliptic PDE rather than Hamiltonian systems was studied by the authors. First a class of admissible curves Γ^+ will defined. Then we will define a normalized functional J on Γ^+. Existence of minimizers of J in Γ^+ will be shown and these minimizers have the desired asymptotic behaviors. Finally we will prove there is a minimizer of J in Γ^+ which is a k energy solution of (LS).

Remark 5.3.1. *For simplicity, from now on we will omit all the sup-index k and sub-index τ in Γ_τ, $\mathcal{M}^k_\tau$ and c^k_τ.*

To define Γ^+, we choose a $u^* \in \mathcal{C}_{S^+}(p_0, v^+(0))$ with $A_{L+k}(u^*) = \Phi_k(p_0, v^+(0)$. Corollary 5.1.5 guarantees the existence of such a u^*. Parts of u^* may lie out of $\mathcal{R}$ or on the boundary of $\mathcal{R}$. However after cutting those pieces off u^* and making a proper time shifting on v^+, we can assume $u^*((0, S^+)) \subset int(\mathcal{R})$ and $u^*(S^+) = v^+(0)$. We will also need the curves $u \in \mathcal{C}_{S^+}(u^*(S^+), u^*(0))$ defined by $u(t) = u^*(S^+ - t)$ and $u_i \in \mathcal{C}([0, S^+], \mathcal{R})$ defined by $u_i(t) = u(t) + i\tau$ for all $t \in [0, S^+]$ and $i \in \mathbb{Z}$. Obviously, all $A_{L+k}(u_i)$'s are finite and the same. Furthermore let $p_i = p_0 + i\tau$ for all $i \in \mathbb{Z}$.

Definition 5.3.1. *We set*

$$\Gamma^+ := \{q \in \mathcal{C}([0, +\infty), \mathbb{R}^2) : q \textit{ satisfies the following conditions }\}$$

- Γ_1^+: *$q(0) \equiv p_0$ and $q([0, +\infty)) \subset \mathcal{R}$;*
- Γ_2^+: *For each q, there is a monotonically increasing sequence $\{t_i(q) : i \in \mathbb{N}\}$, with $t_0(q) \equiv 0$ and $q(t_i(q)) \in u_i([0, S^+])$;*
- Γ_3^+: *For each q and $\{t_i(q), i \in \mathbb{N}\}$, define $s_i(q) \in [0, S^+]$ by $u_i(s_i(q)) = q(t_i(q))$, for all $i \in \mathbb{N}$, moreover it satisfies $s_{i+1}(q) \leq s_i(q)$, for all $i \in \mathbb{N}$.*

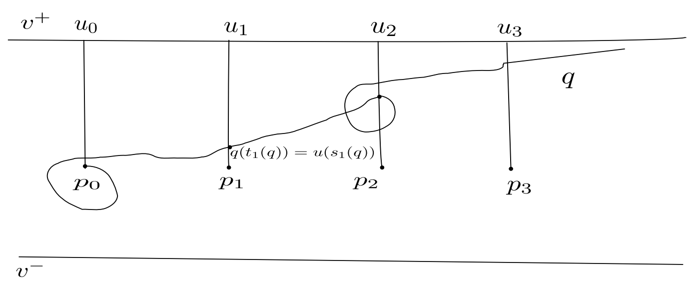

Figure 5.3: Tonelli Lagrangian 3

Obviously Γ^+ is not empty. We define a normalized functional $J : \Gamma^+ \to \mathbb{R} \cup +\infty$ by:

$$J(q) = \sum_{i=0}^{+\infty} a_i(q), \text{ where } a_i(q) = \int_{t_i(q)}^{t_{i+1}(q)} L(dq(t)) + kdt - c.$$

Remark 5.3.2. *For a $q \in \Gamma^+$, we may have different sequences of $\{t_i(q)\}$ satisfying condition Γ_2^+, and $J(q)$ certainly depends on the choice of such a sequence. However if $J(q)$ is finite for a particular choice of sequence $\{t_i(q)\}$, then it is the same for any choice of a sequence satisfying condition Γ_2^+. The reason is exactly the same as Rabinowitz explained in [54], and we omit it here.*

By the superlinearity of L, there is a $B := B(k) > 0$, such that

$$L(x, v) + k + B \geq 0, \quad \forall (x, v) \in \mathbb{R}^4. \tag{5.5}$$

For the rest of this paper B will always be such a positive number which satisfies the above inequality.

Lemma 5.3.2. *Let* $K_1 := \max\{1, A_{L+k}(u) + BS^+\} < +\infty$*, then for any measurable set* $I \subset [0, S^+]$

$$\int_I L(du(t)) + kdt \leq K_1$$

Proof. Because $L(du(t)) + k + B \geq 0$, for all $t \in [0, S^+]$, we have

$$\int_I L(du(t)) + k + Bdt \leq \int_0^{S^+} L(du(t)) + k + Bdt.$$

Let μ be the Lebesgue measure on $\mathbb{R}$, then

$$\begin{aligned}\int_I L(du(t)) + kdt &\leq \int_0^{S^+} L(du(t)) + kdt + B(S^+ - \mu(I)) \\ &\leq A_{L+k}(u)) + BS^+ \\ &\leq K_1.\end{aligned}$$

□

Remark 5.3.3. *Obviously the above lemma still holds with the same constant* K_1 *if we replace* u *by any* u_i*,* $i \in \mathbb{N}$*.*

The proof of the next lemma is almost the same as the proof of Proposition 3.6 in [50] except some estimates.

Lemma 5.3.3. *For any* $q \in \Gamma^+$*, if* K_1 *is the same as defined in Lemma 5.3.2, we have*

1. $\sum_{i=0}^{n-1} a_i(q) \geq -K_1$*,* $\forall n \in \mathbb{Z}^+$*, and* $J(q) \geq -K_1$*;*

2. *If* $J(q) < +\infty$*, then* $\sum_{i=0}^{+\infty} |a_i(q)| \leq J(q) + 2K_1$*.*

Proof. (1) Given a $q \in \Gamma^+$, we set $t_i := t_i(q)$ and $s_i := s_i(q)$, for the sake of simplicity . Now for each $T_i := s_i - s_{i+1} + t_{i+1} - t_i$, $\forall i \in \mathbb{N}$, let's define a $\psi_i \in \mathcal{C}(T_i, \mathcal{R})$ by the following gluing process

$$\psi_i(t) = \begin{cases} u_i(t + s_{i+1}) - i\tau & \text{if } t \in [0, s_i - s_{i+1}] \\ q(t - (s_i - s_{i+1}) + t_i) - i\tau & \text{if } t \in [s_i - s_{i+1}, T_i]. \end{cases}.$$

And for the remaining of this paper we will simply use $*$ to represent such a gluing process, so $\psi = u_i|_{[s_{i+1}, s_i]} * q|_{[t_i, t_{i+1}]} - i\tau$.

