Defects and Stability of Turing Patterns

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Dedication

To my parents Xieming Wu and Yunxian Wu.

Abstract

This paper is concerned with formation mechanisms of patterns and contains two main parts. The first one is about defects of patterns. Specifically, we study grain boundaries in the Swift-Hohenberg equation. Grain boundaries arise as stationary interfaces between roll solutions of different orientations. Our analysis shows that such stationary interfaces exist near onset of instability for *arbitrary* angle between the roll solutions. This extends prior work in [1] where the analysis was restricted to large angles, that is, weak bending near the grain boundary. The main new difficulty stems from possible interactions of the primary modes with other resonant modes. We generalize the normal form analysis in [1] and develop a singular perturbation approach to treat resonances. In the second part, we investigate dynamics near Turing patterns in reaction-diffusion systems posed on the real line. Linear analysis predicts diffusive decay of small perturbations. We construct a "normal form" coordinate system near such Turing patterns which exhibits an approximate discrete conservation law. The key ingredients to the normal form is a conjugation of the reaction-diffusion system on the real line to a lattice dynamical system. At each lattice site, we decompose perturbations into neutral phase shifts and normal decaying components. As an application of our normal form construction, we prove nonlinear stability of Turing patterns with respect to perturbations that are small in $L^1 \cap L^\infty$, with sharp rates, recovering and slightly improving on results in [2, 3].

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Chapter 1

Introduction

1.1 Pattern formation

Patterns are everywhere in nature [4, 5, 6, 7] and have yet been systematically studied for barely a century [8]. Turing's pioneering work [9] established a profound mathematical basis for pattern forming phenomena, featuring the start of the systematical and still on-going research on the dynamics of pattern forming systems within the mathematical community. The application of dynamical systems techniques in evolutionary PDEs, initially developed as methods to crack problems arising in delayed differential equations and generalized to be applicable to general evolutionary PDEs by mathematicians including J. K. Hale and D. Henry in 1970's-1980's [10, 11], has become one of the main tools in this field.

To mathematically illustrate the formation mechanisms of patterns, we are concerned primarily with the existence of solutions representing patterns, their defects and their interfaces, together with their qualitative properties, such as linear and nonlinear stability, instabilities, bifurcations, etc. Specifically, we present two results: Existence of small-amplitude grain boundaries of arbitrary angle in the Swift-Hohenberg equation in Chapter 2, and diffusive stability of Turing patterns via normal forms in Chapter 3. In the following two sections we will briefly introduce the background, motivation and main results. We point out that the notation in Section 1.2, Chapter 2 and Appendix A are independent of those in Section 1.3, Chapter 3 and Appendix B.

1.2 Grain boundaries

Rayleigh-Bénard convection has been one of the prime motivating examples of research in pattern formation. Much insight into the dynamics has been gained by formal or rigorous approximation of the dynamics near onset of instability: convection rolls form when the temperature difference between two plates overcomes the viscosity of the fluid in between. Basic patterns are typically periodic in, say, x, with a characteristic wavelength $L = 2\pi/k$. While such patterns can be readily analyzed mathematically using bifurcation theory, much less is known about imperfections in these perfect periodic patterns that arise in experiments and numerical simulations. Formally such imperfections or defects can be classified phenomenologically, leading to lists including defects such as grain boundaries, dislocations, and disclinations. Very little, however, is known about these defects beyond far-field approximations. Mere existence, for instance, is not known even for simple model approximations. Our goal here is to extend the methods used in [1] and show existence of grain boundaries near onset of instability. We remove a crucial restriction from [1] which forced the angle between grain boundaries to be larger than $\pi/3$, effectively preventing interaction of the primary modes with other possible resonant modes. Our results confirm earlier findings in [12], where an approximation by amplitude equations was analyzed. We show that the approximation there, neglecting higher-order terms and non-adiabatic effects, as well as possible coupling to other, rotated modes, is justified. On the other hand, our results exclude to some extent, at small amplitude, bifurcations such as the ones observed in [13, 14].

Swift-Hohenberg equation We focus our attention on the arguably simplest pattern forming system, the Swift-Hohenberg equation

$$\partial_t u = -(1+\Delta)^2 u + \mu u - u^3. \tag{1.2.1}$$

Here, $u(t,x,y)$ depends on $(x,y) \in \mathbb{R}^2$ and time $t \in \mathbb{R}$, and μ is a real parameter. Simple bifurcation analysis shows the existence of solutions $u_{\mathrm{r}}(kx;k,\mu)$ which are spatially periodic $u_{\mathrm{r}}(\xi;k,\mu) = u_{\mathrm{r}}(\xi + 2\pi;k,\mu)$, and even in ξ for $\mu > 0$, small. We refer to these stationary periodic patterns as roll solutions and denote rotated roll patterns as

$$u_{\mathrm{r}}^{\varphi}(x,y;k) := u_{\mathrm{r}}(k(x\cos\varphi - y\sin\varphi);k,\mu), \quad \varphi \in [0,2\pi). \tag{1.2.2}$$

Grain boundaries solutions Grain boundaries are solutions to (1.2.1) with $u_t = 0$, that are asymptotic to roll solutions of different orientation as $x \to \pm\infty$. In the simplest case that we shall be interested in, here, they possess an additional reflection symmetry $x \mapsto -x$ and periodic in y. This can be seen as a maximal symmetry assumption for a grain boundary, since the pattern imposed by asymptotic roll solutions with different angles accomodates such a reflection symmetry and periodicity.

We construct the grain boundaries as solutions to the equation (1.2.1) which are steady and periodic in y with wavenumber $k = k_*$ for sufficiently small μ. Rescaling y, we need to solve

$$0 = -(1 + \partial_x^2 + k^2\partial_y^2)^2 u + \mu u - u^3, \tag{1.2.3}$$

on $x, y \in \mathbb{R}^2$ with 2π-periodicity in y.

Following [1], we also impose boundary conditions at infinity in the form

$$|u_{\mathrm{gb}}(x - x_\pm, y) - u_{\mathrm{r}}^{\varphi_\pm}(x, y; k_\pm)| \to 0 \text{ for } x \to \pm\infty. \tag{1.2.4}$$

Here, the constants are suitable asymptotic phases $x_\pm$, asymptotic angles $\varphi_\pm$, and asymptotic wavenumbers $k_\pm$ for the roll solutions in the far field. Our symmetry assumption implies that $u_{\mathrm{gb}}(x, y) = u_{\mathrm{gb}}(-x, y)$, $\varphi_- = -\varphi_+$, $k_+ = k_-$ and $x_+ = -x_-$. Possibly translating in y, we may assume that $x_\pm = 0$.

Main result Our main result states existence of such grain boundaries for small values of μ and arbitrary $\varphi_\pm$.

Theorem 1 *For every $\epsilon_\varphi \in (0, \pi/2)$ there exists $\mu_* > 0$ so that for every $\varphi_- - \varphi_+ = \alpha \in (\epsilon_\varphi, \pi - \epsilon_\varphi)$, and any $0 < \mu < \mu_*(\epsilon_\varphi)$, there exists a symmetric grain boundary between rolls of orientations $\varphi_\pm$, for a selected wavenumber $k(\alpha, \mu)$.*

We emphasize that the main difference from the results in [1] is that our theorem here covers *arbitrary* angles, removing the restriction of $\alpha > \pi/3$. We also show existence for small μ, uniformly in the angle α. The limiting cases $\alpha = 0$ and $\alpha = \pi$ are also interesting. Weak bending, $\alpha = \pi$, was studied in [15], basically showing existence in this regime as well. The limit $\alpha \to 0$ appears to be much more challenging. We will give the proof of this result in Chapter 2, largely based on [16].

1.3 Turing patterns

Turing predicted that the simple interplay of reaction and diffusion can lead to stable, spatially periodic patterns [9]. His ideas proved quite influential in the general area of pattern formation, where one seeks to understand the formation and dynamics of self-organized spatio-temporal structures. One can easily envision simple reaction-diffusion systems with two species that exhibit diffusion-driven instabilities of spatially homogeneous equilibria. Typical examples are activator-inhibitor systems such as the Gray-Scott or the Gierer-Meinhard equation; see for instance [17, 18]. Perturbations of the homogeneous unstable equilibrium grow exponentially at an initial stage, with fastest growth for distinct spatial wavenumbers. This wavenumber is roughly independent of boundary conditions in large enough domains. As a final result, one often finds a spatially periodic pattern, up to narrow, exponentially localized boundary layers. In order to understand such nonlinear spatially periodic patterns and the process of wavenumber selection, one is therefore naturally led to considering reaction-diffusion systems on idealized unbouned domains.

Turing patterns in reaction-diffusion systems We consider

$$\mathbf{u}_t = D\triangle\mathbf{u} + \mathbf{f}(\mathbf{u}),$$

for $\mathbf{u}(t,x) \in \mathbb{R}^n$, with $x \in \mathbb{R}^N$, with smooth reaction-kinetics $\mathbf{f}$ and positive diagonal diffusion matrix $D = \mathrm{diag}\,(d_j) > 0$. Here, and in the following, the term "smooth" refers to functions with sufficiently many derivatives. In many circumstances, one can show that there exist families of spatially periodic striped solutions,

$$\mathbf{u}(t,x) = \mathbf{u}_\star(kx_1;k), \qquad \mathbf{u}_\star(\xi;k) = \mathbf{u}_\star(\xi + 2\pi;k),$$

parameterized by the spatial wavenumber $k > 0$. In fact, such families occur for an open class of reaction-diffusion systems, including but not limited to systems of activator-inhibitor type mentioned above.

As a first predictor on the stability of such solutions with respect to perturbations, one analyzes the linearization,

$$\mathbf{v}_t = D\triangle\mathbf{v} + \mathbf{f}'(\mathbf{u}_\star(kx;k))\mathbf{v}. \tag{1.3.1}$$

It turns out that, again for open classes of reaction-diffusion systems including the above examples, solutions to this linear equation are bounded for bounded initial data, for an open subset of patterns $\mathbf{u}_\star(\cdot;k)$ in the family. We refer to such patterns as *linearly stable Turing patterns*. We will discuss detailed assumptions that guarantee such linear stability later in the rest this section.

Nonlinear dynamics The presence of a family of patterns, parameterized by the wavenumber, and, even more obviously, by translations of the pattern in x, implies that solutions to (1.3.1) with general initial conditions will not decay. More explicitly, $\mathbf{v}(t,x) = \partial_x \mathbf{u}_\star(kx;k)$ and $\mathbf{v}(t,x) = \frac{\mathrm{d}}{\mathrm{d}k}\mathbf{u}_\star(kx;k)$ are constant in time and solve (1.3.1).

In fact, one can show that under typical assumptions, initial conditions $\mathbf{v}(t=0,x) \in L^1(\mathbb{R}^N,\mathbb{R}^n)$ will give rise to diffusive decay, $\sup_x |\mathbf{v}(t,x)| \leqslant Ct^{-N/2}$. Such algebraic decay is in general not strong enough to ensure nonlinear decay in dimensions $N \leqslant 3$. The simplest example is the nonlinear heat equation

$$u_t = \triangle u + u^2,$$

which exhibits blowup of arbitrarily small, smooth, positive initial data at finite time in dimensions $N \leqslant 3$ [19, 20]. In the seminal paper [2], Schneider recognized that diffusive decay near Turing patterns is not altered by the presence of nonlinear terms due to cancellations in a Bloch-wave expansion. He studied the most difficult case, $N = 1$, where diffusion is weak and nonlinearity potentially most dangerous, in the specific example of the Swift-Hohenberg equation. His proof has later been generalized, simplified, and adapted; see [21, 22, 23, 24, 25, 3, 26, 27]. Our focus here is, again, on the one-dimensional case, in a general reaction-diffusion setting. Our goal is to find coordinates that show explicitly why nonlinear terms do not alter linear decay near Turing patterns. Going back to the scalar heat equation, the interaction of nonlinear terms with diffusion can be categorized as relevant, critical, or irrelevant; [28, 29]. Explicitly, in the heat equation $u_t = u_{xx} + f(u,u_x,u_{xx})$,

(i) Nonlinear terms such as $f(u,u_x,u_{xx}) = uu_{xx}, u_x^2, u^p$, where $p > 3$ are irrelevant;

(ii) Nonlinear terms such as $f(u,u_x,u_{xx}) = uu_x, u^3$ are critical;

(iii) Nonlinear terms such as $f(u) = u^2$ are relevant.

Without pretending to fully explain this phenomenon, notice that, for L^1-initial data, assuming Gaussian decay, we find $u_{xx} \sim t^{-3/2}$ in L^∞. Irrelevant nonlinear terms decay with rate $t^{-\alpha}$, $\alpha > 3/2$, critical terms have $\alpha = 3/2$, and relevant terms have $\alpha < 3/2$.

Perturbations $\mathbf{v}$ of Turing patterns solve a system

$$\mathbf{v}_t = \partial_{xx}\mathbf{v} + \mathbf{f}'(\mathbf{u}_\star(x))\mathbf{v} + \mathbf{g}(x, \mathbf{v}),$$

where $\mathbf{g}(x, \mathbf{v}) = \mathrm{O}(|\mathbf{v}|^2)$. Note that from here on, we fix the wavenumber $k = 1$, without loss of generality, and write $\mathbf{u}_\star(x) := \mathbf{u}_\star(x; 1)$. In particular, the nonlinearity $\mathbf{g}$ has potentially dangerous quadratic terms. Roughly speaking, our goal is to find coordinates in which the nonlinearity involves at least two "derivatives", which according to the numerology for the scalar heat equation would be sufficient to guarantee nonlinear decay. The reason to hope for derivatives is the presence of a conservation law associated with the translation symmetry, which in turn generates the neutral decay in the linearization.

Main result We now consider reaction diffusion systems

$$\mathbf{u}_t = D\partial_{xx}\mathbf{u} + \mathbf{f}(\mathbf{u}), \tag{1.3.2}$$

where $\mathbf{u}, \mathbf{f} \in \mathbb{R}^n$, $x \in \mathbb{R}$, $t \in (0, +\infty)$, $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with strictly positive diagonal entries and $\mathbf{f}$ is smooth. Firstly, we assume the existence of a Turing pattern of the system.

Hypothesis 1.3.1 (**existence**) *The system of ordinary differential equations* $D\partial_{xx}\mathbf{u} + \mathbf{f}(\mathbf{u}) = 0$ *possesses a smooth periodic even solution* $\mathbf{u}_\star$*.*

Without loss of generality, we assume that the period is 2π. Our aim is to study nonlinear stability of this temporal equilibrium under general small non-periodic perturbations. To this end, we introduce an initial condition

$$\mathbf{u}(0, x) = \mathbf{u}_\star(x) + \mathbf{v}^0(x). \tag{1.3.3}$$

Then assuming that $\mathbf{u}(t, x) = \mathbf{u}_\star(x) + \mathbf{v}(t, x)$ is a solution to (1.3.2) with the given initial condition (1.3.3), we have

$$\begin{cases} \mathbf{v}_t = A\mathbf{v} + \mathbf{g}(x, \mathbf{v}), \\ \mathbf{v}(0) = \mathbf{v}^0, \end{cases} \tag{1.3.4}$$

where

$$\begin{array}{rccc} A: & X^1 & \longrightarrow & X \\ & \mathbf{v} & \longmapsto & D\partial_{xx}\mathbf{v} + \mathbf{f}'(\mathbf{u}_\star)\mathbf{v}. \end{array} \tag{1.3.5}$$

Here we define

$$X = (L^1(\mathbb{R}))^n \cap (L^\infty(\mathbb{R}))^n, \quad X^1 = (W^{2,1}(\mathbb{R}))^n \cap (W^{2,\infty}(\mathbb{R}))^n, \tag{1.3.6}$$

with norms

$$\|\cdot\|_X = \|\cdot\|_{L^1} + \|\cdot\|_{L^\infty}, \quad \|\cdot\|_{X^1} = \|\cdot\|_{W^{2,1}} + \|\cdot\|_{W^{2,\infty}}.$$

Note that from now on, we suppress n and $\mathbb{R}$ if there is no ambiguity. Moreover, $\mathbf{g} : \mathbb{T}_{2\pi} \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth, $\mathbf{g}(x, \mathbf{v}) = \mathbf{f}(\mathbf{u}_\star + \mathbf{v}(x)) - \mathbf{f}(\mathbf{u}_\star) - \mathbf{f}'(\mathbf{u}_\star)\mathbf{v}$, so that $\mathbf{g}(x, 0) \equiv 0$ and $\partial_{\mathbf{v}}\mathbf{g}(x, 0) \equiv 0$.

According to Bloch wave decomposition, let us introduce the family of Bloch operators, for $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$,

$$\begin{array}{rccc} B(\sigma): & (H^2(\mathbb{T}_{2\pi}))^n & \longrightarrow & (L^2(\mathbb{T}_{2\pi}))^n \\ & \mathbf{v} & \longmapsto & D(\partial_x + \mathrm{i}\sigma)^2\mathbf{v} + \mathbf{f}'(\mathbf{u}_\star)\mathbf{v}. \end{array} \tag{1.3.7}$$

For further reading on Bloch wave decomposition and Bloch operators, we refer to Section B.2 and [30]. Note that one obtains $B(\sigma)$ formally by applying A to functions of the form $\mathbf{u} = \mathrm{e}^{\mathrm{i}\sigma x}\mathbf{v}$.

Hypothesis 1.3.2 (**spectral stability**) *The family of Bloch wave operators $B(\sigma)$ has the following properties.*

(i) $\mathrm{spec}(B(\sigma)) \bigcap \{Re\lambda \geqslant 0\} = \emptyset$, *for $\sigma \neq 0$;*

(ii) $\mathrm{spec}(B(0)) \bigcap \{Re\lambda \geqslant 0\} = \{0\}$ *and 0 is simple with* $\mathrm{span}\{\mathbf{u}_\star'\}$ *as its eigenspace;*

(iii) Near $\sigma = 0$, the only eigenvalue λ is a smooth function of σ and the expression of $\lambda(\sigma)$ reads: $\lambda(\sigma) = -d\sigma^2 + \mathrm{O}(|\sigma|^3)$, where $d > 0$ is a constant.

Remark 1.3.3 *The expansion in (iii) is a consequence of the simplicity of $\lambda = 0$ at $\sigma = 0$ and the evenness of $\mathbf{u}_\star$. In fact, we have an "explicit" expression for d; see Section B.4.*

Given the above hypotheses, we can state our main result.

Theorem 2 (**nonlinear stability**) *Assume Hpotheses 1.3.1 and 1.3.2 hold. There are* $C, \sigma > 0$ *so that, for any* $\|\mathbf{v}^0\|_X < \sigma$*, where* $X = (L^1(\mathbb{R}))^n \cap (L^\infty(\mathbb{R}))^n$*, the solution* $\mathbf{v}(t)$ *to the system* (1.3.4) *exists for time* $t \in [0, \infty)$ *and satisfies the estimate*

$$\|\mathbf{v}(t)\|_{(L^\infty(\mathbb{R}))^n} \leqslant C \frac{\|\mathbf{v}^0\|_X}{(1+t)^{\frac{1}{2}}}. \tag{1.3.8}$$

We will give the proof of this theorem in Chapter 3, which is largely based on [31].

Chapter 2

Small-amplitude grain boundaries of arbitrary angle in the Swift-Hohenberg equation

This chapter is occupied with the proof of Theorem 1. Our proof is based on spatial dynamics, rewriting (1.2.3) as an ill-posed dynamical system in the x-variable. We then follow the general strategy of bifurcation theory, using center-manifold reduction and normal form theory to identify a simple reduced differential equation. Within the leading-order terms of this equation, we find heteroclinic orbits that correspond to grain boundaries. We then proceed and show that these heteroclinic orbits persist under higher-order perturbations. We refer to [1] for more background on these methods and references to the literature.

While our methods build on the general strategy introduced in [1], there are several key differences. First, the dimension of the reduced ODE can be arbitrarily large, so that normal form theory is essential to reduce to a tractable system. The increase in dimension stems from the fact that for small k_*, that is, large y-period in the original scale, *many* orientations of roll solutions are compatible with periodicity in y. The analysis therefore needs to incorporate possible resonant interactions between the primary rolls and rotated linear and nonlinear modes. Major technical difficulties are introduced when rolls perpendicular to the grain boundary interface are compatible with y-periodicity.

We refer to this case, $1/k_* \in \mathbb{Z}$, as the resonant case. In fact, varying k_* across such a resonance, the dimension of the center-manifold changes. The linearization exhibits a Jordan block of length 4, introducing a new scale into the system. As a consequence, we need a refined normal form theory. More crucially, the standard scaling $x \sim \mu^{-1/2}$ for the grain boundary introduces singularities in the perpendicular mode, where length scales vary with $\mu^{-1/4}$. Moreover, as we vary k_* near such a resonance, we find a subtle crossover between the $\mu^{-1/4}$- and $\mu^{-1/2}$-scalings. When studying persistence, we then encounter a singularly perturbed differential operator in the linearization. We develop semi-explicit estimates on the Green's function for that operator and employ a frozen Newton method to show persistence.

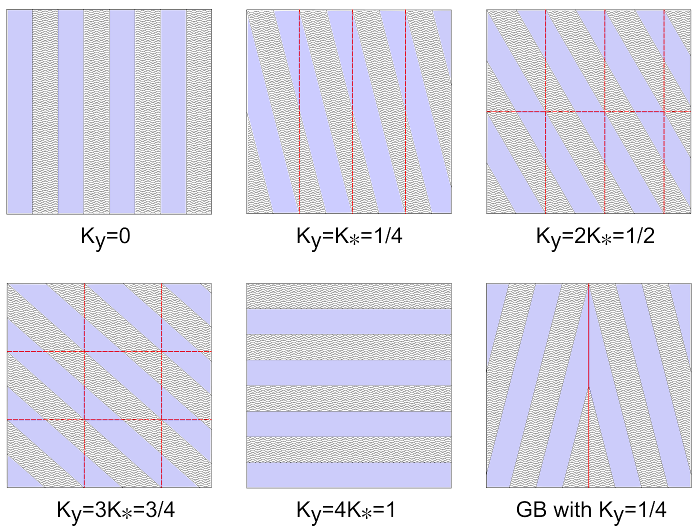

Figure 0.1: The pictures show compatible roll solutions and the grain boundary in the case $k_* = 1/4$. The first five pictures show roll solutions with orientations comptible with the induced vertical y-period 8π. The red dashes show identical periodic cells in each case.

Figure 0.1 illustrates some of these difficulties. The figure shows that, given k_*, there are possibly many different orientations of roll patterns that are compatible with the imposed y-periodicity. In the case shown there, $k_* = 1/4$, there are 5 possible orientations. Note that the horizontal pattern is compatible only for integer choices of $1/k_*$. As k_* decreases past such resonances, more orientations become compatible with the periodicity in y. Note also that the figure shows how the grain boundaries we are finding possess maximal symmetry with respect to y-translations. In fact, we could have also tried to find grain boundaries between other orientations of rolls from Figure 0.1, which would possibly give non-symmetric grain boundaries, or grain boundaries with a y-period less than the imposed period. We will comment on these issues briefly, throughout the proofs.

This chapter is organized as follows. Section 2.1 is concerned with the non-resonant case, $1/k_* \notin \mathbb{Z}$. Section 2.2 treats resonance and near-resonance $1/k_* \in \mathbb{Z}$.

2.1 Grain boundaries for non-resonant angles $1/k_* \notin \mathbb{Z}$

We prove Theorem 1 in the non-resonant case. We first set up the spatial dynamics formulation, Section 2.1.1, then carry out the center-manifold reduction after identifying linearly neutral modes in Section 2.1.2. Normal form transformations put the system in a simple standard form, Section 2.1.3. We finally prove existence and persistence of heteroclinic orbits in Section 2.1.4.

2.1.1 Spatial dynamics

Our basic approach follows [1]. We write the stationary Swift-Hohenberg equation (1.2.3) formally as a first-order differential equation in space x,

$$\frac{dU}{dx} = \mathcal{A}(\mu, k)U + \mathcal{F}(U), \tag{2.1.1}$$

setting

$$U = \begin{pmatrix} u \\ u_1 \\ v \\ v_1 \end{pmatrix}, \quad \mathcal{A}(\mu, k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(1+k^2\partial_y^2) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu & 0 & -(1+k^2\partial_y^2) & 0 \end{pmatrix}, \quad \mathcal{F}(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -u^3 \end{pmatrix}.$$

While ill-posed as an initial-value problem, the operator $\mathcal{A}(\mu, k)$ is bi-sectorial on the function space

$$\mathcal{X} = H^3_{\mathrm{per}}(0, 2\pi) \times H^2_{\mathrm{per}}(0, 2\pi) \times H^1_{\mathrm{per}}(0, 2\pi) \times L^2(0, 2\pi),$$

where

$$H^j_{\mathrm{per}}(0, 2\pi) \;=\; \{u \in H^j_{\mathrm{loc}}(\mathbb{R})\;;\; u(z + 2\pi) = u(z),\ \forall\, z \in \mathbb{R}\}, \quad j \geqslant 1.$$

with domain

$$\mathcal{D}(\mathcal{A}) = \mathcal{Y} = H^4_{\mathrm{per}}(0,2\pi) \times H^3_{\mathrm{per}}(0,2\pi) \times H^2_{\mathrm{per}}(0,2\pi) \times H^1_{\mathrm{per}}(0,2\pi).$$

It is not difficult to verify that the nonlinear map $\mathcal{F} : \mathcal{Y} \to \mathcal{Y}$ is in fact smooth.

We are interested in solutions close to onset, that is, $\mu \sim 0$, $\mu > 0$. We also allow for changes in the wavenumber, setting $k = k_* + \delta$ with $k_* \in (0,1)$ fixed and $\delta \sim 0$ small.

The choice of k implicitly restricts angles and wavenumbers of bifurcating roll solutions. Since bifurcating rolls have wavenumber $(k_x, k_y) = (k_0 \cos\varphi, -k_0 \sin\varphi)$ with $k_0 \sim 1$, the restriction to $2\pi/k_*$-periodicity implies that $k_0 \sin\varphi \in k_*\mathbb{Z}$. Equivalently, only angles

$$\varphi = 0, \pm \arcsin(k_*), \pm \arcsin(2k_*), ..., \pm \arcsin([\frac{1}{k_*}]k_*),$$

where $[z] = \max\{m \in \mathbb{Z},\ m \leqslant z\}$, are compatible with the periodic boundary conditions in y. Clearly, the number of compatible rolls increases when k_* decreases and crosses through a resonance, that is, when $1/k_*$ crosses an integer. We will see later that this fact is mirrored in a change in dimension of the center manifold. We will treat this resonant case later, in Section 2.2, and focus on the nonresonant case from now on. We point out that non-resonance precisely excludes rolls that are horizontal, that is, rolls perpendicular to the grain boundary interface, $\varphi = \pi/2$.

2.1.2 Center manifold reduction

Central space. We view the system (2.1.1) as an infinite-dimensional dynamical system in the form

$$\frac{dU}{dx} = \mathcal{A}_* U + \mathcal{B}(\mu,\delta)U + \mathcal{F}(U), \tag{2.1.2}$$

with

$$\mathcal{A}_* = \mathcal{A}(0, k_*), \quad \mathcal{B}(\mu,\delta) = \mathcal{A}(\mu, k_* + \delta) - \mathcal{A}(0, k_*).$$

Noting that $\mathcal{A}_* : \mathcal{Y} \to \mathcal{X}$ is a continuous linear operator and $\mathcal{Y}$ is dense in and compactly embedded into $\mathcal{X}$, we conclude that the resolvent of $\mathcal{A}_*$ is compact and thus its spectrum $\sigma(\mathcal{A}_*)$ only consists of eigenvalues ν. According to the dispersion relation,

$$\nu^2 = k_*^2 \ell^2 - 1, \quad \ell \in \mathbb{Z},$$

we have

$$\sigma(\mathcal{A}_*) = \{\nu \in \mathbb{C}\ ;\ \nu^2 = k_*^2\ell^2 - 1,\ \ell \in \mathbb{Z}\}. \tag{2.1.3}$$

Moreover,

$$\sigma(\mathcal{A}_*) \cap \mathrm{i}\mathbb{R} = \{\pm \mathrm{i}k_{\ell,x} | \ell \in I_*\}, \tag{2.1.4}$$

where

$$k_{x,\ell} = \sqrt{1-(k_*\ell)^2}, \quad k_{x,\ell} = k_{x,-\ell}, \quad I_* = \{0, \pm 1, \dots \pm \ell_*\}, \quad \ell_* = [\frac{1}{k_*}].$$

The non-resonance condition $1/k_* \notin \mathbb{Z}$ implies that $k_{x,\ell} \neq 0$ for all ℓ. Then it is not hard to see that the central space $\mathcal{X}_c$ of the operator $\mathcal{A}_*$, i.e., the spectral subspace associated with $\sigma(\mathcal{A}_*) \cap \mathrm{i}\mathbb{R}$, is $4(2\ell_*+1)$-dimensional and spanned by the vectors $\{E_\ell, F_\ell, \overline{E_\ell}, \overline{F_\ell} | \ell \in I_*\}$, where, for every $\ell \in I_*$,

$$E_\ell(y) = \begin{pmatrix} 1 \\ \mathrm{i}k_{x,\ell} \\ 0 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\ell y}, \quad F_\ell(y) = \begin{pmatrix} 0 \\ 1 \\ 2\mathrm{i}k_{x,\ell} \\ -2k_{x,\ell}^2 \end{pmatrix} \mathrm{e}^{\mathrm{i}\ell y}.$$

More specifically, the generalized eigenspace associated with the eigenvalue $\mathrm{i}k_{x,\ell}$ (or $-\mathrm{i}k_{x,\ell}$, respectively) is spanned by $E_{\pm\ell}$ and $F_{\pm\ell}$ (or $\overline{E_{\pm\ell}}$ and $\overline{F_{\pm\ell}}$, respectively), which satisfy

$$\mathcal{A}_* E_{\pm\ell} = \mathrm{i}k_{x,\ell} E_{\pm\ell}, \quad \mathcal{A}_* F_{\pm\ell} = \mathrm{i}k_{x,\ell} F_{\pm\ell} + E_{\pm\ell}, \quad \text{for all } \ell \in I_*,$$
$$\mathcal{A}_* \overline{E_{\pm\ell}} = -\mathrm{i}k_{x,\ell} \overline{E_{\pm\ell}}, \quad \mathcal{A}_* \overline{F_{\pm\ell}} = -\mathrm{i}k_{x,\ell} \overline{F_{\pm\ell}} + \overline{E_{\pm\ell}}, \quad \text{for all } \ell \in I_*.$$

We point out here that while the eigenvalues $\pm\mathrm{i}$ are geometrically simple and algebraically double, the eigenvalues $\pm\mathrm{i}k_{x,\ell}$ are geometrically double and algebraically quadruple for $\ell \in I_* \setminus \{0\}$.

The next step is to calculate the spectral projection $\mathcal{P}_c : \mathcal{X} \to \mathcal{X}_c$. We first denote $\mathcal{A}_*^{\mathrm{ad}}$ as the adjoint of $\mathcal{A}_*$ with respect to the scalar product $\langle \cdot, \cdot \rangle$ in $(L^2(0, 2\pi))^4$ and

$$\mathcal{A}_*^{\mathrm{ad}} = \begin{pmatrix} 0 & -(1+k_*^2\partial_y^2) & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -(1+k_*^2\partial_y^2) \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Similar to the $\mathcal{A}_*$ case, we have

$$\mathcal{A}_*^{\mathrm{ad}} F_\ell^{\mathrm{ad}} = -\mathrm{i}k_{x,\ell} F_\ell^{\mathrm{ad}}, \quad \mathcal{A}_*^{\mathrm{ad}} E_\ell^{\mathrm{ad}} = -\mathrm{i}k_{x,\ell} E_\ell^{\mathrm{ad}} + F_\ell^{\mathrm{ad}}, \quad \ell \in I_*,$$

in which

$$E_\ell^{\mathrm{ad}}(y) = \frac{1}{8\pi k_{x,\ell}^3} \begin{pmatrix} 2k_{x,\ell}^3 \\ 2\mathrm{i}k_{x,\ell}^2 \\ 0 \\ \mathrm{i} \end{pmatrix} e^{\mathrm{i}\ell y}, \quad F_\ell^{\mathrm{ad}}(y) = \frac{1}{8\pi k_{x,\ell}^3} \begin{pmatrix} 0 \\ 0 \\ \mathrm{i}k_{x,\ell}^2 \\ -k_{x,\ell} \end{pmatrix} e^{\mathrm{i}\ell y} \quad \ell \in I_*.$$

Secondly, by definition, the spectral projection is given by

$$\mathcal{P}_c U = \sum_{\ell \in I_*} \left(\langle U, E_\ell^{\mathrm{ad}} \rangle E_\ell + \langle U, F_\ell^{\mathrm{ad}} \rangle F_\ell + \langle U, \overline{E_\ell^{\mathrm{ad}}} \rangle \overline{E_\ell} + \langle U, \overline{F_\ell^{\mathrm{ad}}} \rangle \overline{F_\ell} \right),$$

where

$$\langle E_\kappa, E_\ell^{\mathrm{ad}} \rangle = \langle F_\kappa, F_\ell^{\mathrm{ad}} \rangle = \delta_{\kappa\ell}, \quad \langle F_\kappa, E_\ell^{\mathrm{ad}} \rangle = \langle E_\kappa, F_\ell^{\mathrm{ad}} \rangle = 0, \quad \kappa, \ell \in I_*.$$

Reduction to a center manifold. Through a center manifold reduction, we can reduce the PDE system to an ODE system while still keeping all small bounded solutions.

From (2.1.3)–(2.1.4), we note that the spectrum of $\mathcal{A}_*|_{(\mathrm{id} - \mathcal{P}_c)\mathcal{X}}$ is off the imaginary axis. More precisely, we have, for some $\varepsilon > 0$,

$$\sigma\left(\mathcal{A}_*|_{(\mathrm{id} - \mathcal{P}_c)\mathcal{X}} \right) \subset \{\lambda \in \mathbb{C} \,;\, |\Re\lambda| \geqslant \varepsilon\},$$

upon which, it is not hard to see that

$$\left\| (\mathrm{i}\omega - \mathcal{A}_*)^{-1} \right\|_{\mathcal{L}((\mathrm{id} - \mathcal{P}_c)\mathcal{X})} \leqslant \frac{C}{1 + |\omega|}, \quad \forall\, \omega \in \mathbb{R},$$

with some positive constant $C > 0$. Therefore, by applying the center manifold theorem(see [32, §2]), we can prove that there are neighborhoods of the origin $\mathcal{U} \subset \mathcal{X}_c$, $\mathcal{V} \subset (\mathrm{id} - \mathcal{P}_c)\mathcal{Y}$, $\mathcal{W} \subset \mathbb{R}^2$ and, for any $m < \infty$, a C^m-map $\mathbf{\Psi} : \mathcal{U} \times \mathcal{W} \to \mathcal{V}$ having the following properties.

(i) For any $(\mu, \delta) \in \mathcal{W}$, all bounded solutions $U(x)$ of (2.1.2) within $\mathcal{U} \times \mathcal{V}$ are on the center manifold, i.e.,

$$U(x) = U_c(x) + \mathbf{\Psi}(U_c(x), \mu, \delta), \quad \forall\, x \in \mathbb{R}. \tag{2.1.5}$$

(ii) The center manifold is tangent to the center eigenspace, i.e.,

$$\|\mathbf{\Psi}(U_c,\mu,\delta)\|_{\mathcal{Y}} = \mathcal{O}\left(|\mu|\|U_c\| + |\delta|\|U_c\| + \|U_c\|^2\right).$$

We now plug (2.1.5) into (2.1.2) and project it with $\mathcal{P}_c$, obtaining the reduced system for U_c,

$$\frac{dU_c}{dx} = \mathcal{A}_* U_c + \mathcal{P}_c\big(\mathcal{B}(\mu,\delta)U_c + \mathcal{F}(U_c + \mathbf{\Psi}(U_c,\mu,\delta))\big). \tag{2.1.6}$$

Remark 2.1.1 *We have $\mathcal{P}_c\mathcal{B}(\mu,\delta)\mathbf{\Psi}(U_c,\mu,\delta) = 0$ due to the fact that*

$$\mathcal{X}_c = \mathcal{X}_c^{ad} := \{\sum_{\ell\in I_*}\left(A_\ell E_\ell^{ad} + B_\ell F_\ell^{ad} + \overline{A_\ell}\overline{E_\ell^{ad}} + \overline{B_\ell}\overline{F_\ell^{ad}}\right)|A_\ell\in\mathbb{C}, B_\ell\in\mathbb{C}, \textit{for all } \ell\in I_*\}$$

and $B(\mu,\delta)$ maps $(\mathrm{id} - \mathcal{P}_c)\mathcal{Y}$ into $(\mathrm{id} - \mathcal{P}_c)\mathcal{X}$.

