Unramified computation of tensor L-functions on symplectic groups

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Xin Shen

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Abstract

Tensor L-function is one of the important cases in the Langlands conjecture on the analytic properties of L-functions. Using the method of Rankin-Selberg convolution, Ginzburg, Jiang, Rallis and Soudry in [15] found an integral representation of the tensor L-functions for symplectic groups with non-generic representations. In this thesis we calculated the local integrals at the unramified places. First we gave a formula for the Whittaker-Shintani functions for symplectic groups, which is a generalization of the Casselman-Shalika formula for the Whittaker function in the generic case. Then we applied our formula and carried out the unramified calculation. We also investigated the local integrals at the non-archimedean, possibly ramified places and obtain some basic properties, such as convergence, rationalities, and non-vanishing of the local integrals for any given complex number s.

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Chapter 1

Introduction

1.1 Introduction

Given an irreducible cuspidal representation, it is a restricted tensor product of local representations. Almost all of them are unramified. Following Langlands, one can define the local L-function at all unramified places by their Satake parameters. The product of these L-functions is called a partial L-function. Around 1970 Langlands conjectured that one can define local L-functions at all other places, so that we can define a complete L-function as the product of local L-functions at all places, and each complete L-function should satisfy "nice" analytic properties. There are two main approaches toward this conjecture. One is via the integral representation, usually referred to as the "Rankin-Selberg method". One is via the Fourier coefficient of Eisenstein series, usually referred to as "Langlands-Shahidi method". Roughly speaking one first constructs an integral over a group over the Adele ring of a given global field, which we call the global integral. Assume it is a meromorphic function on $\mathbb{C}$. Then we "unfold" the global integral to an Euler product of local integrals, that is, integrals over the groups defined over local fields. At almost every place, the local integral is unramified, i.e., all the vectors in the integral are fixed by some maximal compact subgroups. If one can equate the unramified integrals to the local L-functions at the unramified places, (we call this step "unramified calculation"), then it is expected that one could use the local integrals at other places to define local L-functions too. Then the analytic properties of complete L-function can be obtained from those of the global integral. So the unramified calculation is a criterion for finding the "right" global integral for the desired partial L-functions.

One of the important cases of the Langlands conjecture is the tensor L-functions for $G \times GL_n$ where G is a classical group. Assume that we know enough analytic properties of them, then combining with the Converse theorem by Cogdell and Piatetski-Shapiro [8], the lifting from cuspidal representations of G to those of general linear groups is implied, in accordance with the Langlands functoriality [20]. In [15], Ginzburg, Jiang, Rallis and Soudry used the Fourier-Jacobi models to construct integrals for the tensor L-functions for $Sp_{2n} \times GL_k$. The result of the unramified calculation is stated in [15, Theorem 4.3]. The purpose of this thesis is to give an explicit proof of this theorem with certain restriction on k and n.

The paper is organized as follows. In **Chapter 1**, we will recall the statement of the Langlands conjecture, some definitions of related concepts, and the integral constructed in [15]. In **Chapter 2** and **3** we will cover the two main steps in the unramified calculation. In **Chapter 2** we will find an explicit formula for the unique pairing occurring at the unramified local integrals, which are called the "Whittaker-Shintani" functions. In **Chapter 3** we will use this formula to compute the local integrals at the unramified places. In **Chapter 3** we will also discuss some basic properties of the local integrals, including the convergence, meromorphic continuation, and non-vanishing of the local integrals at non-archimedean places. These properties are early steps of defining the local L-functions at ramified non-archimedean places.

1.2 Automorphic forms and automorphic representations

In this section we recall some basic facts about automorphic forms and automorphic representations following [5] and lecture 2 and 3 of [9].

Let G be a split reductive algebraic group over a number field F. Let $\mathbf{A}$ be the Adele ring of F. We let $G_\infty = \prod_{v|\infty} G(F_v)$ and $\mathfrak{g}_\infty$ be the Lie algebra of G_∞ (viewed as a real group). For finite places v, let $\mathcal{O}_v$ be the maximal compact subring of F_v, and let $K_v = G(\mathcal{O}_v)$. Let $G(\mathbf{A}_f) = \otimes'_{v \text{ finite}} G(F_v)$, where $\otimes'_{v \text{ finite}} G(F_v) = \{\otimes_{v<\infty} g_v | g_v \in K_v$ for almost every $v\}$. Let $G(\mathbf{A}) = G_\infty \otimes G(\mathbf{A}_f)$. A function $f : G(\mathbf{A}) \to \mathbb{C}$ is smooth if $f(x; y)$, where $x \in G_\infty$ and $y \in G(\mathbf{A}_f)$, is a $\mathbf{C}^\infty$ function in x and locally constant function in y.

Definition 1.2.1. *Fix a maximal compact subgroup K_∞ of G_∞. A smooth function f on $G(\mathbf{A})$ is an automorphic form on $G(\mathbf{A})$ if*

1. *f is left $G(F)$-invariant.*
2. *f is right K-finite, where $K = K_\infty \otimes \otimes_{v \text{ finite}} K_v$.*
3. *f is $\mathcal{Z}$-finite, where $\mathcal{Z}$ is the center of the enveloping algebra of $\mathfrak{g}_\infty$.*
4. *f is slowly increasing, i.e., there exists a positive integer n and a constant C such that $||f(x)|| \le C||x||^n$ for all $x \in G(\mathbf{A})$.*

An automorphic form f is called an cuspidal automorphic form, or simply a cusp form, if for any proper F-parabolic subgroup P of G, with N being its unipotent radical, we have

$$\int_{N(F)\backslash N(\mathbf{A})} \mathrm{d}n f(ng) = 0 \tag{1.1}$$

for any $g \in G(\mathbf{A})$.

Now we define the Hecke algebra of $G(\mathbf{A})$. Let $U(\mathfrak{g}_\infty)$ be the enveloping algebra of $\mathfrak{g}_\infty$, and A_{K_∞} the algebra of finite measure on K_∞. Let $\mathfrak{k}$ be the Lie algebra of K_∞. We define $\mathcal{H}_\infty = U(\mathfrak{g}_\infty) \otimes_{U(\mathfrak{k})} A_{K_\infty}$. For each finite place v, we let $\mathcal{H}_v$ be the convolution algebra of smooth and compactly supported functions on $G(F_v)$. We let $\mathcal{H}_f = \otimes'_{v \text{ finite}} \mathcal{H}_v$, where the restricted tensor product means that $\otimes_{v<\infty} h_v \in \otimes'_{v \text{ finite}} \mathcal{H}_v$ if and only if h_v is the characteristic function of K_v for almost every v. The Hecke algebra of $G(\mathbf{A})$ is defined as $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f$.

Definition 1.2.2. *An irreducible representation of $\mathcal{H}$ is automorphic (resp. cuspidal) if it is isomorphic to a sub-quotient of a representation of $\mathcal{H}$ in the space of automorphic (resp. cusp) forms on $G(\mathbf{A})$.*

Remark 1.2.3. *Since the right translation of G_∞ does not preserve K_∞-finiteness, automorphic representations are not $G(\mathbf{A})$-modules, but always $\mathcal{H}_\infty \otimes G(\mathbf{A}_f)$-modules.*

Now we define another set of automorphic forms, called the "smooth automorphic forms". It contains all the automorphic forms we define above, and they form a $G(\mathbf{A})$-module.

Definition 1.2.4. *A smooth function φ on $G(\mathbf{A})$ is called a smooth automorphic form if*

1. *it is left $G(F)$-invariant,*
2. *it is right K_f-finite,*
3. *it is $\mathcal{Z}$-finite,*
4. *there exists a positive integer r such that for all differential operators $X \in \mathcal{U}(\mathfrak{g})$,*
$$|X\varphi(g)| \leq C_X ||g||^r$$

If moreover φ satisfies (1.1), then it is called a smooth cusp form. The space of smooth automorphic forms is denoted by $\mathcal{A}^\infty(G(F)\backslash G(\mathbf{A}))$.

Similarly we can define smooth automorphic representations.

Definition 1.2.5. *A smooth representation of $G(\mathbf{A})$ is automorphic if it is a closed irreducible sub-quotient of $\mathcal{A}^\infty(G(F)\backslash G(\mathbf{A}))$.*

Remark 1.2.6. *Since the action of H or $G(\mathbf{A})$ preserves cuspidality, we can define cuspidal representations as closed sub-quotients of cusp forms in a similar way. However all cusp forms are rapidly decreasing, from this one can deduce that any sub-quotients of the space of cusp forms are actually sub-representations.*

1.3 Langlands L-function

From now on we only consider smooth automorphic forms and smooth automorphic representations, which are $G(\mathbf{A})$-modules. Let π be an irreducible cuspidal representation of $G(\mathbf{A})$. Then we have the following decomposition theorem.

Theorem 1.3.1 (Flath [10])**.** *If (π, V) is a smooth automorphic representation of $G(\mathbf{A})$, then there exist irreducible admissible smooth representations (π_v, V_v) of $G(F_v)$, which are smooth Frechet representations of moderate growth if $v|\infty$, such that $\pi = \pi_\infty \otimes \pi_f$ where*

$$\pi_\infty = \widehat{\otimes}_{v|\infty}\pi_v$$

is the topological tensor product of smooth Frechet representations and

$$\pi_f = \otimes'_{v<\infty}\pi_v$$

is the restricted tensor product of smooth representations of $G(F_v)$.

Here the restricted product is defined as follows. Let V_v be the representation space of π_v. For almost every finite place v, we fix an element $x_v^0 \in V_v$ which is nonzero and fixed by K_v. Then $\otimes'\pi_v$ is spanned by $\otimes x_v$, where $x_v \in V_v$ for all v, and $x_v = x_v^0$ for almost every v.

By this theorem, for almost every finite v, π_v contains a nonzero K_v-invariant element. We call them unramified (or spherical) representations, and the K_v-invariant elements are called unramified (or spherical) vectors. One can associate a complex analytic group ${}^LG$ to G, which is called the L-group of G. Then by the Satake isomorphism each unramified representation π_v can be associated with a conjugacy class z_v in ${}^LG$. Take r be a representation of ${}^LG$ of dimension n. Then the local Langlands L-function at an unramified place v is defined as

$$L_v(s, \pi, r) = \det(I - r(z_v)q_v^{-s})^{-1}.$$

Here q_v is the residue cardinality of F_v. Let S be a finite set of places such that all $v \notin S$ are finite and unramified. Then we define the partial Langlands L-function as

$$L^S(s, \pi, r) = \prod_{v \notin S} L_v(s, \pi, r).$$

Theorem 1.3.2 (Langlands). *The partial Langlands L-function $L^S(s, \pi, r)$ is absolutely convergent for $Re(s) >> 0$.*

Conjecture 1.3.3 (Langlands). *One can define $L_v(s, \pi, r)$ at all other places, so the complete L-function $L(s, \pi, r) = \prod_v L_v(s, \pi, r)$ has good analytic properties, including meromorphic continuation to the whole plane, and a functional equation relating its value at s to $1 - s$.*

1.4 L-function machine

Following [13] we use the term "L-function machine" for the five main steps toward the analytic properties of Langlands L-functions.

Step 1 establishing a global zeta integral, plus an Euler product of local integrals.

Step 2 analyzing the meromorphic behavior of the global integral, and its function equation.

Step 3 equating the unramified local integrals to L-functions desired.

Step 4 analyzing the meromorphic behavior of the local integrals, and their function equations.

Step 5 establishing basic properties of the local zeta-integrals, namely the definition, existence, and properties of the greatest common divisors $L_v(s,\pi,r)$ of the local integrals. (Step 4 and 5 are called "local theory")

Remark 1.4.1. *For the tensor L-functions for $Sp_{2n} \times GL_k$, step 1 and 2 are completed in [15] where the formula for step 3 is given in theorem 4.3.*

1.5 Tensor L-function

A Langlands L-function $L(s,\pi,r)$ is called a "tensor L-function" when $\pi = \pi_1 \times \pi_2$ is the cuspidal representation of $H \times GL_n$, where H is a split classical group, and when r is the standard representation of ${}^LH \times^L GL_n$. We simply denote it by $L(s,\pi_1 \times \pi_2)$.

When $H = GL_m$, the L-function $L(s,\pi_1 \times \pi_2)$ is well studied by Jacquet, Piatetski-Shapiro and Shalika in [17]. One of the important ingredients in the construction of the global integral for the L-function of $GL_n \times GL_m$ is the Whittaker model. For a representation (τ, V) of GL_n (over local fields or Adele ring), the Whittaker functional is defined as follows. Let N be the unipotent of the Borel subgroup B of GL_n, and let ψ_N be a generic character on N. A Whittaker functional on (τ, V) is a functional Λ on V satisfying

$$\Lambda(\tau(n)\xi) = \psi_N(n)\Lambda(\xi)$$

for any $n \in N$. For any $\xi \in V$ and $g \in GL_n$, let $W_\xi(g) = \Lambda(\tau(g)\xi)$ be the Whittaker function of ξ. The map $\xi \mapsto W_\xi(g)$ is a GL_n-homomorphism from (τ, V) to $Ind_N^{GL_n}(\psi_N)$. By Frobenius reciprocity law

$$Hom_{GL_n}(\tau, Ind_N^{GL_n}(\psi)) = Hom_N(\tau, \psi),$$

the existence of non-trivial Whittaker function is equivalent to that of a nontrivial Whittaker functional. In this case we call τ generic. We have

Theorem 1.5.1 (Piatetski-Shapiro [25], Shalika [27]). *Every cuspidal automorphic representation of $GL_m(\mathbf{A})$ is generic.*

The space of functions $W_\xi(g)$ is called the Whittaker model of τ.

On the other hand, we have

Theorem 1.5.2 (Gelfand and Kazhdan [14], Shalika [27]). *If $\pi \cong \otimes'_v \pi_v$ is an irreducible cuspidal representation of $GL(\mathbf{A})$, then the dimension of Whittaker model on π is at most 1.*

The existence and uniqueness of the Whittaker model leads to the decomposition of the global integral into an Euler product. At almost every place one needs to calculate the unramified local integrals, where the Casselman-Shalika formula for unramified Whittaker functions is crucial.

For a representation of a split algebraic group one can define the Whittaker model in a similar way.

Now take $H = Sp_{2r}$, and consider the L-function $L(s, \pi_1 \times \pi_2)$, where π_1 (resp. π_2) is the cuspidal automorphic representation of H (resp. GL_n). When π_1 is generic, the construction of $L(s, \pi_1 \times \pi_2)$ is first given by Gelbart and Piatetski-Shapiro in [12] when $r = n$, and later by Ginzburg, Rallis and Soudry in [16] for general r.

The case when π_1 is non-generic is more complicated. Ginzburg, Jiang, Rallis and Soudry gives a global zeta integral in [15]. The Whittaker models used in the generic case is replaced by the Fourier-Jacobi model, which is a pairing between two representations. By the work of Sun [30] and Liu and Sun [21], such pairing is unique locally up to a scalar, which implies that one can decompose the global integral into an Euler product. For the unramified calculation of local integrals, it is expected in [15] that they are equal to the tensor L-function $L(s, \pi_1 \times \pi_2)$ divided by certain normalizers of the Eisenstein series. An explicit proof will be given in chapter 2 and 3 in this thesis.

1.6 Metaplectic groups

1.6.1 Local metaplectic groups and Weil representation

Let F be a local field with characteristic 0. Let W be a vector space over F with a non-degenerate anti-symmetric form $\langle , \rangle$. Take $X = \{e_1, \dots, e_n\}$ and $Y = \{e_1^*, \dots, e_n^*\}$ be a polarization of W such that $\langle e_i, e_j^* \rangle = \delta_i^j$. Let H_{2n+1} be a Heisenberg group over F of $2n+1$ variables. It is the group consisting of elements in $X \times Y \times F$ with the group structure

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle)$$

It can also be realized as $(x, y, z) = \begin{pmatrix} 1 & x & y & z \\ & 1 & & y' \\ & & 1 & -x' \\ & & & 1 \end{pmatrix}$. Here $x = (x_1, \dots, x_n) \in F^n$ corresponds to the element $\sum x_i e_i \in X$ and $y = (y_1, \dots, y_n) \in F^n$ corresponds to $\sum y_i e_{n+1-i}^* \in Y$, and $z \in F$ and $x' =^t (x_n \dots, x_1)$, $y' =^t (y_n, \dots, y_1)$. The action of $Sp(W)$ on H_{2n+1} (defined by trivial extension of the action of $Sp(W)$ on W to $W \times F$) is compatible with the conjugation of the matrix expression of $Sp(W)$ with respect to the basis $\{e_1, \dots, e_n, e_n^*, \dots, e_1^*\}$ on the matrix expression of H_{2n+1}.

Theorem 1.6.1 (Stone-von Neumann theorem). *For each non-trivial additive character ψ on F, there is a unique representation (up to isomorphism) of H_{2n+1} such that the center acts as ψ.*

The group $Sp(W)$ normalizes H_{2n+1} and stabilizes the center. So by the Stone-Von Neumann theorem, for each $g \in Sp(W)$ there is an automorphism $A(g)$ on the space of the Weil representation, such that $A(g)^{-1} w_\psi(h) A(g) = w_\psi(h^g)$ for any $h \in H_{2n+1}$. This gives a projective representation of $Sp(W)$. It can be lifted to a representation of the double cover $\widetilde{Sp}(W)$, which is also called a Weil representation of $\widetilde{Sp}(W)$.

When $F \neq \mathbb{C}$, $Sp(W)$ (note that W is defined over F) has a unique nontrivial double cover, which we denote by $\widetilde{Sp}(W)$. One can realize $\widetilde{Sp}(W)$ explicitly by using the Rao normalized cocycle $c_W(,) : Sp(W) \times Sp(W) \to \{\pm 1\}$ ([26, Lemma 5.1]). Precisely,

$$\widetilde{Sp}(W) = \{(g, \epsilon) \in Sp(W) \times \{\pm 1\}\}$$

with the group law

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, c_W(g_1, g_2)\epsilon_1\epsilon_2).$$

For F being non-archimedean with characteristic 0, let w_ψ be the Weil representation of $\widetilde{Sp}(W)(F) \ltimes H_{2n+1}(F)$ with respect to the non-trivial additive character ψ. It is realized on the space of Schwartz functions on X. Below are the formulas we will need.

$$\begin{aligned}
&w_\psi(x, 0, 0)\phi(x_0) = \phi(x_0 + x),\\
&w_\psi(0, y, z)\phi(x_0) = \psi(z + x_0 \cdot y')\phi(x_0),\\
&w_\psi \begin{pmatrix} a & \\ & a^* \end{pmatrix} (\phi)(x) = \gamma_\psi(\det a)|\det a|^{\frac{1}{2}}\phi(ax),\\
&w_\psi \begin{pmatrix} I_n & b \\ & I_n \end{pmatrix} (\phi)(x) = \psi(\langle x, bx \rangle)\phi(x).
\end{aligned}$$

Here γ_ψ is the Weil index on $F^*/(F^*)^2$ associated to the second degree character $\psi(x^2)$. It satisfies

$$\gamma_\psi(\det a)\gamma_\psi(\det b) = \gamma_\psi(\det ab)c_W(\begin{pmatrix} a_1 & \\ & a_1^* \end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^* \end{pmatrix}) \tag{1.2}$$

1.6.2 Global metaplectic groups

In this subsection let F be a number field, and F_v its completion. Suppose W is a vector space over F with a anti-symmetric form, an let W_v be its completion at v. We know that the definition of $Sp(W)_{\mathbf{A}}$ is the restricted tensor product of $Sp(W_v)$ with respect to the maximal compact subgroups at each place. To define the double covering $\widetilde{Sp}(\mathbf{A})$, one needs the following fact.

Lemma 1.6.2 (page 43, [22])**.** *For each non-archimedean F_v with odd residual characteristic, the maximal compact subgroup K_v of $Sp(W_v)$ is split in $\widetilde{Sp}(F_v)$, i.e., there exists a group homomorphism $K_v \to \widetilde{Sp}(W_v)$, $k_v \mapsto (k_v, \epsilon_v(k_v))$.*

So we can define $\widehat{Sp}(W)_{\mathbf{A}}$ as the restricted tensor product of $\widetilde{Sp}(F_v)$ with respect to $\{K_v \mid v \text{ is finite and odd }\}$. Let C' be the subgroup $\{\prod_v (I, \epsilon_v) \mid \prod_v \epsilon_v = 1\}$. Then $\widetilde{Sp}(W)_{\mathbf{A}} := C' \backslash \widehat{Sp}(W)_{\mathbf{A}}$ is a double cover of $Sp(W)_{\mathbf{A}}$.

Let $j : \widetilde{Sp}(W)_{\mathbf{A}} \to Sp(W)_{\mathbf{A}}$ be the projection. Now we consider the preimage of $Sp(W)(F)$, the rational points. Let $g \in Sp(W)(F)$, and g_v its image in $Sp(W)(F_v)$, then $g_v \in K_v$ for almost every v. By the properties of ϵ_v in [26], we have

Lemma 1.6.3. *For almost every v that is finite and odd, $\epsilon_v(g_v) = 1$.*

We also have

Lemma 1.6.4 ([26])**.** *For $g_1, g_2 \in Sp(W)(F)$, the product $\prod_v c_{W_v}(g_1, g_2) = 1$.*

With these two lemmas, we can easily deduce that the map $g \mapsto C' \prod_v (g, 1)$ is a homomorphism from $Sp(W)(F)$ to $\widetilde{Sp}(W)_{\mathbf{A}}$. Let N be the maximal unipotent radical of $Sp(W)$. From the definition of c_{W_v} it is not hard to see that $c_{W_v}(n_1, n_2) = 1$ for every $n_1, n_2 \in Sp(W_v)$. So $N(\mathbf{A})$ is split in $\widetilde{Sp}(W)_{\mathbf{A}}$ too.

With these fact about K_v, $Sp(W)(F)$ and $N(\mathbf{A})$ one can define the automorphic form and cusp form on $\widetilde{Sp}(W)_{\mathbf{A}}$ in a similar way.

Let ψ be an additive character on $F \backslash \mathbf{A}$. One can define the Weil representation of $H_{2n+1}(\mathbf{A})\widetilde{Sp}(W)_{\mathbf{A}}$ with respect to ψ on $S(\mathbf{A}^n)$, the space of Schwartz functions on $\mathbf{A}^n$, such that for $\phi = \prod_v \phi_v$, $w_\psi(g)(\phi) = \prod_v w_{\psi_v}(g_v)(\phi_v)$. Then we let

$$\tilde{\theta}_\phi(g) = \sum_{\xi \in F} w_\psi(g)\phi(\xi).$$

It is an automorphic form on $\widetilde{Sp}(W)_{\mathbf{A}} \ltimes H_{2n+1}(\mathbf{A})$.

1.6.3 Local L-function on the metaplectic groups

Let v be finite and odd. Let $\widetilde{B}_v$ be the preimage of the Borel subgroup B in $\widetilde{Sp}(W_v)$. Note that K_v is split in $\widetilde{Sp}(W_v)$, so we have the Iwasawa decomposition $\widetilde{Sp}(W_v) = \widetilde{B}_v K_v$. A representation of $\widetilde{Sp}(W_v)$ is called genuine if $(1,\epsilon)$ acts by multiplying ϵ. For a genuine spherical representation of $\widetilde{Sp}(W_v)$, one can assign the local L-factor with respect to the additive character ψ_v on F_v as follows. Suppose $\tilde{\pi}$ is a spherical representation which can be embedded to $\mathrm{Ind}_{\tilde{B}_v}^{\widetilde{Sp}(W)(F_v)}(\tilde{\chi})$ where

$$\tilde{\chi}(\mathrm{diag}(t_1,\ldots,t_n,t_n^{-1},\ldots,t_1^{-1})n,\epsilon) \mapsto \epsilon \cdot \prod_i \chi_i(t_i)\gamma_{\psi_v}^{-1}(t_1\ldots t_n)$$

where χ is an unramified character on the torus. By (2.39) $\tilde{\chi}$ is a character. Then if we let

$$z_{\tilde{\pi}} = (\chi_1(\varpi),\ldots,\chi_n(\varpi),\chi_n(\varpi)^{-1},\ldots,\chi_1(\varpi)^{-1}),$$

then the local L-factor with respect to ψ_v is defined as

$$L_{\psi_v}(s,\tilde{\pi},r) = \det(I - q^{-s}Tr(r(z_{\tilde{\pi}})))^{-1}$$

where ϖ is the uniformizer in F_v.

Note that when $\tilde{\pi}$ is genuine, the representation $\tilde{\pi}\otimes w_{\psi_v}$ is trivial on ϵ, so it factors through $Sp(W_v)^J = Sp(W_v) \ltimes H_{2n+1,v}$, which we call the Jacobi group. By the discussion in Chapter 2 in [2], $\tilde{\pi} \leftrightarrow \tilde{\pi}\otimes w_{\psi_v}$ is a bijection between genuine representation of $\widetilde{Sp}(W_v)$ and representation of $Sp(W_v)^J$ with central character ψ_v. In particular when $\tilde{\pi} = \mathrm{Ind}_{\tilde{B}_v}^{\widetilde{Sp}(W_v)}$, $\tilde{\pi}\otimes w_{\psi_v}$ is isomorphic to $\mathrm{Ind}_{B_v^J}^{Sp(W_v)^J}(\chi,\psi_v)$, where $B^J = B \ltimes YZ$ (YZ is the group of the elements $(0,y,z)$ in the Heisenberg group), and $(\chi,\psi_v)(b(0,y,z)) = \chi(b)\psi_v(z)$.

1.7 Integral representation using Fourier-Jacobi models

In this section we recall the integral representation for the tensor L-function of symplectic groups using Fourier-Jacobi models as introduced in [15].

Let F be a number field, and $\mathbf{A}$ its ring of adeles. Let $\mathrm{G} = Sp_{2n}$, and $\mathrm{M} = Sp_{2m}$. Let π be an irreducible, cuspidal, automorphic representations of $\mathrm{G}(\mathbf{A})$. For any $r < n$, let P_r^n be the parabolic subgroup of G with Levi decomposition $P_r^n = GL_1^r \times Sp_{2(n-r)} \ltimes V_r^n$. Then one can define the r-th Fourier Jacobi coefficient of $\varphi_\pi \in \pi$ as

$$\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^n}}(\varphi_\pi)(h) = \int_{[V_r^n]} \mathrm{d}v\ \varphi(vh)\widetilde{\theta}^{\phi}_{\psi,n-r}(l_{n-r}(v)h)\psi_{V_r^n}(v) \tag{1.3}$$

with $h \in \widetilde{Sp}_{2(n-r)}$. Here $\phi \in \mathcal{S}(\mathbf{A}^{n-r})$, the space of Schwartz function on $\mathbf{A}^{n-r}$, and ψ is an additive character on $F\backslash\mathbf{A}$. $\widetilde{\theta}^{\phi}_{\psi,n-r}$ is the theta series defined as

$$\widetilde{\theta}^{\phi}_{\psi,n-r}(h) = \sum_{\xi \in F^{n-r}} w_\psi(h)\phi(\xi),$$

and $[V_r^n]$ is the quotient $V_r^n(F)\backslash V_r^n(\mathbf{A})$. The group $H_{2(n-r)+1}$ is the Heisenberg group of $2(n-r)+1$ variables, and map $l_{n-r} : V_r^n \to H_{2(n-r)+1}$ is defined as

$$l_{n-r}(v) = (v_{r,r+1}, v_{r,r+2}, \ldots, v_{r,2k-r}; v_{r,2k-r+1}). \tag{1.4}$$

It is known that

(a) the function $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^n}}(\varphi_\pi)(h)$ is an genuine automorphic function on $\widetilde{Sp}_{2(n-r)}(\mathbf{A})$, and

(b) when r is the largest number such that $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^n}}(\varphi_\pi)(h)$ is not zero for some φ_π, the representation generated by $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^n}}(\varphi_\pi)(h)$ is cuspidal.

When (b) is satisfied, we call $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^n}}(\varphi_\pi)(h)$ the top Fourier-Jacobi coefficient of π. For any genuine automorphic representation π' of $\widetilde{Sp}_{2(n-r)}(\mathbf{A})$ and $\varphi_{\pi'} \in \pi'$, we define the pairing

$$\mathcal{FJP}_{\psi_{V_r^n}}(\varphi_\pi, \phi, \varphi_{\pi'}) = \int_{[Sp_{2(n-r)}]} \mathrm{d}h\ \widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^n}}(\varphi_\pi)(h)\varphi_{\pi'}(h) \tag{1.5}$$

assuming it is convergent.

Assume now that when $r = n - m$, the Fourier Jacobi coefficient is the top one, and $\widetilde{\sigma}$ is a summand in the complex conjugate of the cuspidal automorphic representation generated by $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^n}}(\varphi_\pi)(h)$, so the pairing $\mathcal{FJP}_{\psi_{V_r^n}}(\varphi_\pi, \phi, \varphi_{\widetilde{\sigma}})$ is not identically zero. Then we can consider integral as follows. Let k be a positive integer with $k + m < n$, and let $P_{k,m}$ be the parabolic subgroup of $Sp_{2(m+k)}$ with Levi decomposition $P_{k,m} = GL_k \times \mathrm{M} \ltimes V_{k,m}$. Let τ be an irreducible cuspidal automorphic representation of $GL_k(\mathbf{A})$, and let $\widetilde{E}_{\tau,\widetilde{\sigma}}(h, s)$ be an Eisenstein Series on $\widetilde{Sp}_{2(m+k)}(\mathbf{A})$ associated to a holomorphic section $\widetilde{f}_{\tau,\widetilde{\sigma},s}$ in the induced representation $\mathrm{Ind}_{P_{k,m}(\mathbf{A})}^{\widetilde{Sp}_{2(m+k)}(\mathbf{A})}(\gamma_\psi\tau|\det|^{s-\frac{1}{2}} \otimes \widetilde{\sigma})$. Then with the same notation as above, we consider the integral

$$\begin{aligned} I(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma},s}) :=& \mathcal{FJP}_{\psi_{V^n_{n-m-k}}}(\varphi_\pi, \phi, \widetilde{E}_{\tau,\widetilde{\sigma}}(h, s)) \\ =& \int_{[Sp_{2(m+k)}]} \mathrm{d}h\ \widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V^n_{n-m-k}}}(\varphi_\pi)(h)\ \widetilde{E}_{\tau,\widetilde{\sigma}}(h, s). \end{aligned}$$

By [15, Lemma 3.1], this integral is convergent absolutely and uniformly on vertical strips in $\mathbb{C}$ for s being away from poles of $\widetilde{E}_{\tau,\widetilde{\sigma}}(h, s)$. Then by the calculation in [15, Theorem 3.2], the integral is equal to

$$\begin{aligned} &\int_{\mathrm{M}(\mathbf{A})V_k^{m+k}(\mathbf{A})\backslash Sp_{2(m+k)}(\mathbf{A})} \mathrm{d}h \\ &\int_{R(\mathbf{A})} \mathrm{d}r\ \mathcal{FJP}_{\psi_{V^n_{n-m}}}(\pi(wrh)\varphi_\pi, [\omega_\psi(l_{m+k}(r)h)\phi]_m, \widetilde{f}_{W_{\tau,\widetilde{\sigma},s}}(h)). \end{aligned} \tag{1.6}$$

Here w is a Weyl element in G and $R = \{R(r) \mid r \in Mat_{(n-m-k),k}\}$ is a unipotent subgroup of G such that

$$w = \begin{pmatrix} & I_k & & & \\ I_{n-m-k} & & & & \\ & & I_{2m} & & \\ & & & & I_{n-m-k} \\ & & & I_k & \end{pmatrix}, \quad R(r) = \begin{pmatrix} I_{n-m-k} & r & & & \\ & I_k & & & \\ & & I_{2m} & & \\ & & & I_k & r' \\ & & & & I_{n-m-k} \end{pmatrix},$$

and ω_ψ is the Weil representation on $\mathcal{S}(\mathbf{A}^{m+k})$, and $[\phi]_m \in \mathcal{S}(\mathbf{A}^m)$ is defined as

$$[\phi]_m(y_1, \ldots, y_m) = \phi(\underbrace{0, \ldots, 0}_{k}, y_1, \ldots, y_m).$$

The function $\widetilde{f}_{W_\tau, \widetilde{\sigma}, s}$ is defined on $Sp_{2(m+k)}(\mathbf{A})$ taking the value in V_σ.

The trilinear form $\mathcal{FJP}_{\psi_{V^n_{n-m}}}$ on $V_\pi \times \mathcal{S}(\mathbf{A}^m) \times V_{\widetilde{\sigma}}$ satisfies

$$\mathcal{FJP}_{\psi_{V^n_{n-m}}}(\pi(vh)\varphi_\pi, \omega_\psi(l_m(v)h)\phi, \widetilde{\sigma}(h)\varphi_{\widetilde{\sigma}}) = \psi_{V^n_{n-m}}(v)\mathcal{FJP}_{\psi_{V^n_{n-m}}}(\varphi_\pi, \phi, \varphi_{\widetilde{\sigma}}) \tag{1.7}$$

for all $v \in V^n_{n-m}(\mathbf{A})$ and $h \in \mathrm{M}(\mathbf{A})$. For each v, the restriction of $\mathcal{FJP}_{\psi_{V^n_{n-m}}}$ on local representation spaces satisfies an equation similar to (1.7), and by [30] and [21] such trilinear form is unique up to a scalar. Take $\mathcal{T}_v$ be such trilinear forms for each v, then by suitable normalization, and assume that all the vectors in the integral are factorizable, we obtain an Euler product

$$I(\varphi_\pi, \phi, \widetilde{f}_{\tau, \widetilde{\sigma}, s}) = \prod_v I_v(\varphi_{\pi_v}, \phi_v, \widetilde{f}_{W_\tau, \widetilde{\sigma}, s, v}), \tag{1.8}$$

where

$$\begin{aligned} & I_v(\varphi_{\pi_v}, \phi_v, \widetilde{f}_{W_\tau, \widetilde{\sigma}, s, v}) \\ =& \int_{\mathrm{M}(F_v)V_k^{m+k}(F_v)\backslash Sp_{2(m+k)}(F_v)} \mathrm{d}h \int_{R(F_v)} \mathrm{d}r \\ & \mathcal{T}_v(\pi(wrh)\varphi_{\pi,v}, [\omega_\psi(l_{m+k}(r)h)\phi_v]_m, \widetilde{f}_{W_\tau, \widetilde{\sigma}, s, v}(h)). \end{aligned} \tag{1.9}$$

In the following two chapters, we are going to compute $I_v(\varphi_{\pi_v}, \phi_v, \widetilde{f}_{W_\tau,\widetilde{\sigma},s,v})$ at unramified places, that is, where v is non-archimedean and all the vectors are spherical. First we calculate in Chapter 2 the value of $\mathcal{T}_v(\pi(g)\varphi^\circ_{\pi,v}, \phi^\circ_v, \widetilde{f}^\circ_{\widetilde{\sigma}})$ explicitly. This function is called a **Whittaker-Shintani function**, and we denote it by $\mathrm{W}^0_{\chi,\xi,\psi}(g)$ (after normalization).

Theorem 1.7.1 (Theorem 2.12.1)**.** *For every $(\chi, \xi) \in \mathbb{C}^n \times \mathbb{C}^m$, the normalized Whittaker-Shintani function is given by*

$$\int_{\mathrm{X}^0} \mathrm{d}x \mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}} x \mathfrak{p}^{\mathbf{f}}) = \zeta(1)^{-m} \prod_{i=1}^{m} \zeta(2i) \cdot$$
$$\sum_{w \in W_{\mathrm{G}}, w' \in W_{\mathrm{M}}} b(w\chi, w'\xi) d(w\chi) d'(w'\xi) ((w\chi)^{-1}\delta^{\frac{1}{2}})(\mathfrak{p}^{\mathbf{f}})((w'\xi)^{-1}\delta^{\frac{1}{2}})(\mathfrak{p}^{\mathbf{d}})$$

*for $\mathbf{d} \in \Lambda^+_m$ and $\mathbf{f} \in \Lambda^+_n$. Here $\zeta(s) = (1 - q^{-s})^{-1}$ is the local zeta function. For definitions of notations, see **theorem 2.1.1** and **section 2.2**. If we let $\mathcal{L}(\mathbf{d}', \mathbf{f}') = \int_{\mathrm{X}^0} \mathrm{d}x \mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}'} x \mathfrak{p}^{\mathbf{f}'})$, and let*

$$S(\mathbf{d}, \mathbf{f}) = \{\mathbf{d}' \mid \mathbf{d}' \in \Lambda^+_m, \mathbf{f} + \mathbf{d} - \mathbf{d}' \in \Lambda^+_n, \mathbf{d}' \leq \mathbf{d}\},$$

then for each $\mathbf{d}' \in S(\mathbf{d}, \mathbf{f})$, there exists $a_{\mathbf{d},\mathbf{f}}(\mathbf{d}') \in \mathbb{R}$ independent of (χ, ξ, ψ) such that

$$\mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{\mathbf{f}}) = \sum_{\mathbf{d}' \in S(\mathbf{d},\mathbf{f})} a_{\mathbf{d},\mathbf{f}}(\mathbf{d}') \mathcal{L}(\mathbf{d}', \mathbf{f} + \mathbf{d} - \mathbf{d}')$$

and that $a_{\mathbf{d},\mathbf{f}}(\mathbf{d}) > 0$. In particular, we have

$$\mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{f}}) = \mathcal{L}(\mathbf{0}, \mathbf{f})$$

Note that by the Iwasawa decomposition, we only need the last formula to compute (1.9) assuming all the data are unramified. This is done in **Chapter 3**. We also consider some basic properties of the local integrals at non-archimedean places.

Theorem 1.7.2 (Theorem 3.1.1)**.** *The local integrals $I_v(\varphi_{\pi_v}, \phi_v, \widetilde{f}_{W_\tau,\widetilde{\sigma},s,v})$ at the* *non-archimedean places satisfy*

(A) At the unramified places,

$$I_v(\varphi_{\pi_v}, \phi_v, \widetilde{f}_{W_\tau, \widetilde{\sigma}, s, v}) = \frac{L(\pi_v \times \tau_v, s)}{L_{\psi_v}(\widetilde{\sigma}_v \times \tau_v, s + \frac{1}{2}) L(\tau_v, sym^2, 2s)} \tag{1.10}$$

as expected in [15, Theorem 4.3]

(B) The local integral converges absolutely for $Re(s)$ sufficiently large.

(C) As a function of s, the local integral has a meromorphic continuation to the whole complex plain.

(D) For any given s, there is a choice of data such that the local integral is nonzero.

Chapter 2

Whittaker-Shintani functions

2.1 Introduction

In this chapter we give an explicit formula for the **Whittaker-Shintani function**. This chapter is separately submitted in [29].

Let G and M be symplectic groups, defined over a non-archimedean local field F of rank n and m respectively with $n \geq m+1$. Let $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ be an unramified principal series of G. Let $\mathrm{M^J}$ be the Jacobi group and $\mathrm{B_{M^J}}$ its Borel subgroup as defined in A in **Section 2.2**, and let $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ be an unramified principal series of $\mathrm{M^J}$ as defined in C in **Section 2.2**. Let U be the unipotent radical of a parabolic subgroup P_1^{n-m-1} of G and ψ_{U} be a character on U which is stabilized by $\mathrm{M^J}$ (see 2.5 and 2.6). Then one can define an $\mathrm{M^J}$-invariant, $(\mathrm{U},\psi_{\mathrm{U}})$-equivariant pairing $l_{\chi,\xi,\psi}$ between $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$. Let F_χ^0 and $\mathrm{F}_{\xi,\psi}^0$ be the normalized spherical vectors in $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$, and we define

$$\mathrm{W}_{\chi,\xi,\psi}(g) = l_{\chi,\xi,\psi}(R(g)\mathrm{F}_\chi^0, \mathrm{F}_{\xi,\psi}^0).$$

This function is a Whittaker-Shintani function attached to (χ,ξ,ψ) (see **Definition 2.2.4**). We will show that for given (χ,ξ,ψ) (unramified) such function is unique up to scalar. Denote by $\mathrm{W}^0_{\chi,\xi,\psi}$ the normalized Whittaker-Shintani function which is equal to 1 at the identity. In this paper we prove the following two theorems (for the definition of $\mathrm{X}^0, \mathrm{Z}, \mathrm{K}_{\mathrm{M}^{\mathrm{J}}}, \mathfrak{p}^{\mathbf{d}}, \lambda, \mathfrak{p}^{\mathbf{f}}, \mathrm{K}_{\mathrm{G}}$, see **Section 2.2**).

Theorem 2.1.1. *Let* $(\chi, \xi) \in \mathbb{C}^n \times \mathbb{C}^m$*, and let* $\mathbf{d} \in \Lambda^+_m$*,* $\mathbf{f} \in \Lambda^+_n$*. Let* $\mathrm{W}^0_{\chi,\xi,\psi}$ *be the normalized Whittaker-Shintani function attached to* (χ, ξ, ψ)*. Then we have*

$$\begin{aligned}\int_{\mathrm{X}^0} \mathrm{d}x \mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}} x \mathfrak{p}^{\mathbf{f}}) &= \zeta(1)^{-m} \prod_{i=1}^{m} \zeta(2i) \\ &\cdot \sum_{w \in W_{\mathrm{G}}, w' \in W_{\mathrm{M}}} b(w\chi, w'\xi) d(w\chi) d'(w'\xi) ((w\chi)^{-1}\delta^{\frac{1}{2}})(\mathfrak{p}^{\mathbf{f}}) ((w'\xi)^{-1}\delta^{\frac{1}{2}})(\mathfrak{p}^{\mathbf{d}}).\end{aligned} \tag{2.1}$$

where

$$\mathrm{d}(\chi) = \prod_{1 \le a < b \le n} \zeta(\chi_a \pm \chi_b) \prod_{i=1}^{n} \zeta(\chi_i), \qquad \mathrm{d}'(\xi) = \prod_{1 \le a < b \le m} \zeta(\xi_a \pm \xi_b) \prod_{j=1}^{m} \zeta(2\xi_j),$$

and

$$\begin{aligned}\mathrm{b}(\chi, \xi) = &\prod_{i < j+n-m} \zeta^{-1}(\chi_i - \xi_j + \frac{1}{2}) \cdot \prod_{i > j+n-m} \zeta^{-1}(-\chi_i + \xi_j + \frac{1}{2}) \\ &\prod_{1 \le j \le m} \zeta^{-1}(\xi_j + \frac{1}{2}) \prod_{\substack{1 \le i \le n \\ 1 \le j \le m}} \zeta^{-1}(\chi_i + \xi_j + \frac{1}{2})\end{aligned}$$

Theorem 2.1.2. *Under the same notation and assumptions as in the previous theorem, the support of* $\mathrm{W}^0_{\chi,\xi,\psi}$ *is on*

$$\bigcup_{\mathbf{d} \in \Lambda^+_m, \mathbf{f} \in \Lambda^+_n} \mathrm{Z}\mathrm{U}\mathrm{K}_{\mathrm{M}^{\mathrm{J}}}(\mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{\mathbf{f}}) \mathrm{K}_{\mathrm{G}}. \tag{2.2}$$

If we let $\mathcal{L}(\mathbf{d}', \mathbf{f}') = \int_{\mathrm{X}^0} \mathrm{d}x \mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}'} x \mathfrak{p}^{\mathbf{f}'})$*, and let*

$$S(\mathbf{d}, \mathbf{f}) = \{\mathbf{d}' \mid \mathbf{d}' \in \Lambda^+_m, \mathbf{f} + \mathbf{d} - \mathbf{d}' \in \Lambda^+_n, \mathbf{d}' \le \mathbf{d}\},$$

then for each $\mathbf{d}' \in S(\mathbf{d}, \mathbf{f})$*, there exists* $a_{\mathbf{d},\mathbf{f}}(\mathbf{d}') \in \mathbb{R}$ *independent of* (χ, ξ, ψ) *such* *that*

$$\mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\mathbf{f}}) = \sum_{\mathbf{d}'\in S(\mathbf{d},\mathbf{f})} a_{\mathbf{d},\mathbf{f}}(\mathbf{d}')\mathcal{L}(\mathbf{d}',\mathbf{f}+\mathbf{d}-\mathbf{d}') \tag{2.3}$$

and that $a_{\mathbf{d},\mathbf{f}}(\mathbf{d}) > 0$. *In particular, we have*

$$\mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{f}}) = \mathcal{L}(\mathbf{0},\mathbf{f}) \tag{2.4}$$

The paper is organized as follows. In **Section 2.2** we give the notation we use in this paper. In **Section 2.3** we apply the method of Rankin-Selberg convolution to find an integral expression for the pairing $l_{\chi,\xi,\psi}$ when (χ,ξ) belongs to $\mathcal{Z}_c \subset \mathbb{C}^n \times \mathbb{C}^m$ which contains a Hausdorff open set. In **Section 2.4** we show that the pairing $l_{\chi,\xi,\psi}$ between $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ satisfying Condition A (see **Definition 2.2.2**) is unique up to a scalar. In **Section 2.5** we apply Bernstein's theorem to extend this pairing defined by the integral to generic (χ,ξ). In **Section 2.6** we discuss the double cosets of G on which the Whittaker-Shintani function is supported. By considering the vectors invariant under certain open compact subgroups (in **Section 2.7**) and applying the intertwining operators (in **Section 2.8**) we give an explicit formula in **Section 2.9** for the Whittaker-Shintani function attached to generic (χ,ξ), and we obtain its value at the identity by an combinatorial argument in **Section 2.10**. After showing the uniqueness of the normalized Whittaker-Shintani function in **Section 2.11**, we summarize our results in **Section 2.12**. In **Section 2.13** we use the formula we obtained to give an alternative proof of [24, Theorem 6.1], the unramified calculation of L-functions for $\mathrm{Sp}_{2n} \times \mathrm{GL}_1$.