If we extend the domain of each ψ_i to $\mathbb{R}$ by setting $\psi_i(t+nT_i) = \psi_i(t) + n\tau$ for any $t \in [0, T_i]$ and $n \in \mathbb{Z}$, then $\psi_i \in \Gamma$ and $A_{L+k}(\psi_i) - c \geq 0$. Therefore

$$\begin{aligned}
\sum_{i=0}^{n-1} a_i(q) &= \sum_{i=0}^{n-1} [A_{L+k}(\psi_i) - \int_{s_{i+1}}^{s_i} L(du_i) + kdt - c] \\
&= \sum_{i=0}^{n-1} [A_{L+k}(\psi_i) - c] - \sum_{i=0}^{n-1} \int_{s_{i+1}}^{s_i} L(du_i) + kdt \\
&\geq - \int_{s_n}^{s_0} L(du) + kdt.
\end{aligned}$$

By Lemma 5.3.2, $-\int_{s_n}^{s_0} L(du) + kdt \geq -K_1$,then

$$\sum_{i=0}^{n-1} a_i(q) \geq -K_1.$$

Therefore $J(q) \geq -K_1$.

(2) For any $q \in \Gamma^+$ with $J(q) < +\infty$, let $N^- = N^-(q) = \{i \in \mathbb{N} : a_i(q) < 0\}$, then

$$\sum_{i=0}^{+\infty} |a_i(q)| = J(q) - 2 \sum_{i \in N^-} a_i(q) \tag{5.6}$$

On the other hand,

$$A_{L+k}(\psi_i) - c = \int_{s_{i+1}}^{s_i} L(du_i) + kdt + \int_{t_i}^{t_{i+1}} L(dq) + kdt - c \geq 0,$$

therefore $-a_i(q) \leq \int_{s_{i+1}}^{s_i} L(du_i) + kdt$. Then

$$-\sum_{i \in N^-} a_i(q) \leq \sum_{i \in N^-} \int_{s_{i+1}}^{s_i} L(du_i) + kdt \leq K_1. \tag{5.7}$$

Finally by (5.6) and (5.7), we have

$$\sum_{i=0}^{+\infty} |a_i(q)| \leq J(q) + 2K_1.$$

□

The next lemma plays an important role in our proof of Theorem 5.0.1, because it provides a convenient way to determine the asymptotic behaviors of curves in Γ^+. Compare to the proof of Proposition 3.12 in [50], our main difficulty here is that we can not take advantage of the special form of the Mechanical Lagrangians.

Lemma 5.3.4. *If $q \in \Gamma^+$ with $J(q) < +\infty$, let $\{t_i(q) : i \in \mathbb{N}\}$ and $\{s_i(q) : i \in \mathbb{N}\}$ be as defined in Definition 5.3.1 then*

1. *$t_{i+1}(q) - t_i(q) \to T^+$, as $i \to +\infty$,*
2. *$q(+\infty) = v^+$.*

Proof. For the seek of simplicity, we set $t_i := t_i(q)$, $s_i := s_i(q)$ for all $i \in \mathbb{N}$.

Claim 5.3.1. *If $q \in \Gamma^+$ with $J(q) < +\infty$, then there are $0 < T^0 < T^1 < +\infty$, such that $T^0 \leq t_{i+1} - t_i \leq T^1, \quad \forall i \in \mathbb{N}$.*

Proof of Claim 5.3.1. If $q \in \Gamma^+$ with $J(q) < +\infty$, by Lemma 5.3.3, $\{a_i(q) : i \in \mathbb{N}\}$ is bounded above by $J(q) + 2K_1$. For each $i \in \mathbb{N}$, $q|_{[t_i,t_{i+1}]}$ is an absolutely continuous curve connecting $u_i([0, s^+])$ and $u_{i+1}([0, S^+])$. We define a sequence of curves $q_i \in \mathcal{C}([0, t_{i+1} - t_i], \mathbb{R}^2)$ by $q_i(t) := q(t + t_i) - i\tau$, for all $i \in \mathbb{N}$. Obviously

$$A_{L+k}(q_i) = A_{L+k}(q|_{[t_i,t_{i+1}]}) = a_i(q) + c \leq J(q) + 2K_1 + c,$$

for every q_i. Therefore $\{A_{L+k}(q_i)\}$ has a finite upper bound independent of i. Since all q_i's are curves started from $u_0([0, S^+])$ and ended at $u_1([0, S^+])$ with $u_0([0, S^+]) \cap u_1([0, S^+]) = \emptyset$, our result follows immediately form Corollary 5.1.6. $\square$

Now we will resume our proof of Lemma 5.3.4. For each $i \in \mathbb{N}$, let $\psi_i \in \mathcal{C}(\mathbb{R}, \mathbb{R}^2)$ be the same as defined in Lemma 5.3.3, with period $T_i = t_{i+1} - t_i + s_i - s_{i+1}$.

First we will show that there is a $\psi \in \mathcal{C}(\mathbb{R}, \mathbb{R}^2)$ such that ψ_i converges uniformly to ψ on any compact intervals along a subsequence, and ψ satisfies

$$\psi(t + n\bar{T}) = \psi(t) + n\tau, \forall t \in \mathbb{R} \text{ and } n \in \mathbb{Z}, \text{ for some } \bar{T} > 0, \tag{5.8}$$

Since $\{s_i : i \in \mathbb{N}\}$ is monotonically decreasing, there must be a $s^* \in [0, S^+]$ satisfying $\lim_{i\to+\infty} s_i = s^*$. By Claim 5.3.1 $\{t_{i+1} - t_i : i \in \mathbb{N}\}$ is bounded away from both 0 and

$+\infty$. Then so is $\{T_i = s_i - s_{i+1} + t_{i_1} - t_i : i \in \mathbb{N}\}$. Hence there is a finite $\bar{T} > 0$ with T_i converges to $\bar{T}$ along a subsequence.