Lemma 2.1.2 *For sufficiently small $|\delta|$, we have*

$$\|\mathbf{\Psi}(U_c,\mu,\delta)\|_{\mathcal{Y}} = \mathcal{O}\left(|\mu|\|U_c\| + \|U_c\|^2\right). \tag{2.1.7}$$

Proof. We need to show absence of terms of the form $\mathcal{O}(\delta|U_c|)$. For this, we notice that the tangent space to the center manifold is invariant under the linearization at any point of the center manifold. Since now the center manifold contains all equilibria, in particular the trivial equilibria $U = 0$, δ small but possibly nonzero, we conclude that the tangent space to the center manifold at the origin, for all δ small, is given by the center eigenspace of the linearization at that equilibrium. The center eigenspace at the trivial equilibrium $U = 0$ is in fact independent of δ. To see this, note that any element in the center eigenspace can be represented in the form

$$\sum_{\ell\in I_*}\left(A_\ell E_{\ell,\delta} + B_\ell F_{\ell,\delta} + \overline{A_\ell E_{\ell,\delta}} + \overline{B_\ell F_{\ell,\delta}}\right),$$

where A_ℓ, B_ℓ are complex-valued functions and

$$E_{\ell,\delta}(y) = \begin{pmatrix} 1 \\ \mathrm{i}k_{x,\ell,\delta} \\ 0 \\ 0 \end{pmatrix}\mathrm{e}^{\mathrm{i}\ell y}, \quad F_{\ell,\delta}(y) = \begin{pmatrix} 0 \\ 1 \\ 2\mathrm{i}k_{x,\ell,\delta} \\ -2k_{x,\ell,\delta}^2 \end{pmatrix}\mathrm{e}^{\mathrm{i}\ell y}.$$

Here, $k_{x,\ell,\delta} = \sqrt{1-(k_*+\delta)^2\ell^2}$. A direct computation shows that for $|\delta|$ sufficiently small, the central space

$$\begin{aligned}\mathcal{X}_{c,\delta} :=& \{\sum_{\ell\in I_*}\big(A_\ell E_{\ell,\delta} + B_\ell F_{\ell,\delta} + \overline{A_\ell E_{\ell,\delta}} + \overline{B_\ell F_{\ell,\delta}}\big)\,|A_\ell, B_\ell \in \mathbb{C}\}\\ =& \{\sum_{\ell\in I_*,\ell\geqslant 0}(R_\ell\cos(\ell y) + S_\ell \sin(\ell y))\,|R_\ell, S_\ell \in \mathbb{R}^4\}\end{aligned}$$

is independent of the choice of δ.

■

Reduced system. In this paragraph we calculate the Taylor jet of the vector field of our reduced system (2.1.6) up to order three. First of all, due to the estimate (2.1.7) and the fact that the nonlinearity $\mathcal{F}$ is cubic, a direct computation leads to

$$\frac{dU_c}{dx} = \mathcal{A}_* U_c + \mathcal{P}_c\big(\mathcal{B}(\mu,\delta)U_c + \mathcal{F}(U_c)\big) + \mathcal{O}(|\mu|\|U_c\|^3 + \|U_c\|^4). \tag{2.1.8}$$

Secondly, substituting

$$U_c(x) = \sum_{\ell\in I_*}\big(A_\ell(x)E_\ell + B_\ell(x)F_\ell + \overline{A_\ell}(x)\overline{E_\ell} + \overline{B_\ell}(x)\overline{F_\ell}\big)$$

into (2.1.8) lets us write the leading-order-term reduced system

$$\frac{dU_c}{dx} = \mathcal{A}_* U_c + \mathcal{P}_c\big(\mathcal{B}(\mu)U_c + \mathcal{F}(U_c)\big),$$

in terms of the basis (A_ℓ, B_ℓ), $\ell \in I_*$ as follows,

$$\begin{aligned}A_\ell' =& \mathrm{i}k_{x,\ell}A_\ell + B_\ell - \mathrm{i}\frac{\mu + 2k_{x,\ell}^2(k_{x,\ell}^2 - k_{x,\ell,\delta}^2)}{4k_{x,\ell}^3}a_\ell + \frac{k_{x,\ell}^2 - k_{x,\ell,\delta}^2}{2k_{x,\ell}^2}b_\ell + \frac{\mathrm{i}}{4k_{x,\ell}^3}P_\ell,\\ B_\ell' =& \mathrm{i}k_{x,\ell}B_\ell - \frac{1}{4k_{x,\ell}^2}\mu a_\ell - \mathrm{i}\frac{k_{x,\ell}^2 - k_{x,\ell,\delta}^2}{2k_{x,\ell}}b_\ell + \frac{1}{4k_{x,\ell}^2}P_\ell,\end{aligned} \tag{2.1.9}$$

in which

$$\begin{aligned}&a_\ell = A_\ell + \overline{A_{-\ell}}, && b_\ell = B_\ell - \overline{B_{-\ell}}, \quad \text{for all } \ell \in I_*,\\ &k_{x,\ell,\delta} = \sqrt{1-(k_*+\delta)^2\ell^2}, && P_\ell((A_\ell,\overline{A_\ell})_{\ell\in I_*}) = \sum_{\ell_1+\ell_2+\ell_3=\ell,\ell_1,\ell_2,\ell_3\in I_*} a_{\ell_1}a_{\ell_2}a_{\ell_3}.\end{aligned}$$

Symmetries. In the Swift-Hohenberg equation (1.2.3), there are three reflection symmetries

$$y \mapsto -y, \quad x \mapsto -x, \quad u \mapsto -u,$$

which are preserved all the way from the system (2.1.1) to the reduced system (2.1.6). For simplicity, we denote $I_*^+ = \{0, 1, ..., \ell_*\}$. The reflections $y \mapsto -y$ and $u \mapsto -u$ induce symmetries in the reduced system (2.1.6), for all $\ell \in I_*^+$,

$$\mathcal{S}_1(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell}, A_{-\ell}, B_{-\ell}, \overline{A_{-\ell}}, \overline{B_{-\ell}})^T = (A_{-\ell}, B_{-\ell}, \overline{A_{-\ell}}, \overline{B_{-\ell}}, A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell})^T,$$

$$\mathcal{S}_2(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell}, A_{-\ell}, B_{-\ell}, \overline{A_{-\ell}}, \overline{B_{-\ell}})^T = -(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell}, A_{-\ell}, B_{-\ell}, \overline{A_{-\ell}}, \overline{B_{-\ell}})^T.$$

The reflection $x \mapsto -x$ implies that the reduced vector field (2.1.6) anticommutes with

$$\mathcal{R}(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell}, A_{-\ell}, B_{-\ell}, \overline{A_{-\ell}}, \overline{B_{-\ell}})^T = (\overline{A_{-\ell}}, -\overline{B_{-\ell}}, A_{-\ell}, -B_{-\ell}, \overline{A_\ell}, -\overline{B_\ell}, A_\ell, -B_\ell)^T.$$

Therefore, there are no even order U_c-terms in the reduced system (2.1.8), that is,

$$\frac{dU_c}{dx} = \mathcal{A}_* U_c + \mathcal{P}_c\big(\mathcal{B}(\mu, \delta)U_c + \mathcal{F}(U_c)\big) + \mathcal{O}(|\mu|\|U_c\|^3 + \|U_c\|^5). \tag{2.1.10}$$

Moreover, the invariance of the Swift-Hohenberg equation (1.2.3) under translations in y induces the symmetry for all $\ell \in I_*^+$,

$$\mathcal{T}_\phi(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell}, A_{-\ell}, B_{-\ell}, \overline{A_{-\ell}}, \overline{B_{-\ell}})^T =$$
$$(\mathrm{e}^{i\ell\phi}A_\ell, \mathrm{e}^{i\ell\phi}B_\ell, \mathrm{e}^{-i\ell\phi}\overline{A_\ell}, \mathrm{e}^{-i\ell\phi}\overline{B_\ell}, \mathrm{e}^{-i\ell\phi}A_{-\ell}, \mathrm{e}^{-i\ell\phi}B_{-\ell}, \mathrm{e}^{i\ell\phi}\overline{A_{-\ell}}, \mathrm{e}^{i\ell\phi}\overline{B_{-\ell}})^T.$$

Remark 2.1.3 *For the case $\ell = 0$, the formulas above are somewhat overly complicated since then, of course, $A_\ell = A_{-\ell}$, etc. Still, the formulas hold true in this case as well.*

2.1.3 Normal form transformations

In this section, a composition of linear and nonlinear transformations leads to a normal form of the reduced system (2.1.10).

Versal linear transformation. Consider the linear part of the system (2.1.9), for $\ell \in I_*$,

$$\begin{aligned} A_\ell' =& \mathrm{i}k_{x,\ell}A_\ell + B_\ell - \mathrm{i}\frac{\mu + 2k_{x,\ell}^2(k_{x,\ell}^2 - k_{x,\ell,\delta}^2)}{4k_{x,\ell}^3}a_\ell + \frac{k_{x,\ell}^2 - k_{x,\ell,\delta}^2}{2k_{x,\ell}^2}b_\ell \\ B_\ell' =& \mathrm{i}k_{x,\ell}B_\ell - \frac{1}{4k_{x,\ell}^2}\mu a_\ell - \mathrm{i}\frac{k_{x,\ell}^2 - k_{x,\ell,\delta}^2}{2k_{x,\ell}}b_\ell. \end{aligned} \tag{2.1.11}$$

Lemma 2.1.4 *For all $\ell \in I_*^+$, there exist smooth linear maps $L_\ell(\mu,\delta)$ such that, for sufficiently small μ and δ, the linear changes of variables,*

$$(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell}, A_{-\ell}, B_{-\ell}, \overline{A_{-\ell}}, \overline{B_{-\ell}})^T = L_\ell(C_\ell, D_\ell, \overline{C_\ell}, \overline{D_\ell}, C_{-\ell}, D_{-\ell}, \overline{C_{-\ell}}, \overline{D_{-\ell}})^T \tag{2.1.12}$$

transform the system (2.1.11) *into the normal form*

$$\begin{aligned} C_\ell' &= \mathrm{i}\sqrt{\frac{1}{2}\left(k_{x,\ell,\delta}^2 + \sqrt{k_{x,\ell,\delta}^4 - \mu}\right)}C_\ell + D_\ell \\ D_\ell' &= -\frac{1}{2}\left(k_{x,\ell,\delta}^2 - \sqrt{k_{x,\ell,\delta}^4 - \mu}\right)C_\ell + \mathrm{i}\sqrt{\frac{1}{2}\left(k_{x,\ell,\delta}^2 + \sqrt{k_{x,\ell,\delta}^4 - \mu}\right)}D_\ell. \end{aligned} \tag{2.1.13}$$

Moreover, we have the following properties for $L_\ell(\mu,\delta)$:

- *$L_\ell(\mu,\delta)$ commutes with the symmetries $\mathcal{S}_1, \mathcal{S}_2, \mathcal{R}$, and $\mathcal{T}_\phi$, $\phi \in \mathbb{R}$;*
- *$L_\ell(0,0) = \mathrm{id}$ and, we can choose*

$$L_\ell(0,\delta) = \begin{pmatrix} \alpha_1 & \mathrm{i}\alpha_3 & 0 & 0 & 0 & 0 & \alpha_4 & \mathrm{i}\alpha_3 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & \alpha_1 & -\mathrm{i}\alpha_3 & \alpha_4 & -\mathrm{i}\alpha_3 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & \alpha_5 & 0 & 0 \\ 0 & 0 & \alpha_4 & \mathrm{i}\alpha_3 & \alpha_1 & \mathrm{i}\alpha_3 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 & 0 & \alpha_2 & 0 & 0 \\ \alpha_4 & -\mathrm{i}\alpha_3 & 0 & 0 & 0 & 0 & \alpha_1 & -\mathrm{i}\alpha_3 \\ 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & \alpha_2 \end{pmatrix}, \tag{2.1.14}$$

where

$$\alpha_1 = \frac{k_{x,\ell,\delta} + k_{x,\ell}}{2k_{x,\ell}}, \quad \alpha_2 = \frac{k_{x,\ell,\delta}(k_{x,\ell,\delta} + k_{x,\ell})}{2k_{x,\ell}^2}, \quad \alpha_3 = \frac{k_{x,\ell,\delta}^2 - k_{x,\ell}^2}{2k_{x,\ell}^3},$$
$$\alpha_4 = -\frac{k_{x,\ell,\delta} - k_{x,\ell}}{k_{x,\ell}}, \quad \alpha_5 = \frac{k_{x,\ell,\delta}(k_{x,\ell,\delta} - k_{x,\ell})}{2k_{x,\ell}^2}.$$

Proof. We will apply [33, Theorem 4.4] to simplify the linear part. To do that, we rewrite the systems (2.1.11) and (2.1.13) separately as follows. For all $\ell \in I_*^+$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell}, A_{-\ell}, B_{-\ell}, \overline{A_{-\ell}}, \overline{B_{-\ell}})^T = M_\ell(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell}, A_{-\ell}, B_{-\ell}, \overline{A_{-\ell}}, \overline{B_{-\ell}})^T,$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(C_\ell, D_\ell, \overline{C_\ell}, \overline{D_\ell}, C_{-\ell}, D_{-\ell}, \overline{C_{-\ell}}, \overline{D_{-\ell}})^T = N_\ell(C_\ell, D_\ell, \overline{C_\ell}, \overline{D_\ell}, C_{-\ell}, D_{-\ell}, \overline{C_{-\ell}}, \overline{D_{-\ell}})^T.$$

By the symmetries of these matrices, we conclude that $M_\ell(\mu,\delta), N_\ell(\mu,\delta) \in \mathcal{M}$, where

$$\mathcal{M} := \{M \in \mathrm{M}^{8\times 8} | M\mathcal{S}_j = \mathcal{S}_j M, M\mathcal{R} = -\mathcal{R}M, M\mathcal{T}_\phi = \mathcal{T}_\phi M, \text{for all } j = 1,2, \phi \in \mathbb{R}\}.$$

Now the key part is to find a versal deformation of $M_\ell(0,0)$ in $\mathcal{M}$. On one hand, we note that

$$M_\ell(0,0) = \begin{pmatrix} \mathrm{i}k_{x,\ell} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{i}k_{x,\ell} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i}k_{x,\ell} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i}k_{x,\ell} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{i}k_{x,\ell} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i}k_{x,\ell} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}k_{x,\ell} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}k_{x,\ell} \end{pmatrix}$$

is already in its Jordan form. The commutator of $M_\ell^*(0,0)$ in $\mathcal{M}$ consists of matrices of the general form

$$\widetilde{N} = \begin{pmatrix} \mathrm{i}\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_2 & \mathrm{i}\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i}\gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_2 & -\mathrm{i}\gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{i}\gamma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2 & \mathrm{i}\gamma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\gamma_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & -\mathrm{i}\gamma_1 \end{pmatrix},$$

where $\gamma_{1\backslash 2} \in \mathbb{R}$. Therefore, $M_\ell(0,0) + \widetilde{N}(\gamma_1,\gamma_2)$ is a versal deformation of $M_\ell(0,0)$. Moreover, by Theorem 4.4 in [33], there exists a smooth deformation of the identity matrix, denoted as $L_\ell(\mu,\delta)$ and two smooth functions $\gamma_{1\backslash 2}(\mu,\delta)$ such that, for sufficiently small μ and δ,

$$M_\ell(\mu,\delta) = L_\ell(\mu,\delta)\left(M_\ell(0,0) + \widetilde{N}(\gamma_1(\mu,\delta),\gamma_2(\mu,\delta))\right)L_\ell^{-1}(\mu,\delta).$$

From the original system, it is straightforward to see that the characteristic polynomial $P(\lambda)$ of $M_\ell(\mu,\delta)$ is

$$P(\lambda) = \left(\mu - (1+\lambda^2-(k_*+\delta)^2\ell^2)^2\right)^2,$$

On the other hand,

$$P(\lambda) = \det(\lambda - M_\ell(0,0) - \widetilde{N}) = \left(\left(\lambda^2 + (k_{x,\ell}+\gamma_1)^2\right)^2 - 2\gamma_2\left(\lambda^2 - (k_{x,\ell}+\gamma_1)^2\right) + \gamma_2^2\right)^2.$$

Comparing the coefficients of the above two polynomials, we have

$$\begin{aligned}\gamma_1 + k_{x,\ell} &= \sqrt{\frac{1}{2}\left(k_{x,\ell,\delta}^2 + \sqrt{k_{x,\ell,\delta}^4 - \mu}\right)},\\ \gamma_2 &= -\frac{1}{2}\left(k_{x,\ell,\delta}^2 - \sqrt{k_{x,\ell,\delta}^4 - \mu}\right).\end{aligned}$$

Moreover, $N_\ell(\mu,\delta) = M_\ell(0,0) + \widetilde{N}(\gamma_1(\mu,\delta),\gamma_2(\mu,\delta))$. As for the choice of $L_\ell(0,\delta)$, we just note that $M_\ell(0,\delta)$ and $N_\ell(0,\delta)$ are the linear systems separately under two sets of bases $\{E_\ell, F_\ell\}_{\ell\in I_*}$ and $\{E_{\ell,\delta}, F_{\ell,\delta}\}_{\ell\in I_*}$. Thus, $L_\ell(0,\delta)$ is chosen naturally as the corresponding transition matrix. ■

Based on Lemma 2.1.4, our reduced system (2.1.9)–(2.1.10) has the following expression. For all $\ell \in I_*$,

$$\begin{aligned}C_\ell' &= \mathrm{i}\sqrt{\frac{1}{2}\left(k_{x,\ell,\delta}^2 + \sqrt{k_{x,\ell,\delta}^4 - \mu}\right)}C_\ell + D_\ell + \frac{\mathrm{i}}{4k_{x,\ell,\delta}^3}P_\ell,\\ D_\ell' &= -\frac{1}{2}\left(k_{x,\ell,\delta}^2 - \sqrt{k_{x,\ell,\delta}^4 - \mu}\right)C_\ell + \mathrm{i}\sqrt{\frac{1}{2}\left(k_{x,\ell,\delta}^2 + \sqrt{k_{x,\ell,\delta}^4 - \mu}\right)}D_\ell + \frac{1}{4k_{x,\ell,\delta}^2}P_\ell,\end{aligned} \tag{2.1.15}$$

with higher order terms in the form

$$\mathcal{O}(|\mu|(\sum_{\ell\in I_*}|C_\ell| + |D_\ell|)^3 + (\sum_{\ell\in I_*}|C_\ell| + |D_\ell|)^5).$$

Cubic transformation. Now we consider the leading order of system (2.1.15) for $\mu = 0$ and $\ell \in I_*$,

$$\begin{aligned}A_\ell' &= \mathrm{i}k_{x,\ell,\delta}A_\ell + B_\ell + \frac{\mathrm{i}}{4k_{x,\ell,\delta}^3}P_\ell,\\ B_\ell' &= \mathrm{i}k_{x,\ell,\delta}B_\ell + \frac{1}{4k_{x,\ell,\delta}^2}P_\ell.\end{aligned} \tag{2.1.16}$$

In order to find a normal form of (2.1.16), we first prove the following lemma.

Lemma 2.1.5 *For $z_i \in \mathbb{C}, |z_i| = 1, i = 1,2,3,4,$ we have $z_1 + z_2 + z_3 + z_4 = 0$ if and only if $z_1 = -z_2$ and $z_3 = -z_4$, possibly after permuting the indices.*

Proof. The lemma is a formulation of the somewhat folklore fact that the only equilateral quadrilateral is the rhombus. In fact, the "if" part is clear. To prove the "only if" part, we interpret the z_j as vectors, so that the equation $z_1+z_2+z_3+z_4=0$ corresponds to the fact that the vectors form a quadrilateral when attached at endpoints. Now intersecting the quadrilateral with a diagonal, we obtain two congruent triangles since sides have equal lengths. This implies that the equilateral quadrilateral is a parallelogram and thus a rhombus. ■

Lemma 2.1.6 *For all $\ell \in I_*$, there exist smooth families of homogeneous polynomials $\{\Phi_\ell(\delta), \Psi_\ell(\delta)\}_{\ell\in I_*}$ of degree 3 in the complex variables $(C_\kappa, D_\kappa, \overline{C_\kappa}, \overline{D_\kappa})_{\kappa\in I_*}$, such that the change of variables*

$$\begin{aligned} A_\ell =& C_\ell + \Phi_\ell(\delta)((C_\kappa, D_\kappa, \overline{C_\kappa}, \overline{D_\kappa})_{\kappa\in I_*}), \\ B_\ell =& D_\ell + \Psi_\ell(\delta)((C_\kappa, D_\kappa, \overline{C_\kappa}, \overline{D_\kappa})_{\kappa\in I_*}), \end{aligned} \tag{2.1.17}$$

is well-defined in a neighborhood of the origin and transforms the system (2.1.16) into the normal form

$$\begin{aligned} C_\ell' =& \mathrm{i}k_{x,\ell,\delta}C_\ell + D_\ell + \mathcal{O}((\sum_{\kappa\in I_*}|C_\kappa| + |D_\kappa|)^5) \\ D_\ell' =& \mathrm{i}k_{x,\ell,\delta}D_\ell + \frac{3}{4k_{x,\ell,\delta}^2}C_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2) + \frac{3\mathrm{i}}{4k_{x,\ell,\delta}^3}D_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2) \\ &+ \frac{3\mathrm{i}}{4k_{x,\ell,\delta}^3}C_\ell\left(-(C_\ell\overline{D_\ell} + \overline{C_\ell}D_\ell) + 2\sum_{\kappa\in I_*}(C_\kappa\overline{D_\kappa} + \overline{C_\kappa}D_\kappa)\right) \\ &+ \mathcal{O}((\sum_{\kappa\in I_*}|C_\kappa| + |D_\kappa|)^5) \end{aligned} \tag{2.1.18}$$

Proof. Denote the cubic terms in the right hand sides of the systems (2.1.16) and (2.1.18) by $(F_\ell, G_\ell)_{\ell\in I_*}$ and $(0, N_\ell)_{\ell\in I_*}$, respectively. Following the proof of the normal form theorem (see for instance [32, §3]), substituting (2.1.17) into (2.1.16), and taking into account (2.1.18) at order 3 in the resulting equalities we find that the polynomials $\{\Phi_\ell, \Psi_\ell\}_{\ell\in I_*}$ satisfy the equations, for all $\ell \in I_*$,

$$(\mathcal{D} - \mathrm{i}k_{x,\ell,\delta})\Phi_\ell = \Psi_\ell + F_\ell, \quad (\mathcal{D} - \mathrm{i}k_{x,\ell,\delta})\Psi_\ell = G_\ell - N_\ell, \tag{2.1.19}$$

in which

$$\mathcal{D} = \quad \sum_{\ell\in I_*}\left((\mathrm{i}k_{x,\ell,\delta}C_\ell + D_\ell)\frac{\partial}{\partial C_\ell} + \mathrm{i}k_{x,\ell,\delta}D_\ell\frac{\partial}{\partial D_\ell} + c.c.\right).$$

Here *c.c.* represents the complex conjugate of the part before *c.c.* in the pair of parentheses. The polynomials Φ_ℓ, Ψ_ℓ exist provided the right hand sides in the equations (2.1.19) belong to the range of $\mathcal{D} - \mathrm{i}k_{x,\ell,\delta}$.

We have

$$F_\ell = \frac{\mathrm{i}}{4k_{x,\ell,\delta}^3}P_\ell, \quad G_\ell = \frac{1}{4k_{x,\ell,\delta}^2}P_\ell,$$

with

$$P_\ell((C_\ell, \overline{C_\ell})_{\ell\in I_*}) = \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell,\\ \ell_1,\ell_2,\ell_3\in I_*}} c_{\ell_1}c_{\ell_2}c_{\ell_3}, \quad c_\ell = C_\ell + \overline{C_{-\ell}}, \quad \overline{c_\ell} = c_{-\ell}, \quad \text{for all } \ell\in I_*.$$

In order to see which monomials in P_ℓ are in the range of $\mathcal{D} - \mathrm{i}k_{x,\ell}$, we first notice that

$$\begin{aligned}
c_{\ell_1}c_{\ell_2}c_{\ell_3} &= (C_{\ell_1} + \overline{C_{-\ell_1}})(C_{\ell_2} + \overline{C_{-\ell_2}})(C_{\ell_3} + \overline{C_{-\ell_3}})\\
&= C_{\ell_1}C_{\ell_2}C_{\ell_3} + (C_{\ell_1}C_{\ell_2}\overline{C_{-\ell_3}} + C_{\ell_1}\overline{C_{-\ell_2}}C_{\ell_3} + \overline{C_{-\ell_1}}C_{\ell_2}C_{\ell_3})\\
&\quad +(C_{\ell_1}\overline{C_{-\ell_2}C_{-\ell_3}} + \overline{C_{-\ell_1}C_{-\ell_2}}C_{\ell_3} + \overline{C_{-\ell_1}}C_{\ell_2}\overline{C_{-\ell_3}}) + \overline{C_{-\ell_1}C_{-\ell_2}C_{-\ell_3}},
\end{aligned}$$

which indicates that we have essentially only 4 types of monomials regardless of permutations

$$C_{\ell_1}C_{\ell_2}C_{\ell_3}, \quad C_{\ell_1}C_{\ell_2}\overline{C_{-\ell_3}}, \quad C_{\ell_1}\overline{C_{-\ell_2}C_{-\ell_3}}, \quad \overline{C_{-\ell_1}C_{-\ell_2}C_{-\ell_3}}.$$

We discuss all possibly cases, next.

- Monomials of type "$C_{\ell_1}C_{\ell_2}C_{\ell_3}$" are in the range of $\mathcal{D} - \mathrm{i}k_{x,\ell}$. First, we notice that

$$\begin{aligned}
(\mathcal{D} - \mathrm{i}k_{x,\ell,\delta})C_{\ell_1}C_{\ell_2}C_{\ell_3} &= \mathrm{i}(k_{x,\ell_1,\delta} + k_{x,\ell_2,\delta} + k_{x,\ell_3,\delta} - k_{x,\ell,\delta})C_{\ell_1}C_{\ell_2}C_{\ell_3} +\\
&\quad D_{\ell_1}C_{\ell_2}C_{\ell_3} + C_{\ell_1}D_{\ell_2}C_{\ell_3} + C_{\ell_1}C_{\ell_2}D_{\ell_3},\\
(\mathcal{D} - \mathrm{i}k_{x,\ell,\delta})D_{\ell_1}C_{\ell_2}C_{\ell_3} &= \mathrm{i}(k_{x,\ell_1,\delta} + k_{x,\ell_2,\delta} + k_{x,\ell_3,\delta} - k_{x,\ell,\delta})D_{\ell_1}C_{\ell_2}C_{\ell_3} +\\
&\quad D_{\ell_1}D_{\ell_2}C_{\ell_3} + D_{\ell_1}C_{\ell_2}D_{\ell_3},\\
(\mathcal{D} - \mathrm{i}k_{x,\ell,\delta})C_{\ell_1}D_{\ell_2}C_{\ell_3} &= \mathrm{i}(k_{x,\ell_1,\delta} + k_{x,\ell_2,\delta} + k_{x,\ell_3,\delta} - k_{x,\ell,\delta})C_{\ell_1}D_{\ell_2}C_{\ell_3} +\\
&\quad D_{\ell_1}D_{\ell_2}C_{\ell_3} + C_{\ell_1}D_{\ell_2}D_{\ell_3},\\
(\mathcal{D} - \mathrm{i}k_{x,\ell,\delta})C_{\ell_1}C_{\ell_2}D_{\ell_3} &= \mathrm{i}(k_{x,\ell_1,\delta} + k_{x,\ell_2,\delta} + k_{x,\ell_3,\delta} - k_{x,\ell,\delta})C_{\ell_1}C_{\ell_2}D_{\ell_3} +\\
&\quad D_{\ell_1}C_{\ell_2}D_{\ell_3} + C_{\ell_1}D_{\ell_2}D_{\ell_3},
\end{aligned}$$

$$\begin{aligned}
(\mathcal{D}-\mathrm{i}k_{x,\ell,\delta})D_{\ell_1}D_{\ell_2}C_{\ell_3} &= \mathrm{i}(k_{x,\ell_1,\delta}+k_{x,\ell_2,\delta}+k_{x,\ell_3,\delta}-k_{x,\ell,\delta})D_{\ell_1}D_{\ell_2}C_{\ell_3}+\\
&\quad D_{\ell_1}D_{\ell_2}D_{\ell_3},\\
(\mathcal{D}-\mathrm{i}k_{x,\ell,\delta})D_{\ell_1}C_{\ell_2}D_{\ell_3} &= \mathrm{i}(k_{x,\ell_1,\delta}+k_{x,\ell_2,\delta}+k_{x,\ell_3,\delta}-k_{x,\ell,\delta})D_{\ell_1}C_{\ell_2}D_{\ell_3}+\\
&\quad D_{\ell_1}D_{\ell_2}D_{\ell_3},\\
(\mathcal{D}-\mathrm{i}k_{x,\ell,\delta})C_{\ell_1}D_{\ell_2}D_{\ell_3} &= \mathrm{i}(k_{x,\ell_1,\delta}+k_{x,\ell_2,\delta}+k_{x,\ell_3,\delta}-k_{x,\ell,\delta})C_{\ell_1}D_{\ell_2}D_{\ell_3}+\\
&\quad D_{\ell_1}D_{\ell_2}D_{\ell_3},\\
(\mathcal{D}-\mathrm{i}k_{x,\ell,\delta})D_{\ell_1}D_{\ell_2}D_{\ell_3} &= \mathrm{i}(k_{x,\ell_1,\delta}+k_{x,\ell_2,\delta}+k_{x,\ell_3,\delta}-k_{x,\ell,\delta})D_{\ell_1}D_{\ell_2}D_{\ell_3}.
\end{aligned}$$

Moreover, according to lemma 2.1.5 and the fact that

$$\begin{cases} |k_{x,\ell_j,\delta}+i(k_*+\delta)\ell_j|=1, \quad j=1,2,3\\ |k_{x,\ell,\delta}+i(k_*+\delta)\ell|=1\\ \ell_1+\ell_2+\ell_3-\ell=0\\ k_{x,\ell,\delta},k_{x,\ell_j,\delta}>0, \quad j=1,2,3, \end{cases}$$

we conclude

$$k_{x,\ell_1,\delta}+k_{x,\ell_2,\delta}+k_{x,\ell_3,\delta}-k_{x,\ell,\delta}\neq 0.$$

As a result, monomials of type "$C_{\ell_1}C_{\ell_2}C_{\ell_3}$" are in the range of $\mathcal{D}-\mathrm{i}k_{x,\ell,\delta}$.

- Monomials of types "$C_{\ell_1}\overline{C_{-\ell_2}C_{-\ell_3}}$" and "$\overline{C_{-\ell_1}C_{-\ell_2}C_{-\ell_3}}$" are also in the range of $\mathcal{D}-\mathrm{i}k_{x,\ell}$. The proof is similar to the above case, essentially due to the fact that

$$k_{x,\ell_1,\delta}-k_{x,\ell_2,\delta}-k_{x,\ell_3,\delta}-k_{x,\ell,\delta}\neq 0, \quad -k_{x,\ell_1,\delta}-k_{x,\ell_2,\delta}-k_{x,\ell_3,\delta}-k_{x,\ell,\delta}\neq 0.$$

- Monomials of type "$C_{\ell_1}C_{\ell_2}\overline{C_{-\ell_3}}$" are not in the range of $\mathcal{D}-\mathrm{i}k_{x,\ell,\delta}$ precisely when

$$\begin{cases}\ell_1=\ell\\ \ell_2=-\ell_3\end{cases} \quad \text{or} \quad \begin{cases}\ell_1=-\ell_3\\ \ell_2=\ell.\end{cases}$$

The proof again is similar to the first case but now relies on Lemma 2.1.5, which implies

$$\begin{cases}\ell_1+\ell_2+\ell_3-\ell=0\\ k_{x,\ell_1,\delta}+k_{x,\ell_2,\delta}-k_{x,\ell_3,\delta}-k_{x,\ell,\delta}=0\end{cases} \quad \text{iff} \quad \begin{cases}\ell_1=\ell\\ \ell_2=-\ell_3\end{cases} \quad \text{or} \quad \begin{cases}\ell_1=-\ell_3\\ \ell_2=\ell.\end{cases}$$

Consequently, upon choosing

$$\Psi_\ell = -\frac{3\mathrm{i}}{4k_{x,\ell,\delta}^3} C_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2) + \widetilde{\Psi_\ell}, \tag{2.1.20}$$

with $\widetilde{\Psi_\ell}$ any element of the range of $\mathcal{D} - \mathrm{i}k_{x,\ell,\delta}$, the polynomial $\Psi_\ell + F_\ell$ belongs to the range of $\mathcal{D} - \mathrm{i}k_{x,\ell,\delta}$, so that there exists Φ_ℓ satisfying the first equality in (2.1.19). Substituting (2.1.20) into the second equation in (2.1.19) we find

$$(\mathcal{D} - \mathrm{i}k_{x,\ell,\delta})\widetilde{\Psi_\ell} \quad = \quad \frac{3\mathrm{i}}{4k_{x,\ell,\delta}^3}(\mathcal{D} - \mathrm{i}k_{x,\ell})\left(C_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2)\right) + G_\ell - N_\ell.$$

Taking

$$N_\ell \quad = \quad \frac{3\mathrm{i}}{4k_{x,\ell,\delta}^3}(\mathcal{D} - \mathrm{i}k_{x,\ell} - \mathrm{i}k_{x,\ell,\delta})\left(C_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2)\right),$$

we find the equation

$$(\mathcal{D} - \mathrm{i}k_{x,\ell,\delta})\widetilde{\Psi_\ell} = \frac{1}{4k_{x,\ell,\delta}^2}\left(P((C_\ell, \overline{C_\ell})_{\kappa\in I_*}) - 3C_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2)\right),$$

with right hand side belonging to the range of $\mathcal{D} - \mathrm{i}k_{x,\ell,\delta}$. Consequently, $\widetilde{\Psi_\ell}$ exists, and it is not difficult to see that it also belongs to the range of $\mathcal{D} - \mathrm{i}k_{x,\ell,\delta}$. ■

Normal form of the reduced system. We apply the change of variables to the reduced system (2.1.10) and obtain the normal form to leading order as follows,

$$\begin{aligned}
C_\ell' =& \mathrm{i}\sqrt{\frac{1}{2}(k_{x,\ell,\delta}^2 + \sqrt{k_{x,\ell,\delta}^4 - \mu})}C_\ell + D_\ell \\
D_\ell' =& -\frac{1}{2}(k_{x,\ell,\delta}^2 - \sqrt{k_{x,\ell,\delta}^4 - \mu})C_\ell + \mathrm{i}\sqrt{\frac{1}{2}(k_{x,\ell,\delta}^2 + \sqrt{k_{x,\ell,\delta}^4 - \mu})}D_\ell + \\
& \frac{3}{4k_{x,\ell,\delta}^2}C_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2) + \\
& \frac{3\mathrm{i}}{4k_{x,\ell,\delta}^3}D_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2) + \\
& \frac{3\mathrm{i}}{4k_{x,\ell,\delta}^3}C_\ell\left(-(C_\ell\overline{D_\ell} + \overline{C_\ell}D_\ell) + 2\sum_{\kappa\in I_*}(C_\kappa\overline{D_\kappa} + \overline{C_\kappa}D_\kappa)\right)
\end{aligned} \tag{2.1.21}$$

The higher order terms in this normal form are of order

$$|\mu|(\sum_{\kappa\in I_*}(|C_\kappa| + |D_\kappa|))^3 + (\sum_{\kappa\in I_*}(|C_\kappa| + |D_\kappa|))^5.$$

2.1.4 Existence of heteroclinic orbits

We next pass to a corotating frame with respect to the normal form symmetry, at leading order,

$$C_\ell(x) = \exp(\mathrm{i}\tilde{k}_{\ell,\delta,\mu}x)\widetilde{C_\ell}, \quad D_\ell(x) = \exp(\mathrm{i}\tilde{k}_{\ell,\delta,\mu}x)\widetilde{D_\ell}, \quad \ell \in I_*,$$

where $\tilde{k}_{\ell,\mu,\delta} = \sqrt{\frac{1}{2}(k_{x,\ell,\delta}^2 + \sqrt{k_{x,\ell,\delta}^4 - \mu})}$. We can now scale the equation, explicitly exhibiting leading-order terms:

$$\widehat{x} = |\mu|^{1/2}x, \quad \widetilde{C_\ell} = |\mu|^{1/2}\widehat{C_\ell}, \quad \widetilde{D_\ell} = |\mu|\widehat{D_\ell}, \quad \ell \in I_*,$$

yields the new system, for $\ell \in I_*$,

$$\begin{aligned} C_\ell' =& D_\ell + \mathcal{O}(|\mu|^{1/2}) \\ D_\ell' =& -\frac{1}{4k_{x,\ell,\delta}^2}C_\ell\left(\mathrm{sign}(\mu) - 3(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2)\right) + \mathcal{O}(|\mu|^{1/2}), \end{aligned} \tag{2.1.22}$$

in which we have dropped the hats. Taking $\mu > 0$, we can rewrite this as a second-order equation, for $\ell \in I_*$,

$$C_\ell'' = -\frac{1}{4k_{x,\ell,\delta}^2}C_\ell + \frac{3}{4k_{x,\ell,\delta}^2}C_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2) + \mathcal{O}(\mu^{1/2}). \tag{2.1.23}$$

We will proceed and find grain boundaries to leading order as heteroclinic solutions to (2.1.23). Note that this equation is non-autonomous at higher order, with non-autonomous terms induced by the corotating frame to terms that are not in normal form. Our approach therfore is based on three steps. We first identify asymptotic solutions *exactly*. In fact, (2.1.23) possesses a heteroclinic orbit connecting equilibria at leading order. These equilibria continue to periodic orbits at *all orders* since they correspond to rotated roll solutions. We will make this precise in the next paragraph. We will then identify the heteroclinic orbits at leading order and study the linearization at those. In the last step, we carry out perturbation theory by decomposing perturbations into "exactly" known terms at infinity, namely some representative of the family of roll solutions, and an unknown exponentially localized perturbation. Persistence then will follow from the implicit function theorem.