2.2 Notation

In this paper, we let F be a non-archimedean local field of characteristic 0. Let $\mathcal{O}$ be its maximal compact subring and $\mathfrak{p}$ the uniformizer. Suppose the order of the residue field is q which is not a power of 2. All the groups are defined over F.

Through out the paper we fix ψ to be an additive character on F with conductor 0.

(A) **Groups.** Let $\mathrm{G} = \mathrm{Sp}_{2n}$, $\mathrm{H} = \mathrm{Sp}_{2m+2}$ and $\mathrm{M} = \mathrm{Sp}_{2m}$, where m, n are two positive integers with $n \geq m+1$. Let $\mathcal{M}_{2n\times 2n}(F)$ be the set of all $2n$ by $2n$ matrices over F. Identify G as a subset of $\mathcal{M}_{2n\times 2n}(F)$, and identify $g \in \mathrm{M}$ (or $g \in \mathrm{H}$) with $diag(I_{n-m}, g, I_{n-m}) \in \mathrm{G}$ (or $diag(I_{n-m-1}, g, I_{n-m-1}) \in \mathrm{G}$). For any subgroup P of G, and any $i \geq 0$, we define P^i as

$$P^i = P \cap \left(I_{2n} + \mathcal{M}_{2n\times 2n}(\mathfrak{p}^i\mathcal{O})\right),$$

and we always normalize the measure on P so that $\mu_P(P^0) = 1$. Let $\mathrm{K_G} = \mathrm{G}^0$, and $\mathrm{K_M} = \mathrm{M} \cap \mathrm{K_G}$. For $x, y \in F^m$ and $z \in F$ let

$$\mathrm{J}(x,y,z) = \begin{pmatrix} 1 & x & y & z \\ & I_m & & {}^t y \\ & & I_m & -{}^t x \\ & & & 1 \end{pmatrix} \in \mathrm{H}.$$

$\mathrm{J} = \{J(x,y,z) \mid x, y \in F^m, z \in F\}$ is a Heisenberg group of dimension $2m+1$. Let $\mathrm{M^J} = \mathrm{M} \ltimes \mathrm{J}$, and $\mathrm{K_{M^J}} = \mathrm{K_M} \ltimes \mathrm{J}^0$. Let $\mathrm{X}(x) = \mathrm{J}(x,0,0)$, $\mathrm{Y}(y) = \mathrm{J}(0,y,0)$, $\mathrm{Z}(z) = \mathrm{J}(0,0,z)$ and denote the respective images of the functions X, Y and Z by X, Y and Z as well. Let $\mathrm{B_G}$, $\mathrm{B_H}$ and $\mathrm{B_M}$ be the standard Borel subgroups of $\mathrm{G}, \mathrm{H}, \mathrm{M}$, and $\mathrm{B_{M^J}} = \mathrm{B_M} \ltimes (\mathrm{Y} \times \mathrm{Z})$, and let $\mathrm{N_G}, \mathrm{N_H}, \mathrm{N_M}, \mathrm{N_{M^J}}$ be their unipotent radicals respectively. Let $\mathrm{T_G}$ be the toral part of of $\mathrm{B_G}$, and let

$$\Lambda_k^+ = \{(d_1, ..., d_k) \in \mathbb{Z}^k \mid d_1 \geq d_2 \geq ... \geq d_k \geq 0\}$$
$$\mathrm{T_G^+} = \{diag(t_1, ..., t_n, t_n^{-1}, ..., t_1^{-1}) \mid |t_1| \leq ... \leq |t_n| \leq 1\}$$
$$\mathrm{T_G^-} = \{t^{-1} \mid t \in T_{\mathrm{G}}^+\}.$$

Define T_{M}^{+} and T_{M}^{-} analogously. Let P_1^{n-m-1} be the standard parabolic subgroup of G with Levi decomposition

$$P_1^{n-m-1} = \mathrm{GL}_1^{n-m-1} \times \mathrm{H} \ltimes \mathrm{U}. \tag{2.5}$$

Let ψ_{U} be the character on U given by

$$\psi_{\mathrm{U}}(u) = \psi(\sum_{i=1}^{n-m-1} u_{i,i+1}). \tag{2.6}$$

Then M^{J} stabilizes ψ_{U}. We denote by I_{G}, I_{M} the Iwahori subgroups of G and M respectively. Let $\bar{\mathrm{I}}_{\mathrm{M}} = \mathrm{I}_{\mathrm{M}} \ltimes \mathrm{J}^0$, and let $\mathrm{I}_{\mathrm{M}^{\mathrm{J}}} = \mathrm{I}_{\mathrm{M}} \ltimes (\mathrm{X}^1\mathrm{Y}^0\mathrm{Z}^0)$. Let W_{G}, W_{M} be the Weyl groups of G and M with respect to T_{G} and T_{M}.

(B) **Elements.** Let w_0^{G} be the longest Weyl element in G. By abuse of notation we also use w_0^{G} to denote one of its representatives

$$\begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}$$

in G. For $k \leq n$, and for given $t_1, ..., t_k \in F^*$, we let

$$d_k(t_1, ..., t_k) = diag(I_{n-k}, t_1, \ldots, t_k, t_k^{-1}, \ldots, t_1^{-1}, I_{n-k}) \in \mathrm{T}_{\mathrm{G}}.$$

Let $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$, and let v be the normalized valuation from F to $\overline{\mathbb{Z}}$. We define an order $\geq$ on $\overline{\mathbb{Z}}^k$ such that for $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_k) \in \overline{\mathbb{Z}}^k$, $\mathbf{a} \geq \mathbf{b}$ if and only if $a_i \geq b_i$ for $1 \leq i \leq k$. We define $\min(\mathbf{a}, \mathbf{b}) = (\min(a_1, b_1), \ldots, \min(a_k, b_k))$. When $\mathbf{a} \in \overline{\mathbb{Z}}^m$, we let $\lambda(\mathbf{a}) = \mathrm{X}(\mathfrak{p}^{a_1}, \ldots, \mathfrak{p}^{a_m})$.

Here $\mathfrak{p}^\infty = 0$. Let $\lambda = \lambda(\mathbf{0}) = \mathrm{X}(1, 1, \ldots, 1)$. For $\mathbf{a} = (a_1, ..., a_k) \in \mathbb{Z}^k$, we let $\mathfrak{p}^{\mathbf{a}} = d_k(\mathfrak{p}^{a_1}, ..., \mathfrak{p}^{a_k})$.

For α being a root in G, we define n_α, N_α as follows. For $1 \leq i, j \leq 2n$, let $E_{i,j} \in \mathcal{M}_{2n\times 2n}$ with 1 at the (i, j)-entry and 0 at others. Then for $x \in F$ and $i \neq j$, we let

$$n_\alpha(x) = \begin{cases} x(E_{i,j} - E_{2n+1-i,2n+1-j}) & \text{if } \alpha = e_i - e_j, \\ x(E_{i,2n+1-j} + E_{j,2n+1-i}) & \text{if } \alpha = e_i + e_j, \\ xE_{n,n+1} & \text{if } \alpha = 2e_n, \\ x(E_{2n+1-i,j} + E_{2n+1-j,i}) & \text{if } \alpha = -e_i - e_j, \\ xE_{n+1,n} & \text{if } \alpha = -2e_n. \end{cases}$$

Let N_α be the group consisting of $n_\alpha(x)$ for $x \in F$. When α is a simple root, we define T_α and w_α as follows. For $1 \leq i \leq n$, let $D_i(t) = d_n(1, \ldots, 1, \overset{i-th}{t}, 1, \ldots, 1)$. Then we let

$$\mathrm{T}_\alpha(t) = \begin{cases} D_i(t)D_{i+1}(t^{-1}) & \text{if } \alpha = e_i - e_{i+1}, \\ D_n(t) & \text{if } \alpha = 2e_n, \end{cases}$$

and

$$w_\alpha = \begin{cases} E_{i,i+1} - E_{i+1,i} - E_{2n+1-j,2n+1-i} + E_{2n+1-i,2n+1-j} & \text{if } \alpha = e_i - e_{i+1}, \\ E_{n,n+1} - E_{n+1,n} & \text{if } \alpha = 2e_n. \end{cases}$$

By abuse of notation we also denote by w_α its image in the Weyl group. For β being a root in M, we define n_β, N_β, w_β and T_β as elements in or subgroups of M in a similar way.

(C) **Representations.** Let χ and ξ be unramified characters on T_{G} and T_{M}. We parametrize them as $\chi = (\chi_1, ..., \chi_n) \in \mathbb{C}^n$ and $\xi = (\xi_1, ..., \xi_m) \in \mathbb{C}^m$

such that $\chi(d_n(t_1,...,t_n)) = \prod_{i=1}^n |t_i|^{\chi_i}$ and $\xi(d_m(t_1,...,t_m)) = \prod_{j=1}^m |t_j|^{\xi_j}$. By abuse of notation we also let χ_i, ξ_j be unramified characters on $F^\times$ so that $\chi_i(\mathfrak{p}) = q^{-\chi_i}$ and $\xi_j(\mathfrak{p}) = q^{-\xi_j}$. Let $e_i \in Hom(\mathrm{T_G}, GL_1)$, for $1 \le i \le n$, and $e'_j \in Hom(\mathrm{T_M}, GL_1)$, for $1 \le j \le m$ such that $e_i(d_n(t_1,\dots,t_n)) = t_i$ and $e'_j(d_m(t_1,\dots,t_m)) = t_j$. Let $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ be a representation of $\mathrm{M^J}$ consisting of smooth functions on $\mathrm{M^J}$ such that

$$f(b_\mathrm{M}(0,y,z)m^\mathrm{J}) = \xi\delta_{\mathrm{B_{M^J}}}^{\frac{1}{2}}(b_m)\psi(z)f(m^\mathrm{J}),$$

with $\mathrm{M^J}$ acting by right translation. Sometimes we write $\xi\psi$ as a character on $\mathrm{B_{M^J}}$ such that

$$\xi\psi(b_\mathrm{M}\mathrm{J}(0,y,z)) = \xi(b_\mathrm{M})\psi(z).$$

Remark 2.2.1. *Although* $\mathrm{B_{M^J}}\backslash\mathrm{M^J}$ *is not compact, functions in* $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ *are compactly supported on* $\mathrm{B_{M^J}}\backslash\mathrm{M^J}$ *by smoothness. In fact by Iwasawa decomposition on* M*, we have*

$$\mathrm{M^J} = \mathrm{B_{M^J}XK_M}.$$

Suppose $f \in \mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ *which is right* K_f *invariant for some open compact subgroup* K_f*. Note that* $\mathrm{K}_f\backslash\mathrm{K_{M^J}}$ *is finite. Let* k *be a representative in one of the cosets and suppose* $f(xk) \neq 0$*. Then by the smoothness of* f*,* $f(xk) = f(xyk)$ *when* y *is in a neighborhood of* $0 \in F^m$*. But then* $f(xyk) = \psi(2\langle x,y\rangle)f(xk)$*, so* $\psi(2\langle x,y\rangle) = 1$ *for all such* y*, which implies that* x *belongs to a compact subset.*

In particular, if f *is* $\mathrm{K_{M^J}}$*-invariant, then the restriction of* f *on* X *is up to a scalar equal to the characteristic function of* X^0*. So the spherical vector in* $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ *is supported on* $\mathrm{B_{M^J}K_{M^J}}$*, and hence is unique up to a scalar.*

(D) **Functions and Functionals.** Let $\zeta(s) = (1-q^{-s})^{-1}$ be the local zeta function. For any set $\mathcal{X}$ we denote by $Ch_\mathcal{X}$ the characteristic function of $\mathcal{X}$. For $\varphi_1 \in C_c^\infty(\mathrm{G})$, we let

$$\mathrm{F}_\chi(\varphi_1)(g) = \int_{\mathrm{B_G}} \chi^{-1}\delta_{\mathrm{B_G}}^{\frac{1}{2}}(b_\mathrm{G})\varphi_1(b_\mathrm{G}g)\,\mathrm{d}_l b_\mathrm{G}. \tag{2.7}$$

Then the map $\varphi_1 \mapsto \mathrm{F}_\chi(\varphi_1)$ is surjective from $C_c^\infty(\mathrm{G})$ to $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$. Similarly for $\varphi_2 \in C_c^\infty(\mathrm{M^J})$, we let

$$\mathrm{F}_{\xi,\psi}(\varphi_2)(m^{\mathrm{J}}) = \int_{\mathrm{B_{M^J}}} (\xi\psi)^{-1}\delta_{\mathrm{B_{M^J}}}^{\frac{1}{2}}(b_{\mathrm{M^J}})\varphi_2(b_{\mathrm{M^J}}m^{\mathrm{J}})\,\mathrm{d}_l b_{\mathrm{M^J}}. \tag{2.8}$$

The map $\varphi_2 \mapsto \mathrm{F}_{\xi,\psi}(\varphi_2)$ is surjective from $C_c^\infty(\mathrm{M^J})$ to $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$. Let

$$\mathcal{K}_{\chi,\xi,\psi}(b_{\mathrm{G}}w_0^{\mathrm{G}}\lambda\mathrm{J}(0,y,z)b_{\mathrm{M}}u) = \chi^{-1}\delta_{\mathrm{B_G}}^{\frac{1}{2}}(b_{\mathrm{G}})\xi\delta_{\mathrm{B_{M^J}}}^{-\frac{1}{2}}(b_{\mathrm{M}})\psi(z)\psi_{\mathrm{U}}^{-1}(u). \tag{2.9}$$

We will prove in section 3 that the formula (2.9) gives a well-defined function on $\mathrm{B_G}w_0^{\mathrm{G}}\lambda\mathrm{B_{M^J}}\mathrm{U}$, which can be extended to a function on G by setting $\mathcal{K}_{\chi,\xi,\psi}(g) = 0$ for all g outside of this set. For $\varphi_1 \in C_c^\infty(\mathrm{G})$ and $\varphi_2 \in C_c^\infty(\mathrm{M^J})$, let

$$\mathrm{I}_{\chi,\xi,\psi}(\varphi_1,\varphi_2)(g) = \int_{\mathrm{G}} dg' \int_{\mathrm{M^J}} dm^{\mathrm{J}}\varphi_1(g')\mathcal{K}_{\chi,\xi,\psi}(g'g^{-1}(m^{\mathrm{J}})^{-1})\varphi_2(m^{\mathrm{J}}), \tag{2.10}$$

and let $\mathrm{I}^0_{\chi,\xi,\psi}(g) = \mathrm{I}_{\chi,\xi,\psi}(Ch_{\mathrm{K_G}}, Ch_{\mathrm{K_{M^J}}})(g)$. Let $\mathrm{F}^0_\chi = \mathrm{F}_\chi(Ch_{\mathrm{K_G}})$ and $\mathrm{F}^0_{\xi,\psi} = \mathrm{F}_{\xi,\psi}(Ch_{\mathrm{K_{M^J}}})$, which are spherical in $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ respectively. Let $\mathcal{H}_{\mathrm{G}}$ be the spherical Hecke algebra of G, and $\mathcal{H}_{\mathrm{M^J},\psi}$ be the spherical Hecke algebra of $\mathrm{M^J}$ with respect to ψ as defined in section 4 of [23], and let them act on $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)^{\mathrm{K_G}}$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)^{\mathrm{K_{M^J}}}$ by characters ω_χ and ω_ξ respectively. For any function f on G, let $(L(g_0)f)(g) = f(g_0^{-1}g)$, and $(R(g_0)f)(g) = f(gg_0)$.

Definition 2.2.2. *A pairing $l_{\chi,\xi,\psi}$ between $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ will be said to satisfy* ***Condition A*** *if*

(i) $l_{\chi,\xi,\psi}(\mathrm{F}_\chi,\mathrm{F}_{\xi,\psi}) = l_{\chi,\xi,\psi}(R(m^{\mathrm{J}})\mathrm{F}_\chi, R(m^{\mathrm{J}})\mathrm{F}_{\xi,\psi})$ *for any* $m^{\mathrm{J}} \in \mathrm{M^J}$.

(ii) $l_{\chi,\xi,\psi}(R(u)\mathrm{F}_\chi,\mathrm{F}_{\xi,\psi}) = \psi_{\mathrm{U}}(u)l_{\chi,\xi,\psi}(\mathrm{F}_\chi,\mathrm{F}_{\xi,\psi})$ *for any* $u \in \mathrm{U}$.

Remark 2.2.3. *By the definition of $\mathcal{K}_{\chi,\xi,\psi}$,* (2.10) *actually gives a pairing between $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ satisfying* ***Condition A****, for those (χ,ξ) such that the integral is convergent for all choices of (φ_1,φ_2), (see proposition 2.3.2).*

Definition 2.2.4. *For* $(\chi, \xi) \in \mathbb{C}^n \times \mathbb{C}^m$*, a function* $\mathrm{W}_{\chi,\xi,\psi} \in \mathbf{C}^\infty(\mathrm{G})$ *is called a Whittaker-Shintani Function attached to* (χ, ξ) *if*

(i) $\mathrm{W}_{\chi,\xi,\psi}(zuk_{\mathrm{M^J}}gk_{\mathrm{G}}) = \psi^{-1}(z)\psi_{\mathrm{U}}(u)\mathrm{W}_{\chi,\xi,\psi}(g)$.

(ii) $L(\varphi_{\mathrm{M^J}})R(\varphi_{\mathrm{G}})\mathrm{W}_{\chi,\xi,\psi} = \omega_\xi(\varphi_{\mathrm{M^J}})\omega_\chi(\varphi_{\mathrm{G}}) \cdot \mathrm{W}_{\chi,\xi,\psi}$ *for any* $\varphi_{\mathrm{M^J}} \in \mathcal{H}_{\mathrm{M^J},\psi}$ *and* $\varphi_{\mathrm{G}} \in \mathcal{H}_{\mathrm{G}}$.

The space of Whittaker-Shintani functions attached to (χ, ξ, ψ) *is denoted by* $\mathcal{WS}_{\chi,\xi,\psi}$*. Sometimes we omit* ψ *because it is fixed in this paper. A Whittaker-Shintani function is called a Normalized Whittaker-Shintani function if it equals 1 at the identity. We denote it by* $\mathrm{W}^0_{\chi,\xi,\psi}$.

2.3 Integral expression for the pairing

We use the function $\mathcal{K}_{\chi,\xi,\psi}$, as defined in (2.9) to construct a pairing between $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{B_G}}(\chi)$ and $\mathrm{Ind}^{\mathrm{M^J}}_{\mathrm{B_{M^J}}}(\xi, \psi)$ satisfying Condition A. First we show that

Lemma 2.3.1. *The function* $\mathcal{K}_{\chi,\xi,\psi}(g)$ *as defined in (2.9) is well-defined.*

Proof. To show that $\mathcal{K}_{\chi,\xi,\psi}(g)$ is well-defined we only need to show that the way to write an element $g \in \mathrm{B_G}w_0^{\mathrm{G}}\lambda\mathrm{YZB_M U}$ as $g = b_{\mathrm{G}}w_0^{\mathrm{G}}\lambda yzb_{\mathrm{M}}u$ is unique. Take $g = b_{\mathrm{G}}w_0^{\mathrm{G}}\lambda yzb_{\mathrm{M}}u$, and let $b_{\mathrm{M}} = tn$ be the Levi decomposition, then
$g = (b_{\mathrm{G}}t^{-1})w_0^{\mathrm{G}}(t^{-1}\lambda t)(t^{-1}yt)(t^{-1}zt)nu$. Note that

(1) $(b_{\mathrm{G}}t^{-1}) \in \mathrm{B_G}$ and $(t^{-1}\lambda t)(t^{-1}yt)(t^{-1}zt)nu \in \mathrm{N_G}$.

(2) The way to write an element $g \in \mathrm{B_G}w_0^{\mathrm{G}}\mathrm{N_G}$ as $g = bw_0^{\mathrm{G}}n'$ is unique.

(3) $\lambda^{t^{-1}}$, $y^{t^{-1}}$, $z^{t^{-1}}$, n, u belongs to different root subgroups of $\mathrm{N_G}$.

(4) For $t, t' \in \mathrm{T_M}$, $\lambda^t = \lambda^{t'}$ if and only if $t = t'$.

Then our lemma is implied. □

Observe that the function $\mathrm{I}_{\chi,\xi,\psi}$ defined on $C_c^\infty(\mathrm{G}) \times C_c^\infty(\mathrm{M^J}) \times \mathrm{G}$ by equation (2.10) factors through the mappings $\mathrm{F}_\chi : C_c^\infty(\mathrm{G}) \to \mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{F}_{\xi,\psi}(\varphi_2) : C_c^\infty(\mathrm{M^J}) \to \mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$. Thus, we may define a pairing $l_{\chi,\xi,\psi}$ on $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi) \times \mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ by setting $l_{\chi,\xi,\psi}(\mathrm{F}_\chi(\varphi_1), \mathrm{F}_{\xi,\psi}(\varphi_2)) = \mathrm{I}_{\chi,\xi,\psi}(\varphi_1,\varphi_2)(e)$. Explicitly:

$$l_{\chi,\xi,\psi}(\mathrm{F}_\chi(\varphi_1), \mathrm{F}_{\xi,\psi}(\varphi_2)) = \int_{\mathrm{G}} \int_{\mathrm{M^J}} \varphi_1(g')\mathcal{K}_{\chi,\xi,\psi}(g'(m^{\mathrm{J}})^{-1})\varphi_2(m^{\mathrm{J}})\,\mathrm{d}m^{\mathrm{J}}\,\mathrm{d}g'. \quad (2.11)$$

It is easy to see that the pairing $l_{\chi,\xi,\psi}(\mathrm{F}_\chi(\varphi_1), \mathrm{F}_{\xi,\psi}(\varphi_2))$ satisfies Condition A if the integral is convergent, and the integral is convergent if $\mathcal{K}_{\chi,\xi,\psi}$ is continuous on G. In the rest of this section we will prove the following proposition.

Propostion 2.3.2. *Let $\mathcal{Z}_c$ be the set of unramified characters (χ,ξ) satisfying*

$$\begin{cases} Re(\chi_i - \chi_{i+1}) > 1 & \text{for } 1 \le i \le n-m-1 \\ Re(\chi_{n-m-1+j} - \xi_j) > \frac{1}{2} & \text{for } 1 \le j \le m \\ Re(-\chi_{n-m+j} + \xi_j) > \frac{1}{2} & \text{for } 1 \le j \le m \\ Re(\chi_n) > 1 \end{cases} \quad (2.12)$$

then when $(\chi,\xi) \in \mathcal{Z}_c$, the function $\mathcal{K}_{\chi,\xi,\psi}$ is continuous on G*, and as a consequence, the integral (2.11) is convergent.*

Since $\mathcal{K}_{\chi,\xi,\psi}$ is defined continuously on $\mathrm{B_G} w_0^{\mathrm{G}} \lambda \mathrm{B_{M^J}} \mathrm{U}$, which is a Zariski open subset of G (we will see this soon) and extended by 0 to G, we only need to show the continuity outside the Zariski open set, for the function $|\mathcal{K}_{\chi,\xi,\psi}|$. The method we use here is similar to that in [19].

First by the Bruhat decomposition we have

$$\mathrm{G} = \bigcup_{w \in \mathrm{W_G}} \mathrm{B_G} w \mathrm{N_G}.$$

And we know that $\mathrm{B_G} w_0^{\mathrm{G}} \mathrm{N_G}$ is Zariski open in G. In fact we have

Lemma 2.3.3. *There exists* $\alpha_k \in \mathfrak{o}[\mathrm{G}]$ *for* $1 \le k \le n$ *such that*

$$\mathrm{B_G} w_0^{\mathrm{G}} \mathrm{N_G} = \{g \mid \alpha_k(g) \neq 0 \textit{ for all } k \ \}.$$

Proof. For $g \in \mathrm{G}$, let its matrix be $g = (g_{ij})_{1 \le i,j \le 2n}$. Let $\mathcal{N}_{2n} = \{1, 2, \ldots, 2n\}$. For $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ both belonging to $\mathcal{N}_{2n}^k$, we Let $g_{IJ} = (g_{i_s, j_t})_{1 \le s,t \le k}$. We define

$$\Delta_{IJ}(g) = \det g_{IJ}. \tag{2.13}$$

For $1 \le k \le n$, let $I_k = \{2n+1-k, 2n+1-(k-1), \ldots, 2n\}$, and $J_k = \{1, 2, \ldots, k\}$, and we take

$$\alpha_k(g) = \Delta_{I_k, J_k}(g). \tag{2.14}$$

Then one can check that

1. For any $n_1, n_2 \in \mathrm{N_G}$, $\alpha_k(n_1 g n_2) = \alpha_k(g)$.
2. $\alpha_k(d_n(t_1, \ldots, t_n) g d_n(s_1, \ldots, s_n)) = \prod_{i=1}^k t_i^{-1} s_i \cdot \alpha_k(g)$.
3. Let $w \in \mathrm{W_G}$ and w' one of its representative in G. If $\alpha_k(w') \neq 0$ for all $1 \le k \le n$, then $w = w_0^{\mathrm{G}}$.

Combining these properties with the Bruhat decomposition of G, we have our lemma. □

Next we have

Lemma 2.3.4. *There exists* $\beta_l \in \mathfrak{o}[\mathrm{G}]$ *for* $1 \le l \le m$ *such that*

$$\mathrm{B_G} w_0^{\mathrm{G}} \lambda \mathrm{YZB_M U} = \{g \in \mathrm{G} \mid \alpha_k(g) \neq 0, \beta_l(g) \neq 0 \textit{ for all } 1 \le k \le n, 1 \le l \le m\},$$

where α_k *is as defined in lemma 2.3.3.*

Proof. Note that for any $w \in \mathrm{W}$, $\mathrm{B_G} w \mathrm{N_G} = \mathrm{B_G} w \mathrm{XUN_{M^J}}$. For any $\mathrm{X}(x_1, \ldots, x_m) \in \mathrm{X}$, we have $\mathrm{X}(x_1, \ldots, x_m) = \mathbf{s}^{-1} \mathrm{X}(r_1, \ldots, r_m) \mathbf{s}$, where $\mathbf{s} = d_m(s_1, \ldots, s_m) \in \mathrm{T_M}$

such that

$$(s_i, r_i) = \begin{cases} (x_i, 1) & \text{if } x_i \neq 0; \\ (1, 0) & \text{if } x_i = 0 \end{cases}$$

From this we can see that

$$\mathrm{B_G} w \mathrm{N_G} = \bigcup_{\mathbf{r} \in \{0,1\}^m} \mathrm{B_G} w \mathrm{X}(\mathbf{r}) \mathrm{B_{M^J}} \mathrm{U}, \tag{2.15}$$

and when $w = w_0^{\mathrm{G}}$, the union is disjoint. For $1 \leq l \leq m$, we let

$$J_l' = \{1, 2, \ldots, \widehat{(n-m)}, \ldots, n-m+l\},$$

and we define

$$\beta_l(g) = \Delta_{I_{n-m+l-1}, J_l'}(g). \tag{2.16}$$

Then β_l satisfies

1. $\beta_l(n_1 g) = \beta_l(g)$ for any $n_1 \in \mathrm{N_G}$.
2. $\beta_l(g n_2 u) = \beta_l(g)$ for any $n_2 \in \mathrm{N_{M^J}}$ and $u \in \mathrm{U}$.
3. $\beta_l(d_n(t_1, \ldots, t_n) g d_m(s_1, \ldots, s_m)) = \prod_{i=1}^{n-m} t_i^{-1} \cdot \prod_{j=1}^{l-1} t_{n-m+j}^{-1} \cdot \prod_{j=1}^{l} s_j \cdot \beta_l(g)$.
4. $\beta_l(w_0^{\mathrm{G}} \mathrm{X}(\mathbf{r})) = \pm r_l$. The sign in front of r_l depends on l, which is not important since we are only interested in $|\beta_l|$.

In fact, (1) is by the definition of I_k while (3) and (4) are by direct calculation. For (2), note that if we only consider the first n column of (g_{ij}), multiplying elements in $\mathrm{N_{M^J}U}$ from the right corresponds to column operations adding multiples of column k_1 to column k_2 where $1 \leq k_1 < k_2 \leq n$ with $k_1 \neq n-m$. On the other hand elements in J_l' are consecutive from 1 to $n-m+l$ with $n-m$ missing, so $\Delta_{I_{n-m+l-1}, J_l'}(g)$ is invariant under such column operations.

So for $g \in \mathrm{B_G} w \mathrm{X}(\mathbf{r}) \mathrm{B_{M^J}} \mathrm{U}$ with $\mathbf{r} \in \{0,1\}^m$, $\alpha_k(g) \neq 0$ and $\beta_l(g) \neq 0$ for all $1 \leq k \leq n$ and $1 \leq l \leq m$ if and only if $w = w_0^{\mathrm{G}}$ (by lemma 2.3.3) and $r = (1, 1, \ldots, 1)$ (by the property of β_l's), completing our proof. $\square$

Remark 2.3.5. *If we let* $\varpi_i = e_1 + e_2 + \cdots + e_i$*, and* $\varpi'_j = e'_1 + e'_2 + \cdots + e'_j$ *be dominant weights of* G *and* M *with respect to* $\mathrm{B_G}$ *and* $\mathrm{B_M}$*, then the properties of* α_k *and* β_l *actually shows that under the* $\mathrm{B_G} \times \mathrm{B_{M^J}}$ *action,* α_i *has the highest weight* $(\varpi_k, 0)$ *when* $1 \leq k \leq n-m$*, and* $(\varpi_k, \varpi'_{k-(n-m)})$ *when* $n-m+1 \leq k \leq n$*, and* β_l *has the highest weight* $(\varpi_{n-m+l-1}, \varpi'_l)$*.*

Now we can expressed $\mathcal{K}_{\chi,\xi,\psi}$ by α_k and β_l. First we have

Lemma 2.3.6. *Let* $g = d_n(t_1, \ldots, t_n) n_{\mathrm{G}} w_0^{\mathrm{G}} \lambda d_m(s_1, \ldots, s_m) n_{\mathrm{M^J}} u \in \mathrm{B_G} w_0^{\mathrm{G}} \lambda \mathrm{B_{M^J}} \mathrm{U}$*, we have*

$$|t_i| = \begin{cases} |\alpha_1(g)^{-1}| & \text{if } i = 1; \\ |\alpha_{i-1}\alpha_i^{-1}(g)| & \text{if } 2 \leq i \leq n-m; \\ |\beta_{i-(n-m)}\alpha_i^{-1}(g)| & \text{if } i > n-m \end{cases}$$

and

$$|s_j| = |\beta_j \alpha_{n-m+j-1}^{-1}(g)| \qquad \text{for } 1 \leq j \leq m$$

By this we have

Lemma 2.3.7. *For* $g \in \mathrm{B_G} w_0^{\mathrm{G}} \lambda \mathrm{B_{M^J}} \mathrm{U}$*, we have*

$$\begin{aligned} |\mathcal{K}_{\chi,\xi,\psi}(g)| = &\prod_{i=1}^{n-m-1} |(\chi_i \chi_{i+1}^{-1}| \cdot |^{-1})(\alpha_i(g))| \cdot \prod_{j=1}^{m} |(\chi_{n-m-1+j}\xi_j^{-1})| \cdot |^{-\frac{1}{2}}(\alpha_{n-m+j-1}(g))| \\ &\cdot |(\chi_n| \cdot |^{-1})(\alpha_n(g))| \cdot \prod_{j=1}^{m} |(\chi_{n-m+j}^{-1}\xi_j| \cdot |^{-\frac{1}{2}})(\beta_j(g))|. \end{aligned}$$

The proof of these are by direct calculation. By lemma 2.3.7 we can see that when the assumptions in proposition 2.3.2 are satisfied, the extension of $|\mathcal{K}_{\chi,\xi,\psi}(g)|$ by letting $\mathcal{K}_{\chi,\xi,\psi}(g) = 0$ for $g \notin \mathrm{B_G} w_0^{\mathrm{G}} \lambda \mathrm{B_{M^J}} \mathrm{U}$ is continuous, and so $\mathcal{K}_{\chi,\xi,\psi}(g)$ is continuous.

2.4 Uniqueness of the pairing for generic (χ, ξ)

Each pairing $l_{\chi,\xi,\psi}$ satisfying Condition A corresponds to an element of

$$\mathrm{Hom}_{\mathrm{B_{M^J}U}}(\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi), \xi^{-1}\psi^{-1}\delta_{\mathrm{B_{M^J}}}^{\frac{1}{2}} \otimes \psi_{\mathrm{U}}). \tag{2.17}$$

In this section we prove that

Propostion 2.4.1. *For generic* (χ, ξ)*,*

$$\mathrm{dimHom}_{\mathrm{B_{M^J}U}}(\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi), \xi^{-1}\psi^{-1}\delta_{\mathrm{B_{M^J}}}^{\frac{1}{2}} \otimes \psi_{\mathrm{U}}) \leq 1.$$

By (2.15), the action of $\mathrm{B_G} \times \mathrm{B_{M^J}U}$ on G via $(b_1, b_2) \circ x = b_1 x b_2^{-1}$ has finite many orbits. According to 1.5 in [3], (see also 5.1(3) in [4]), there is a numbering $Z_1, Z_2, \ldots, Z_k$ of $\mathrm{B_G} \times \mathrm{B_{M^J}U}$ orbits on G, such that $Z'_i = \cup_{j=1}^{i} Z_j$ are open in G for every i. Note that $Z_1 = \mathrm{B_G} w_0^{\mathrm{G}} \lambda \mathrm{B_{M^J}U}$. Given $\mathcal{U}$, a union of Z_i's, let $I_c^{\infty}(\chi\delta_{\mathrm{B_G}}^{\frac{1}{2}}|\mathrm{B_G}, \mathcal{U})$ be the set of functions $f : \mathcal{U} \to \mathbb{C}$ which are locally constant, compactly supported modulo $\mathrm{B_G}$ and satisfying $f(bg) = \chi\delta_{\mathrm{B_G}}^{\frac{1}{2}}(b)f(g)$. It is a $\mathrm{B_{M^J}U}$-module. Since Z'_{i-1} is open in Z'_i, by [7, Lemma 6.1.1], we have the exact sequence of $\mathrm{B_{M^J}U}$-modules:

$$0 \to I(\chi\delta_{\mathrm{B_G}}^{\frac{1}{2}}|\mathrm{B_G}, Z'_{i-1}) \to I(\chi\delta_{\mathrm{B_G}}^{\frac{1}{2}}|\mathrm{B_G}, Z'_i) \to I(\chi\delta_{\mathrm{B_G}}^{\frac{1}{2}}|\mathrm{B_G}, Z_i) \to 0.$$

Observe that

1. $I(\chi\delta_{\mathrm{B_G}}^{\frac{1}{2}}|\mathrm{B_G}, Z'_k) = \mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ as $\mathrm{B_{M^J}U}$-modules, and
2. $\dim \mathrm{Hom}_{\mathrm{B_{M^J}U}}(I(\chi\delta_{\mathrm{B_G}}^{\frac{1}{2}}|\mathrm{B_G}, Z_1), \xi^{-1}\psi^{-1}\delta_{\mathrm{B_{M^J}}}^{\frac{1}{2}} \otimes \psi_{\mathrm{U}}) \leq 1$,

so to prove proposition 2.4.1 it suffices to prove the following lemma.

Lemma 2.4.2. *Let* $\mathcal{U}_g = \mathrm{B_G} g \mathrm{B_{M^J}U}$. *Suppose* (χ, ξ) *is generic, and* $\mathcal{U}_g \neq Z_1$ *, then*

$$\dim \mathrm{Hom}_{\mathrm{B_{M^J}U}}(I(\chi\delta_{\mathrm{B_G}}^{\frac{1}{2}}|\mathrm{B_G}, \mathcal{U}_g), \xi^{-1}\psi^{-1}\delta_{\mathrm{B_{M^J}}}^{\frac{1}{2}} \otimes \psi_{\mathrm{U}}) = 0.$$

Proof. Let $\mathrm{G}_g = \mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{U} \cap g^{-1}\mathrm{B}_{\mathrm{G}}g$, then

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{U}}(I(\chi\delta_{\mathrm{B}_{\mathrm{G}}}^{\frac{1}{2}}|\mathrm{B}_{\mathrm{G}},\mathcal{U}_g),\xi^{-1}\psi^{-1}\delta_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{\frac{1}{2}}\otimes\psi_{\mathrm{U}})\\ =&\mathrm{Hom}_{\mathrm{G}_g}(g^{-1}(\chi\delta_{\mathrm{B}_{\mathrm{G}}}^{\frac{1}{2}})\otimes\xi\psi\delta_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{-\frac{1}{2}}\otimes\psi_{\mathrm{U}}^{-1},\delta_g)\end{aligned}$$

where δ_g is the modulus character of G_g. So we need to show that

$$g^{-1}(\chi\delta_{\mathrm{B}_{\mathrm{G}}}^{\frac{1}{2}})\cdot\xi\psi\delta_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{-\frac{1}{2}}\cdot\psi_{\mathrm{U}}^{-1}\cdot\delta_g^{-1}\not\equiv 1 \tag{2.18}$$

when restricted to G_g for any generic (χ,ξ). Recall from (2.15) that

$$\mathrm{G} = \bigcup_{w\in\mathrm{W}_{\mathrm{G}},\mathbf{r}\in\{0,1\}^m}\mathrm{B}_{\mathrm{G}}w\mathrm{X}(\mathbf{r})\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}.$$

So when $\mathcal{U}_g \neq Z_1$ we can assume $g = w\mathrm{X}(\mathbf{r})$ with either $w \neq w_0^{\mathrm{G}}$ or $\mathbf{r} \neq (1,1,\ldots,1)$. To prove (2.18) we need to find $b_1 \in \mathrm{B}_{\mathrm{G}}$ and $b_2 \in \mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{U}$ such that

$$\begin{aligned}&b_1g = gb_2\\ &(\chi\delta_{\mathrm{B}_{\mathrm{G}}}^{\frac{1}{2}})(b_1)\neq\xi\psi\delta_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{-\frac{1}{2}}\psi_{\mathrm{U}}\cdot\delta_g(b_2).\end{aligned} \tag{2.19}$$

First suppose $\mathbf{r} \neq (1,1,\ldots,1)$. In this case we claim that $\mathrm{T}_{\mathrm{G}} \cap (g\mathrm{T}_{\mathrm{M}}g^{-1})$ contains a nontrivial torus. Let $\mathbf{t} = d_m(t_1,\ldots t_m) \in \mathrm{T}_{\mathrm{M}}$. Note that T_{G} is stabilized by the adjoint action of W_{G}, so it suffices to show that there exists a nontrivial torus T_s of T_{M} such that when $\mathbf{t} \in \mathrm{T}_s$, $\mathrm{X}(\mathbf{r})^{-1}\mathbf{t}\mathrm{X}(\mathbf{r}) \in \mathrm{T}_{\mathrm{G}}$. Note that

$$\mathrm{X}(\mathbf{r})^{-1}\mathbf{t}\mathrm{X}(\mathbf{r}) = \mathbf{t}\cdot\mathrm{X}((1-t_1)r_1,(1-t_2)r_2,\ldots,(1-t_m)r_m),$$

so when $r_j = 0$ for some j, we can let $\mathrm{T}_s = \{\mathbf{t} = d_m(1,\ldots,\overset{\text{j-th}}{t},\ldots,1)\}$ be the torus we claimed. Then since (χ,ξ) is generic, one can find some $b_2 \in \mathrm{T}_s$ and $b_1 = gb_2g^{-1} \in \mathrm{T}_{\mathrm{M}}$ so that (2.19) is satisfied, completing the proof for this case.

Now suppose $\mathbf{r} = (1,\ldots,1) \in F^m$ and $w \neq w_0^{\mathrm{G}}$, so $\mathrm{X}(\mathbf{r}) = \lambda$ by our notation.

In this case there is a simple root α in G such that $w\mathrm{N}_\alpha w^{-1} \in \mathrm{N}_\mathrm{G}$.

When $\alpha = e_i - e_{i+1}$ with $1 \le i \le n-m-1$, note that $\lambda \in \mathrm{M}^\mathrm{J}$ stabilizes ψ_U, so $\psi_\mathrm{U}(\lambda^{-1}n_\alpha(t)\lambda) = \psi_\mathrm{U}(n_\alpha(t)) \neq 1$ for some $t \in F$. On the other hand, $wn_\alpha(t)w^{-1} \in \mathrm{N}_\mathrm{G}$ by our assumption. So let $b_1 = wn_\alpha(t)w^{-1}$ and $b_2 = \lambda^{-1}n_\alpha(t)\lambda$ we have (2.19).

When $\alpha = e_i - e_{i+1}$ with $i \ge n-m$, we let $\mathbf{r}(t,i) = \mathrm{X}(1, \ldots, \overset{i'\text{-th}}{1}, 1+t, \ldots, 1)$, where $i' = i-(n-m)$. Then for $t \neq -1$ we have

$$\begin{aligned} wn_\alpha(t)\lambda &= w\mathrm{X}(\mathbf{r}(t,i))n_\alpha(t) \\ &=(d_m^{-1}(\mathbf{r}(t,i))\,)^w w\lambda d_m(\mathbf{r}(t,i))n_\alpha(t). \end{aligned}$$

So let $b_1 = (d_m^{-1}(\mathbf{r}(t,i))n_\alpha(t))^w$ and $b_2 = d_m(\mathbf{r}(t,i))n_\alpha(t)$. For generic (χ, ξ), we can always find some $t \in F$ so that (2.19) is satisfied.

When $\alpha = 2e_n$, we have

$$wn_\alpha(t)\lambda = w\lambda\mathrm{Y}(t)\mathrm{Z}(-t)n_\alpha(t).$$

So let $b_1 = (n_\alpha(t))^w$ and $b_2 = \mathrm{Y}(t)\mathrm{Z}(-t)n_\alpha(t)$, we have $(\chi\delta_{\mathrm{B_G}}^{\frac{1}{2}})(b_1) = 1$, and we can find some $t \in F$ so that $\psi(\mathrm{Z}(-t)) \neq 1$, so (2.19) is satisfied. □

2.5 Rationality(I)

In this section we prove the following proposition.