Given an arbitrarily $T^* > 0$ we can find an smallest $n_i \in \mathbb{N}$ for each T_i, such that $T^* \in ((n_i - 1)T_i, n_i T_i]$. It is easy to see $\{n_i : i \in \mathbb{N}\}$ has a finite upper bound $\bar{n} := \sup\{[\frac{T^*}{T_i}] + 1 : i \in \mathbb{N}\}$. Then

$$\begin{aligned} \int_{-T^*}^{T^*} L(d\psi_i) + k + Bdt &\leq \int_{-n_i T_i}^{n_i T_i} L(d\psi_i) + k + Bdt \\ &\leq \int_{-\bar{n} T_i}^{\bar{n} T_i} L(d\psi_i) + k + Bdt \\ &= 2\bar{n}(\int_0^{T_i} L(d\psi_i) + kdt) + 2\bar{n} T_i B. \end{aligned}$$

By the definition of ψ_i, Lemma 5.3.2 and Lemma 5.3.3,

$$\begin{aligned} \int_0^{T_i} L(d\psi_i) + kdt &= a_i(q) + c + \int_{s_{i+1}}^{s_i} L(du) + kdt \\ &\leq J(q) + 2K_1 + c + K_1, \end{aligned}$$

where the right hand side of this inequality is a finite number independent of i. Hence $\{\int_{-T^*}^{T^*} L(d\psi_i)dt : i \in \mathbb{N}\}$ also has an finite upper bound independent of i. Then Lemma 5.1.1 tells us there is a $\psi \in \mathcal{C}([-T^*, T^*], \mathbb{R}^2)$, such that ψ_i converges uniformly to ψ along a subsequence.

Now if we choose a monotonically increasing sequence $T_n^* \nearrow +\infty$, applying the above argument to every T_n^* and after a diagonal extraction, we get a subsequence $\{\psi_{n_i}\}$ converges uniformly to a $\psi \in \mathcal{C}(\mathbb{R}, \mathbb{R}^2)$ on any compact subset of $\mathbb{R}$. Without loss of generality, we assume $\{T_{n_i}\}$, periods of ψ_{n_i}, converges to a $\bar{T} \in \mathbb{R}$, as $i \to +\infty$.

Therefore

$$\psi(\bar{T}) = \lim_{i \to +\infty} \psi_{n_i}(\bar{T}) = \lim_{i \to +\infty} \psi_{n_i}(T_{n_i}) = \lim_{i \to +\infty} \psi_{n_i}(0) + \tau = \psi(0) + \tau.$$

Similar it can be shown that

$$\psi(t + n\bar{T}) = \psi(t) + n\tau, \forall t \in \mathbb{R} \text{ and } n \in \mathbb{Z},$$

which gives us equation (5.8).

It is not hard to see for any $\varepsilon > 0$ small enough,

$$\int_0^{\bar{T}} L(d\psi) + kdt = \lim_{\varepsilon \to 0^+} \int_0^{\bar{T}-\varepsilon} L(d\psi) + kdt.$$

Because $T_{n_i} \geq \bar{T} - \varepsilon$, when i is large enough,

$$\begin{aligned}\liminf_{i \to +\infty} \int_0^{T_{n_i}} L(d\psi_{n_i}) + k + Bdt &\geq \liminf_{i \to +\infty} \int_0^{\bar{T}-\varepsilon} L(d\psi_{n_i}) + k + Bdt \\ &\geq \int_0^{\bar{T}-\varepsilon} L(d\psi) + k + Bdt\end{aligned}$$

The last inequality is because of the lower semi continuity of the action of $L + k + B$. Therefore,

$$\begin{aligned}\int_0^{\bar{T}-\varepsilon} L(d\psi) + kdt &\leq \liminf_{i \to +\infty} \int_0^{T_{n_i}} L(d\psi_{n_i}) + k + Bdt - B(\bar{T} - \varepsilon) \\ &\leq \liminf_{i \to +\infty} \int_0^{T_{n_i}} L(d\psi_{n_i}) + kdt + \lim_{i \to +\infty} BT_{n_i} - B(\bar{T} - \varepsilon) \\ &= \liminf_{i \to +\infty} \int_0^{T_{n_i}} L(d\psi_{n_i}) + kdt + B\epsilon.\end{aligned}$$

Since ε can be chosen arbitrarily small, we get

$$\int_0^{\bar{T}} L(d\psi) + kdt \leq \liminf_{i \to +\infty} \int_0^{T_{n_i}} L(d\psi_{n_i}) + kdt. \tag{5.9}$$

Now we will prove $\int_0^{\bar{T}} L(d\psi) + kdt = c$.

Because $J(q) < +\infty$, by lemma 5.3.3

$$\lim_{i \to +\infty} a_i(q) = \lim_{i \to +\infty} \int_{t_i}^{t_{i+1}} L(dq) + kdt - c = 0.$$

On the other hand, since $s_i - s_{i+1} \to 0$ as $i \to +\infty$,

$$\lim_{i \to +\infty} \int_{s_{i+1}}^{s_i} L(du_i) + kdt = 0.$$

By the definition of ψ_i,

$$\int_0^{T_{n_i}} L(d\psi_{n_i}) + kdt = a_i(q) + c + \int_{s_{i+1}}^{s_i} L(du_{n_i}) + kdt,$$

therefore

$$\lim_{i\to+\infty}\int_0^{T_{n_i}} L(d\psi_i)+kdt=c,$$

and by (5.9) we get $\int_0^{\bar{T}} L(d\psi)+kdt\le c$.

On the other hand (5.8) implies $\psi\in\Gamma$ and $\int_0^{\bar{T}} L(d\psi)+kdt\ge c$. Therefore $\int_0^{\bar{T}} L(d\psi)+kdt=c$ and ψ belongs to $\mathcal{M}$.

Because ψ lies entirely in $\mathcal{R}$ and v^-,v^+ is a pair of neighboring minimal curves, ψ must be one of them. By the definition of Γ^+, it is easy to see it can not be v^-. Hence $\psi=v^+$ up to a time-shift and $\bar{T}=T^+$.

Therefore along a subsequence, $\{t_{i+1}-t_i\}$ converges to T^+ and ψ_i converges uniformly to v^+ on any compact intervals. However the uniqueness of the limit implies it is true along the entire sequence, which proves our lemma. □

Proposition 5.3.5. *Let $c^+:=\inf\{J(q):q\in\Gamma^+\}$ then c^+ is finite and there is a $Q\in\Gamma^+$ with $J(Q)=c^+$.*

Proof. First we will show c^+ is finite, by Lemma 5.3.3 $c^+\ge -K_1>-\infty$, here K_1 is defined as in Lemma 5.3.2. Let $\hat{q}:=u^*|_{[0,S^+]}*v^+|_{[0,+\infty)}\in\Gamma^+$, obviously $\hat{q}\in\Gamma^+$ and $J(\hat{q})<+\infty$, so $c^+<+\infty$.

By the definition of c^+, we can choose a minimizing sequence $\{q^m\in\Gamma^+:n\in\mathbb{N}\}$, i.e., $\lim_{m\to+\infty}J(q^m)=c^+$. Without loss of generality, we fix a $E>c^+$ and assume

$$J(q^m)<E,\quad \forall m\in\mathbb{N}. \tag{5.10}$$

Claim 5.3.2. *There is a $E^*>0$ and $0<T^0<T^1<+\infty$, such that*

1. $\int_{t_i(q^m)}^{t_{i+1}(q^m)} L(dq^m)+kdt\le E^*,\quad \forall i,m\in\mathbb{N}$;
2. $T^0\le t_{i+1}(q^m)-t_i(q^m)\le T^1,\quad \forall i,m\in\mathbb{N}$.