Rotated rolls. The Swift-Hohenberg equation (1.2.1) possesses roll solutions

$$u_{\mu,\kappa}(x) = \sqrt{4(\mu-\kappa^2)/3}\cos(\sqrt{1+\kappa}\,x) + \mathcal{O}(|\mu-\kappa^2|^{3/2}), \tag{2.1.24}$$

for small $\mu \in (0,\mu_0]$ and $\kappa^2 < \mu$ (see for instance [34]). By isotropy of the equation, we obtain a family of rotated rolls,

$$\begin{aligned} u_{\mu,\delta,\varepsilon,\ell}(x,y) =& \frac{1}{\sqrt{3}}\mu^{\frac{1}{2}}(1-4k_{x,\ell,\delta}^2\varepsilon^2)^{\frac{1}{2}}\Big(\mathrm{e}^{\mathrm{i}(\tilde{k}_{\ell,\mu,\delta}+\varepsilon\mu^{1/2})x}\,\mathrm{e}^{\mathrm{i}(k_*+\delta)\ell y} \\ &+\mathrm{e}^{-\mathrm{i}(\tilde{k}_{\ell,\mu,\delta}+\varepsilon\mu^{1/2})x}\,\mathrm{e}^{-\mathrm{i}(k_*+\delta)\ell y}\Big) + \mathcal{O}(\mu^{3/2}+\varepsilon\mu), \end{aligned} \tag{2.1.25}$$

where we set

$$\kappa = (\tilde{k}_{\ell,\mu,\delta}+\varepsilon\mu^{1/2})^2 - k_{x,\ell,\delta}^2, \quad \ell_x = \frac{\tilde{k}_{\ell,\mu,\delta}+\varepsilon\mu^{1/2}}{\sqrt{1+\kappa}}, \quad \ell_y = \frac{(k_*+\delta)\ell}{\sqrt{1+\kappa}}, \quad \ell \in I_*,$$

so that $\ell_x^2+\ell_y^2 = 1$.

These roll solutions correspond to solutions on the center manifold, which one readily finds in the form

$$\mathbf{P}_{\mu,\delta,\varepsilon,\ell}(x) = \left(0,...,\frac{1}{\sqrt{3}}(1-4k_{x,\ell,\delta}^2\varepsilon^2)^{1/2}\mathrm{e}^{\mathrm{i}\varepsilon x},...,0\right) + \mathcal{O}(\mu^{1/2}). \tag{2.1.26}$$

Here, the nonzero term is at the ℓ-th position in a $2\ell_*+1$-dimensional vector with position ordering $-\ell_*,...,\ell_*$ from left to right. Notice that these periodic orbits are not reversible. In particular, the reversibility symmetry $\mathcal{R}$ generates a second family of periodic orbits

$$\mathbf{Q}_{\mu,\delta,\varepsilon,\ell}(x) = (\mathcal{R}\mathbf{P}_{\mu,\delta,\varepsilon,\ell})(-x) = \left(0,...,\frac{1}{\sqrt{3}}(1-4k_{x,\ell,\delta}^2\varepsilon^2)^{1/2}\mathrm{e}^{\mathrm{i}\varepsilon x},...,0\right) + O(\mu^{1/2}),$$

which corresponds to the reflected rolls $u_{\mu,\delta,\varepsilon,\ell}(-x,y)$ and the nonzero term is now at the $(-\ell)$-th position.

Heteroclinic orbit of the leading order system. We set $\mu=\delta=0$ in (2.1.23). A priori, one can look for a large variety of heteroclinic solutions in this high-dimensional system of ODEs. Note that setting any of the modes $C_\ell = 0$ provides us with invariant subspaces, so that we could in principal choose any two modes, C_{ℓ_1} and C_{ℓ_2}, set $C_\ell \equiv 0$ for $\ell \notin \{\ell_1,\ell_2\}$, and attempt to find heteroclinic orbits connecting $\mathbf{P}_{\mu,\delta,\varepsilon,\ell_1}$ and $\mathbf{Q}_{\mu,\delta,\varepsilon,\ell_2}$. We focus on $\ell_1 = -\ell_2 = 1$, here, which gives the simplest possible grain boundaries. In fact, requiring reflection symmetry imposes $\ell_1 = -\ell_2$. Grain boundaries with $\ell_1 > 1$ correspond to patterns where multiple rolls fit inside the fixed-width strip in the y-direction, and can be found by changing $k_* \mapsto \ell_1 k_*$ and subsequently considering $\ell_1 = 1$. From now on, we therefore just consider $\mathbf{P}_{\mu,\delta,\varepsilon,1}$ and $\mathbf{Q}_{\mu,\delta,\varepsilon,1}$.

We can then set $C_\ell = 0$ for $|\ell| \neq 1$ and find

$$\begin{aligned} C_1'' &= -\frac{1}{4k_{x,1}^2}C_1 + \frac{3}{4k_{x,1}^2}C_1(|C_1|^2 + 2|C_{-1}|^2) \\ C_{-1}'' &= -\frac{1}{4k_{x,1}^2}C_{-1} + \frac{3}{4k_{x,1}^2}C_{-1}(2|C_1|^2 + |C_{-1}|^2). \end{aligned} \tag{2.1.27}$$

According to [35] this system possesses a real, reversible heteroclinic orbit (C_+^*, C_-^*) with the following properties:

(i) $\lim_{x\to\infty}(C_+^*(x), C_-^*(x)) = (1/\sqrt{3}, 0)$ and $\lim_{x\to-\infty}(C_+^*(x), C_-^*(x)) = (0, 1/\sqrt{3})$;

(ii) $C_+^*(x) \geqslant 0$ and $C_-^*(x) \geqslant 0$, for all $x \in \mathbb{R}$;

(iii) $C_+^*(x) = C_-^*(-x)$, for all $x \in \mathbb{R}$;

(iv) $C_+^*(x) = C_-^*(x)$ if and only if $x = 0$;

(v) $C_+^{*2}(x) + C_-^{*2}(x) \leqslant 1/3$ and $C_+^*(x) + C_-^*(x) \geqslant 1/\sqrt{3}$, for all $x \in \mathbb{R}$.

Next, we study the linearization at this heteroclinic, which is

$$\mathcal{L}_* \begin{pmatrix} C_{-\ell_*} \\ \dots \\ C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \\ \dots \\ C_{\ell_*} \end{pmatrix} = \begin{pmatrix} C_{-\ell_*}'' + \frac{1}{4k_{x,-\ell_*}^2}C_{-\ell_*} - \frac{3}{2k_{x,-\ell_*}^2}(C_+^{*2} + C_-^{*2})C_{-\ell_*} \\ \dots \\ C_{-2}'' + \frac{1}{4k_{x,-2}^2}C_{-2} - \frac{3}{2k_{x,-2}^2}(C_+^{*2} + C_-^{*2})C_{-2} \\ \mathcal{L}_{*,-1}(C_{-1}, \overline{C_{-1}}, C_1, \overline{C_1}) \\ C_0'' + \frac{1}{4k_{x,0}^2}C_\ell - \frac{3}{2k_{x,0}^2}(C_+^{*2} + C_-^{*2})C_0 \\ \mathcal{L}_{*,1}(C_{-1}, \overline{C_{-1}}, C_1, \overline{C_1}) \\ C_2'' + \frac{1}{4k_{x,2}^2}C_2 - \frac{3}{2k_{x,2}^2}(C_+^{*2} + C_-^{*2})C_2 \\ \dots \\ C_{\ell_*}'' + \frac{1}{4k_{x,\ell_*}^2}C_{\ell_*} - \frac{3}{2k_{x,\ell_*}^2}(C_+^{*2} + C_-^{*2})C_{\ell_*} \end{pmatrix},$$

where

$$\mathcal{L}_{*,-1} = C''_{-1} + \frac{1}{4k_{x,1}^2}C_{-1} - \frac{3}{4k_{x,1}^2}\left(2(C_+^{*2}+C_-^{*2})C_{-1} + C_-^{*2}\overline{C_{-1}} + 2C_+^*C_-^*(C_1+\overline{C_1})\right),$$
$$\mathcal{L}_{*,1} = C''_{1} + \frac{1}{4k_{x,1}^2}C_{1} - \frac{3}{4k_{x,1}^2}\left(2(C_+^{*2}+C_-^{*2})C_{1} + C_+^{*2}\overline{C_{1}} + 2C_+^*C_-^*(C_{-1}+\overline{C_{-1}})\right).$$

We will see that $\mathcal{L}_* : \mathcal{Y}_\eta^r \to \mathcal{X}_\eta^r$ is Fredholm, where

$$\mathcal{X}_\eta^r = \{(C_\ell, \overline{C_\ell})_{\ell\in I_*} \in \mathcal{X}_\eta \ ;\ C_\ell(x) = \overline{C_{-\ell}}(-x),\ x\in\mathbb{R}, \ell\in I_*\},$$

$$\mathcal{X}_\eta = \{(C_\ell, \overline{C_\ell})_{\ell\in I_*} \in (L_\eta^2)^{4\ell+2}\}, \quad L_\eta^2 = \{f:\mathbb{R}\to\mathbb{C}\ ;\ \int_{\mathbb{R}} \mathrm{e}^{2\eta|x|}|f(x)|^2 < \infty\},$$
$$\mathcal{Y}_\eta = \{(C_\ell, \overline{C_\ell})_{\ell\in I_*} \in (H_\eta^2)^{4\ell+2}\}, \quad H_\eta^2 = \{f:\mathbb{R}\to\mathbb{C}\ ;\ f, f', f'' \in L_\eta^2\},$$

and $\mathcal{Y}_\eta^r = \mathcal{X}_\eta^r \cap \mathcal{Y}_\eta$.

Lemma 2.1.7 *Assume $\eta > 0$ is sufficiently small. Then the operator $\mathcal{L}_* : \mathcal{Y}_\eta^r \to \mathcal{X}_\eta^r$ is Fredholm with trivial kernel and one-dimensional co-kernel, spanned by*

$$(0,...,0,-\mathrm{i}C_-^*,0,\mathrm{i}C_+^*,0,...,0;0,...,0,\mathrm{i}C_-^*,0,-\mathrm{i}C_+^*,0,...,0).$$

Proof. Since $C_\pm^*$ is real, the linearization is diagonal after separating real and imaginary parts, $C_\kappa = U_\kappa + \mathrm{i}V_\kappa, \quad \kappa\in I_*$, with diagonal entries

$$\mathcal{M}_\ell \begin{pmatrix} U_\ell \\ V_\ell \end{pmatrix} = \begin{pmatrix} U''_\ell + \frac{1}{4k_{x,\ell}^2}U_\ell - \frac{3}{2k_{x,\ell}^2}(C_+^{*2}+C_-^{*2})U_\ell \\ V''_\ell + \frac{1}{4k_{x,\ell}^2}V_\ell - \frac{3}{2k_{x,\ell}^2}(C_+^{*2}+C_-^{*2})V_\ell \end{pmatrix}, \quad \text{for all } \ell\in I_*\setminus\{\pm1\}$$

$$\mathcal{M}_r \begin{pmatrix} U_1 \\ U_{-1} \end{pmatrix} = \begin{pmatrix} U''_1 + \frac{1}{4k_{x,1}^2}U_1 - \frac{3}{4k_{x,1}^2}\left((3C_+^{*2}+2C_-^{*2})U_1 + 4C_+^*C_-^*U_{-1}\right) \\ U''_{-1} + \frac{1}{4k_{x,1}^2}U_{-1} - \frac{3}{4k_{x,1}^2}\left((2C_+^{*2}+3C_-^{*2})U_{-1} + 4C_+^*C_-^*U_1\right) \end{pmatrix},$$
$$\mathcal{M}_i \begin{pmatrix} V_1 \\ V_{-1} \end{pmatrix} = \begin{pmatrix} V''_1 + \frac{1}{4k_{x,1}^2}V_1 - \frac{3}{4k_{x,1}^2}(C_+^{*2}+2C_-^{*2})V_1 \\ V''_{-1} + \frac{1}{4k_{x,1}^2}V_{-1} - \frac{3}{4k_{x,1}^2}(2C_+^{*2}+C_-^{*2})V_{-1} \end{pmatrix}.$$

We can now follow [1] to conclude that all $\mathcal{M}_\ell$ are invertible, $\mathcal{M}_i$ is invertible, and $\mathcal{M}_r$ is Fredholm of index -1. In fact, $\mathcal{M}_{r,i}$ coincide with the operators considered there, and we therefore also get an explicit description of the cokernel. This proves the lemma. ■

Persistence of the heteroclinic orbit. We are now ready to prove our main persistence result.

Theorem 3 *Assume that* $1/k_* \notin \mathbb{Z}$. *Then for all* $\mu > 0$ *small and small angle variations* δ, *there exists a smooth wavenumber correction* $\varepsilon = \varepsilon(\sqrt{\mu}, \delta)$, $\varepsilon(0,0) = 0$, *such that the system* (2.1.23) *possesses a heteroclinic orbit* $\mathbf{C}_{\mu,\delta}$ *connecting the periodic orbit* $\mathbf{P}_{\mu,\delta,\varepsilon,1}$ *to* $\mathbf{Q}_{\mu,\delta,\varepsilon,1}$.

Proof. We outline the proof which essentially follows the strategy in [36, 1]. We substitute into (2.1.23) together with its conjugate the ansatz

$$\mathbf{C}(x) = \mathrm{e}^{\mathrm{i}\varepsilon x}\mathbf{C}^*(x) + \chi(x)\widetilde{\mathbf{P}}_{\mu,\delta,\varepsilon,1}(x) + \left(\mathcal{R}(\chi\widetilde{\mathbf{P}}_{\mu,\delta,\varepsilon,1})\right)(-x) + \mathbf{V}(x), \tag{2.1.28}$$

in which heteroclinic orbit and correction to the periodic orbit are given as

$$\mathbf{C}^* = (0,...,0,C_-^*,0,C_+^*,0,...,0), \quad \widetilde{\mathbf{P}}_{\mu,\delta,\varepsilon,1} = \mathbf{P}_{\mu,\delta,\varepsilon,1} - \left(0,...,0,0,0,\frac{1}{\sqrt{3}},0,...,0\right)\mathrm{e}^{\mathrm{i}\varepsilon x},$$

$\chi : \mathbb{R} \to [0,1]$ is a smooth cut-off function with

$$\chi(x) = 1, \text{ if } x \geqslant M, \qquad \chi(x) = 0, \text{ if } x \leqslant m, \qquad \chi(x) + \chi(-x) \equiv 1,$$

for some positive constants $m < M$, and $(\mathbf{V}, \overline{\mathbf{V}}) \in \mathcal{Y}_\eta^r$. The substitution then leads to a nonlinear equation of the form

$$\mathcal{T}(\mathbf{V}, \overline{\mathbf{V}}, \varepsilon, \mu^{1/2}, \delta) = 0.$$

In fact, $\mathcal{T}$ is smooth on the weighted spaces $\mathcal{Y}_\eta^r$ (here we use the particular form of the ansatz) and we have

$$\mathcal{T}(0,0,0,0,0) = \mathcal{F}(\mathbf{C}^*, \overline{\mathbf{C}^*}, 0, 0) = 0, \quad D_{\mathbf{V},\overline{\mathbf{V}}}\mathcal{T}(0,0,0,0,0) = \mathcal{L}_*$$

and

$$D_\varepsilon\mathcal{T}(0,0,0,0,0) = \mathcal{L}_*\begin{pmatrix} ix\mathbf{C}^* \\ -ix\mathbf{C}^* \end{pmatrix} = \begin{pmatrix} 2i\mathbf{C}^{*\prime} \\ -2i\mathbf{C}^{*\prime} \end{pmatrix}.$$

Using the explicit form of the co-kernel, we find that the linearization with respect to $\mathbf{V}$, $\overline{\mathbf{V}}$ and ε, jointly, is invertible, so that we can use the implicit function theorem to conclude persistence as stated in the theorem. ∎

The result in Theorem 1 is an immediate consequence of Theorem 3. Since $\varepsilon = \mathcal{O}(\mu^{1/2} + |\delta|)$, we find that the angle of the selected grain boundary is $\alpha + \mathcal{O}(\mu + |\delta|)$, and that the wavenumber of the asymptotic rolls is $k_x = k_{x,1} + \mathcal{O}(\mu + \mu^{1/2}|\delta|)$. As a result, to leading order in μ and δ, the constant wavenumber k has the expansion $k = \sqrt{k_x^2 + (k_* + \delta)^2} = 1 + \mathcal{O}(\mu + \mu^{1/2}|\delta|)$, which, combined with the compactness of the interval $[\epsilon_\varphi, \pi - \epsilon_\varphi]$ for fixed $\epsilon_\varphi \in (0, \pi)$, shows that Theorem 1 is true.

2.2 Grain boundaries for resonant angles $1/k_* \in \mathbb{Z}$

In this section, we treat the resonant case; that is, the wavenumber $k = k_*$ satisfies the condition that

$$\frac{1}{k_*} \in \mathbb{Z}^+ \setminus \{1\}. \tag{2.2.1}$$

Recall that this condition guarantees rolls with angles $\varphi = 0, \pm \arcsin(k_*), \ldots, \pm \arcsin(1 - k_*), \pm\frac{\pi}{2}$ are compatible with the periodic boundary conditions in y. The basic idea follows the resonant case with some major differences that we will emphasize throughout, sometimes relegating the reader to Section 2.1 for aspects that are similar to the non-resonant case.

We start with the general center-manifold reduction and linear analysis in Section 2.2.1, calculate the normal form in Section 2.2.2, and prove persistence of heteroclinics in Section 2.2.3.

2.2.1 Center manifold reduction

Central space. We first recall the system (2.1.2)

$$\frac{dU}{dx} = \mathcal{A}_* U + \mathcal{B}(\mu, \delta)U + \mathcal{F}(U).$$

Taking into account the restriction (2.2.1) we have

$$\sigma(\mathcal{A}_*) \cap \mathrm{i}\mathbb{R} = \{\pm \mathrm{i} k_{\ell,x} | \ell = 0, \pm 1, \ldots \pm \ell_*\}, \quad k_{x,\ell} = \sqrt{1 - (k_* \ell)^2} \text{ and } \ell_* = \frac{1}{k_*}. \tag{2.2.2}$$

For simplicity, we denote $\widetilde{I}_* = \{0, \pm 1, \ldots \pm (\ell_* - 1)\}$ and $J_* = \{\pm \ell_*\}$. For every $\ell \in \widetilde{I}_*$, we define the generalized eigenspaces exactly as those in the nonresonant case.

For $\jmath \in J$, the eigenvalue 0 is geometrically double and algebraically octuple. The generalized eigenspace associated with 0 is spanned by

$$E_{\pm\ell_*}(y) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathrm{e}^{\pm \mathrm{i}\ell_* y}, \quad F_{\pm\ell_*,1}(y) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mathrm{e}^{\pm \mathrm{i}\ell_* y},$$

$$F_{\pm\ell_*,2}(y) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{\pm \mathrm{i}\ell_* y}, \quad F_{\pm\ell_*,3}(y) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{\pm \mathrm{i}\ell_* y},$$

which satisfy

$$\mathcal{A}_* E_{\pm\ell_*} = 0, \quad \mathcal{A}_* F_{\pm\ell_*,1} = E_{\pm\ell_*}, \quad \mathcal{A}_* F_{\pm\ell_*,2} = F_{\pm\ell_*,1}, \quad \mathcal{A}_* F_{\pm\ell_*,3} = F_{\pm\ell_*,2}.$$

The central space $\mathcal{X}_c$ of the operator $\mathcal{A}_*$ is $4(2\ell_*+1)$-dimensional and spanned by

$$\{E_\ell, F_\ell, \overline{E_\ell}, \overline{F_\ell} | \ell \in \widetilde{I}_*\} \bigcup \{E_{\pm\ell_*}, F_{\pm\ell_*,j} | j = 1,2,3\}.$$

Moreover, the spectral projection $\mathcal{P}_c : \mathcal{X} \to \mathcal{X}_c$ is given by

$$\begin{aligned} \mathcal{P}_c U = & \sum_{\ell \in \widetilde{I}_*} \Big(\langle U, E_\ell^{\mathrm{ad}} \rangle E_\ell + \langle U, F_\ell^{\mathrm{ad}} \rangle F_\ell + \langle U, \overline{E_\ell^{\mathrm{ad}}} \rangle \overline{E_\ell} + \langle U, \overline{F_\ell^{\mathrm{ad}}} \rangle \overline{F_\ell} \Big) + \\ & \frac{1}{2\pi} \sum_{\jmath = \pm\ell_*} \left(\langle U, E_\jmath \rangle E_\jmath + \langle U, F_{\jmath,1} \rangle F_{\jmath,1} + \langle U, F_{\jmath,2} \rangle F_{\jmath,2} + \langle U, F_{\jmath,3} \rangle F_{\jmath,3} \right), \end{aligned}$$

where $\{E_\ell^{\mathrm{ad}}, F_\ell^{\mathrm{ad}}\}_{\ell \in \widetilde{I}_*}$ is defined exactly as those in the nonresonant case.

Reduction to a center manifold and reduced system. We reduce to a center manifold and expand the reduced system. A direct computation confirms that we have the same Taylor expansion as in the previous case

$$\frac{dU_c}{dx} = \mathcal{A}_* U_c + \mathcal{P}_c \big(\mathcal{B}(\mu, \delta) U_c + \mathcal{F}(U_c) \big) + O(|\mu| \|U_c\|^3 + \|U_c\|^4). \tag{2.2.3}$$

Next, using the basis of $\mathcal{X}_c$ constructed above we set

$$\begin{aligned} U_c(x) = & \sum_{\ell \in \widetilde{I}_*} \big(A_\ell(x) E_\ell + B_\ell(x) F_\ell + \overline{A_\ell}(x) \overline{E_\ell} + \overline{B_\ell}(x) \overline{F_\ell} \big) + \\ & \sum_{\jmath = \pm\ell_*} \left(A_\jmath(x) E_\jmath + B_{\jmath,1} F_{\jmath,1} + B_{\jmath,2} F_{\jmath,2} + B_{\jmath,3} F_{\jmath,3} \right), \end{aligned}$$

where $A_\ell, B_\ell, A_{\pm\ell_*}, B_{\pm\ell_*,1}, B_{\pm\ell_*,2}, B_{\pm\ell_*,3},\ \ell \in \widetilde{I}_*$, are complex-valued functions and $A_{-\ell_*} = \overline{A_{\ell_*}}, B_{-\ell_*,j} = \overline{B_{\ell_*,j}},\ j = 1,2,3$. A straightforward but lengthy calculation gives the leading order terms in the reduced system

$$\frac{dU_c}{dx} = \mathcal{A}_* U_c + \mathcal{P}_c\big(\mathcal{B}(\mu,\delta)U_c + \mathcal{F}(U_c)\big),$$

expressed in the basis $\{A_\ell, B_\ell, A_{\ell_*}, B_{\ell_*,1}, B_{\ell_*,2}, B_{\ell_*,3} | \ell \in \widetilde{I}_*\}$ of the center eigenspace,

$$\begin{aligned}
A'_\ell =& \mathrm{i}k_{x,\ell}A_\ell + B_\ell - \mathrm{i}\frac{\mu + 2k_{x,\ell}^2(k_{x,\ell}^2 - k_{x,\ell,\delta}^2)}{4k_{x,\ell}^3}a_\ell + \frac{k_{x,\ell}^2 - k_{x,\ell,\delta}^2}{2k_{x,\ell}^2}b_\ell + \frac{\mathrm{i}}{4k_{x,\ell}^3}P_\ell \\
B'_\ell =& \mathrm{i}k_{x,\ell}B_\ell - \frac{1}{4k_{x,\ell}^2}\mu a_\ell - \mathrm{i}\frac{k_{x,\ell}^2 - k_{x,\ell,\delta}^2}{2k_{x,\ell}}b_\ell + \frac{1}{4k_{x,\ell}^2}P_\ell \\
A'_{\ell_*} =& B_{\ell_*,1} \\
B'_{\ell_*,1} =& -k_{x,\ell_*,\delta}^2 A_{\ell_*} + B_{\ell_*,2} \\
B'_{\ell_*,2} =& B_{\ell_*,3} \\
B'_{\ell_*,2} =& \mu A_{\ell_*} - k_{x,\ell_*,\delta}^2 B_{\ell_*,2} - P_{\ell_*}
\end{aligned} \tag{2.2.4}$$

in which

$$k_{x,\ell_*,\delta}^2 = 1 - (k_* + \delta)^2\ell_*^2, \quad P_\ell((A_\ell, \overline{A_\ell})_{\ell\in I_*}) = \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell,\\ \ell_1,\ell_2,\ell_3\in I_*}} a_{\ell_1}a_{\ell_2}a_{\ell_3}, \quad \text{for all } \ell \in I_*,$$

where

$$k_{x,\ell_*,\delta} = \begin{cases} \sqrt{1-(k_*+\delta)^2\ell_*^2}, & \delta \leqslant 0 \\ -\mathrm{i}\sqrt{(k_*+\delta)^2\ell_*^2 - 1}, & \delta > 0, \end{cases} \qquad a_\ell = \begin{cases} A_\ell + \overline{A_{-\ell}}, & \ell \in \widetilde{I}_* \\ A_{\ell_*}, & \ell = \ell_* \\ A_{-\ell_*}, & \ell = -\ell_*. \end{cases}$$

Note that here $|\delta|$ is sufficiently small and $k_{x,\ell_*,\delta}$ is not smooth with respect to δ while $k_{x,\ell_*,\delta}^2$ is.

Symmetries. The three reflection symmetries in the Swift-Hohenberg equation

$$y \mapsto -y, \quad x \mapsto -x, \quad u \mapsto -u.$$

are inherited through the center manifold reduction, that is, the reduced system (2.2.3) inherits the induced symmetries. While the symmetries with respect to the vector

$(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell})_{\ell\in\widetilde{I}_*}$ are the same as those in the nonresonant case, we have to take care of the other part as follows. The reflection $y \mapsto -y$, $u \mapsto -u$ and the y-translation invariance of the equation (1.2.3) imply symmetries in the reduced system (2.2.3) that

$$\begin{aligned}
&\mathcal{S}_1(A_{\ell_*}, B_{\ell_*,1}, B_{\ell_*,2}, B_{\ell_*,3}, A_{-\ell_*}, B_{-\ell_*,1}, B_{-\ell_*,2}, B_{-\ell_*,3})^T\\
&= (A_{-\ell_*}, B_{-\ell_*,1}, B_{-\ell_*,2}, B_{-\ell_*,3}, A_{\ell_*}, B_{\ell_*,1}, B_{\ell_*,2}, B_{\ell_*,3})^T,\\
&\mathcal{S}_2(A_{\ell_*}, B_{\ell_*,1}, B_{\ell_*,2}, B_{\ell_*,3}, A_{-\ell_*}, B_{-\ell_*,1}, B_{-\ell_*,2}, B_{-\ell_*,3})^T\\
&= -(A_{\ell_*}, B_{\ell_*,1}, B_{\ell_*,2}, B_{\ell_*,3}, A_{-\ell_*}, B_{-\ell_*,1}, B_{-\ell_*,2}, B_{-\ell_*,3})^T,\\
&\mathcal{T}_\phi(A_{\ell_*}, B_{\ell_*,1}, B_{\ell_*,2}, B_{\ell_*,3}, A_{-\ell_*}, B_{-\ell_*,1}, B_{-\ell_*,2}, B_{-\ell_*,3})^T\\
&= \Big(\mathrm{e}^{i\ell_*\phi}(A_{\ell_*}, B_{\ell_*,1}, B_{\ell_*,2}, B_{\ell_*,3}), \mathrm{e}^{-i\ell_*\phi}(A_{-\ell_*}, B_{-\ell_*,1}, B_{-\ell_*,2}, B_{-\ell_*,3})\Big)^T.
\end{aligned}$$

The reflection $x \mapsto -x$ implies the vector field of the reduced system anticommutes with

$$\begin{aligned}
&\mathcal{R}(A_{\ell_*}, B_{\ell_*,1}, B_{\ell_*,2}, B_{\ell_*,3}, A_{-\ell_*}, B_{-\ell_*,1}, B_{-\ell_*,2}, B_{-\ell_*,3})^T\\
=&(A_{\ell_*}, -B_{\ell_*,1}, B_{\ell_*,2}, -B_{\ell_*,3}, A_{-\ell_*}, -B_{-\ell_*,1}, B_{-\ell_*,2}, -B_{-\ell_*,3})^T.
\end{aligned}$$

As a consequence, the higher order terms in the reduced system (2.2.3) are such that

$$\frac{dU_c}{dx} = \mathcal{A}_* U_c + \mathcal{P}_c\big(\mathcal{B}(\mu,\delta)U_c + \mathcal{F}(U_c)\big) + \mathcal{O}(|\mu|\|U_c\|^3 + \|U_c\|^5). \tag{2.2.5}$$

2.2.2 Normal form transformations

Just as in Section 2.1.3, to arrive at a normal form for (2.2.5), we sequentially simplifying the linear terms and the cubic terms.

Linear versal transformation. Consider the linear part of the system (2.2.4), for $\ell \in \widetilde{I}_*$,

$$\begin{aligned}
A_\ell' =&\mathrm{i}k_{x,\ell}A_\ell + B_\ell - \mathrm{i}\frac{\mu + 2k_{x,\ell}^2(k_{x,\ell}^2 - k_{x,\ell,\delta}^2)}{4k_{x,\ell}^3}a_\ell + \frac{k_{x,\ell}^2 - k_{x,\ell,\delta}^2}{2k_{x,\ell}^2}b_\ell\\
B_\ell' =&\mathrm{i}k_{x,\ell}B_\ell - \frac{1}{4k_{x,\ell}^2}\mu a_\ell - \mathrm{i}\frac{k_{x,\ell}^2 - k_{x,\ell,\delta}^2}{2k_{x,\ell}}b_\ell\\
A_{\ell_*}' =&B_{\ell_*,1}\\
B_{\ell_*,1}' =&-k_{x,\ell_*,\delta}^2 A_{\ell_*} + B_{\ell_*,2}\\
B_{\ell_*,2}' =&B_{\ell_*,3}\\
B_{\ell_*,2}' =&\mu A_{\ell_*} - k_{x,\ell_*,\delta}^2 B_{\ell_*,2} - P_{\ell_*}
\end{aligned} \tag{2.2.6}$$

We derive the linear normal form for the non-resonant modes $(A_\ell, B_\ell, \overline{A_\ell}, \overline{B_\ell})_{\ell\in\widetilde{I}_*}$ as in Lemma 2.1.4. For $(A_\jmath, B_{\jmath,1}, B_{\jmath,2}, B_{\jmath,3})_{\jmath=\pm\ell_*}$, we have the following unfolding.

Lemma 2.2.1 *There exist linear maps $L_{\ell_*}(\mu,\delta)$ such that, for sufficiently small μ and δ, the linear change of variables*

$$\begin{aligned}&(A_{\ell_*}, B_{\ell_*,1}, B_{\ell_*,2}, B_{\ell_*,3}, A_{-\ell_*}, B_{-\ell_*,1}, B_{-\ell_*,2}, B_{-\ell_*,3})^T\\ =&L_{\ell_*}(\mu,\delta)(C_{\ell_*}, D_{\ell_*,1}, C_{\ell_*,2}, D_{\ell_*,3}, C_{-\ell_*}, D_{-\ell_*,1}, C_{-\ell_*,2}, D_{-\ell_*,3})^T\end{aligned}\tag{2.2.7}$$

transforms the corresponding part of system (2.2.6) together with its complex conjugate into the normal form

$$C'_\jmath = D_{\jmath,1}, \quad D'_{\jmath,1} = D_{\jmath,2}, \quad D'_{\jmath,2} = D_{\jmath,3}, \quad D'_{\jmath,3} = (\mu - k^4_{x,\ell_*,\delta})C_\jmath - 2k^2_{x,\ell_*,\delta}D_{\jmath,2}, \tag{2.2.8}$$

where $\jmath = \pm\ell_*$. Moreover, we can choose $L_\ell(\mu,\delta)$ as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ k^2_{x,\ell_*,\delta} & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & k^2_{x,\ell_*,\delta} & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & k^2_{x,\ell_*,\delta} & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & k^2_{x,\ell_*,\delta} & 0 & 1 \end{pmatrix}. \tag{2.2.9}$$

Proof. We proceed as in the proof of Lemma 2.2.2. The versal transformation we use here is just the Sylvester family (see [33] for details). To obtain $L_{\ell_*}(\mu,\delta)$ as claimed, just note the fact that $L_{\ell_*}(\mu,\delta)$ commutes with $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{R}$ and $\mathcal{T}_\phi$ for all $\phi\in\mathbb{R}$. ■

Based on Lemma 2.2.1, our reduced system (2.2.4,2.2.5) has the following expression.

$$\begin{aligned}C'_\ell =&\mathrm{i}\sqrt{\frac{1}{2}(k^2_{x,\ell,\delta} + \sqrt{k^4_{x,\ell,\delta} - \mu})}C_\ell + D_\ell + \frac{\mathrm{i}}{4k^3_{x,\ell,\delta}}P_\ell, \quad \ell\in\widetilde{I}_*,\\ D'_\ell =&-\frac{1}{2}(k^2_{x,\ell,\delta} - \sqrt{k^4_{x,\ell,\delta} - \mu})C_\ell +\\ &\mathrm{i}\sqrt{\frac{1}{2}(k^2_{x,\ell,\delta} + \sqrt{k^4_{x,\ell,\delta} - \mu})}D_\ell + \frac{1}{4k^2_{x,\ell,\delta}}P_\ell, \quad \ell\in\widetilde{I}_*,\\ C'_{\ell_*} =&D_{\ell_*,1}, \quad D'_{\ell_*,1} = D_{\ell_*,2}, \quad D'_{\ell_*,2} = D_{\ell_*,3},\\ D'_{\ell_*,3} =&(\mu - k^4_{x,\ell_*,\delta})C_{\ell_*} - 2k^2_{x,\ell_*,\delta}D_{\ell_*,2} - P_{\ell_*}.\end{aligned}\tag{2.2.10}$$

with higher order terms in the form

$$\begin{aligned}\mathcal{O}(|\mu|(\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|+|D_\kappa|+\sum_{\zeta=\pm\ell_*}|C_\zeta|+|D_{\zeta,1}|+|D_{\zeta,2}|+|D_{\zeta,3}|)^3\\ +(\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|+|D_\kappa|+\sum_{\zeta=\pm\ell_*}|C_\zeta|+|D_{\zeta,1}|+|D_{\zeta,2}|+|D_{\zeta,3}|)^5).\end{aligned}$$

Cubic transformation. Taking $\mu=\delta=0$, we obtain the leading order of system (2.1.9),

$$\begin{aligned}A'_\ell =&\mathrm{i}k_{x,\ell}A_\ell+B_\ell+\frac{\mathrm{i}}{4k_{x,\ell}^3}P_\ell,\\ B'_\ell =&\mathrm{i}k_{x,\ell}B_\ell+\frac{1}{4k_{x,\ell}^2}P_\ell\\ A'_{\ell_*} =&B_{\ell_*,1}\\ B'_{\ell_*,1} =&B_{\ell_*,2}\\ B'_{\ell_*,2} =&B_{\ell_*,3}\\ B'_{\ell_*,2} =&-P_{\ell_*}.\end{aligned}\tag{2.2.11}$$