Propostion 2.5.1. *1. For fixed $\varphi_1 \in C_c^\infty(\mathrm{G})$ and $\varphi_2 \in C_c^\infty(\mathrm{M^J})$, the function*

$$(\chi, \xi) \mapsto l_{\chi,\xi,\psi}(\mathrm{F}_\chi(\varphi_1), \mathrm{F}_\xi(\varphi_2))$$

extends to $\mathbb{C}^n \times \mathbb{C}^m$ as a rational function of $(q^{\chi_1}, \ldots, q^{\chi_n}, q^{\xi_1}, \ldots, q^{\xi_m})$.

2. The pairing $l_{\chi,\xi,\psi}$ extends by meromorphic continuation to a pairing between $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi, \psi)$ which is defined for all χ, ξ outside of a countable *union of hyperplanes in $\mathbb{C}^n \times \mathbb{C}^m$. Where it is defined, this pairing satisfies* ***Condition A****.*

3. *For χ, ξ outside of this same countable union of hyperplanes, the space of pairings between $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi, \psi)$ which satisfy* ***Condition A*** *is one-dimensional.*

To prove this proposition we apply Bernstein's theorem, which we recall as follows. Let V be a vector space over $\mathbb{C}$, and $V^* = Hom_{\mathbb{C}}(V, \mathbb{C})$. We call a set $\Xi \subset V \times \mathbb{C}$ a system of equations in $V \times \mathbb{C}$, or simply a system, and we call $\lambda \in V^*$ a solution to Ξ if $\lambda(v) = x$ for every $(v, x) \in \Xi$. Let $\mathcal{D}$ be an algebraic variety over $\mathbb{C}$, and $\mathbb{C}[\mathcal{D}]$ the algebra of regular functions $\mathcal{D} \to \mathbb{C}$. Let $V_{\mathbb{C}[\mathcal{D}]} = V \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{D}]$. For $d \in \mathcal{D}$, let $\xi_d : V_{\mathbb{C}[\mathcal{D}]} \to V$ defined by $v \otimes \phi \mapsto \phi(d)v$. A function $\mathcal{D} \to V$ is called regular if it is given by $d \mapsto \xi_d(\hat{v})$ for some $\hat{v} \in V_{\mathbb{C}[\mathcal{D}]}$.

Let $\Xi_{\mathcal{D}} = \{\Xi_d | d \in \mathcal{D}\}$ be a family of system, where each Ξ_d is a system $\{(v_{r,d}, x_{r,d}) \in V \times \mathbb{C} | r \in \mathcal{R}\}$. Here $\mathcal{R}$ is a common set for all $d \in \mathcal{D}$. We call $\Xi_{\mathcal{D}}$ regular if for each $r \in \mathcal{R}$, both $d \mapsto v_{r,d}$ and $d \mapsto x_{r,d}$ are regular, which means we can choose $\hat{v}_r \in V_{\mathbb{C}[\mathcal{D}]}$ and $x_r \in \mathbb{C}[\mathcal{D}]$ for each r, so that $\xi_d(\hat{v}_r) = v_{r,d}$ and $x_r(d) = x_{r,d}$ for all $d \in \mathcal{D}$.

Let $F = \mathbb{C}(\mathcal{D})$, the field of fractions of $\mathbb{C}[\mathcal{D}]$. Let $V_F = V \otimes_{\mathbb{C}} F$, and $V_F^* = Hom_F(V_F, F)$. The family of system $\Xi_{\mathcal{D}}$ can be treated as a system $\Xi = \{(\hat{v}_r, x_r) \in V_F \times F | r \in \mathcal{R}\}$. If $\lambda \in V_F^*$ is a solution to Ξ, then λ_d, defined as $\lambda_d(v) = \lambda(v \otimes 1)(d)$, is a solution to Ξ_d when it has no poles at d.

With the notation above, we can state Bernstein's theorem.

Theorem 2.5.2 (Bernstein)**.** *If V is countable dimensional over $\mathbb{C}$, and for a Hausdorff open subset $\mathcal{D}_0$ of $\mathcal{D}$, Ξ_d has a unique solution for all $d \in \mathcal{D}_0$, then*

1. *Ξ has a unique solution $\lambda \in V_F^*$.*

2. *There exists $\mathcal{D}' \subset \mathcal{D}$, the complement of which is a countable union of hyperplanes, such that $\lambda_d \in V^*$ is defined and is the unique solution.*

In [1], the author gives a stronger result on the poles of λ.

Corollary 2.5.3. *With the same notations as above, suppose moreover that $\mathcal{D}$ is non-singular. If Ξ_d has a unique solution, then λ has no poles at d. In particular, if Ξ_d has a unique solution for all $d \in \mathcal{D}$, then $\lambda : V_{\mathbb{C}[\mathcal{D}]} \to \mathbb{C}[\mathcal{D}]$.*

To apply Bernstein's theorem to our case, we need to explain what V, $\mathcal{D}$, Ξ are, verify the assumptions in Bernstein's theorem, and show that the conclusion in Bernstein's theorem implies the rationality of the pairing $l_{\chi,\xi,\psi}$.

First we define V. Take $f \in \mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$. By the Iwasawa decomposition, $\mathrm{G} = \mathrm{B_G K_G}$, so the restriction of f on $\mathrm{K_G}$ determines f uniquely. Let V_1 be the space of smooth, $\mathbb{C}$-valued, left $\mathrm{B_G^0}$-invariant functions on $\mathrm{K_G}$, then $V_1 = \{f|_{\mathrm{K_G}} : f \in \mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)\}$. Let G act on V_1 by

$$(g \circ f|_{\mathrm{G_K}})(k) = \chi\delta_{\mathrm{B_G}}^{\frac{1}{2}}(b')f_R(k') \tag{2.20}$$

where $b'k' = kg$. Note that the way to write kg as $b'k'$ is not unique, but by the definition of V_1 the right hand side of the above formula is well-defined. Then $f \mapsto f|_{\mathrm{K_G}}$ gives an isomorphism of G-representations.

We could realize $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi, \psi)$ in a similar way. We know that $\mathrm{M^J} = \mathrm{B_M YZXK_M}$, but the way to decompose is not unique. In fact, $b_1y_1z_1x_1k_1 = b_2y_2z_2x_2k_2$ if any only if $b_1^{-1}b_2 = k_1k_2^{-1}$ and $b_1^{-1}b_2(y_2z_2x_2)b_2^{-1}b_1 = y_1z_1x_1$. For each (x_1, k_1) and (x_2, k_2) satisfying this with some $b_i, y_i, z_i (i = 1, 2)$, we denote $(x_1, k_1) \sim (x_2, k_2)$. It is not hard to check that $(x_1, k_1) \sim (x_2, k_2)$ if and only if there exists $b \in \mathrm{B_M^0}$ such that $bk_2 = k_1$ and $bx_2b^{-1} \sim x_1$ (Here we treat X as J/YZ, and $\mathrm{B_M}$ acts on it by conjugation). Let V_2 be the space of smooth, $\mathbb{C}$-valued functions on $\mathrm{X} \times \mathrm{K_M}$ which are invariant on each such orbit. Then we let $\mathrm{M^J}$ act on V_2 by

$$(m^{\mathrm{J}} \circ f')(xk) = \xi\delta_{\mathrm{B_M}}^{\frac{1}{2}}(b')\psi(z')f'(x'k') \tag{2.21}$$

where $xkm^J = b'y'z'x'k'$. By the definition of V_2 this is also well-defined. It is not hard to check that under this action V_2 is isomorphic to $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi, \psi)$ as a representation of $\mathrm{M^J}$. Note that for each $k \in \mathrm{K_M}$, the mapping $x \mapsto f'(x,k)$ is compactly supported by the smoothness of f'.

Let $V = V_1 \otimes V_2$. V_1 has a countable basis which consists of characteristic functions of open compact subsets of $\mathrm{K_G}$. Let $\{f_i : i \in \mathbb{Z}_{\geq 0}\}$ be such a basis. V_2 has a countable basis consisting of characteristic functions of open compact subsets of $\mathrm{X} \times \mathrm{K_M}$. Let $\{f'_j : j \in \mathbb{Z}_{\geq 0}\}$ be such a basis. Then it follows that $\{f_i \otimes f'_j : i,j \in \mathbb{Z}_{\geq 0}\}$ is a countable basis of V.

Let $\mathcal{D}$ be $(\mathbb{C}/\frac{2\pi i}{log(q)})^{n+m} \cong (\mathbb{C}^\times)^{n+m}$ via

$$(\chi_1, \ldots, \chi_n, \xi_1, \ldots, \xi_m) \mapsto (q^{\chi_1}, \ldots, q^{\chi_n}, q^{\xi_1}, \ldots, q^{\xi_m}).$$

Then $\mathbb{C}[\mathcal{D}]$ (resp. $F = \mathbb{C}(\mathcal{D})$) is the polynomial ring (resp. field of rational functions) in $q^{\pm\chi_1}, \ldots, q^{\pm\chi_n}, q^{\pm\xi_1}, \ldots, q^{\pm\xi_m}$.

To define Ξ, we use the lemma 2.8.5, which does not depend on our current discussion. For $(\chi, \xi) \in \mathcal{Z}_c$,

$$l_{\chi,\xi,\psi}(\mathrm{F}_\chi(\mathrm{Ch}_{\mathrm{I_G}}), \mathrm{F}_{\xi,\psi}(\mathrm{Ch}_{\mathrm{I_{M^J}}})) = \mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\mathrm{I_{M^J}}). \tag{2.22}$$

By direct calculation, $\mathrm{F}_\chi(\mathrm{Ch}_{\mathrm{I_G}}) \mid_{\mathrm{K_G}} = \mathrm{Ch}_{\mathrm{I_G}}$, and $\mathrm{F}_{\xi,\psi}(\mathrm{Ch}_{\mathrm{I_{M^J}}}) \mid_{\mathrm{X}\times\mathrm{K_M}} = \mathrm{Ch}_{\mathrm{X^1}\times\mathrm{I_M}}$.

Now we let the system Ξ be

$$\begin{aligned}
&\{(R_\chi(m^J) f_i \otimes R_{\xi,\psi}(m^J)(f'_j) - f_i \otimes f_j, 0) \mid m^{\mathrm{J}} \in \mathrm{M^J}, i,j \in \mathbb{Z}_{\geq 0}\}\\
\cup&\{(R_\chi(u) f_i \otimes f'_j - \psi_{\mathrm{U}}(u) f_i \otimes f'_j, 0) \mid u \in \mathrm{U}, i,j \in \mathbb{Z}_{\geq 0}\}\\
\cup&\{(\mathrm{Ch}_{\mathrm{I_G}} \otimes \mathrm{Ch}_{\mathrm{X^1}\times\mathrm{I_M}}, \mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\mathrm{I_{M^J}}))\}
\end{aligned} \tag{2.23}$$

Now we can prove our proposition by applying Bernstein's theorem.

Proof of proposition 2.5.1. We already showed that V is of countable dimension over $\mathbb{C}$. Both $R_\chi(g) f_i$ and $R_{\xi,\psi}(m^J) f'_j$ are linear combinations of $\{f_i\}$ and $\{f'_j\}$ respectively, with coefficients in $\mathbb{C}[q^\chi]$ and $\mathbb{C}[q^\xi]$. So the system Ξ is regular in $\mathbb{C}[\mathcal{D}]$. For $(\chi, \xi) \in \mathcal{Z}_c$ and generic, $l_{\chi,\xi,\psi}$ as defined in (2.11) is the only solution to

$\Xi_{(q^{\chi_1},\dots,q^{\chi_n},q^{\xi_1},\dots,q^{\xi_m})}$. In fact a solution to the first two parts of the union of (2.23) is a pairing satisfying Condition A, which is unique up to a constant for generic (χ,ξ) by Proposition 2.4.1, and the third part of (2.23) makes the solution unique for $(\chi,\xi)\in\mathcal{Z}_c$. Note that $\mathcal{Z}_c$ contains a Hausdorff open set, so by Bernstein's theorem, if we fix two families of elements in $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ and $\mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ with the same restrictions to $\mathrm{K_G}$ and $\mathrm{X}\times\mathrm{K_M}$ respectively and let (χ,ξ) vary, the pairing $l_{\chi,\xi,\psi}$ on them for $(\chi,\xi)\in\mathcal{Z}_c$ as defined in (2.11) extends to a rational function in $\mathbb{C}(q^{\chi},q^{\xi})$. Moreover, it has no poles for generic (χ,ξ) by corollary 2.5.3. Now in our proposition what we fix is not the restrictions, but the φ_1 and φ_2. So we need a little more discussion. Since φ_1 is smooth, there is a partition $\mathrm{K_G}=\cup\mathrm{K}_i$ of $\mathrm{K_G}$ independent of χ such that $\mathrm{F}_\chi(\varphi_1)(k)$ is constant in each K_i. On the other hand, since φ_1 is compactly supported, for each $k_i\in\mathrm{K}_i$, the integral $\mathrm{F}_\chi(\varphi_1)(k_i)$ is a linear combination of values of φ_1 with coefficient in $\mathbb{C}[q^{\pm\chi}]$. So $\mathrm{F}_\chi(\varphi_1)\mid_{\mathrm{K_G}}$ varies polynomially on $q^{\pm\chi}$. For a similar reason $\mathrm{F}_{\xi,\psi}(\varphi_2)$ varies polynomially on $q^{\pm\xi}$. So for given φ_1 and φ_2, the pairing $l_{\chi,\xi,\psi}(\mathrm{F}_\chi(\varphi_1),\mathrm{F}_{\xi,\psi}(\varphi_2))$ is also rational in (q^{χ},q^{ξ}), and has no poles for generic (χ,ξ). $\square$

From now on, we use $l_{\chi,\xi,\psi}$ to denote both the original pairing given by (2.11) in the domain of convergence, and the meromorphic continuation of this pairing to other values of (χ,ξ) which are generic. We treat $\mathrm{I}_{\chi,\xi,\psi}(g)$ and $\mathrm{I}^0_{\chi,\xi,\psi}(g)$ similarly.

2.6 Double coset decomposition

Let $\mathrm{W}_{\chi,\xi,\psi}(g)$ be a Whittaker-Shintani function. In this section we discuss the support of $\mathrm{W}_{\chi,\xi,\psi}(g)$ on the double cosets $\mathrm{ZUK_{M^J}}\backslash\mathrm{G}/\mathrm{K_G}$. We write $g_1\sim g_2$ when g_1 and g_2 belong to the same double coset.

Theorem 2.6.1. *The support of* $\mathrm{W}_{\chi,\xi,\psi}(g)$ *is contained in the double cosets*

$$\bigcup_{\mathbf{d}\in\Lambda_m^+,\mathbf{f}\in\Lambda_n^+}\mathrm{ZUK_{M^J}}(\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\mathbf{f}})\mathrm{K_G}.$$

First, by the Iwasawa decomposition on G and the Cartan decomposition on M, we have

$$\mathrm{G} = (\mathrm{UJ} \rtimes \mathrm{T}_1^{n-m} \times \mathrm{M})\mathrm{K}_{\mathrm{G}} = \mathrm{ZUK}_{\mathrm{M}^{\mathrm{J}}}(\mathrm{XYT}_1^{n-m}\mathrm{T}_{\mathrm{M}}^{+})\mathrm{K}_{\mathrm{G}}.$$

Here T_1^{n-m} is the embedding of GL_1^{n-m} to G as

$$\mathrm{T}_1^{n-m}(t_1, ..., t_{n-m}) = d_n(t_1, ..., t_{n-m}, 1, 1, \dots, 1).$$

So we only need to consider the support of $\mathrm{W}_{\chi,\xi,\psi}$ on the set $\mathrm{XY}\mathfrak{p}^{(\mathbf{a};\mathbf{b})}$ where $\mathbf{a} \in \mathbb{Z}^{n-m}$ and $\mathbf{b} \in \Lambda_m^+$. We have

Lemma 2.6.2. *Suppose* $\mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) \neq 0$. *Then* $y \in \mathrm{Y}^0$ *and* $\mathbf{a} \in \Lambda_{n-m}^+$

Proof. The proof is similar in lemma 2.1 in [24]. First for any $z \in \mathrm{Z}^0$, we have

$$\mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) = \mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}z) = \psi(\mathfrak{p}^{2a_{n-m}}z)\mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}).$$

So $a_{n-m} \geq 0$. To show $y \in \mathrm{Y}^0$, we argue by contradiction. Let $y = \mathrm{Y}(y_1, ..., y_m)$ with $|y_j| = q^r$ for some $1 \leq j \leq m$ and $r > 0$. Then for any $t \in \mathcal{O}^*$,

$$\begin{aligned} &\mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) = \mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}n_{-2e'_j}(\mathfrak{p}^{2b_j+r}t)) \\ =&\mathrm{W}_{\chi,\xi,\psi}(xyn_{-2e'_j}(\mathfrak{p}^r t))\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) = \psi(\mathfrak{p}^r t y_j^2)\mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) \end{aligned}$$

But note that $\mathfrak{p}^r t y_j^2 \in \mathfrak{p}^{-r}\mathcal{O}^*$, one can choose $t \in \mathcal{O}^*$ so that $\psi(\mathfrak{p}^r t y_j^2) \neq 1$, which is a contradiction. So $y \in \mathrm{Y}^0$.

To show $\mathbf{a} \in \Lambda_{n-m}^+$, we also argue by contradiction. Assume that $a_i < a_{i+1}$ for some $i \leq n-m-1$. Note that when $i \leq n-m-1$, $N_{e_i - e_{i+1}} \subset \mathrm{U}$. So for any $t \in \mathcal{O}$,

$$\begin{aligned} &\mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) = \mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}n_{e_i-e_{i+1}}(t)) \\ =&\psi(\mathfrak{p}^{a_i-a_{i+1}}t)\mathrm{W}_{\chi,\xi,\psi}(xy\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) \end{aligned}$$

When $a_i < a_{i+1}$, we can always find some $t \in \mathcal{O}$ such that $\psi(\mathfrak{p}^{a_i - a_{i+1}} t) \neq 1$, contradicting the assumption. So $a_i \geq a_{i+1}$ for every i. Note that we already show $a_{n-m} \geq 0$. So $\mathbf{a} \in \Lambda^+_{n-m}$.

$\square$

By this lemma the support of $\mathrm{W}_{\chi,\xi,\psi}$ is on $\mathrm{ZUK}_{\mathrm{M}^{\mathrm{J}}}(\mathrm{X}\mathrm{T}^+_{n-m}\mathrm{T}^+_m)\mathrm{K}_{\mathrm{G}}$. Next we narrow the support further on the X-part. Note that if $v(x_i) = c_i$ for $1 \leq i \leq m$, then $\mathrm{X}((x_1, \dots, x_m))t \sim \lambda(\mathbf{c})t$ for any $t \in \mathrm{T}_{\mathrm{G}}$. So it suffices to determine the set of $\mathbf{c} \in \mathbb{Z}^m$ such that $\lambda(\mathbf{c})\mathrm{T}^+_{n-m}\mathrm{T}^+_m$ is contained in the support.

Lemma 2.6.3. *Suppose $x = \lambda(\mathbf{c})$, then for any $t \in \mathrm{T}_n$,*

$$x\,t \sim \lambda(min(\mathbf{c}, \mathbf{0}))t$$

Proof. Note that $\lambda \in \mathrm{K}_{\mathrm{M}^{\mathrm{J}}}$, so $x\,t \sim \lambda x\,t \in \mathrm{T}^0_{\mathrm{M}}\lambda(\min(\mathbf{c}, \mathbf{0}))t\mathrm{T}^0_{\mathrm{M}}$. $\square$

By this lemma, we just need to consider the support of $\mathrm{W}_{\chi,\xi,\psi}$ on $x\mathfrak{p}^{(\mathbf{a};\mathbf{b})}$ with $x = \lambda(-\mathbf{d})$ for some $\mathbf{d} \geq \mathbf{0}$.

Lemma 2.6.4. *Let $x = \lambda(-\mathbf{d})$ with $\mathbf{d} \geq \mathbf{0}$. Let $\mathbf{a} \in \Lambda^+_{n-m}$ and $\mathbf{b} \in \Lambda^+_m$. Suppose $\mathrm{W}_{\chi,\xi,\psi}(x\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) \neq 0$, then $\mathbf{d} \leq \mathbf{b}$.*

Proof. Suppose not, so we let $d_j > b_j$ for some j. For any $t \in \mathcal{O}^*$,

$$\begin{aligned}
&\mathrm{W}_{\chi,\xi,\psi}(x\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) = \mathrm{W}_{\chi,\xi,\psi}(x\mathfrak{p}^{(\mathbf{a},\mathbf{b})}n_{2e'_j}(t)) = \mathrm{W}_{\chi,\xi,\psi}(xn_{2e'_j}(\mathfrak{p}^{2b_j}t)\mathfrak{p}^{(\mathbf{a},\mathbf{b})})\\
=&\psi(\mathfrak{p}^{2b_j-2d_j}t)\mathrm{W}_{\chi,\xi,\psi}(Y(0,\dots,0,\underbrace{\mathfrak{p}^{2b_j-d_j}}_{j-th},0,\dots,0)x\mathfrak{p}^{(\mathbf{a},\mathbf{b})})
\end{aligned}$$

By lemma 2.6.2, when $\mathrm{W}_{\chi,\xi,\psi}(x\mathfrak{p}^{(\mathbf{a},\mathbf{b})}) \neq 0$, we have $\mathfrak{p}^{2b_j-d_j} \in \mathcal{O}$, so the last formula equals $\psi(\mathfrak{p}^{2b_j-2d_j}t)\mathrm{W}_{\chi,\xi,\psi}(x\mathfrak{p}^{(\mathbf{a},\mathbf{b})})$. But when $d_j > b_j$ we can choose some $t \in \mathcal{O}^*$ such that $\psi(\mathfrak{p}^{2b_j-2d_j}t) \neq 1$, contradicting the assumption. $\square$

Note that when $x = \lambda(-\mathbf{d})$ with $\mathbf{d} \geq \mathbf{0}$, $x\mathfrak{p}^{(\mathbf{a},\mathbf{b})} = \mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{(\mathbf{a},\mathbf{b}-\mathbf{d})}$. Let $\mathbf{r} = \mathbf{b} - \mathbf{d}$. Then combining lemma 2.6.22.6.3 and 2.6.4, the support of $\mathrm{W}_{\chi,\xi,\psi}$ is on the union of $\mathrm{ZUK}_{\mathrm{M^J}}(\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{(\mathbf{a},\mathbf{r})})\mathrm{K_G}$ for all $\mathbf{a} \in \Lambda^+_{n-m}$, $\mathbf{d} \geq \mathbf{0}$, $\mathbf{r} \geq \mathbf{0}$ and $\mathbf{d}+\mathbf{r} \in \mathbf{\Lambda}^+_{\mathbf{m}}$. The following two lemmas help us to narrow our choice of $\mathbf{a}, \mathbf{b}, \mathbf{d}$ so that we get theorem 2.6.1. We also need to use them in the later calculations for the Whittaker-Shintani function.

Lemma 2.6.5. *Suppose* $g = \mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{(\mathbf{a},\mathbf{r})}$ *with* $\mathbf{a} \in \Lambda^+_{n-m}$, $\mathbf{d} \geq \mathbf{0}$, $\mathbf{r} \geq \mathbf{0}$, *and* $\mathbf{d}+\mathbf{r} \in \Lambda^+_m$.

1. *Suppose* $a_{n-m} < r_1$. *Let* $\bar{\mathbf{r}} = (a_{n-m}, r_2, \ldots, r_m)$. *Then*

$$\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{(\mathbf{a},\mathbf{r})} \sim \mathfrak{p}^{\mathbf{d}+\mathbf{r}-\bar{\mathbf{r}}}\lambda\mathfrak{p}^{(\mathbf{a},\bar{\mathbf{r}})}.$$

   Moreover, if $\mathbf{d} \in \Lambda^+_m$, *then so is* $\mathbf{d}+\mathbf{r}-\bar{\mathbf{r}}$. *We call the process from* $(\mathbf{d};\mathbf{a},\mathbf{r})$ *to* $(\mathbf{d}+\mathbf{r}-\bar{\mathbf{r}};\mathbf{a},\bar{\mathbf{r}})$ ***Operation 1***.

2. *Fix* i, *let* $\tilde{\mathbf{r}} = (\tilde{r}_1, ..., \tilde{r}_m)$ *where*

$$\tilde{r}_j = \begin{cases} r_i & \textit{if } j > i \textit{ and } r_j > r_i \\ r_j & \textit{otherwise,} \end{cases}$$

   then

$$\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{(\mathbf{a},\mathbf{r})} \sim \mathfrak{p}^{\mathbf{d}+\mathbf{r}-\tilde{\mathbf{r}}}\lambda\mathfrak{p}^{(\mathbf{a},\tilde{\mathbf{r}})}.$$

   Moreover, if $a_{n-m} \geq r_1$, *then* $a_{n-m} \geq \tilde{r}_1$; *if* $\mathbf{d} \in \Lambda^+_m$, *then so is* $\mathbf{d}+\mathbf{r}-\tilde{\mathbf{r}}$. *For given* i, *we call the process from* $(\mathbf{d};\mathbf{a},\mathbf{r})$ *to* $(\mathbf{d}+\mathbf{r}-\tilde{\mathbf{r}};\mathbf{a},\tilde{\mathbf{r}})$ ***Operation (2,i)***.

3. *Given* i, *let* $\tilde{\mathbf{d}} = (\tilde{d}_1, ..., \tilde{d}_m)$ *where*

$$\tilde{d}_j = \begin{cases} d_i & \textit{if } j < i \textit{ and } d_j < d_i \\ d_j & \textit{otherwise,} \end{cases}$$

   then

$$\mathrm{W}_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{(\mathbf{a},\mathbf{r})}) = \mathrm{W}_{\chi,\xi,\psi}(\mathfrak{p}^{\tilde{\mathbf{d}}}\lambda\mathfrak{p}^{(\mathbf{a},\mathbf{r}+\mathbf{d}-\tilde{\mathbf{d}})}).$$

Moreover, if $(\mathbf{a}, \mathbf{r}) \in \Lambda_n^+$*, then so is* $(\mathbf{a}, \mathbf{r} + \mathbf{d} - \tilde{\mathbf{d}})$*. For given* i*, we call the process from* $(\mathbf{d}; \mathbf{a}, \mathbf{r})$ *to* $(\tilde{\mathbf{d}}; \mathbf{a}, \mathbf{r} + \mathbf{d} - \tilde{\mathbf{d}})$ ***Operation (3,i)***

Proof. For part 1, note that when $a_{n-m} < r_1$,

$$\begin{aligned} &\mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{(\mathbf{a},\mathbf{r})} \sim \mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{(\mathbf{a},\mathbf{r})} \lambda \\ \sim &\mathfrak{p}^{\mathbf{d}} \lambda((a_{n-m} - r_1, 0, \ldots, 0)) \mathfrak{p}^{(\mathbf{a},\mathbf{r})} \sim \mathfrak{p}^{\mathbf{d}+\mathbf{r}-\bar{\mathbf{r}}} \lambda \mathfrak{p}^{(\mathbf{a},\bar{\mathbf{r}})}, \end{aligned}$$

so we have part 1. For part 2, we fix i, then we have

$$\begin{aligned} &\mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{(\mathbf{a},\mathbf{r})} \sim \mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{(\mathbf{a},\mathbf{r})} \cdot \prod_{j \,|j>i, r_j>r_i} n_{e_i'-e_j'}(1) \\ \sim &\prod_{j\,|j>i, r_j>r_i} n_{e_i'-e_j'}(\mathfrak{p}^{\mathbf{d_i}+\mathbf{r_i}-\mathbf{d_j}-\mathbf{r_j}}) \mathfrak{p}^{\mathbf{d}} \lambda(\tilde{\mathbf{r}} - \mathbf{r}) \mathfrak{p}^{(\mathbf{a},\mathbf{r})} \sim \mathfrak{p}^{\mathbf{d}+\mathbf{r}-\tilde{\mathbf{r}}} \lambda \mathfrak{p}^{(\mathbf{a},\tilde{\mathbf{r}})} \end{aligned}$$

the rest of part 2 is correct since $\mathbf{d} + \mathbf{r} \in \Lambda_m^+$. Part 3 is similar to part 2. For fix i, we have

$$\begin{aligned} &\mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{(\mathbf{a},\mathbf{r})} \sim \prod_{j|j<i, d_j<d_i} n_{e_i'-e_j'}(1) \mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{(\mathbf{a},\mathbf{r})} \\ \sim &\mathfrak{p}^{\mathbf{d}} \lambda(\mathbf{d} - \tilde{\mathbf{d}}) \mathfrak{p}^{(\mathbf{a},\mathbf{r})} \prod_{j|j<i, d_j<d_i} n_{e_i'-e_j'}(\mathfrak{p}^{\mathbf{d_j}+\mathbf{r_j}-\mathbf{d_i}-\mathbf{r_i}}) \sim \mathfrak{p}^{\tilde{\mathbf{d}}} \lambda \mathfrak{p}^{(\mathbf{a},\mathbf{r}+\mathbf{d}-\tilde{\mathbf{d}})}. \end{aligned}$$

The rest of part 3 is correct since $\mathbf{d} + \mathbf{r} \in \Lambda_m^+$ and $\tilde{r}_1 \leq r_1$. □

Propostion 2.6.6. *Suppose* $\mathbf{a} \in \Lambda_{n-m}^+$*,* $\mathbf{d} \geq \mathbf{0}$*,* $\mathbf{r} \geq \mathbf{0}$ *and* $\mathbf{d} + \mathbf{r} \in \Lambda_m^+$*. Starting from* $(\mathbf{d}; \mathbf{a}, \mathbf{r})$*, if we take the* ***Operation 1****, and then* ***Operation (2,i)****, for i from* 1 *to m, and then* ***Operation (3,i)****, for i from m to* 1*, we will end the process with* $(\overline{\mathbf{d}}; \mathbf{a}, \overline{\mathbf{r}})$ *such that*

1. $\overline{\mathbf{d}} + \overline{\mathbf{r}} = \mathbf{d} + \mathbf{r}$.
2. $\overline{\mathbf{d}} \geq \mathbf{d}$*, so equivalently,* $\overline{\mathbf{r}} \leq \mathbf{r}$.
3. $\overline{\mathbf{d}} \in \Lambda_m^+$*, and* $(\mathbf{a}, \overline{\mathbf{r}}) \in \Lambda_n^+$.
4. *For any triple* $(\mathbf{d}'; \mathbf{a}, \mathbf{r}')$ *satisfying (1),(2) and (3) above,* $\overline{\mathbf{d}} \leq \mathbf{d}'$*.*

In other words, for $(\mathbf{d};\mathbf{a},\mathbf{r})$ such that $\mathbf{a}\in\Lambda^+_{n-m}$, $\mathbf{d}\geq\mathbf{0}$, $\mathbf{r}\geq\mathbf{0}$ and $\mathbf{d}+\mathbf{r}\in\Lambda^+_m$, there exists $(\overline{\mathbf{d}};\mathbf{a},\overline{\mathbf{r}})$ satisfying (1),(2) and (3) above such that $\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{(\mathbf{a},\mathbf{r})}\sim\mathfrak{p}^{\overline{\mathbf{d}}}\lambda\mathfrak{p}^{(\mathbf{a},\overline{\mathbf{r}})}$. Moreover, among all the triples satisfying (1), (2) and (3), $(\overline{\mathbf{d}};\mathbf{a},\overline{\mathbf{r}})$ is the one with the smallest $\overline{\mathbf{d}}$.

Proof. Part (1) is obvious. Part (2) is true because in all the operations, either $\mathbf{d}$ increases or $\mathbf{r}$ decreases, and by (1) they are equivalent. Part (3) is true because by operation 1, $a_{n-m}\geq\overline{r}_1$, and after operation (2,i), $\overline{r}_i\geq\overline{r}_j$ for all $j>i$, and after operation (3,i), $\overline{d}_i\leq\overline{d}_j$ for all $j<i$.

To prove part (4), we let $(\mathbf{d}';\mathbf{a},\mathbf{r}')$ satisfy (1),(2) and (3). So $\mathbf{d}\leq\mathbf{d}'$ by (1). We will show below that suppose $(\mathbf{D},\mathbf{a},\mathbf{R})$ satisfies $\mathbf{D}+\mathbf{R}=\mathbf{d}'+\mathbf{r}'$ and $\mathbf{D}\leq\mathbf{d}'$, and $(\mathbf{D}',\mathbf{a},\mathbf{R}')$ is the result of it by one of the operations in lemma 2.6.6, then $\mathbf{D}'+\mathbf{R}'=\mathbf{d}'+\mathbf{r}'$ and $\mathbf{D}'\leq\mathbf{d}'$. This implies that $\overline{\mathbf{d}}\leq\mathbf{d}'$ because $(\overline{\mathbf{d}},\mathbf{a},\overline{\mathbf{r}})$ is the result of $2m+1$ operations on $(\mathbf{d},\mathbf{a},\mathbf{r})$.

It is clear that $\mathbf{D}'+\mathbf{R}'=\mathbf{D}+\mathbf{R}$ by the definitions of the operations, so $\mathbf{D}'+\mathbf{R}'=\mathbf{d}'+\mathbf{r}'$. To prove $\mathbf{D}'\leq\mathbf{d}'$, first suppose the operation we took is (3,i), then for any j, either $\mathbf{D}'_\mathbf{j}=\mathbf{D}_\mathbf{j}\leq\mathbf{d}'_\mathbf{j}$, or $\mathbf{D}'_\mathbf{j}=\mathbf{D}_\mathbf{i}$, in which case $j<i$, implying $\mathbf{D}'_\mathbf{j}=\mathbf{D}_\mathbf{i}\leq\mathbf{d}'_\mathbf{i}\leq\mathbf{d}'_\mathbf{j}$. So we always have $\mathbf{D}'_\mathbf{j}\leq\mathbf{d}'_\mathbf{j}$ and hence $\mathbf{D}'\leq\mathbf{d}'$. If the operation we took is (1) or (2,i), we can similarly prove that $\mathbf{R}'\geq\mathbf{r}'$. Since $\mathbf{D}'+\mathbf{R}'=\mathbf{d}'+\mathbf{r}'$, so $\mathbf{D}'\leq\mathbf{d}'$. So in any case we have $\mathbf{D}'\leq\mathbf{d}'$, completing the proof. $\square$

Following lemma 2.6.5 and 2.6.6, theorem 2.6.1 is implied.

2.7 Vectors invariant under certain open compact subgroups

2.7.1 The $\bar{\mathrm{I}}_{\mathrm{M}}$-Invariant vectors in $\mathrm{Ind}_{\mathrm{B}_{\mathrm{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$.

Consider $\mathrm{Ind}_{\mathrm{B}_{\mathrm{M^J}}}^{\mathrm{M^J}}(\xi,\psi)^{\bar{\mathrm{I}}_{\mathrm{M}}}$.

Lemma 2.7.1. *Any* $f \in \mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{\mathrm{M}^{\mathrm{J}}}(\xi, \psi)^{\bar{\mathrm{I}}_{\mathrm{M}}}$ *is supported on* $\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{W}_{\mathrm{M}}\bar{\mathrm{I}}_{\mathrm{M}}$.

Proof. Since

$$\mathrm{M} = \mathrm{B}_{\mathrm{M}}\mathrm{W}_{\mathrm{M}}\mathrm{I}_{\mathrm{M}},$$

we have

$$\mathrm{M}^{\mathrm{J}} = \mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{X}\mathrm{W}_{\mathrm{M}}\mathrm{I}_{\mathrm{M}}.$$

Suppose f $\in \mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{\mathrm{M}^{\mathrm{J}}}(\xi, \psi)^{\bar{\mathrm{I}}_{\mathrm{M}}}$ with $\mathrm{f}(\mathrm{X}(x)w) \neq 0$, then since W_{M} normalizes J^0, we have

$$f(\mathrm{X}(x)w) = f(\mathrm{X}(x)\mathrm{Y}(y)w)$$

for any $y \in \mathcal{O}^m$. But

$$\begin{aligned} &f(\mathrm{X}(x)\mathrm{Y}(y)w) \\ =&f(\mathrm{Y}(y)\mathrm{X}(x)\mathrm{Z}(2\langle x, y\rangle)w) \\ =&\psi(2\langle x, y\rangle)f(\mathrm{X}(x)w). \end{aligned}$$

So $\psi(2\langle x, y\rangle) = 1$ for any $y \in \mathcal{O}^m$, which means $x \in \mathcal{O}^m$, completing our proof since $\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{X}^0\mathrm{W}_{\mathrm{M}}\mathrm{I}_{\mathrm{M}} = \mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{W}_{\mathrm{M}}\bar{\mathrm{I}}_{\mathrm{M}}$. $\square$

By this lemma, we have

$$\dim \mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{\mathrm{M}_{\mathrm{J}}}(\xi, \psi)^{\bar{\mathrm{I}}_{\mathrm{M}}} \leq \mathrm{Card}(\mathrm{W}_{\mathrm{M}}).$$

For any character ξ on T_{M}, let $\mathrm{T}_w^{\xi,\psi}$ be the intertwining operator from $\mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{\mathrm{M}_{\mathrm{J}}}(\xi, \psi)$ to $\mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{\mathrm{M}_{\mathrm{J}}}(w\xi, \psi)$ defined as

$$\mathrm{T}_w^{\xi,\psi}(f)(g) = \int_{\mathrm{N}_{\mathrm{M}^{\mathrm{J}}} \cap w\mathrm{N}_{\mathrm{M}^{\mathrm{J}}}w^{-1} \backslash \mathrm{N}_{\mathrm{M}^{\mathrm{J}}}} f(w^{-1}ng)\,\mathrm{d}n. \tag{2.24}$$

It is equal to

$$\int_{\mathrm{N}_{\mathrm{M}} \cap w\mathrm{N}_{\mathrm{M}}w^{-1} \backslash \mathrm{N}_{\mathrm{M}}} \int_{\mathrm{Y} \cap w\mathrm{Y}w^{-1} \backslash \mathrm{Y}} f(w^{-1}yng)\,\mathrm{d}y\,\mathrm{d}n. \tag{2.25}$$

One can identify $\mathrm{Y} \cap w\mathrm{Y}w^{-1}\backslash\mathrm{Y}$ with $\mathrm{Y} \cap w\mathrm{X}w^{-1}$. So (2.25) is equal to

$$\int_{\mathrm{N_M} \cap w\mathrm{N_M}w^{-1}\backslash\mathrm{N_M}} \int_{\mathrm{X} \cap w^{-1}\mathrm{Y}w} f(xw^{-1}ng)\,\mathrm{d}x\,\mathrm{d}n.$$

For fixed $n \in \mathrm{N_M}$ and $g \in \mathrm{G}$, the function $x \mapsto f(xw^{-1}ng)$ is of compact support on X as discussed in remark 2.2.1. So similar to the usual intertwining operators on $\mathrm{Ind}_{\mathrm{B_M}}^{\mathrm{M}}(\xi)$, the integral (2.24) is convergent when $Re(\xi)$ is sufficiently large, and continues holomorphically to generic ξ. For generic ξ, the $\mathrm{M^J}$-intertwining operator from $Ind_{\mathrm{B_{M^J}}}^{\mathrm{M_J}}(\xi, \psi)$ to $Ind_{\mathrm{B_{M^J}}}^{\mathrm{M_J}}(w\xi, \psi)$ is unique, so we have

$$\mathrm{T}_{w_1}^{w_2\xi,\psi} \circ \mathrm{T}_{w_2}^{\xi,\psi} = c \cdot \mathrm{T}_{w_1w_2}^{\xi,\psi}$$

for some constant c. To normalize $\mathrm{T}_w^{\xi,\psi}$'s, we need to find $\mathrm{T}_w^{\xi,\psi}(\mathrm{F}_{\xi,\psi})(e)$. First we have the following lemma.