Proof of Claim 5.3.2. (1) By (5.10) and Lemma 5.3.3, $\sum_0^{+\infty}|a_i(q^m)|\le E+2K_1$, then $a_i(q^m)\le E+2K_1$, which means

$$\int_{t_i(q^m)}^{t_{i+1}(q^m)} L(dq^m)+kdt\le E+2K_1+c:=E^*\quad \forall i,m\in\mathbb{N}.$$

(2) For any $i, m \in \mathbb{N}$, we define a $q_i^m \in \mathcal{C}([0, t_{i+1}(q^m) - t_i(q^m)], \mathcal{R})$ by $q_i^m(t) = q^m(t + t_i(q^m)) - i\tau$, for $t \in [0, t_{i+1}(q^m) - t_i(q^m)]$. Then $\{q_i^m : m, i \in \mathbb{N}\}$ is a set of curves connecting $u_0([0, S^+])$ and $u_1([0, S^+])$. From what we just proved, the actions of $L + k$ on all q_i^m's are bounded by E^* and corollary 5.1.6 gives the desired result. □

Now we resume our proof of Proposition 5.3.5.

Choose an arbitrary $T^* > 0$, by Claim 5.3.2, for each q^m there is a smallest $j_m \in \mathbb{N}$, such that $T^* \in (t_{j_m-1}(q^m), t_{j_m}(q^m)]$ and j_m has an finite upper bound $n^* = [\frac{T^*}{T^0}] + 1$ independent of m, so

$$\begin{aligned}
\int_0^{T^*} L(dq^m) + k + Bdt &\le \int_0^{t_{j_m}(q^m)} L(dq^m) + k + Bdt, \\
\int_0^{T^*} L(dq^m) + kdt &\le \int_0^{t_{j_m}(q^m)} L(dq^m) + kdt + B(t_{j_m}(q^m) - T^*) \\
&\le \sum_{i=0}^{j_m-1} a_i(q^m) + j_m c + B(t_{j_m}(q^m) - t_{j_m-1}(q^m)) \\
&\le \sum_{i=0}^{+\infty} |a_i(q^m)| + n^* c + BT^1 \\
&\le E + 2K_1 + n^* c + BT^1.
\end{aligned}$$

Therefore the actions $A_{L+k}(q^m|_{[0,T^*]})$ have a finite upper bound independent of m. By Lemma 5.1.1, there is a $Q_{T^*} \in \mathcal{C}([0, T^*], \mathbb{R}^2)$, such that q^m converges uniformly to Q_{T^*} on $[0, T^*]$ along a subsequence. Since this is true for any T^*, after a diagonal extraction as in the proof of Lemma 5.3.4, we have q^m converges uniformly to a $Q \in \mathcal{C}([0, +\infty), \mathbb{R}^2)$ along a subsequence on any compact sub-interval of $[0, +\infty)$. We rename the subsequence as $\{q^m\}$.

Now we will show that Q satisfies conditions Γ_1^+, Γ_2^+ and Γ_3^+. The first one is obvious, since q^m converges uniformly to Q on any compact sub-interval.

By Claim 5.3.2, for any fixed i, $\{t_i(q^m) : m \in \mathbb{N}\}$ is bounded, so along a subsequence $t_i(q^m)$ converges to a finite t_i^* as $m \to +\infty$. Again by a diagonal extraction and renaming the subsequence as $\{q^m\}$, we may assume the minimizing sequence $\{q^m\}$ satisfies

$$\lim_{m \to +\infty} t_i(q^m) = t_i^* \in \mathbb{R}, \quad \forall i \in \mathbb{N},$$

.

Let $t_i(Q) := t_i^*$, obviously $\{t_i(Q) : i \in \mathbb{N}\}$ is monotonically increasing, since $\{t_i(q^m) : i \in \mathbb{N}\}$ is monotonically increasing for any m.

$Q(t_i(Q)) = \lim_{m\to+\infty} q^m(t_i(q^m))$ lies on $u_i([0, S^+])$, and therefore Γ_2^+ is satisfied.

For each $i \in \mathbb{N}$, we define a $s_i(Q) \in [0, S^+]$ by $u_i(s_i(Q)) = Q(t_i(Q))$. Since

$$u_i(s_i(Q)) = Q(t_i(Q)) = \lim_{m\to+\infty} q^m(t_i(q^m)) = \lim_{m\to+\infty} u_i(s_i(q^m)),$$

and $s_{i+1}(q^m) \leq s_i(q^m), \forall i \in \mathbb{N}$, we have $s_{i+1}(Q) \leq s_i(Q)$. Therefore Q satisfies Γ_3^+.

The only thing left is to show that $J(Q) = c^+$. To prove this we define $J_l(q) := \sum_{i=0}^{l-1} a_i(q)$, for any $l \in \mathbb{Z}^+$ and $q \in \Gamma^+$.

Claim 5.3.3. *For Q and q^m defined as above, we have*

$$J_l(Q) \leq \liminf_{m\to\infty} J_l(q^m). \quad \forall l \in \mathbb{Z}^+$$

Proof of Claim 5.3.3. From the above proof, we have $t_l(Q) = \lim_{m\to+\infty} t_l(q^m)$, so for any small enough $\varepsilon > 0$, $t_l(q^m) \geq t_l(Q) - \varepsilon$ for m large enough. By the lower semi-continuity of the action A_{L+k},

$$\begin{aligned}\int_0^{t_l(Q)-\varepsilon} L(dQ) + k + Bdt &\leq \liminf_{m\to+\infty} \int_0^{t_l(Q)-\varepsilon} L(dq^m) + k + Bdt \\ &\leq \liminf_{m\to+\infty} \int_0^{t_l(q^m)} L(dq^m) + k + Bdt \\ &\leq \liminf_{m\to+\infty} \int_0^{t_l(q^m)} L(dq^m) + kdt + \lim_{m\to+\infty} Bt_l(q^m) \\ &\leq \liminf_{m\to+\infty} \int_0^{t_l(q^m)} L(dq^m) + kdt + Bt_l(Q).\end{aligned}$$

Therefore

$$\int_0^{t_l(Q)-\varepsilon} L(dQ) + kdt - B\varepsilon \leq \liminf_{m\to+\infty} \int_0^{t_l(q^m)} L(dq^m) + kdt.$$

Since we can choose ε arbitrarily small

$$\int_0^{t_l(Q)} L(dQ) + kdt \leq \liminf_{m\to+\infty} \int_0^{t_l(q^m)} L(dq^m) + kdt.$$