Lemma 2.2.2 *There exist homogeneous polynomials*

$$\{\Phi_\ell,\Psi_\ell,\Phi_{\ell_*},\Psi_{\ell_*,1},\Psi_{\ell_*,2},\Psi_{\ell_*,3}|\ell\in\widetilde{I}_*\}$$

of degree 3 in the complex variables $(C_\kappa,D_\kappa,\overline{C_\kappa},\overline{D_\kappa})_{\kappa\in I_*}$ *and* $(C_\zeta,D_{\zeta,1},D_{\zeta,2},D_{\zeta,3})_{\zeta=\pm\ell_*}$, *such that the change of variables, for all* $\ell\in\widetilde{I}_*$,

$$\begin{aligned}A_\ell =&C_\ell+\Phi_\ell((C_\kappa,D_\kappa,\overline{C_\kappa},\overline{D_\kappa})_{\kappa\in\widetilde{I}_*},(C_\zeta,D_{\zeta,1},D_{\zeta,2},D_{\zeta,3})_{\zeta=\pm\ell_*}),\\ B_\ell =&D_\ell+\Psi_\ell((C_\kappa,D_\kappa,\overline{C_\kappa},\overline{D_\kappa})_{\kappa\in\widetilde{I}_*},(C_\zeta,D_{\zeta,1},D_{\zeta,2},D_{\zeta,3})_{\zeta=\pm\ell_*}),\\ A_{\ell_*} =&C_{\ell_*}+\Phi_{\ell_*}((C_\kappa,D_\kappa,\overline{C_\kappa},\overline{D_\kappa})_{\kappa\in\widetilde{I}_*},(C_\zeta,D_{\zeta,1},D_{\zeta,2},D_{\zeta,3})_{\zeta=\pm\ell_*}),\\ B_{\ell_*,1} =&D_{\ell_*,1}+\Psi_{\ell_*,1}((C_\kappa,D_\kappa,\overline{C_\kappa},\overline{D_\kappa})_{\kappa\in\widetilde{I}_*},(C_\zeta,D_{\zeta,1},D_{\zeta,2},D_{\zeta,3})_{\zeta=\pm\ell_*}),\\ B_{\ell_*,2} =&D_{\ell_*,2}+\Psi_{\ell_*,2}((C_\kappa,D_\kappa,\overline{C_\kappa},\overline{D_\kappa})_{\kappa\in\widetilde{I}_*},(C_\zeta,D_{\zeta,1},D_{\zeta,2},D_{\zeta,3})_{\zeta=\pm\ell_*}),\\ B_{\ell_*,3} =&D_{\ell_*,3}+\Psi_{\ell_*,3}((C_\kappa,D_\kappa,\overline{C_\kappa},\overline{D_\kappa})_{\kappa\in\widetilde{I}_*},(C_\zeta,D_{\zeta,1},D_{\zeta,2},D_{\zeta,3})_{\zeta=\pm\ell_*}),\end{aligned}\tag{2.2.12}$$

is well-defined in a neighborhood of the origin and transforms the system (2.1.16) *into* *the normal form with* $\ell \in \widetilde{I}_*$,

$$\begin{aligned}
C'_\ell =& \mathrm{i}k_{x,\ell}C_\ell + D_\ell + \\
& \mathcal{O}((\sum_{\kappa\in\widetilde{I}_*}|C_\kappa| + |D_\kappa| + \sum_{\zeta=\pm\ell_*}|C_\zeta| + |D_{\zeta,1}| + |D_{\zeta,2}| + |D_{\zeta,3}|)^5) \\
D'_\ell =& \mathrm{i}k_{x,\ell}D_\ell + \frac{3}{4k_{x,\ell}^2}C_\ell(|C_{\ell_*}|^2 - |C_\ell|^2 + 2\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|^2) + \\
& \frac{3\mathrm{i}}{4k_{x,\ell}^3}D_\ell(|C_{\ell_*}|^2 - |C_\ell|^2 + 2\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|^2) + \frac{3\mathrm{i}}{4k_{x,\ell}^3}C_\ell\left((C_{\ell_*}\overline{D_{\ell_*,1}}\right. \\
& \left.+\overline{C_{\ell_*}}D_{\ell_*,1}) - (C_\ell\overline{D_\ell} + \overline{C_\ell}D_\ell) + 2\sum_{\kappa\in I_*}(C_\kappa\overline{D_\kappa} + \overline{C_\kappa}D_\kappa)\right) + \\
& \mathcal{O}((\sum_{\kappa\in\widetilde{I}_*}|C_\kappa| + |D_\kappa| + \sum_{\zeta=\pm\ell_*}|C_\zeta| + |D_{\zeta,1}| + |D_{\zeta,2}| + |D_{\zeta,3}|)^5) \\
C'_{\ell_*} =& D_{\ell_*,1} \\
D'_{\ell_*,1} =& D_{\ell_*,2} \\
D'_{\ell_*,2} =& D_{\ell_*,3} \\
D'_{\ell_*,3} =& -3C_{\ell_*}(|C_{\ell_*}|^2 + 2\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|^2) + \\
& \mathcal{O}((\sum_{\kappa\in\widetilde{I}_*}|C_\kappa| + |D_\kappa| + \sum_{\zeta=\pm\ell_*}|C_\zeta| + |D_{\zeta,1}| + |D_{\zeta,2}| + |D_{\zeta,3}|)^5).
\end{aligned} \tag{2.2.13}$$

Proof. Denote the cubic terms in the right hand sides of the systems (2.2.11) and (2.2.13), respectively, by $(F_\ell, G_\ell)_{\ell\in\widetilde{I}_*}$ and $(0,0,0,G_{\ell_*,3})$, $(0, N_\ell)_{\ell\in\widetilde{I}_*}$ and $(0,0,0,N_{\ell_*,3})$. Using the same argument as in the nonresonant case, we derive $(0, N_\ell)_{\ell\in\widetilde{I}_*}$. The constant of $|C_{\ell_*}|^2$ is 1 instead of 2 due to the fact that $a_{\pm\ell_*} = A_{\pm\ell_*}$ instead of $a_{\pm\ell_*} = A_{\pm\ell_*} + \overline{A_{\mp\ell_*}}$. On the other hand, it is straightforward to see that

$$\Phi_{\ell_*} = 0, \quad \Psi_{\ell_*,1} = 0, \quad \Psi_{\ell_*,2} = 0.$$

Moreover, we have $\mathcal{D}\Psi_{\ell_*,3} = G_{\ell_*,3} - N_{\ell_*,3}$, where

$$\begin{aligned}
\mathcal{D} \;=\; & \sum_{\ell\in I_*}\left((\mathrm{i}k_{x,\ell}C_\ell + D_\ell)\frac{\partial}{\partial C_\ell} + \mathrm{i}k_{x,\ell}D_\ell\frac{\partial}{\partial D_\ell} + c.c.\right) + \\
& \sum_{\jmath=\pm\ell_*}\left(D_{\jmath,1}\frac{\partial}{\partial C_\jmath} + D_{\jmath,2}\frac{\partial}{\partial D_{\jmath,1}} + D_{\jmath,2}\frac{\partial}{\partial D_{\jmath,3}}\right).
\end{aligned}$$

Then it is not hard to see that $N_{\ell_*,3} = -3C_{\ell_*}(|C_{\ell_*}|^2 + 2\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|^2)$. ■

Normal form of the reduced system. Applying the change of variables to the reduced system (2.2.5) gives the normal form to leading order, for $\ell \in \widetilde{I}_*$,

$$
\begin{aligned}
C'_\ell =& \mathrm{i}\sqrt{\frac{1}{2}(k^2_{x,\ell,\delta} + \sqrt{k^4_{x,\ell,\delta} - \mu})}C_\ell + D_\ell \\
D'_\ell =& -\frac{1}{2}(k^2_{x,\ell,\delta} - \sqrt{k^4_{x,\ell,\delta} - \mu})C_\ell + \mathrm{i}\sqrt{\frac{1}{2}(k^2_{x,\ell,\delta} + \sqrt{k^4_{x,\ell,\delta} - \mu})}D_\ell + \\
& \frac{3}{4k^2_{x,\ell}}C_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2) + \frac{3\mathrm{i}}{4k^3_{x,\ell}}D_\ell(-|C_\ell|^2 + 2\sum_{\kappa\in I_*}|C_\kappa|^2) + \\
& \frac{3\mathrm{i}}{4k^3_{x,\ell}}C_\ell\left(-(C_\ell\overline{D_\ell} + \overline{C_\ell}D_\ell) + 2\sum_{\kappa\in I_*}(C_\kappa\overline{D_\kappa} + \overline{C_\kappa}D_\kappa)\right) \qquad (2.2.14) \\
C'_{\ell_*} =& D_{\ell_*,1} \\
D'_{\ell_*,1} =& D_{\ell_*,2} \\
D'_{\ell_*,2} =& D_{\ell_*,3} \\
D'_{\ell_*,3} =& (\mu - k^4_{x,\ell_*,\delta})C_{\ell_*} - 2k^2_{x,\ell_*,\delta}D_{\ell_*,2} - 3C_{\ell_*}(|C_{\ell_*}|^2 + 2\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|^2).
\end{aligned}
$$

The higher order terms in this normal form are of order

$$
\begin{aligned}
&|\delta|(\textstyle\sum_{\kappa\in\widetilde{I}_*}(|C_\kappa| + |D_\kappa|) + \sum_{\zeta=\pm\ell_*}(|C_\zeta| + |D_{\zeta,1}| + |D_{\zeta,2}| + |D_{\zeta,3}|))^3 + \\
&|\mu|(\textstyle\sum_{\kappa\in\widetilde{I}_*}(|C_\kappa| + |D_\kappa|) + \sum_{\zeta=\pm\ell_*}(|C_\zeta| + |D_{\zeta,1}| + |D_{\zeta,2}| + |D_{\zeta,3}|))^3 + \\
&(\textstyle\sum_{\kappa\in\widetilde{I}_*}(|C_\kappa| + |D_\kappa|) + \sum_{\zeta=\pm\ell_*}(|C_\zeta| + |D_{\zeta,1}| + |D_{\zeta,2}| + |D_{\zeta,3}|))^5.
\end{aligned}
$$

Specifically, the higher order terms for $(C_{\ell_*}, D_{\ell_*,1}, D_{\ell_*,2}, D_{\ell_*,3})$ are of order

$$
\begin{aligned}
&|\mu|(\textstyle\sum_{\kappa\in\widetilde{I}_*}(|C_\kappa| + |D_\kappa|) + \sum_{\zeta=\pm\ell_*}(|C_\zeta| + |D_{\zeta,1}| + |D_{\zeta,2}| + |D_{\zeta,3}|))^3 + \\
&(\textstyle\sum_{\kappa\in\widetilde{I}_*}(|C_\kappa| + |D_\kappa|) + \sum_{\zeta=\pm\ell_*}(|C_\zeta| + |D_{\zeta,1}| + |D_{\zeta,2}| + |D_{\zeta,3}|))^5.
\end{aligned}
$$

2.2.3 Existence of heteroclinic orbits

We look for solutions of the system (2.2.14) in the form

$$
\begin{aligned}
C_\ell(x) = \mathrm{e}^{\mathrm{i}\tilde{k}_{\ell,\mu,\delta}x}\widetilde{C_\ell}, \quad D_\ell(x) = \mathrm{e}^{\mathrm{i}\tilde{k}_{\ell,\mu,\delta}x}\widetilde{D_\ell}, \quad \ell \in I_*; \\
C_{\ell_*}(x) = \widetilde{C_{\ell_*}}(x), \quad D_{\ell_*,j}(x) = \widetilde{D_{\ell_*,j}}(x), \quad j = 1, 2, 3.
\end{aligned}
$$

With the scaling

$$\begin{gathered}\widehat{x}=|\mu|^{1/2}x,\quad \widetilde{C_\ell}=|\mu|^{1/2}\widehat{C_\ell},\quad \widetilde{D_\ell}=|\mu|\widehat{D_\ell},\quad \ell\in\widetilde{I}_*,\\ \widetilde{C_{\ell_*}}=|\mu|^{1/2}\widehat{C_{\ell_*}},\quad \widetilde{D_{\ell_*,1}}=|\mu|^{1/2}\widehat{D_{\ell_*,1}},\\ \widetilde{D_{\ell_*,2}}=|\mu|^{1/2}\widehat{D_{\ell_*,2}},\quad \widetilde{D_{\ell_*,3}}=|\mu|^{1/2}\widehat{D_{\ell_*,3}},\end{gathered}$$

and taking $\mu>0$, we obtain the new system, for $\ell\in\widetilde{I}_*$,

$$\begin{aligned}C_\ell' =&D_\ell+\mathcal{O}(\mu+\mu^{1/2}|\delta|)\\ D_\ell' =&-\frac{1}{4k_{x,\ell}^2}C_\ell\left(1-3(|C_{\ell_*}|^2-|C_\ell|^2+2\sum_{\kappa\in I_*}|C_\kappa|^2)\right)+\mathcal{O}(\mu^{1/2}+|\delta|),\\ \mu^{1/2}C_{\ell_*}' =&D_{\ell_*,1}+\mathcal{O}(\mu^2)\\ \mu^{1/2}D_{\ell_*,1}' =&D_{\ell_*,2}+\mathcal{O}(\mu^2)\\ \mu^{1/2}D_{\ell_*,2}' =&D_{\ell_*,3}+\mathcal{O}(\mu^2)\\ \mu^{1/2}D_{\ell_*,3}' =&-2k_{x,\ell_*,\delta}^2D_{\ell_*,2}-k_{x,\ell_*,\delta}^4C_{\ell_*}+\\ &\mu C_{\ell_*}\left(1-3(|C_{\ell_*}|^2+2\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|^2)\right)+\mathcal{O}(\mu^2).\end{aligned}\tag{2.2.15}$$

in which we have dropped the hats. Since the equations for $(C_{\ell_*},D_{\ell_*,1},D_{\ell_*,2},D_{\ell_*,3})$ are singular, the strategy here is to break (2.2.15) up into a regular and a singular perturbation problem. The regular perturbation part

$$C_\ell''=-\frac{1}{4k_{x,\ell}^2}C_\ell+\frac{3}{4k_{x,\ell}^2}C_\ell(|C_{\ell_*}|^2-|C_\ell|^2+2\sum_{\kappa\in I_*}|C_\kappa|^2)+\mathcal{O}(\mu^{1/2}+|\delta|),\quad \ell\in I_*,\tag{2.2.16}$$

can be dealt with just as in the nonresonant case. As for the singular part, we rewrite it as follows.

$$\begin{aligned}\mu^{1/2}C_{\ell_*}' =&D_{\ell_*,1}\\ \mu^{1/2}D_{\ell_*,1}' =&D_{\ell_*,2}\\ \mu^{1/2}D_{\ell_*,2}' =&D_{\ell_*,3}\\ \mu^{1/2}D_{\ell_*,3}' =&-2k_{x,\ell_*,\delta}^2D_{\ell_*,2}-k_{x,\ell_*,\delta}^4C_{\ell_*}+\\ &\mu C_{\ell_*}\left(1-3(|C_{\ell_*}|^2+2\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|^2)\right)+\mathcal{O}(\mu^2),\end{aligned}\tag{2.2.17}$$

which can be achieved straightforwardly by an iterative family of smooth invertible coordinate changes.

Heteroclinic orbit of the leading order system Consider the systems (2.2.16) and (2.2.17) with $\mu = 0$ and $\delta = 0$. Restricting to $C_\ell = 0$ for $|\ell| \neq 1$ and real-valued solutions C_1 and C_{-1} we find the system

$$\begin{aligned} C_1'' &= -\frac{1}{4k_{x,1}^2} C_1 + \frac{3}{4k_{x,1}^2} C_1(C_1^2 + 2C_{-1}^2) \\ C_{-1}'' &= -\frac{1}{4k_{x,1}^2} C_{-1} + \frac{3}{4k_{x,1}^2} C_{-1}(2C_1^2 + C_{-1}^2). \end{aligned} \tag{2.2.18}$$

This system possesses a heteroclinic orbit (C_+^*, C_-^*) with properties as those in the nonresonant case.

The system (2.2.17) is a singular perturbation problem, which, viewed as an operator, is defined on different spaces when $(\mu, \delta) \neq (0, 0)$ as compared to the case $(\mu, \delta) = (0, 0)$. Our approach to this singular perturbation problem relies on the following lemma, which shows invertibility of the linearization uniformly for small $\mu \neq 0$.

Lemma 2.2.3 *For fixed $\mu > 0$ and fixed $\delta \in \mathbb{R}$, the linear operator*

$$\begin{aligned} \mathcal{S} : (H_\eta^1)^4 &\longrightarrow (L_\eta^2)^4 \\ \begin{pmatrix} W_1(x) \\ W_2(x) \\ W_3(x) \\ W_4(x) \end{pmatrix} &\longmapsto \begin{pmatrix} \mu^{1/2}\partial_x & -1 & 0 & 0 \\ 0 & \mu^{1/2}\partial_x & -1 & 0 \\ 0 & 0 & \mu^{1/2}\partial_x & -1 \\ g(x) + k_{x,\ell_*,\delta}^4/\mu & 0 & 2k_{x,\ell_*,\delta}^2/\mu & \mu^{-1/2}\partial_x \end{pmatrix} \begin{pmatrix} W_1(x) \\ W_2(x) \\ W_3(x) \\ W_4(x) \end{pmatrix}, \end{aligned}$$

where $g(x) = 6(C_+^{*2} + C_-^{*2}) - 1$, is bounded and invertible, for sufficiently small $\eta > 0$. Moreover, for fixed $\eta > 0$ small, and for $\mu > 0$, $\delta \sim 0$ small, there exists a positive constant $C \geqslant 1$ independent of μ and δ such that

$$\|\mathcal{S}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ f \end{pmatrix} \|_{(H_\eta^1)^4} \leqslant C \|f\|_{H_\eta^1}, \tag{2.2.19}$$

for all $f \in H_\eta^1$.

Proof. We define the equivalent scalar fourth-order differential operator

$$\begin{aligned} \widetilde{\mathcal{S}}: H^4_\eta &\longrightarrow \qquad\qquad\qquad L^2_\eta \\ C(x) &\longmapsto \frac{1}{\mu}(\mu\partial_x^2 + k^2_{x,\ell_*,\delta})^2 C(x) + g(x)C(x). \end{aligned} \tag{2.2.20}$$

From [37], we know that $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ share the same Fredholm index. Moreover, $\mathrm{Ker}\,(\widetilde{\mathcal{S}}) = \{0\}$ if and only if $\mathrm{Ker}\,(\mathcal{S}) = \{0\}$. Thus, it is enough to prove the same properties for $\widetilde{\mathcal{S}}$. It is straightforward to see that $\widetilde{\mathcal{S}}$ is bounded. To show that $\widetilde{\mathcal{S}}$ is invertible, we just need to show $\widetilde{\mathcal{S}_\eta}$ is invertible, where

$$\begin{aligned} \widetilde{\mathcal{S}_\eta}: H^4 &\longrightarrow L^2 \\ C(x) &\longmapsto \cosh(\eta x)\mathcal{S}(\frac{1}{\cosh(\eta x)}C(x)). \end{aligned}$$

Note that $\widetilde{\mathcal{S}_\eta}$ is a family of smooth bounded operators with respect to η and $\widetilde{\mathcal{S}_\eta} = \widetilde{\mathcal{S}_0} + \eta\cdot\mathcal{O}(1)$. Therefore, to show $\widetilde{\mathcal{S}_\eta}$ is invertible for sufficiently small η, we only have to show that $\widetilde{\mathcal{S}_0}$ is invertible. In fact, according to property v of (C_+^*, C_-^*),

$$\frac{1}{6} \leqslant C_+^{*2} + C_-^{*2} \leqslant \frac{1}{3},$$

where the left equality holds if and only if $(C_+^*, C_-^*) = (\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$. If the curve (C_+^*, C_-^*) touches $(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$, by properties (iii,iv), $C_+^* + C_-^*$ is an even function and $C_+^*(0) + C_-^*(0) = \frac{1}{\sqrt{3}}$. Moreover, according to system (2.1.27), we have

$$(C_+^* + C_-^*)''|_{x=0} = -\frac{1}{4\sqrt{3}} < 0.$$

As a result, for sufficiently small $|x| \neq 0$, $C_+^*(x) + C_-^*(x) < \frac{1}{\sqrt{3}}$, which is a contradiction with property (v). Thus, there exists a positive constant ξ such that

$$\frac{1}{6} + \xi \leqslant C_+^{*2} + C_-^{*2} \leqslant \frac{1}{3}.$$

Now we look into $\widetilde{\mathcal{S}}_0$, which is self-adjoint and

$$\langle \widetilde{\mathcal{S}}_0 f, f\rangle_{L^2} \geqslant 6\xi \|f\|^2_{L^2},$$

which indicates that, for sufficiently small μ and δ, $\mathcal{S}_0$ is invertible and

$$\|(\widetilde{\mathcal{S}}_0)^{-1} f\|_{L^2} \leqslant \frac{1}{6\xi}\|f\|_{L^2}, \tag{2.2.21}$$

which shows that $\mathcal{S}(\mu,\delta)$ is bounded and invertible for sufficiently small $\eta > 0$. In addition, there exists some positive constant $N(\xi)$ such that

$$\|(\widetilde{\mathcal{S}}(\mu,\delta))^{-1} f\|_{H^1_\eta} \leqslant N\|f\|_{H^1_\eta}. \tag{2.2.22}$$

In fact, we have

$$\partial_x(\widetilde{\mathcal{S}}^{-1} f) = [\partial_x, \widetilde{\mathcal{S}}^{-1}]f + \widetilde{\mathcal{S}}^{-1}(\partial_x f) = \widetilde{\mathcal{S}}^{-1}[\widetilde{\mathcal{S}}, \partial_x]\widetilde{\mathcal{S}}^{-1} f + \widetilde{\mathcal{S}}^{-1}(\partial_x f), \tag{2.2.23}$$

where $[\widetilde{\mathcal{S}}, \partial_x] = g'(x)$. This new expression (2.2.23), combined with the L^2_η estimate (2.2.21), gives the H^1_η estimate in (2.2.22). We now notice that

$$\mathcal{S}^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \\ f \end{pmatrix} = \begin{pmatrix} \widetilde{\mathcal{S}}^{-1} f \\ \mu^{1/2}\partial_x \widetilde{\mathcal{S}}^{-1} f \\ \mu\partial_x^2 \widetilde{\mathcal{S}}^{-1} f \\ \mu^{3/2}\partial_x^3 \widetilde{\mathcal{S}}^{-1} f \end{pmatrix}, \tag{2.2.24}$$

which, combined with (2.2.23), shows that there exists some $N \geqslant 1$ such that

$$\begin{aligned} \|\mathcal{S}^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \\ f \end{pmatrix}\|_{(H^1_\eta)^4} \leqslant & N\sum_{j=0}^{3} \mu^{j/2}\Big(\|\partial_x^j(\widetilde{\mathcal{S}})^{-1} f\|_{L^2_\eta} + \|\partial_x^j(\widetilde{\mathcal{S}})^{-1}\partial_x f\|_{L^2_\eta} + \\ & \|\partial_x^j(\widetilde{\mathcal{S}})^{-1}(g(x)\widetilde{\mathcal{S}}^{-1} f)\|_{L^2_\eta}\Big). \end{aligned} \tag{2.2.25}$$

By the inequality (2.2.25), in order to prove that (2.2.19) holds for all $f \in H^1_\eta$, we only have to show that for all $f \in L^2_\eta$, there exists a positive constant $N \geqslant 1$ such that

$$\begin{aligned} \|\partial_x(\widetilde{\mathcal{S}}(\mu,\delta))^{-1} f\|_{L^2_\eta} &\leqslant \frac{N}{\mu^{1/2}}\|f\|_{L^2_\eta}, \\ \|\partial_x^2(\widetilde{\mathcal{S}}(\mu,\delta))^{-1} f\|_{L^2_\eta} &\leqslant \frac{N}{\mu}\|f\|_{L^2_\eta}, \\ \|\partial_x^3(\widetilde{\mathcal{S}}(\mu,\delta))^{-1} f\|_{L^2_\eta} &\leqslant \frac{N}{\mu^{3/2}}\|f\|_{L^2_\eta}, \end{aligned} \tag{2.2.26}$$

which, by Lemma A.0.5, is true. ■

For sufficiently small $\mu \geqslant 0$ and $\delta \in \mathbb{R}$, we now rewrite the system (2.2.16,2.2.17) as follows.

$$C_\ell'' + \frac{1}{4k_{x,\ell}^2} C_\ell \left(1 - 3(|C_{\ell_*}|^2 - |C_\ell|^2 + 2\sum_{\kappa \in I_*} |C_\kappa|^2)\right) + \mathcal{O}(\mu^{1/2} + |\delta|) = 0, \tag{2.2.27}$$

$$\mathcal{S}(\mu,\delta)\begin{pmatrix} C_{\ell_*} \\ D_{\ell_*,1} \\ D_{\ell_*,2} \\ D_{\ell_*,3} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(x) \end{pmatrix} = 0, \tag{2.2.28}$$

where $\ell \in \widetilde{I}_*$ and $f(x) = 3C_{\ell_*}(|C_{\ell_*}|^2 + 2\sum_{\kappa \in \widetilde{I}_*} |C_\kappa|^2 - 2C_+^{*2} - 2C_-^{*2}) + \mathcal{O}(|\mu|)$.

Rotated rolls. Similar to the non-resonant case, the family of rolls in (2.1.25) gives a family of periodic orbits for (2.2.27,2.2.28), still denoted as $\mathbf{P}_{\mu,\delta,\varepsilon,\ell}$ and $\mathbf{Q}_{\mu,\delta,\varepsilon,\ell}$ for all $\ell \in \widetilde{I}_*$. Again, we just consider $\ell = 1$.

Persistence of the heteroclinic orbit Our main goal now is to show that the heteroclinic actually persists as a solution for the reduced equation (2.2.27,2.2.28). In particular, we want to show that there is a heteroclinic orbit for (2.2.27,2.2.28), for small μ and δ, which connects two periodic orbits $\mathbf{P}_{\mu,\delta,\varepsilon,1}$, as $x \to \infty$, and $\mathbf{Q}_{\mu,\delta,\varepsilon,1}$, as $x \to -\infty$.

The systems (2.2.27) and (2.2.28) together with the complex conjugated equations are separately of the form

$$\mathcal{F}_1(\mathbf{C}, \overline{\mathbf{C}}, \mu^{1/2}, \delta) = 0, \tag{2.2.29}$$

$$\mathcal{F}_2(\mathbf{C}, \overline{\mathbf{C}}, \mu^{1/2}, \delta) = 0, \tag{2.2.30}$$

where

$$\mathbf{C} = (C_{-(\ell_*-1)}, ..., C_{-2}, C_{-1}, C_0, C_1, C_2, ..., C_{\ell_*}, D_{\ell_*,1}, D_{\ell_*,2}, D_{\ell_*,3}),$$
$$\widetilde{\mathbf{C}} = (C_{-(\ell_*-1)}, ..., C_{-2}, C_{-1}, C_0, C_1, C_2, ..., C_{\ell_*-1}).$$

They have the periodic solutions $\mathbf{C} = \mathbf{P}_{\mu,\delta,\varepsilon,1}$ and $\mathbf{C} = \mathbf{Q}_{\mu,\delta,\varepsilon,1}$ for μ,δ and ε sufficiently small, and the heteroclinic solution $\mathbf{C} = (0, ..., 0, C_-^*, 0, C_+^*, 0, ..., 0)$ for $\mu = \delta = 0$.

We consider the ansatz

$$\mathbf{C}(x) = \mathrm{e}^{\mathrm{i}\varepsilon x}\mathbf{C}^*(x) + \chi(x)\widetilde{\mathbf{P}}_{\mu,\delta,\varepsilon,1}(x) + \left(\mathcal{R}(\chi\widetilde{\mathbf{P}}_{\mu,\delta,\varepsilon,1})\right)(-x) + \mathbf{V}(x), \tag{2.2.31}$$

in which heteroclinic orbit and correction to the periodic orbit are given as

$$\mathbf{C}^* = (0, ..., 0, C_-^*, 0, C_+^*, 0, ..., 0), \quad \widetilde{\mathbf{P}}_{\mu,\delta,\varepsilon,1} = \mathbf{P}_{\mu,\delta,\varepsilon,1} - \left(0, ..., 0, 0, 0, \frac{1}{\sqrt{3}}, 0, ..., 0\right) \mathrm{e}^{\mathrm{i}\varepsilon x},$$

$\chi : \mathbb{R} \to [0, 1]$ is a smooth cut-off function with

$$\chi(x) = 1, \text{ if } x \geqslant M, \qquad \chi(x) = 0, \text{ if } x \leqslant m, \qquad \chi(x) + \chi(-x) \equiv 1,$$

for some positive constants $m < M$. Moreover, we have

$$\mathbf{V} = (\widetilde{\mathbf{V}}, \mathbf{W}), \quad \widetilde{\mathbf{V}} = (V_{-(\ell_*-1)}, ..., V_{-1}, V_0, V_1, ..., V_{\ell_*-1}), \quad \mathbf{W} = (V_{\ell_*}, V_{\ell_*,1}, V_{\ell_*,2}, V_{\ell_*,3}).$$

We next modify (2.2.29,2.2.30), by adding zero terms, where the functions $\mathcal{F}_j$ are evaluated on the periodic-orbit solutions, in order to make mapping properties in exponentially weighted spaces more explicit later on. Therefore, define

$$\mathcal{F}_1(\mathbf{C}, \overline{\mathbf{C}}) - \chi(x) F_1(\mathbf{P}_{\mu,\delta,\varepsilon,1}, \overline{\mathbf{P}_{\mu,\delta,\varepsilon,1}}) - \chi(-x) F_1(\mathbf{Q}_{\mu,\delta,\varepsilon,1}, \overline{\mathbf{Q}_{\mu,\delta,\varepsilon,1}}) = 0, \tag{2.2.32}$$

$$\mathcal{F}_2(\mathbf{C}, \overline{\mathbf{C}}) - \chi(x) F_2(\mathbf{P}_{\mu,\delta,\varepsilon,1}, \overline{\mathbf{P}_{\mu,\delta,\varepsilon,1}}) - \chi(-x) F_2(\mathbf{Q}_{\mu,\delta,\varepsilon,1}, \overline{\mathbf{Q}_{\mu,\delta,\varepsilon,1}}) = 0, \tag{2.2.33}$$

and substitute our ansatz (2.2.31) into (2.2.32,2.2.33), to obtain a system of the form

$$\mathcal{T}_1(\widetilde{\mathbf{V}}, \overline{\widetilde{\mathbf{V}}}, \mathbf{W}, \overline{\mathbf{W}}, \varepsilon, \mu^{1/2}, \delta) = 0, \tag{2.2.34}$$

$$\mathcal{T}_2(\widetilde{\mathbf{V}}, \overline{\widetilde{\mathbf{V}}}, \mathbf{W}, \overline{\mathbf{W}}, \varepsilon, \mu^{1/2}, \delta) = 0. \tag{2.2.35}$$

For $\eta > 0$, we define the spaces of exponentially decaying functions

$$\begin{aligned}
\widetilde{\mathcal{X}_\eta} = \{(C_\ell, \overline{C_\ell})_{\ell \in \widetilde{I_*}} \in (L^2_\eta)^{4\ell_*-2}\}, \quad \widetilde{\mathcal{Y}_\eta} &= \{(C_\ell, \overline{C_\ell})_{\ell \in \widetilde{I_*}} \in (H^2_\eta)^{4\ell_*-2}\}, \\
\widetilde{\mathcal{X}^r_\eta} &= \{(C_\ell, \overline{C_\ell})_{\ell \in \widetilde{I_*}} \in \widetilde{\mathcal{X}_\eta} \ ; \ C_\ell(x) = \overline{C_{-\ell}}(-x), \ x \in \mathbb{R}, \ell \in \widetilde{I}_*\}, \\
\widetilde{\mathcal{Y}^r_\eta} &= \{(C_\ell, \overline{C_\ell})_{\ell \in \widetilde{I_*}} \in \widetilde{\mathcal{Y}_\eta} \ ; \ C_\ell(x) = \overline{C_{-\ell}}(-x), \ x \in \mathbb{R}, \ell \in \widetilde{I}_*\}, \\
\mathcal{Z}^r_\eta &= \{(W_j, \overline{W_j})_{j=1,2,3,4} \in (H^1_\eta)^8 | W_j(x) = W_j(-x), j = 1, 2, 3, 4\}, \\
\widetilde{\mathcal{Z}^r_\eta} &= \{(W_j, \overline{W_j})_{j=1,2,3,4} \in (L^2_\eta)^8 | W_j(x) = W_j(-x), j = 1, 2, 3, 4\},
\end{aligned}$$

It is now not hard to verify that

$$\begin{aligned}
\mathcal{T}_1 : \widetilde{\mathcal{Y}^r_\eta} \times \mathcal{Z}^r_\eta \times \mathbb{R}^3 &\longrightarrow \widetilde{\mathcal{X}^r_\eta}, \\
\mathcal{T}_2 : \widetilde{\mathcal{Y}^r_\eta} \times \mathcal{Z}^r_\eta \times \mathbb{R}^3 &\longrightarrow \widetilde{\mathcal{Z}^r_\eta}
\end{aligned}$$

are well-defined nonlinear operators. We start by solving the first of these two equations, which is in fact similar to the non-resonant case.

Proposition 2.2.4 *For* $(\mathbf{W}, \overline{\mathbf{W}}, \mu, \delta) \in \mathcal{Z}^r_\eta \times [0, \infty) \times \mathbb{R}$ *sufficiently small, there exist smooth functions* $\widetilde{\mathbf{V}}(\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta)$ *and* $\varepsilon(\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta)$*, solving equation* (2.2.34)*, with* $\widetilde{\mathbf{V}}(0,0,0,0) = 0$ *and* $\varepsilon(0,0,0,0) = 0$*.*

Proof. The proof is based on the inverse-function-theorem argument from the nonresonant case. A key role is played by the linear operator, found by linearizing the system (2.2.27) at

$$(C_{-\ell_*+1}, ..., C_{-2}, C_{-1}, C_0, C_1, C_2, ..., C_{\ell_*-1}) = (0, ..., 0, C^*_-, 0, C^*_+, 0, ..., 0),$$

with $\mu = \delta = 0$ and $C_{\ell_*} = D_{\ell_*,1} = D_{\ell_*,2} = D_{\ell_*,3} = 0$, i.e.,

$$\widetilde{\mathcal{L}_*} \begin{pmatrix} C_{-\ell_*+1} \\ ... \\ C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \\ ... \\ C_{\ell_*-1} \end{pmatrix} = \begin{pmatrix} C''_{-\ell_*+1} + \frac{1}{4k^2_{x,-\ell_*+1}} C_{-\ell_*+1)} - \frac{3}{2k^2_{x,-\ell_*+1}} (C^{*2}_+ + C^{*2}_-) C_{-\ell_*+1} \\ ... \\ C''_{-2} + \frac{1}{4k^2_{x,-2}} C_{-2} - \frac{3}{2k^2_{x,-2}} (C^{*2}_+ + C^{*2}_-) C_{-2} \\ \mathcal{L}_{*,-1}(C_{-1}, \overline{C_{-1}}, C_1, \overline{C_1}) \\ C''_0 + \frac{1}{4k^2_{x,0}} C_\ell - \frac{3}{2k^2_{x,0}} (C^{*2}_+ + C^{*2}_-) C_0 \\ \mathcal{L}_{*,1}(C_{-1}, \overline{C_{-1}}, C_1, \overline{C_1}) \\ C''_2 + \frac{1}{4k^2_{x,2}} C_2 - \frac{3}{2k^2_{x,2}} (C^{*2}_+ + C^{*2}_-) C_2 \\ ... \\ C''_{\ell_*-1} + \frac{1}{4k^2_{x,\ell_*-1}} C_{\ell_*-1} - \frac{3}{2k^2_{x,\ell_*-1}} (C^{*2}_+ + C^{*2}_-) C_{\ell_*-1} \end{pmatrix},$$

where

$$\mathcal{L}_{*,-1} = C''_{-1} + \frac{1}{4k^2_{x,1}} C_{-1} - \frac{3}{4k^2_{x,1}} \left(2(C^{*2}_+ + C^{*2}_-) C_{-1} + C^{*2}_- \overline{C_{-1}} + 2C^*_+ C^*_- (C_1 + \overline{C_1}) \right),$$

$$\mathcal{L}_{*,1} = C''_1 + \frac{1}{4k^2_{x,1}} C_1 - \frac{3}{4k^2_{x,1}} \left(2(C^{*2}_+ + C^{*2}_-) C_1 + C^{*2}_+ \overline{C_1} + 2C^*_+ C^*_- (C_{-1} + \overline{C_{-1}}) \right).$$

For $\eta > 0$ sufficiently small, the operator $\widetilde{\mathcal{L}_*}$ acting in $\widetilde{\mathcal{X}^r_\eta}$ is Fredholm with trivial kernel and one-dimensional co-kernel, spanned by

$$(0, ..., 0, -\mathrm{i}C^*_-, 0, \mathrm{i}C^*_+, 0, ..., 0; 0, ..., 0, \mathrm{i}C^*_-, 0, -\mathrm{i}C^*_+, 0, ..., 0).$$

Next, notice that

$$\mathcal{T}_1(0,0,0,0,0,0,0) = 0, \quad D_{\widetilde{\mathbf{V}}, \widetilde{\overline{\mathbf{V}}}} \mathcal{T}_1(0,0,0,0,0,0,0) = \widetilde{\mathcal{L}_*}$$

and

$$D_\varepsilon \mathcal{T}(0,0,0,0,0,0,0) = \widetilde{\mathcal{L}_*}\begin{pmatrix} ix\widetilde{\mathbf{C}^*} \\ -ix\widetilde{\mathbf{C}^*} \end{pmatrix} = \begin{pmatrix} 2i\widetilde{\mathbf{C}^{*\prime}} \\ -2i\widetilde{\mathbf{C}^{*\prime}} \end{pmatrix}.$$

Using the explicit form of the co-kernel, we find that the linearization with respect to $\widetilde{\mathbf{V}}$, $\widetilde{\overline{\mathbf{V}}}$ and ε, jointly, is invertible, so that we can use the implicit function theorem to conclude persistence as stated in the theorem. ■

We substitute the solutions $\widetilde{\mathbf{V}}(\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta)$ and $\varepsilon(\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta)$ from Proposition 2.2.4 into (2.2.35) and obtain

$$\mathcal{T}_3(\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta) = 0. \tag{2.2.36}$$

Theorem 4 *For $\mu > 0$ and δ sufficiently small, there exists a continuous function $\mathbf{W}(\mu^{1/2}, \delta)$ solving equation (2.2.36), with $\mathbf{W}(0,0) = 0$.*

Proof. We define the map

$$\begin{aligned} \mathcal{T}_4 : \mathcal{Z}^r_\eta \times [0,\infty) \times \mathbb{R} &\longrightarrow \mathcal{Z}^r_\eta \\ (\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta) &\longmapsto (\mathbf{W}, \overline{\mathbf{W}}) - (\mathcal{S}(\mu,\delta))^{-1}\mathcal{T}_3(\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta), \end{aligned} \tag{2.2.37}$$

and extend to $\mu = 0$ by setting

$$(\mathcal{S}(0,\delta))^{-1} := 0. \tag{2.2.38}$$

To prove the theorem, we only have to show that there exist two small neighborhoods $\mathcal{W} = \{(\mathbf{W}, \overline{\mathbf{W}}) \in \mathcal{Z}^r_\eta | \|(\mathbf{W}, \overline{\mathbf{W}}\|_{\mathcal{Z}^r_\eta} < a\}$ of $(\mathbf{W}, \overline{\mathbf{W}}) = (0,0)$ and $\mathcal{U} \subset [0,\infty) \times \mathbb{R}$ of $(\mu^{1/2}, \delta) = (0,0)$ such that $\mathcal{T}_4 : \overline{\mathcal{W}} \times \mathcal{U} \longrightarrow \overline{\mathcal{W}}$ is a uniform contraction. We only have to show that $\mathcal{T}_4$ has the following properties.