Lemma 2.7.2. *Suppose* $|x| = q^{-i}$*, then*

$$\int_{|t|=q^{-j}} \psi(t^{-1}x)dt = \begin{cases} q^{-j}(1-q^{-1}) & \text{if } j \leq i \\ -q^{-(i+2)} & \text{if } j = i+1 \\ 0 & \text{if } j \geq i+2 \end{cases}$$

Proof. One can prove this lemma by direct calculation. □

Lemma 2.7.3. *For* α *being a simple root in* M*, let*

$$\tilde{c}_\alpha(\xi) = \frac{\zeta(\langle \xi, \alpha \rangle)}{\zeta(\langle \xi, \alpha \rangle + 1)}.$$

(Note that the pairing is with α *itself, not* $\check{\alpha}$*. The pairing is not canonical since both* ξ *and* α *are characters of* $\mathrm{T_M}$*. But we can identify them as elements in* $\mathbb{Z}^m$*, and define the pairing simply as the standard dot product.) Let*

$$\tilde{c}_w(\xi) = \prod_{\alpha>0, w\alpha<0} \tilde{c}_\alpha(\xi),$$

then,

$$\mathrm{T}_w^{\xi,\psi}(\mathrm{F}^0_{\xi,\psi})(\mathrm{e}) = \tilde{c}_w(\xi). \tag{2.26}$$

Proof. By a similar method as in step 3 of [6, Theorem 3.1], we only need to prove (2.26) for simple reflections $w = w_\alpha$ where α is a simple root in M. When $\alpha = e'_i - e'_{i+1}$ for $i = 1, \dots, m-1$, $\mathrm{X} \cap w_\alpha^{-1}\mathrm{Y}w_\alpha$ is trivial, so (2.26) follows from considering the corresponding simple reflections in M, (note that $\alpha = \check{\alpha}$ in this case). When $\alpha = 2e'_m$, $\mathrm{X} \cap w_\alpha^{-1}\mathrm{Y}w_\alpha = \{\mathrm{X}(0, \dots, 0, x) \mid x \in F\}$, and $\mathrm{N_M} \cap w_\alpha \mathrm{N_M} w_\alpha^{-1} \backslash \mathrm{N_M}$ can be identified with N_α. Let $\mathrm{X}_m(x) = \mathrm{X}(0, \dots, 0, x)$. We have, for $\alpha = 2e'_m$,

$$\mathrm{T}_{w_\alpha}^{\xi,\psi}(\mathrm{F}^0_{\xi,\psi})(\mathrm{e}) = \int_{F^2} \mathrm{d}x\, \mathrm{d}y \mathrm{F}^0_{\xi,\psi}(\mathrm{X}_m(x) w_\alpha^{-1} n_\alpha(y)). \tag{2.27}$$

When $|y| \leq 1$, $\mathrm{F}^0_{\xi,\psi}(\mathrm{X}_m(x) w_\alpha^{-1} n_\alpha(y)) = \mathrm{F}^0_{\xi,\psi}(\mathrm{X}_m(x))$, which equals $\mathrm{Ch}_{\mathcal{O}}(x)$ by remark 2.2.1. So

$$\int_{|y|\leq 1} \mathrm{d}y \int_F \mathrm{d}x \mathrm{F}^0_{\xi,\psi}(\mathrm{X}_m(x) w_\alpha^{-1} n_\alpha(y)) = 1. \tag{2.28}$$

When $|y| > 1$, $w_\alpha^{-1} n_\alpha(y) = n_\alpha(-y^{-1}) \mathrm{T}_\alpha(y^{-1}) n_{-\alpha}(y^{-1})$. So

$$\begin{aligned}
\mathrm{X}_m(x) w_\alpha^{-1} n_\alpha(y) &= \mathrm{X}_m(x) n_\alpha(-y^{-1}) \mathrm{T}_\alpha(y^{-1}) n_{-\alpha}(y^{-1}) \\
&= n_\alpha(-y^{-1}) \mathrm{T}(y^{-1}) \mathrm{J}((0, \dots, 0, xy^{-1}), (-x, 0, \dots, 0), 0) n_{-\alpha}(y^{-1}) \\
&= n_\alpha(-y^{-1}) \mathrm{T}(y^{-1}) \mathrm{Y}(-x, 0, \dots, 0) Z(x^2 y^{-1}) \mathrm{X}_m(xy^{-1}) n_{-\alpha}(y^{-1}).
\end{aligned}$$

So

$$\mathrm{F}^0_{\xi,\psi}(\mathrm{X}_m(x) w_\alpha^{-1} n_\alpha(y)) = |y|^{-(\xi_m + \frac{3}{2})} \psi(x^2 y^{-1}) \mathrm{Ch}_{\mathcal{O}}(xy^{-1}).$$

For $i, j \in \mathbb{Z}$, let

$$S_{i,j} = q^{j(\xi_m + \frac{3}{2})} \cdot \int_{|x| = q^{-i}} \mathrm{d}x \int_{|y| = q^{-j}} \mathrm{d}y \psi(x^2 y^{-1}).$$

Then

$$\int_{|y|>1} \mathrm{d}y \int_F \mathrm{d}x \ \mathrm{F}^0_{\xi,\psi}(\mathrm{X}_m(x) w_\alpha^{-1} n_\alpha(y))$$
$$= \sum_{i=-\infty}^{\infty} \sum_{j\leq -1, j\leq i} S_{i,j}. \tag{2.29}$$

For fixed x, consider the integral on y and apply lemma 2.7.2, we have $S_{i,j} = 0$ when $j \geq 2i+2$. So (2.29) is equal to

$$\sum_{i=-\infty}^{\infty} \sum_{j\leq min\{-1,i,2i+1\}} S_{i,j}. \tag{2.30}$$

Note that $-1 < i < 2i+1$ when $i \geq 0$, and $2i+1 \leq i \leq -1$ when $i \leq -1$, so (2.30) equals

$$\sum_{i\geq 0} \sum_{j\leq -1} S_{i,j} + \sum_{i\leq -1} \sum_{j\leq 2i+1} S_{i,j}. \tag{2.31}$$

Applying lemma 2.7.2,

$$\begin{aligned} \sum_{i\geq 0} \sum_{j\leq -1} S_{i,j} &= \sum_{i\geq 0} \sum_{j\leq -1} q^{j(\xi_m+\frac{1}{2})} q^{-i} (1-q^{-1})^2 \\ &= \frac{(1-q^{-1}) \cdot q^{-(\xi_m+\frac{1}{2})}}{1-q^{-(\xi_m+\frac{1}{2})}}, \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} \sum_{i\leq -1} \sum_{j\leq 2i+1} S_{i,j} &= \sum_{i\leq -1} \sum_{j\leq 2i} S_{i,j} + \sum_{i\leq -1} S_{i,2i+1} \\ &= \sum_{i\leq -1} \sum_{j\leq 2i} q^{j(\xi_m+\frac{1}{2})} q^{-i}(1-q^{-1})^2 - \sum_{i\leq -1} q^{2i\xi_m} q^{\xi_m-\frac{1}{2}}(1-q^{-1}) \\ &= \frac{(1-q^{-1})^2 \cdot q^{-2\xi_m}}{(1-q^{-(\xi_m+\frac{1}{2})})(1-q^{-2\xi_m})} - \frac{(1-q^{-1}) \cdot q^{-(\xi_m+\frac{1}{2})}}{1-q^{-2\xi_m}}. \end{aligned} \tag{2.33}$$

Combining (2.27), (2.28), (2.32) and (2.33), we have

$$\mathrm{T}_{w_\alpha}^{\xi,\psi}(\mathrm{F}_{\xi,\psi}^0)(\mathrm{e}) = \frac{1-q^{-(2\xi_m+1)}}{1-q^{-2\xi_m}} = \frac{\zeta(\langle \xi_m, \alpha\rangle)}{\zeta(\langle \xi_m, \alpha\rangle + 1)} \tag{2.34}$$

for $\alpha = 2e'_m$, completing the proof of the lemma. □

So if we define

$$\overline{\mathrm{T}}_w^{\xi,\psi} = (\tilde{c}_w(\xi))^{-1}\mathrm{T}_w^{\xi,\psi},$$

then

$$\overline{\mathrm{T}}_{w_1}^{w_2\xi,\psi} \circ \overline{\mathrm{T}}_{w_2}^{\xi,\psi} = \overline{\mathrm{T}}_{w_1w_2}^{\xi,\psi}$$

Now let

$$\Psi_1^{\xi,\psi} = \mathrm{F}_{\xi,\psi}(\mathrm{Ch}_{\bar{\mathrm{I}}_\mathrm{M}}),$$

then we have

$$\Psi_1^{\xi,\psi}(g) = \begin{cases} \xi\psi\delta_{\mathrm{B}_{\mathrm{M}^\mathrm{J}}}^{\frac{1}{2}}(b) & \text{if } g = bn^-x,\ b \in \mathrm{B}_{\mathrm{M}^\mathrm{J}},\ n^- \in \mathrm{N}_\mathrm{M}^{-,1},\ x \in \mathrm{X}^0. \\ 0 & \text{if } g \notin \mathrm{B}_{\mathrm{M}^\mathrm{J}}\mathrm{N}_\mathrm{M}^{-,1}\mathrm{X}^0 \end{cases}$$

Lemma 2.7.4. *The set* $\{\overline{\mathrm{T}}_{w^{-1}}\Psi_1^{w\xi,\psi}\}_{w\in W_\mathrm{M}}$ *forms a basis of* $\mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^\mathrm{J}}}^{\mathrm{M}_\mathrm{J}}(\xi,\psi)^{\bar{\mathrm{I}}_\mathrm{M}}$

Proof. Since $\dim \mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^\mathrm{J}}}^{\mathrm{M}_\mathrm{J}}(\xi,\psi)^{\bar{\mathrm{I}}_\mathrm{M}} \leq \mathrm{Card}(\mathrm{W}_\mathrm{M})$, it suffices to prove that the elements in $\{\overline{\mathrm{T}}_{w^{-1}}\Psi_1^{w\xi,\psi}\}_{w\in W_\mathrm{M}}$ are linearly independent. Consider

$$\overline{\mathrm{T}}_w(\Psi_1^{\xi,\psi})(w_0^\mathrm{M}) = (\tilde{c}_w(\xi))^{-1}\int_{\mathrm{N}_{\mathrm{M}^\mathrm{J}}\cap w\mathrm{N}_{\mathrm{M}^\mathrm{J}}w^{-1}\backslash\mathrm{N}_{\mathrm{M}^\mathrm{J}}} \mathrm{d}n^\mathrm{J}\Psi_1^{\xi,\psi}(w^{-1}n^\mathrm{J}w_0^\mathrm{M})$$

By the definition of $\Psi_1^{\xi,\psi}$, if the integral is non-zero then

$$w^{-1}\mathrm{N}_{\mathrm{M}^\mathrm{J}}w_0^\mathrm{M} \cap \mathrm{B}_{\mathrm{M}^\mathrm{J}}\mathrm{N}_{1,\mathrm{M}}^{-}\mathrm{X}^0 \neq \emptyset,$$

which implies

$$w^{-1}\mathrm{N}_{\mathrm{M}^{\mathrm{J}}} \cap \mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{N}^{-}_{1,\mathrm{M}}\mathrm{X}^{0}(w_0^{\mathrm{M}})^{-1} \neq \emptyset.$$

Note that $w^{-1}\mathrm{N}_{\mathrm{M}^{\mathrm{J}}} \subseteq \mathrm{B}_{\mathrm{M}}w^{-1}\mathrm{N}_{\mathrm{M}} \ltimes \mathrm{J}$, and that $\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{N}^{-}_{1,\mathrm{M}}\mathrm{X}^{0}(w_0^{\mathrm{M}})^{-1} \subseteq \mathrm{B}_{\mathrm{M}}(w_0^{\mathrm{M}})^{-1}\mathrm{N}_{\mathrm{M}} \ltimes \mathrm{J}$. By considering the Bruhat-decomposition on M we have

$$w = w_0^{\mathrm{M}}$$

if $\overline{\mathrm{T}}_w(\Psi_1^{\xi,\psi})(w_0^{\mathrm{M}}) \neq 0$. On the other hand, when $w = w_0^{\mathrm{M}}$,

$$\overline{\mathrm{T}}_{w_0^{\mathrm{M}}}(\Psi_1^{\xi,\psi})(w_0^{\mathrm{M}}) = (\tilde{c}_{w_0^{\mathrm{M}}}(\xi))^{-1}\int_{\mathrm{N}^{-}\mathrm{X}} \mathrm{d}n\,\mathrm{d}x\,\Psi_1^{\xi,\psi}(nx)$$

which equals

$$(\tilde{c}_{w_0^{\mathrm{M}}}(\xi))^{-1}\mu_{\mathrm{N}^{-}_{\mathrm{M}}}(\mathrm{N}^{-,1}_{\mathrm{M}})$$

by the definition of $\Psi_1^{\xi,\psi}$. Let $\overline{\mathrm{T}}_{w_0^{\mathrm{M}}w}$ act on all elements in $\{\overline{\mathrm{T}}_{w^{-1}}\Psi_1^{w\xi,\psi}\}_{w\in W_{\mathrm{M}}}$ and evaluate them at w_0^{M}. Only $\overline{\mathrm{T}}_{w_0^{\mathrm{M}}w} \circ \overline{\mathrm{T}}_{w^{-1}}\Psi_1^{w\xi,\psi}$ is non-zero. So they are linearly independent, and hence form a basis of $\mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{\mathrm{M}^{\mathrm{J}}}(\xi,\psi)^{\overline{I}_{\mathrm{M}}}$. □

So then we have

Lemma 2.7.5. *We can write* $\mathrm{F}^0_{\xi,\psi}$ *as a linear combination of* $\{\overline{\mathrm{T}}_{w^{-1}}\Psi_1^{w\xi,\psi}\}_{w\in W_{\mathrm{M}}}$ *as*

$$\mathrm{F}^0_{\xi,\psi} = \frac{1}{\mu_{\mathrm{N}^{-}_{\mathrm{M}}}(\mathrm{N}^{-,1}_{\mathrm{M}})}\sum_{w\in \mathrm{W}_{\mathrm{M}}} \tilde{c}_{w_0^{\mathrm{M}}}(w\xi)\overline{\mathrm{T}}_{w^{-1}}\Psi_1^{w\xi,\psi}. \tag{2.35}$$

Proof. Since $\{\overline{\mathrm{T}}_{w^{-1}}\Psi_1^{w\xi,\psi}\}_{w\in W_{\mathrm{M}}}$ is the basis of $\mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}^{\mathrm{M}^{\mathrm{J}}}(\xi,\psi)^{\overline{I}_{\mathrm{M}}}$, we assume

$$\mathrm{F}^0_{\xi,\psi} = \sum_{w\in \mathrm{W}_{\mathrm{M}}} b_w\overline{\mathrm{T}}_{w^{-1}}\Psi_1^{w\xi,\psi}.$$

with $b_w \in \mathbb{C}$. Let $\overline{\mathrm{T}}_{w_0^{\mathrm{M}}w}$ act on both sides of the equation and take the value at w_0^{M}, we have

$$1 = b_w(\tilde{c}_{w_0^{\mathrm{M}}}(w\xi))^{-1}\mu_{\mathrm{N}^{-}_{\mathrm{M}}}(\mathrm{N}^{-,1}_{\mathrm{M}})$$

completing our proof. □

Next we consider the action $R(\mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}}a\bar{\mathrm{I}}_{\mathrm{M}}})$ on $\mathrm{F}^0_{\xi,\psi}$ for $a \in \mathrm{T}^-_{\mathrm{M}}$. We have

Propostion 2.7.6.

$$R(\mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}}a\bar{\mathrm{I}}_{\mathrm{M}}})\mathrm{F}^0_{\xi,\psi} = \frac{\mu_{\mathrm{M}^{\mathrm{J}}}(\bar{\mathrm{I}}_{\mathrm{M}}a\bar{\mathrm{I}}_{\mathrm{M}})}{\mu_{\mathrm{N}^-_{\mathrm{M}}}(\mathrm{N}^{-,1}_{\mathrm{M}})} \sum_{w \in W_{\mathrm{M}}} \tilde{c}_{w_0^{\mathrm{M}}}(w\xi) \cdot (w\xi)\delta^{-\frac{1}{2}}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}(a) \cdot \overline{\mathrm{T}}_{w^{-1}} \Psi_1^{w\xi,\psi}$$

Proof. First calculate $R(\mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}}a\bar{\mathrm{I}}_{\mathrm{M}}})\Psi_1^{\xi,\psi}$. Note that it belongs to $\mathrm{Ind}^{\mathrm{M}^{\mathrm{J}}}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}}(\xi,\psi)^{\bar{\mathrm{I}}_{\mathrm{M}}}$, so by lemma 2.7.1, we only need to consider its value on W_{M}. Note that when $a \in \mathrm{T}^-_{\mathrm{M}}$, we have the decomposition

$$\bar{\mathrm{I}}_{\mathrm{M}}a\bar{\mathrm{I}}_{\mathrm{M}} = \mathrm{N}^{-,1}_{\mathrm{M}}\mathrm{X}^0 a\mathrm{B}^0_{\mathrm{M}^{\mathrm{J}}}.$$

Since $\Psi_1^{\xi,\psi}$ is $\bar{\mathrm{I}}_{\mathrm{M}}$-invariant,

$$R(\mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}}a\bar{\mathrm{I}}_{\mathrm{M}}})\Psi_1^{\xi,\psi}(w) = \frac{\mu_{\mathrm{M}^{\mathrm{J}}}(\bar{\mathrm{I}}_{\mathrm{M}}a\bar{\mathrm{I}}_{\mathrm{M}})}{\mu_{\mathrm{N}^-_{\mathrm{M}}}(\mathrm{N}^{-,1}_{\mathrm{M}})} \int_{N^{-,1}_{\mathrm{M}}} \int_{\mathrm{X}^0} \Psi_1^{\xi,\psi}(wnxa)\,\mathrm{d}x\,\mathrm{d}n$$

Suppose it is non-zero. Then by considering the support of $\Psi_1^{\xi,\psi}$,

$$w\mathrm{N}^-_{1,\mathrm{M}}a\mathrm{X} \cap \mathrm{B}_{\mathrm{M}^{\mathrm{J}}}\mathrm{N}^{-,1}_{\mathrm{M}}\mathrm{X}^0 \neq \emptyset$$

so

$$w\mathrm{N}^-_{1,\mathrm{M}}a\mathrm{X}w_0^{\mathrm{M}} \cap \mathrm{B}_{\mathrm{M}^{\mathrm{J}}}w_0^{\mathrm{M}}\mathrm{N}^1_{\mathrm{M}}\mathrm{Y}^0 \neq \emptyset$$

Note that

$$w\mathrm{N}^-_{1,\mathrm{M}}a\mathrm{X}w_0^{\mathrm{M}} \subseteq \mathrm{B}_{\mathrm{M}}ww_0^{\mathrm{M}}\mathrm{N}_{\mathrm{M}} \ltimes \mathrm{J}$$

and

$$\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}w_0^{\mathrm{M}}\mathrm{N}^1_{\mathrm{M}}\mathrm{Y}^0 \subseteq \mathrm{B}_{\mathrm{M}}w_0^{\mathrm{M}}\mathrm{N}_{\mathrm{M}} \ltimes \mathrm{J}.$$

So by Bruhat decomposition of M we have

$$w = \mathrm{e}.$$

This implies that $R(\mathrm{Ch}_{\bar{\mathrm{I}}_\mathrm{M} a \bar{\mathrm{I}}_\mathrm{M}})\Psi_1^{\xi,\psi}$ is proportional to $\Psi_1^{\xi,\psi}$. Consider $R(\mathrm{Ch}_{\bar{\mathrm{I}}_\mathrm{M} a \bar{\mathrm{I}}_\mathrm{M}})\Psi_1^{\xi,\psi}(\mathrm{e})$, which is equal to

$$\frac{\mu_{\mathrm{M}^\mathrm{J}}(\bar{\mathrm{I}}_\mathrm{M} a \bar{\mathrm{I}}_\mathrm{M})}{\mu_{\mathrm{N}_\mathrm{M}^-}(\mathrm{N}_\mathrm{M}^{-,1})}\int_{N_\mathrm{M}^{-,1}\mathrm{X}^0} \mathrm{d}n\, \mathrm{d}x \Psi_1^{\xi,\psi}(nxa).$$

Note that $\Psi_1^{\xi,\psi} \in \mathrm{Ind}_{\mathrm{B}_{\mathrm{M}^\mathrm{J}}}^{\mathrm{M}^\mathrm{J}}(\xi,\psi)$, the formula above is equal to

$$\frac{\mu_{\mathrm{M}^\mathrm{J}}(\bar{\mathrm{I}}_\mathrm{M} a \bar{\mathrm{I}}_\mathrm{M})}{\mu_{\mathrm{N}_\mathrm{M}^-}(\mathrm{N}_\mathrm{M}^{-,1})}\xi\delta_{\mathrm{B}_{\mathrm{M}^\mathrm{J}}}^{\frac{1}{2}}(a)\int_{\mathrm{N}_\mathrm{M}^{-,1}\mathrm{X}^0} \mathrm{d}n\, \mathrm{d}x\, \Psi_1^{\xi,\psi}(a^{-1}nxa).$$

Considering the support of $\Psi_1^{\xi,\psi}$,

$$\xi\delta_{\mathrm{B}_{\mathrm{M}^\mathrm{J}}}^{\frac{1}{2}}(a)\int_{\mathrm{N}_\mathrm{M}^{-,1}\mathrm{X}^0} \mathrm{d}n\, \Psi_1^{\xi,\psi}(a^{-1}nxa) = \xi\delta_{\mathrm{B}_{\mathrm{M}^\mathrm{J}}}^{\frac{1}{2}}(a)\mu_{\mathrm{N}_\mathrm{M}^-\mathrm{X}}(\mathrm{N}_\mathrm{M}^{-,1}\mathrm{X}^0 \cap a\mathrm{N}_\mathrm{M}^{-,1}\mathrm{X}^0 a^{-1})$$

When $a \in \mathrm{T}_\mathrm{M}^-$, $a\mathrm{N}_{1,\mathrm{M}}^-\mathrm{X}^0 a^{-1} \subset \mathrm{N}_\mathrm{M}^{-,1}\mathrm{X}^0$, so

$$\begin{aligned}
&\mu_{\mathrm{N}_\mathrm{M}^-\mathrm{X}}(\mathrm{N}_\mathrm{M}^{-,1}\mathrm{X}^0 \cap a\mathrm{N}_{1,\mathrm{M}}^-\mathrm{X}^0 a^{-1})\\
=&\mu_{\mathrm{N}_\mathrm{M}^-\mathrm{X}}(a\mathrm{N}_{1,\mathrm{M}}^-\mathrm{X}^0 a^{-1})\\
=&\delta_{\mathrm{B}_{\mathrm{M}^\mathrm{J}}}^{-1}(a)\mu_{\mathrm{N}_\mathrm{M}^-}(\mathrm{N}_\mathrm{M}^{-,1}).
\end{aligned}$$

So we have

$$R(\mathrm{Ch}_{\bar{\mathrm{I}}_\mathrm{M} a \bar{\mathrm{I}}_\mathrm{M}})\Psi_1^{\xi,\psi} = \mu_{\mathrm{M}^\mathrm{J}}(\bar{\mathrm{I}}_\mathrm{M} a \bar{\mathrm{I}}_\mathrm{M})\xi\delta_{\mathrm{B}_{\mathrm{M}^\mathrm{J}}}^{-\frac{1}{2}}(a)\Psi_1^{\xi,\psi}.$$

Applying this to both sides of equation (2.35), our proposition is proved. □

2.7.2 The I_G-invariant vectors in $\mathrm{Ind}_{\mathrm{B}_\mathrm{G}}^{\mathrm{G}}(\chi)$

For a Weyl element w and a character χ on the T_G, let T_w^χ be an intertwining operator from $\mathrm{Ind}_{\mathrm{B}_\mathrm{G}}^{\mathrm{G}}(\chi)$ to $\mathrm{Ind}_{\mathrm{B}_\mathrm{G}}^{\mathrm{G}}(w\chi)$ defined by

$$\mathrm{T}_w^\chi \mathrm{f}(g) = \int_{\mathrm{N_G} \cap w\mathrm{N_G}w^{-1} \backslash \mathrm{N_G}} \mathrm{d}n\ \mathrm{f}(w^{-1}ng).$$

The integral is convergent when $Re(\chi)$ is sufficiently large, and by [7] it continues holomorphically to generic χ. We write T_w^χ as T_w when there is no risk of confusion. For generic χ, the G-intertwining operator from $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)$ to $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(w\chi)$ is unique for every $w \in \mathrm{W_G}$. So

$$\mathrm{T}_{w_1}^{w_2\chi} \circ \mathrm{T}_{w_2}^{\chi} = c \cdot \mathrm{T}_{w_1 w_2}^{\chi}$$

for some constant c. Moreover,if we let

$$c_\alpha(\chi) = \frac{\zeta(\langle \chi, \check{\alpha} \rangle)}{\zeta(\langle \chi, \check{\alpha} \rangle + 1)}$$
$$c_w(\chi) = \prod_{\alpha > 0, w\alpha < 0} c_\alpha(\chi),$$

then by theorem 3.1 in [6]

$$\mathrm{T}_w^\chi \mathrm{F}_\chi^0 = c_w(\chi) \mathrm{F}_{w\chi}^0.$$

So if we let

$$\overline{\mathrm{T}}_w^\chi = c_w(\chi)^{-1} \mathrm{T}_w^\chi,$$

then

$$\overline{\mathrm{T}}_{w_1}^{w_2\chi} \overline{\mathrm{T}}_{w_2}^{\chi} = \overline{\mathrm{T}}_{w_1 w_2}^{\chi}.$$

Let $\Phi_1^\chi = \mathrm{F}_\chi(Ch_{\mathrm{I_G}})$. Then Φ_1^χ is $\mathrm{I_G}$-invariant, and $\Phi_1^\chi(e) = 1$. The next proposition encapsulates the analogs of lemma 2.7.4 and 2.7.5, as well as proposition 2.7.6. The proof is similar to the proofs of these results, so some details are omitted.

Propostion 2.7.7. *Elements in* $\{\overline{\mathrm{T}}_{w^{-1}} \Phi_1^{w\chi}\}_{w \in W_{\mathrm{G}}}$ *form a basis of* $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)^{\mathrm{I_G}}$*, and we have*

$$\mathrm{F}_\chi^0 = \frac{1}{\mu_{\mathrm{N_G^-}}(\mathrm{N_G^{-,1}})} \sum_{w \in W_{\mathrm{G}}} c_{w_0^{\mathrm{G}}}(w\chi) \overline{\mathrm{T}}_{w^{-1}} \Phi_1^{w\chi}. \tag{2.36}$$

For $a \in \mathrm{T}_G^-$*,*

$$R(\mathrm{Ch}_{I_G a I_G})\Phi_1^\chi = \mu_{\mathrm{G}}(\mathrm{I_G} a \mathrm{I_G})\chi\delta_{\mathrm{B_G}}^{-\frac{1}{2}}(a)\Phi_1^\chi. \tag{2.37}$$

As a consequence,

$$R(\mathrm{Ch}_{\mathrm{I_G} a \mathrm{I_G}})\mathrm{F}_\chi^0 = \frac{\mu_{\mathrm{G}}(\mathrm{I_G} a \mathrm{I_G})}{\mu_{\mathrm{N_G^-}}(\mathrm{N_G^{-,1}})} \sum_{w \in \mathrm{W_G}} c_{w_0^{\mathrm{G}}}(w\chi) \cdot (w\chi)\delta_{\mathrm{B_G}}^{-\frac{1}{2}}(a) \cdot \overline{\mathrm{T}}_{w^{-1}}\Phi_1^{w\chi}. \tag{2.38}$$

Proof. Consider $\overline{\mathrm{T}}_w\Phi_1^\chi(w_0^{\mathrm{G}})$. By definition it is equal to

$$c_w^{-1}(\chi)\int_{\mathrm{N_G} \cap w\mathrm{N_G}w^{-1}\backslash \mathrm{N_G}} \mathrm{d}n\ \Phi_1^\chi(w^{-1}nw_0^{\mathrm{G}}). \tag{2.39}$$

Note that Φ_1^χ is supported on $\mathrm{B_G I_G}$. If the formula above is nonzero, we have $w^{-1}\mathrm{N_G}w_0^{\mathrm{G}} \cap \mathrm{B_G I_G} \neq \emptyset$, so $w^{-1}\mathrm{N_G} \cap \mathrm{B_G I_G} w_0^{\mathrm{G}} \neq \emptyset$. But note that $w^{-1}\mathrm{N_G} \subset \mathrm{B_G}w^{-1}\mathrm{N_G}$ and $\mathrm{B_G I_G} w_0^{\mathrm{G}} = \mathrm{B_G N_G^{-,1}} w_0^{\mathrm{G}} \subset \mathrm{B_G} w_0^{\mathrm{G}}\mathrm{N_G}$. So by considering the Bruhat decomposition on G, we have $w^{-1} = w_0^{\mathrm{G}}$. On the other hand when $w^{-1} = w_0^{\mathrm{G}}$,

$$\overline{\mathrm{T}}_w\Phi_1^\chi(w_0^{\mathrm{G}}) = c_{w_0^{\mathrm{G}}}^{-1}(\chi)\int_{\mathrm{N_G^-}} \mathrm{d}n\ \Phi_1^\chi(n),$$

which is equal to $c_{w_0^{\mathrm{G}}}^{-1}(\chi)\mu_{\mathrm{N_G^-}}(\mathrm{N_G^{-,1}})$.

From this, if we apply $\overline{\mathrm{T}}_{w'}$ to all elements in $\{\overline{\mathrm{T}}_{w^{-1}}\Phi_1^{w\chi}\}_{w\in W_{\mathrm{G}}}$ and take the value at w_0^{G}, only the one with $w = w_0^{\mathrm{G}}w'$ is non-zero. So the elements of $\{\overline{\mathrm{T}}_{w^{-1}}\Phi_1^{w\chi}\}_{w\in W_{\mathrm{G}}}$ are linearly independent. Note that the dimension of $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi)^{\mathrm{I_G}}$ is at most $Card(\mathrm{W_G})$, so $\{\overline{\mathrm{T}}_{w^{-1}}\Phi_1^{w\chi}\}_{w\in W_{\mathrm{G}}}$ forms a basis. Write

$$\mathrm{F}_\chi^0 = \sum_{w\in \mathrm{W_G}} a_w \overline{\mathrm{T}}_{w^{-1}}\Phi_1^{w\chi}.$$

Apply $\overline{\mathrm{T}}_{w'}$ for all $w' \in \mathrm{W_G}$ and take the value at w_0^{G}, we obtain (2.36) by comparing the values on both sides.

The calculation for (3.9) is similar to that in proposition 2.7.6. We skip the details here. It implies (2.38) since intertwining operators commute with right translations. □

2.8 γ-factor

In this section we assume (χ, ξ) is generic, and denote $\mathcal{K}_{\chi,\xi,\psi}$ simply by $\mathcal{K}$. We are going to show that

Theorem 2.8.1. *Let $\chi = (\chi_1, ..., \chi_n)$ and $\xi = (\xi_1, ..., \xi_m)$. Let $\Gamma(\chi, \xi)$ be a function on (χ, ξ) given by*

$$\begin{aligned}\Gamma(\chi, \xi) = &\prod_{1\le a<b\le n} \zeta^{-1}(\chi_a - \chi_b + 1)\zeta^{-1}(\chi_a + \chi_b + 1) \cdot \prod_{i=1}^{n} \zeta^{-1}(\chi_i + 1)\\ &\cdot \prod_{1\le a<b\le n} \zeta^{-1}(\xi_a - \xi_b + 1)\zeta^{-1}(\xi_a + \xi_b + 1) \cdot \prod_{j=1}^{m}(1 + \xi_j(\mathfrak{p})q^{-\frac{1}{2}})\\ &\cdot \prod_{j=1}^{m} \prod_{i=1}^{(n-m)+j-1} \frac{\zeta(\chi_i - \xi_j + \frac{1}{2})}{\zeta(-\chi_i + \xi_j + \frac{1}{2})} \cdot \prod_{i=1}^{n}\prod_{j=1}^{m} \zeta(\chi_i + \xi_j + \frac{1}{2})\zeta(-\chi_i + \xi_j + \frac{1}{2})\end{aligned}$$

Then for any fixed $g \in \mathrm{G}$,

$$\frac{l_{\chi,\xi,\psi}(R(g)\mathrm{F}^0_{\chi}, \mathrm{F}^0_{\xi,\psi})}{\Gamma(\chi, \xi)}$$

is $\mathrm{W_G} \times \mathrm{W_M}$-invariant.

If we can prove that for $(w, w') = (w_\alpha, 1)$ and $(w, w') = (1, w_\beta)$ where α is a simple root in G and β is a simple root in M,

$$\frac{\Gamma(w\chi, w'\xi)}{\Gamma(\chi, \xi)} = \frac{l_{w\chi,w'\xi,\psi}(R(g)\mathrm{F}^0_{w\chi}, \mathrm{F}^0_{w'\xi,\psi})}{l_{\chi,\xi,\psi}(R(g)\mathrm{F}^0_{\chi}, \mathrm{F}^0_{\xi,\psi})}$$

then theorem 2.8.1 is implied. Since we can calculate the left hand side above directly, we only need to consider the ratio

$$\frac{l_{w\chi,w'\xi,\psi}(R(g)\mathrm{F}^0_{w\chi}, \mathrm{F}^0_{w'\xi,\psi})}{l_{\chi,\xi,\psi}(R(g)\mathrm{F}^0_{\chi}, \mathrm{F}^0_{\xi,\psi})}. \tag{2.40}$$

We obtain its value by the uniqueness of the pairing $l_{\chi,\xi,\psi}$. To be precise, for

$(w, w') \in \mathrm{W_G} \times \mathrm{W_M}$, let

$$\tilde{l}_{\chi,\xi,w,w'}(\mathrm{F}_\chi, \mathrm{F}_{\xi,\psi}) = l_{w\chi,w'\xi,\psi}(\mathrm{T}_w^\chi \mathrm{F}_\chi, \mathrm{T}_{w'}^{\xi,\psi} \mathrm{F}_{\xi,\psi}).$$

Then $\tilde{l}_{\chi,\xi,w,w'}$ is also a pairing on $\mathrm{Ind}_{\mathrm{B_G}}^{\mathrm{G}}(\chi) \otimes \mathrm{Ind}_{\mathrm{B_{M^J}}}^{\mathrm{M^J}}(\xi,\psi)$ satisfying Condition A. Since such pairing is unique up to scalar, there exists a constant, which we denote by $\gamma(\chi,\xi,w,w')$, such that

$$\tilde{l}_{\chi,\xi,w,w'} = \gamma(\chi,\xi,w,w') l_{\chi,\xi,\psi}.$$

Then, since $\mathrm{T}_w^\chi \mathrm{F}_\chi^0 = c_w(\chi)\mathrm{F}_{w\chi}^0$ and $\mathrm{T}_{w'}^{\xi,\psi}\mathrm{F}_{\xi,\psi}^0 = \tilde{c}_{w'}(\xi)\mathrm{F}_{w'\xi,\psi}^0$, we have

$$\begin{aligned}
&\frac{l_{w\chi,w'\xi,\psi}(R(g)\mathrm{F}_{w\chi}^0, \mathrm{F}_{w'\xi,\psi}^0)}{l_{\chi,\xi,\psi}(R(g)\mathrm{F}_\chi^0, \mathrm{F}_{\xi,\psi}^0)} \\
=&c_w^{-1}(\chi)\tilde{c}_{w'}^{-1}(\xi) \cdot \frac{\tilde{l}_{\chi,\xi,w,w'}(R(g)\mathrm{F}_\chi^0, \mathrm{F}_{\xi,\psi}^0)}{l_{\chi,\xi,\psi}(R(g)\mathrm{F}_\chi^0, \mathrm{F}_{\xi,\psi}^0)} \\
=&c_w^{-1}(\chi)\tilde{c}_{w'}^{-1}(\xi)\gamma(\chi,\xi,w,w').
\end{aligned}$$

In the rest of this section we calculate $\gamma(\chi,\xi,w_\alpha,1)$ and $\gamma(\chi,\xi,1,w_\beta)$. The result will be stated in theorem 2.8.4 and 2.8.8, which implies the theorem 2.8.1.

2.8.1 The calculation of $\gamma(\chi,\xi,w_\alpha,1)$

First we state a lemma which is useful for our calculation.

Lemma 2.8.2. *Let α and β be simple roots in* G *and* M *respectively. For all choices of ± 1's on both sides, we have*

$$\mathrm{I_G}(w_0^{\mathrm{G}})^{\pm 1}(\lambda^{\pm 1})\mathrm{I_{M^J}}\mathrm{U}^0 = \mathrm{B_G^0}(w_0^{\mathrm{G}})^{\pm 1}(\lambda^{\pm 1})\mathrm{B_{M^J}^0}\mathrm{U}^0, \tag{2.41}$$

$$\mathrm{I_G} w_\alpha \mathrm{I_G}(w_0^{\mathrm{G}})^{\pm 1}(\lambda^{\pm 1})\mathrm{I_{M^J}}\mathrm{U}^0 = \mathrm{B_G^0} w_\alpha \mathrm{N}_\alpha^0 (w_0^{\mathrm{G}})^{\pm 1}(\lambda^{\pm 1})\mathrm{B_{M^J}^0}\mathrm{U}^0, \tag{2.42}$$

and

$$\mathrm{I_G}(w_0^{\mathrm{G}})^{\pm1}(\lambda^{\pm1})\bar{\mathrm{I}}_{\mathrm{M}}w_\beta\bar{\mathrm{I}}_{\mathrm{M}}\mathrm{U}^0 = \mathrm{B}^0_{\mathrm{G}}(w_0^{\mathrm{G}})^{\pm1}\mathrm{N}^0_\beta w_\beta \mathrm{X}^0\mathrm{B}^0_{\mathrm{M^J}}\mathrm{U}^0. \tag{2.43}$$

Proof. Since $\lambda^{-1} \in \mathrm{T}^0_{\mathrm{M}}\lambda\mathrm{T}^0_{\mathrm{M}}$, and $(w_0^{\mathrm{G}})^{-1} = -Iw_0^{\mathrm{G}}$, it suffices to prove these 3 equations when all ±1's are 1's. First we prove (2.41).

step1. Since $\mathrm{I_G} = \mathrm{B}^0_{\mathrm{G}}\mathrm{N}^{-,1}_{\mathrm{G}}$ and $\mathrm{I_{M^J}} = \mathrm{X}^1\mathrm{N}^{-,1}_{\mathrm{M}}\mathrm{B}^0_{\mathrm{M^J}}$, it suffices to prove that

$$\mathrm{N}^{-,1}_{\mathrm{G}}w_0^{\mathrm{G}}\lambda\mathrm{X}^1\mathrm{N}^{-,1}_{\mathrm{M}} \subset \mathrm{B}^0_{\mathrm{G}}w_0^{\mathrm{G}}\lambda\mathrm{B}^0_{\mathrm{M^J}}\mathrm{U}^0.$$

step2. Note that $\lambda\mathrm{X}^1 \subset \mathrm{T}^0_{\mathrm{M}}\lambda\mathrm{T}^0_{\mathrm{M}}$. So we only need to prove

$$\mathrm{N}^{-,1}_{\mathrm{G}}w_0^{\mathrm{G}}\lambda\mathrm{N}^{-,1}_{\mathrm{M}} \subset \mathrm{B}^0_{\mathrm{G}}w_0^{\mathrm{G}}\lambda\mathrm{B}^0_{\mathrm{M^J}}\mathrm{U}^0.$$

step3. For $n \in \mathrm{N}^{-,1}_{\mathrm{M}}$, $n\lambda n^{-1} \in \mathrm{T}^0_{\mathrm{M}}\lambda\mathrm{T}^0_{\mathrm{M}}$, and note that $w_0^{\mathrm{G}}\mathrm{N}^{-,1}_{\mathrm{M}}(w_0^{\mathrm{G}})^{-1} = \mathrm{N}^1_{\mathrm{M}}$, so it suffices to prove that

$$\mathrm{N}^{-,1}_{\mathrm{G}}\mathrm{N}^1_{\mathrm{M}}w_0^{\mathrm{G}}\lambda \subset \mathrm{B}^0_{\mathrm{G}}w_0^{\mathrm{G}}\lambda\mathrm{B}^0_{\mathrm{M^J}}\mathrm{U}^0.$$

step4. Note that $\mathrm{N}^{-,1}_{\mathrm{G}}\mathrm{N}^1_{\mathrm{M}} \subset \mathrm{I_G} = \mathrm{B}^0_{\mathrm{G}}\mathrm{N}^{-,1}_{\mathrm{G}}$, and that $(w_0^{\mathrm{G}})^{-1}\mathrm{N}^{-,1}_{\mathrm{G}}w_0^{\mathrm{G}} = \mathrm{N}^1_{\mathrm{G}}$. So we only need to prove that

$$w_0^{\mathrm{G}}\mathrm{N}^1_{\mathrm{G}}\lambda \subset \mathrm{B}^0_{\mathrm{G}}w_0^{\mathrm{G}}\lambda\mathrm{B}^0_{\mathrm{M^J}}\mathrm{U}^0.$$

step5. Since $\mathrm{N}^1_{\mathrm{G}} = \mathrm{U}^1\mathrm{J}^1\mathrm{N}^1_{\mathrm{M}}$, and for $n \in \mathrm{N}^1_{\mathrm{M}}$, $n\lambda n^{-1} \in \mathrm{J}^1\lambda$, so it suffices to prove that

$$w_0^{\mathrm{G}}\mathrm{U}^1\mathrm{J}^1\lambda \subset \mathrm{B}^0_{\mathrm{G}}w_0^{\mathrm{G}}\lambda\mathrm{B}^0_{\mathrm{M^J}}\mathrm{U}^0. \tag{2.44}$$

step6. Note that λ normalizes U^1 and J^1. So $w_0^{\mathrm{G}}\mathrm{U}^1\mathrm{J}^1\lambda = w_0^{\mathrm{G}}\lambda\mathrm{J}^1\mathrm{U}^1 = w_0^{\mathrm{G}}\lambda\mathrm{X}^1\mathrm{Y}^1\mathrm{Z}^1\mathrm{U}^1$. But $\lambda\mathrm{X}^1 \subset \mathrm{T}^0_{\mathrm{M}}\lambda\mathrm{T}^0_{\mathrm{M}}$, so (2.44) is correct, completing the proof of (2.41).

Now we prove (2.42). First we have

$$\mathrm{I_G} w_\alpha \mathrm{I_G} w_0^{\mathrm{G}} \lambda \mathrm{I_{M^J}} \mathrm{U}^0 = \mathrm{B_G^0} w_\alpha \mathrm{I_G} w_0^{\mathrm{G}} \lambda \mathrm{I_{M^J}} \mathrm{U}^0.$$

Then by (2.41), the right hand side is equal to

$$\mathrm{B_G^0} w_\alpha \mathrm{B_G^0} w_0^{\mathrm{G}} \lambda \mathrm{B_{M^J}^0} \mathrm{U}^0,$$

which is equal to

$$\mathrm{B_G^0} w_\alpha \mathrm{N}_\alpha^0 w_0^{\mathrm{G}} \lambda \mathrm{B_{M^J}^0} \mathrm{U}^0,$$

which is the right hand side of (2.42).

To prove (2.43), note that $\bar{\mathrm{I}}_{\mathrm{M}} w_\beta \bar{\mathrm{I}}_{\mathrm{M}} = \mathrm{I_{M^J}} w_\beta \mathrm{X}^0 \mathrm{B_{M^J}^0}$, so

$$\mathrm{I_G} w_0^{\mathrm{G}} \lambda \bar{\mathrm{I}}_{\mathrm{M}} w_\beta \bar{\mathrm{I}}_{\mathrm{M}} \mathrm{U}^0 = \mathrm{I_G} w_0^{\mathrm{G}} \lambda \mathrm{I_{M^J}} w_\beta \mathrm{X}^0 \mathrm{B_{M^J}^0} \mathrm{U}^0.$$

By (2.41), and the fact that $w_\beta \mathrm{X}^0 \mathrm{B_{M^J}^0}$ normalizes U^0, the right hand side is equal to

$$\mathrm{B_G^0} w_0^{\mathrm{G}} \lambda \mathrm{B_{M^J}^0} w_\beta \mathrm{X}^0 \mathrm{B_{M^J}^0} \mathrm{U}^0,$$

which is equal to

$$\mathrm{B_G^0} w_0^{\mathrm{G}} \mathrm{N}_\beta^0 w_\beta \mathrm{X}^0 \mathrm{B_{M^J}^0} \mathrm{U}^0,$$

completing the proof of (2.43). $\square$

Let $\mathrm{F}_{\xi,\psi}^1 = \mathrm{F}_{\xi,\psi}(\mathrm{Ch}_{\mathrm{I_{M^J}}})$. For $w \in \mathrm{W_G}$, let $\Phi_{\chi,w} = \mathrm{F}_\chi(\mathrm{Ch}_{\mathrm{I_G} w \mathrm{I_G}})$. Then by theorem 3.4 in [6]

$$\mathrm{T}_{w_\alpha}(\Phi_{\chi,1} + \Phi_{\chi,w_\alpha}) = c_\alpha(\chi)(\Phi_{w_\alpha\chi,1} + \Phi_{w_\alpha\chi,w_\alpha}).$$

So we have

$$\gamma(\chi,\xi,w_\alpha,1) = c_\alpha(\chi) \cdot \frac{l_{w_\alpha\chi,\xi,\psi}(R(\lambda w_0^{\mathrm{G}}) \cdot (\Phi_{w_\alpha\chi,1} + \Phi_{w_\alpha\chi,w_\alpha}), \mathrm{F}_{\xi,\psi}^1)}{l_{\chi,\xi,\psi}(R(\lambda w_0^{\mathrm{G}}) \cdot (\Phi_{\chi,1} + \Phi_{\chi,w_\alpha}), \mathrm{F}_{\xi,\psi}^1)}. \tag{2.45}$$

Let $i' = i - (n-m)$. The result of the calculation is stated as the proposition below:

Propostion 2.8.3. *For generic* (χ, ξ), $l_{\chi,\xi,\psi}(R(\lambda w_0^{\mathrm{G}}) \cdot (\Phi_{\chi,1} + \Phi_{\chi,w_\alpha}), \mathrm{F}^1_{\xi,\psi})$ *equals*

$$\mu_{\mathrm{G}}(\mathrm{I}_{\mathrm{G}})\mu_{\mathrm{M}^{\mathrm{J}}}(\mathrm{I}_{\mathrm{M}^{\mathrm{J}}})(q - (\chi_i\chi_{i+1}^{-1})(\mathfrak{p}))$$

for $\alpha = e_i - e_{i+1}, 1 \le i \le n-m-1$, *and equals*

$$\mu_{\mathrm{G}}(\mathrm{I}_{\mathrm{G}})\mu_{\mathrm{M}^{\mathrm{J}}}(\mathrm{I}_{\mathrm{M}^{\mathrm{J}}})(q-1)\frac{1 - q^{-1}(\chi_i\chi_{i+1}^{-1})(\mathfrak{p})}{(1 - (\chi_i\xi_{i'+1}^{-1})(\mathfrak{p})q^{-\frac{1}{2}})(1 - (\chi_{i+1}^{-1}\xi_{i'+1})(\mathfrak{p})q^{-\frac{1}{2}})}$$

for $\alpha = e_i - e_{i+1}, n-m \le i \le n-1$, *and equals*

$$\mu_{\mathrm{G}}(\mathrm{I}_{\mathrm{G}})\mu_{\mathrm{M}^{\mathrm{J}}}(\mathrm{I}_{\mathrm{M}^{\mathrm{J}}})(q - \chi_n(\mathfrak{p}))$$

for $\alpha = 2e_n$.