After minus both side by lc, we get the desired inequality. $\square$

Back to the proof of Proposition 5.3.5, by Claim 5.3.3, for any $l \in \mathbb{N}$,

$$J_l(Q) \leq \liminf_{m\to\infty} J_l(q^m) \leq \liminf_{m\to+\infty} \sum_{i=0}^{+\infty} |a_i(q^m)|.$$

Because $J(q^m) < E$, for any m, by Lemma 5.3.3,

$$\sum_{i=0}^{+\infty} |a_i(q^m)| \leq E + 2K_1.$$

Therefore for any $l \in \mathbb{N}$,

$$J_l(Q) \leq E + 2K_1,$$

and

$$J(Q) \leq E + 2K_1 < +\infty.$$

Again by Lemma 5.3.3, we have

$$\sum_{i=0}^{+\infty} |a_i(Q)| \leq E + 4K_1.$$

Fixing an arbitrary $\varepsilon > 0$, because $Q \in \Gamma^+$ and $J(Q) < +\infty$, by the proof of Lemma 5.3.4, we have $s_i(Q) \to 0$ as $i \to +\infty$. There is an $l = l(\varepsilon)$, such that

$$s_j(Q) \leq \varepsilon \text{ and } J(Q) \leq J_j(Q) + \varepsilon, \quad \forall j \geq l. \tag{5.11}$$

By Claim 5.3.3, $J_j(Q) \leq \liminf_{m\to+\infty} J_j(q^m)$, so there is a $n_0 = n_0(\varepsilon)$, such that for all $m > n_0$,

$$J_l(Q) \leq J_l(q^m) + \varepsilon. \tag{5.12}$$

Since q^m is a minimizing sequence, there is a $n_1 = n_1(\varepsilon)$ with

$$J(q^m) \leq c^+ + \varepsilon, \quad \forall m \geq n_1. \tag{5.13}$$

By (5.11), (5.12) and (5.13), for $m \geq \max\{n_0, n_1\}$, we have

$$J(Q) \leq J(q^m) - \sum_{i>l} a_i(q^m) + 2\varepsilon \leq c^+ + 3\varepsilon - \sum_{i>l} a_i(q^m).$$

On the other hand, for any $i, m \in \mathbb{N}$,

$$a_i(q^m) + \int_{s_{i+1}(q^m)}^{s_i(q^m)} L(du_i) + kdt \geq 0,$$

so

$$-a_i(q^m) \leq \int_{s_{i+1}(q^m)}^{s_i(q^m)} L(du_i) + kdt,$$

and

$$-\sum_{i>l} a_i(q^m) \leq \int_0^{s_{l+1}(q^m)} L(du) + kdt.$$

Therefore

$$\begin{aligned} J(Q) &\leq c^+ + 3\varepsilon + \int_0^{s_{l+1}(q^m)} L(du) + kdt \\ &\leq c^+ + 3\varepsilon + \int_0^{s_{l+1}(q^m)} L(du) + k + Bdt \\ &\leq c^+ + 3\varepsilon + \int_0^{2\varepsilon} L(du) + k + Bdt \\ &\to c^+ \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Because $Q \in \Gamma^+$, the other direction of the inequality is trivial.

□

The next lemma shows the minimizers we found in Propostion 5.3.5 satisfies certain monotonicity.

Lemma 5.3.6. *For any $Q \in \Gamma^+$, if $J(Q) = c^+$, then $s_i(Q) \neq 0$ and $s_{i+1}(Q) < s_i(Q)$, for all $i \in \mathbb{N}$.*

Proof. First we will prove $s_i(Q) \neq 0, \forall i \in \mathbb{N}$. Assuming this is not true, then there is a smallest $i_0 \in \mathbb{Z}^+$ with $s_{i_0}(Q) = 0$. Therefore $s_i(Q) = s_{i_0}(Q) = 0$ for all $i \geq i_0$, which means

$$Q(t_i(Q)) = u_i(s_i(Q)) = u_i(0) = v^+(iT^+), \text{ for all } i \geq i_0.$$

.

Because Q is a minimizer of J in Γ^+, we must have

$$A_{L+k}(Q|_{[t_i(Q), t_{i+1}(Q)]}) \leq A_{L+k}(v^+|_{[iT^+, (i+1)T^+]}) = c, \forall i \geq i_0.$$

Therefore $A_{L+k}(Q|_{[t_i(Q),t_{i+1}(Q)]}) = c$, for all $i \geq i_0$. On the other hand, v^-, v^+ is a pair of neighboring minimal curves, so we must have $Q|_{[t_{i_0}(Q),+\infty)}$ equal to $v^+|_{[i_0T^+,+\infty)}$ up to a time-shift.

There is a $t^* \in (t_{i_0-1}(Q), t_{i_0}(Q)]$ satisfies $Q([t^*, t_{i_0}(Q)]) \subset v^+(\mathbb{R})$ and $Q(t) \notin v^+(\mathbb{R})$ for any $t < t^*$ but close enough to t^*.

Hence Q is not a solution of (LS) at t^* and we can find a small neighborhood $[\alpha, \beta]$ of t^* and a $\gamma \in \mathcal{C}_{[\alpha,\beta]}(Q(\alpha), Q(\beta))$, such that

$$A_{L+k}(\gamma|_{[\alpha,\beta]}) < A_{L+k}(Q|_{[\alpha,\beta]}).$$

Because v^+ is a free-time minimizer, we can assume $\gamma([\alpha, \beta]) \subset \mathcal{R}$. Replacing $Q|_{[\alpha,\beta]}$ by γ, we get a new curve Q^* which also belongs to Γ^+ , if we choose $[\alpha, \beta]$ small enough. By the way Q^* is defined, we have $J(Q^*) < J(Q) = c^+$, which contradicts the fact that $Q^* \in \Gamma^+$. We finished the first part of this lemma.

For the second part, it is enough to show that $s_{i+1}(Q) \neq s_i(Q)$ for any $i \in \mathbb{N}$. Assume this is not true, then there is a $i_1 \in \mathbb{N}$ with $s_{i_1+1}(Q) = s_{i_1}(Q)$. From the above proof, we know $Q(t_{i_1}(Q)) = u_{i_1}(s_{i_1}(Q)) \notin v^+(\mathbb{R})$, then

$$a_{i_1}(Q) = A_{L+k}(Q|_{[t_{i_1},t_{i_i+1}]}) - c > 0.$$

We define a new curve $Q^* \in \mathcal{C}([0, +\infty), \mathbb{R})$ by

$$Q^*(t) = \begin{cases} Q(t) & \text{if } t \in [0, t_{i_1}(Q)], \\ Q(t + t_{i_1+1}(Q) - t_{i_1}(Q)) - \tau & \text{if } t \in [t_i(Q), +\infty). \end{cases}$$

Obviously $Q^* \in \Gamma^+$ and $J(Q^*) = J(Q) - (a_{i_1}(Q) - c) < c^+$, since $a_{i_1}(Q) - c > 0$. This is a contradiction, which proves the second part of the lemma. □

Now we will show that among the minimizers of J in Γ^+, there is a solution of (LS). This is achieved by showing that such a minimizer is a local free-time minimizer of $L+k$.