(i) $\|\mathcal{T}_4(\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta)\|_{\mathcal{Z}^r_\eta} \leqslant a$, for all $(\mathbf{W}, \overline{\mathbf{W}}) \in \overline{\mathcal{W}}$ and $(\mu^{1/2}, \delta) \in \mathcal{U}$.

(ii)

$$\|\mathcal{T}_4(\mathbf{W}_1, \overline{\mathbf{W}_1}, \mu^{1/2}, \delta) - \mathcal{T}_4(\mathbf{W}_2, \overline{\mathbf{W}_2}, \mu^{1/2}, \delta)\|_{\mathcal{Z}^r_\eta} \leqslant \frac{1}{2}\|(\mathbf{W}_1, \overline{\mathbf{W}_1}) - (\mathbf{W}_2, \overline{\mathbf{W}_2})\|_{\mathcal{Z}^r_\eta},$$

for all $(\mathbf{W}_1, \overline{\mathbf{W}_1}), (\mathbf{W}_2, \overline{\mathbf{W}_2}) \in \mathcal{W}$ and $(\mu^{1/2}, \delta) \in \mathcal{U}$.

To see that, we first recall that equation (2.2.36) is equation (2.2.33)

$$\mathcal{F}_2(\mathbf{C},\overline{\mathbf{C}})-\chi(x)F_2(\mathbf{P}_{\mu,\delta,\varepsilon,1},\overline{\mathbf{P}_{\mu,\delta,\varepsilon,1}})-\chi(-x)F_2(\mathbf{Q}_{\mu,\delta,\varepsilon,1},\overline{\mathbf{Q}_{\mu,\delta,\varepsilon,1}})=0,$$

together with its conjugate in terms of $(\mathbf{W},\overline{\mathbf{W}},\mu^{1/2},\delta)$. Moreover, we recall that

$$\mathcal{F}_2(\mathbf{C},\overline{\mathbf{C}},\mu^{1/2},\delta)=\mathcal{S}(\mu,\delta)\begin{pmatrix}C_{\ell_*}\\ D_{\ell_*,1}\\ D_{\ell_*,2}\\ D_{\ell_*,3}\end{pmatrix}+\begin{pmatrix}0\\0\\0\\f(x)\end{pmatrix},$$

in which $f(x)=3C_{\ell_*}(|C_{\ell_*}|^2+2\sum_{\kappa\in\widetilde{I}_*}|C_\kappa|^2-2C_+^{*2}-2C_-^{*2})+\mathcal{O}(|\mu|)$ and that $\mathcal{O}(\mu)$ here represents the higher order terms of order

$$\mu(\sum_{\kappa\in\widetilde{I}_*}(|C_\kappa|+|D_\kappa|)+\sum_{\zeta=\pm\ell_*}(|C_\zeta|+|D_{\zeta,1}|+|D_{\zeta,2}|+|D_{\zeta,3}|))^3.$$

A lengthy calculation shows that

$$\mathcal{T}_4(\mathbf{W},\overline{\mathbf{W}},\mu^{1/2},\delta)=\begin{pmatrix}U(\varepsilon,\mu^1/2,\delta)\\ \overline{U}(\varepsilon,\mu^1/2,\delta)\end{pmatrix}+\begin{pmatrix}(\mathcal{S}(\mu,\delta))^{-1}\begin{pmatrix}(0\\0\\0\\\widetilde{f})\end{pmatrix}\\ (\mathcal{S}(\mu,\delta))^{-1}\begin{pmatrix}(0\\0\\0\\\overline{\widetilde{f}})\end{pmatrix}\end{pmatrix},\tag{2.2.39}$$

in which

$$\|(U,\overline{U})\|_{\mathcal{Z}_\eta^r}=\mathcal{O}(\mu^{1/2}),\quad \|\widetilde{f}\|_{H_\eta^1}=\mathcal{O}(\|(\mathbf{W},\overline{\mathbf{W}})\|^2_{\mathcal{Z}_\eta^r}+(\mu^{1/2}+|\delta|)\|(\mathbf{W},\overline{\mathbf{W}})\|_{\mathcal{Z}_\eta^r}+\mu^{1/2}).\tag{2.2.40}$$

We recall the estimate (2.2.19) in Lemma 2.2.3,

$$\|\mathcal{S}^{-1}\begin{pmatrix}0\\0\\0\\f\end{pmatrix}\|_{(H_\eta^1)^4}\leqslant C\|f\|_{H_\eta^1},$$

which, combined with (2.2.40), shows that there exist positive constants $C_1, C_2 \geqslant 1$ such that

$$\|\mathcal{T}_4(\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta)\|_{\mathcal{Z}_\eta^r} \leqslant C_1 \|(\mathbf{W}, \overline{\mathbf{W}})\|^2_{\mathcal{Z}_\eta^r} + (\mu^{1/2} + |\delta|) C_1 \|(\mathbf{W}, \overline{\mathbf{W}})\|_{\mathcal{Z}_\eta^r} + C_2 \mu^{1/2}.$$

Noting that $h(x) = C_1 x^2 + (\mu^{1/2} + \delta) C_1 x + C_2 \mu^{1/2}$ is positive and increasing on $[0, \infty)$ and solving the inequality $C_1 a^2 + (\mu^{1/2} + \delta) C_1 a + C_2 \mu^{1/2} \leqslant a$, we conclude that property (i) of $\mathcal{T}_4$ holds as long as we take

$$\begin{aligned} &\mu^{1/2} \leqslant \frac{1}{16 C_1 C_2}, \quad |\delta| \leqslant \frac{7}{16 C_1 C_2}, \\ &a = \frac{1 - C_1(\mu^{1/2} + |\delta|) - \sqrt{(1 - C_1(\mu^{1/2} + |\delta|))^2 - 4 C_1 C_2 \mu^{1/2}}}{2 C_1}. \end{aligned} \tag{2.2.41}$$

On the other hand, there exist positive constants $C_3, C_4 \geqslant 1$ such that

$$\begin{aligned} &\|\mathcal{T}_4(\mathbf{W}_1, \overline{\mathbf{W}_1}, \mu^{1/2}, \delta) - \mathcal{T}_4(\mathbf{W}_2, \overline{\mathbf{W}_2}, \mu^{1/2}, \delta)\|_{\mathcal{Z}_\eta^r} \\ \leqslant & C_3 \mu^{1/2} \|(\mathbf{W}_1, \overline{\mathbf{W}_1}) - (\mathbf{W}_2, \overline{\mathbf{W}_2})\|_{\mathcal{Z}_\eta^r} + \\ & C_4 \|\widetilde{f}(\mathbf{W}_1, \overline{\mathbf{W}_1}, \mu^{1/2}, \delta) - \widetilde{f}(\mathbf{W}_2, \overline{\mathbf{W}_2}, \mu^{1/2}, \delta)\|_{L^2_\eta}. \end{aligned} \tag{2.2.42}$$

A straightforward calculation shows that there exists a positive constant $C_5 \geqslant 1$ such that

$$\begin{aligned} &\|\widetilde{f}(\mathbf{W}_1, \overline{\mathbf{W}_1}, \mu^{1/2}, \delta) - \widetilde{f}(\mathbf{W}_2, \overline{\mathbf{W}_2}, \mu^{1/2}, \delta)\|_{H^1_\eta} \\ \leqslant \quad & C_5 (a + \mu^{1/2} + |\delta|) \|(\mathbf{W}_1, \overline{\mathbf{W}_1}) - (\mathbf{W}_2, \overline{\mathbf{W}_2})\|_{\mathcal{Z}_\eta^r}, \end{aligned}$$

which, plugged into equation (2.2.42), gives that

$$\begin{aligned} &\|\mathcal{T}_4(\mathbf{W}_1, \overline{\mathbf{W}_1}, \mu^{1/2}, \delta) - \mathcal{T}_4(\mathbf{W}_2, \overline{\mathbf{W}_2}, \mu^{1/2}, \delta)\|_{\mathcal{Z}_\eta^r} \\ \leqslant & (C_3 \mu^{1/2} + C_4 C_5 (a + \mu^{1/2} + |\delta|)) \|(\mathbf{W}_1, \overline{\mathbf{W}_1}) - (\mathbf{W}_2, \overline{\mathbf{W}_2})\|_{\mathcal{Z}_\eta^r} \\ \leqslant & \frac{1}{2} \|(\mathbf{W}_1, \overline{\mathbf{W}_1}) - (\mathbf{W}_2, \overline{\mathbf{W}_2})\|_{\mathcal{Z}_\eta^r}, \end{aligned} \tag{2.2.43}$$

provided μ and δ are sufficiently small and choosing a as in equation (2.2.41).

Therefore, we have proved that for sufficiently small μ and δ, we can take a properly such that $\mathcal{T}_4$ is a uniform contraction. By fixed point theorem, for μ and δ sufficiently small, there exists a function $\mathbf{W}(\mu^{1/2}, \delta)$ solving equation

$$\mathcal{T}_4(\mathbf{W}, \overline{\mathbf{W}}, \mu^{1/2}, \delta) = (\mathbf{W}, \overline{\mathbf{W}}).$$

Due to the smoothness of $\mathcal{T}_4$ away from $\mu = 0$, we conclude that $\mathbf{W}(\mu^{1/2}, \delta)$ is smooth for $\mu \neq 0$. On the other hand, from equation (2.2.41), we have

$$\lim_{\mu \longrightarrow 0+} a = 0,$$

which implies that

$$\lim_{\mu \longrightarrow 0+} \mathbf{W}(\mu^{1/2}, \delta) = 0.$$

Noting that $\mathbf{W}(0, \delta) = 0$, we thus proved the continuity of $\mathbf{W}(\mu^{1/2}, \delta)$. ■

The result in Theorem 1 is an immediate consequence of Theorem 3. Since $\varepsilon(\mu^{1/2}, \delta)$ is a continuous function with $\varepsilon(0, 0) = 0$, we find that the angle of the selected grain boundary is a continuous function $\alpha(\mu^{1/2}, \delta)$ with $\alpha(0, 0) = 1$, and that the wavenumber of the asymptotic rolls is $k_x \sim k_{x,1}$. As a result, the wavenumber function $k(\mu^{1/2}, \delta) = \sqrt{k_x^2 + (k_* + \delta)^2}$ with $k(0, 0) = 1$ is continuous, which, combined with the compactness of the interval $[\epsilon_\varphi, \pi - \epsilon_\varphi]$ for fixed $\epsilon_\varphi \in (0, \pi)$, shows that Theorem 1 is true.

Chapter 3

Diffusive stability of Turing patterns via normal forms

In this chapter, we will prove the nonlinear stability result–Theorem 2 as shown in the Introduction. It contains three main contributions. First, we construct normal form coordinates, where the neutral mode is represented by a discrete phase θ_j, which decays according to a linear discrete diffusion equation $\dot{\theta}_j = d(\theta_{j+1} - 2\theta_j + \theta_{j-1})$. The idea is to capture the leading order dynamics of perturbations using an ansatz of the type $\mathbf{u}(t,x) = \mathbf{u}_\star(x - \theta_j) + \mathbf{w}_j(t,x)$ on intervals $x \in [2\pi(j-1/2), 2\pi(j+1/2)]$, where $\mathbf{w}_j(t,x)$ lies in a linear strong stable fiber. The coordinate change mimics the much simpler coordinate change in [26], where strong stable fibers of a *temporally periodic*, but spatially homogeneous solution were straightened out.

Our second main contribution are decay estimates for the linearization in these coordinates. In particular, we show that the $\mathbf{w}_j$ indeed decay with higher algebraic rate than the θ_j.

Our third main contribution is the computation of nonlinear terms in the new coordinate systems. Leading nonlinear terms turn out to involve *discrete derivatives*, associated with the discrete translational symmetry near the periodic pattern. Similarly to the scalar case, these discrete derivatives render the nonlinearity irrelevant. From a different view point, dependence on derivatives, only, indicates the presence of a conservation law: An equation $u_t = u_{xx} + f(u_x)$ can be rewritten as $v_t = v_{xx} + (f(v))_x$, for $v = u_x$, and the gain in decay is now clear from an integration by parts in the variation of constant formula. An analogous observation applies to the $\theta - \mathbf{W}$ system, where discrete derivatives in the nonlinearity reflect a discrete conservation law.

This chapter is organized as follows. In Section 3.1, we construct the normal form. Section 3.2 contains linear estimates in Fourier-Bloch space. Section 3.3 converts those decay estimates into $L^p - L^q$ decay estimates in physical space. Section 3.4 contains the proof of the nonlinear stability result. We relegate a detailed description of the nonlinearity, and the spectral properties and the analytic semigroup results of the linear operator to the appendix.

Notation Throughout Chapter 3 and Appendix B we will use the following notation.

- $(\cdot,\cdot)$ is the standard inner product on $\mathbb{R}^n$ given by

$$(\mathbf{u},\mathbf{v}) = \sum_{j=1}^{n} u_j v_j, \text{ for any } \mathbf{u} = \{u_j\}_{j=1}^n, \mathbf{v} = \{v_j\}_{j=1}^n \in \mathbb{R}^n.$$

- $\langle\cdot,\cdot\rangle$ is the standard inner product on the Hilbert space $(L^2(-\pi,\pi))^n$ given by

$$\langle \mathbf{u},\mathbf{v}\rangle = \int_{-\pi}^{\pi} (\mathbf{u}(x),\mathbf{v}(x))\mathrm{d}x, \text{ for any } \mathbf{u},\mathbf{v} \in (L^2(-\pi,\pi))^n.$$

- $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is the standard inner products on $(\ell^2)^n$, or the $(\ell^p)^n$–$(\ell^q)^n$ pairing, given by

$$\langle\!\langle \underline{\mathbf{u}},\underline{\mathbf{v}}\rangle\!\rangle = \sum_{j\in\mathbb{Z}} (\mathbf{u}_j,\mathbf{v}_j), \text{ for any } \mathbf{u} = \{u_j\}_{j\in\mathbb{Z}}, \mathbf{v} = \{v_j\}_{j\in\mathbb{Z}}.$$

We denote the Euclidean norm in Euclidean spaces as $|\cdot|$, the norm in a general Banach space $\mathscr{X}$ as $\|\cdot\|_{\mathscr{X}}$, and the norm of a linear operator from a Banach space $\mathscr{X}$ to $\mathscr{Y}$ as $|||\cdot|||_{\mathscr{X}\to\mathscr{Y}}$. For the case $\mathscr{Y} = \mathscr{X}$, the last norm notation simply becomes $|||\cdot|||_{\mathscr{X}}$.

3.1 Normal form

As we pointed out in the introduction, the linear system for the perturbation

$$\mathbf{v}_t = \partial_{xx}\mathbf{v} + \mathbf{f}'(\mathbf{u}_\star(x))\mathbf{v}, \tag{3.1.1}$$

is expected to exhibit diffusive decay for the linear part. This weak decay is not obviously strong enough to conclude nonlinear decay because of quadratic and cubic terms in the nonlinearity. Our approach here converts the system (3.1.1) into an infinite-dimensional lattice dynamical system for $\underline{\mathbf{V}} = \{\mathbf{V}_j\}_{j\in\mathbb{Z}}$, where $\mathbf{V}_j = (\theta_j, \mathbf{W}_j) \in \mathbb{R}\times L^p(-\pi,\pi)$ for all $j \in \mathbb{Z}$. Here, the scalar component θ_j of $\mathbf{V}_j$ measures *local shifts* of the primary periodic pattern, and the infinite-dimensional component, $\mathbf{W}_j$, represents *local complements*. In such a representation, one expects diffusive decay of θ_j and faster decay of $\mathbf{W}_j$. We will make this precise in Section 3.3. In fact, the linear asymptotics of θ_j are equivalent to the discrete diffusion

$$\dot{\theta}_j = d(\theta_{j+1} - 2\theta_j + \theta_{j-1}).$$

The key idea is that in this lattice system, nonlinear terms in the θ-equations involve discrete derivatives, $\theta_{j+1} - \theta_j$, rather than θ_j alone. Roughly speaking, we expect θ-dependence to disappear when $\theta_j = \theta_{j+1}$ for all $j \in \mathbb{Z}$ due to shift invariance of the original system. Just like in the continuous scalar heat equation, these derivatives decay faster, so that terms like $(\theta_{j+1} - \theta_j)^2$ are now irrelevant, that is, they do not alter linear diffusive decay.

In summary, we will find a system, where the linear part exhibits diffusive decay, and where nonlinearities are *explicitly* irrelevant. In this sense, our transformation has eliminated lower-order terms in the system, that turn out not to contribute to leading order dynamics. The term normal form alludes to this elimination of lower-order terms by comparing with normal form theory in ODEs, where coordinate changes are used to simplify equations and systems at least locally, mostly through removing lower-order terms in the Taylor jet of the equation or system.

The remainder of this section is organized as follows. We discuss local well-posedness and "chopping-up", the first key step in the transformation to a lattice system in Section 3.1.1. The ultimate transformation towards a quasilinear lattice dynamical system is constructed in Section 3.1.2. Key steps involve separation of the neutral phase θ_j and a smoothing procedure at the chopping boundaries.

3.1.1 Well-posedness: spatially extended system and lattice system

We first show local well-posedness of the system (1.3.4) on the space $X = L^1 \cap L^\infty$.

Lemma 3.1.1 *The initial value problem of the semi-linear parabolic system* (1.3.4) *is locally well-posed in X. To be precise, the following assertions hold:*

(i) *(**existence and uniqueness**) For any given $\mathbf{v}^0 \in X$, there exists some $T > 0$, depending only on $\|\mathbf{v}^0\|_X$, such that the system* (1.3.4) *admits a unique mild solution*

$$\mathbf{v} \in C^0([0,T],(L^1(\mathbb{R}))^n) \cap C^0((0,T],(L^\infty(\mathbb{R}))^n).$$

Here a mild solution solves the integral-equation variant of (1.3.4).

(ii) *(**regularity**) The solution $\mathbf{v}(t,x)$ to* (1.3.4) *is smooth for $t \in (0,T]$. Moreover, there exists $C > 0$ such that, for all $t \in (0,T]$,*

$$\|\mathbf{v}(t)\|_{H^2} \leqslant Ct^{-1}\|\mathbf{v}^0\|_X.$$

Proof. The existence and uniqueness follow directly from [10] and [38]. To show that $\|\mathbf{v}(t)\|_{H^2} \leqslant Ct^{-1}\|\mathbf{v}^0\|_X$, we first note that for any $T_0 \in (0,T)$, there exists $C(T_0) > 0$ such that

$$\|\mathbf{v}(t)\|_{H^2} \leqslant C(T_0)\|\mathbf{v}^0\|_{L^2}, \text{ for all } t \in (T_0,T). \tag{3.1.2}$$

Moreover, by [38, Thm. 7.1.5], for every $\mathbf{v}^0 \in L^2$, there are $T_1 > 0$ and $C(T_1) > 0$ such that, for all $t \in (0,T_1)$

$$\|\mathbf{v}(t/2)\|_{H^1} \leqslant C(T_1)(t/2)^{-1/2}\|\mathbf{v}^0\|_{L^2}, \quad \|\mathbf{v}(t)\|_{H^2} \leqslant C(T_1)(t/2)^{-1/2}\|\mathbf{v}(t/2)\|_{H^1},$$

which implies that

$$\|\mathbf{v}(t)\|_{H^2} \leqslant \frac{C(T_1)}{2}t^{-1}\|\mathbf{v}^0\|_{L^2}, \text{ for all } t \in (0,T_1). \tag{3.1.3}$$

Combining (3.1.2) and (3.1.3), we conclude our proof. ■

Remark 3.1.2 *By Lemma 3.1.1, we can assume without loss of generality that, in the proof of Theorem 2, the initial perturbation is small in $X \cap H^2$.*

Now suppose that $\mathbf{v}(t,x)$ is a solution to (1.3.4), close to 0. In particular, $\mathbf{v}(t,x)$ is close to 0 on all intervals $[2\pi(j-1/2), 2\pi(j+1/2)]$, $j \in \mathbb{Z}$. Then instead of solving

(1.3.4), we claim that it is equivalent to solve the infinite-dimensional system, for all $j \in \mathbb{Z}$,

$$\begin{cases} \partial_t \mathbf{v}_j = D\partial_{xx}\mathbf{v}_j + \mathbf{f}'(\mathbf{u}_\star)\mathbf{v}_j + \mathbf{g}(x, \mathbf{v}_j) \\ \mathbf{v}_j(t,\pi) = \mathbf{v}_{j+1}(t,-\pi) \\ \partial_x \mathbf{v}_j(t,\pi) = \partial_x \mathbf{v}_{j+1}(t,-\pi). \end{cases} \tag{3.1.4}$$

In order to justify the well-posedness of (3.1.4), we first introduce the chopped space

$$X_{\text{ch}} = \ell^1(\mathbb{Z}, (L^1(-\pi,\pi))^n) \cap \ell^\infty(\mathbb{Z}, (L^\infty(-\pi,\pi))^n), \tag{3.1.5}$$

with the norm defined as

$$\|\underline{\mathbf{w}}\|_{X_{\text{ch}}} = \sum_{j\in\mathbb{Z}} \|\mathbf{w}_j\|_{L^1} + \sup_{j\in\mathbb{Z}} \|\mathbf{w}_j\|_{L^\infty}, \text{ for any } \underline{\mathbf{w}} = \{\mathbf{w}_j\}_{j\in\mathbb{Z}} \in X_{\text{ch}}.$$

We then consider the chopping map

$$\begin{array}{rccc} \mathscr{T}_{\text{ch}}: & X_{\text{ch}} & \longrightarrow & X \\ & \underline{\mathbf{v}} & \longmapsto & \mathscr{T}_{\text{ch}}(\underline{\mathbf{v}}), \end{array} \tag{3.1.6}$$

where X is defined in (1.3.6) and $\mathscr{T}_{\text{ch}}(\underline{\mathbf{v}})(2\pi j + x) = \mathbf{v}_j(x)$, for all $x \in [-\pi,\pi]$ and $j \in \mathbb{Z}$. It is not hard to see that $\mathscr{T}_{\text{ch}}$ is an isomorphism and thus we have the diagram

$$\begin{array}{ccc} X^1 & \xrightarrow{A} & X \\ \mathscr{T}_{\text{ch}} \uparrow & & \mathscr{T}_{\text{ch}} \uparrow \\ X^1_{\text{ch}} & \xrightarrow{A_{\text{ch}}} & X_{\text{ch}}, \end{array}$$

where $X^1_{\text{ch}} := \mathscr{T}_{\text{ch}}^{-1}(X^1)$ and

$$\begin{array}{rccc} A_{\text{ch}}: & X^1_{\text{ch}} & \longrightarrow & X_{\text{ch}} \\ & \underline{\mathbf{v}} & \longmapsto & \mathscr{T}_{\text{ch}}^{-1} A \mathscr{T}_{\text{ch}} \underline{\mathbf{v}}. \end{array} \tag{3.1.7}$$

More specifically, $(A_{\text{ch}}\underline{\mathbf{v}})_j = D\partial_{xx}\mathbf{v}_j + \mathbf{f}'(\mathbf{u}_\star)\mathbf{v}_j$. To describe X^1_{ch}, we define

$$\widetilde{\mathscr{D}}(A_{\text{ch}}, X_{\text{ch}}) := \ell^1(W^{2,1}(-\pi,\pi)) \cap \ell^\infty(W^{2,\infty}(-\pi,\pi)),$$
$$\mathscr{D}(A_{\text{ch}}, X_{\text{ch}}) := \{\underline{\mathbf{v}} \in \widetilde{\mathscr{D}}(A_{\text{ch}}, X_{\text{ch}}) \mid \mathbf{v}_j^{(k)}(t,\pi) = \mathbf{v}_{j+1}^{(k)}(t,-\pi), t \geqslant 0, j \in \mathbb{Z}, k = 0, 1\}.$$

Lemma 3.1.3 *We have* $X^1_{\text{ch}} = \mathscr{D}(A_{\text{ch}}, X_{\text{ch}})$.

Proof. From the definition, we find $X^1_{\text{ch}} \subseteq \mathscr{D}(A_{\text{ch}}, X_{\text{ch}})$. We only need to show that for any given $\underline{\mathbf{v}} \in \mathscr{D}(A_{\text{ch}}, X_{\text{ch}})$, we have $\mathbf{v} = \mathscr{T}_{\text{ch}}(\underline{\mathbf{v}}) \in X^1$. In fact, for arbitrary $\mathbf{w} \in C^\infty_c$, we obtain

$$\begin{aligned}\langle \mathbf{v}, \mathbf{w}' \rangle_{L^2(\mathbb{R})} &= \textstyle\sum_{j\in\mathbb{Z}} \langle \mathbf{v}_j(x), \mathbf{w}'(2\pi j + x)\rangle \\ &= -\textstyle\sum_{j\in\mathbb{Z}} \langle \mathbf{v}'_j(x), \mathbf{w}(2\pi j + x)\rangle \\ &= -\langle \mathscr{T}_{\text{ch}}(\{\mathbf{v}'_j\}_{j\in\mathbb{Z}}), \mathbf{w}\rangle_{L^2(\mathbb{R})},\end{aligned}$$

which shows $\mathbf{v}' = \mathscr{T}_{\text{ch}}(\{\mathbf{v}'_j\}_{j\in\mathbb{Z}}) \in X$. Similarly, we have $\mathbf{v}'' \overset{a.e.}{=} \mathscr{T}_{\text{ch}}(\{\mathbf{v}''_j\}_{j\in\mathbb{Z}}) \in X$. ■

In all, we conclude that our initial value problem for a spatially extended system (1.3.4) is equivalent to an initial value problem for a lattice system as follows.

$$\begin{cases} \partial_t \underline{\mathbf{v}} = A_{\text{ch}}\underline{\mathbf{v}} + \mathbf{G}(\underline{\mathbf{v}}), \quad x \in (-\pi, \pi), t > 0, \\ \mathbf{v}^{(k)}_j(t, \pi) = \mathbf{v}^{(k)}_{j+1}(t, -\pi), \quad k = 0, 1, j \in \mathbb{Z}, t \geqslant 0, \\ \mathbf{v}_j(0, x) = \mathbf{v}^0(2\pi j + x), \quad x \in [-\pi, \pi], j \in \mathbb{Z}, \end{cases} \tag{3.1.8}$$

where $\mathbf{G}(\underline{\mathbf{v}}) = \{\mathbf{g}(x, \mathbf{v}_j)\}_{j\in\mathbb{Z}}$.

Remark 3.1.4 *For any solution $\underline{\mathbf{v}}$ to (3.1.8), we have all higher matching boundary conditions, that is, $\partial^m_x \mathbf{v}_j(t, \pi) = \partial^m_x \mathbf{v}_{j+1}(t, -\pi)$, for all $t > 0$, $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$.*

3.1.2 Phase decomposition and boundary-condition matching

We start with sketching the construction of the normal form step by step without rigorous justification. We first decompose each 2π-long piece $\mathbf{v}_j(x) = \mathbf{v}(2\pi j + x)$ into a linearly neutral phase and a stable phase and then match the boundary conditions for the stable phase. This two-step smooth phase decomposition procedure will be summarized and justified rigorously in a lemma at the end of this section.

We now decompose each $\mathbf{v}_j$ according to

$$\begin{cases} \mathbf{v}_j(x) = \mathbf{w}_j(x) + \mathbf{u}_\star(x - \theta_j) - \mathbf{u}_\star(x) \\ \langle \mathbf{w}_j(x), \mathbf{u}_{\text{ad}}(x - \theta_j)\rangle = 0, \end{cases}$$

where $\mathbf{u}_{\text{ad}}$ is an element in the kernel of the adjoint operator of $B(0)$ with $\langle \mathbf{u}'_\star, \mathbf{u}_{\text{ad}}\rangle = 1$. Substituting this expression into (1.3.4), we can therefore formally derive a system for

θ_j and $\mathbf{w}_j$, which takes the explicit form

$$\begin{cases} \dot{\theta}_j = & \dfrac{1}{-1+\langle \mathbf{w}_j(x), \mathbf{u}'_{\text{ad}}(x-\theta_j)\rangle}[-(\mathbf{w}_j(\pi)-\mathbf{w}_j(-\pi), D\mathbf{u}'_{\text{ad}}(\pi-\theta_j)) \\ & +(\partial_x \mathbf{w}_j(\pi)-\partial_x \mathbf{w}_j(-\pi), D\mathbf{u}_{\text{ad}}(\pi-\theta_j))+\langle \tilde{\mathbf{g}}(\theta_j,\mathbf{w}_j), \mathbf{u}_{\text{ad}}(x-\theta_j)\rangle] \\ \dot{\mathbf{w}}_j = & D\partial_{xx}\mathbf{w}_j+\mathbf{u}_{\star,\theta}(x-\theta_j)\dot{\theta}_j+\mathbf{f}(\mathbf{w}_j+\mathbf{u}_\star(x-\theta_j))-\mathbf{f}(\mathbf{u}_\star(x-\theta_j)), \end{cases} \tag{3.1.9}$$

with boundary conditions

$$\begin{cases} \partial_x^m \mathbf{w}_j(\pi)-\partial_x^m \mathbf{w}_{j+1}(-\pi)=\mathbf{u}_\star^{(m)}(\pi-\theta_{j+1})-\mathbf{u}_\star^{(m)}(\pi-\theta_j), \text{ for } m=0,1 \\ \langle \mathbf{w}_j(x), \mathbf{u}_{\text{ad}}(x-\theta_j)\rangle = 0, \end{cases}$$

where

$$\tilde{\mathbf{g}}(\theta_j,\mathbf{w}_j)=\mathbf{f}(\mathbf{w}_j+\mathbf{u}_\star(x-\theta_j))-\mathbf{f}(\mathbf{u}_\star(x-\theta_j))-\mathbf{f}'(\mathbf{u}_\star(x-\theta_j))\mathbf{w}_j.$$

Remark 3.1.5 *In the second equation of (3.1.9), $\dot{\theta}_j$ represents the right hand side of the first equation.*

Note that $\mathbf{w}_j$ is in a codimension-one subspace depending on θ_j. More formally, we mapped every $\mathbf{v}_j$ into a vector bundle. Also, the boundary conditions are now nonlinear. These facts generate technical difficulties so that we find it easier to work with a further modified system, where, for all $j \in \mathbb{Z}$, we substitute

$$\mathbf{w}_j(x)=\mathbf{W}_j(x)+\mathbf{H}(x,\theta_{j-1},\theta_j,\theta_{j+1},\mathbf{W}_j). \tag{3.1.10}$$

For simplicity, we denote $\mathbf{H}_j(x)=\mathbf{H}(x,\theta_{j-1},\theta_j,\theta_{j+1},\mathbf{W}_j)$. In the new coordinates $\underline{\mathbf{V}}=(\underline{\theta},\underline{\mathbf{W}})$, where $\underline{\theta}=\{\theta_j\}_{j\in\mathbb{Z}}$ and $\underline{\mathbf{W}}=\{\mathbf{W}_j\}_{j\in\mathbb{Z}}$, we will have again "homogeneous matching boundary conditions" and all $\mathbf{W}_j$'s are in a fixed codimension-1 subspace, that is, for all $j \in \mathbb{Z}$,

$$\partial_x^m \mathbf{W}_j(\pi)-\partial_x^m \mathbf{W}_{j+1}(-\pi)=0, \text{ for } m=0,1, \tag{3.1.11}$$

$$\langle \mathbf{W}_j(x), \mathbf{u}_{\text{ad}}(x)\rangle = 0. \tag{3.1.12}$$

We now construct $\underline{\mathbf{H}}=\{\mathbf{H}_j(x)\}_{j\in\mathbb{Z}}$ explicitly in the form

$$\mathbf{H}_j=\mathbf{H}_j^1+\mathbf{H}_j^2, \tag{3.1.13}$$

where $\mathbf{H}_j^1$ accomplishes "homogeneous matching boundary conditions" (3.1.11) and $\mathbf{H}_j^2$ corrects so that every $\mathbf{W}_j$ is perpendicular to $\mathbf{u}_{\mathrm{ad}}$ (3.1.12). First, we construct $\mathbf{H}_j^1$. In order to accomplish (3.1.11), one readily verifies that we need

$$\partial_x^m \mathbf{H}_j^1(\pi) - \partial_x^m \mathbf{H}_{j+1}^1(-\pi) = \mathbf{u}_\star^{(m)}(\pi - \theta_{j+1}) - \mathbf{u}_\star^{(m)}(\pi - \theta_j), \text{ for } m = 0, 1,$$

which can be achieved by choosing

$$\begin{cases} \mathbf{H}_j^1(x) = \dfrac{1}{2}(\mathbf{u}_\star(x - \theta_{j+1}) - \mathbf{u}_\star(x - \theta_j)), \text{for } x \sim \pi, \\ \mathbf{H}_j^1(x) = \dfrac{1}{2}(\mathbf{u}_\star(x - \theta_{j-1}) - \mathbf{u}_\star(x - \theta_j)), \text{for } x \sim -\pi. \end{cases}$$

In light of this observation, we let

$$\begin{aligned} \mathbf{H}_j^1(x) = &\frac{1}{2}\phi(x)(\mathbf{u}_\star(x - \theta_{j+1}) - \mathbf{u}_\star(x - \theta_{j-1})) + \\ &\frac{1}{4}(\mathbf{u}_\star(x - \theta_{j+1}) + \mathbf{u}_\star(x - \theta_{j-1}) - 2\mathbf{u}_\star(x - \theta_j)), \end{aligned} \tag{3.1.14}$$

where ϕ is a smooth odd, increasing function on $[-\pi, \pi]$ such that

$$\phi(x) = \begin{cases} \dfrac{1}{2}, & \text{for } x > \dfrac{\pi}{2}, \\ -\dfrac{1}{2}, & \text{for } x < -\dfrac{\pi}{2}. \end{cases}$$

To be specific, we can choose

$$\phi(x) = [\eta * \chi_{[0,\infty)}](x) \cdot \chi_{[-\pi,\pi]}(x) - \frac{1}{2},$$

where χ_J is the characteristic function of the interval J and η is a smooth nonnegative even mollifier such that

$$\int_{\mathbb{R}} \eta(x)\mathrm{d}x = 1, \text{and } |\eta(x)| = 0, \text{ for all } |x| > \frac{\pi}{2}.$$

In order to keep $\mathbf{H}_j$ identical with $\mathbf{H}_j^1$ near $\pm\pi$, $\mathbf{H}_j^2$ has to be 0 near $\pm\pi$. We first note that there exists an odd function $\psi \in (C_c^\infty(-\pi, \pi))^n$ such that $\langle \psi, \mathbf{u}_{\mathrm{ad}} \rangle = 1$ since $(C_c^\infty(-\pi, \pi))^n$ is dense in $(L^2(-\pi, \pi))^n$ and $\langle \mathbf{u}_\star', \mathbf{u}_{\mathrm{ad}} \rangle = 1$. We then define

$$\mathbf{H}_j^2 = c_j \psi(x - \theta_j), \tag{3.1.15}$$

where

$$c_j = -\langle \mathbf{H}_j^1, \mathbf{u}_{\mathrm{ad}}(x - \theta_j) \rangle - \langle \mathbf{W}_j, \mathbf{u}_{\mathrm{ad}}(x - \theta_j) - \mathbf{u}_{\mathrm{ad}}(x) \rangle. \tag{3.1.16}$$

Noting that θ_j and $\mathbf{W}_j$ are small, this concludes the construction of $\mathbf{H}_j$.

Defining $X_{\mathrm{ch}}^{\perp} = \{\underline{\mathbf{v}} \in X_{\mathrm{ch}} \mid \langle \mathbf{v}_j, \mathbf{u}_{\mathrm{ad}} \rangle = 0, \text{ for all } j \in \mathbb{Z}\}$, where X_{ch} is defined in (3.1.5), we summarize the "smooth phase decomposition" procedure, denoted as $\mathscr{T}_{\mathrm{phd}}$, in the following lemma.