Substituting this in (2.45), we have

Theorem 2.8.4. *For generic* (χ, ξ), *the* γ*-factor* $\gamma(\chi, \xi, w_\alpha, 1)$ *is equal to*

$$c_\alpha(\chi) \cdot \frac{\zeta(\chi_i - \chi_{i+1} + 1)}{\zeta(\chi_{i+1} - \chi_i + 1)}$$

for $\alpha = e_i - e_{i+1}, 1 \le i \le n-m-1$, *and*

$$c_\alpha(\chi) \cdot \frac{\zeta(\chi_i - \chi_{i+1} + 1)}{\zeta(\chi_{i+1} - \chi_i + 1)} \frac{\zeta(\chi_{i+1} - \xi_{i'+1} + \frac{1}{2})\zeta(-\chi_i + \xi_{i'+1} + \frac{1}{2})}{\zeta(\chi_i - \xi_{i'+1} + \frac{1}{2})\zeta(-\chi_{i+1} + \xi_{i'+1} + \frac{1}{2})}$$

for $\alpha = e_i - e_{i+1}, n-m \le i \le n-1$, *and*

$$c_\alpha(\chi) \cdot \frac{\zeta(\chi_n + 1)}{\zeta(-\chi_n + 1)}$$

for $\alpha = 2e_n$.

Recall that

$$l_{\chi,\xi,\psi}(R(g)\mathrm{F}_{\chi}(\varphi_1), \mathrm{F}_{\xi,\psi}(\varphi_2)) = \mathrm{I}_{\chi,\xi,\psi}(\varphi_1, \varphi_2)(g).$$

First we calculate $\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}}, \mathrm{Ch}_{\mathrm{I_{M^J}}})(\lambda w_0^{\mathrm{G}})$.

Lemma 2.8.5. *For $(\chi, \xi) \in \mathcal{Z}_c$, we have*

$$\mathrm{I}_{\chi,\xi}(\mathrm{Ch}_{\mathrm{I_G}}, \mathrm{Ch}_{\mathrm{I_{M^J}}})(\lambda w_0^{\mathrm{G}}) = \mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\mathrm{I_{M^J}}). \tag{2.46}$$

Proof. By definition,

$$\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}}, \mathrm{Ch}_{\mathrm{I_{M^J}}})(\lambda w_0^{\mathrm{G}}) = \int_{\mathrm{I_G}\times \mathrm{I_{M^J}}} \mathcal{K}(x(w_0^{\mathrm{G}})^{-1}\lambda^{-1}x')dxdx'$$

By lemma 2.8.2, $\mathcal{K}(x(w_0^{\mathrm{G}})^{-1}\lambda^{-1}x') = \mathcal{K}(w_0^{\mathrm{G}}\lambda) = 1$ for $x \in \mathrm{I_G}$ and $x' \in \mathrm{I_{M^J}}$, so our lemma is implied. □

Now we calculate $\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}w_\alpha\mathrm{I_G}}, \mathrm{Ch}_{\mathrm{I_{M^J}}})(\lambda w_0^{\mathrm{G}})$. By definition it is equal to

$$\int_{\mathrm{I_G}w_\alpha\mathrm{I_G}} \mathrm{d}x \int_{\mathrm{I_{M^J}}} \mathrm{d}x'\mathcal{K}(x(w_0^{\mathrm{G}})^{-1}\lambda^{-1}x').$$

By lemma 2.8.2 it is equal to

$$\mu_{\mathrm{G}}(\mathrm{I_G}w_\alpha\mathrm{I_G})\mu_{\mathrm{M^J}}(\mathrm{I_{M^J}}) \cdot \int_{N_\alpha^0} dn_\alpha\mathcal{K}(w_\alpha n_\alpha w_0^{\mathrm{G}}\lambda).$$

It is not hard to see that

$$\mu_{\mathrm{G}}(\mathrm{I_G}w_\alpha\mathrm{I_G}) = q\mu_{\mathrm{G}}(\mathrm{I_G}).$$

So combining with (2.46), we have

$$\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}}+\mathrm{Ch}_{\mathrm{I_G}w_\alpha\mathrm{I_G}}, \mathrm{Ch}_{\mathrm{I_{M^J}}})(\lambda w_0^{\mathrm{G}}) = \mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\mathrm{I_{M^J}})(1+q\int_{N_\alpha^0} dn_\alpha\mathcal{K}(w_\alpha n_\alpha w_0^{\mathrm{G}}\lambda))$$

Note that for $t \in F$, $w_\alpha n_\alpha(t) \in \mathrm{T}^0_{\mathrm{G}}\mathrm{T}_\alpha(t^{-1})n_\alpha(-t)n_{-\alpha}(t^{-1})$, so it is equal to

$$\mu_{\mathrm{G}}(\mathrm{I}_{\mathrm{G}})\mu_{\mathrm{M^J}}(\mathrm{I}_{\mathrm{M^J}}) \cdot \left(1 + q\int_{|t|\leq 1} dt\mathcal{K}(T_\alpha(t^{-1})w_0 n_\alpha(t^{-1})\lambda)\right).$$

We consider the calculation of this case by case.

Case a. When $\alpha = e_i - e_{i+1}$ with $1 \leq i \leq n-m-1$ we have $n_\alpha(t^{-1})\lambda = \lambda n_\alpha(t^{-1})$ if $i < n-m-1$, and $n_\alpha(t^{-1})\lambda = \lambda n_\alpha(t^{-1})u$ for some $u \in \mathrm{U}$ with $\psi_{\mathrm{U}}(u) = 1$ if $i = n-m-1$. So

$$\mathcal{K}(T_\alpha(t^{-1})w_0 n_\alpha(t^{-1})\lambda) = \mathcal{K}(T_\alpha(t^{-1})w_0\lambda n_\alpha(t^{-1})),$$

and so

$$\int_{|t|\leq 1} dt\mathcal{K}(T_\alpha(t^{-1})w_0 n_\alpha(t^{-1})\lambda) = \int_{|t|\leq 1} dt(\chi_i\chi_{i+1}^{-1})(t)|t|^{-1}\psi(t^{-1}).$$

We let

$$\begin{aligned}\mathrm{I}(n) =& \int_{|t|=q^{-n}} dt(\chi_i\chi_{i+1}^{-1})(t)|t|^{-1}\psi(t^{-1}) \\ =&(\chi_i\chi_{i+1}^{-1}(\mathfrak{p}))^n q^n \int_{|t|=q^{-n}} dt\psi(t^{-1}).\end{aligned}$$

By lemma 2.7.2 we have $\mathrm{I}(0) = 1 - q^{-1}$, $\mathrm{I}(1) = -q^{-1}(\chi_i\chi_{i+1}^{-1})(\mathfrak{p})$, and $\mathrm{I}(n) = 0$ if $n \geq 2$. Combining these we have

$$\begin{aligned}&\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}} + \mathrm{Ch}_{\mathrm{I_G}w_\alpha\mathrm{I_G}}, \mathrm{Ch}_{\mathrm{I_{M^J}}})(\lambda w_0^{\mathrm{G}}) \\ =&\mu_{\mathrm{G}}(\mathrm{I}_{\mathrm{G}})\mu_{\mathrm{M^J}}(\mathrm{I}_{\mathrm{M^J}})(q - (\chi_i\chi_{i+1}^{-1})(\mathfrak{p})),\end{aligned}$$

which is the first part of proposition 2.8.3.

Case b. When $\alpha = e_i - e_{i+1}$ with $n-m \leq i \leq n-1$, we let
$\mathbf{t}_j = (1, 1, ..., \overset{\text{j-th}}{1 + t^{-1}}, 1, ..., 1) \in \mathbf{F}^m$. Then $n_\alpha(t^{-1})\lambda = \mathrm{X}(\mathbf{t}_1)$ if $i = n-m$, and $n_\alpha(t^{-1})\lambda = \mathrm{X}(\mathbf{t}_{i'+1})n_\alpha(t^{-1})$ if $i \geq n-m+1$. Here $i' = i-(n-m)$. Note that

$w_0^{\mathrm{G}}\mathrm{X}(\mathbf{t}_j) = d_m(\mathbf{t}_j)w_0^{\mathrm{G}}\lambda d_m(\mathbf{t}_j)$, so

$$\mathcal{K}(T_\alpha(t^{-1})w_0 n_\alpha(t^{-1})\lambda) = \int_{|t|\leq 1} \mathrm{d}t\, |t|^{\chi_i - \xi_{i'+1} - \frac{1}{2}} |t+1|^{-\chi_{i+1}+\xi_{i'+1}-\frac{1}{2}}.$$

To calculate this we apply the lemma 8.6 in [19].

Lemma 2.8.6. *Suppose η and η' are two unramified characters on F^*. Then*

$$1 + q\int_{\mathcal{O}} \mathrm{d}t\ \eta(t)\eta'(1+t) = (q-1)\frac{1 - q^{-2}(\eta\eta')(\mathfrak{p})}{(1-q^{-1}\eta(\mathfrak{p}))(1-q^{-1}\eta'(\mathfrak{p}))}.$$

Applying the lemma for $\chi = \chi_i\xi_{i'+1}^{-1}|\cdot|^{-\frac{1}{2}}$ and $\chi' = \chi_{i+1}^{-1}\xi_{i'+1}|\cdot|^{-\frac{1}{2}}$, we have, when $\alpha = e_i - e_{i+1}$ with $n - m \leq i \leq n-1$,

$$\begin{aligned}&\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}} + \mathrm{Ch}_{\mathrm{I_G}w_\alpha\mathrm{I_G}}, \mathrm{Ch}_{\mathrm{I_{M^J}}})(\lambda w_0^{\mathrm{G}})\\ =&\mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\mathrm{I_{M^J}})(q-1)\frac{1-q^{-1}(\chi_i\chi_{i+1}^{-1})(\mathfrak{p})}{(1-(\chi_i\xi_{i'+1}^{-1})(\mathfrak{p})q^{-\frac{1}{2}})(1-(\chi_{i+1}^{-1}\xi_{i'+1})(\mathfrak{p})q^{-\frac{1}{2}})},\end{aligned}$$

which is the second part of proposition 2.8.3.

Case c. When $\alpha = 2e_n$, we have

$$n_\alpha(t^{-1})\lambda = \lambda\mathrm{Z}(t^{-1})\mathrm{Y}_1(-t^{-1})n_\alpha(t^{-1})$$

So

$$\int_{|t|\leq 1} \mathrm{d}t\, \mathcal{K}(T_\alpha(t^{-1})w_0 n_\alpha(t^{-1})\lambda) = \int_{|t|\leq 1} \mathrm{d}t\, \chi_n(t)|t|^{-1}\psi(t^{-1}).$$

Similar to **Case a**, if we let

$$\begin{aligned}\tilde{I}(i) =& \int_{|t|=q^{-i}} dt\chi_n(t)|t|^{-1}\psi(t^{-1})\\ =&\chi_n(\mathfrak{p})^i q^i \int_{|t|=q^{-i}} dt\psi(t^{-1}),\end{aligned}$$

then by lemma 2.7.2,

$$\tilde{I}(i) = \begin{cases} 1 - q^{-1} & \text{if } i = 0; \\ -q^{-1}\chi_n(\mathfrak{p}) & \text{if } i = 1; \\ 0 & \text{if } i \geq 2. \end{cases}$$

So

$$\begin{aligned} &\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}} + \mathrm{Ch}_{\mathrm{I_G} w_\alpha \mathrm{I_G}}, \mathrm{Ch}_{\mathrm{I_{M^J}}})(\lambda w_0^{\mathrm{G}}) \\ =&\mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\mathrm{I_{M^J}})(q - \chi_n(\mathfrak{p})), \end{aligned}$$

which is part 3 of proposition 2.8.3.

So by the calculation in the **Case a, b** and **c**, we have proved proposition 2.8.3, which implies theorem 2.8.4.

2.8.2 Calculation of $\gamma(\chi, \xi, 1, w_\beta)$

Let $\tilde{\chi}_i = \chi_{n-m+i}$ for $1 \leq i \leq m$. Let $\mathit{\Phi}_{\chi,1} = \mathrm{F}_\chi(\mathrm{Ch}(\mathrm{I_G}))$ as before. Recall that $\bar{\mathrm{I}}_{\mathrm{M}} = \mathrm{I_M} \ltimes \mathrm{J}^0$. Let

$$\Psi_{\xi,\psi,w} = \mathrm{F}_{\xi,\psi}(\mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}} w \bar{\mathrm{I}}_{\mathrm{M}}}).$$

Similar to theorem 3.4 in [6], we have

$$\mathrm{T}_{w_\beta}(\Psi_{\xi,\psi,1} + \Psi_{\xi,\psi,w_\beta}) = \tilde{c}_\beta(\xi)(\Psi_{w_\beta\xi,\psi,1} + \Psi_{w_\beta\xi,\psi,w_\beta}). \tag{2.47}$$

So we have

$$\gamma(\chi, \xi, 1, w_\beta) = \tilde{c}_\beta(\xi) \cdot \frac{l_{\chi,w_\beta\xi,\psi}(R(\lambda w_0^{\mathrm{G}})\mathit{\Phi}_{\chi,1}, \Psi_{w_\beta\xi,\psi,1} + \Psi_{w_\beta\xi,\psi,w_\beta})}{l_{\chi,\xi,\psi}(R(\lambda w_0^{\mathrm{G}})\mathit{\Phi}_{\chi,1}, \Psi_{\xi,\psi,1} + \Psi_{\xi,\psi,w_\beta})}. \tag{2.48}$$

What we are going to show is

Propostion 2.8.7. *For generic* (χ,ξ)*, the value of* $l_{\chi,\xi,\psi}(R(\lambda w_0^{\mathrm{G}})\varPhi_{\chi,1},\Psi_{\xi,\psi,1}+\Psi_{\xi,\psi,w_\beta})$ *equals*

$$
\begin{aligned}
&q(1-q^{-1})^m\mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_{\mathrm{M}})\prod_{j\neq i,i+1}\zeta(\frac{1}{2}-\tilde{\chi}_j+\xi_j)\\
&\cdot\frac{\zeta(\xi_i-\tilde{\chi}_i+\frac{1}{2})\zeta(\xi_{i+1}-\tilde{\chi}_{i+1}+\frac{1}{2})\zeta(\xi_i-\tilde{\chi}_{i+1}+\frac{1}{2})\zeta(-\xi_{i+1}+\tilde{\chi}_i+\frac{1}{2})}{\zeta(\tilde{\chi}_i-\tilde{\chi}_{i+1}+1)\zeta(\xi_i-\xi_{i+1}+1)}
\end{aligned}
$$

when $\beta=e'_i-e'_{i+1}$*, and equals*

$$
\begin{aligned}
&q(1-q^{-1})^m\mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_{\mathrm{M}})\prod_{j=1}^{m-1}\zeta(-\tilde{\chi}_j+\xi_j+\frac{1}{2})\\
&\cdot\frac{(1-\tilde{\chi}_m(\mathfrak{p})q^{-1})(1+\xi_m(\mathfrak{p})q^{-\frac{1}{2}})}{(1-\tilde{\chi}_m^{-1}\xi_m(\mathfrak{p})q^{-\frac{1}{2}})(1-\tilde{\chi}_m\xi_m(\mathfrak{p})q^{-\frac{1}{2}})}
\end{aligned}
$$

when $\beta=2e'_m$.

Substituting this in (2.48) we have

Theorem 2.8.8. *For generic* (χ,ξ)*, the* γ*-factor* $\gamma(\chi,\xi,1,w_\beta)$ *equals*

$$
\tilde{c}_\beta(\xi)\frac{\zeta(\xi_i-\xi_{i+1}+1)\zeta(-\tilde{\chi}_i+\xi_{i+1}+\frac{1}{2})\zeta(\tilde{\chi}_i-\xi_i+\frac{1}{2})}{\zeta(-\xi_i+\xi_{i+1}+1)\zeta(\tilde{\chi}_i-\xi_{i+1}+\frac{1}{2})\zeta(-\tilde{\chi}_i+\xi_i+\frac{1}{2})}
$$

when $\beta=e'_i-e'_{i+1}$*, and equals*

$$
\tilde{c}_\beta(\xi)\frac{\zeta(-\xi_m-\tilde{\chi}_m+\frac{1}{2})\zeta(-\xi_m+\tilde{\chi}_m+\frac{1}{2})\zeta(2\xi_m+1)\zeta(-\xi_m+\frac{1}{2})}{\zeta(\xi_m-\tilde{\chi}_m+\frac{1}{2})\zeta(\xi_m+\tilde{\chi}_m+\frac{1}{2})\zeta(-2\xi_m+1)\zeta(\xi_m+\frac{1}{2})}
$$

when $\beta=2e'_m$.

To prove proposition 2.8.7, first we have

Propostion 2.8.9. *For* $(\chi,\xi)\in\mathscr{Z}_C$,

$$
\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}},\mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}}})(\lambda w_0^{\mathrm{G}})=\mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_{\mathrm{M}})\prod_{j=1}^{m}\frac{1-q^{-1}}{1-\tilde{\chi}_j^{-1}\xi_j(\mathfrak{p})q^{-\frac{1}{2}}}\tag{2.49}
$$

Proof. By definition, we have

$$\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}}, \mathrm{Ch}_{\bar{\mathrm{I}}_\mathrm{M}})(\lambda w_0^\mathrm{G}) = \int_{\mathrm{I_G}} \mathrm{d}g \int_{\bar{\mathrm{I}}_\mathrm{M}} \mathrm{d}m \mathcal{K}(g(w_0^\mathrm{G})^{-1}\lambda^{-1}m).$$

Applying lemma 2.8.2, it is equal to

$$\mu_\mathrm{G}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_\mathrm{M}) \int_{\mathrm{X}^0} \mathrm{d}x' \mathcal{K}(w_0^\mathrm{G} x').$$

Parametrize X^0 as $\mathrm{X}^0 = \{\mathrm{X}(\mathbf{x}) = \mathrm{X}(x_1, ..., x_m) | x_i \in \mathcal{O} \text{ for all } i\}$, then, $w_0^\mathrm{G}\mathrm{X}(x_1, ..., x_m) = d_m(x_1, ..., x_m) w_0^\mathrm{G} \lambda d_m(x_1, ..., x_m)$ if all x_i are non-zero, and $\mathcal{K}(w_0^\mathrm{G} x') = 0$ otherwise. So

$$\begin{aligned}\int_{\mathrm{X}^0} dx' \mathcal{K}(w_0^\mathrm{G} x') &= \prod_{j=1}^m \int_{x_j \in \mathcal{O}} \mathrm{d}x_j (\tilde{\chi}_j^{-1}\xi_j|\cdot|^{-\frac{1}{2}})(x_j) \\ &= \prod_{j=1}^m \frac{1-q^{-1}}{1-\tilde{\chi}_j^{-1}\xi_j(\mathfrak{p})q^{-\frac{1}{2}}},\end{aligned}$$

where the last equality follows from the following lemma. □

Lemma 2.8.10. *Let η be an unramified character on F^*. Then*

$$\int_{x\in\mathcal{O}} \eta(x)\, \mathrm{d}x = \frac{1-q^{-1}}{1-q^{-1}\eta(\mathfrak{p})}.$$

Proof. One can prove this by direct calculation. □

Next we consider $\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}}, \mathrm{Ch}_{\bar{\mathrm{I}}_\mathrm{M} w_\beta \bar{\mathrm{I}}_\mathrm{M}})(\lambda w_0^\mathrm{G})$. By definition, it is equal to

$$\int_{\mathrm{I_G}} \mathrm{d}x \int_{\bar{\mathrm{I}}_\mathrm{M} w_\beta \bar{\mathrm{I}}_\mathrm{M}} \mathrm{d}x' \mathcal{K}(x(w_0^\mathrm{G})^{-1}\lambda^{-1}x').$$

By lemma 2.8.2 it is equal to

$$\mu_\mathrm{G}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_\mathrm{M} w_\beta \bar{\mathrm{I}}_\mathrm{M}) \int_{|t|\le 1} \mathrm{d}t \int_{\mathrm{X}^0} \mathrm{d}x \mathcal{K}(w_0^\mathrm{G} w_\beta n_{-\beta}(t) x). \tag{2.50}$$

Since

$$\mu_{\mathrm{M}^{\mathrm{J}}}(\bar{\mathrm{I}}_{\mathrm{M}} w_\beta \bar{\mathrm{I}}_{\mathrm{M}}) = q\mu_{\mathrm{M}^{\mathrm{J}}}(\bar{\mathrm{I}}_{\mathrm{M}}),$$

combining (2.50) with (2.49) we have

$$\begin{aligned}
&\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}}, \mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}}} + \mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}} w_\beta \bar{\mathrm{I}}_{\mathrm{M}}})(\lambda w_0^{\mathrm{G}}) \\
=&\mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M}^{\mathrm{J}}}(\bar{\mathrm{I}}_{\mathrm{M}}) \left(\prod_{j=1}^{m} \frac{1-q^{-1}}{1-\tilde{\chi}_j^{-1}\xi_j(\mathfrak{p})q^{-\frac{1}{2}}} + q\int_{|t|\le 1} \mathrm{d}t \int_{\mathrm{X}^0} \mathrm{d}x \mathcal{K}(w_0^{\mathrm{G}} w_\beta n_{-\beta}(t)x) \right).
\end{aligned} \tag{2.51}$$

Now we calculate

$$\int_{|t|\le 1} \mathrm{d}t \int_{\mathrm{X}^0} \mathrm{d}x \mathcal{K}(w_0^{\mathrm{G}} w_\beta n_{-\beta}(t)x).$$

Note that $w_\beta n_{-\beta}(t) = n_{-\beta}(-t^{-1})\mathrm{N}_\beta(t)T_\beta(t)$, so

$$\begin{aligned}
&\int_{|t|\le 1} \mathrm{d}t \int_{\mathrm{X}^0} \mathrm{d}x \mathcal{K}(w_0^{\mathrm{G}} w_\beta n_{-\beta}(t)x) \\
=&\int_{|t|\le 1} \mathrm{d}t \int_{\mathrm{X}^0} \mathrm{d}x \mathcal{K}(w_0^{\mathrm{G}} n_{-\beta}(-t^{-1}) n_\beta(t) T_\beta(t) x) \\
=&\int_{|t|\le 1} \mathrm{d}t \int_{\mathrm{X}^0} \mathrm{d}x \mathcal{K}(w_0^{\mathrm{G}} n_\beta(t) T_\beta(t) x T_\beta^{-1}(t) n_\beta^{-1}(t))(\xi\delta_{\mathrm{B}_{\mathrm{M}}^{\mathrm{J}}}^{-\frac{1}{2}})(T_\beta(t))
\end{aligned} \tag{2.52}$$

Below we discuss this integral case by case.

Case a. When $\beta = e_i' - e_{i+1}'$.

We parametrize X^0 as $\mathrm{X}(x) = \mathrm{X}(x_1, ..., x_m)$ with $x_j \in \mathcal{O}$. Then we have

$$\begin{aligned}
&n_\beta(t)T_\beta(t)\mathrm{X}(x_1,...,x_m)T_\beta^{-1}(t)n_\beta^{-1}(t) \\
=&\mathrm{X}(x_1,...,x_{i-1}, t^{-1}x_i, -x_i + tx_{i+1}, x_{i+2}, \ldots, x_m)
\end{aligned}$$

and

$$(\xi\delta_{\mathrm{B}_{\mathrm{M}}^{\mathrm{J}}}^{-\frac{1}{2}})(T_\beta(t)) = (\xi_i\xi_{i+1}^{-1}|\cdot|^{-1})(t).$$

Let $\mathrm{x}(i,t) = (x_1, ..., x_{i-1}, t^{-1}x_i, -x_i + tx_{i+1}, x_{i+2}, ..., x_m)$, then (2.52) is equal to

$$\int_{|t|\le 1} \mathrm{d}t \int_{|x_i|\le 1, i=1,2,...,m} \mathrm{d}x_i \ (\xi_i\xi_{i+1}^{-1})(t)|t|^{-1}\mathcal{K}(w_0^{\mathrm{G}}\mathrm{X}(\mathrm{x}(i,t))).$$

Note that if none of the components of $\mathrm{x}(i,t)$ is zero, then

$$w_0^{\mathrm{G}}\mathrm{X}(\mathrm{x}(i,t)) = d_m(\mathrm{x}(i,t))w_0^{\mathrm{G}}\lambda d_m(\mathrm{x}(i,t))$$

and $\mathcal{K}(w_0^{\mathrm{G}}\mathrm{X}(\mathrm{x}(i,t))) = 0$ if otherwise. So the integral above is equal to

$$\begin{gathered}\prod_{j\in\{1,2,...,i-1,i+2,...,m\}} \int_{x_j\in\mathcal{O}} (\tilde{\chi}_j^{-1}\xi_j|\cdot|^{-\frac{1}{2}})(x_j)\,\mathrm{d}x_j\cdot \\ \int_{|t|\le 1, |x_i|\le 1, |x_{i+1}|\le 1} \mathrm{d}t\,\mathrm{d}x_i\,\mathrm{d}x_{i+1}(\xi_i\xi_{i+1}^{-1}|\cdot|^{-1})(t) \\ (\tilde{\chi}_i^{-1}\xi_i|\cdot|^{-\frac{1}{2}})(t^{-1}x_i)\,(\tilde{\chi}_{i+1}^{-1}\xi_{i+1}|\cdot|^{-\frac{1}{2}})(-x_i+tx_{i+1})\end{gathered}$$

By lemma 2.8.10, it is equal to

$$\begin{gathered}\prod_{j\in\{1,2,...,i-1,i+2,...,m\}} \frac{1-q^{-1}}{1-\tilde{\chi}_j^{-1}\xi_j(\mathfrak{p})q^{-\frac{1}{2}}} \\ \int_{|t|\le 1, |x_i|\le 1, |x_{i+1}|\le 1} \mathrm{d}t\,\mathrm{d}x_i\,\mathrm{d}x_{i+1}(\xi_i\xi_{i+1}^{-1}|\cdot|^{-1})(t) \\ (\tilde{\chi}_i^{-1}\xi_i|\cdot|^{-\frac{1}{2}})(t^{-1}x_i)\,(\tilde{\chi}_{i+1}^{-1}\xi_{i+1}|\cdot|^{-\frac{1}{2}})(-x_i+tx_{i+1})\end{gathered} \tag{2.53}$$

Let $\mathrm{I}(a,b)$ be part of the integral above for $|t| = q^{-a}$ and $|x_i| = q^{-b}$ with $a, b \ge 0$, then the integral is equal to $\sum_{a,b\ge 0}\mathrm{I}(a,b)$. When $a > b$. i.e., $|x_i| > |t|$, we have $|-x_i + tx_{i+1}| = |x_i|$. So

$$\mathrm{I}(a,b) = (1-q^{-1})^2 q^{-\frac{a}{2}}(\tilde{\chi}_i^{a-b}\tilde{\chi}_{i+1}^{-b}\xi_i^b\xi_{i+1}^{-a+b})(\mathfrak{p}).$$

When $a \leq b$, i.e., $|x_i| \leq |t|$, we can change the variable $x_{i+1} \to x_{i+1} + \frac{x_i}{t}$, we have

$$\mathrm{I}(a,b) = \sum_{c \geq 0} (1-q^{-1})^3 q^{-\frac{(b+c)}{2}} \tilde{\chi}_i^{a-b} (\tilde{\chi}_{i+1}^{-a-c} \xi_i^b \xi_{i+1}^c)(\mathfrak{p}),$$

where the summand for c corresponds to $|x_{i+1}| = q^{-c}$ after the change of variable. Applying this to equation (2.51), we have

$$\begin{aligned}
&\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}}, \mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}}} + \mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}} w_\beta \bar{\mathrm{I}}_{\mathrm{M}}})(\lambda w_0^{\mathrm{G}}) = \mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_{\mathrm{M}}) \\
&\cdot \left(\prod_{j=1}^{m} \frac{1-q^{-1}}{1-\tilde{\chi}_j^{-1}\xi_j(\mathfrak{p})q^{-\frac{1}{2}}} + q \prod_{j \neq i,i+1} \frac{1-q^{-1}}{1-\tilde{\chi}_j^{-1}\xi_j(\mathfrak{p})q^{-\frac{1}{2}}} \sum_{a,b \geq 0} \mathrm{I}(a,b) \right) \\
=& q(1-q^{-1})^m \mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_{\mathrm{M}}) \prod_{j \neq i,i+1} \zeta(\frac{1}{2} - \tilde{\chi}_j + \xi_j) \\
&\cdot \frac{\zeta(\xi_i - \tilde{\chi}_i + \frac{1}{2})\zeta(\xi_{i+1} - \tilde{\chi}_{i+1} + \frac{1}{2})\zeta(\xi_i - \tilde{\chi}_{i+1} + \frac{1}{2})\zeta(-\xi_{i+1} + \tilde{\chi}_i + \frac{1}{2})}{\zeta(\tilde{\chi}_i - \tilde{\chi}_{i+1} + 1)\zeta(\xi_i - \xi_{i+1} + 1)}
\end{aligned}$$

which is the first part of proposition 2.8.7

Case b. When $\beta = 2e'_m$, we have

$$(\xi \delta_{\mathrm{B_{M^J}}}^{-\frac{1}{2}})(T_\beta(t)) = \xi_m(t)|t|^{-\frac{3}{2}},$$

and

$$\begin{aligned}
&n_\beta(t) T_\beta(t) \mathrm{X}(x_1, ..., x_m) T_\beta^{-1}(t) n_\beta^{-1}(t) \\
=& \mathrm{X}(x_1, ..., x_{m-1}, t^{-1}x_m)\mathrm{Y}(-x_m, 0, \ldots, 0)\mathrm{Z}(t^{-1}x_m^2).
\end{aligned}$$

So (2.52) is equal to

$$\begin{aligned}
&\int_{|t| \leq 1} \mathrm{d}t \int_{x_i \in \mathcal{O}} \mathrm{d}x_i \, \psi(t^{-1}x_m^2)\xi_m(t)|t|^{-\frac{3}{2}} \mathcal{K}(w_0^{\mathrm{G}} \mathrm{X}(x_1, ..., x_{m-1}, t^{-1}x_m)) \\
=& \prod_{j=1}^{m-1} \frac{1-q^{-1}}{1-(\tilde{\chi}_j^{-1}\xi_j)(\mathfrak{p})q^{-\frac{1}{2}}} \\
&\cdot \int_{t, x_m \in \mathcal{O}} \mathrm{d}t \, \mathrm{d}x_m \, (\tilde{\chi}_m^{-1}\xi_m|\cdot|^{-\frac{1}{2}})(t^{-1}x_m)(\xi_m|\cdot|^{-\frac{3}{2}})(t)\psi(t^{-1}x_m^2).
\end{aligned}$$

Let $\mathrm{I}(a,b)$ be part of the integral above when $|t| = q^{-a}$ and $|x_m| = q^{-b}$. By lemma 2.7.2, $\mathrm{I}(a,b) = (1-q^{-1})^2(\tilde{\chi}_m^{a-b}\xi_m^b)(\mathfrak{p})q^{-\frac{b}{2}}$ when $0 \le a \le 2b$, and $\mathrm{I}(a,b) = -(1-q^{-1})(\tilde{\chi}_m\xi_m^b)(\mathfrak{p})q^{-\frac{b}{2}-1}$ when $a = 2b+1$, and $\mathrm{I}(a,b) = 0$ when $a \ge 2b+2$. Applying these to equation (2.51), we have

$$\begin{aligned}&\mathrm{I}_{\chi,\xi,\psi}(\mathrm{Ch}_{\mathrm{I_G}}, \mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}}} + \mathrm{Ch}_{\bar{\mathrm{I}}_{\mathrm{M}} w_\beta \bar{\mathrm{I}}_{\mathrm{M}}})(\lambda w_0^{\mathrm{G}})\\ =&q(1-q^{-1})^m \mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_{\mathrm{M}})\prod_{j=1}^{m-1}\zeta(-\tilde{\chi}_j + \xi_j + \frac{1}{2})\\ &\cdot \frac{(1-\tilde{\chi}_m(\mathfrak{p})q^{-1})(1+\xi_m(\mathfrak{p})q^{-\frac{1}{2}})}{(1-\tilde{\chi}_m^{-1}\xi_m(\mathfrak{p})q^{-\frac{1}{2}})(1-\tilde{\chi}_m\xi_m(\mathfrak{p})q^{-\frac{1}{2}})}\end{aligned}$$

which is the second part of the proposition 2.8.7. Combining **Case a** and **b**, we have proposition 2.8.7, which implies theorem 2.8.8.

2.9 Formula for generic (χ, ξ)

In this section we discuss the value of $\mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\mathbf{f}})$ for generic (χ,ξ) where $\mathbf{d} \in \Lambda_m^+$ and $\mathbf{f} \in \Lambda_n^+$.

Consider

$$\Gamma^{-1}(\chi,\xi) \cdot l_{\chi,\xi,\psi}(R(\lambda w_0^{\mathrm{G}})R(Ch_{\mathrm{I_G}\mathfrak{p}^{-\mathbf{f}}\mathrm{I_G}})\mathrm{F}^0_\chi, R(Ch_{\bar{\mathrm{I}}_{\mathrm{M}}\mathfrak{p}^{-\mathbf{d}}\bar{\mathrm{I}}_{\mathrm{M}}})\mathrm{F}^0_{\xi,\psi}). \tag{2.54}$$

By lemma 2.8.2 it is equal to

$$\Gamma^{-1}(\chi,\xi)\mu_{\mathrm{G}}(\mathrm{I_G}\mathfrak{p}^{-\mathbf{f}}\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_{\mathrm{M}}\mathfrak{p}^{-\mathbf{d}}\bar{\mathrm{I}}_{\mathrm{M}})\int_{\mathrm{X}^0}\mathrm{d}x\,\mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}x\mathfrak{p}^{\mathbf{f}}).$$

On the other hand, by proposition 2.7.7 and 2.7.6, (2.54) is equal to

$$\Gamma^{-1}(\chi,\xi)\frac{\mu_{\mathrm{G}}(\mathrm{I_G}\mathfrak{p}^{-\mathbf{f}}\mathrm{I_G})\mu_{\mathrm{M^J}}(\overline{\mathrm{I}}_{\mathrm{M}}\mathfrak{p}^{-\mathbf{d}}\overline{\mathrm{I}}_{\mathrm{M}})}{\mu_{\mathrm{N_G^-}}(\mathrm{N_G^{-,1}})\mu_{\mathrm{N_M^-}}(\mathrm{N_M^{-,1}})}$$
$$\cdot\sum_{w\in W_{\mathrm{G}},w'\in W_{\mathrm{M}}}\Big[c_{w_0^{\mathrm{G}}}(w\chi)(w\chi\delta_{\mathrm{B_G}}^{-\frac{1}{2}})(\mathfrak{p}^{-\mathbf{f}})\tilde{c}_{w_0^{\mathrm{M}}}(w'\xi)(w'\xi\delta_{\mathrm{B_{M^J}}}^{-\frac{1}{2}})(\mathfrak{p}^{-\mathbf{d}})$$
$$\cdot\, l_{\chi,\xi,\psi}(R(\lambda w_0^{\mathrm{G}})\overline{T}_{w^{-1}}\varPhi_{w\chi,1},\overline{T}_{(w')^{-1}}\Psi_{w'\xi,\psi,1})\Big].$$

So we have

$$\begin{aligned}\Gamma^{-1}(\chi,\xi)\int_{\mathrm{X}^0}\mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}x\mathfrak{p}^{\mathbf{f}}) &= \frac{1}{\mu_{\mathrm{N_G^-}}(\mathrm{N_G^{-,1}})\mu_{\mathrm{N_M^-}}(\mathrm{N_M^{-,1}})}\\ \cdot\sum_{w\in W_{\mathrm{G}},w'\in W_{\mathrm{M}}}\Big[c_{w_0^{\mathrm{G}}}(w\chi)(w\chi\delta_{\mathrm{B_G}}^{-\frac{1}{2}})(\mathfrak{p}^{-\mathbf{f}})&\tilde{c}_{w_0^{\mathrm{M}}}(w'\xi)(w'\xi\delta_{\mathrm{B_{M^J}}}^{-\frac{1}{2}})(\mathfrak{p}^{-\mathbf{d}})\\ &\cdot\, l_{\chi,\xi,\psi}(R(\lambda w_0)\overline{T}_{w^{-1}}\varPhi_{w\chi,1},\overline{T}_{(w')^{-1}}\Psi_{w'\xi,\psi,1})\cdot\Gamma^{-1}(\chi,\xi)\Big].\end{aligned} \tag{2.55}$$

To calculate the right hand side of the equation above, we use the following lemma.

Lemma 2.9.1. *Let $\mathcal{A}(\chi,\xi,w,w')$ be an arbitrary function from $\mathbb{C}^n\times\mathbb{C}^m\times\mathrm{W_G}\times\mathrm{W_M}$ to $\mathbb{C}$, and let $(\mathbf{d},\mathbf{f})\in\Lambda_m^+\times\Lambda_n^+$. Suppose (χ,ξ) is generic. Let*

$$\mathcal{B}(\chi,\xi,\mathbf{d},\mathbf{f})=\sum_{w\in\mathrm{W_G},w'\in\mathrm{W_M}}\mathcal{A}(\chi,\xi,w,w')(w\chi)(\mathfrak{p}^{\mathbf{f}})(w'\xi)(\mathfrak{p}^{\mathbf{d}}).$$

If $\mathcal{B}(\chi,\xi,\mathbf{d},\mathbf{f})$ is $\mathrm{W_G}\times\mathrm{W_M}$-invariant for all $(\mathbf{d},\mathbf{f})\in\Lambda_m^+\times\Lambda_n^+$, then

$$\mathcal{A}(\chi,\xi,w,w')=\mathcal{A}(w\chi,w'\xi,e,e)$$

for all $(w,w')\in\mathrm{W_G}\times\mathrm{W_M}$.

Proof. By considering the Vandermonde determinant, one easily see that for any finite set $\{z_1,\dots z_k\}$ of distinct complex numbers, the functions $n\mapsto z_i^n (n\in\mathbb{Z}_{\geq 0})$ are linearly independent. Next, using induction, one easily shows that for any positive integer r and for any finite set $\{\underline{z}_1,\dots,\underline{z}_k\}$ of distinct r-tuples $\underline{z}_i=(z_{i,1},\dots,z_{i,r})\in\mathbb{C}^r$, the functions $\underline{n}\mapsto\prod_{i=1}^r z_i^{n_i}, (\underline{n}=(n_1,\dots,n_r)\in\Lambda_r^+)$ are all linearly independent. For each w, w' we define

$$z_w = \left(q^{-(w\chi)_1}, \ldots, q^{-(w\chi)_n}\right) \in \mathbb{C}^n$$

and

$$z_{w'} = \left(q^{-(w\xi)_1}, \ldots, q^{-(w\xi)_m}\right) \in \mathbb{C}^m.$$

(Here $(w\chi)_i$ (resp. $(w'\xi)_i$) denotes the i-th entry of $w\chi$ (resp. $w'\xi$).) When (χ, ξ) is generic, all tuples in z_w and $z_{w'}$ are distinct. So functions $(\mathbf{f}, \mathbf{d}) \mapsto (w\chi)(\mathfrak{p}^{\mathbf{f}})(w'\xi)(\mathfrak{p}^{\mathbf{d}})$ ($\mathbf{f} \in \Lambda_n^+$, $\mathbf{d} \in \Lambda_m^+$), for all $w \in \mathrm{W_G}$ and $w' \in \mathrm{W_M}$, are linearly independent. The lemma follows. $\square$

By this lemma, the summation on the right hand side of (2.55) is determined by its summand at $(w, w') = (e, e)$. By proposition 2.8.9,

$$\begin{aligned}
&c_{w_0^{\mathrm{G}}}(\chi)(\chi\delta_{\mathrm{B_G}}^{-\frac{1}{2}})(\mathfrak{p}^{-\mathbf{f}})\tilde{c}_{w_0^{\mathrm{M}}}(\xi)(\xi\delta_{\mathrm{B_{M^J}}}^{-\frac{1}{2}})(\mathfrak{p}^{-\mathbf{d}}) \cdot l_{\chi,\xi,\psi}(R(\lambda w_0)\Phi_1^{\chi}, \Psi_1^{\xi,\psi}) \cdot \Gamma^{-1}(\chi, \xi) \\
=&(1-q^{-1})^m \mu_{\mathrm{G}}(\mathrm{I_G})\mu_{\mathrm{M^J}}(\bar{\mathrm{I}}_{\mathrm{M}})\mathrm{b}(\chi, \xi)\mathrm{d}(\chi)\mathrm{d}'(\xi)(\chi\delta_{\mathrm{B_G}}^{-\frac{1}{2}})(\mathfrak{p}^{-\mathbf{f}})(\xi\delta_{\mathrm{B_{M^J}}}^{-\frac{1}{2}})(\mathfrak{p}^{-\mathbf{d}}),
\end{aligned}$$

where

$$\mathrm{d}(\chi) = \prod_{1\le a<b\le n} \zeta(\chi_a - \chi_b)\zeta(\chi_a + \chi_b) \prod_{i=1}^{n} \zeta(\chi_i),$$

$$\mathrm{d}'(\xi) = \prod_{1\le a<b\le m} \zeta(\xi_a - \xi_b)\zeta(\xi_a + \xi_b) \prod_{j=1}^{m} \zeta(2\xi_j),$$

and

$$\begin{aligned}
\mathrm{b}(\chi, \xi) = &\prod_{i<j+n-m} \zeta^{-1}(\chi_i - \xi_j + \frac{1}{2}) \cdot \prod_{i>j+n-m} \zeta^{-1}(-\chi_i + \xi_j + \frac{1}{2}) \\
&\prod_{1\le j\le m} \zeta^{-1}(\xi_j + \frac{1}{2}) \cdot \prod_{\substack{1\le i\le n\\ 1\le j\le m}} \zeta^{-1}(\chi_i + \xi_j + \frac{1}{2})
\end{aligned}$$

Applying the lemma 2.9.1, we have

$$\begin{aligned}\int_{\mathrm{X}^0} \mathrm{d}x\, \mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}} x \mathfrak{p}^{\mathbf{f}}) &= (1-q^{-1})^m \frac{\mu_{\mathrm{G}}(\mathrm{I}_{\mathrm{G}})\mu_{\mathrm{M}^{\mathrm{J}}}(\bar{\mathrm{I}}_{\mathrm{M}})}{\mu_{\mathrm{N}^-_{\mathrm{G}}}(\mathrm{N}^{-,1}_{\mathrm{G}})\mu_{\mathrm{N}^-_{\mathrm{M}}}(\mathrm{N}^{-,1}_{\mathrm{M}})}\Gamma(\chi,\xi) \\ \cdot \sum_{w\in \mathrm{W}_{\mathrm{G}}, w'\in \mathrm{W}_{\mathrm{M}}} & \mathrm{b}(w\chi, w'\xi)\mathrm{d}(w\chi)\mathrm{d}'(w'\xi)(w\chi\delta^{-\frac{1}{2}}_{\mathrm{B}_{\mathrm{G}}})(\mathfrak{p}^{-\mathbf{f}})(w'\xi\delta^{-\frac{1}{2}}_{\mathrm{B}_{\mathrm{M}^{\mathrm{J}}}})(\mathfrak{p}^{-\mathbf{d}}).\end{aligned} \tag{2.56}$$

Though this is not a direct formula for $\mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\mathbf{f}})$, we have the following theorem.