Proposition 5.3.7. *There is a $Q^+ \in \Gamma^+$ with $J(Q^+) = c^+$ which is a solution of (LS) with $Q^+(0) = p_0$ and $Q(+\infty) = v^+$.*

Proof. We will find a $Q^+ \in \Gamma^+$ with $J(Q^+) = c^+$ and show that for any $t^* \in \mathbb{R}$ there is a small enough neighborhood $[\alpha, \beta]$ of t^*, such that $Q^+|_{[\alpha,\beta]}$ is a local free-time minimizer of $L + k$.

First by Propostion 5.3.5, there is a $Q \in \Gamma^+$ with $J(Q) = c^+$. For simplicity we set $t_i := t_i(Q)$ and $s_i := s_i(Q)$ for all $i \in \mathbb{N}$. For any $t^* \in \mathbb{R}$, if $Q(t^*) \in int(\mathcal{R})$, we can choose a small interval $[\alpha, \beta]$ with $t^* \in (\alpha, \beta)$, and a small open neighborhood U of $Q([\alpha, \beta])$ with $U \subset int(\mathcal{R})$. If $Q|_{[\alpha,\beta]}$ is not a local free-time minimizer in U, we can find a $\gamma \in \mathcal{C}([a, b], U)$ with $\gamma(a) = Q(\alpha), \gamma(b) = Q(\beta)$ and $A_{L+k}(\gamma) < A_{L+k}(Q|_{[\alpha,\beta]})$, we can replace $Q|_{[\alpha,\beta]}$ by γ to get a new curve Q^*. Because of Lemma 5.3.6, it is not hard to see that if we choose $[\alpha, \beta]$ and U small enough, Q^* will still satisfies Γ_3^+ in Definition 5.3.1. Then $Q^* \in \Gamma^+$ and $J(Q^*) < J(Q) = c^+$, which is a contradiction.

If $Q(t^*)$ lies on the boundary of $\mathcal{R}$, there are two different cases: the first case is Q intersects v^+ (or v^-) at $Q(t^*)$ from the same direction of v^+ (or v^-); the second case is Q intersects v^+ (or v^-) at $Q(t^*)$ from the opposite direction of v^+ (or v^-). We will discuss the two cases separately.

Case 1: Assume there is a small enough neighborhood $[\alpha, \beta]$ of t^* such that $Q|_{[\alpha,\beta]}$ is a not a local free-time minimizer of $L + k$. Then there is a $\gamma \in \mathcal{C}([a, b], \mathbb{R}^2)$ with $\gamma(a) = Q(\alpha), \gamma(b) = Q(\beta)$ and $A_{L+k}(\gamma) < A_{L+k}(Q|_{[\alpha,\beta]})$. We can always assume γ lies entirely in $\mathcal{R}$, because proposition 5.2.1 tells us $v^\pm$ are free-time minimizers of $L + k$. Therefore we can replace the pieces of γ lying outside of $\mathcal{R}$ by pieces of v^+ or v^- with the same end points without increasing the action of $L + k$. Now we can construct a $Q^* \in \Gamma^+$ and draw a contradiction as before.

Case 2: Different arguments will be used for $Q(t^*) \in v^+(\mathbb{R})$ and $Q(t^*) \in v^-(\mathbb{R})$.

If $Q(t^*) \in v^+(\mathbb{R})$, then there is a $j \in \mathbb{N}$ with $t^* \in (t_j, t_{j+1})$, and an $m \in \mathbb{Z}$ with $Q(t^*) \in v^+([mT^+, (m+1)T^+))$. Since $J(Q) < +\infty$, Lemma 5.3.4 implies $Q(+\infty) = v^+$. Hence there is a $l > j$ large enough, such that $Q(a) = Q(b) - (l-m)\tau$ for some $a \in (t_j, t^*)$ and $b \in (t_l, t_{l+1})$, as indicated in the following graph.

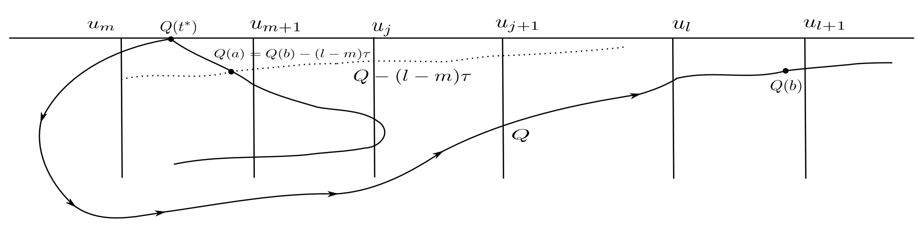

Figure 5.4: Tonelli Lagrangian 4

Let $Q^* = Q|_{[0,a]} * (Q|_{[b,+\infty)} - (l-m)\tau)$, i,e.,

$$Q^*(t) = \begin{cases} Q(t), & \text{if } t \in [0,a] \\ Q(t-a+b) - (l-m)\tau, & \text{if } t \in [a,+\infty) \end{cases}$$

It is not hard to see $Q^* \in \Gamma^+$ with $t_i(Q^*) = t_i(Q)$ if $0 \leq i \leq j$ and $t_i(Q^*) = t_{i+l-m}(Q) - b + a$ if $i > j$. Hence

$$\begin{aligned} J(Q^*) = \sum_{i=0}^{j-1} a_i(Q) + \sum_{i=j+l-m+1}^{+\infty} a_i(Q) + \int_{t_j(Q)}^{a} L(dQ) + kdt \\ + \int_{b}^{t_{j+l-m+1}(Q)} L(dQ) + kdt - c. \end{aligned} \tag{5.14}$$

On the other hand

$$\begin{aligned} J(Q) = &\sum_{i=0}^{j-1} a_i(Q) + \sum_{i=j+l-m+1}^{+\infty} a_i(Q) + \int_{t_j(Q)}^{a} L(dQ) + kdt \\ &+ \int_a^b L(dQ) + kdt + \int_{b}^{t_{j+l-m+1}(Q)} L(dQ) + kdt - (l-m+1)c. \end{aligned}$$