Lemma 3.1.6 *The "smooth phase decomposition" operator $\mathscr{T}_{\mathrm{phd}}$, as constructed above, is a smooth local diffeomorphism. More precisely, there are two neighborhoods of zero $\mathscr{U} \in X_{\mathrm{ch}}$, $\mathscr{V} \in \ell^1 \times X_{\mathrm{ch}}^{\perp}$ such that the nonlinear transformation*

$$
\begin{array}{rccc}
\mathscr{T}_{\mathrm{phd}} : & \mathscr{V} & \longrightarrow & \mathscr{U} \\
& \underline{\mathbf{V}} = (\underline{\theta}, \underline{\mathbf{W}}(x)) & \longmapsto & \{\mathbf{W}_j(x) + \mathbf{H}_j(x) + \mathbf{u}_\star(x-\theta_j) - \mathbf{u}_\star(x)\}_{j\in\mathbb{Z}}
\end{array}
$$

is invertible with $\mathscr{T}_{\mathrm{phd}}$ and $\mathscr{T}_{\mathrm{phd}}^{-1}$ smooth. Its derivative at the origin is

$$
\begin{array}{rccc}
\mathscr{L}_{\mathrm{phd}} := \mathscr{T}'_{\mathrm{phd}}(0) : & \ell^1 \times X_{\mathrm{ch}}^{\perp} & \longrightarrow & X_{\mathrm{ch}} \\
& \underline{\mathbf{V}} = (\underline{\theta}, \underline{\mathbf{W}}) & \longmapsto & \underline{\mathbf{W}} + \underline{\mathbf{E}} * \underline{\theta},
\end{array}
\tag{3.1.17}
$$

where $\underline{\mathbf{E}}$ is defined in (3.1.18) below.

Proof. We claim that

(i) $\mathscr{T}_{\mathrm{phd}}(0) = 0$;

(ii) $\mathscr{T}_{\mathrm{phd}}$ is C^∞;

(iii) $\mathscr{T}'_{\mathrm{phd}}(0)$, denoted as $\mathscr{L}_{\mathrm{phd}}$, is an invertible bounded linear operator.

Property (i) is straightforward. As for (ii), $\mathscr{T}_{\mathrm{phd}}$ is smooth with respect to $\underline{\mathbf{W}}$ due to the fact that $\mathscr{T}_{\mathrm{phd}}$ is linear in $\underline{\mathbf{W}}$ for fixed $\underline{\theta}$. On the other hand, the smoothness of $\mathscr{T}_{\mathrm{phd}}$ with respect to $\underline{\theta}$ can be readily reduced to the smoothness of the mapping $\underline{\theta} \mapsto \{\mathbf{u}_*(x-\theta_j) - \mathbf{u}_*(x)\}_{j\in\mathbb{Z}}$. A direct calculation shows that, for given $m \in \mathbb{Z}^+$, the mth-derivative mapping at $\underline{\theta}$ is $\underline{\eta} \mapsto \{\frac{1}{m!}\mathbf{u}_*^{(m)}(\theta_j - x)\eta_j\}_{j\in\mathbb{Z}}$. We now only have to show that (iii) is true. In fact, the linear part of $\{\mathbf{H}_j + \mathbf{u}_\star(x-\theta_j) - \mathbf{u}_\star(x)\}_{j\in\mathbb{Z}}$ with respect to $(\underline{\theta}, \underline{\mathbf{W}})$ around $(0,0)$ is $\underline{\mathbf{E}} * \underline{\theta} = \{\sum_{k\in\mathbb{Z}} \mathbf{E}_{j-k}\theta_k\}_{j\in\mathbb{Z}}$, where $\underline{\mathbf{E}} = \{\mathbf{E}_j\}_{j\in\mathbb{Z}}$ with

$$
\mathbf{E}_j = \begin{cases}
\frac{1}{4}\psi(x) - (\frac{1}{4} + \frac{1}{2}\phi(x))\mathbf{u}'_\star(x), & j = -1, \\
-\frac{1}{2}(\psi(x) + \mathbf{u}'_\star(x)), & j = 0, \\
\frac{1}{4}\psi(x) - (\frac{1}{4} - \frac{1}{2}\phi(x))\mathbf{u}'_\star(x), & j = 1, \\
0, & \textit{others}.
\end{cases}
\tag{3.1.18}
$$

Then we have the linear phase decomposition operator

$$\begin{array}{rccc} \mathscr{L}_{\text{phd}}: & \ell^1 \times X_{\text{ch}}^{\perp} & \longrightarrow & X_{\text{ch}} \\ & \underline{\mathbf{V}} = (\underline{\theta}, \underline{\mathbf{W}}) & \longmapsto & \underline{\mathbf{W}} + \underline{\mathbf{E}} * \underline{\theta}. \end{array}$$

Moreover, through direct calculation, it is not hard to obtain the bounded inverse of $\mathscr{L}_{\text{phd}}$

$$\begin{array}{rccc} \mathscr{L}_{\text{phd}}^{-1}: & X_{\text{ch}} & \longrightarrow & \ell^1 \times X_{\text{ch}}^{\perp} \\ & \underline{\mathbf{v}} & \longmapsto & (F\underline{\mathbf{v}}, \underline{\mathbf{v}} - \mathbf{E} * F\underline{\mathbf{v}}), \end{array}$$

where

$$\begin{array}{rccc} F: & X_{\text{ch}} & \longrightarrow & \ell^1 \\ & \underline{\mathbf{v}} = \{\mathbf{v}_j\}_{j\in\mathbb{Z}} & \longmapsto & \{-\langle \mathbf{v}_j, \mathbf{u}_{\text{ad}}\rangle\}_{j\in\mathbb{Z}}. \end{array} \tag{3.1.19}$$

By (i), (ii) and the inverse function theorem, the conclusion of the lemma follows. ■

Remark 3.1.7 *The above lemma still holds when replacing* X_{ch} *with* $\mathscr{T}_{\text{ch}}(X \cap H^2)$ *and the proof is similar.*

In the new coordinates, the system contains lengthy expressions. We therefore introduce some simplifying notation first.

$$\begin{array}{rccc} \delta_+: & \mathbb{C}^{\mathbb{Z}} & \longrightarrow & \mathbb{C}^{\mathbb{Z}} \\ & \underline{x} = \{x_j\}_{j\in\mathbb{Z}} & \longmapsto & \{x_{j+1} - x_j\}_{j\in\mathbb{Z}}. \\ \\ \delta_-: & \mathbb{C}^{\mathbb{Z}} & \longrightarrow & \mathbb{C}^{\mathbb{Z}} \\ & \underline{x} & \longmapsto & \{x_j - x_{j-1}\}_{j\in\mathbb{Z}}. \\ \\ \Gamma: & (C([-\pi,\pi],\mathbb{R}^n))^{\mathbb{Z}} & \longrightarrow & \mathbb{R}^{\mathbb{Z}} \\ & \underline{\mathbf{v}} & \longmapsto & \{(\mathbf{v}_j(-\pi), D\mathbf{u}'_{\text{ad}}(\pi))\}_{j\in\mathbb{Z}}. \end{array} \tag{3.1.20}$$

Now, sorting out the linear terms, our lattice system is

$$\begin{pmatrix} \dot{\underline{\theta}} \\ \dot{\underline{\mathbf{W}}} \end{pmatrix} = A_{\text{nf}} \begin{pmatrix} \underline{\theta} \\ \underline{\mathbf{W}} \end{pmatrix} + \begin{pmatrix} \mathbf{N}^{\theta}(\underline{\theta}, \underline{\mathbf{W}}) \\ \mathbf{N}^{\mathbf{w}}(\underline{\theta}, \underline{\mathbf{W}}) \end{pmatrix} \tag{3.1.21}$$

with boundary-matching and phase-decomposition conditions (3.1.11), (3.1.12), where $\mathbf{N}^{\theta/\mathbf{w}}$ represent the nonlinear terms of the system and

$$A_{\text{nf}} = \mathscr{L}_{\text{phd}}^{-1} A_{\text{ch}} \mathscr{L}_{\text{phd}} = \begin{pmatrix} 0 & \delta_+\Gamma \\ A_{\text{ch}}\mathbf{E}* & A_{\text{ch}} - \mathbf{E} * \delta_+\Gamma \end{pmatrix}, \tag{3.1.22}$$

where A_{ch} is the linear operator acting on the chopped variables; see (3.1.7).

Remark 3.1.8 *(i) Due to the fact that $\mathscr{T}_{\mathrm{ch}}$ and $\mathscr{L}_{\mathrm{phd}}$ are isomorphisms, $A_{\mathrm{nf}} = \mathscr{L}_{\mathrm{phd}}^{-1}\mathscr{T}_{\mathrm{ch}}^{-1}A\mathscr{T}_{\mathrm{ch}}\mathscr{L}_{\mathrm{phd}}$ shares many properties with A. For example, A_{nf} is sectorial in $\ell^1 \times X_{\mathrm{ch}}^{\perp}$ since A is sectorial in X. Here we use the definition of a sectorial operator from [38] which does not require the operator to have a dense domain.*

(ii) We relegate the detailed estimates of the nonlinear terms to Lemma B.1.1 in the appendix since expressions are lengthy. We have, roughly,

$$\begin{cases} |\mathbf{N}^{\theta}| \sim |(\delta_+\underline{\theta})^2| + |\underline{\theta}^3||\delta_+\underline{\theta}| + (|\underline{\theta}| + |\underline{\mathbf{W}}|)(|\underline{\mathbf{W}}| + |\delta_+\partial_{xx}\underline{\mathbf{W}}|) + |\underline{\mathbf{W}}^2| \\ |\mathbf{N}^{\mathbf{w}}| \sim |\underline{\theta}||\delta_+\underline{\theta}| + (|\underline{\theta}| + |\underline{\mathbf{W}}|)(|\underline{\mathbf{W}}| + |\delta_+\partial_{xx}\underline{\mathbf{W}}|) + |\underline{\mathbf{W}}^2|. \end{cases}$$

(iii) Since the branch of continuous spectrum connected to $\lambda = 0$ may intersect the branches of continuous spectrum in $\operatorname{Re}\lambda < 0$, it is in general not clear how to globally separate neutral from stable modes even linearly. Phase decompositions have been achieved globally in the case of weak pulse interaction, that is, in the regime where $\mathbf{u}_\star(x)$ is close to a homoclinic orbit in the ordinary differential system $D\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}) = 0$; see [39] for a linear analysis and [40] for a nonlinear reduction.

3.2 Linear Fourier-Bloch estimates

In Section 3.2 and Section 3.3, we derive linear diffusive decay in our linear normal form

$$\dot{\underline{\mathbf{V}}} = A_{\mathrm{nf}}\underline{\mathbf{V}}.$$

To illustrate the idea, we again use the linear heat equation $u_t(t,x) = \triangle u(t,x)$. In order to obtain the diffusive decay on $\mathrm{e}^{\triangle t}$, we apply the Fourier transform and obtain the "diagonalized" equation $\widehat{u}_t(t,k) = -k^2\widehat{u}(t,k)$. Then we have that $|\widehat{u}(t,k)| = \mathrm{e}^{-k^2t}|\widehat{u}(0,k)|$, for all $t > 0$ and $k \in \mathbb{R}$, which, combined with Young's inequality, will give us diffusive decay for the scalar heat equation.

In light of this procedure, we exploit Fourier transforms and the Bloch wave decomposition of A to construct an isomorphism diagram, from which we obtain a direct integral representation of A_{nf}, that is, $\widehat{A}_{\mathrm{nf}} = \int_{-1/2}^{1/2}\widehat{A}_{\mathrm{nf}}(\sigma)\mathrm{d}\sigma$. Unlike the explicit expression of e^{-k^2t}, the estimates on $\mathrm{e}^{\widehat{A}_{\mathrm{nf}}(\sigma)t}$ are more intricate and their derivation will occupy most of this section.

To show the conjugacy between the linear normal form and its counterpart in a Fourier-Bloch space, we build a commutative isomorphism diagram involving the underlying spaces for these two operators, the linear operator A and its Bloch wave decomposition. To this end, we recall the definitions of the linearized operator A in (1.3.5), the chopping operator $\mathscr{T}_{\mathrm{ch}}$ in (3.1.6), and the linear phase decomposition operator $\mathscr{L}_{\mathrm{phd}}$ in (3.1.17) from above. We now consider these operators on L^2/ℓ^2-based spaces, that is, with new notation,

$$\begin{aligned}
\widetilde{A}: &\quad (H^2(\mathbb{R}))^n &\longrightarrow&\quad (L^2(\mathbb{R}))^n,\\
\widetilde{\mathscr{T}_{\mathrm{ch}}}: &\quad \ell^2(\mathbb{Z},(L^2(\mathbb{T}_{2\pi}))^n) &\longrightarrow&\quad (L^2(\mathbb{R}))^n,\\
\widetilde{\mathscr{L}_{\mathrm{phd}}}: &\quad \ell^2\times\ell^2_\perp(\mathbb{Z},(L^2(\mathbb{T}_{2\pi}))^n) &\longrightarrow&\quad \ell^2(\mathbb{Z},(L^2(\mathbb{T}_{2\pi}))^n),
\end{aligned} \tag{3.2.1}$$

where $\mathbb{T}_\alpha=\mathbb{R}/\alpha\mathbb{Z}$ is the one-dimensional torus of length α and

$$\ell^2_\perp(\mathbb{Z},(L^2(\mathbb{T}_{2\pi}))^n)=\{\underline{\mathbf{w}}\in\ell^2(\mathbb{Z},(L^2(\mathbb{T}_{2\pi}))^n)\mid\langle\mathbf{w}_j,\mathbf{u}_{\mathrm{ad}}\rangle=0,\ \text{for all } j\in\mathbb{Z}\}.$$

We write $\widehat{\mathbf{u}}=\int_{\mathbb{R}}\mathbf{u}(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$ and introduce several Fourier transform variants as follows:

$$\begin{aligned}
\mathscr{F}: &\quad \ell^2 &\longrightarrow&\quad L^2(\mathbb{T}_1)\\
&\quad \underline{\theta}=\{\theta_j\}_{j\in\mathbb{Z}} &\longmapsto&\quad \textstyle\sum_{j\in\mathbb{Z}}\theta_j\mathrm{e}^{-\mathrm{i}2\pi j\sigma},\\
\mathscr{F}_n: &\quad (L^2(\mathbb{T}_{2\pi}))^n &\longrightarrow&\quad (\ell^2)^n\\
&\quad \mathbf{u}(x) &\longmapsto&\quad \underline{\mathbf{u}}=\{\textstyle\int_{-\pi}^{\pi}\mathbf{u}(x)\mathrm{e}^{-\mathrm{i}\ell x}\mathrm{d}x\}_{\ell\in\mathbb{Z}},\\
\mathscr{F}_{\mathrm{ch}}: &\quad \ell^2(\mathbb{Z},(L^2(\mathbb{T}_{2\pi}))^n) &\longrightarrow&\quad L^2(\mathbb{T}_1,(\ell^2)^n)\\
&\quad \underline{\mathbf{u}}(x)=\{\mathbf{u}_j(x)\}_{j\in\mathbb{Z}} &\longmapsto&\quad \widehat{\underline{\mathbf{u}}}(\sigma)=\{\textstyle\sum_{j\in\mathbb{Z}}\int_{\mathbb{T}_{2\pi}}\mathbf{u}_j(x)\mathrm{e}^{-\mathrm{i}(\sigma+\ell)(2\pi j+x)}\}_{\ell\in\mathbb{Z}},\\
\mathscr{F}_{\mathrm{nf}}: &\quad \ell^2\times\ell^2_\perp(\mathbb{Z},(L^2(\mathbb{T}_{2\pi}))^n) &\longrightarrow&\quad L^2(\mathbb{T}_1)\times L^2_\perp(\mathbb{T}_1,(\ell^2)^n)\\
&\quad (\underline{\theta},\underline{\mathbf{u}})^T &\longmapsto&\quad (\mathscr{F}(\underline{\theta}),\mathscr{F}_{\mathrm{ch}}(\underline{\mathbf{u}}))^T,
\end{aligned} \tag{3.2.2}$$

where

$$L^2_\perp(\mathbb{T}_1,(\ell^2)^n)=\{\underline{\mathbf{w}}\in L^2(\mathbb{T}_1,(\ell^2)^n)\mid\langle\!\langle\underline{\mathbf{w}}(\sigma),\mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}})\rangle\!\rangle=0,\ \text{for all }\sigma\in\mathbb{T}_1\},$$

We then have a commutative diagram of isomorphisms as follows,

$$\begin{array}{ccccc} (L^2(\mathbb{R}))^n & \xleftarrow{\widetilde{\mathscr{T}}_{\mathrm{ch}}} & \ell^2(\mathbb{Z},(L^2(\mathbb{T}_{2\pi}))^n) & \xleftarrow{\widetilde{\mathscr{L}}_{\mathrm{phd}}} & \ell^2\times\ell^2_{\perp}(\mathbb{Z},(L^2(\mathbb{T}_{2\pi}))^n) \\ \downarrow \mathscr{B}^{-1} & & \downarrow \mathscr{F}_{\mathrm{ch}} & & \downarrow \mathscr{F}_{\mathrm{nf}} \\ L^2(\mathbb{T}_1,(L^2(\mathbb{T}_{2\pi}))^n) & \xleftarrow{\widehat{\mathscr{T}}_{\mathrm{ch}}} & L^2(\mathbb{T}_1,(\ell^2)^n) & \xleftarrow{\widehat{\mathscr{L}}_{\mathrm{phd}}} & L^2(\mathbb{T}_1)\times L^2_{\perp}(\mathbb{T}_1,(\ell^2)^n), \end{array} \tag{3.2.3}$$

where $\mathscr{B}^{-1}$ is the inverse of the direct integral defined in (B.2.1), Section B.2 and

$$\begin{array}{rccc} \widehat{\mathscr{T}}_{\mathrm{ch}}: & L^2(\mathbb{T}_1,(\ell^2)^n) & \longrightarrow & L^2(\mathbb{T}_1,(L^2(\mathbb{T}_{2\pi}))^n) \\ & \underline{\mathbf{u}}(\sigma)=\{\mathbf{u}_j(\sigma)\}_{j\in\mathbb{Z}} & \longmapsto & \mathbf{u}(\sigma)=(2\pi)^{\frac{1}{2}}\mathscr{F}_n^{-1}\underline{\mathbf{u}}(\sigma), \end{array}$$

$$\begin{array}{rccc} \widehat{\mathscr{L}}_{\mathrm{phd}}: & L^2(\mathbb{T}_1)\times L^2_{\perp}(\mathbb{T}_1,(\ell^2)^n) & \longrightarrow & L^2(\mathbb{T}_1,(\ell^2)^n) \\ & (\theta(\sigma),\underline{\mathbf{w}}(\sigma)) & \longmapsto & \theta(\sigma)\widehat{\underline{\mathbf{E}}}(\sigma)+\underline{\mathbf{w}}(\sigma). \end{array}$$

Here we have

$$\widehat{\underline{\mathbf{E}}}(\sigma)=\mathscr{F}_{\mathrm{ch}}(\underline{\mathbf{E}}), \tag{3.2.4}$$

with $\underline{\mathbf{E}}$ defined in (3.1.18). The inverse of $\widehat{\mathscr{L}}_{\mathrm{phd}}$, which will be used later, has the expression

$$\begin{array}{rccc} \widehat{\mathscr{L}}_{\mathrm{phd}}^{-1}: & L^2(\mathbb{T}_1,(\ell^2)^n) & \longrightarrow & L^2(\mathbb{T}_1)\times L^2_{\perp}(\mathbb{T}_1,(\ell^2)^n) \\ & \underline{\mathbf{w}}(\sigma) & \longmapsto & (\widehat{F}(\underline{\mathbf{w}}(\sigma)),\underline{\mathbf{w}}(\sigma)-\widehat{\underline{\mathbf{E}}}(\sigma)\widehat{F}(\underline{\mathbf{w}}(\sigma))), \end{array}$$

where

$$\begin{array}{rccc} \widehat{F}: & L^2(\mathbb{T}_1,(\ell^2)^n) & \longrightarrow & L^2(\mathbb{T}_1) \\ & \underline{\mathbf{w}}(\sigma) & \longmapsto & -\langle\!\langle \underline{\mathbf{w}}(\sigma),\mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}})\rangle\!\rangle. \end{array}$$

We now use tildes for operators in physical space and hats for their conjugates in Fourier space. The index “ch” refers to the chopped operators, the index “phd” refers to the smooth phase decomposition operators, and the index “nf” refers to the normal form operators. We then define

$$\widetilde{A}_{\mathrm{ch}}:=\widetilde{\mathscr{T}}_{\mathrm{ch}}^{-1}\widetilde{A}\widetilde{\mathscr{T}}_{\mathrm{ch}},\qquad \widehat{A}_{\mathrm{ch}}:=\widehat{\mathscr{T}}_{\mathrm{ch}}^{-1}\widehat{A}\widehat{\mathscr{T}}_{\mathrm{ch}}, \tag{3.2.5}$$

$$\widetilde{A}_{\mathrm{nf}}:=\widetilde{\mathscr{L}}_{\mathrm{phd}}^{-1}\widetilde{\mathscr{T}}_{\mathrm{ch}}^{-1}\widetilde{A}\widetilde{\mathscr{T}}_{\mathrm{ch}}\widetilde{\mathscr{L}}_{\mathrm{phd}},\quad \widehat{A}_{\mathrm{nf}}:=\widehat{\mathscr{L}}_{\mathrm{phd}}^{-1}\widehat{\mathscr{T}}_{\mathrm{ch}}^{-1}\widehat{A}\widehat{\mathscr{T}}_{\mathrm{ch}}\widehat{\mathscr{L}}_{\mathrm{phd}}, \tag{3.2.6}$$

where, according to the Bloch wave decomposition from Theorem 5 in the appendix, we have

$$\mathscr{B}^{-1}\widetilde{A}\mathscr{B}=\widehat{A}=\int_{-\frac{1}{2}}^{\frac{1}{2}}B(\sigma)\mathrm{d}\sigma, \tag{3.2.7}$$

with $B(\sigma)$ defined in (1.3.7). Therefore, by the commutative diagram (3.2.3) and the equivalence relations in (3.2.6), (3.2.7), we find the conjugacy

$$\widetilde{A}_{\mathrm{nf}} = \mathscr{F}_{\mathrm{nf}}^{-1} \widehat{A}_{\mathrm{nf}} \mathscr{F}_{\mathrm{nf}}. \tag{3.2.8}$$

Just as we pointed out at the beginning of this section, based on this conjugacy, in order to obtain estimates on $\mathrm{e}^{\widetilde{A}_{\mathrm{nf}}t}$, we only need to derive estimates on $\mathrm{e}^{\widehat{A}_{\mathrm{nf}}t}$. To this end, we first derive an explicit direct integral expression of $\widehat{A}_{\mathrm{nf}}$. From the equivalence relations in (3.2.5),(3.2.7), it is straightforward to see that

$$\widehat{A}_{\mathrm{ch}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{A}_{\mathrm{ch}}(\sigma)\mathrm{d}\sigma, \text{ with } \widehat{A}_{\mathrm{ch}}(\sigma) := \mathscr{F}_n B(\sigma) \mathscr{F}_n^{-1}, \text{ for all } \sigma \in [-1/2, 1/2]. \tag{3.2.9}$$

Moreover, for any given $(\theta(\sigma), \underline{\mathbf{w}}(\sigma)) \in L^2(\mathbb{T}_1) \times L^2_{\perp}(\mathbb{T}_1, (\ell^2)^n)$ and fixed $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, by definition, we have,

$$\begin{aligned}
\left(\widehat{A}_{\mathrm{nf}} \begin{pmatrix} \theta \\ \underline{\mathbf{w}} \end{pmatrix}\right)(\sigma) &= \left(\widehat{\mathscr{L}}_{\mathrm{phd}}^{-1} \widehat{\mathscr{T}}_{\mathrm{ch}}^{-1} \widehat{A} \widehat{\mathscr{T}}_{\mathrm{ch}} \widehat{\mathscr{L}}_{\mathrm{phd}} \begin{pmatrix} \theta \\ \underline{\mathbf{w}} \end{pmatrix}\right)(\sigma) \\
&= \begin{pmatrix} \widehat{F}(\sigma) \\ \mathrm{id} - \underline{\widehat{\mathbf{E}}}(\sigma)\widehat{F}(\sigma) \end{pmatrix} \widehat{A}_{\mathrm{ch}}(\sigma) \begin{pmatrix} \underline{\widehat{\mathbf{E}}}(\sigma) & \mathrm{id} \end{pmatrix} \begin{pmatrix} \theta(\sigma) \\ \underline{\mathbf{w}}(\sigma) \end{pmatrix} \\
&= \begin{pmatrix} 0 & R(\sigma) \\ \widehat{A}_{\mathrm{ch}}(\sigma)\underline{\widehat{\mathbf{E}}}(\sigma) & \widehat{A}_{\mathrm{ch}}(\sigma) - \underline{\widehat{\mathbf{E}}}(\sigma)R(\sigma) \end{pmatrix} \begin{pmatrix} \theta(\sigma) \\ \underline{\mathbf{w}}(\sigma) \end{pmatrix} \\
&=: \widehat{A}_{\mathrm{nf}}(\sigma) \begin{pmatrix} \theta(\sigma) \\ \underline{\mathbf{w}}(\sigma) \end{pmatrix},
\end{aligned} \tag{3.2.10}$$

where

$$\begin{aligned}
\widehat{F}(\sigma): \quad (\ell^2)^n &\longrightarrow \mathbb{C} \\
\underline{\mathbf{w}} &\longmapsto -\langle\!\langle \underline{\mathbf{w}}, \mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}})\rangle\!\rangle, \\
R(\sigma): \quad (\ell^1)^n &\longrightarrow \mathbb{C} \\
\underline{\mathbf{w}} &\longmapsto \mathrm{i}\tfrac{\sin \pi\sigma}{\pi}\left(\textstyle\sum_\ell (-1)^\ell \mathbf{w}_\ell, D\mathbf{u}_{\mathrm{ad}}'(\pi)\right).
\end{aligned} \tag{3.2.11}$$

We now conclude that

$$\widehat{A}_{\mathrm{nf}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{A}_{\mathrm{nf}}(\sigma)\mathrm{d}\sigma = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{L}_{\mathrm{phd}}(\sigma)^{-1} \widehat{A}_{\mathrm{ch}}(\sigma) \mathscr{L}_{\mathrm{phd}}(\sigma)\mathrm{d}\sigma, \tag{3.2.12}$$

where

$$\begin{array}{rccc}\mathscr{L}_{\mathrm{phd}}(\sigma): & \mathbb{C}\times(\ell^2)^n(\sigma) & \longrightarrow & (\ell^2)^n \\ & (\theta,\underline{\mathbf{w}}) & \longmapsto & \theta\underline{\widehat{\mathbf{E}}}(\sigma)+\underline{\mathbf{w}}.\end{array}$$

Here $(\ell^2)^n(\sigma) = \{\underline{\mathbf{w}} \in (\ell^2)^n \mid \langle\!\langle \underline{\mathbf{w}}, \mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}})\rangle\!\rangle = 0\}$. We also recall that $\widehat{A}_{\mathrm{ch}}(\sigma)$ is defined in (3.2.9) and $\underline{\widehat{\mathbf{E}}}(\sigma)$ defined in (3.2.4).

Remark 3.2.1 *We note that for any* $\underline{\mathbf{u}} \in L^2(\mathbb{T}_1, (\ell^2)^n)$ *and* $\underline{\mathbf{v}} \in \mathscr{D}(\widehat{A}_{\mathrm{ch}})$*,*

$$(\widehat{F}\underline{\mathbf{u}})(\sigma) = \widehat{F}(\sigma)\underline{\mathbf{u}}(\sigma), \quad (\mathscr{F}\delta_+\Gamma\mathscr{F}_{\mathrm{ch}}^{-1}\underline{\mathbf{v}})(\sigma) = R(\sigma)\underline{\mathbf{v}}(\sigma),\ \textit{for a.e. } \sigma\in[-\frac{1}{2},\frac{1}{2}].$$

In addition, for any $(\theta,\underline{\mathbf{w}}) \in L^2(\mathbb{T}_1)\times L^2_\perp(\mathbb{T}_1,(\ell^2)^n)$*,*

$$\left(\widehat{\mathscr{L}_{\mathrm{phd}}}\begin{pmatrix}\theta\\ \underline{\mathbf{w}}\end{pmatrix}\right)(\sigma) = \mathscr{L}_{\mathrm{phd}}(\sigma)\begin{pmatrix}\theta(\sigma)\\ \underline{\mathbf{w}}(\sigma)\end{pmatrix},\ \textit{for a.e. } \sigma\in[-\frac{1}{2},\frac{1}{2}].$$

We now consider the family of linear systems,

$$\begin{pmatrix}\dot{\theta}\\ \dot{\underline{\mathbf{w}}}\end{pmatrix} = \widehat{A}_{\mathrm{nf}}(\sigma)\begin{pmatrix}\theta\\ \underline{\mathbf{w}}\end{pmatrix},\ \text{for all } \sigma\in[-\frac{1}{2},\frac{1}{2}]. \tag{3.2.13}$$

While we obtained these operators based on L^2/ℓ^2 spaces, we can also consider them on L^q/ℓ^q-based spaces. To be more precise, we first define a family of projections

$$\begin{array}{rccc}\widetilde{P}_q(\sigma): & Y_q & \longrightarrow & Y_q \\ & \underline{\mathbf{w}} & \longmapsto & \underline{\mathbf{w}} - \frac{1}{2\pi}\langle\!\langle \underline{\mathbf{w}}, \mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}})\rangle\!\rangle\mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}'_\star),\end{array} \tag{3.2.14}$$

where

$$Y_q = \begin{cases}(\ell^q)^n, & \text{for } 1\leqslant q<\infty,\\ (\ell^\infty_0)^n, & \text{for } q=\infty.\end{cases} \tag{3.2.15}$$

Here we have $\ell^\infty_0 = \{x\in\ell^\infty \mid \lim_{|n|\to\infty}|x_n| = 0\}$ with the supremum norm. For any $q\in[1,\infty]$, the projection $P_q(\sigma)$ is well-defined. In fact, $\mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}}), \mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}'_\star)\in Y_1$ since $\mathbf{u}_{\mathrm{ad}}(\pm\pi) = \mathbf{u}'_\star(\pm\pi) = 0$. We now denote $\widetilde{Y}_{q,\mathrm{s}}(\sigma) = \mathrm{Rg}\,\widetilde{P}_q(\sigma)$, and, in the following lemma, define $\widehat{A}_{\mathrm{nf}}(\sigma)$ on L^q/ℓ^q-based space.

Lemma 3.2.2 *For* $q\in[1,\infty]$ *and* $\sigma\in[-\frac{1}{2},\frac{1}{2}]$*,*

$$\begin{array}{rccc}\mathscr{L}_{\mathrm{phd}}(\sigma): & \mathbb{C}\times\widetilde{Y}_{q,\mathrm{s}}(\sigma) & \longrightarrow & Y_q \\ & (\theta,\underline{\mathbf{w}}) & \longmapsto & \theta\underline{\widehat{\mathbf{E}}}(\sigma)+\underline{\mathbf{w}},\end{array}$$

is uniformly bounded and invertible with its inverse

$$\begin{array}{rccc}\mathscr{L}_{\mathrm{phd}}(\sigma)^{-1}: & Y_q & \longrightarrow & \mathbb{C}\times \widetilde{Y}_{q,\mathrm{s}}(\sigma) \\ & \underline{\mathbf{v}} & \longmapsto & (\widehat{F}(\sigma)\underline{\mathbf{v}}, \underline{\mathbf{v}} - \widehat{\underline{E}}(\sigma)\widehat{F}(\sigma)\underline{\mathbf{v}}).\end{array}$$

Moreover,

$$\widehat{A}_{\mathrm{nf}}(\sigma) : \mathbb{C}\times (\widetilde{Y}_{q,\mathrm{s}}(\sigma)\cap \mathscr{D}_q(\widehat{A}_{\mathrm{ch}}(\sigma))) \quad\rightarrow\quad \mathbb{C}\times \widetilde{Y}_{q,\mathrm{s}}(\sigma)$$

is well-defined and sectorial. Here $\mathscr{D}_q(\widehat{A}_{\mathrm{ch}}(\sigma)) = \{\underline{\mathbf{w}}\in Y_q \mid \{(1+m^2)\mathbf{w}_m\}_{m\in\mathbb{Z}}\in Y_q\}$ *is the domain of* $\widehat{A}_{\mathrm{ch}}(\sigma)$ *in* Y_q*.*

Proof. The assertions for $\mathscr{L}_{\mathrm{phd}}(\sigma)$ are straightforward. In order to show that $\widehat{A}_{\mathrm{nf}}(\sigma)$ is well-defined, we recall the definition of $\widehat{A}_{\mathrm{nf}}(\sigma)$ in (3.2.10), which indicates that we only need to show

$$\widehat{A}_{\mathrm{ch}}(\sigma)\widehat{\underline{\mathbf{E}}}(\sigma)\in \widetilde{Y}_{q,\mathrm{s}}(\sigma), \quad \mathrm{Rg}\,(\widehat{A}_{\mathrm{ch}}(\sigma) - \widehat{\underline{\mathbf{E}}}(\sigma)R(\sigma))\subseteq \widetilde{Y}_{q,\mathrm{s}}(\sigma).$$

We claim that $\widehat{A}_{\mathrm{ch}}(\sigma)\widehat{\underline{\mathbf{E}}}(\sigma)\in \mathrm{Rg}\,(\widehat{A}_{\mathrm{ch}}(\sigma) - \widehat{\underline{\mathbf{E}}}(\sigma)R(\sigma))$. In fact, recall the definition of $R(\sigma)$ in (3.2.11) and define $\mathbf{E}(\sigma,x) := (\sum_j \mathbf{E}_j(x)\mathrm{e}^{-\mathrm{i}2\pi j\sigma})\mathrm{e}^{-\mathrm{i}\sigma x}\in (C^\infty)^n(\mathscr{T}_{2\pi})$, we have

$$R(\sigma)\widehat{\underline{\mathbf{E}}}(\sigma) = 2\mathrm{i}\sin\pi\sigma(\mathbf{E}(\sigma,\pi), D\mathbf{u}'_{\mathrm{ad}}(\pi)) = 0,$$

which means that $\widehat{A}_{\mathrm{ch}}(\sigma)\widehat{\underline{\mathbf{E}}}(\sigma) = (\widehat{A}_{\mathrm{ch}}(\sigma) - \widehat{\underline{\mathbf{E}}}(\sigma)R(\sigma))\widehat{\underline{\mathbf{E}}}(\sigma)\in \mathrm{Rg}\,(\widehat{A}_{\mathrm{ch}}(\sigma) - \widehat{\underline{\mathbf{E}}}(\sigma)R(\sigma))$.

We now only have to show $\mathrm{Rg}\,(\widehat{A}_{\mathrm{ch}}(\sigma) - \widehat{\underline{\mathbf{E}}}(\sigma)R(\sigma))\subseteq \widetilde{Y}_{q,\mathrm{s}}(\sigma)$. Actually, for any $\underline{\mathbf{w}}\in \mathscr{D}_q(\widehat{A}_{\mathrm{ch}}(\sigma))$ with finitely many nonzero components, we have

$$\begin{aligned}&\langle\!\langle \widehat{A}_{\mathrm{ch}}(\sigma)\underline{\mathbf{w}} - \widehat{\underline{\mathbf{E}}}(\sigma)R(\sigma)\underline{\mathbf{w}}, \mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}})\rangle\!\rangle \\ =&\langle\!\langle \widehat{A}_{\mathrm{ch}}(\sigma)\underline{\mathbf{w}}, \mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}})\rangle\!\rangle + 2\pi R(\sigma)\underline{\mathbf{w}} \\ =&2\pi\langle A(\mathrm{e}^{\mathrm{i}\sigma x}\mathscr{F}_n^{-1}\underline{\mathbf{w}}), \mathbf{u}_{\mathrm{ad}}\rangle + 2\pi(\mathrm{e}^{\mathrm{i}\sigma x}\mathscr{F}_n^{-1}\underline{\mathbf{w}}, D\mathbf{u}'_{\mathrm{ad}}(x))|_{-\pi}^{\pi} \\ =&2\pi\langle \mathrm{e}^{\mathrm{i}\sigma x}\mathscr{F}_n^{-1}\underline{\mathbf{w}}, B^*(0)\mathbf{u}_{\mathrm{ad}}\rangle = 0,\end{aligned}$$

and $\{\underline{\mathbf{w}}\in D_q(\widehat{A}_{\mathrm{ch}}(\sigma))|\underline{\mathbf{w}}$ has finite many nonzero elements$\}$ is dense in $\mathscr{D}_q(\widehat{A}_{\mathrm{ch}}(\sigma))$ under the graph norm of $\widehat{A}_{\mathrm{ch}}(\sigma)$. Therefore, $\widehat{A}_{\mathrm{nf}}(\sigma)$ is well-defined.

Next, $\widehat{A}_{\mathrm{nf}}(\sigma)$ is sectorial, due to the facts that $\widehat{A}_{\mathrm{nf}}(\sigma) = \mathscr{L}_{\mathrm{phd}}(\sigma)^{-1}\widehat{A}_{\mathrm{ch}}(\sigma)\mathscr{L}_{\mathrm{phd}}(\sigma)$ and $\widehat{A}_{\mathrm{ch}}(\sigma)$ is sectorial (for details, see Section B.3 in the appendix). ■

Now we are ready to obtain the estimates for the time evolution of system (3.2.13), for any given $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$. Our discussion is split into the case σ close to 0 and the case σ away from 0.