Theorem 2.9.2. *Let* $l(\mathbf{d},\mathbf{f}) = \mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\mathbf{f}})$*, and let*

$$L(\mathbf{d},\mathbf{f}) = \int_{\mathrm{X}^0} \mathrm{d}x\, \mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}} x \mathfrak{p}^{\mathbf{f}}), \tag{2.57}$$

then $l(\mathbf{0},\mathbf{f}) = L(\mathbf{0},\mathbf{f})$*. Moreover, let* $S(\mathbf{d},\mathbf{f}) = \{\mathbf{d}' \mid \mathbf{d}' \leq \mathbf{d}, \mathbf{d}' \in \Lambda^+_m, \mathbf{f}+\mathbf{d}-\mathbf{d}' \in \Lambda^+_n\}$*.(Here we identify* $\mathbf{d} = (d_1,\dots,d_m) \in \mathbb{Z}^m$ *with* $(0,\dots,0,d_1,\dots d_m) \in \mathbb{Z}^n$*. So* $\mathbf{f}+\mathbf{d}-\mathbf{d}'$ *is defined.) Then for each* $\mathbf{d}' \in S(\mathbf{d},\mathbf{f})$*, there exists* $a_{\mathbf{d},\mathbf{f}}(\mathbf{d}') \in \mathbb{R}$ *such that*

$$l(\mathbf{d},\mathbf{f}) = \sum_{\mathbf{d}'\in S(\mathbf{d},\mathbf{f})} a_{\mathbf{d},\mathbf{f}}(\mathbf{d}')L(\mathbf{d}',\mathbf{f}+\mathbf{d}-\mathbf{d}'),$$

and that $a_{\mathbf{d},\mathbf{f}}(\mathbf{d}) > 0$*.*

Proof. We have $l(\mathbf{0},\mathbf{f}) = L(\mathbf{0},\mathbf{f})$ by the $\mathrm{K}_{\mathrm{M}^{\mathrm{J}}}$-invariance on the left. Because of this, if we can prove

$$L(\mathbf{d},\mathbf{f}) = \sum_{\mathbf{d}'\in S(\mathbf{d},\mathbf{f})} b(\mathbf{d}')l(\mathbf{d}',\mathbf{f}+\mathbf{d}-\mathbf{d}') \tag{2.58}$$

where b is a function $S(\mathbf{d},\mathbf{f}) \to \mathbb{R}$ such that $b(\mathbf{d}) > 0$, then the theorem is implied. Expand the right hand side of equation (2.57), and then by lemma 2.6.3, $L(\mathbf{d},\mathbf{f})$ is a linear combination of $\mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\mathrm{X}(\mathbf{c})\mathfrak{p}^{\mathbf{f}})$ for $\mathbf{0} \leq \mathbf{c} \leq \mathbf{d}$ with coefficients in $\mathbb{R}_{\geq 0}$. Note that $\mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\mathrm{X}(\mathbf{c})\mathfrak{p}^{\mathbf{f}}) = \mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}-\mathbf{c}}\lambda\mathfrak{p}^{\mathbf{f}+\mathbf{c}})$, and that $(\mathbf{d}-\mathbf{c};\mathbf{f}+\mathbf{c})$ satisfies the assumption in proposition 2.6.6. So by proposition 2.6.6, $\mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}-\mathbf{c}}\lambda\mathfrak{p}^{\mathbf{f}+\mathbf{c}}) = \mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}'}\lambda\mathfrak{p}^{\mathbf{f}+\mathbf{d}-\mathbf{d}'})$ for some $\mathbf{d}' \in S(\mathbf{d},\mathbf{f})$. So we have (2.58). When $x \in (\mathcal{O}^*)^m$,

$\mathrm{I}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\mathrm{X}(x)\mathfrak{p}^{\mathbf{f}}) = l(\mathbf{d},\mathbf{f})$. So $b(\mathbf{d}) \geq (1-q^{-1})^m > 0$, completing our proof. □

2.10 Normalization

By equation (2.56), we have

$$\begin{aligned}
&\mathrm{I}^0_{\chi,\xi,\psi}(e)\\
=&(1-q^{-1})^m \frac{\mu_{\mathrm{G}}(\mathrm{I}_{\mathrm{G}})\mu_{\mathrm{M}^{\mathrm{J}}}(\bar{\mathrm{I}}_{\mathrm{M}})}{\mu_{\mathrm{N}^-_{\mathrm{G}}}(\mathrm{N}^{-,1}_{\mathrm{G}})\mu_{\mathrm{N}^-_{\mathrm{M}}}(\mathrm{N}^{-,1}_{\mathrm{M}})}\Gamma(\chi,\xi)\cdot \sum_{w\in \mathrm{W}_{\mathrm{G}}, w'\in \mathrm{W}_{\mathrm{M}}} \mathbf{b}(w\chi, w'\xi)\mathbf{d}(w\chi)\mathbf{d}'(w'\xi).
\end{aligned}$$

Now we calculate the value of $\mathrm{I}^0_{\chi,\xi,\psi}(e)$. The method is similar to that in section 11 in [19].

Theorem 2.10.1. *The value of* $\mathrm{I}^0_{\chi,\xi,\psi}$ *at identity is given by*

$$\mathrm{I}^0_{\chi,\xi,\psi}(e) = \frac{\mu_{\mathrm{G}}(\mathrm{I}_{\mathrm{G}})\mu_{\mathrm{M}^{\mathrm{J}}}(\bar{\mathrm{I}}_{\mathrm{M}})}{\mu_{\mathrm{N}^-_{\mathrm{G}}}(\mathrm{N}^{-,1}_{\mathrm{G}})\mu_{\mathrm{N}^-_{\mathrm{M}}}(\mathrm{N}^{-,1}_{\mathrm{M}})}\Gamma(\chi,\xi)\prod_{i=1}^{m}\zeta^{-1}(2i).$$

Let $\mathbf{C} = \sum_{w\in \mathrm{W}_{\mathrm{G}}, w'\in \mathrm{W}_{\mathrm{M}}} \mathbf{b}(w\chi, w'\xi)\mathbf{d}(w\chi)\mathbf{d}'(w'\xi)$, then what we need to show is

$$\mathbf{C} = \zeta(1)^m \prod_{i=1}^{m}\zeta^{-1}(2i).$$

For $\mathbf{b}_1(\chi)$ being a function defined on $\chi \in \mathbb{C}^n$ and $\mathbf{b}_2(\xi)$ a function on $\xi \in \mathbb{C}^m$, we define

$$\mathcal{A}_{\mathrm{W}_{\mathrm{G}}}(\mathbf{b}_1(\chi)) = \sum_{w\in \mathrm{W}_{\mathrm{G}}} sgn(w)\mathbf{b}_1(w\chi).$$

and

$$\mathcal{A}_{\mathrm{W}_{\mathrm{M}}}(\mathbf{b}_2(\xi)) = \sum_{w'\in \mathrm{W}_{\mathrm{M}}} sgn(w')\mathbf{b}_2(w'\xi).$$

Then for $w \in \mathrm{W}_{\mathrm{G}}$ and $w' \in \mathrm{W}_{\mathrm{M}}$,

$$\mathcal{A}_{\mathrm{W}_{\mathrm{G}}}(\mathbf{b}_1(w\chi)) = sgn(w)\mathcal{A}_{\mathrm{W}_{\mathrm{G}}}(\mathbf{b}_1(\chi)),$$
$$\mathcal{A}_{\mathrm{W}_{\mathrm{M}}}(\mathbf{b}_2(w'\xi)) = sgn(w')\mathcal{A}_{\mathrm{W}_{\mathrm{M}}}(\mathbf{b}_2(\xi)).$$

For $\epsilon \in \mathbb{Z}^n$ and $\mu \in \mathbb{Z}^m$, we say $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ (resp. $\mu = (\mu_1, \ldots, \mu_m)$) is regular if $w\epsilon = \epsilon$ (resp. $w'\mu = \mu$) implies $w = e$ (resp. $w' = e$). Then ϵ being regular is equivalent to that $\epsilon_i \neq \pm\epsilon_j$ and $\epsilon_i \neq 0$ for all $i \neq j$. It is similar for μ.

Lemma 2.10.2. *For non-regular ϵ and μ,*

$$\mathcal{A}_{\mathrm{W_G}}(q^{-\chi\cdot\epsilon}) = 0, \qquad \mathcal{A}_{\mathrm{W_M}}(q^{-\xi\cdot\mu}) = 0.$$

Let $\rho_1 = (n - \frac{1}{2}, n - \frac{3}{2}, ..., \frac{1}{2})$ and $\rho_2 = (m, m-1, ..., 1)$, and let

$$\mathrm{P}(\chi) = q^{-\chi\cdot\rho_1}, \qquad \mathrm{Q}(\xi) = q^{-\xi\cdot\rho_2}.$$

Note that ρ_1 is the half sum of positive roots in $\hat{\mathrm{G}} = SO_{2n+1}(\mathbb{C})$, and ρ_2 the half sum of positive roots in $Sp_{2m}(\mathbb{C})$, and that $\mathrm{W_G} = \mathrm{W_{\hat{G}}}$. So by a well-known identity (see, for example Lemma 24.3 of [11]),

$$\mathbf{d}(\chi) = \frac{(-1)^n}{\mathrm{P}(\chi)\mathcal{A}_{\mathrm{W_G}}(\mathrm{P}(\chi))}, \quad \mathbf{d}'(\xi) = \frac{(-1)^m}{\mathrm{Q}(\xi)\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))}.$$

From which we know that

$$\mathbf{d}(w\chi) = \frac{(-1)^n sgn(w)}{\mathrm{P}(w\chi)\mathcal{A}_{\mathrm{W_G}}(\mathrm{P}(\chi))}, \quad \mathbf{d}'(w'\xi) = \frac{(-1)^m sgn(w')}{\mathrm{Q}(w'\xi)\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))}.$$

So we have

$$\mathbf{C} = (-1)^{m+n} \frac{(\mathcal{A}_{\mathrm{W_G}} \circ \mathcal{A}_{\mathrm{W_M}})(\mathbf{b}(\chi, \xi)\mathrm{P}(\chi^{-1})\mathrm{Q}(\xi^{-1}))}{\mathcal{A}_{\mathrm{W_G}}(\mathrm{P}(\chi))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))}. \tag{2.59}$$

Note that $sgn(w_0^{\mathrm{G}})sgn(w_0^{\mathrm{M}}) = (-1)^{m+n}$, $w_0^{\mathrm{G}}(\chi) = \chi^{-1}$ and $w_0^{\mathrm{M}}(\xi) = \xi^{-1}$. So

$$\mathbf{C} = \frac{(\mathcal{A}_{\mathrm{W_G}} \circ \mathcal{A}_{\mathrm{W_M}})(\mathbf{b}(\chi^{-1}, \xi^{-1})\mathrm{P}(\chi)\mathrm{Q}(\xi))}{\mathcal{A}_{\mathrm{W_G}}(\mathrm{P}(\chi))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))}.$$

To calculate $\mathbf{C}$, we need to simplify $(\mathcal{A}_{\mathrm{W_G}} \circ \mathcal{A}_{\mathrm{W_M}})(\mathbf{b}(\chi^{-1}, \xi^{-1})\mathrm{P}(\chi)\mathrm{Q}(\xi))$.

Lemma 2.10.3. *Let*

$$A(\chi,\xi)=\frac{\mathbf{b}(\chi^{-1},\xi^{-1})\mathrm{P}(\chi)\mathrm{Q}(\xi)}{\prod_{i=1}^{n-m}\prod_{j=1}^{m}\zeta^{-1}(-\chi_i+\xi_j+\frac{1}{2})\zeta^{-1}(-\chi_i-\xi_j+\frac{1}{2})}$$

then

$$(\mathcal{A}_{\mathrm{W_G}}\circ\mathcal{A}_{\mathrm{W_M}})(A(\chi,\xi))=(\mathcal{A}_{\mathrm{W_G}}\circ\mathcal{A}_{\mathrm{W_M}})(\mathbf{b}(\chi^{-1},\xi^{-1})\mathrm{P}(\chi)\mathrm{Q}(\xi)),$$

and hence

$$\mathbf{C}=\frac{(\mathcal{A}_{\mathrm{W_G}}\circ\mathcal{A}_{\mathrm{W_M}})(A(\chi,\xi))}{\mathcal{A}_{\mathrm{W_G}}(\mathrm{P}(\chi))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))}.$$

Proof. $\mathbf{b}(\chi^{-1},\xi^{-1})\mathrm{P}(\chi)\mathrm{Q}(\xi)$ equals

$$\prod_{i<j+n-m}\zeta^{-1}(-\chi_i+\xi_j+\frac{1}{2})\cdot\prod_{i>j+n-m}\zeta^{-1}(+\chi_i-\xi_j+\frac{1}{2})$$
$$\prod_{1\le j\le m}\zeta^{-1}(-\xi_j+\frac{1}{2})\cdot\prod_{\substack{1\le i\le n\\1\le j\le m}}\zeta^{-1}(-\chi_i-\xi_j+\frac{1}{2})\cdot q^{-(\chi\cdot\rho_1+\xi\cdot\rho_2)}.$$

Let

$$\mathbf{b}(\chi^{-1},\xi^{-1})\mathrm{P}(\chi)\mathrm{Q}(\xi)=\sum_{\epsilon\in\mathbb{Z}^n,\mu\in\mathbb{Z}^m}c_{\epsilon,\mu}\cdot q^{-(\chi\cdot\epsilon+\xi\cdot\mu)},$$

then by lemma 2.10.2,

$$(\mathcal{A}_{\mathrm{W_G}}\circ\mathcal{A}_{\mathrm{W_M}})(\mathbf{b}(\chi^{-1},\xi^{-1})\mathrm{P}(\chi)\mathrm{Q}(\xi))=\sum_{\epsilon,\mu\text{ regular}}c_{\epsilon,\mu}\cdot\mathcal{A}_{\mathrm{W_G}}(q^{-\chi\cdot\epsilon})\mathcal{A}_{\mathrm{W_M}}(q^{-\xi\cdot\mu}).$$

By direct calculation, if $c_{\epsilon,\mu}\neq 0$,

$$\frac{1}{2}+(n-i)-2m\le\epsilon_i\le\frac{1}{2}+(n-i)\qquad\text{when }1\le i\le n-m,$$
$$\frac{1}{2}-m\le\epsilon_i\le m-\frac{1}{2}\qquad\text{when }n-m+1\le i\le n.$$

Suppose ϵ is regular, then for $n-m+1\le i\le n$, $\{|\epsilon_i|\}$ is a permutation of $\{\frac{1}{2},\frac{3}{2},...,m-\frac{1}{2}\}$. This implies $\epsilon_{n-m}=\frac{1}{2}+m$, which then implies $\epsilon_{n-m-1}=\frac{1}{2}+(m+1)$, and so on. So $\epsilon_i=\frac{1}{2}+n-i$ for $1\le i\le n-m$. In other words, for $1\le i\le n-m$,

ϵ_i attains its upper bound. This implies that $(\mathcal{A}_{\mathrm{W_G}} \circ \mathcal{A}_{\mathrm{W_M}})(\mathbf{b}(\chi^{-1}, \xi^{-1})\mathrm{P}(\chi)\mathrm{Q}(\xi))$ remains the same if we divide $\prod_{i=1}^{n-m}\prod_{j=1}^{m}\zeta^{-1}(-\chi_i+\xi_j+\frac{1}{2})\zeta^{-1}(-\chi_i-\xi_j+\frac{1}{2})$ from $\mathbf{b}(\chi^{-1}, \xi^{-1})\mathrm{P}(\chi)\mathrm{Q}(\xi)$, and our lemma follows. $\square$

Let $\tilde{\chi} = (\tilde{\chi}_1, \dots, \tilde{\chi}_m) \in \mathbb{C}^m$ with $\tilde{\chi}_i = \chi_{n-m+i}$. Let

$$\begin{aligned}\tilde{\mathrm{A}}(\tilde{\chi}, \xi) =& \frac{A(\chi, \xi)}{\prod_{i=1}^{n-m} q^{-(\frac{1}{2}+(n-i))\chi_i}} \\ =& \prod_{i<j}\zeta^{-1}(-\tilde{\chi}_i + \xi_j + \frac{1}{2})\prod_{i>j}\zeta^{-1}(\tilde{\chi}_i - \xi_j + \frac{1}{2}) \\ &\cdot \prod_{1\le j\le m}\zeta^{-1}(-\xi_j + \frac{1}{2})\prod_{1\le i,j\le m}\zeta^{-1}(-\tilde{\chi}_i - \xi_j + \frac{1}{2})q^{-(\tilde{\chi}\cdot\tilde{\rho}_1 + \xi\cdot\rho_2)},\end{aligned} \tag{2.60}$$

where $\tilde{\rho}_1 = (m - \frac{1}{2}, n - \frac{3}{2}, ..., \frac{1}{2})$.

Lemma 2.10.4. *Let* $\tilde{\mathrm{P}}(\tilde{\chi}) = \tilde{\chi}^{\tilde{\rho}_1}$, *and let*

$$(\mathcal{A}_{\mathrm{W_M}} \circ \mathcal{A}_{\mathrm{W_M}})(\tilde{\mathrm{A}}(\tilde{\chi}, \xi)) = \sum_{w_1, w_2 \in \mathrm{W_M}} sgn(w_1)sgn(w_2)\tilde{\mathrm{A}}(w_1\tilde{\chi}, w_2\xi),$$

then

$$\mathbf{C} = \frac{(\mathcal{A}_{\mathrm{W_M}} \circ \mathcal{A}_{\mathrm{W_M}})(\tilde{\mathrm{A}}(\tilde{\chi}, \xi))}{\mathcal{A}_{\mathrm{W_M}}(\tilde{\mathrm{P}}(\tilde{\chi}))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))}.$$

Moreover, the value of $\mathbf{C}$ *is independent of* (χ, ξ).

Proof. Let $A(\chi, \xi) = \sum_{\epsilon\in\mathbb{Z}^n, \mu\in\mathbb{Z}^m} d_{\epsilon,\mu}q^{-(\chi\cdot\epsilon+\xi\cdot\mu)}$. Then by the discussion in lemma 2.10.3, for regular (ϵ, μ) with $d_{\epsilon,\mu} \neq 0$,

$$\epsilon_i = \frac{1}{2} + n - i \quad \text{ for } 1 \le i \le n - m,$$

and $\{|\epsilon_{n-m+1}|, \dots, |\epsilon_n|\}$ is a permutation of $\{\frac{1}{2}, \frac{3}{2}, \dots, m - \frac{1}{2}\}$. So there exists $w_1 \in \mathrm{W_M}$ so that $\epsilon = w_1\rho_1$. If we let $\epsilon' = (\epsilon_{n-m+1}, \dots, \epsilon_n) \in \mathbb{Z}^m$, then $\epsilon' = w_1\tilde{\rho}_1$, and

$$\epsilon = (n - \frac{1}{2}, n - \frac{3}{2}, \dots, m + \frac{1}{2}; \epsilon'),$$

and hence

$$\tilde{\mathrm{A}}(\tilde{\chi},\xi)=\sum_{\epsilon,\mu} d_{\epsilon,\mu} q^{-(\tilde{\chi}\cdot\epsilon'+\xi\cdot\mu)}.$$

By a similar consideration on μ we have

$$\{|\mu_1|,\ldots,|\mu_m|\} \text{ is a permutation of } \{1,\ldots,m\}.$$

So $\mu = w_2\rho_2$ for some $w_2 \in \mathrm{W_M}$. Since ϵ,μ are regular, w_1, w_2 are uniquely determined by ϵ,μ. Let $d_{w_1,w_2} = d_{\epsilon,\mu}$, then

$$\begin{aligned}
(\mathcal{A}_{\mathrm{W_G}}\circ\mathcal{A}_{\mathrm{W_M}})(A(\chi,\xi)) &= \sum_{\epsilon,\mu \text{ regular}} d_{\epsilon,\mu}\cdot(\mathcal{A}_{\mathrm{W_G}}\circ\mathcal{A}_{\mathrm{W_M}})(q^{-(\chi\cdot\epsilon+\xi\cdot\mu)}) \\
=&\sum_{w_1,w_2\in\mathrm{M}} d_{w_1,w_2}\cdot(\mathcal{A}_{\mathrm{W_G}}\circ\mathcal{A}_{\mathrm{W_M}})\left(q^{-(\chi\cdot(w_1\rho_1)+\xi\cdot(w_2\rho_2))}\right) \\
=&\mathcal{A}_{\mathrm{W_G}}(\mathrm{P}(\chi))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))\sum_{w_1,w_2\in\mathrm{M}} d_{w_1,w_2} sgn(w_1)sgn(w_2).
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{A}_{\mathrm{W_M}}\circ\mathcal{A}_{\mathrm{W_M}})(\tilde{\mathrm{A}}(\tilde{\chi},\xi)) &= \sum_{\epsilon,\mu \text{ regular}} d_{\epsilon,\mu}(\mathcal{A}_{\mathrm{W_M}}\circ\mathcal{A}_{\mathrm{W_M}})(q^{-(\tilde{\chi}\cdot\epsilon'+\xi\cdot\mu)}) \\
=&\sum_{w_1,w_2\in\mathrm{M}} d_{w_1,w_2}(\mathcal{A}_{\mathrm{W_M}}\circ\mathcal{A}_{\mathrm{W_M}})\left(q^{-(\tilde{\chi}\cdot(w_1\tilde{\rho}_1)+\xi\cdot(w_2\rho_2))}\right) \\
=&\mathcal{A}_{\mathrm{W_M}}(\tilde{\mathrm{P}}(\tilde{\chi}))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))\sum_{w_1,w_2\in\mathrm{M}} d_{w_1,w_2} sgn(w_1)sgn(w_2).
\end{aligned}$$

So

$$\begin{aligned}
\mathbf{C} =& \frac{(\mathcal{A}_{\mathrm{W_G}}\circ\mathcal{A}_{\mathrm{W_M}})(A(\chi,\xi))}{\mathcal{A}_{\mathrm{W_G}}(\mathrm{P}(\chi))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))} \\
=& \sum_{w_1,w_2\in\mathrm{M}} d_{w_1,w_2} sgn(w_1)sgn(w_2) \\
=& \frac{(\mathcal{A}_{\mathrm{W_M}}\circ\mathcal{A}_{\mathrm{W_M}})(\tilde{\mathrm{A}}(\tilde{\chi},\xi))}{\mathcal{A}_{\mathrm{W_M}}(\tilde{\mathrm{P}}(\tilde{\chi}))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))},
\end{aligned}$$

completing our proof since d_{w_1,w_2} is independent of (χ,ξ). $\square$

So to find the value of **C** one only needs to find the value of

$$\frac{(\mathcal{A}_{\mathrm{W_M}} \circ \mathcal{A}_{\mathrm{W_M}})(\tilde{\mathrm{A}}(\tilde{\chi}, \xi))}{\mathcal{A}_{\mathrm{W_M}}(\tilde{\mathrm{P}}(\tilde{\chi}))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))}$$

for a special (χ, ξ). From now on we let $\tilde{\chi}_i = -(m+1-i)$ and $\xi_j = -(m+\frac{1}{2}-j)$, then we have the following lemma

Lemma 2.10.5. *If* $\tilde{\mathrm{A}}(w\tilde{\chi}, w'\xi) \neq 0$*, then* $w = w' = e$*.*

Proof. We let $\tilde{\chi}_{-i} = -\tilde{\chi}_i$ and $\xi_{-j} = -\xi_j$. For any Weyl element $w, w' \in W_{\mathrm{M}}$, there exists σ, σ' which are permutations of the set $\{\pm 1, \pm 2, \ldots, \pm m\}$ such that

1. $\sigma(-i) = -\sigma(i)$, $\sigma'(-j) = -\sigma'(j)$.

2. $(w\tilde{\chi})_i = \tilde{\chi}_{\sigma(i)}$, $(w'\xi)_j = \xi_{\sigma'(j)}$.

So $w = w' = e$ if and only if $\sigma = \sigma' = \mathrm{id}$. Suppose $\tilde{\mathrm{A}}(w\tilde{\chi}, w'\xi) \neq 0$. Since $\zeta^{-1}(0) = 0$, σ and σ' should satisfy the following properties:

1. For any $i < j$, $\tilde{\chi}_{\sigma(i)} - \xi_{\sigma'(j)} \neq \frac{1}{2}$.

2. For any $i > j$, $\tilde{\chi}_{\sigma(i)} - \xi_{\sigma'(j)} \neq -\frac{1}{2}$.

3. For any $1 \leq j \leq m$, $\xi_{\sigma'(j)} \neq \frac{1}{2}$.

4. For any $1 \leq i, j \leq m$, $\tilde{\chi}_{\sigma(i)} + \xi_{\sigma'(j)} \neq \frac{1}{2}$.

These four conditions actually imply $\sigma = \sigma' = \mathrm{id}$. To see this we consider $A = \{\tilde{\chi}_{\sigma(i)}, 1 \leq i \leq m\}$ and $B = \{\xi_{\sigma'(j)}, 1 \leq j \leq m\}$. Note that $\{|\tilde{\chi}_{\sigma(1)}|, \ldots, |\tilde{\chi}_{\sigma(m)}|\} = \{1, 2, \ldots, m\}$ and $\{|\xi_{\sigma(1)}|, \ldots, |\xi_{\sigma(m)}|\} = \{\frac{1}{2}, \frac{3}{2}, \ldots, (m-\frac{1}{2})\}$. By property (3), $\frac{1}{2} \notin B$, which implies $-\frac{1}{2} \in B$. Then by property (4), $1 \notin A$. So then $-1 \in A$. Then again by property (4), $\frac{3}{2} \notin B$, so then $-\frac{3}{2} \in B$. And then by property (4), $2 \notin A$, so $-2 \in A$. Continuing this process, we will eventually have $A = \{-m, -(m-1), ..., -1\}$ and $B = \{-m+\frac{1}{2}, -m+\frac{3}{2}, ..., -\frac{1}{2}\}$. So σ and σ' are actually permutations of $\{1, 2, ..., m\}$. Note that $\tilde{\chi}_k - \xi_k = -\frac{1}{2}$, so by property (2), $\sigma(i) \neq \sigma'(j)$ for any $i > j$, which implies that

$$\sigma^{-1}(k) \leq (\sigma')^{-1}(k) \tag{2.61}$$

when $1 \leq k \leq m$. On the other hand, since $\tilde{\chi}_{k+1} - \xi_k = \frac{1}{2}$, we have

$$\sigma^{-1}(k+1) \geq (\sigma')^{-1}(k). \tag{2.62}$$

by property (1). Combining equation (2.61) and (2.62), we have

$$(\sigma')^{-1}(m) \geq \sigma^{-1}(m) \geq (\sigma')^{-1}(m-1) \geq \sigma^{-1}(m-1) \geq ... \geq (\sigma')^{-1}(1) \geq \sigma^{-1}(1),$$

which implies that $\sigma = \sigma' = \mathrm{id}$. □

By this lemma,

$$\mathbf{C} = \frac{\tilde{\mathrm{A}}(\tilde{\chi}, \xi)}{\mathcal{A}_{\mathrm{W_M}}(\tilde{\mathrm{P}}(\tilde{\chi}))\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi))}$$

with $\tilde{\chi}_i = -(n+1-i)$ and $\xi_j = -(m + \frac{1}{2} - j)$. By a well-known identity (see, for example Lemma 24.3 of [11]),

$$\mathcal{A}_{\mathrm{W_M}}(\tilde{\mathrm{P}}(\tilde{\chi})) = \prod_{1 \leq a < b \leq m} \zeta^{-1}(-\tilde{\chi}_a + \tilde{\chi}_b)\zeta^{-1}(-\tilde{\chi}_a - \tilde{\chi}_b) \prod_{i=1}^{n} \zeta^{-1}(-\tilde{\chi}_i)\tilde{\mathrm{P}}(\tilde{\chi});$$
$$\mathcal{A}_{\mathrm{W_M}}(\mathrm{Q}(\xi)) = \prod_{1 \leq a < b \leq m} \zeta^{-1}(-\xi_a + \xi_b)\zeta^{-1}(-\xi_a - \xi_b) \prod_{j=1}^{m} \zeta^{-1}(-2\xi_j)\mathrm{Q}(\xi).$$

By direct calculation we have

$$\mathbf{C} = \zeta(1)^m \prod_{i=1}^{m} \zeta^{-1}(2i).$$

2.11 Uniqueness of the Whittaker Shintani function

In this section we prove the following theorem.

Theorem 2.11.1. *Let* $\mathrm{W}_{\chi,\xi,\psi}$ *be a Whittaker Shintani function on* G. *Let* $\mathcal{W}(\mathbf{d},\mathbf{f}) = \mathrm{W}_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\mathbf{f}})$ *with* $\mathbf{d} \in \Lambda_m^+$ *and* $\mathbf{f} \in \Lambda_n^+$. *If* $\mathcal{W}(\mathbf{0},\mathbf{0}) = 0$, *then* $\mathcal{W}(\mathbf{d},\mathbf{f}) = 0$ *for every* $\mathbf{d} \in \Lambda_m^+$ *and* $\mathbf{f} \in \Lambda_n^+$.

Combining this with theorem 2.6.1 we know that for all (χ,ξ), the space of Whittaker Shintani function $\mathcal{WS}_{\chi,\xi,\psi}$ is at most one dimensional. The method of proof is from [19] and [24]. First we define an order on $\Lambda_n^+ \times \Lambda_m^+$ as

Definition 2.11.2. *Let* ϖ_k, ϖ'_l *be the dominant weights of* G *and* M. *For any* $(\mathbf{d}',\mathbf{f}'),(\mathbf{d},\mathbf{f}) \in \Lambda_m^+ \times \Lambda_n^+$, *we write* $(\mathbf{d},\mathbf{f}) \geq_{\mathcal{WS}} (\mathbf{d}',\mathbf{f}')$ *if*

1. $\varpi_l \cdot \mathbf{f} \geq \varpi_l \cdot \mathbf{f}'$ *for* $1 \leq l \leq n-m$,

2. $\varpi_l \cdot \mathbf{f} + \varpi'_{l-(n-m)} \cdot \mathbf{d} \geq \varpi_l \cdot \mathbf{f}' + \varpi'_{l-(n-m)} \cdot \mathbf{d}'$ *for* $n-m+1 \leq l \leq n$,

3. $\varpi_{n-m+l-1} \cdot \mathbf{f} + \varpi'_l \cdot \mathbf{d} \geq \varpi_{n-m+l-1} \cdot \mathbf{f}' + \varpi'_l \cdot \mathbf{d}'$ *for* $1 \leq l \leq m$,

Here "$\cdot$" are the standard dot products in $\mathbb{Z}^n$ *and* $\mathbb{Z}^m$.

Then we have the following lemma

Lemma 2.11.3. *Suppose* $(\mathbf{d},\mathbf{f}) \in \Lambda_m^+ \times \Lambda_n^+$.

1. *If* $(\mathbf{d}',\mathbf{f}') \in \Lambda_m^+ \times \Lambda_n^+$ *satisfies*

$$\mathrm{K}_{\mathrm{M}^{\mathrm{J}}}\mathfrak{p}^{\mathbf{d}}\mathrm{K}_{\mathrm{G}}\mathfrak{p}^{\mathbf{f}}\mathrm{K}_{\mathrm{G}} \cap \mathrm{ZUK}_{\mathrm{M}^{\mathrm{J}}}\mathfrak{p}^{\mathbf{d}'}\lambda\mathfrak{p}^{\mathbf{f}'}\mathrm{K}_{\mathrm{G}} \neq \emptyset,$$

   then $(\mathbf{d},\mathbf{f}) \geq_{\mathcal{WS}} (\mathbf{d}',\mathbf{f}')$.

2. *If* $u \in \mathrm{U}$ *and* $z \in \mathrm{Z}$ *satisfies*

$$\mathrm{K}_{\mathrm{M}^{\mathrm{J}}}\mathfrak{p}^{\mathbf{d}}\mathrm{K}_{\mathrm{G}}\mathfrak{p}^{\mathbf{f}}\mathrm{K}_{\mathrm{G}} \cap zu\mathrm{K}_{\mathrm{M}^{\mathrm{J}}}\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\mathbf{f}}\mathrm{K}_{\mathrm{G}} \neq \emptyset,$$

then $\psi_{\mathrm{U}}(u) = 1$ *and* $\psi(z) = 1$.

Before proving the lemma 2.11.3, we first show it implies theorem 2.11.1.

Proof of theorem 2.11.1 assuming lemma 2.11.3. Consider $\int_{\mathrm{K}_{\mathrm{M^J}} \mathfrak{p}^{\mathbf{d}} \mathrm{K}_{\mathrm{G}} \mathfrak{p}^{\mathbf{f}} \mathrm{K}_{\mathrm{G}}} \mathrm{d}g \mathrm{W}_{\chi,\xi,\psi}(g)$. By lemma 2.11.3 and theorem 2.6.1 we have

$$\begin{aligned} &\omega_\chi(\mathrm{Ch}_{\mathrm{K}_{\mathrm{G}} \mathfrak{p}^{\mathbf{f}} \mathrm{K}_{\mathrm{G}}}) \omega_\xi(\mathrm{Ch}_{\mathrm{K}_{\mathrm{M^J}} \mathfrak{p}^{\mathbf{d}} \mathrm{K}_{\mathrm{M^J}}}) \mathcal{W}(\mathbf{0}, \mathbf{0}) \\ = &\sum_{(\mathbf{d}', \mathbf{f}') \le_{\mathcal{WS}} (\mathbf{d}, \mathbf{f}), (\mathbf{d}', \mathbf{f}') \in \Lambda_m^+ \times \Lambda_n^+} C^{\mathbf{d}, \mathbf{f}}_{\mathbf{d}', \mathbf{f}'} \mathcal{W}(\mathbf{d}', \mathbf{f}'), \end{aligned} \tag{2.63}$$

where $C^{\mathbf{d}, \mathbf{f}}_{\mathbf{d}', \mathbf{f}'}$ is positive by the second part of the lemma 2.11.3. So if $\mathcal{W}(\mathbf{0}, \mathbf{0}) = 0$, we have

$$\sum_{(\mathbf{d}', \mathbf{f}') \le_{\mathcal{WS}} (\mathbf{d}, \mathbf{f}), (\mathbf{d}', \mathbf{f}') \in \Lambda_m^+ \times \Lambda_n^+} C^{\mathbf{d}, \mathbf{f}}_{\mathbf{d}', \mathbf{f}'} \mathcal{W}(\mathbf{d}', \mathbf{f}') = 0$$

for all $(\mathbf{d}, \mathbf{f}) \in \Lambda_m^+ \times \Lambda_n^+$. Taking the induction on $(\mathbf{d}, \mathbf{f})$ with respect to the order $\le_{\mathcal{WS}}$ we have

$$\mathcal{W}(\mathbf{d}, \mathbf{f}) = 0$$

for all $(\mathbf{d}, \mathbf{f}) \in \Lambda_m^+ \times \Lambda_n^+$, completing our proof. $\square$

So in the rest of this section we only need to prove lemma 2.11.3. To prove the first part of lemma 2.11.3, we need the following lemma.

Lemma 2.11.4. *Let* $\mathcal{N}_{2n} = \{1, 2, \ldots, 2n\}$, *and let* $g, g^1, g^2, g^3 \in \mathrm{G}$. *For* $I = (i_1, \ldots, i_k)$, $J = (j_1, \ldots, j_k) \in (\mathcal{N}_{2n})^k$, *we denote* $f_{I,J}(g) = \prod_{s=1}^k g_{i_s, j_s}$, *and*

$$\Delta_{I,J}(g) = \det(g_{I,J}) = \sum_{\sigma \in S_k} sgn(\sigma) \prod_{s=1}^k g_{i_s, j_{\sigma(s)}}.$$

If $g = g^1 g^2 g^3$, *then we have*

$$\Delta_{I,J}(g) = \sum_{A, C \in \mathcal{N}_{2n}^k} f_{I,A}(g^1) \Delta_{A,C}(g^2) f_{C,J}(g^3). \tag{2.64}$$

Proof. Since $g = g^1 g^2 g^3$, we have

$$g_{i,j} = \sum_{a,b} g^1_{i,a} g^2_{a,b} g^3_{b,j}, \tag{2.65}$$

where a, b runs over $\mathcal{N}^k_{2n}$. So

$$\Delta_{I,J}(g) = \sum_{\sigma \in S_k} sgn(\sigma) \sum_{(a_1,\ldots,a_k),(b_1,\ldots,b_k)\in(\mathcal{N}_{2n})^k} \prod_{s=1}^{k} g^1_{i_s,a_s} g^2_{a_s,b_s} g^3_{b_s,j_{\sigma(s)}}. \tag{2.66}$$

Note that S_k acts on $(\mathcal{N}_{2n})^k$. If we define $c_s = b_{\sigma^{-1}(s)}$, then

$$\begin{aligned}
&\sum_{(a_1,\ldots,a_k),(b_1,\ldots,b_k)\in(\mathcal{N}_{2n})^k} \prod_{s=1}^{k} g^1_{i_s,a_s} g^2_{a_s,b_s} g^3_{b_s,j_{\sigma(s)}} \\
= &\sum_{(a_1,\ldots,a_k),(c_1,\ldots,c_k)\in(\mathcal{N}_{2n})^k} \prod_{s=1}^{k} g^1_{i_s,a_s} g^2_{a_s,c_{\sigma(s)}} g^3_{c_{\sigma(s)},j_{\sigma(s)}}.
\end{aligned}$$

Note that for any $\sigma \in S_k$,

$$\prod_{s=1}^{k} g^3_{c_{\sigma(s)},j_{\sigma(s)}} = \prod_{s=1}^{k} g^3_{c_s,j_s}.$$

So (2.66) is equal to

$$\begin{aligned}
&\sum_{(a_1,\ldots,a_k),(c_1,\ldots,c_k)\in(\mathcal{N}_{2n})^k} \prod_{s=1}^{k} \left(g^1_{i_s,a_s} g^3_{c_s,j_s}\right) \cdot \sum_{\sigma\in S_k} sgn(\sigma) \prod_{s=1}^{k} g^2_{a_s,c_{\sigma(s)}} \\
= &\sum_{(a_1,\ldots,a_k),(c_1,\ldots,c_k)\in(\mathcal{N}_{2n})^k} \prod_{s=1}^{k} \left(g^1_{i_s,a_s} g^3_{c_s,j_s}\right) \cdot \Delta_{(a_1,\ldots,a_k),(c_1,\ldots,c_k)}(g^2).
\end{aligned}$$

which is the formula we want to prove. □

By this lemma we can prove the first part of lemma 2.11.3.

Proof of first part of lemma 2.11.3. Suppose

$$\mathrm{K}_{\mathrm{M}^{\mathrm{J}}}\mathfrak{p}^{\mathbf{d}}\mathrm{K}_{\mathrm{G}}\mathfrak{p}^{\mathbf{f}}\mathrm{K}_{\mathrm{G}} \cap \mathrm{ZUK}_{\mathrm{M}^{\mathrm{J}}}\mathfrak{p}^{\mathbf{d}'}\lambda\mathfrak{p}^{\mathbf{f}'}\mathrm{K}_{\mathrm{G}} \neq \emptyset,$$

then $\mathfrak{p}^{\mathbf{f}'}w_0^{\mathrm{G}}\lambda\mathfrak{p}^{-\mathbf{d}'}uz = k_1\mathfrak{p}^{-\mathbf{f}}k_2\mathfrak{p}^{-\mathbf{d}}k$ for some $k_1, k_2 \in \mathrm{K}_{\mathrm{G}}$ and $k \in \mathrm{K}_{\mathrm{M}^{\mathrm{J}}}$. Apply α_l and β_l, which are defined in (2.14) and (2.16) to both sides of the equation. When $l \geq n-m+1$,

$$v(\alpha_l(\mathfrak{p}^{\mathbf{f}'}w_0^{\mathrm{G}}\lambda\mathfrak{p}^{-\mathbf{d}'}uz)) = -\langle\varpi_l, \mathbf{f}'\rangle - \langle\varpi'_{l-(n-m)}, \mathbf{d}'\rangle. \tag{2.67}$$

On the other hand, note that

$$\alpha_l(g) = \Delta_{I_l,J_l}(g). \tag{2.68}$$

where $I_l = (2n+1-l, 2n+1-(l-1), \ldots, 2n)$, and $J_l = (1, 2, \ldots, l)$. By (2.64), we have

$$\alpha_l(k_1\mathfrak{p}^{-\mathbf{f}}k_2\mathfrak{p}^{-\mathbf{d}}k) = \sum_{A,C\in(\mathcal{N}_{2n})^l} f_{I,A}(k_1)\Delta_{A,C}(\mathfrak{p}^{-\mathbf{f}}k_2\mathfrak{p}^{-\mathbf{d}})f_{C,J}(k).$$

Note that for $f_{I,A}(k_1)\Delta_{A,C}(\mathfrak{p}^{-\mathbf{f}}k_2\mathfrak{p}^{-\mathbf{d}})f_{C,J}(k) \neq 0$, both A and C should contain distinct coordinates. Moreover, note that $k \in \mathrm{K}_{\mathrm{M}^{\mathrm{J}}}$, so $f_{C,J}(k) \neq 0$ implies $c_j = j$ for all $1 \leq j \leq n-m$. Under these two restrictions it is not hard to see that

$$v(\Delta_{A,C}(\mathfrak{p}^{-\mathbf{f}}k_2\mathfrak{p}^{-\mathbf{d}})) \geq -\langle\varpi_l, \mathbf{f}\rangle - \langle\varpi'_{l-(n-m)}, \mathbf{d}\rangle.$$

So

$$v(\alpha_l(k_1\mathfrak{p}^{-\mathbf{f}}k_2\mathfrak{p}^{-\mathbf{d}}k)) \geq -\langle\varpi_l, \mathbf{f}\rangle - \langle\varpi'_{l-(n-m)}, \mathbf{d}\rangle. \tag{2.69}$$

Comparing (2.67) and (2.69) we have

$$\langle\varpi_l, \mathbf{f}'\rangle + \langle\varpi'_{l-(n-m)}, \mathbf{d}'\rangle \leq \langle\varpi_l, \mathbf{f}\rangle + \langle\varpi'_{l-(n-m)}, \mathbf{d}\rangle$$

for all $n-m+1 \leq l \leq n$. Similarly, if we apply α_l for $1 \leq l \leq n-m$ or β_l for $1 \leq l \leq m$, we will have

$$\langle \varpi_l, \mathbf{f}' \rangle \leq \langle \varpi_l, \mathbf{f} \rangle$$

for all $1 \leq l \leq n-m$ and

$$\langle \varpi_{n-m+l-1}, \mathbf{f}' \rangle + \langle \varpi_l', \mathbf{d}' \rangle \leq \langle \varpi_{m-m+l-1}, \mathbf{f} \rangle + \langle \varpi_l', \mathbf{d} \rangle$$

for all $1 \leq l \leq m$. So we have the first part of the lemma 2.11.3. $\square$

Next we prove the second part of the lemma 2.11.3.