As a homotopy class $[\pi(Q|_{[a,b]})] = (l-m)\tau$ and it is easy to see $Q(a) = Q(b) - (l-m)\tau \in int(\mathcal{R})$, therefore

$$\int_a^b L(dQ) + kdt - (l-m)c > 0.$$

Plugging this into the previous equation we get

$$J(Q) > \sum_{i=0}^{j-1} a_i(Q) + \sum_{i=j+l-m+1}^{+\infty} a_i(Q) + \int_{t_j}^{a} L(dQ) + kdt + \int_{b}^{t_{j+l-m+1}} L(dQ) + kdt - c.$$

Then (5.14) implies $J(Q^*) < J(Q) = c^+$, it is a contradiction to the fact that $Q^* \in \Gamma^+$

If $Q(t^*) \in v^-(\mathbb{R})$, then there is a $j \in \mathbb{N}$ with $t^* \in (t_j, t_{j+1})$. Let $\alpha_j = t^*$, because $u_j|_{[0,s_j]} * Q|_{[t_j,\alpha_j]}$ separate $\mathcal{R}$ into two disconnected regions, there must be a $\beta_j \in (\alpha_j, t_{j+1})$ with either $Q(\beta_j) \in Q((t_j,\alpha_j))$ (see the following left graph) or $Q(\beta_j) \in u_j([0,s_j])$ (see the following right graph).

If $Q(\beta_j) \in Q([t_j,\alpha_j))$, assume $Q(\beta_j) = Q(\eta_j)$ for some $\eta_j \in [t_j,\alpha_j)$, then $Q|_{[\eta_j,\beta_j]}$ is a closed curve. Since $k > c(L)$, by the definition of $c(L)$, $A_{L+k}(Q|_{[\eta_j,\beta_j]}) \geq 0$. Therefore if we cut off $Q|_{[\eta_j,\beta_j]}$ from Q, we get a $Q^* \in \Gamma^+$ and obviously $J(Q^*) \leq J(Q)$.

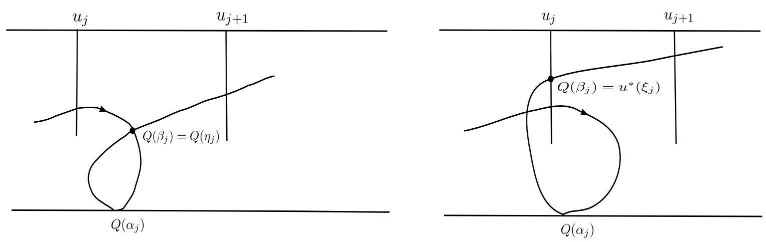

Figure 5.5: Tonelli Lagrangian 5

If $Q(\beta_j) \in u_j([0, s_j])$, then $Q(\beta_j) = u_j^*(\xi_j)$ for some $\xi_j \in [S^+ - s_j, S^+]$, because u_j^* is a free-time minimizer, $A_{L+k}(u_j^*|_{[S^+ - s_j, \xi_j]}) \leq A_{L+k}(Q|_{[t_j, \beta_j]})$. Replacing $Q|_{[t_j,\beta_j]}$ by $u_j^*|_{[S^+ s_j, \xi_j]}$, we get a $Q^* \in \Gamma^+$. Obviously $J(Q^*) \leq J(Q)$.

By the above arguments we can get rid of all the points on $Q(\mathbb{R})$ which intersect the boundary of $\mathcal{R}$ from the opposite direction to get a $Q^+ \in \Gamma^+$ with $J(Q^+) = c^+$, such a Q^+ is a k energy solution of (LS) with $Q^+(0) = p_0$ and $Q^+(+\infty) = v^+$. □

5.4 Heteroclinic orbits

We will prove Theorem 5.0.2 in this section. The detailed proof will be given for the existence of $\bar{Q}^+$, while the existence of $\bar{Q}^-$ can be shown similarly. The idea of the proof is similar to the one we used in the previous section. This time we will define an admissible class of curves $\bar{\Gamma}^+ \subset \mathcal{C}(\mathbb{R}, \mathbb{R}^2)$.

By Corollary 5.1.5, there is a $S > 0$ and $w \in \mathcal{C}_S(v^+(0), v^-(0))$ with $A_{L+k}(w) = \Phi_k(v^+(0), v^-(0))$. Similar to what we said about u^* in the previous section, we may assume $w((0, S)) \subset int(\mathcal{R})$.

Like Lemma 5.3.2, we have

Lemma 5.4.1. *Let* $K_2 := \max\{1, A_{L+k}(w) + BS\} > 0$*, then for any measurable set* $I \subset [0, S]$*, we have*

$$\int_I L(dw) + kdt \leq K_2$$

Proof. The proof is similar to Lemma 5.3.2 and we omit it here. □

Now we can give the precise definition of our admissible class of curves $\bar{\Gamma}^+$.

Definition 5.4.1. *Let* $\bar{\Gamma}^+ := \{q \in \mathcal{C}(\mathbb{R}, \mathbb{R}^2) : q$ *satisfies conditions* $\bar{\Gamma}_1^+$ *to* $\bar{\Gamma}_4^+\}$ *where,*

- $\bar{\Gamma}_1^+$: *$q(\mathbb{R}) \subset \mathcal{R}$;*

- $\bar{\Gamma}_2^+$: *For each q, there is a monotonically increasing sequence $\{t_i(q) : i \in \mathbb{Z}\}$, with $q(t_i(q)) \in w_i([0, S])$ and $t_0(q) = 0$;*

- $\bar{\Gamma}_3^+$: *For each q and $t_i(q), i \in \mathbb{N}$, define $s_i(q)$ by $w_i(s_i(q)) = q(t_i(q))$, for all $i \in \mathbb{Z}$, furthermore they satisfies $s_{i+1}(q) \leq s_i(q)$, for all $i \in \mathbb{Z}$;*

- $\bar{\Gamma}_4^+$: *$q(+\infty) = v^+$ and $q(-\infty) = v^-$.*

We set $\hat{\Gamma}^+ := \{q \in \mathcal{C}(\mathbb{R}, \mathbb{R}^2) : q$ satisfies conditions $\bar{\Gamma}_1^+, \bar{\Gamma}_2^+$ and $\bar{\Gamma}_3^+\}$. The normalized functional J in the previous section can be extended to this new class $\hat{\Gamma}^+$ in the following way

$$J(q) = \sum_{i=-\infty}^{+\infty} a_i(q), \text{ where } a_i(q) = \int_{t_i(q)}^{t_{i+1}(q)} L(dq) + kdt - c.$$

Lemma 5.4.2. *For any $q \in \hat{\Gamma}^+$, if K_2 is the same as defined in Lemma 5.4.1, we have*

1. *$\sum_{i=-n}^{n-1} a_i(q) \geq -K_2$, $\forall n \in \mathbb{N}$, therefore $J(q) \geq -K_2$;*

2. *If $J(q) \leq +\infty$, then $\sum_{i=-\infty}^{+\infty} |a_i(q)| \leq J(q) + 2K_2$.*

Lemma 5.4.3. *If $q \in \hat{\Gamma}^+$ with $J(q) < +\infty$, set $t_i := t_i(q)$, for all $i \in \mathbb{Z}$, then*

1. *$t_{i+1} - t_i \to T^{\pm}$, as $i \to \pm\infty$.*

2. *$q(\pm\infty) = v^+$ or v^-.*

The proofs of the above two lemmas are similar to the proofs of Lemma 5.3.3 and Lemma 5.3.4, so we omit them here. Since $\bar{\Gamma}^+$ is a subset of $\hat{\Gamma}^+$, the above two lemmas apply to any $q \in \bar{\Gamma}^+$ as well.