For the case $\sigma \sim 0$, the derivation of the estimate relies on a diagonalized normal form, that is, a complete separation of the netural and stable phase. First, we notice that $\operatorname{spec}(\widehat{A}_{\text{nf}}(\sigma)) = \operatorname{spec}(\widehat{A}_{\text{ch}}(\sigma))$ is independent of the choice of $q \in [1, \infty]$ and $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, which we will prove in Proposition B.3.2. Moreover, for σ sufficiently small, there is a unique continuation of the eigenvalue 0, denoted as $\lambda(\sigma)$. The set $\Lambda_1 := \{\lambda(\sigma)\}$ is a spectral set; see Section B.4, B.5 for detailed treatment. Hence, let

$$\begin{array}{rccc} P_q(\sigma): & Y_q & \longrightarrow & Y_q \\ & \underline{\mathbf{w}} & \longmapsto & \underline{\mathbf{w}} - \frac{1}{2\pi} \langle\!\langle \underline{\mathbf{w}}, \mathscr{F}_n(\mathbf{e}^*(\sigma)) \rangle\!\rangle \mathscr{F}_n(\mathbf{e}(\sigma)) \end{array} \tag{3.2.16}$$

be the spectral projection associated with $\Lambda_2 := \operatorname{spec}(\widehat{A}_{\text{ch}}(\sigma)) \backslash \{\lambda(\sigma)\}$. Here $\mathbf{e}(\sigma)$ (respectively, $\mathbf{e}^*(\sigma)$) is the eigenvector of the Bloch wave operator $B(\sigma)$ (respectively, the adjoint operator $B^*(\sigma)$) according to $\lambda(\sigma)$ with

$$\mathbf{e}(0) = \mathbf{u}'_\star, \quad \mathbf{e}^*(0) = \mathbf{u}_{\text{ad}}, \quad \langle \mathbf{e}(\sigma), \mathbf{e}^*(\sigma) \rangle = 1. \tag{3.2.17}$$

We refer to Section B.4 in the appendix for more details on $\mathbf{e}(\sigma)$ and $\mathbf{e}^*(\sigma)$. We now denote

$$\begin{aligned} Y_{q,\text{c}}(\sigma) &= \operatorname{span}\{\mathbf{e}(\sigma)\}, \quad Y_{q,\text{s}}(\sigma) = \operatorname{Rg} P_q(\sigma), \\ \widehat{A}_{\text{ch}}(\sigma)|_{Y_{q,\text{c}}(\sigma)} &= \widehat{A}_{\text{c}}(\sigma), \quad \widehat{A}_{\text{ch}}(\sigma)|_{Y_{q,\text{s}}} = \widehat{A}_{\text{s}}(\sigma). \end{aligned} \tag{3.2.18}$$

We then introduce the following diagonalized operator

$$\widehat{A}_{\text{dg}}(\sigma) = \begin{pmatrix} \lambda(\sigma) & 0 \\ 0 & \widehat{A}_{\text{s}}(\sigma) \end{pmatrix}. \tag{3.2.19}$$

It is not hard to conclude that for σ sufficiently small,

$$\widehat{A}_{\text{dg}}(\sigma) : \mathbb{C} \times (Y_{q,\text{s}}(\sigma) \cap \mathscr{D}^q(\widehat{A}_{\text{ch}}(\sigma))) \to \mathbb{C} \times Y_{q,\text{s}}(\sigma)$$

is a well-defined operator.

The key step here is to find an invertible bounded linear transformation

$$\widehat{\mathscr{T}_{\text{dg}}}(\sigma) = \begin{pmatrix} \widehat{T}_{00}(\sigma) & \widehat{T}_{01}(\sigma) \\ \widehat{T}_{10}(\sigma) & \widehat{T}_{11}(\sigma) \end{pmatrix} : \mathbb{C} \times \widetilde{Y}_{q,\text{s}}(\sigma) \to \mathbb{C} \times Y_{q,\text{s}}(\sigma) \tag{3.2.20}$$

such that $\widehat{\mathscr{T}_{\rm dg}}(\sigma)\widehat{A}_{\rm nf}(\sigma) = \widehat{A}_{\rm dg}(\sigma)\widehat{\mathscr{T}_{\rm dg}}(\sigma)$. We note that the choice of $\widehat{\mathscr{T}_{\rm dg}}(\sigma)$ is not unique since there are nontrivial invertible operators that commute with $\widehat{A}_{\rm dg}(\sigma)$.

Lemma 3.2.3 *For σ sufficiently small (that is, $|\sigma| \leqslant \gamma_0$) and $q \in [1, \infty]$,*

$$\widehat{\mathscr{T}_{\rm dg}}(\sigma) = \begin{pmatrix} \mu(\delta) & S(\sigma)|_{\widetilde{Y}_{q,\rm s}(\sigma)} \\ P_q(\sigma)\underline{\widehat{\mathbf{E}}}(\sigma) & P_q(\sigma)|_{\widetilde{Y}_{q,\rm s}(\sigma)} \end{pmatrix} \tag{3.2.21}$$

satisfies $\widehat{\mathscr{T}_{\rm dg}}(\sigma)\widehat{A}_{\rm nf}(\sigma) = \widehat{A}_{\rm dg}(\sigma)\widehat{\mathscr{T}_{\rm dg}}(\sigma)$, where $\mu(\sigma) = -\frac{1}{2\pi}\langle\!\langle \underline{\widehat{\mathbf{E}}}(\sigma), \mathscr{F}_n(\mathbf{e}^*(\sigma))\rangle\!\rangle$ and

$$\begin{array}{rccc} S(\sigma): & Y_q & \longrightarrow & \mathbb{C} \\ & \underline{\mathbf{w}} & \longmapsto & -\frac{1}{2\pi}\langle\!\langle \underline{\mathbf{w}}, \mathscr{F}_n(\mathbf{e}^*(\sigma))\rangle\!\rangle. \end{array}$$

Moreover, we have

$$\widehat{\mathscr{T}_{\rm dg}}^{-1} = (\widehat{T}_{00} - \widehat{T}_{01}\widehat{T}_{11}^{-1}\widehat{T}_{10})^{-1} \begin{pmatrix} 1 & -\widehat{T}_{01}\widehat{T}_{11}^{-1} \\ -\widehat{T}_{11}^{-1}\widehat{T}_{10} & (\widehat{T}_{00} - \widehat{T}_{01}\widehat{T}_{11}^{-1}\widehat{T}_{10})\widehat{T}_{11}^{-1} + \widehat{T}_{11}^{-1}\widehat{T}_{10}\widehat{T}_{01}\widehat{T}_{11}^{-1} \end{pmatrix}, \tag{3.2.22}$$

in which we suppress σ-dependence for simplicity.

Proof. We recall from (3.2.10) that

$$\widehat{A}_{\rm nf}(\sigma) = \begin{pmatrix} \widehat{F}(\sigma) \\ \mathrm{id} - \underline{\widehat{\mathbf{E}}}(\sigma)\widehat{F}(\sigma) \end{pmatrix} \widehat{A}_{\rm ch}(\sigma) \begin{pmatrix} \underline{\widehat{\mathbf{E}}}(\sigma) & \mathrm{id} \end{pmatrix}.$$

Therefore, in order to find a $\widehat{\mathscr{T}_{\rm dg}}$ as required, we only need to find an invertible bounded linear operator

$$\widehat{\mathscr{T}_{\rm int}}(\sigma) = \begin{pmatrix} \widehat{\mathscr{T}_1}(\sigma) \\ \widehat{\mathscr{T}_2}(\sigma) \end{pmatrix} : Y_q \longrightarrow \mathbb{C} \times Y_{q,\rm s}(\sigma)$$

such that

$$\widehat{\mathscr{T}_{\rm int}}(\sigma)\widehat{A}_{\rm ch}(\sigma) = \widehat{A}_{\rm dg}(\sigma)\widehat{\mathscr{T}_{\rm int}}(\sigma),$$

which is equivalent to

$$\begin{cases} \widehat{\mathscr{T}_1}(\sigma)(\lambda(\sigma) - \widehat{A}_{\rm ch}(\sigma)) = 0, \\ \widehat{\mathscr{T}_2}(\sigma)\widehat{A}_{\rm ch}(\sigma) - \widehat{A}_{\rm ch}(\sigma)\widehat{\mathscr{T}_2}(\sigma) = 0. \end{cases}$$

While the choice of $\widehat{\mathscr{T}}_{1/2}(\sigma)$ satisfying the above equation is apparently not unique, we choose that $\widehat{\mathscr{T}}_1(\sigma) = S(\sigma)$ and $\widehat{\mathscr{T}}_2(\sigma) = P_q(\sigma)$. As a result, we have

$$\widehat{\mathscr{T}}_{\mathrm{dg}}(\sigma) = \begin{pmatrix} \widehat{\mathscr{T}}_1(\sigma) \\ \widehat{\mathscr{T}}_2(\sigma) \end{pmatrix} \begin{pmatrix} \underline{\widehat{\mathbf{E}}}(\sigma) & \mathrm{id}\,|_{\widetilde{Y}_{q,\mathrm{s}}(\sigma)} \end{pmatrix} = \begin{pmatrix} \mu(\delta) & S(\sigma)|_{\widetilde{Y}_{q,\mathrm{s}}(\sigma)} \\ P_q(\sigma)\underline{\widehat{\mathbf{E}}}(\sigma) & P_q(\sigma)|_{\widetilde{Y}_{q,\mathrm{s}}(\sigma)} \end{pmatrix}.$$

To show that $\widehat{\mathscr{T}}_{\mathrm{dg}}(\sigma)^{-1}$ in (3.2.22) is correct, we only need to verify that

$$\widehat{\mathscr{T}}_{\mathrm{dg}}(\sigma)^{-1}\widehat{\mathscr{T}}_{\mathrm{dg}}(\sigma) = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id}\,|_{\widetilde{Y}_{q,\mathrm{s}}(\sigma)} \end{pmatrix}, \quad \widehat{\mathscr{T}}_{\mathrm{dg}}(\sigma)\widehat{\mathscr{T}}_{\mathrm{dg}}(\sigma)^{-1} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id}\,|_{Y_{q,\mathrm{s}}(\sigma)} \end{pmatrix},$$

which is clearly true. ∎

Based on this lemma, we now derive the estimate for $\mathrm{e}^{\widehat{A}_{\mathrm{nf}}(\sigma)t}$ when σ is close to zero. We first introduce new notation $M(t,\sigma) := \mathrm{e}^{\widehat{A}_{\mathrm{nf}}t}$ and $\mathscr{M}(t) := \mathrm{e}^{\widetilde{A}_{\mathrm{nf}}t}$ with

$$\begin{aligned} M(t,\sigma) = \begin{pmatrix} M_{00}(t,\sigma) & M_{01}(t,\sigma) \\ M_{10}(t,\sigma) & M_{11}(t,\sigma) \end{pmatrix} &: \qquad \mathbb{C}\times \widetilde{Y}_{q,\mathrm{s}}(\sigma) \longrightarrow \mathbb{C}\times \widetilde{Y}_{q,\mathrm{s}}(\sigma), \\ \mathscr{M}(t) = \begin{pmatrix} \mathscr{M}_{00}(t) & \mathscr{M}_{01}(t) \\ \mathscr{M}_{10}(t) & \mathscr{M}_{11}(t) \end{pmatrix} &: \qquad \ell^2\times \ell^2_{\perp}(\mathbb{Z}, L^2(\mathbb{T}_{2\pi})) \longrightarrow \ell^2\times \ell^2_{\perp}(\mathbb{Z}, L^2(\mathbb{T}_{2\pi})). \end{aligned} \tag{3.2.23}$$

To make sense of the derivatives and Taylor expansions with respect to σ of entries in $\mathscr{T}_{\mathrm{dg}}(\sigma)$, we extend $\widehat{T}_{01}(\sigma)$ and $\widehat{T}_{11}(\sigma)$ continuously as operators on Y_q, that is,

$$\widehat{T}_{01}(\sigma) = S(\sigma)\widetilde{P}_q(\sigma), \quad \widehat{T}_{11}(\sigma) = P_q(\sigma)\widetilde{P}_q(\sigma). \tag{3.2.24}$$

The same argument applies to operators $\widehat{\mathscr{T}}_{\mathrm{dg}}^{-1}(\sigma)$ and $M(t,\sigma)$.

Lemma 3.2.4 *For σ sufficiently small (that is, $|\sigma| \leqslant \gamma_0$) and $q \in [1,\infty]$, there exist positive constants $C(q)$ and $\widetilde{d}$ such that, for all $t \geqslant 0$,*

$$\begin{pmatrix} |M_{00}(t,\sigma)| & |||M_{01}(t,\sigma)|||_{Y_q\to\mathbb{C}} \\ |||M_{10}(t,\sigma)|||_{\mathbb{C}\to Y_q} & |||M_{11}(t,\sigma)|||_{Y_q} \end{pmatrix} \leqslant C(q) \begin{pmatrix} 1 & \frac{1}{\sqrt{1+t}} \\ \frac{1}{\sqrt{1+t}} & \frac{1}{1+t} \end{pmatrix} \mathrm{e}^{-\widetilde{d}\sigma^2 t}. \tag{3.2.25}$$

Moreover, we have a higher regularity result for $M_{11}(t,\sigma)$, that is, for any given $\sigma \in [-\gamma_0,\gamma_0]$, $q\in[1,\infty]$ and $\alpha > 0$, there exists $C(q,\alpha) > 0$ such that

$$|||M_{11}(t,\sigma)|||_{Y_q\to Y_q^{\alpha}} \leqslant C(q,\alpha)[(1+t^{-\alpha})\mathrm{e}^{-\frac{\gamma_1}{2}t} + \frac{1}{1+t}\mathrm{e}^{-\frac{d}{2}\sigma^2 t}].$$

Proof. The idea is to evaluate $M(t,\sigma) = \mathrm{e}^{\widehat{A}_{\mathrm{nf}}(\sigma)t}$ based on

$$\mathrm{e}^{\widehat{A}_{\mathrm{nf}}(\sigma)t} = \widehat{\mathscr{T}_{\mathrm{dg}}}(\sigma)^{-1}\mathrm{e}^{\widehat{A}_{\mathrm{dg}}(\sigma)t}\widehat{\mathscr{T}_{\mathrm{dg}}}(\sigma).$$

We first state the following estimate from Proposition B.5.1: For all $q \in [1,\infty]$ and $\sigma \in [-\gamma_0,\gamma_0]$, there exists a constant $C(q)>0$ such that

$$|\mathrm{e}^{\lambda(\sigma)t}| \leqslant C(q)\mathrm{e}^{-\frac{d}{2}\sigma^2 t}, \quad |||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)t}|||_q \leqslant C(q)\mathrm{e}^{-\frac{\gamma_1}{2}t}.$$

To obtain estimates on $\widehat{\mathscr{T}_{\mathrm{dg}}}$ and its inverse, we start by computing the Taylor expansions of entries in $\widehat{\mathscr{T}_{\mathrm{dg}}}(\sigma)$. A straightforward calculation using (3.2.17), (3.2.4) and (3.1.18) shows that

$$\begin{aligned} \mathbf{e}(\sigma) &= \mathbf{u}_\star' + \mathrm{i}\sigma\mathbf{e}_1 + \mathcal{O}(\sigma^2), & \mathbf{e}^*(\sigma) &= \mathbf{u}_{\mathrm{ad}} + \mathrm{i}\sigma\mathbf{e}_1^* + \mathcal{O}(\sigma^2),\\ \mathrm{e}^{-\mathrm{i}\sigma x} &= 1 - \mathrm{i}\sigma x + \mathcal{O}(\sigma^2), & \widehat{\underline{\mathbf{E}}}(\sigma) &= \mathscr{F}_n(-\mathbf{u}_\star' - 2\pi\mathrm{i}\sigma\phi\mathbf{u}_\star' + \mathrm{i}\sigma x\mathbf{u}_\star') + \mathcal{O}(\sigma^2), \end{aligned}$$

where $\mathbf{e}_1$(respectively, $\mathbf{e}_1^*$) is even and nonzero due to the fact that $B(0)\mathbf{e}_1 = -2D\mathbf{u}_\star''$ (respectively, $B^*(0)\mathbf{e}_1^* = -2D\mathbf{u}_{\mathrm{ad}}'$). Then, plugging these expansions into $\widehat{\mathscr{T}_{\mathrm{dg}}}$, and using (3.2.24), we obtain

$$\begin{aligned} \widehat{\mathscr{T}_{\mathrm{dg}}}(\sigma) &= \begin{pmatrix} \mu(\sigma) & S(\sigma)\widetilde{P}_q(\sigma) \\ P_q(\sigma)\widehat{\underline{\mathbf{E}}}(\sigma) & P_q(\sigma)\widetilde{P}_q(\sigma) \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2\pi\mathrm{i}\sigma\Psi \\ -2\pi\mathrm{i}\sigma\mathscr{F}_n(\Phi) & P_q(0) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\sigma^2) & \mathcal{O}(\sigma^2) \\ \mathcal{O}(\sigma^2) & \mathcal{O}(\sigma) \end{pmatrix}, \end{aligned} \tag{3.2.26}$$

where $\Phi(x) = -\frac{x}{2\pi}\mathbf{u}_\star' + \phi\mathbf{u}_\star' - \frac{\mathbf{e}_1}{2\pi}$ and

$$\begin{array}{rccc} \Psi: & Y_q & \longrightarrow & \mathbb{C} \\ & \underline{\mathbf{w}} & \longmapsto & \frac{1}{4\pi^2}\langle\!\langle \underline{\mathbf{w}}, \mathscr{F}_n(x\mathbf{u}_{\mathrm{ad}} + \mathbf{e}_1^*)\rangle\!\rangle. \end{array}$$

Therefore, for any $\sigma \in [-\gamma_0,\gamma_0]$, $q \in [1,\infty]$, there exist positive constants $C(q)$ and $\widetilde{d}$ such that, for all $(\theta,\underline{\mathbf{w}}) \in \mathbb{C}\times Y_q$, we have the following estimate.

$$\begin{aligned} &\begin{pmatrix} |M_{00}(t,\sigma)\theta| & |M_{01}(t,\sigma)\underline{\mathbf{w}}| \\ \|M_{10}(t,\sigma)\theta\|_{Y_q} & \|M_{11}(t,\sigma)\underline{\mathbf{w}}\|_{Y_q} \end{pmatrix} \\ \leqslant & C(q)\begin{pmatrix} 1 & |\sigma| \\ |\sigma| & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\frac{d}{2}\sigma^2 t} & 0 \\ 0 & \mathrm{e}^{-\frac{\gamma_1}{2}t} \end{pmatrix}\begin{pmatrix} 1 & |\sigma| \\ |\sigma| & 1 \end{pmatrix}\begin{pmatrix} |\theta| \\ \|\underline{\mathbf{w}}\|_{Y_q} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&\leqslant C(q)\left[\begin{pmatrix} 1 & |\sigma| \\ |\sigma| & |\sigma|^2 \end{pmatrix} \mathrm{e}^{-\frac{d}{2}\sigma^2 t} + \begin{pmatrix} |\sigma|^2 & |\sigma| \\ |\sigma| & 1 \end{pmatrix} \mathrm{e}^{-\frac{\gamma_1}{2}t}\right] \begin{pmatrix} |\theta| \\ \|\underline{\mathbf{w}}\|_{Y_q} \end{pmatrix} \\
&\leqslant C(q)\mathrm{e}^{-\tilde{d}\sigma^2 t} \begin{pmatrix} 1 & \frac{1}{\sqrt{1+t}} \\ \frac{1}{\sqrt{1+t}} & \frac{1}{1+t} \end{pmatrix} \begin{pmatrix} |\theta| \\ \|\underline{\mathbf{w}}\|_{Y_q} \end{pmatrix}.
\end{aligned}
$$

Using (3.2.20), (3.2.22) and (3.2.23), we now expand $M_{11}(t,\sigma)$ and obtain

$$
\begin{aligned}
\|M_{11}(t,\sigma)\underline{\mathbf{W}}\|_{Y_q^\alpha} \leqslant & C(\|\widehat{T}_{11}(\sigma)^{-1}\widehat{T}_{10}(\sigma)\|_{Y_q^\alpha}|\mathrm{e}^{\lambda(\sigma)t}\widehat{T}_{01}(\sigma)\underline{\mathbf{W}}|+ \\
& \|\widehat{T}_{11}(\sigma)^{-1}\mathrm{e}^{\widehat{A}_s(\sigma)t}\widehat{T}_{11}(\sigma)\underline{\mathbf{W}}\|_{Y_q^\alpha}+ \\
& \|\widehat{T}_{11}^{-1}(\sigma)\widehat{T}_{10}(\sigma)\|_{Y_q^\alpha}|\widehat{T}_{01}(\sigma)\widehat{T}_{11}(\sigma)^{-1}\mathrm{e}^{\widehat{A}_s(\sigma)t}\widehat{T}_{11}(\sigma)\underline{\mathbf{W}}|) \\
\leqslant & C(q,\alpha)[|\sigma|^2\mathrm{e}^{-\frac{d}{2}\sigma^2 t} + (t^{-\alpha}+1)\mathrm{e}^{-\frac{\gamma_1}{2}t} + |\sigma|^2\mathrm{e}^{-\frac{\gamma_1}{2}t}]\|\underline{\mathbf{W}}\|_{Y_q} \\
\leqslant & C(q,\alpha)[(t^{-\alpha}+1)\mathrm{e}^{-\frac{\gamma_1}{2}t} + \frac{1}{1+t}\mathrm{e}^{-\frac{d}{2}\sigma^2 t}]\|\underline{\mathbf{W}}\|_{Y_q},
\end{aligned}
$$

where in the second inequality we used (3.2.26), and Proposition B.5.1. ■

Remark 3.2.5 *We point out that, in the above lemma, the estimate for $M_{10}(\sigma)$ can not be improved, since $\mathscr{F}_n(\Phi) \neq 0$. In fact, due to the fact that $\Phi(x) \in (\mathbf{C}^\infty(\mathbb{T}_{2\pi}))^n$ and $\phi\mathbf{u}'_\star$ is a nonzero even function, we have*

$$
B(0)\Phi = \frac{1}{2\pi}[-2D\mathbf{u}''_\star - B(0)\mathbf{e}_1 + 2\pi B(0)(\phi\mathbf{u}'_\star)] = B(0)(\phi\mathbf{u}'_\star) \neq 0.
$$

On the other hand, the estimate for $M_{01}(\sigma)$ can be improved given suitable additional assumptions. For example, if we assume that $\mathbf{u}'_{\mathrm{ad}}(\pm\pi) = 0$, then $x\mathbf{u}_{\mathrm{ad}} + \mathbf{e}_1^*$ is zero, which leads to a better estimate.

For the case σ away from 0, we have the following result.

Lemma 3.2.6 *For σ away from zero (i.e., for $\gamma_0 \leqslant |\sigma| \leqslant \frac{1}{2}$) and $q \in [1,\infty]$, there exist constants $C(q), \gamma_2 > 0$ such that*

$$
|||M(t,\sigma)|||_{\mathbb{C}\times Y_q} \leqslant C(q)\mathrm{e}^{-\gamma_2 t} \tag{3.2.27}
$$

Moreover, we also have a higher regularity estimate for $M_{11}(t,\sigma)$, that is, for any given $\gamma_0 \leqslant |\sigma| \leqslant \frac{1}{2}$, $q \in [1,\infty]$ and $\alpha > 0$, there exists $C(q,\alpha) > 0$ such that

$$
|||M_{11}(t,\sigma)|||_{Y_q \to Y_q^\alpha} \leqslant C(q,\alpha)(1+t^{-\alpha})\mathrm{e}^{-\gamma_2 t}.
$$

Proof. Recall that $\mathrm{e}^{\widehat{A}_{\mathrm{nf}}(\sigma)t} = \mathscr{L}_{\mathrm{phd}}(\sigma)^{-1}\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)t}\mathscr{L}_{\mathrm{phd}}(\sigma)$. The inequality (3.2.27) is true due to the uniform boundedness of $\mathscr{T}(\sigma)$ in Lemma 3.2.2 and the fact that $|||\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)t}|||_{Y_q} \leqslant C(q)\mathrm{e}^{-\gamma_2 t}$, for σ away from 0, in Proposition B.5.1. Moreover, by the expressions of $\mathscr{L}_{\mathrm{phd}}(\sigma)$ and its inverse in Lemma 3.2.2, we have $M_{11}(t,\sigma) = (\,\mathrm{id} - \underline{\widehat{E}}(\sigma)\widehat{F}(\sigma))\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)t}$. Applying Proposition B.5.1, we conclude that

$$|||M_{11}(t,\sigma)|||_{Y_q\to Y_q^\alpha} = |||(\,\mathrm{id} - \underline{\widehat{E}}(\sigma)\widehat{F}(\sigma))\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)t}|||_{Y_q\to Y_q^\alpha} \leqslant C(q,\alpha)(1+t^{-\alpha})\mathrm{e}^{-\gamma_2 t}.$$

■

Lemma 3.2.4 and 3.2.6 give the following proposition.

Proposition 3.2.7 (Fourier-Bloch estimates) *For any $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, $q \in [1,\infty]$, there exist constants $C(q)$, $c > 0$ such that $\widehat{A}_{\mathrm{nf}}(\sigma)$ is sectorial and for all $t \geqslant 0$,*

$$\begin{pmatrix} |M_{00}(t,\sigma)| & |||M_{01}(t,\sigma)|||_{Y_q\to\mathbb{C}} \\ |||M_{10}(t,\sigma)|||_{\mathbb{C}\to Y_q} & |||M_{11}(t,\sigma)|||_{Y_q} \end{pmatrix} \leqslant C(q)\begin{pmatrix} 1 & \frac{1}{\sqrt{t+1}} \\ \frac{1}{\sqrt{t+1}} & \frac{1}{t+1} \end{pmatrix}\mathrm{e}^{-c\sigma^2 t}. \tag{3.2.28}$$

Moreover, we have a higher regularity estimate on $M_{11}(t,\sigma)$, that is, for any $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, $q \in [1,\infty]$ and $\alpha > 0$, there exist constants $C(q,\alpha)$, $\gamma > 0$ such that

$$|||M_{11}(t,\sigma)|||_{Y_q\to Y_q^\alpha} \leqslant C(q,\alpha)\left((1+t^{-\alpha})\mathrm{e}^{-\gamma t} + \frac{1}{1+t}\mathrm{e}^{-\frac{d}{2}\sigma^2 t}\right), \text{ for all } t > 0. \tag{3.2.29}$$

We also need the Fourier-Bloch estimates for the derivative $\partial_\sigma M(t,\sigma)$ in the following lemma.

Proposition 3.2.8 (Fourier-Bloch estimates for derivatives) *For any $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, $q \in [1,\infty]$ and $\beta \in (\frac{1}{2}(1-\frac{1}{q}), 1)$, there exist positive constants $C(q,\beta)$ and $\widetilde{c}$ such that, for all $t \geqslant 0$,*

$$\begin{aligned}&\begin{pmatrix} |(\partial_\sigma M)_{00}(t,\sigma)| & |||(\partial_\sigma M)_{01}(t,\sigma)|||_{Y_q\to\mathbb{C}} \\ |||(\partial_\sigma M)_{10}(t,\sigma)|||_{\mathbb{C}\to Y_q} & |||(\partial_\sigma M)_{11}(t,\sigma)|||_{Y_q} \end{pmatrix} \\ \leqslant & C(q,\beta)\begin{pmatrix} 1 & \frac{1}{\sqrt{1+t}} \\ \frac{1}{\sqrt{1+t}} & \frac{1}{1+t} \end{pmatrix}(t^{\frac{1}{2}} + t^{1-\beta})\mathrm{e}^{-\widetilde{c}\sigma^2 t}.\end{aligned} \tag{3.2.30}$$

Moreover, we have a higher regularity estimate on $(\partial_\sigma M)_{11}(t,\sigma)$, that is, for $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, $q \in [1,\infty]$, $\beta \in (\frac{1}{2}(1-\frac{1}{q}), 1)$ and $\alpha \in (0,1)$, there exist $C(q,\alpha,\beta) > 0$ and $\widetilde{\gamma} > 0$ such *that, for all* $t>0$,

$$|||(\partial_\sigma M)_{11}(t,\sigma)|||_{Y_q\to Y_q^\alpha} \leqslant C(q,\alpha,\beta)\left(\frac{t^{\frac{1}{2}}+t^{1-\beta}}{1+t}\mathrm{e}^{-\frac{d}{2}\sigma^2 t}+(t^{\frac{1}{2}-\alpha}+t^{1-\beta})\mathrm{e}^{-\widetilde{\gamma}t}\right).$$

Proof. On the one hand, we take the partial derivative of the following system with respect to σ

$$\begin{pmatrix}\theta(t,\sigma)\\ \underline{\widehat{\mathbf{W}}}(t,\sigma)\end{pmatrix}=M(t,\sigma)\begin{pmatrix}\theta(0,\sigma)\\ \underline{\widehat{\mathbf{W}}}(0,\sigma)\end{pmatrix},$$

and obtain

$$\begin{pmatrix}\partial_\sigma\theta(t,\sigma)\\ \partial_\sigma\underline{\widehat{\mathbf{W}}}(t,\sigma)\end{pmatrix}=M(t,\sigma)\begin{pmatrix}\partial_\sigma\theta(0,\sigma)\\ \partial_\sigma\underline{\widehat{\mathbf{W}}}(0,\sigma)\end{pmatrix}+(\partial_\sigma M(t,\sigma))\begin{pmatrix}\theta(0,\sigma)\\ \underline{\widehat{\mathbf{W}}}(0,\sigma)\end{pmatrix}.$$

On the other hand, we have that

$$\begin{pmatrix}\dot{\underline{\theta}}(t,\sigma)\\ \dot{\underline{\widehat{\mathbf{W}}}}(t,\sigma)\end{pmatrix}=\widehat{A}_{\mathrm{nf}}(\sigma)\begin{pmatrix}\underline{\theta}(t,\sigma)\\ \underline{\widehat{\mathbf{W}}}(t,\sigma)\end{pmatrix}.$$

Taking the partial derivative with respect to σ, the equation becomes

$$\begin{pmatrix}(\partial_\sigma\dot{\underline{\theta}})(t,\sigma)\\ (\partial_\sigma\dot{\underline{\widehat{\mathbf{W}}}})(t,\sigma)\end{pmatrix}=\widehat{A}_{\mathrm{nf}}(\sigma)\begin{pmatrix}\partial_\sigma\underline{\theta}(t,\sigma)\\ \partial_\sigma\underline{\widehat{\mathbf{W}}}(t,\sigma)\end{pmatrix}+\widehat{A}'_{\mathrm{nf}}(\sigma)\begin{pmatrix}\underline{\theta}(t,\sigma)\\ \underline{\widehat{\mathbf{W}}}(t,\sigma)\end{pmatrix},$$

for which the variation of constant formula gives

$$\begin{pmatrix}\partial_\sigma\theta(t,\sigma)\\ \partial_\sigma\underline{\widehat{\mathbf{W}}}(t,\sigma)\end{pmatrix}=M(t,\sigma)\begin{pmatrix}\partial_\sigma\theta(0,\sigma)\\ \partial_\sigma\underline{\widehat{\mathbf{W}}}(0,\sigma)\end{pmatrix}+\int_0^t M(t-s,\sigma)\widehat{A}'_{\mathrm{nf}}(\sigma)M(s,\sigma)\begin{pmatrix}\theta(0,\sigma)\\ \underline{\widehat{\mathbf{W}}}(0,\sigma)\end{pmatrix}\mathrm{d}s.$$

Therefore, we have

$$\begin{aligned}&\partial_\sigma M(t,\sigma)=\int_0^t M(t-s,\sigma)\widehat{A}'_{\mathrm{nf}}(\sigma)M(s,\sigma)\mathrm{d}s\\ =&\int_0^t \mathscr{L}_{\mathrm{phd}}(\sigma)^{-1}\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)(t-s)}\mathscr{L}_{\mathrm{phd}}(\sigma)\cdot\\ &(\mathscr{L}_{\mathrm{phd}}(\sigma)^{-1}\widehat{A}_{\mathrm{ch}}(\sigma)\mathscr{L}_{\mathrm{phd}}(\sigma))'\mathscr{L}_{\mathrm{phd}}(\sigma)^{-1}\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)s}\mathscr{L}_{\mathrm{phd}}(\sigma)\mathrm{d}s\\ =&\int_0^t \mathscr{L}_{\mathrm{phd}}(\sigma)^{-1}\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)(t-s)}\mathscr{N}(\sigma)\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)s}\mathscr{L}_{\mathrm{phd}}(\sigma)\mathrm{d}s,\end{aligned}\tag{3.2.31}$$

where

$$\mathscr{N}(\sigma)=\widehat{A}'_{\mathrm{ch}}(\sigma)+\widehat{A}_{\mathrm{ch}}(\sigma)\underline{\widehat{E}}'(\sigma)\widehat{F}(\sigma)-\underline{\widehat{E}}'(\sigma)\widehat{F}(\sigma)\widehat{A}_{\mathrm{ch}}(\sigma).\tag{3.2.32}$$

We recall that $\widehat{\underline{E}}(\sigma)$ is defined in (3.2.4), $\widehat{F}(\sigma)$ in (3.2.11) and $\widehat{A}_{\mathrm{ch}}(\sigma)$ in (3.2.9).

For $|\sigma| \leqslant \gamma_0$, by Lemma 3.2.3 and the above equation (3.2.31), we have

$$\partial_\sigma M(t,\sigma) = \int_0^t \widehat{\mathscr{T}_{\mathrm{dg}}}(\sigma)^{-1} \widetilde{\mathscr{N}}(\sigma,t,s) \widehat{\mathscr{T}_{\mathrm{dg}}}(\sigma) \mathrm{d}s, \tag{3.2.33}$$

where

$$\begin{aligned}
\widetilde{\mathscr{N}}(\sigma,t,s) &= \mathrm{e}^{\widehat{A}_{\mathrm{dg}}(\sigma)(t-s)} \begin{pmatrix} S(\sigma) \\ P_q(\sigma) \end{pmatrix} \mathscr{N}(\sigma) \Big(-\mathscr{F}_n \mathbf{e}(\sigma),\, \mathrm{id} \Big) \mathrm{e}^{\widehat{A}_{\mathrm{dg}}(\sigma)s} \\
&= \begin{pmatrix} -\mathrm{e}^{\lambda(\sigma)t} S(\sigma)\mathscr{N}(\sigma)\mathscr{F}_n \mathbf{e}(\sigma) & \mathrm{e}^{\lambda(\sigma)(t-s)} S(\sigma)\mathscr{N}(\sigma)\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s} \\ -\mathrm{e}^{\lambda(\sigma)s}\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)(t-s)} P_q(\sigma)\mathscr{N}(\sigma)\mathscr{F}_n \mathbf{e}(\sigma) & \mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)(t-s)} P_q(\sigma)\mathscr{N}(\sigma)\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s} \end{pmatrix} \\
&=: \begin{pmatrix} \widetilde{N}_{00}(\sigma,t,s) & \widetilde{N}_{01}(\sigma,t,s) \\ \widetilde{N}_{10}(\sigma,t,s) & \widetilde{N}_{11}(\sigma,t,s) \end{pmatrix}.
\end{aligned} \tag{3.2.34}$$

We now evaluate the entries of $\widetilde{\mathscr{N}}$ with expansions combining (3.2.32) and (3.2.34). First, recall the definitions of $\widehat{A}_{\mathrm{dg}}(\sigma)$ in (3.2.9), $S(\sigma)$ in Lemma 3.2.3, $P_q(\sigma)$ in (3.2.16), $\mathscr{F}_n$ in (3.2.2), and $\mathbf{e}(\sigma)$ in (3.2.17).