Proof of second part of the lemma 2.11.3. Suppose

$$\mathfrak{p}^{\mathbf{d}} k^1 \mathfrak{p}^{\mathbf{f}} = zuk\mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{\mathbf{f}} k^2 \tag{2.70}$$

for some $u \in \mathrm{U}$, $z \in \mathrm{Z}$, $k \in \mathrm{K}_{\mathrm{M}^{\mathrm{J}}}$ and $k_1, k_2 \in \mathrm{K}_{\mathrm{G}}$. We need to show that $\psi(z) = \psi_{\mathrm{U}}(u) = 1$. Consider the element $r_{2n} = (0, 0, \ldots, 0, 1) \in F^{2n}$. Multiplying r_{2n} from the left to both sides of (2.70), we have

$$\mathbf{k}_{2n}^1 \mathfrak{p}^{\mathbf{f}} = \mathfrak{p}^{-f_1} \mathbf{k}_{2n}^2.$$

Here $\mathbf{k}_{2n}^1$, $\mathbf{k}_{2n}^2$ are the 2n-th row of k^1 and k^2. Suppose $\mathbf{k}_{2n}^1 = (k_{2n,1}^1, \ldots, k_{2n,2n}^1)$, then

$$\begin{aligned} &\mathbf{k}_{2n}^2 \\ =&(\mathfrak{p}^{f_1+f_1} k_{2n,1}^1, \mathfrak{p}^{f_1+f_2} k_{2n,2}^1, \ldots, \mathfrak{p}^{f_1+f_n} k_{2n,n}^1, \mathfrak{p}^{f_1-f_n} k_{2n,n+1}^1, \ldots, \mathfrak{p}^{f_1-f_2} k_{2n,2n-1}^1, k_{2n,2n}^1). \end{aligned} \tag{2.71}$$

Note that when $k^2 \in \mathrm{K}_{\mathrm{G}}$, each row of it is primitive, that is, it belongs to $\mathcal{O}^{2n}$, but not $(\mathfrak{p}\mathcal{O})^{2n}$. So suppose $f_1 = f_2 = \cdots f_k > f_{k+1} \geq \cdots \geq f_n$ for some k, then by (2.71), at least one element in $\{k_{2n,2n-k+1}^1, \ldots, k_{2n,2n}^1\}$ belongs to $\mathcal{O}^*$. Let's say it is $k_{2n,2n-i+1}^1$. Let w be an Weyl element of G transposing 1 and i, then $k^1 w^{-1}$

has element in $\mathcal{O}^*$ at the $(2n, 2n)$ position. Then, by the Bruhat decomposition of K $(mod\ \mathfrak{p})$, there exists $x_1, x_2, y_1, y_2 \in \mathcal{O}^{n-1}$, and $z_1, z_2 \in \mathcal{O}$, such that

$$k^1 w^{-1} = E_1(x_1, y_1, z_1) \begin{pmatrix} \epsilon & & \\ & k' & \\ & & \epsilon^{-1} \end{pmatrix} E_1(x_2, y_2, z_2)^{w_0^G},$$

where $\epsilon \in \mathcal{O}^*$, $k' \in \mathrm{K}_{Sp_{2n-2}}$, and

$$E_1(x, y, z) = \begin{pmatrix} 1 & x & y & z \\ & & & {}^t y \\ & I_{2n-2} & & -{}^t x \\ & & & 1 \end{pmatrix}$$

So (2.70) becomes

$$E_1(x_1, y_1, z_1)^{\mathfrak{p}^{\mathbf{d}}} \left(\mathfrak{p}^{\mathbf{d}} \begin{pmatrix} \epsilon & & \\ & k' & \\ & & \epsilon^{-1} \end{pmatrix} \mathfrak{p}^{\mathbf{f}} \right) \mathfrak{p}^{-\mathbf{f}} E_1(x_2, y_2, z_2)^{w_0^G} w \mathfrak{p}^{\mathbf{f}} = uzk\mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{\mathbf{f}} k^2.$$

By the definition of w, it commutes with $\mathfrak{p}^{\mathbf{f}}$, so $\mathfrak{p}^{-\mathbf{f}} E_1(x_2, y_2, z_2)^{w_0^G} w \mathfrak{p}^{\mathbf{f}} \in \mathrm{K_G}$. So we just need to show that

$$E_1(x_1, y_1, z_1)^{\mathfrak{p}^{\mathbf{d}}} \left(\mathfrak{p}^{\mathbf{d}} \begin{pmatrix} 1 & & \\ & k' & \\ & & 1 \end{pmatrix} \mathfrak{p}^{\mathbf{f}} \right) = uzk\mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{\mathbf{f}} k^2 \tag{2.72}$$

implies $\psi(z) = \psi_{\mathrm{U}}(u) = 1$. We prove this by induction on $n - m$.

When $n - m = 1$, U is trivial, and $E_1(x, y, z) = \mathrm{J}(x, y, z)$. By (2.72), we have $\mathfrak{p}^{\mathbf{f}} k^2 \mathfrak{p}^{-\mathbf{f}} \in \mathrm{M^J}$. So $k^2 \in \mathrm{K_{M^J}}$. Suppose $k^2 = n_1 k''$ where $n_1 \in \mathrm{J}^0$ and $k'' \in \mathrm{K_M}$, and suppose $\tilde{\mathbf{f}} = (f_2, \dots, f_n)$, then we have

$$\mathrm{J}(x_1, y_1, z_1)^{\mathfrak{p}^{\mathbf{d}}} \left(\mathfrak{p}^{\mathbf{d}} \begin{pmatrix} 1 & & \\ & k' & \\ & & 1 \end{pmatrix} \mathfrak{p}^{\tilde{\mathbf{f}}} \right) = zk\mathfrak{p}^{\mathbf{d}} \lambda n_1^{\mathfrak{p}^{\mathbf{f}}} \mathfrak{p}^{\tilde{\mathbf{f}}} k''$$

Note that now both sides belongs to M^{J}, we write both sides in the form of $\mathrm{J} \rtimes \mathrm{M}$. Then by comparing the J-part of both sides we have

$$\mathrm{J}(x_1, y_1, z_1)^{\mathfrak{p}^{\mathbf{d}}} = z(\lambda n_1^{\mathfrak{p}^{\mathbf{f}}})^{k\mathfrak{p}^{\mathbf{d}}}.$$

When $\mathbf{f} \in \Lambda_n^+$, $n_1^{\mathfrak{p}^{\mathbf{f}}} \in \mathrm{J}^0$. So we assume $\lambda n_1^{\mathfrak{p}^{\mathbf{f}}} = \mathrm{J}(x_3, y_3, z_3)$ with $x_3, y_3 \in \mathcal{O}^{n-1}$ and $z_3 \in \mathcal{O}$. Then we have

$$z \cdot \mathrm{Z}(z_3 - z_1) = \mathrm{J}(x_1, y_1, 0)^{\mathfrak{p}^{\mathbf{d}}} \mathrm{J}(-x_3, -y_3, 0)^{k\mathfrak{p}^{\mathbf{d}}}$$

When $\mathrm{J}(x_1, y_1, 0)^{\mathfrak{p}^{\mathbf{d}}} \mathrm{J}(-x_3, -y_3, 0)^{k\mathfrak{p}^{\mathbf{d}}} \in \mathrm{Z}$, we have

$$\mathrm{J}(-x_3, -y_3, 0)^{k\mathfrak{p}^{\mathbf{d}}} = \mathrm{J}(-x_1, -y_1, 0)^{\mathfrak{p}^{\mathbf{d}}}.$$

So $z \cdot \mathrm{Z}(z_3 - z_1) = x_1 \cdot {}^t y_1 \in \mathrm{Z}^0$. Since $z_1, z_3 \in \mathcal{O}$, so $z \in \mathrm{Z}^0$, and hence $\psi(z) = 1$, completing the proof for $n - m = 1$.

Assume the lemma is true for $n - m = r - 1$, and suppose now $n - m = r > 1$. Then $E_1(x_1, y_1, z_1)^{\mathfrak{p}^{\mathbf{d}}} \in \mathrm{U}$. Since $E_1(x_1, y_1, z_1) \in \mathrm{U}^0$ and $\mathfrak{p}^{\mathbf{d}}$ stabilizes ψ_{U}, we have $\psi_{\mathrm{U}}(E_1(x_1, y_1, z_1)^{\mathfrak{p}^{\mathbf{d}}}) = 1$. So from (2.72) reduces to show that

$$\mathfrak{p}^{\mathbf{d}} \begin{pmatrix} 1 & & \\ & k' & \\ & & 1 \end{pmatrix} \mathfrak{p}^{\mathbf{f}} = uzk\mathfrak{p}^{\mathbf{d}} \lambda \mathfrak{p}^{\mathbf{f}} k_2 \tag{2.73}$$

implies $\psi(z) = \psi_{\mathrm{U}}(u) = 1$. Let

$$\mathrm{G}' = \{g \in \mathrm{G} \mid g = \begin{pmatrix} 1 & * & * \\ & g' & * \\ & & 1 \end{pmatrix}\} \cong \mathrm{Sp}_{2n-2} \ltimes \mathcal{H}_{2n-1}.$$

Then by (2.73), $\mathfrak{p}^{\mathbf{f}} k_2 \mathfrak{p}^{-\mathbf{f}} \in \mathrm{G}'$, so $k_2 \in \mathrm{K}_{\mathrm{G}'}$. Let $k_2 = \tilde{u}k''$ where $\tilde{u} \in \mathcal{H}_{2n-1}$ and $k'' \in \mathrm{K}_{\mathrm{Sp}_{2n-2}}$. Then we have

$$\mathfrak{p}^{\mathbf{d}} \begin{pmatrix} 1 & & \\ & k' & \\ & & 1 \end{pmatrix} \mathfrak{p}^{\tilde{\mathbf{f}}} = uzk\mathfrak{p}^{\mathbf{d}}\lambda \cdot (\tilde{u})^{\mathfrak{p}^{\mathbf{f}}} \mathfrak{p}^{\tilde{\mathbf{f}}} k''$$

Suppose $u = u^1 u^2$ where $u^1 \in \mathcal{H}_{2n-1}$ and $u^2 \in \mathrm{Sp}_{2n-2}$. Now both sides belongs to G'. Write them in the form $\mathcal{H}_{2n-1}\mathrm{Sp}_{2n-2}$ we have

$$u^1 (\tilde{u})^{zk\mathfrak{p}^{\mathbf{d}}\mathfrak{p}^{\mathbf{f}}} = 1$$

and

$$u^2 zk\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\tilde{\mathbf{f}}}k'' = \mathfrak{p}^{\mathbf{d}} \begin{pmatrix} 1 & & \\ & k' & \\ & & 1 \end{pmatrix} \mathfrak{p}^{\tilde{\mathbf{f}}}.$$

Note that $\tilde{u} \in \mathrm{U}^0$, $\mathbf{f} \in \Lambda_n^+$, and $zk\mathfrak{p}^{\mathbf{d}} \in \mathrm{M}^J$ which stabilizes ψ_{U},we have

$$\psi_{\mathrm{U}}(u_1) = \psi_{\mathrm{U}}^{-1}(\tilde{u}^{zu\mathfrak{p}^{\mathbf{d}}\mathfrak{p}^{\mathbf{f}}}) = \psi_{\mathrm{U}}^{-1}(\tilde{u}^{\mathfrak{p}^{\mathbf{f}}}) = 1.$$

On the other hand, by inductive hypothesis, when

$$u^2 zk\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\tilde{\mathbf{f}}}k'' = \mathfrak{p}^{\mathbf{d}} \begin{pmatrix} 1 & & \\ & k' & \\ & & 1 \end{pmatrix} \mathfrak{p}^{\tilde{\mathbf{f}}},$$

we have $\psi_{\mathrm{U}}(u_2) = \psi(z) = 1$. So $\psi_{\mathrm{U}}(u) = \psi_{\mathrm{U}}(u_1 u_2) = 1$ and $\psi(z) = 1$, completing our proof for $n - m = r > 1$. $\square$

2.12 The formula for the normalized Whittaker-Shintani function

Let

$$\mathrm{W}^0_{\chi,\xi,\psi}(g) = \left[\frac{\mu_{\mathrm{G}}(\mathrm{I}_{\mathrm{G}})\mu_{\mathrm{M}^{\mathrm{J}}}(\bar{\mathrm{I}}_{\mathrm{M}})}{\mu_{\mathrm{N}^-_{\mathrm{G}}}(\mathrm{N}^{-,1}_{\mathrm{G}})\mu_{\mathrm{N}^-_{\mathrm{M}}}(\mathrm{N}^{-,1}_{\mathrm{M}})}\Gamma(\chi,\xi)\prod_{i=1}^{m}\zeta^{-1}(2i)\right]^{-1} l_{\chi,\xi,\psi}(R(g)\mathrm{F}^0_{\chi},\mathrm{F}^0_{\xi}).$$

Then $\mathrm{W}^0_{\chi,\xi,\psi}(g)$ satisfies all conditions in definition 2.2.4 as a rational function in (q^{χ}, q^{ξ}), and $\mathrm{W}^0_{\chi,\xi,\psi}(e) = 1$. Applying (2.63) and taking induction on $(\mathbf{d}, \mathbf{f})$ with respect to the order $\leq_{\mathcal{WS}}$, we can see that for any fixed g, $\mathrm{W}^0_{\chi,\xi,\psi}(g)$ is regular, i.e., $\mathrm{W}^0_{\chi,\xi,\psi}(g) \in \mathbb{C}[q^{\pm\chi_1},\ldots,q^{\pm\chi_n},q^{\pm\xi_1},\ldots,q^{\pm\xi_m}]$. So for *every* $(\chi,\xi) \in \mathbb{C}^n \times \mathbb{C}^m$, $\mathrm{W}^0_{\chi,\xi,\psi}$ is a normalized Whittaker-Shintani function.

Now we summarize our result.

Theorem 2.12.1. *For every $(\chi,\xi) \in \mathbb{C}^n \times \mathbb{C}^m$, the normalized Whittaker-Shintani function is given by*

$$\int_{\mathrm{X}^0} \mathrm{d}x \mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}} x \mathfrak{p}^{\mathbf{f}}) = \zeta(1)^{-m}\prod_{i=1}^{m}\zeta(2i)\cdot$$
$$\sum_{w\in W_{\mathrm{G}}, w'\in W_{\mathrm{M}}} b(w\chi, w'\xi)d(w\chi)d'(w'\xi)((w\chi)^{-1}\delta^{\frac{1}{2}})(\mathfrak{p}^{\mathbf{f}})((w'\xi)^{-1}\delta^{\frac{1}{2}})(\mathfrak{p}^{\mathbf{d}})$$

for $\mathbf{d} \in \Lambda^+_m$ and $\mathbf{f} \in \Lambda^+_n$. If we let $\mathcal{L}(\mathbf{d}',\mathbf{f}') = \int_{\mathrm{X}^0} \mathrm{d}x \mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}'} x \mathfrak{p}^{\mathbf{f}'})$, and let $S(\mathbf{d},\mathbf{f}) = \{\mathbf{d}' \mid \mathbf{d}' \in \Lambda^+_m, \mathbf{f}+\mathbf{d}-\mathbf{d}' \in \Lambda^+_n, \mathbf{d}' \leq \mathbf{d}\}$. Then for each $\mathbf{d}' \in S(\mathbf{d},\mathbf{f})$ there exists $a_{\mathbf{d},\mathbf{f}}(\mathbf{d}') \in \mathbb{R}$ independent of (χ,ξ,ψ), such that

$$\mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{d}}\lambda\mathfrak{p}^{\mathbf{f}}) = \sum_{\mathbf{d}'\in S(\mathbf{d},\mathbf{f})} a_{\mathbf{d},\mathbf{f}}(\mathbf{d}')\mathcal{L}(\mathbf{d}',\mathbf{f}+\mathbf{d}-\mathbf{d}'),$$

and that $a_{\mathbf{d},\mathbf{f}}(\mathbf{d}) > 0$*. In particular,*

$$\mathrm{W}^0_{\chi,\xi,\psi}(\mathfrak{p}^{\mathbf{f}}) = \mathcal{L}(\mathbf{0}, \mathbf{f}).$$

2.13 Application

Using the formula for the Shintani function, we can give an alternative proof of the **Theorem 6.1** in [24]. We rewrite the theorem below.

Theorem 2.13.1 (Theorem 6.1 in [24], conjectured by T.Shintani)**.** *Let* $\mathrm{G} = \mathrm{Sp}_{2n}$ *and* $\mathrm{M} = \mathrm{Sp}_{2m}$ *as defined in our paper, and suppose* $n = m+1$*. Let* π *and* $\tilde{\sigma}$ *be unramified representations of* $\mathrm{G}(F_v)$ *and* $\tilde{\mathrm{M}}(F_v)$ *respectively. Let* $z_\pi = (q^{-\chi_1}, \dots, q^{-\chi_{m+1}}, 1, q^{\chi_{m+1}}, \dots, q^{\chi_1})$ *be the Satake parameters of* π *and* $z_{\tilde{\sigma}} = (q^{-\xi_1}, \dots, q^{-\xi_m}, q^{\xi_m}, \dots, q^{\xi_1})$ *be the Satake parameters of* $\tilde{\sigma}$ *with respect to* ψ *so that* $\tilde{\sigma} \otimes \omega_\psi \cong \mathrm{Ind}^{\mathrm{M^J}}_{\mathrm{B_{M^J}}}(\xi, \psi)$*. Let* $\mathrm{W}^0_{\chi,\xi,\psi}$ *be the Whittaker-Shintani function as defined in this paper. Then we have*

$$\int_{\mathrm{GL}_1} \mathrm{W}^0_{\chi,\xi,\psi}\begin{pmatrix} t & & \\ & I_{2m} & \\ & & t^{-1}\end{pmatrix} |t|^{s-m-1}\,\mathrm{d}t = \frac{L(\pi, s)}{L_\psi(\tilde{\sigma}, s+\frac{1}{2})\zeta(2s)} \tag{2.74}$$

We denote by LHS and RHS the left hand side and right hand side of above respectively. First we have

Lemma 2.13.2. *Recall that* $\rho_1 = (n-\frac{1}{2}, n-\frac{3}{2}, \dots, \frac{1}{2})$*, where* $n = m+1$ *now, corresponding to the half sum of positive roots in* $SO_{2n+1} = SO_{2m+3}$*. Then*

$$\mathrm{LHS} = \sum_{l\geq 0} \frac{\mathcal{A}_{\mathrm{W_G}}(\prod_{1\leq j\leq m}(1-q^{\chi_1 \pm \xi_j - \frac{1}{2}})q^{-l\chi_1 - \langle \chi, \rho_1\rangle})}{\mathcal{A}_{\mathrm{W_G}}(q^{-\langle \chi, \rho_1\rangle})} \cdot q^{-ls}.$$

Proof. Since all the data are unramified, the integral on the left is actually a sum over $t = \mathfrak{p}^l$ with $l \in \mathbb{Z}$. By the discussion of section 2.6, the Whittaker-Shintani function vanishes unless $l \geq 0$. Substituting $\mathrm{W}^0_{\chi,\xi,\psi}$ by the formula developed in the previous sections, and note that $\delta_{\mathrm{B_G}}^{\frac{1}{2}}\begin{pmatrix} t & & \\ & I_{2m} & \\ & & t^{-1}\end{pmatrix} = |t|^{m+1}$, we have

$$\mathrm{LHS} = \mathbf{C}^{-1}\cdot\sum_{l\geq 0}\left[q^{-sl}\cdot\sum_{w\in W_{\mathrm{G}},w'\in W_{\mathrm{M}}} b(w\chi,w'\xi)d(w\chi)d'(w'\xi)((w\chi)^{-1})(\mathfrak{p}^{(l,0,\ldots,0)})\right].$$

Here $\mathbf{C}=\zeta(1)^m\prod_{i=1}^m\zeta^{-1}(2i)$. Recall that $\rho_2=(m,m-1,\ldots,1)$. Then

$$\begin{aligned} d(\chi) &= (-1)^{m+1}q^{\langle\chi,\rho_1\rangle}\mathcal{A}_{W_{\mathrm{G}}}^{-1}(q^{-\langle\chi,\rho_1\rangle}).\\ d'(\xi) &= (-1)^{m}q^{\langle\xi,\rho_2\rangle}\mathcal{A}_{W_{\mathrm{M}}}^{-1}(q^{-\langle\xi,\rho_2\rangle}).\end{aligned}$$

So

$$\begin{aligned}\mathrm{LHS} =&\frac{\mathbf{C}^{-1}}{\mathcal{A}_{W_{\mathrm{G}}}(q^{-\langle\chi,\rho_1\rangle})\mathcal{A}_{W_{\mathrm{M}}}(q^{-\langle\xi,\rho_2\rangle})}\\ &\cdot\sum_{l\geq 0}\left[q^{-ls}\sum_{w\in W_{\mathrm{G}},w'\in W_{\mathrm{M}}}(-1)^{2m+1}sgn(w)sgn(w')q^{l(w\chi)_1+\langle w\chi,\rho_1\rangle}q^{\langle w'\xi,\rho_2\rangle}b(w\chi,w'\xi)\right].\end{aligned} \tag{2.75}$$

In fact the summation over $\mathrm{W_G}\times\mathrm{W_M}$ is equal to, by a change of variable $w\mapsto ww_0^{\mathrm{G}}$ and $w'\mapsto w'w_0^{\mathrm{M}}$,

$$\sum_{w\in W_{\mathrm{G}},w'\in W_{\mathrm{M}}} sgn(w)sgn(w')q^{-l(w\chi)_1-\langle w\chi,\rho_1\rangle}q^{-\langle w'\xi,\rho_2\rangle}b(-w\chi,-w'\xi)$$

We can further simplify this summation by a similar discussion as in **Section** 2.10. By the definition of $b(\chi,\xi)$, we have

$$\begin{aligned}&q^{-l\chi_1+\langle w\chi,\rho_1\rangle}q^{-\langle w'\xi,\rho_2\rangle}b(-w\chi,-w'\xi)\\ =&\left[\prod_{j=1}^m(1-q^{\chi_1\pm\xi_j-\frac{1}{2}}q^{-(l+m+\frac{1}{2})\chi_1})\right]\left[\tilde{A}(\tilde{\chi},\xi)\right].\end{aligned}$$

Here $\tilde{A}(\tilde{\chi}, \xi)$ is defined in (2.60). Denote by $\circ$ the action of $\mathrm{W_G}$ (resp. $\mathrm{W_M}$) on functions $\mathbb{C}^n \to \mathbb{C}$ (resp. $\mathbb{C}^m \to \mathbb{C}$) induced by its action on $\mathbb{C}^n$ (resp. $\mathbb{C}^m$). Let W_1, W_2 be stabilizers of $\{e_2^*, \dots, e_n^*\}$ and e_1^* in $\mathrm{W_G}$ respectively, and let W_0 be a set of representatives in $(\mathrm{W}_1 \times \mathrm{W}_2)\backslash\mathrm{W_G}$. (Here e_i^* is the function $\chi \mapsto \chi_i$ for $1 \le i \le n$.) Then the first expression in brackets is invariant under $\mathrm{W}_2 \times \mathrm{W_M}$, and the second expression in brackets is invariant under W_1. So the summation over $\mathrm{W_G} \times \mathrm{W_M}$ is equal to

$$\sum_{w_0 \in \mathrm{W}_0, w_1 \in \mathrm{W}_1} sgn(w_0 w_1)[(w_1 w_0) \circ (\prod_{j=1}^{m} (1 - q^{\chi_1 \pm \xi_j - \frac{1}{2}} q^{-(l+m+\frac{1}{2})\chi_1}))]$$
$$\cdot [\sum_{w_2 \in \mathrm{W}_2, w_\mathrm{M} \in \mathrm{W_M}} sgn(w_2 w_\mathrm{M}) \tilde{A}(w_2 w_0 \tilde{\chi}, w_\mathrm{M}\xi)]$$

By **Lemma 2.10.4**, the second bracket is equal to

$$\mathbf{C} \cdot \sum_{w_2 \in \mathrm{W}_2, w_\mathrm{M} \in \mathrm{W_M}} sgn(w_2 w_\mathrm{M}) q^{-\langle w_2 w_0 \chi, \tilde{\rho}_1 \rangle} q^{-\langle w_\mathrm{M} \xi, \rho_2 \rangle},$$

where $\tilde{\rho}_1 = (0, m - \frac{1}{2}, m - \frac{3}{2}, \dots, \frac{1}{2})$. From this it is not hard to see that the summation over $\mathrm{W_G} \times \mathrm{W_M}$ in (2.75) is equal to

$$\mathbf{C} \cdot \left[\sum_{w \in \mathrm{W_G}} sgn(w) \cdot w \circ \left(\prod_{j=1}^{m} (1 - q^{\chi_1 \pm \xi_j - \frac{1}{2}}) q^{-l\chi_1 - \langle \chi, \rho_1 \rangle} \right) \right] \mathcal{A}_{\mathrm{W_M}}(q^{-\langle \xi, \rho_2 \rangle})$$

Substituting the formula to equation (2.75) we obtain our lemma. $\square$

Now we can prove **Theorem 2.13.1**.

*(Proof of **Theorem 2.13.1**).* By Weyl's character formula, for the representation of $SO_{2N+1}(\mathbb{C})$ whose highest weight is $\lambda = (\lambda_1, \dots, \lambda_N) \in \Lambda_N^+$, the trace of $x = diag(x_1, \dots, x_N, 1, x_N^{-1}, \dots, x_1^{-1})$ is $\tilde{\mathrm{T}}_N(\lambda; x) = \frac{\det(x_i^{\lambda_j + N - j + \frac{1}{2}} - x_i^{-(\lambda_j + N - j + \frac{1}{2})})}{\det(x_i^{N-j+\frac{1}{2}} - x_i^{-(N-j+\frac{1}{2})})}$. The function $\tilde{\mathrm{T}}_N(\lambda; x)$ is in fact defined for all $\lambda \in \mathbb{Z}^N$. For any set $A = \{a_1, \dots, a_N\}$, we let $\wedge^i(A) = \sum_{S \subset A, |S| = i} (\prod_{s \in S} a_s)$. Using these notation, we can express LHS as

$$\text{LHS} = \sum_{l\geq 0, r\in\{0,1,\dots,2m\}} (-1)^r \wedge^r (\Gamma_{\tilde{\sigma}}) \cdot \tilde{\mathrm{T}}_{m+1}((l-r,0,\dots,0);z_\pi)q^{-ls}. \tag{2.76}$$

Here $\Gamma_{\tilde{\sigma}}$ is the set $\{q^{\pm\xi_1-\frac{1}{2}},\dots,q^{\pm\xi_m-\frac{1}{2}}\}$. Next we consider RHS. By the discussion in [16, Theorem 3.1], we have

$$\frac{L(\pi,s)}{\zeta(2s)} = \sum_{a\geq 0} \tilde{\mathrm{T}}_{m+1}((a,0,\dots,0);z_\pi)q^{-as}.$$

So by the notation introduced above, we have

$$\text{RHS} = \sum_{a\geq 0, r\in\{0,1,\dots,2m\}^m} (-1)^r \wedge^r (\Gamma_{\tilde{\sigma}})\tilde{\mathrm{T}}_{m+1}((a,0,\dots,0);z_\pi)q^{-(a+r)s}. \tag{2.77}$$

To show that (2.76) equals (2.77), note that in (2.76) if $l < r$, then $l-r \in \{-1,\dots,-2m\}$ since $0 \leq r \leq 2m$. Then it is not hard to see that $\tilde{\mathrm{T}}_{m+1}(l-r,0,\dots,0) = 0$ by its definition. So one can replace the summation from $l \geq 0$ to $l \geq r$. Then by a change of the variable $l = a+r$ with $a \geq 0$ we have (2.76) equals (2.77). $\square$

Chapter 3

Local theory

3.1 Introduction

This part of the thesis is in the author's preprint [28].

In this chapter we will prove the following theorem.

Theorem 3.1.1. *The local integrals* $I_v(\varphi_{\pi_v}, \phi_v, \widetilde{f}_{W_\tau,\widetilde{\sigma},s,v})$ *as defined in* (1.9) *at the non-archimedean places satisfy the following conditions.*

(1) At the unramified places,

$$I_v(\varphi_{\pi_v}, \phi_v, \widetilde{f}_{W_\tau,\widetilde{\sigma},s,v}) = \frac{L(\pi_v \times \tau_v, s)}{L_{\psi_v}(\widetilde{\sigma}_v \times \tau_v, s+\frac{1}{2})L(\tau_v, sym^2, 2s)}. \qquad (3.1)$$

(2) It converges absolutely for $Re(s)$ *sufficiently large.*

(3) As a function of s*, it continues meromorphically to* $\mathbb{C}$ *as a rational function in* q^s*, where* q *is the cardinality of the residue field.*

(4) Given s*, there is a choice* $(\varphi_{\pi_v}, \phi_v, \tilde{f}_{W_\tau,\tilde{\sigma},s,v})$ *so that the local integral is nonzero.*

3.2 Notation

From now on everything we consider is local, so we drop the subscriptions v when there is no risk of confusion. Let k be an positive integer such that $k \leq n-m-1$. Recall that $\mathrm{G} = Sp_{2n}$. By abusing the notation we let w denote an Weyl element as well as one of its representatives in G as

$$w = \begin{pmatrix} & I_k & & & \\ I_{n-m-k} & & & & \\ & & I_{2m} & & \\ & & & & I_{n-m-k} \\ & & & I_k & \end{pmatrix}.$$

Let $R = \{R(r) | r \in \mathcal{M}_{(n-m-k)\times k}(F)\}$ where

$$R(r) = \begin{pmatrix} I_{n-m-k} & r & & & \\ & I_k & & & \\ & & I_{2m} & & \\ & & & I_k & r' \\ & & & & I_{n-m-k}. \end{pmatrix} \tag{3.2}$$

Here r' is determined by r so that $R(r) \in \mathrm{G}$. For $r_i \in F^k$ being a row vector ($1 \leq i \leq n-m-k$), we define $R_i(r_i)$ as $R(r)$ where the i-th row of r is r_i and all other rows of r are zeroes. Let $R_i = \{R_i(r_i)\}$, and $R^i = \prod_{j=1}^{i} R_j$. Let $Y(r) = R(r)^w$, $Y_i(r_i) = R_i(r_i)^w$, $Y_i = R_i^w$ and $Y^i = (R^i)^w$. Let $L = \{L(c) | c \in \mathcal{M}_{k\times(n-m-k)}(F)\}$ where

$$L(c) = \begin{pmatrix} I_k & c & & & \\ & I_{n-m-k} & & & \\ & & I_{2m} & & \\ & & & I_{n-m-k} & c' \\ & & & & I_k. \end{pmatrix} \tag{3.3}$$

Here c' is determined by c so that $L(c) \in$ G. For $c_i \in F^k$ being a column vector ($1 \leq i \leq n-m-k$), we define $L_i(c_i) = L(c)$ where the i-th column of c is c_i and all other columns of c are zeroes. Let $L_i = \{L_i(c_i) \mid c_i \in F^k\}$ and $L_i = \prod_{j=1}^{i} L_i$. For $l = (l_1, \ldots, l_k) \in F^k$, let $l' = (\underbrace{0, \ldots, 0}_{n-m-k}, l_k, l_{k-1}, \ldots, l_1)$ and $l'' = (l_1, \ldots, l_k, \underbrace{0, \ldots, 0}_{n-m-k})^T$, let $Z(l) = \begin{pmatrix} \mathrm{I}^{n-m} & & l'' & 0_{n-m-1} \\ & & 0 & l' \\ & I_{2m} & & \\ & & & \mathrm{I}^{n-m} \end{pmatrix}$.

Then,

$$[Y_{n-m-k}(y_1, \ldots y_k), Z(l_1, \ldots, l_k)] = I_{2n} + \sum y_j l_j \cdot E_{n-m,n+m+1}. \tag{3.4}$$

Let $T_k(t) = diag(I_{n-m-k}, t, I_{2m}, t^{-1}, I_{n-m-k})$ for $t \in GL_1^k$, and $T'_k(t) = wT_k(t)w^{-1}$, and let $T_k = \{T_k(t) \mid t \in GL_1^k\}$ and $T'_k = wT_kw^{-1}$. For $a_i \in GL_1(F)$, let $H_i(a_i) = diag(a_iI_i, I_{2n-2i}, a_i^{-1}I_i)$ for $1 \leq i \leq k$, and let $H_i = \{H_i(a_i) \mid a_i \in GL_1\}$. So $T'_k \cong \prod_{i=1}^{k} H_i$. For $a = (a_1, \ldots, a_k)$, if we let $\hat{a} = (a_1 \cdots a_k, a_2 \cdots a_k, \ldots, a_{k-1}a_k, a_k)$, then $T'_k(\hat{a}) = \prod_{i=1}^{k} H_i(a_i)$.

3.3 Two transformation of the local integral

3.3.1 First transformation

Let $P_k^{m+k} = (M \times GL_1^k) \ltimes V_k^{m+k}$ be a parabolic subgroup of $Sp_{2(m+k)}$. Then by the Iwasawa decomposition,

$$Sp_{2(m+k)} = V_k^{m+k} \cdot \mathrm{M} \cdot GL_1^k \cdot K_{Sp_{2(m+k)}}. \tag{3.5}$$

Let $h = vgtk$ with respect to this decomposition, and $\dot{h}$ its image in $\mathrm{M}V_k^{m+k}\backslash Sp_{2(m+k)}$. Then $d\dot{h} = \delta_{V_k^{m+k}}^{-1}(t)\,\mathrm{d}t\,\mathrm{d}k$. Then from equation (1.9),

$$\begin{aligned}
&I(\varphi_\pi, \phi, \widetilde{f}_{W_\tau,\widetilde{\sigma},s})\\
=&\int_{T_k}\mathrm{d}t\int_R \mathrm{d}r\int_{K_{Sp_{2(m+k)}}}\mathrm{d}k\ \delta_{V_k^{m+k}}^{-1}(t)\\
&\mathcal{T}(\pi(wrtk)\varphi_\pi, [\omega_\psi(l_{m+k}(r)tk)\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(tk)).
\end{aligned}\tag{3.6}$$

Note that T_k normalizes R. By a change of variable $r' = t^{-1}rt$,

$$\begin{aligned}
&I(\varphi_\pi, \phi, \widetilde{f}_{W_\tau,\widetilde{\sigma},s})\\
=&\int_{T_k}\mathrm{d}t\int_R \mathrm{d}r\int_{K_{Sp_{2(m+k)}}}\mathrm{d}k\ |t|^{-(n-m-k)}\delta_{V_k^{m+k}}^{-1}(t)\\
&\mathcal{T}(\pi(wtrk)\varphi_\pi, [\omega_\psi(tl_{m+k}(r)k)\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(tk)).
\end{aligned}\tag{3.7}$$

Lemma 3.3.1. *To prove part (A), (B), and (C) of* ***Theorem(3.1.1)****, we only need to prove them for the integration*

$$\begin{aligned}
&J(\varphi_\pi, \phi, \widetilde{f}_{W_\tau,\widetilde{\sigma},s})\\
=&\int_{T'_k}\mathrm{d}t\ |t|^s\delta_{B_{GL_k}}^{-\frac{1}{2}}(t)\delta_{B_{\mathrm{G}}}^{-\frac{1}{2}}(t)W_\tau(t)\mathcal{T}(\pi(t)\varphi_\pi, [\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(1)),
\end{aligned}\tag{3.8}$$

where B_{GL_k} and B_{G} are Borel subgroups of GL_k and G *respectively.*

Proof. First consider the integration over $K_{Sp_{2(m+k)}}$ in (3.6). At unramified places, the integration on $K_{Sp_{2(m+k)}}$ is trivial. For part (B) and (C), by smoothness of vectors in the integral, the integration on $K_{Sp_{2(m+k)}}$ is in fact a linear combination of right translations by finite many elements in $K_{Sp_{2(m+k)}}$. So it suffices to prove part (A), (B) and (C) for

$$\begin{aligned}
&\int_{T_k}\mathrm{d}t\int_R \mathrm{d}r|t|^{-(n-m-k)}\delta_{V_k^{m+k}}^{-1}(t)\\
&\mathcal{T}(\pi(wtr)\varphi_\pi, [\omega_\psi(tl_{m+k}(r))\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t)).
\end{aligned}\tag{3.9}$$

Consider the integration over R in (3.9). For fixed t,

$$\begin{aligned}
&\int_R \mathrm{d}r\ |t|^{-(n-m-k)}\delta^{-1}_{V_k^{m+k}}(t) \\
&\mathcal{T}(\pi(wtr)\varphi_\pi, [\omega_\psi(tl_{m+k}(r))\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t)) \\
&= \int_{R^{n-m-k-1}} \mathrm{d}r' \int_{F^k} \mathrm{d}r \\
&\mathcal{T}(\pi(wtr'R_{n-m-k}(r))\varphi_\pi, [\omega_\psi(tl_{m+k}(R_{n-m-k}(r)))\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t)).
\end{aligned} \tag{3.10}$$

First consider the integration on F^k. Note that

$$[\omega_\psi(tl_{m+k}(R_{n-m-k}(r)))\phi]_m(\cdot) = |\det t|^{\frac{1}{2}}\gamma_\psi(\det t)\phi(y_1,\ldots,y_k;\cdot). \tag{3.11}$$

At the unramified places where $\phi = Ch_{\mathcal{O}^{m+k}}$, the action of $l_{m+k}(R_{n-m-k}(r))$ in the Weil representation is trivial when $r \in \mathcal{O}^k$, and it vanishes otherwise. At other non-archimedean places, since ϕ is a Schwartz function on F^{m+k}, the integration on F^k is a linear combination of values of finite many elements in R_{n-m-k}. So one only needs to prove part (A), (B) and (C) for

$$\int_{R^{n-m-k-1}} \mathrm{d}r'\ \mathcal{T}(\pi(wtr'))\varphi_\pi, [\omega_\psi(tl_{m+k})\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t)). \tag{3.12}$$

Consider the integral on $R^{n-m-k-1}$. Since φ_π is arbitrary and $w \in K_{\mathrm{M}}$, it suffices to prove the same result if one replaces φ_π by $\pi(w^{-1})\varphi_\pi$. By changing the variables, (3.12) becomes

$$\begin{aligned}
&\int_{T'_k} \mathrm{d}t \int_{Y^{n-m-k-1}} \mathrm{d}y\ |t|^{-(n-m-k)}|\det(\mathrm{Ad}\,t, V_k^{m+k})|^{-1} \\
&\mathcal{T}(\pi(ty)\varphi_\pi, [\omega_\psi(t)\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t)).
\end{aligned} \tag{3.13}$$

If $n-m-k-1=0$, then there is nothing to prove. If $n-m-k-1>0$, consider $L_{n-m-k}(l)$ with $l=(l_1,\ldots,l_k)$. Note that $T_kY^{n-m-k-1}$ normalizes L_{n-m-k}. For all l in a neighborhood of 0 in F^k so that φ_π is invariant under $L_{n-m-k}(l)$, we have

$$\begin{aligned}
&\mathcal{T}(\pi(ty)\varphi_\pi, [\omega_\psi(t)\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t)) \\
=&\mathcal{T}(\pi(tyL_{n-m-k}(l))\varphi_\pi, [\omega_\psi(t)\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t)) \\
=&\psi(l\cdot y_{n-m-k-1})\mathcal{T}(\pi(ty)\varphi_\pi, [\omega_\psi(t)\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t)).
\end{aligned} \tag{3.14}$$

where $y_{n-m-k-1}$ is the last row of y. So when the trilinear form above is nonzero, we have

$$\psi(l\cdot y_{n-m-k-1}) = 1 \tag{3.15}$$

for all l in a neighborhood of 0. This means $y_{n-m-k-1}$ is bounded, so the integrand is compactly supported as a function on $Y_{n-m-k-1}$. For the same reason as before one can remove the integration on $Y_{n-m-k-1}$ when trying to prove part (B) and (C). At unramified places, (3.14) holds for all $l\in\mathcal{O}^k$. Since ψ has conductor 0, (3.15) holds for all $l\in\mathcal{O}^k$ implies that $y_{n-m-k-1}\in\mathcal{O}^k$. So the integration over $Y^{n-m-k-1}$ is trivial at unramified places. Using the same method we can remove the integration over Y_i for $1\le i\le n-m-l-2$ (by using L_{i+1}). So eventually we only need to consider the integral

$$\begin{aligned}
&\int_{T'_k} \mathrm{d}t|t|^{-(n-m-k)}|\det(\mathrm{Ad}\,t), V_k^{m+k})|^{-1} \\
&\mathcal{T}(\pi(t)\varphi_\pi, [\omega_\psi(t)\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t)).
\end{aligned} \tag{3.16}$$

By definition

$$[\omega_\psi(t)\phi]_m = |\det t|^{\frac{1}{2}}\gamma_\psi(\det t)[\phi]_m \tag{3.17}$$

$$\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(t) = \gamma_\psi^{-1}(\det t)|\det t|^{s-\frac{1}{2}}W_\tau(t)\delta_{P_{k,m}}^{\frac{1}{2}}(t)\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(1). \tag{3.18}$$

And by direct calculation we have

$$|t|^{-(n-m-k)}|\det(\mathrm{Ad}\,t), V_k^{m+k})|^{-1}\delta_{P_{k,m}}^{\frac{1}{2}}(t) = \delta_{B_{GL_k}}^{-\frac{1}{2}}(t)\delta_{B_{\mathrm{G}}}^{-\frac{1}{2}}(t). \tag{3.19}$$

So we obtain our lemma. $\square$

3.3.2 Second transformation

Now we come back to equation (1.9). Note that

$$\begin{aligned}\mathrm{MV}_k^{m+k}\backslash Sp_{2(m+k)} &= \left(\mathrm{MV}_k^{m+k}\backslash P_{k,m}\right)\cdot\left(P_{k,m}\backslash Sp_{2(m+k)}\right)\\ &\cong (N_k\backslash GL_k)\cdot\left((P_{k,m}\cap K_{Sp_{2(m+k)}})\backslash K_{Sp_{2(m+k)}}\right). \qquad (3.20)\end{aligned}$$

Here N_k is the maximal unipotent subgroup of GL_k. For $h = gk'$ with respect to the above decomposition, we have $\mathrm{d}h = \delta_{P_{k,m}}^{-1}(g)\,\mathrm{d}g\,\mathrm{d}k'$. With this decomposition, we have

$$\begin{aligned}&I(\varphi_\pi,\phi,\widetilde{f}_{W_\tau,\widetilde{\sigma},s})\\ =&\int_{N_k\backslash GL_k}\mathrm{d}g\int_{(P_{k,m}\cap K_{Sp_{2(m+k)}})\backslash K_{Sp_{2(m+k)}}}\mathrm{d}k'\int_R \mathrm{d}r\delta_{P_{k,m}}^{-1}(g)\\ &\mathcal{T}(\pi(wrgk')\varphi_\pi,[\omega_\psi(l_{m+k}(r)gk')\phi]_m,\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(gk')). \qquad (3.21)\end{aligned}$$

We will use this formula in the proof of part (D) of **Theorem (3.1.1)**.

3.4 Unramified Computation

In this section we calculate the local integral at (non-archimedean) unramified places. Recall from Lemma 3.3.1 that we only need to calculate

$$\begin{aligned}&J(\varphi_\pi,\phi,\widetilde{f}_{W_\tau,\widetilde{\sigma},s})\\ =&\int_{T_k'}\mathrm{d}t\ |t|^s\delta_{B_{GL_k}}^{-\frac{1}{2}}(t)\delta_{B_G}^{-\frac{1}{2}}(t)W_\tau(t)\mathcal{T}(\pi(t)\varphi_\pi,[\phi]_m,\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(1)). \qquad (3.22)\end{aligned}$$

Lemma 3.4.1. *Assuming both W_τ and $\mathcal{T}$ are normalized, we have*

$$J(\varphi_\pi,\phi,\widetilde{f}_{W_\tau,\widetilde{\sigma},s}) = \frac{L(\pi\times\tau,s)}{L_\psi(\widetilde{\sigma}\times\tau,s+\frac{1}{2})L(\tau,sym^2,2s)} \qquad (3.23)$$

Proof. Assume that π and $\tilde{\sigma} \otimes w_\psi$ are spherical components of $Ind_{B_{\mathrm{G}}}^{\mathrm{G}}(\chi)$ and $Ind_{B_{\mathrm{M}^J}}^{\mathrm{M}^{\mathrm{J}}}(\xi \otimes \psi)$ respectively. Then

$$\mathcal{T}(\pi(t)\varphi_\pi, [\phi]_m, \widetilde{f}_{W_\tau, \widetilde{\sigma}, s}(1))$$

is the normalized Whittaker-Shintani function $\mathrm{W}^0_{\chi,\xi,\psi}(t)$ as defined in **Definition 2.2.4**. We will prove (3.23) with the following steps.

Step 1 Recall that $\Lambda_k^+ = \{\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k | n_1 \geq n_2 \geq \ldots \geq n_k \geq 0\}$. Identify $\mathbf{n} \in \Lambda_k^+$ as $(\mathbf{n}; 0, 0, \ldots, 0) \in \Lambda_n^+$. By the formula of the Whittaker-Shintani functions in theorem 2.12.1 and the Casselman-Shalika formula for Whittaker functions, we have

$$J(\varphi_\pi, \phi, \widetilde{f}_{W_\tau, \widetilde{\sigma}, s}) = \zeta^{-m}(1) \prod_{i=1}^{m} \zeta(2i) \cdot$$
$$\sum_{\mathbf{n} \in \Lambda_k^+} \left[q^{-s|\mathbf{n}|} T_k(\mathbf{n}; z_\tau) \cdot \sum_{w \in W_{\mathrm{G}}, w' \in W_{\mathrm{M}}} b(w\chi, w'\xi) d(w\chi) d'(w'\xi)((w\chi)^{-1})(\mathfrak{p}_{\mathbf{n}}) \right]. \tag{3.24}$$

Here $T_k(\mathbf{n}; z_\tau)$ is the trace of z_τ in the irreducible representation of $GL_k(\mathbb{C})$ with highest weight $\mathbf{n}$ when $\mathbf{n} \in \Lambda_k^+$, and is extended to all $\mathbf{n} \in \mathbb{Z}^k$ by meromorphic continuation, and $|\mathbf{n}| = \sum_{i=1}^k n_i$.