Proposition 5.4.4. *Let $\bar{c}^+ := \inf\{J(q) : q \in \bar{\Gamma}^+\}$, then $\bar{c}^+$ is finite and there is a $Q \in \bar{\Gamma}^+$ with $J(Q) = \bar{c}^+$.*

Proof. First we will show $\bar{c}^+$ is finite. Define a curve $\hat{q} \in \mathcal{C}(\mathbb{R}, \mathbb{R}^2)$ by $\hat{q} := v^-|_{(-\infty,0]} * w^*|_{[0,S]} * v^+|_{[0,+\infty)}$, where $w^* \in \mathcal{C}_S(v^-(0), v^+(0))$ satisfies $w^*(t) := w(S - t), \forall t \in [0, S]$.

Obviously $\hat{q} \in \bar{\Gamma}$ and $\bar{c}^+ \leq A_{L+k}(\hat{q}) < +\infty$. Together with Lemma 5.4.2 we showed that $\bar{c}^+$ is finite.

Let $q^m \in \bar{\Gamma}^+$ be a minimizing sequence, i.e., $\lim_{m\to+\infty} J(q^m) = \bar{c}^+$, following almost exact the same argument as in the proof of Proposition 5.3.5, we find a $Q \in \hat{\Gamma}^+$, i.e., Q satisfies conditions $\bar{\Gamma}_1^+, \bar{\Gamma}_2^+$ and $\bar{\Gamma}_3^+$, and q^m converges uniformly to Q on any compact sub-interval along a subsequence.

Without loss of generality, we will assume $J(q^m) < E$ for some $E > 0$, $\forall m \in \mathbb{N}$.

Because $J(q^m)$ is invariant when we shift q^m by any integer multiplying τ, for an arbitrary $\delta \in (0, S)$, we may assume the following is true for any q^m

$$s_i(q^m) \in [S-\delta, S] \text{ if } i \leq 0,$$
$$s_i(q^m) \in [0, S-\delta] \text{ if } i > 0.$$

Because $s_i(Q) = \lim_{m\to+\infty} s_i(q^m)$ for all $i \in \mathbb{Z}$, we have

$$s_i(Q) \in [S-\delta, S] \text{ if } i \leq 0, \tag{5.15}$$

$$s_i(Q) \in [0, S-\delta] \text{ if } i > 0. \tag{5.16}$$

By arguments similar to those in Proposition 5.3.5, we can show

$$J(Q) \leq E + 4K_2 < +\infty.$$

Then by Lemma 5.4.3, $Q(\pm\infty) = v^+$ or v^-. However (5.15), (5.16) guarantee $Q(+\infty) = v^+$ and $Q(-\infty) = v^-$. Therefore Q also satisfies condition $\bar{\Gamma}_4^+$ and $Q \in \bar{\Gamma}^+$.

Again following the same argument as in the proof of Proposition 5.3.5, we can show that $J(Q) = \bar{c}^+$.

□

Like Lemma 5.3.6, the minimizers of J in $\bar{\Gamma}^+$ satisfy a similar monotone property.

Lemma 5.4.5. *If $Q \in \bar{\Gamma}^+$ with $J(Q) = \bar{c}^+$, then $s_{i+1}(Q) < s_i(Q)$ and $s_i(Q) \neq 0$ or S, for all $i \in \mathbb{Z}$.*

Proposition 5.4.6. *There is a $\bar{Q}^+ \in \bar{\Gamma}^+$ with $J(\bar{Q}^+) = \bar{c}^+$, which is a solution of (LS) with energy k and a heteroclinic orbit from v^- to v^+.*

Proof. As in the proof of Proposition 5.3.7, we will find a $\bar{Q}^+ \in \bar{\Gamma}^+$ with $J(\bar{Q}^+) = \bar{c}^+$, such that for any $t^* \in \mathbb{R}$, there is a small enough neighborhood $[\alpha, \beta]$ of t^*, where $\bar{Q}^+|_{[\alpha,\beta]}$ is a locally free-time minimizer.

First by Proposition 5.4.4, there is a $Q \in \bar{\Gamma}^+$ with $J(Q) = \bar{c}^+$. We will consider different cases for $Q(t^*), t^* \in \mathbb{R}$ as we did in the proof of Proposition 5.3.7, and the arguments are exact the same, except for the case $Q(t^*) \in v^+(\mathbb{R})$ and Q intersects v^+ at $Q(t^*)$ from the opposite direction. Therefore we will only show the proof for this case.

First, there is a $j \in \mathbb{N}$ with $t^* \in (t_j, t_{j+1})$. Then $w_j([s_j, S]) \cup Q([t_j, t^*])$ divides $\mathcal{R}$ into two disconnected regions. Then there is a $\beta_j \in (t^*, t_{j+1})$ such that $Q(\beta_j) \in Q([t_j, t^*])$ or $w_j([s_j, S])$.

If $Q(\beta_j) \in Q([t_j, t^*])$, then there is a $\alpha_j \in [t_j, t^*)$, such that $Q(\alpha_j) = Q(\beta_j)$. Because $k > c(L)$, $A_{L+k}(Q|_{[\alpha_j, \beta_j]}) \geq 0$, we can cut off $Q|_{[\alpha_j,\beta_j]}$ from Q, and get a $Q^* \in \bar{\Gamma}^+$. Obviously $J(Q^*) \leq J(Q)$, then $J(Q^*) = \bar{c}^+$.

If $Q(\beta_j) \in w_j([s_j, S])$, then there is a $\alpha_j \in (s_j, S)$, such that $w_j(\alpha_j) = Q(\beta_j)$. Since w_j is a free-time minimizer of $L + k$, we can replace $Q|_{[t_j,\beta_j]}$ by $w_j|_{[s_j,\alpha_j]}$ and get a $Q^* \in \bar{\Gamma}^+$ with $J(Q^*) \leq J(Q)$. Hence $J(Q^*) = \bar{c}^+$. $\square$

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