Step 2 By the properties of $b(\chi, \xi)$, we can simplify the right hand side above as

$$\sum_{\mathbf{n} \in \Lambda_k^+, \mathbf{m} \in \{0,1,\ldots,2m\}^k} (-1)^{|\mathbf{m}|} \wedge^{\mathbf{m}} (\Gamma_{\tilde{\sigma}}) \tilde{\mathrm{T}}_n((\mathbf{n} - \mathbf{m}); z_\pi) \mathrm{T}_k(\mathbf{n}; z_\tau) q^{-s|\mathbf{n}|} \tag{3.25}$$

where $\tilde{\mathrm{T}}_n$ is defined as follows. Let $\lambda \in \Lambda_n^+$ and

$$x = diag(x_1, \ldots, x_n, 1, x_n^{-1}, \ldots, x_1^{-1}) \in SO_{2n+1}(\mathbb{C}),$$

then $\tilde{\mathrm{T}}_n(\lambda, x)$ is the trace of x in the irreducible representation with highest weight λ. Extends $\tilde{\mathrm{T}}_n$ to all $\lambda \in Z^n$ by meromorphic continuation.

Step 3 On the other hand, the right hand side of (3.23) is equal to

$$\sum_{\substack{\mathbf{a} \in \Lambda_k^+ \\ \mathbf{b} \in \{0,1,\ldots,2m\}^k}} (-1)^{|\mathbf{b}|} \wedge^{\mathbf{b}} (\Gamma_{\tilde{\sigma}}) \tilde{\mathrm{T}}_n(\mathbf{a}; z_\pi) \mathrm{T}_k(\mathbf{a}+\mathbf{b}; z_\tau) q^{-s|\mathbf{a}+\mathbf{b}|}. \tag{3.26}$$

Step 4 Finally, we will prove that (3.25) and (3.26) are actually equal.

[Proof of Step 1] For $t = diag(t_1, \ldots, t_k)$, and $\mathbf{n} = (n_1, \ldots n_k) \in \mathbb{Z}^k$, we write $t \sim \mathbf{n}$ if $|t_i| = |\mathfrak{p}|^{n_i}$. We use $\mathfrak{p}_{\mathbf{n}}$ to denote

$$diag(\mathfrak{p}^{n_1}, \ldots, \mathfrak{p}^{n_k}, I_{2(n-k)}, \mathfrak{p}^{-n_k}, \ldots, \mathfrak{p}^{-n_1}).$$

We also use it to denote $diag(\mathfrak{p}^{n_1}, \ldots, \mathfrak{p}^{n_k})$ when it is considered as an element in GL_k. By abusing the notation χ_i is either a character on $F^\times$ or a complex number, so that $\chi_i(\mathfrak{p}) = q^{-\chi_i}$, and similar for ξ_j. Then by the formula for the Whittaker-Shintani function in theorem 2.12.1, for $t \sim \mathbf{n}$, we have

$$\mathrm{W}^0_{\chi,\xi,\psi}(t) = \zeta^{-m}(1) \prod_{i=1}^{m} \zeta(2i) \sum_{w \in W_{\mathrm{G}}, w' \in W_{\mathrm{M}}} b(w\chi, w'\xi) d(w\chi) d'(w'\xi)((w\chi)^{-1}\delta^{\frac{1}{2}}_{B_{\mathrm{G}}})(\mathfrak{p}_{\mathbf{n}}) \tag{3.27}$$

if $\mathbf{n} \in \Lambda_k^+$, and $\mathrm{W}^0_{\chi,\xi,\psi}(t) = 0$ if $\mathbf{n} \notin \Lambda_k^+$. Here

$$\mathrm{d}(\chi) = \prod_{1 \le a < b \le n} \zeta(\chi_a \pm \chi_b) \prod_{i=1}^{n} \zeta(\chi_i), \quad \mathrm{d}'(\xi) = \prod_{1 \le a < b \le m} \zeta(\xi_a \pm \xi_b) \prod_{j=1}^{m} \zeta(2\xi_j),$$

$$\mathrm{b}(\chi, \xi) = \prod_{i < j+n-m} \zeta^{-1}(\chi_i - \xi_j + \frac{1}{2}) \cdot \prod_{i > j+n-m} \zeta^{-1}(-\chi_i + \xi_j + \frac{1}{2})$$
$$\prod_{1 \le j \le m} \zeta^{-1}(\xi_j + \frac{1}{2}) \cdot \prod_{\substack{1 \le i \le n \\ 1 \le j \le m}} \zeta^{-1}(\chi_i + \xi_j + \frac{1}{2}).$$

On the other hand, by the Casselman-Shalika formula,

$$W_\tau(t) = \mathrm{T}_k(\mathbf{n}; z_\tau)\delta_{B_{\mathrm{M}}}^{\frac{1}{2}}(\mathfrak{p}_{\mathbf{n}}) \tag{3.28}$$

if $\mathbf{n} \in \Lambda_k^+$, and $W_\tau(t) = 0$ if $\mathbf{n} \notin \Lambda_k^+$. Combining these two formulas, we obtain (3.24). □

Proof for Step 2. For any function $\Phi(\chi)$, as in section 2.10 we let $\mathcal{A}_{W_{\mathrm{G}}}(\Phi(\chi)) = \sum_{w\in W_{\mathrm{G}}} sgn(w)\Phi(w\chi)$, and define $\mathcal{A}_{W_{\mathrm{M}}}$ similarly for functions on ξ. Let $\rho_1 = (n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{1}{2})$ and $\rho_2 = (m, m-1, \ldots, 1)$. Then

$$d(\chi) = (-1)^n q^{\chi\cdot\rho_1}\mathcal{A}_{W_{\mathrm{G}}}^{-1}(q^{-\chi\cdot\rho_1}) \quad d'(\xi) = (-1)^m q^{\xi\cdot\rho_2}\mathcal{A}_{W_{\mathrm{M}}}^{-1}(q^{-\xi\cdot\rho_2}).$$

So (3.24) is equal to

$$\begin{aligned} &\frac{\zeta^{-m}(1)\prod_{i=1}^m \zeta(2i)}{\mathcal{A}_{W_{\mathrm{G}}}(q^{-\chi\cdot\rho_1})\mathcal{A}_{W_{\mathrm{M}}}(q^{-\xi\cdot\rho_2})} \cdot \sum_{\mathbf{n}\in\Lambda_k^+} \Big[\mathrm{T}_k(\mathbf{n}; z_\tau)q^{-s|\mathbf{n}|} \\ &\sum_{w\in W_{\mathrm{G}}, w'\in W_{\mathrm{M}}} (-1)^{m+n} sgn(w)sgn(w')q^{w\chi\cdot(\rho_1+\mathbf{n})}q^{w'\xi\cdot\rho_2}b(w\chi, w'\xi)\Big], \end{aligned} \tag{3.29}$$

by a change of variable $w \mapsto ww_0^{\mathrm{G}}$ and $w' \mapsto w'w_0^{\mathrm{M}}$, it is equal to

$$\begin{aligned} &\frac{\zeta^{-m}(1)\prod_{i=1}^m \zeta(2i)}{\mathcal{A}_{W_{\mathrm{G}}}(q^{-\chi\cdot\rho_1})\mathcal{A}_{W_{\mathrm{M}}}(q^{-\xi\cdot\rho_2})} \cdot \sum_{\mathbf{n}\in\Lambda_k^+} \Big[\mathrm{T}_k(\mathbf{n}; z_\tau)q^{-s|\mathbf{n}|} \\ &\sum_{w\in W_{\mathrm{G}}, w'\in W_{\mathrm{M}}} sgn(w)sgn(w')q^{-w\chi\cdot(\rho_1+\mathbf{n})}q^{-w'\xi\cdot\rho_2}b(-w\chi, -w'\xi)\Big], \end{aligned} \tag{3.30}$$

Let $\tilde{\chi} = (\chi_{n-m+1}, \ldots, \chi_n)$, and let $\rho_1^* = (n-\frac{1}{2}, n-\frac{3}{2}, \ldots, m+\frac{1}{2}, 0, \ldots, 0)$. Then by the definition of $b(\chi, \xi)$ it is not hard to see that

$$q^{-\chi\cdot(\rho_1+\mathbf{n})}q^{-\xi\cdot\rho_2}b(-\chi, -\xi)\left[\prod_{i=1}^{n-m}\prod_{j=1}^{m}(1-q^{\chi_i\pm\xi_j-\frac{1}{2}})\cdot q^{-\chi\cdot(\mathbf{n}+\rho_1^*)}\right]^{-1} \tag{3.31}$$

is actually a function on $(\tilde{\chi}, \xi)$. We denote it by $\tilde{A}(\tilde{\chi}, \xi)$ as in (2.60). Then (3.30) is equal to

$$\frac{\zeta^{-m}(1)\prod_{i=1}^{m}\zeta(2i)}{\mathcal{A}_{W_{\mathrm{G}}}(q^{-\chi\cdot\rho_1})\mathcal{A}_{W_{\mathrm{M}}}(q^{-\xi\cdot\rho_2})} \cdot \sum_{\mathbf{n}\in\Lambda_k^+}\Big[\mathrm{T}_k(\mathbf{n};z_\tau)q^{-s|\mathbf{n}|}$$
$$\sum_{w\in W_{\mathrm{G}},w'\in W_{\mathrm{M}}} sgn(w)sgn(w')(w,w')\circ(\prod_{i=1}^{n-m}\prod_{j=1}^{m}(1-q^{\chi_i\pm\xi_j-\frac{1}{2}})\cdot q^{-\chi\cdot(\mathbf{n}+\rho_1^*)}\tilde{A}(\tilde{\chi},\xi))]. \tag{3.32}$$

Let e_i^* be the function $\chi \mapsto \chi_i$ for $1 \le i \le n$. Let $\mathrm{W}_1, \mathrm{W}_2$ be subgroups of W_{G} stabilizing $\{e_{n-m+1}^*, \dots, e_n^*\}$ and $\{e_1^*, \dots, e_{n-m}^*\}$ respectively. Then $\tilde{A}(\tilde{\chi}, \xi)$ is stabilized by W_1 and $\prod_{i=1}^{n-m}\prod_{j=1}^{m}(1-q^{\chi_i\pm\xi_j-\frac{1}{2}})\cdot q^{-\chi\cdot(\mathbf{n}+\rho_1^*)}$ is stabilized by W_2 and W_{M}. Let W_0 be a set of representatives of $(\mathrm{W}_1 \times \mathrm{W}_2)\backslash\mathrm{W}_{\mathrm{G}}$, then (3.32) is equal to

$$\frac{\zeta^{-m}(1)\prod_{i=1}^{m}\zeta(2i)}{\mathcal{A}_{W_{\mathrm{G}}}(q^{-\chi\cdot\rho_1})\mathcal{A}_{W_{\mathrm{M}}}(q^{-\xi\cdot\rho_2})}\cdot$$
$$\cdot\sum_{\mathbf{n}\in\Lambda_k^+}\Bigg[q^{-s|\mathbf{n}|}\cdot\mathrm{T}_k(\mathbf{n};z_\tau)\sum_{w_0,w_1} sgn(w_1w_0)(w_1w_0)\circ(\prod_{i=1}^{n-m}\prod_{j=1}^{m}(1-q^{\chi_i\pm\xi_j-\frac{1}{2}}q^{-\chi\cdot(\mathbf{n}+\rho_1)}))$$
$$\sum_{w_2,w_{\mathrm{M}}} sgn(w_2w_{\mathrm{M}})\tilde{A}(w_2w_0\tilde{\chi}, w_{\mathrm{M}}\xi)\Bigg]. \tag{3.33}$$

By lemma 2.10.4 we have

$$\begin{aligned}&\sum_{w_2,w_{\mathrm{M}}} sgn(w_2w_{\mathrm{M}})\tilde{A}(w_2w_0\tilde{\chi}, w_{\mathrm{M}}\xi)\\ =&\zeta^m(1)\prod_{i=1}^{m}\zeta^{-1}(2i)\cdot\sum_{w_2} sgn(w_2)q^{-w_2w_0\chi\cdot\tilde{\rho}_1}\mathcal{A}_{W_{\mathrm{M}}}(q^{-\xi\cdot\rho_2})\end{aligned} \tag{3.34}$$

Substituting this to (3.33) we get

$$\sum_{\mathbf{n}\in\Lambda_k^+} \frac{\mathcal{A}_{\mathrm{W_G}}(\prod_{1\le i\le n-m}\prod_{1\le j\le m}(1-q^{\chi_i\pm\xi_j-\frac{1}{2}})q^{-\chi\cdot(\mathbf{n}+\rho_1)})}{\mathcal{A}_{\mathrm{W_G}}(q^{-\chi\cdot\rho_1})}\cdot \mathrm{T}_k(\mathbf{n};z_\tau)q^{-s|\mathbf{n}|}. \quad (3.35)$$

Note that for any $i, a\in\mathbb{Z}$ with $1\le a\le m$ and $k+1\le i\le n-m$, we have $\mathcal{A}_{\mathrm{W_G}}(q^{a\chi_i}q^{-\chi\cdot(\mathbf{n}+\rho_1)})=0$ for any $\mathbf{n}\in\mathbb{Z}^k$. So (3.35) is equal to

$$\sum_{\mathbf{n}\in\Lambda_k^+} \frac{\mathcal{A}_{\mathrm{W_G}}(\prod_{1\le i\le k}\prod_{1\le j\le m}(1-q^{\chi_i\pm\xi_j-\frac{1}{2}})q^{-\chi\cdot(\mathbf{n}+\rho_1)})}{\mathcal{A}_{\mathrm{W_G}}(q^{-\chi\cdot\rho_1})}\cdot \mathrm{T}_k(\mathbf{n};z_\tau)q^{-s|\mathbf{n}|}. \quad (3.36)$$

For any set $A=\{a_1,\dots,a_N\}$, we let

$$\wedge^i(A)=\sum_{S\subset A,|S|=i}(\prod_{s\in S}a_s). \quad (3.37)$$

For $\mathbf{k}=(k_1,\dots,k_N)$, we let

$$\wedge^{\mathbf{k}}(A)=\prod_{s=1}^{N}\wedge^{k_s}(A). \quad (3.38)$$

With these notation, and applying the Weyl character formula for $SO_{2n+1}(\mathbb{C})$, (3.35) is equal to

$$\sum_{\mathbf{n}\in\Lambda_k^+,\mathbf{m}\in\{0,1,\dots,2m\}^k}(-1)^{|\mathbf{m}|}\wedge^{\mathbf{m}}(\Gamma_{\tilde{\sigma}})\tilde{\mathrm{T}}_n(\mathbf{n}-\mathbf{m};z_\pi)\mathrm{T}_k(\mathbf{n};z_\tau)q^{-s|\mathbf{n}|}. \quad (3.39)$$

where $\Gamma_{\tilde{\sigma}}$ is the set $\{q^{\pm\xi_1-\frac{1}{2}},\dots,q^{\pm\xi_m-\frac{1}{2}}\}$. This is step 2. □

Proof of Step 3. By the discussion in **theorem 3.1** in [16], we have

$$\frac{L(\pi\otimes\tau,s)}{L(\tau,sym^2,2s)}=\sum_{\mathbf{a}\in\Lambda_k^+}\tilde{\mathrm{T}}_n(\mathbf{a};z_\pi)\mathrm{T}_k(\mathbf{a};z_\tau)q^{-s|\mathbf{a}|}.$$

Let $z_\tau = (q^{-\alpha_1}, \dots, q^{-\alpha_k})$, then

$$\begin{aligned}&\frac{L(\pi \times \tau, s)}{L(\tilde{\sigma} \times \tau, s+\frac{1}{2})L(\tau, sym^2, 2s)}\\ &= \prod_{\substack{1\le i\le k\\ 1\le j\le m}} (1 - q^{-\alpha_i \pm \xi_j - s - \frac{1}{2}}) \sum_{\mathbf{a}\in\Lambda_k^+} \tilde{\mathrm{T}}_n(\mathbf{a}; z_\pi)\mathrm{T}_k(\mathbf{a}; z_\tau) q^{-s|\mathbf{a}|}.\end{aligned}$$

Using the notation defined in (3.37) and (3.38), the right hand side can be written as

$$\sum_{\substack{\mathbf{a}\in\Lambda_k^+\\ \mathbf{b}\in\{0,1,\dots,2m\}^k}} (-1)^{|\mathbf{b}|} \wedge^{\mathbf{b}} (\Gamma_{\tilde{\sigma}}) (\prod_{j=1}^{k} q^{-\alpha_j b_j}) \tilde{\mathrm{T}}_n(\mathbf{a}; z_\pi)\mathrm{T}_k(\mathbf{a}; z_\tau) q^{-s|\mathbf{a}+\mathbf{b}|}. \tag{3.40}$$

Let S_k be the permutation group of k elements. Then S_k acts on $\{0, 1, \dots, 2m\}^k$ as it acts on $\mathbb{Z}^k$. Let B be the set of representatives in each orbit, and for each $\mathbf{b} \in B$, let $n(\mathbf{b})$ be the order of its stabilizer in S_k. Then(3.40) is equal to

$$\sum_{\substack{\mathbf{a}\in\Lambda_k^+\\ \mathbf{b}\in B}} (-1)^{|\mathbf{b}|} \wedge^{\mathbf{b}} (\Gamma_{\tilde{\sigma}}) q^{-s|\mathbf{a}+\mathbf{b}|} n(\mathbf{b})^{-1} \sum_{\epsilon\in S_k} (\prod_{j=1}^{k} q^{-\alpha_j b_{\epsilon(j)}}) \tilde{\mathrm{T}}_n(\mathbf{a}; z_\pi)\mathrm{T}_k(\mathbf{a}; z_\tau). \tag{3.41}$$

To continue we use the following equation

$$\sum_{\epsilon\in S_k} \mathrm{T}_k(\mathbf{a}; z_\tau)(\prod_{j=1}^{k} q^{-\alpha_j b_{\epsilon(j)}}) = \sum_{\epsilon\in S_k} \mathrm{T}_k(\mathbf{a} + \mathbf{b}^\epsilon; z_\tau). \tag{3.42}$$

To prove this equation, note that for generic z_τ,

$$\mathrm{T}_k(\mathbf{a}; z_\tau) = \frac{1}{\prod_{1\le i<j\le k}(q^{-\alpha_i} - q^{-\alpha_j})} \cdot \sum_{\epsilon'\in S_k} sgn(\epsilon') q^{-\alpha_i(a_{\epsilon'(i)} + k - \epsilon'(i))}. \tag{3.43}$$

So the left hand side of (3.42) is equal to

$$\frac{1}{\prod_{1\le i<j\le k}(q^{-\alpha_i} - q^{-\alpha_j})} \cdot \sum_{\epsilon,\epsilon'\in S_k} sgn(\epsilon') q^{-\alpha_i(a_{\epsilon'(i)} + k - \epsilon'(i) + b_{\epsilon(i)})} \tag{3.44}$$

By changing ϵ to $\epsilon'\epsilon$ we obtain the right hand side of (3.42). By (3.42), and note that $\wedge^{\mathbf{b}}(\Gamma_{\tilde{\sigma}})$ and $|b|$, as well as $n(\mathbf{b})$, are invariant under the action of ϵ, so (3.41) is equal to

$$\sum_{\substack{\mathbf{a}\in\Lambda_k^+ \\ \mathbf{b}\in\{0,1,\ldots,2m\}^k}} (-1)^{|\mathbf{b}|} \wedge^{\mathbf{b}} (\Gamma_{\tilde{\sigma}})\tilde{\mathrm{T}}_n(\mathbf{a}; z_\pi)\mathrm{T}_k(\mathbf{a}+\mathbf{b}; z_\tau)q^{-s|\mathbf{a}+\mathbf{b}|}. \tag{3.45}$$

This is step 3. □

Proof of Step 4. To prove step 4, we need to show that (3.25) equals (3.26). We will show that there is a one-to-one correspondence between most terms on each side, and all the remaining terms vanish. First we let

$$\mathcal{S}_1(\mathbf{n}-\mathbf{m};\mathbf{n}) = (-1)^{|\mathbf{m}|} \wedge^{\mathbf{m}} (\Gamma_{\tilde{\sigma}})\tilde{\mathrm{T}}_n((\mathbf{n}-\mathbf{m}, 0, \ldots, 0); z_\pi)\mathrm{T}_k(\mathbf{n}; z_\tau)q^{-s|\mathbf{n}|} \tag{3.46}$$

and

$$\mathcal{S}_2(\mathbf{a}, \mathbf{a}+\mathbf{b}) = (-1)^{|\mathbf{b}|} \wedge^{\mathbf{b}} (\Gamma_{\tilde{\sigma}})\tilde{\mathrm{T}}_n(\mathbf{a}; z_\pi)\mathrm{T}_k(\mathbf{a}+\mathbf{b}; z_\tau)q^{-s|\mathbf{a}+\mathbf{b}|} \tag{3.47}$$

So we need to show that

$$\sum_{\substack{\mathbf{n}\in\Lambda_k^+ \\ \mathbf{m}\in\{0,1,\ldots,2m\}^k}} \mathcal{S}_1(\mathbf{n}-\mathbf{m}, \mathbf{n}) = \sum_{\substack{\mathbf{a}\in\Lambda_k^+ \\ \mathbf{b}\in\{0,1,\ldots,2m\}^k}} \mathcal{S}_2(\mathbf{a}, \mathbf{a}+\mathbf{b}). \tag{3.48}$$

We are going to construct four sets A_1, A_2, B_1, B_2 satisfying

1. $\Lambda_k^+ \times \{0, 1, \ldots, 2m\}^k = A_1 \sqcup A_2 = B_1 \sqcup B_2$.
2. For $(\mathbf{n}, \mathbf{m}) \in A_1$, $\mathcal{S}_1(\mathbf{n}-\mathbf{m}, \mathbf{n}) = 0$. For $(\mathbf{a}, \mathbf{b}) \in B_1$, $\mathcal{S}_2(\mathbf{a}, \mathbf{a}+\mathbf{b}) = 0$.
3. There exists a bijection $I_1 : A_2 \to B_2$, such that when $I_1(\mathbf{n}, \mathbf{m}) = (\mathbf{a}, \mathbf{b})$, $\mathcal{S}_1(\mathbf{n}-\mathbf{m}, \mathbf{n}) = \mathcal{S}_2(\mathbf{a}, \mathbf{a}+\mathbf{b})$.

The existence of such sets implies (3.48).

Let $\rho_k = (n-\frac{1}{2}, n-\frac{3}{2}, \dots, n-k+\frac{1}{2}) \in \mathbb{C}^k$. Let $\mathcal{D}_1(\mathbf{n},\mathbf{m}) = \{n_i - m_i + \rho_{k,i}\}_{i=1}^k$. Let A_2 be the set of $(\mathbf{n},\mathbf{m}) \in \Lambda_k^+ \times \{0,1,\dots,2m\}^k$ such that elements in $\mathcal{D}_1(\mathbf{n},\mathbf{m})$ are distinct and positive, and let A_1 be the complement of A_2. By the definition of $\tilde{\mathrm{T}}_n$, if any two terms in $\mathcal{D}_1$ are equal, we have $\mathcal{S}_1(\mathbf{n}-\mathbf{m},\mathbf{n}) = 0$. Note that $n_i - m_i + \rho_{k,i} \geq -2m + n - k + \frac{1}{2} \geq -m + \frac{1}{2}$. If $n_i - m_i + \rho_{k,i} \leq 0$ for some $1 \leq i \leq k$, then $n_i - m_i + \rho_{1,i} \in \{-(m-\frac{1}{2}), \dots, -\frac{1}{2}\}$. Then by the definition of $\tilde{\mathrm{T}}_n$ we also have $\mathcal{S}_1(\mathbf{n}-\mathbf{m},\mathbf{n}) = 0$. So A_1 satisfies (2).

Similarly we let $\mathcal{D}_2(\mathbf{a},\mathbf{b}) = \{a_1 + b_1 + \rho_{k,1}, \dots, a_k + b_k + \rho_{k,k}\}$. Let B_2 be the set of $(\mathbf{a},\mathbf{b}) \in \Lambda_k^+ \times \{0,1,\dots,2m\}^k$ such that elements in $\mathcal{D}_2(\mathbf{a},\mathbf{b})$ are distinct, and B_1 be the complement of B_2. Then by the definition of T_k, B_1 satisfies (2).

Condition (1) is obviously satisfied by our construction, so now we consider condition (3). For $(\mathbf{n},\mathbf{m}) \in A_2$, since elements in $\mathcal{D}_1$ are distinct and positive, there exists a unique $\epsilon_1 \in S_k$, such that $(\mathbf{n}-\mathbf{m}+\rho_k)^{\epsilon_1} \in \Lambda_k^+$. Let $I_1(\mathbf{n},\mathbf{m}) = ((\mathbf{n}-\mathbf{m}+\rho_k)^{\epsilon_1} - \rho_k, \mathbf{m}^{\epsilon_1})$. One can check that it is in B_2. Similarly since elements in $\mathcal{D}_2$ are distinct when $(\mathbf{a},\mathbf{b}) \in B_2$, there exists a unique $\epsilon_2 \in S_k$ such that $(\mathbf{a}+\mathbf{b}+\rho_k)^{\epsilon_2} \in \Lambda_k^+$. We let $I_2(\mathbf{a},\mathbf{b}) = ((\mathbf{a}+\mathbf{b}+\rho_k)^{\epsilon_2} - \rho_k, \mathbf{b}^{\epsilon_2})$. One can check that it is in A_2, and that I_1 and I_2 are inverse to each other, so they are bijections. When $I_1(\mathbf{n},\mathbf{m}) = (\mathbf{a},\mathbf{b})$, we have

$$
\begin{aligned}
&\mathcal{S}_2(\mathbf{a},\mathbf{a}+\mathbf{b})\\
=&(-1)^{|\mathbf{b}|} \wedge^{\mathbf{b}} (\Gamma_{\tilde{\sigma}})\tilde{\mathrm{T}}_n(\mathbf{a}; z_\pi)\mathrm{T}_k(\mathbf{a}+\mathbf{b}; z_\tau)q^{-s|\mathbf{a}+\mathbf{b}|}\\
=&(-1)^{|\mathbf{m}|} \wedge^{\mathbf{b}} (\Gamma_{\tilde{\sigma}})\tilde{\mathrm{T}}_n((\mathbf{n}-\mathbf{m}+\rho_k)^{\epsilon_1} - \rho_k; z_\pi)\mathrm{T}_k((\mathbf{n}+\rho_k)^{\epsilon_1} - \rho_k; z_\tau)q^{-s|\mathbf{n}|}
\end{aligned}
$$

By the definition of $\tilde{\mathrm{T}}_n$ and T_k, we have $\tilde{\mathrm{T}}_n((\mathbf{n}-\mathbf{m}+\rho_1)^{\epsilon_1} - \rho_1; z_\pi) = sgn(\epsilon_1)\tilde{\mathrm{T}}_n(\mathbf{n}-\mathbf{m}; z_\pi)$ and $\mathrm{T}_k((\mathbf{n}+\rho_1)^{\epsilon_1} - \rho_1; z_\tau) = sgn(\epsilon_1)\mathrm{T}_k(\mathbf{n}; z_\tau)$. So the formula above is equal to $\mathcal{S}_1(\mathbf{n}-\mathbf{m},\mathbf{n})$. So (3) is satisfied, completing our proof. $\square$

3.5 Estimation

Now we come back to (3.8). In this section we prove part (B) and (C) of Theorem 3.1.1. Fix $[\phi]_m$ and $\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(1)$, we let $W(\varphi_\pi, g) = \mathcal{T}\left(\pi(g)\varphi_\pi, [\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(1)\right)$. Then we are going to prove that

Lemma 3.5.1. *There exists finite functions λ_i, where $i \in \mathcal{I}$ which is a finite set, such that for any $\varphi_\pi \in \mathcal{V}_\pi$, there are Schwartz functions $\phi_i \in \mathcal{S}(\mathrm{F}^k)$, such that*

$$W(\varphi_\pi, T'_k(\hat{a})) = \sum_{i\in\mathcal{X}} \lambda_i(a_1, \dots, a_k)\phi_i(a_1, \dots, a_k). \tag{3.49}$$

By this lemma and the similar result for W_τ ([18, Proposition 2.2]) it is not hard to see that when $Re(s) >> 0$, the integral $J(\varphi_\pi, \phi, \widetilde{f}_{W_\tau,\widetilde{\sigma},s})$ in equation (3.8) is convergent and belongs to $\mathbb{C}[q^s, q^{-s}]$, which implies part (B) and (C) of **Theorem 3.1.1**.

The method we use is the same as in [18, Section 2]. Consider the mapping

$$W_\pi : \varphi_\pi \mapsto W(\varphi_\pi, g) \tag{3.50}$$

We denote the image by $W(\pi)$, and assume it to be nonzero for a suitable choice of $[\phi]_m$ and $f_{\widetilde{\sigma}}$. Note that W_π is a G-homomorphism. Since π is irreducible, W_π is an isomorphism from π to $W(\pi)$. We identify them by this isomorphism. We prove lemma (3.5.1) by applying [18, **Lemma 2.2.1**], which we recall here

Lemma 3.5.2 ([18],Lemma 2.2.1)**.** *Let $\mathcal{V}$ be a space of smooth functions on the group $H = \prod_{i=1}^k H_i$, $H_i \cong GL_1$. Assume $\mathcal{V}$ is invariant under translation. Suppose each ϕ in $\mathcal{V}$ vanishes on $a \in H$ when $|a_i|$ is sufficiently large for each i. On the other hand let $\mathcal{V}_i$ be the space of $\phi \in \mathcal{V}$ which vanish for $|a_i|$ small enough and σ_i the representation of H_i on $\mathcal{V}/\mathcal{V}_i$. Suppose the algebra $\mathcal{A}_i$ spanned by the operator $\sigma_i(a), a \in H_i$, is finite dimensional for each i. Then there is a finite set X of finite functions on H such that every $\phi \in \mathcal{V}$ can be written*

$$\phi(a) = \sum \chi(a)\phi_\chi(a), \quad \phi_\chi \in \mathcal{S}(F^k), \quad \chi \in X. \tag{3.51}$$

Note that the conclusion in this lemma is exactly what we want to prove in lemma 3.5.1 if we let $H = T'_k$, $\mathcal{V} = W(\pi)$, H_i be as we defined, and $\mathcal{V}_i$ as follows. Recall that $P_{i,n-i}$ is the parabolic subgroup of G with Levi decomposition $P_{i,n-i} = GL_i \times Sp_{2(n-i)} \ltimes V_{i,n-i}$. Let $W(\pi, V_{i,n-i}) = span\{W(\varphi_\pi, gv) - W(\varphi_\pi, g) | \varphi_\pi \in \mathcal{V}_\pi, v \in V_{i,n-i}\}$, and $W(\pi)_{V_{i,n-i}} = W(\pi)/W(\pi, V_{i,n-i})$. They are $GL_i \times Sp_{2(n-i)}$-modules. Treat $W(\pi)_{V_{i,n-i}}$ as a H_i-module, and let it be $\mathcal{V}_i$. What we need is to verify the conditions in the lemma, that is, to show that for any given φ_π,

(a) $W(\varphi_\pi, T'_k(a)) = 0$ when $|a_i|$ is sufficiently large for some i.

(b) $W(\varphi_\pi, T'_k(a)) = 0$ when $|a_i|$ is sufficiently small for all $W(\varphi_\pi, g) \in W(\pi, V_{i,n-i})$.

(c) The algebra $\mathcal{A}_i$ spanned by the operator $\sigma_i(a), a \in H_i$, is finite dimensional for each i.

Verification of (a),(b) and (c). For (a) let $u_i(x) = I_{2n} + x \cdot n_{e_i - e_{i+1}}$ for $1 \leq i \leq k$. Note that $W(\varphi_\pi, u_i(x)g) = \psi(x)W(\varphi_\pi, g)$, and that $T'_k(a)u_i(x)T'_k(a^{-1}) = u_i(ax)$. So when $|a_i|$ is sufficiently large one can pick $|x|$ so small that $\pi(u_i(x))$ stabilizes φ_π but $\psi(a_i x) \neq 1$, then we have

$$W(\varphi_\pi, T'_k(a)) = W(\varphi_\pi, T'_k(a)u_i(x)) = \psi(a_i x)W(\varphi_\pi, T'_k(a)), \tag{3.52}$$

which implies $W(\varphi_\pi, T'_k(a)) = 0$. For (b) we take

$$W(\varphi_\pi, T'_k(a)) = \sum_{j=1}^{N} W(\varphi_j, T'_k(a)v^j_{i,n-i}) - W(\varphi_j, T'_k(a))$$

for some $\varphi_j \in \pi$ and $v^j_{i,n-i} \in V_{i,n-i}$. Then when $|a_i|$ is sufficiently small we have $\psi_{V^n_{n-m}}(T'_k(a)v^j_{i,n-i}T'_k(a^{-1})) = 1$ for all j, so

$$W(\varphi_\pi, T'_k(a)) = \sum_{j=1}^{N} \psi_{V^n_{n-m}}(T'_k(a)v^j_{i,n-i}T'_k(a^{-1}))W(\varphi_j, T'_k(a)) - W(\varphi_j, T'_k(a)) = 0. \tag{3.53}$$

For (c), note that since π is irreducible, its Jacquet module $W(\pi)_{V_{i,n-i}}$ is of finite-length. Now H_i is the center of $GL_i \times Sp_{2(n-i)}$, so its action on $W(\pi)_{V_{i,n-i}}$, generates a finite dimensional algebra. $\square$

3.6 Non-vanishing theorem

In this section we prove part (D).

Propostion 3.6.1. *For any given s, there exists a choice of $(\varphi_\pi, \phi, \widetilde{f}_{W_\tau,\widetilde{\sigma},s})$ so that $I(\varphi_\pi, \phi, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}) \neq 0$.*

Proof. We argue by contradictions. Suppose for any choice of data we have

$$I(\varphi_\pi, \phi, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}) \equiv 0. \tag{3.54}$$

Recall from (3.21) that

$$\begin{aligned}
&I(\varphi_\pi, \phi, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}) \\
=&\int_{N_k\backslash GL_k} \mathrm{d}g \int_{(P_{k,m}\cap K_{Sp_{2(m+k)}}\backslash K_{Sp_{2(m+k)}})} \mathrm{d}k' \int_R \mathrm{d}r \delta_{P_{k,m}}^{-1}(g) \\
&\mathcal{T}(\pi(wrgk')\varphi_\pi, [\omega_\psi(l_{m+k}(r)gk')\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(gk')).
\end{aligned}$$

Step 1. First, (3.54) implies that

$$\int_{N_k\backslash GL_k} \mathrm{d}g \int_R \mathrm{d}r \delta_{P_{k,m}}^{-1}(g) \mathcal{T}(\pi(wrg)\varphi_\pi, [\omega_\psi(l_{m+k}(r)g)\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(g)) \equiv 0. \tag{3.55}$$

To see this, let $\mathit{\Phi} \in \mathcal{S}(P_{k,m} \cap K_{Sp_{2(m+k)}} \backslash K_{Sp_{2(m+k)}})$. We define

$$\widetilde{f}^{\mathit{\Phi}}_{W_\tau,\widetilde{\sigma},s}(g) = \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(g) \cdot \mathit{\Phi}(k'), \tag{3.56}$$

where $g = p'k'$ is the Iwasawa decomposition with respect to $\mathrm{G} = P_{k,m}K_{\mathrm{G}}$. Note that $\widetilde{f}^{\Phi}_{W_\tau,\widetilde{\sigma},s}(g) \in \mathrm{Ind}^{\widetilde{Sp}_{2(m+k)}(\mathbb{A})}_{P_{k,m}(\mathbb{A})}(\gamma_\psi W_\tau |\det|^{s-\frac{1}{2}} \otimes \widetilde{\sigma})$. So (3.54) actually implies that

$$\int_{N_k\backslash GL_k} \mathrm{d}g \int_{(P_{k,m}\cap K_{Sp_{2(m+k)}}\backslash K_{Sp_{2(m+k)}}} \mathrm{d}k' \int_R \mathrm{d}r \delta^{-1}_{P_{k,m}}(g)$$
$$\mathcal{T}(\pi(wrgk')\varphi_\pi, [\omega_\psi(l_{m+k}(r)gk')\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(gk'))\Phi(k')$$

for any $\Phi \in \mathcal{S}(P_{k,m}\cap K_{Sp_{2(m+k)}}\backslash K_{Sp_{2(m+k)}})$, which implies (3.55).

Step 2. Next, (3.55) implies that

$$\int_{N_k\backslash GL_k} \mathrm{d}g \delta^{-\frac{1}{2}}_{P_{k,m}}(g)|g|^{s-(n-m-k)} W_\tau(g)\mathcal{T}(\pi(g)\varphi_\pi, [\phi]_m, \widetilde{f}_{\widetilde{\sigma}}) \equiv 0. \tag{3.57}$$

Here GL_k embeds to G as $g \mapsto diag(g, I_{2(n-k)}, g^{-1})$, and $\widetilde{f}_{\widetilde{\sigma}} = \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(1)$. If we change the order of r and g by a change of variable, and then replace φ_π by $\pi(w^{-1})\varphi_\pi$, and then conjugate g and r by w in (3.55), then

$$\int_{N_k\backslash GL_k} \mathrm{d}g \int_Y \mathrm{d}y$$
$$\delta^{-1}_{P_{k,m}}(g)|g|^{-(n-m-k)}\mathcal{T}(\pi(gy)\varphi_\pi, [\omega_\psi(gl_{m+k}(w^{-1}yw))\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(g)) \equiv 0. \tag{3.58}$$

Note that $Y = Y^{n-m-k-1} \times Y_{n-m-k}$. Separate the integral on Y as $\int_{Y^{n-m-k-1}} \int_{Y_{n-m-k}}$ and let $y = y^{n-m-k-1}Y_{n-m-k}(x)$ with $y_{n-m-k-1} \in Y_{n-m-k-1}$ and $x \in F^k$. Let Φ be any Schwartz function on F^k. Replace φ_π by $\int_{F^k} \Phi(l)\pi(Z(l))\varphi_\pi \,\mathrm{d}l$. Then (3.58) implies

$$\begin{aligned}&\int_{N_k\backslash GL_k} \mathrm{d}g \int_{Y^{n-m-k-1}} \mathrm{d}y \int_{F^k} \mathrm{d}x \int_{F^k} \mathrm{d}l \delta^{-1}_{P_{k,m}}(g)|g|^{-(n-m-k)}\\ &\Phi(l)\mathcal{T}(\pi(gyY_{n-m-k}(x)Z(l))\varphi_\pi, [\omega_\psi(gx)\phi]_m, \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(g)) \equiv 0.\end{aligned} \tag{3.59}$$

Note that $Y_{n-m-k}(x)Z(l) = (I_{2n} + x\cdot lE_{n-m,n+m+1})Z(l)Y_{n-m-k}(x)$, and that $GL_k \ltimes Y^{n-m-k-1}$ stabilizes both $\psi_{V^m_{n-m}}$ and the projection l_m from V^m_{n-m} to H_{2m+1}, so by the property of $\mathcal{T}$,

$$\begin{aligned}&\mathcal{T}(\pi(gyY_{n-m-k}(x)Z(l))\varphi_\pi,[\omega_\psi(gx)\phi]_m,\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(g))\\=&\psi(-x\cdot l)\mathcal{T}(\pi(gyY_{n-m-k}(x))\varphi_\pi,[\omega_\psi(gx)\phi]_m,\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(g))\end{aligned}\tag{3.60}$$

Then (3.59) implies that

$$\begin{aligned}&\int_{N_k\backslash GL_k}\mathrm{d}g\int_{Y^{n-m-k-1}}\mathrm{d}y\int_{F^k}\mathrm{d}x\delta_{P_{k,m}}^{-1}(g)|g|^{-(n-m-k)}\\&\hat{\Phi}'(-x)\mathcal{T}(\pi(gyY_{n-m-k}(x))\varphi_\pi,[\omega_\psi(gx)\phi]_m,\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(g))\equiv 0.\end{aligned}\tag{3.61}$$

Since Φ' is arbitrary, we have

$$\begin{aligned}&\int_{N_k\backslash GL_k}\mathrm{d}g\int_{Y^{n-m-k-1}}\mathrm{d}y\delta_{P_{k,m}}^{-1}(g)|g|^{-(n-m-k)}\\&\mathcal{T}(\pi(gy)\varphi_\pi,[\omega_\psi(g)\phi]_m,\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(g))\equiv 0.\end{aligned}\tag{3.62}$$

Using the same method we can remove the integral on Y_i (by using L_{i+1}) for i from $n-m-k-1$ to 1. So eventually,

$$\int_{N_k\backslash GL_k}\mathrm{d}g\delta_{P_{k,m}}^{-1}(g)|g|^{-(n-m-k)}\mathcal{T}(\pi(g)\varphi_\pi,[\omega_\psi(g)\phi]_m,\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(g))\equiv 0,\tag{3.63}$$

which is equivalent to (3.57).

Step 3. Let $v=(v_1,\ldots,v_k)$ and $\Phi'(v)$ be a Schwartz function on F^k. Then (3.57) implies

$$\int_{N_k\backslash GL_k}\mathrm{d}g\int_{F^k}\mathrm{d}v\delta_{P_{k,m}}^{-\frac{1}{2}}(g)|g|^{s-(n-m-k)}\Phi'(v)W_\tau(g)\mathcal{T}(\pi(gL_1(v))\varphi_\pi,[\phi]_m,\widetilde{f}_{\widetilde{\sigma}})\equiv 0.\tag{3.64}$$

Note that by the property of $\mathcal{T}$,

$$\begin{aligned}&\mathcal{T}(\pi(gL_1(v))\varphi_\pi,[\phi]_m,\widetilde{f}_{\widetilde{\sigma}})\\=&\psi_{V_{n-m}^n}(gL_1(v)g^{-1})\mathcal{T}(\pi(g)\varphi_\pi,[\phi]_m,\widetilde{f}_{\widetilde{\sigma}}).\end{aligned}\tag{3.65}$$

Let $(g_1, \dots, g_k)$ be the last row of g, then $\psi_{V^n_{n-m}}(gL_1(v)g^{-1}) = \psi(\sum_{i=1}^k g_i v_i)$. So then we have

$$\int_{N_k\backslash GL_k} \mathrm{d}g \delta_{P_{k,m}}^{-\frac{1}{2}}(g)|g|^{s-(n-m-k)}\hat{\Phi}'(\epsilon_k g) W_\tau(g)\mathcal{T}(\pi(g)\varphi_\pi, [\phi]_m, \widetilde{f}_{\widetilde{\sigma}}) \equiv 0. \tag{3.66}$$

Here $\epsilon_k = (0, \dots, 0, 1) \in F^k$. Let GL_k acts on F^k in the standard way, and let Q_k be the mirabolic subgroup of GL_k stabilizing ϵ_k. Since Φ is arbitrary, (3.66) implies that

$$\int_{N_k\backslash Q_k} \mathrm{d}g \delta_{P_{k,m}}^{-\frac{1}{2}}(g)|g|^{s-(n-m-k)}\ W_\tau(g)\mathcal{T}(\pi(g)\varphi_\pi, [\phi]_m, \widetilde{f}_{\widetilde{\sigma}}) \equiv 0. \tag{3.67}$$

Note that $N_k\backslash Q_k \cong N_{k-1}\backslash GL_{k-1}$. Use the same method on GL_{k-1} one can reduce the integral from $N_{k-1}\backslash GL_{k-1}$ to $N_{k-2}\backslash GL_{k-2}$, and so on. Eventually we have $W_\tau(1)\mathcal{T}(\pi(1)\varphi_\pi, [\phi]_m, \widetilde{f}_{\widetilde{\sigma}}) \equiv 0$, which is a contradiction. So our theorem is proved. $\square$

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