For $\widetilde{N}_{00}$, note that it is smooth with respect to σ and

$$\widetilde{N}_{00}(0,t,s) = -S(0) \Big(\widehat{A}'_{\mathrm{ch}}(0) + \widehat{A}_{\mathrm{ch}}(0)\widehat{\underline{E}}'(0)\widehat{F}(0) - \widehat{\underline{E}}'(0)\widehat{F}(0)\widehat{A}_{\mathrm{ch}}(0) \Big) \mathscr{F}_n \mathbf{e}(0).$$

We claim that $\widetilde{N}_{00}(0,t,s) = 0$. In fact, since $\widehat{A}'_{\mathrm{ch}}(0)\mathscr{F}_n(\mathbf{e}(0))$ and $\widehat{A}_{\mathrm{ch}}(0)\widehat{\underline{E}}'(0)$ are orthogonal to $\mathscr{F}_n(\mathbf{u}_{\mathrm{ad}})$ in Y_2, $S(0)\widehat{A}'_{\mathrm{ch}}(0)\mathscr{F}_n(\mathbf{e}(0)) = 0$ and $S(0)\widehat{A}_{\mathrm{ch}}(0)\widehat{\underline{E}}'(0) = 0$. Moreover, $\widehat{F}(\sigma)\widehat{A}_{\mathrm{ch}}(\sigma) = R(\sigma)$, which is defined in (3.2.11) with $R(0) = 0$. Therefore, there exists a positive constant C such that

$$|\widetilde{N}_{00}| \leqslant C|\sigma|\mathrm{e}^{-\frac{d}{2}\sigma^2 t}.$$

For $\widetilde{N}_{10}$, due to Proposition B.5.1 and the fact that $\widetilde{N}_{10}$ is smooth in σ with $\widetilde{N}_{10}(0,t,s) \neq 0$, there exists a positive constant C such that

$$|||\widetilde{N}_{10}|||_{\mathbb{C}\to Y_{q,\mathrm{s}}(\sigma)} \leqslant C\mathrm{e}^{-\frac{\gamma_1}{2}(t-s)}\mathrm{e}^{-\frac{d}{2}\sigma^2 s} \leqslant C\mathrm{e}^{-\frac{d}{2}\sigma^2 t}\mathrm{e}^{-\frac{\gamma_1}{4}(t-s)}.$$

For $\widetilde{N}_{01}$, we have, for any $q \in [1,\infty]$ and $\beta > \frac{1}{2}(1-\frac{1}{q})$,

$$\begin{aligned}
|||\widetilde{N}_{01}|||_{Y_{q,\mathrm{s}}(\sigma)\to\mathbb{C}} \leqslant& C|\mathrm{e}^{\lambda(\sigma)(t-s)}| \Big(|||\widehat{A}'_{\mathrm{ch}}(\sigma)\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)\to Y_q} + \\
& |S(\sigma)\widehat{A}_{\mathrm{ch}}(\sigma)\widehat{\underline{E}}'(\sigma)| |||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)} + |||\widehat{F}(\sigma)\widehat{A}_{\mathrm{ch}}(\sigma)\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)\to\mathbb{C}} \Big)
\end{aligned}$$

$$\leqslant C\mathrm{e}^{-\frac{d}{2}\sigma^2(t-s)}\left(|||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)\to Y_q^{\frac{1}{2}}}+|\sigma||||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)}+|\sigma||||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)\to Y_1}\right)$$
$$\leqslant C(\beta)\mathrm{e}^{-\frac{d}{2}\sigma^2(t-s)}\left(|||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)\to Y_q^{\frac{1}{2}}}+|||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)}+|\sigma||||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)\to Y_q^{\beta}}\right),$$

where the last inequality results from the fact that, for any $q\in[1,\infty]$ and $\beta>\frac{1}{2}(1-\frac{1}{q})$, we have a continuous imbedding

$$Y_q^{\beta}\hookrightarrow Y_1.$$

Now, using Proposition B.5.1 and B.4.1, we can further conclude that

$$\begin{aligned}|||\widetilde{N}_{01}|||_{Y_{q,\mathrm{s}}(\sigma)\to\mathbb{C}}\leqslant&C(q,\beta)(s^{-\frac{1}{2}}+1+|\sigma|s^{-\beta})\mathrm{e}^{-\frac{d}{2}\sigma^2(t-s)}\mathrm{e}^{-\frac{\gamma_1}{2}s}\\ \leqslant&C(q,\beta)\mathrm{e}^{-\frac{d}{2}\sigma^2t}(s^{-\frac{1}{2}}+|\sigma|+|\sigma|s^{-\beta})\mathrm{e}^{-\frac{\gamma_1}{4}s}.\end{aligned}$$

For $\widetilde{N}_{11}$, we have, for any $q\in[1,\infty]$ and $\beta>\frac{1}{2}(1-\frac{1}{q})$,

$$\begin{aligned}|||\widetilde{N}_{11}|||_{Y_{q,\mathrm{s}}(\sigma)}\leqslant&C|||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)(t-s)}|||_{Y_{q,\mathrm{s}}(\sigma)}\Big(|||\widehat{A}'_{\mathrm{ch}}(\sigma)\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)\to Y_q}\\ &+|||\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)}+|||\widehat{F}(\sigma)\widehat{A}_{\mathrm{ch}}(\sigma)\mathrm{e}^{\widehat{A}_{\mathrm{s}}(\sigma)s}|||_{Y_{q,\mathrm{s}}(\sigma)\to\mathbb{C}}\Big)\\ \leqslant&C(q,\beta)\left(s^{-\frac{1}{2}}+1+|\sigma|s^{-\beta}\right)\mathrm{e}^{-\frac{\gamma_1}{2}t}.\end{aligned}$$

Therefore, combining (3.2.33), (3.2.26), and the above estimates for entries, we conclude that, for $|\sigma|\in[-\gamma_0,\gamma_0]$, $q\in[1,\infty]$ and $\beta\in(\frac{1}{2}(1-\frac{1}{q}),1)$, there exist positive constants $C(q,\beta)$ and $c_1\leqslant\frac{d}{2}$ such that

$$\begin{aligned}&\begin{pmatrix}|(\partial_\sigma M)_{00}(t,\sigma)| & |||(\partial_\sigma M)_{01}(t,\sigma)|||_{Y_q\to\mathbb{C}}\\ |||(\partial_\sigma M)_{10}(t,\sigma)|||_{\mathbb{C}\to Y_q} & |||(\partial_\sigma M)_{11}(t,\sigma)|||_{Y_q}\end{pmatrix}\\ \leqslant&\begin{pmatrix}1 & |\sigma|\\ |\sigma| & 1\end{pmatrix}\int_0^t\begin{pmatrix}|\widetilde{N}_{00}(\sigma,t,s)| & |||\widetilde{N}_{01}(\sigma,t,s)|||_{Y_{q,\mathrm{s}}(\sigma)\to\mathbb{C}}\\ |||\widetilde{N}_{10}(\sigma,t,s)|||_{\mathbb{C}\to Y_{q,\mathrm{s}}(\sigma)} & |||\widetilde{N}_{11}(\sigma,t,s)|||_{Y_{q,\mathrm{s}}(\sigma)}\end{pmatrix}\mathrm{d}s\begin{pmatrix}1 & |\sigma|\\ |\sigma| & 1\end{pmatrix}\\ \overset{*}{\leqslant}&C(q,\beta)\begin{pmatrix}1 & |\sigma|\\ |\sigma| & 1\end{pmatrix}\begin{pmatrix}|\sigma|t & \frac{t^{\frac{1}{2}}}{\sqrt{1+t}}+|\sigma|\frac{t^{1-\beta}}{(1+t)^{1-\beta}}\\ \frac{t^{\frac{1}{2}}}{\sqrt{1+t}} & \frac{t^{\frac{1}{2}}+t^{1-\beta}}{1+t}\end{pmatrix}\begin{pmatrix}1 & |\sigma|\\ |\sigma| & 1\end{pmatrix}\mathrm{e}^{-\frac{d}{2}\sigma^2t}\\ \leqslant&C(q,\beta)\begin{pmatrix}1 & \frac{1}{\sqrt{1+t}}\\ \frac{1}{\sqrt{1+t}} & \frac{1}{1+t}\end{pmatrix}(t^{\frac{1}{2}}+t^{1-\beta})\mathrm{e}^{-c_1\sigma^2t}.\end{aligned} \tag{3.2.35}$$

Here the inequality $(*)$ relies on the fact that for any $\beta \in (0,1)$, there exists a positive constant $C(\beta)$ such that

$$\int_0^t \mathrm{e}^{-\frac{\gamma_1}{4}s}\mathrm{d}s \leqslant C(\beta)\frac{t^\beta}{(1+t)^\beta}, \quad \int_0^t s^{-\beta}\mathrm{e}^{-\frac{\gamma_1}{4}s}\mathrm{d}s \leqslant C(\beta)\frac{t^{1-\beta}}{(1+t)^{1-\beta}}.$$

On the other hand, for $\gamma_0 \leqslant |\sigma| \leqslant \frac{1}{2}$, $q \in [1,\infty]$ and $\beta \in (\frac{1}{2}(1-\frac{1}{q}),1)$, by the expression (3.2.31) and Proposition B.5.1, there exist positive constants $C(q,\beta)$ and c_2 such that

$$\begin{aligned}
|||\partial_\sigma M(t,\sigma)|||_{\mathbb{C}\times Y_q} \leqslant & C(q)\int_0^t |||\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)(t-s)}\mathscr{N}(\sigma)\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)s}|||_q\mathrm{d}s \\
\leqslant & C(q,\beta)\mathrm{e}^{-\gamma_2 t}\int_0^t (s^{-\frac{1}{2}}+1+|\sigma|s^{-\beta})\mathrm{d}s \\
\leqslant & C(q,\beta)(t^{\frac{1}{2}}+t+|\sigma|t^{1-\beta})\mathrm{e}^{-\gamma_2 t}.
\end{aligned} \tag{3.2.36}$$

By (3.2.35) and (3.2.36),we now conclude that, for any $\sigma \in [-\frac{1}{2},\frac{1}{2}]$, $q \in [1,\infty]$ and $\beta \in (\frac{1}{2}(1-\frac{1}{q}),1)$, there exists positive constant $C(q,\beta)$ and $\widetilde{c}$ such that

$$\begin{aligned}
&\begin{pmatrix} |(\partial_\sigma M)_{00}(t,\sigma)| & |||(\partial_\sigma M)_{01}(t,\sigma)|||_{Y_q\to\mathbb{C}} \\ |||(\partial_\sigma M)_{10}(t,\sigma)|||_{\mathbb{C}\to Y_q} & |||(\partial_\sigma M)_{11}(t,\sigma)|||_{Y_q} \end{pmatrix} \\
\leqslant & C(q,\beta)\begin{pmatrix} 1 & \frac{1}{\sqrt{1+t}} \\ \frac{1}{\sqrt{1+t}} & \frac{1}{1+t} \end{pmatrix}(t^{\frac{1}{2}}+t^{1-\beta})\mathrm{e}^{-\widetilde{c}\sigma^2 t}.
\end{aligned}$$

We now consider $(\partial_\sigma M)_{11}(t,\sigma)$. For $\sigma \in [-\gamma_0,\gamma_0]$, we plug $\widehat{\mathscr{T}_{\mathrm{dg}}}(\sigma)$ from (3.2.20), $\widehat{\mathscr{T}_{\mathrm{dg}}}(\sigma)^{-1}$ from (3.2.22), and $\widetilde{N}(\sigma,t,s)$ from the last equality in (3.2.34) into (3.2.33). We then obtain

$$\begin{aligned}
(\partial_\sigma M)_{11}(t,\sigma) = & \int_0^t -\widehat{T}_{11}^{-1}\widehat{T}_{10}\left(\widetilde{N}_{00}\widehat{T}_{01}+\widetilde{N}_{01}\widehat{T}_{11}\right)+ \\
& \widehat{T}_{11}^{-1}\left(\widetilde{N}_{10}\widehat{T}_{01}+\widetilde{N}_{11}\widehat{T}_{11}\right)\mathrm{d}s\cdot(1+\mathrm{O}(\sigma))
\end{aligned}$$

More precisely, for $q \in [1,\infty]$ and $\alpha \in (0,1)$, there exists C such that

$$\begin{aligned}
\|(\partial_\sigma M)_{11}(t,\sigma)\underline{\mathbf{W}}\|_{Y_q^\alpha} \leqslant & C\int_0^t \Big[\|\widehat{T}_{11}^{-1}\widehat{T}_{10}\|_{Y_q^\alpha}\left(|\widetilde{N}_{00}||\widehat{T}_{01}\underline{\mathbf{W}}|+|\widetilde{N}_{01}\widehat{T}_{11}\underline{\mathbf{W}}|\right) \\
& +\|\widehat{T}_{11}^{-1}\widetilde{N}_{10}\|_{Y_q^\alpha}|\widehat{T}_{01}\underline{\mathbf{W}}|+\|\widehat{T}_{11}^{-1}\widetilde{N}_{11}\widehat{T}_{11}\underline{\mathbf{W}}\|_{Y_q^\alpha}\Big]\mathrm{d}s \\
\overset{**}{\leqslant} & C\int_0^t \Big[\|\widehat{T}_{10}\|_{Y_q^\alpha}\left(|\widetilde{N}_{00}||\widehat{T}_{01}\underline{\mathbf{W}}|+|\widetilde{N}_{01}\widehat{T}_{11}\underline{\mathbf{W}}|\right) \\
& +\|\widetilde{N}_{10}\|_{Y_q^\alpha}|\widehat{T}_{01}\underline{\mathbf{W}}|+\|\widetilde{N}_{11}\widehat{T}_{11}\underline{\mathbf{W}}\|_{Y_q^\alpha}\Big]\mathrm{d}s.
\end{aligned}$$

Here the inequality $(**)$ relies on the fact that

$$\begin{array}{rccl} \widehat{T}_{11}(\sigma): & Y_q^\alpha & \longrightarrow & Y_q^\alpha \\ & \underline{\mathbf{v}} & \longmapsto & \underline{\mathbf{v}} - \langle \mathbf{e}(\sigma), \mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}}\rangle^{-1}\langle\!\langle \underline{\mathbf{v}}, \mathscr{F}_n(\mathrm{e}^{-\mathrm{i}\sigma x}\mathbf{u}_{\mathrm{ad}})\rangle\!\rangle \mathscr{F}_n\mathbf{e}(\sigma) \end{array}$$

is a uniformly bounded operator for $q \in [1,\infty]$ and $\alpha \in (0,1)$. Using the explicit expressions of the entries of $\widehat{\mathscr{T}_{\mathrm{dg}}}(\sigma)$ in (3.2.21) and the estimates on the entries of $\widetilde{\mathscr{N}}$ as shown above, we derive the following estimates,

$$\begin{aligned}
&\int_0^t \|\widehat{T}_{10}\|_{Y_q^\alpha}|\widetilde{N}_{00}||\widehat{T}_{01}\underline{\mathbf{W}}|\mathrm{d}s \leqslant C(q,\alpha)|\sigma|^3 t\mathrm{e}^{-\frac{d}{2}\sigma^2 t}\|\underline{\mathbf{W}}\|_{Y_q},\\
&\int_0^t \|\widehat{T}_{10}\|_{Y_q^\alpha}|\widetilde{N}_{01}\widehat{T}_{11}\underline{\mathbf{W}}|\mathrm{d}s \leqslant C(q,\alpha,\beta)|\sigma|\left(\frac{t^{\frac{1}{2}}}{\sqrt{1+t}} + |\sigma|\frac{t^{1-\beta}}{(1+t)^{1-\beta}}\right)\mathrm{e}^{-\frac{d}{2}\sigma^2 t}\|\underline{\mathbf{W}}\|_{Y_q},\\
&\int_0^t \|\widetilde{N}_{10}\|_{Y_q^\alpha}|\widehat{T}_{01}\underline{\mathbf{W}}|\mathrm{d}s \leqslant C(q,\alpha)\frac{t^{\frac{1}{2}}}{\sqrt{1+t}}\mathrm{e}^{-\frac{d}{2}\sigma^2 t}\|\underline{\mathbf{W}}\|_{Y_q},\\
&\int_0^t \|\widetilde{N}_{11}\widehat{T}_{11}\underline{\mathbf{W}}\|_{Y_q^\alpha}\mathrm{d}s \leqslant C(q,\alpha,\beta)\left(t + t^{1-\beta} + \int_0^t (t-s)^{-\alpha}s^{-1/2}\mathrm{d}s\right)\mathrm{e}^{-\frac{\gamma_1}{2}t}\|\underline{\mathbf{W}}\|_{Y_q}.
\end{aligned}$$

We now conclude that, for $\sigma \in [-\gamma_0,\gamma_0]$, $q \in [1,\infty]$, $\beta \in (\frac{1}{2}(1-\frac{1}{q}),1)$ and $\alpha \in (0,1)$, there exists $C(q,\alpha,\beta) > 0$ such that

$$|\!|\!|(\partial_\sigma M)_{11}(t,\sigma)|\!|\!|_{Y_q\to Y_q^\alpha} \leqslant C(q,\alpha,\beta)[\frac{t^{\frac{1}{2}}+t^{1-\beta}}{1+t}\mathrm{e}^{-\frac{d}{2}\sigma^2 t} + (t^{\frac{1}{2}-\alpha}+t^{1-\beta})\mathrm{e}^{-\frac{\gamma_1}{2}t}].$$

For $\gamma_0 \leqslant |\sigma| \leqslant \frac{1}{2}$, $q \in [1,\infty]$, $\beta \in (\frac{1}{2}(1-\frac{1}{q}),1)$ and $\alpha \in (0,1)$, there exists $C(q,\alpha,\beta) > 0$ such that

$$\begin{aligned}
|\!|\!|(\partial_\sigma M)_{11}(t,\sigma)|\!|\!|_{Y_q\to Y_q^\alpha} =& |\!|\!|(\,\mathrm{id}\, - \underline{\widehat{E}}(\sigma)\widehat{F}(\sigma))\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)(t-s)}\mathscr{N}(\sigma)\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)s}|\!|\!|_{Y_q\to Y_q^\alpha}\\
\leqslant& C(q,\alpha,\beta)\mathrm{e}^{-\gamma_2 t}\int_0^t (t-s)^{-\alpha}s^{-\frac{1}{2}} + s^{-\frac{1}{2}} + 1 + |\sigma|s^{1-\beta}\mathrm{d}s\\
\leqslant& C(q,\alpha,\beta)(t^{\frac{1}{2}-\alpha}+t^{1-\beta})\mathrm{e}^{-\frac{\gamma_2}{2}t}.
\end{aligned}$$

Altogether, for $\sigma \in [-\frac{1}{2},\frac{1}{2}]$, $q \in [1,\infty]$, $\beta \in (\frac{1}{2}(1-\frac{1}{q}),1)$ and $\alpha \in (0,1)$, there exist $C(q,\alpha,\beta) > 0$ and $\widetilde{\gamma} > 0$ such that, for all $t > 0$,

$$|\!|\!|(\partial_\sigma M)_{11}(t,\sigma)|\!|\!|_{Y_q\to Y_q^\alpha} \leqslant C(q,\alpha,\beta)[\frac{t^{\frac{1}{2}}+t^{1-\beta}}{1+t}\mathrm{e}^{-\frac{d}{2}\sigma^2 t} + (t^{\frac{1}{2}-\alpha}+t^{1-\beta})\mathrm{e}^{-\widetilde{\gamma}t}].$$

■

3.3 Linear estimates in physical space

According to the outline at the beginning of Section 3.2, we are now ready to derive the linear estimates for $\mathrm{e}^{A_{\mathrm{nf}}t}$. To be more precise, we first show by Fubini's Theorem that

$$\mathscr{M}(t)\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{W}}\end{pmatrix} = \check{M}(t) * \begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{W}}\end{pmatrix},$$

where $\check{M}(t)$ is the generalized "inverse Fourier transform" of $M(t,\sigma)$. We then employ an argument similar to, but more intricate than, Young's inequality for the case of the scalar heat equation, exploiting the linear Fourier-Bloch estimates in Proposition 3.2.7 and 3.2.8, to obtain the general L^p–L^q estimate on our linear normal form $\mathrm{e}^{A_{\mathrm{nf}}t}$.

To this end, we first note that $A_{\mathrm{nf}} = \widetilde{A}_{\mathrm{nf}}|_{\ell^1\times X_{\mathrm{ch}}^{\perp}}$ and thus we have, by (3.2.8), for any $(\underline{\theta}, \underline{\mathbf{W}}) \in \ell^1 \times X_{\mathrm{ch}}^{\perp}$,

$$\mathscr{M}(t)\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{W}}\end{pmatrix} = \mathrm{e}^{A_{\mathrm{nf}}t}\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{W}}\end{pmatrix} = \mathrm{e}^{\widetilde{A}_{\mathrm{nf}}t}\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{W}}\end{pmatrix} = \mathscr{F}_{\mathrm{nf}}^{-1}\mathrm{e}^{\widehat{A}_{\mathrm{nf}}t}\mathscr{F}_{\mathrm{nf}}\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{W}}\end{pmatrix}, \text{ for all } t>0.$$

Recall the notation $\mathscr{M}(t) = \mathrm{e}^{A_{\mathrm{nf}}t}$, the definition of $\mathscr{F}_{\mathrm{nf}}$ from (3.2.2), and the definition of $\widetilde{A}_{\mathrm{nf}}$, $\widehat{A}_{\mathrm{nf}}$ from (3.2.6). In addition, by (3.2.12), for any $(\theta(\sigma), \widehat{\underline{\mathbf{W}}}(\sigma)) \in L^2(\mathbb{T}_1) \times L^2_{\perp}(\mathbb{T}_1, \ell^2)$, the following equation holds for a.e. $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$:

$$\left(\mathrm{e}^{\widehat{A}_{\mathrm{nf}}t}\begin{pmatrix}\theta\\ \widehat{\underline{\mathbf{W}}}\end{pmatrix}\right)(\sigma) = \mathrm{e}^{\widehat{A}_{\mathrm{nf}}(\sigma)t}\begin{pmatrix}\theta(\sigma)\\ \widehat{\underline{\mathbf{W}}}(\sigma)\end{pmatrix} = \mathscr{L}_{\mathrm{phd}}(\sigma)^{-1}\mathrm{e}^{\widehat{A}_{\mathrm{ch}}(\sigma)t}\mathscr{L}_{\mathrm{phd}}(\sigma)\begin{pmatrix}\theta(\sigma)\\ \widehat{\underline{\mathbf{W}}}(\sigma)\end{pmatrix}.$$

To show that $\mathrm{e}^{A_{\mathrm{nf}}t}$ is a generalized convolution, we first define $M(t,\sigma)$'s "generalized inverse Fourier transform" $\check{M}(t) := \begin{pmatrix}\underline{\check{M}}_{00} & \underline{\check{M}}_{01}\\ \underline{\check{M}}_{10} & \underline{\check{M}}_{11}\end{pmatrix}$, with expressions

$$\begin{aligned} \underline{\check{M}}_{00}(t) &:= \{\check{M}_{00}(t,j)\}_{j\in\mathbb{Z}}, & \underline{\check{M}}_{01}(t,y) &:= \{\check{M}_{01}(t,y,j)\}_{j\in\mathbb{Z}},\\ \underline{\check{M}}_{10}(t,x) &:= \{\check{M}_{10}(t,x,j)\}_{j\in\mathbb{Z}}, & \underline{\check{M}}_{11}(t,x,y) &:= \{\check{M}_{11}(t,x,y,j)\}_{j\in\mathbb{Z}}, \end{aligned} \tag{3.3.1}$$

where

$$\check{M}_{00}(t,j) := \int_{-\frac{1}{2}}^{\frac{1}{2}} M_{00}(t,\sigma)\mathrm{e}^{\mathrm{i}2\pi\sigma j}\mathrm{d}\sigma,$$

$$\check{M}_{01}(t,y,j) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{\ell\in\mathbb{Z}} (M_{01})_{\ell}(t,\sigma)\mathrm{e}^{-\mathrm{i}(\sigma+\ell)y}\mathrm{e}^{\mathrm{i}2\pi j\sigma}\mathrm{d}\sigma,$$

$$\check{M}_{10}(t,x,j) := \frac{1}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{\ell\in\mathbb{Z}} (M_{10})_\ell(t,\sigma)\mathrm{e}^{\mathrm{i}(\sigma+\ell)x}\mathrm{e}^{\mathrm{i}2\pi j\sigma}\mathrm{d}\sigma,$$

$$\check{M}_{11}(t,x,y,j) := \frac{1}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{\ell,\eta\in\mathbb{Z}} (M_{11})_{\ell\eta}(t,\sigma)\mathrm{e}^{\mathrm{i}(\sigma+\ell)x-\mathrm{i}(\sigma+\eta)y}\mathrm{e}^{\mathrm{i}2\pi j\sigma}\mathrm{d}\sigma.$$

We then have the following lemma.

Lemma 3.3.1 *For any* $(\underline{\theta},\underline{\mathbf{W}})\in\ell^1\times X_{\mathrm{ch}}^{\perp}$ *and all* $t>0$,

$$\mathscr{M}(t)\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{W}}\end{pmatrix} = \check{M}(t)*\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{W}}\end{pmatrix} = \begin{pmatrix}\underline{\check{M}}_{00}*\underline{\theta} & \underline{\check{M}}_{01}*\underline{\mathbf{W}}\\ \underline{\check{M}}_{10}*\underline{\theta} & \underline{\check{M}}_{11}*\underline{\mathbf{W}}\end{pmatrix}, \tag{3.3.2}$$

where

$$\begin{aligned}
\underline{\check{M}}_{00}*\underline{\theta} =& \{\sum_{k\in\mathbb{Z}} \check{M}_{00}(t,j-k)\theta_k\}_{j\in\mathbb{Z}},\\
\underline{\check{M}}_{01}*\underline{\mathbf{W}} =& \{\sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \check{M}_{01}(t,y,j-k)\mathbf{W}_k(y)\mathrm{d}y\}_{j\in\mathbb{Z}},\\
\underline{\check{M}}_{10}*\underline{\theta} =& \{\sum_{k\in\mathbb{Z}} \check{M}_{10}(t,x,j-k)\theta_k\}_{j\in\mathbb{Z}},\\
\underline{\check{M}}_{11}*\underline{\mathbf{W}} =& \{\sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \check{M}_{11}(t,x,y,j-k)\mathbf{W}_k(y)\mathrm{d}y\}_{j\in\mathbb{Z}}.
\end{aligned}$$

Proof. The proof is a straightforward application of Fubini's theorem. ■

We are now ready to obtain the general $L^p - L^q$ linear estimates on $\mathscr{M}(t)$. We denote

$$X_q = (L^q(\mathbb{Z}, L^q(\mathbb{T}_{2\pi})))^n, \text{ for any } q\in[1,\infty],$$

and prove the following proposition.

Proposition 3.3.2 (**general L^p–L^q estimates**) *For any* $1\leqslant q\leqslant p\leqslant\infty$ *and* $(\underline{\theta},\underline{\mathbf{W}})\in\ell^1\times X_{\mathrm{ch}}^{\perp}$, *there exists a positive constant* C *such that, for all* $t>0$,

$$\begin{aligned}
&\begin{pmatrix}\|\mathscr{M}_{00}(t)\underline{\theta}\|_{\ell^p} & \|\mathscr{M}_{01}(t)\underline{\mathbf{W}}\|_{\ell^p}\\ \|\mathscr{M}_{10}(t)\underline{\theta}\|_{X_p} & \|\mathscr{M}_{11}(t)\underline{\mathbf{W}}\|_{X_p}\end{pmatrix}\\
\leqslant& C\begin{pmatrix}(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|\underline{\theta}\|_{\ell^q} & (1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|\underline{\mathbf{W}}\|_{X_q}\\ (1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|\underline{\theta}\|_{\ell^q} & t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}(1+t)^{-1}\|\underline{\mathbf{W}}\|_{X_q}\end{pmatrix}.
\end{aligned}\tag{3.3.3}$$

Proof. We illustrate the derivation of the estimates on $\mathscr{M}_{01}$ and sketch the estimates on $\mathscr{M}_{00}$ and $\mathscr{M}_{10}$. Lastly, we show the estimates for $\mathscr{M}_{11}$.

We first notice that, for any $\underline{\mathbf{W}} \in X_{\mathrm{ch}}^{\perp}$ and $1 \leqslant q, r \leqslant p \leqslant \infty$ satisfying $1+\frac{1}{p} = \frac{1}{q}+\frac{1}{r}$, there exists a positive constant C such that

$$\|\mathscr{M}_{01}(t)\underline{\mathbf{W}}\|_{\ell^p} \leqslant C\|\underline{\check{M}}_{01}(t)\|_{X_\infty}^{\frac{1}{q}-\frac{1}{p}} \left(\sum_j \sup_{|y|\leqslant\pi} |\check{M}_{01}(t,y,j)|\right)^{\frac{1}{r}} \|\underline{\mathbf{W}}\|_{X_q}. \tag{3.3.4}$$

In fact, by Hölder's inequality, we have

$$\begin{aligned}
&\|\mathscr{M}_{01}(t)\underline{\mathbf{W}}\|_{\ell^p}^p = \|\underline{\check{M}}_{01} * \underline{\mathbf{W}}\|_{\ell^p}^p = \sum_{j\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \check{M}_{01}(t,y,j-k)\mathbf{W}_k(y)\mathrm{d}y|^p \\
\leqslant& \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} |\check{M}_{01}(t,y,j-k)|^{1-\frac{r}{p}} |\mathbf{W}_k(y)|^{1-\frac{q}{p}} \left(|\check{M}_{01}(t,y,j-k)|^r |\mathbf{W}_k(y)|^q\right)^{\frac{1}{p}} \mathrm{d}y\right)^p \\
\leqslant& \|\underline{\check{M}}_{01}(t)\|_{X_r}^{p-r} \|\underline{\mathbf{W}}\|_{X_q}^{p-q} \sum_{j,k\in\mathbb{Z}} \int_{-\pi}^{\pi} |\underline{\check{M}}_{01}(t,y,j-k)|^r |\mathbf{W}_k(y)|^q \mathrm{d}y \\
\leqslant& \|\underline{\check{M}}_{01}(t)\|_{X_r}^{p-r} \left(\sup_{|y|\leqslant\pi} \sum_{j\in\mathbb{Z}} |\check{M}_{01}(t,y,j)|^r\right) \|\underline{\mathbf{W}}\|_{X_q}^p \\
\leqslant& \left[(2\pi)^{1-\frac{1}{q}} \|\underline{\check{M}}_{01}(t)\|_{X_\infty}^{\frac{1}{q}-\frac{1}{p}} \left(\sup_{|y|\leqslant\pi} \sum_{j\in\mathbb{Z}} |\check{M}_{01}(t,y,j)|\right)^{\frac{1}{r}} \|\underline{\mathbf{W}}\|_{X_q}\right]^p.
\end{aligned}$$

Moreover, by (3.3.1), we have

$$\begin{aligned}
\|\underline{\check{M}}_{01}(t)\|_{X_\infty} \leqslant& \sup_{|y|\leqslant\pi} |\int_{-\frac{1}{2}}^{\frac{1}{2}} |\sum_{\ell\in\mathbb{Z}} (M_{01})_\ell(t,\sigma)\mathrm{e}^{-\mathrm{i}(\sigma+\ell)y}|\mathrm{d}\sigma| \\
\leqslant& \frac{C(\infty)}{\sqrt{1+t}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{-c\sigma^2 t}\mathrm{d}\sigma \leqslant \frac{C}{1+t}.
\end{aligned} \tag{3.3.5}$$

Here we use the fact that any bounded linear functional on ℓ_0^∞ can be viewed as a bounded linear functional on ℓ^∞ with the same norm. We now estimate the X_1 norm of $\{\check{M}_{01}(t,y,j)\}_{j\in\mathbb{Z}}$. By using Proposition 3.2.8, there exists $C > 0$, independent of the choice of $y \in [-\pi, \pi]$, such that

$$
\begin{aligned}
\sum_{j\neq 0} |\check{M}_{01}(t,y,j)| =& \sum_{j\neq 0} \left(1+\frac{(j-\frac{y}{2\pi})^2}{t}\right)^{-\frac{1}{2}} \left(1+\frac{(j-\frac{y}{2\pi})^2}{t}\right)^{\frac{1}{2}} |\check{M}_{01}(t,y,j)| \\
\leqslant& C\left(\int_{\mathbb{R}} \frac{1}{1+\frac{x^2}{t}} \mathrm{d}x\right)^{\frac{1}{2}} \left[\sum_j (1+\frac{(j-\frac{y}{2\pi})^2}{t}) |\check{M}_{01}(t,y,j)|^2\right]^{\frac{1}{2}} \\
\leqslant& Ct^{\frac{1}{4}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{\alpha=0}^{1} t^{-\alpha} |\sum_{\ell\in\mathbb{Z}} (\partial_\sigma^\alpha (M_{01})_\ell(t,\sigma)) \mathrm{e}^{-\mathrm{i}(\sigma+\ell)y}|^2 \mathrm{d}\sigma\right)^{\frac{1}{2}} \\
\overset{***}{\leqslant}& Ct^{\frac{1}{4}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{e}^{-2c\sigma^2 t}}{1+t} \mathrm{d}\sigma + \frac{1}{t}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(t^{\frac{1}{2}}+t^{1-\frac{3}{4}})^2 \mathrm{e}^{-2\tilde{c}\sigma^2 t}}{1+t} \mathrm{d}\sigma\right)^{\frac{1}{2}} \\
\leqslant& C\frac{t^{\frac{1}{4}}+1}{(1+t)^{\frac{3}{4}}} \leqslant \frac{C}{\sqrt{1+t}}, \text{ for all } t>0.
\end{aligned} \tag{3.3.6}
$$

Here in the inequality (***), we applied Proposition 3.2.8 with $q=\infty$ and $\beta=\frac{3}{4}$ (actually, any fixed $\beta \in (\frac{1}{2},\frac{3}{4}]$). Combining (3.3.4), (3.3.5), and (3.3.6), we have that, for all $1 \leqslant q \leqslant p \leqslant \infty$ and $\underline{\mathbf{W}} \in X_{\mathrm{ch}}^{\perp}$, there exists a positive constant C such that

$$
\|\sqrt{1+t}\mathscr{M}_{01}(t)\underline{\mathbf{W}}\|_{\ell^p} \leqslant \frac{C}{(1+t)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}} \|\underline{\mathbf{W}}\|_{X_q}, \text{ for all } t \geqslant 0.
$$

For $\mathscr{M}_{00}$, the steps are the same as above but easier. For $\mathscr{M}_{10}$, we point out two main differences to the above calculation. First, instead of (3.3.4), we use

$$
\|\mathscr{M}_{10}(t)\underline{\theta}\|_{X_p} \leqslant C\|\underline{\check{M}}_{10}(t)\|_{X_\infty}^{\frac{1}{q}-\frac{1}{p}} \left(\int_{-\pi}^{\pi} \left(\sum_j |\check{M}_{10}(t,x,j)|\right)^{\frac{p}{r}} \mathrm{d}x\right)^{\frac{1}{p}} \|\underline{\theta}\|_{\ell^q}.
$$

Second, to estimate the Y_1 norm of $\{\check{M}_{10}(t,x,j)\}_{j\in\mathbb{Z}}$, we use Proposition 3.2.8 with $q=1$ and $\beta=\frac{1}{2}$(actually, any fixed $\beta \in (0,\frac{3}{4}]$), instead of $q=\infty$ and $\beta=\frac{3}{4}$.

The last step of the proof consists of deriving the estimates for $\mathscr{M}_{11}$. We first have

$$
\begin{aligned}
\|\mathscr{M}_{11}(t)\underline{\mathbf{W}}\|_{X_p} \leqslant& (2\pi)^{\frac{1}{r}} \left(\sup_{|x|,|y|\leqslant|\pi|} \sup_{j\in\mathbb{Z}} |\check{M}_{11}(t,x,y,j)|\right)^{\frac{1}{q}-\frac{1}{p}} \cdot \\
& \left(\sup_{|x|,|y|\leqslant\pi} \sum_{j\in\mathbb{Z}} |\check{M}_{11}(t,x,y,j)|\right)^{\frac{1}{r}} \|\underline{\mathbf{W}}\|_{X_q}.
\end{aligned}
$$

On the one hand, we apply Proposition 3.2.7 with $q=\infty$ and $\alpha>\frac{1}{2}$ and have

$$\begin{aligned}\sup_{|x|,|y|\leqslant|\pi|}\sup_{j\in\mathbb{Z}}|\check{M}_{11}(t,x,y,j)|\leqslant&\sup_{|x|,|y|\leqslant|\pi|}\int_{-\frac{1}{2}}^{\frac{1}{2}}|\sum_{\ell,\eta\in\mathbb{Z}}(M_{11})_{\ell\eta}(t,\sigma)\mathrm{e}^{\mathrm{i}(\sigma+\ell)x}\mathrm{e}^{-\mathrm{i}(\sigma+\eta)y}|\mathrm{d}\sigma\\ \leqslant&C(\alpha)\int_{-\frac{1}{2}}^{\frac{1}{2}}|||M_{11}(t,\sigma)|||_{Y_\infty\to Y_\infty^\alpha}\mathrm{d}\sigma\\ \leqslant&\frac{C(\alpha)}{t^\alpha(1+t)^{\frac{3}{2}-\alpha}}.\end{aligned}$$

On the other hand, by applying Proposition 3.2.7 and 3.2.8 with $q=\infty$, $\alpha\in(\frac{1}{2},1)$ and $\beta=\frac{3}{4}$, there exists $C(\alpha)$, independent of choices of $x,y\in[-\pi,\pi]$, such that

$$\begin{aligned}\sum_{|j|>1}|\check{M}_{11}(t,x,y,j)|=&\sum_{|j|>1}\left(1+\frac{(j+\frac{x-y}{2\pi})^2}{t}\right)^{-\frac{1}{2}}\left(1+\frac{(j+\frac{x-y}{2\pi})^2}{t}\right)^{\frac{1}{2}}|\check{M}_{11}(t,x,y,j)|\\ \leqslant&Ct^{\frac{1}{4}}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{\alpha=0}^{1}t^{-\alpha}|\sum_{\ell,\eta\in\mathbb{Z}}\left(\partial_\sigma^\alpha(M_{11})_{\ell\eta}(t,\sigma)\right)\mathrm{e}^{\mathrm{i}(\sigma+\ell)x}\mathrm{e}^{-\mathrm{i}(\sigma+\eta)y}|^2\mathrm{d}\sigma\right)^{\frac{1}{2}}\\ \leqslant&C(\alpha)t^{\frac{1}{4}}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}|||M_{11}(t,\sigma)|||^2_{Y_\infty\to Y_\infty^\alpha}\mathrm{d}\sigma+\int_{-\frac{1}{2}}^{\frac{1}{2}}|||(\partial_\sigma M)_{11}(t,\sigma)|||^2_{Y_\infty\to Y_\infty^\alpha}\mathrm{d}\sigma\right)^{\frac{1}{2}}\\ \leqslant&C(\alpha)t^{\frac{1}{4}}\left(\frac{1}{t^{2\alpha}(1+t)^{\frac{5}{2}-2\alpha}}\right)^{\frac{1}{2}}\\ \leqslant&C(\alpha)\frac{1}{t^{\alpha-\frac{1}{4}}(1+t)^{\frac{5}{4}-\alpha}},\quad\text{for all } t>0.\end{aligned}$$

Moreover, combining the above two estimates, we have that, for given $\alpha\in(\frac{1}{2},1)$, there exists $C(\alpha)>0$ such that

$$\sup_{|x|,|y|\leqslant\pi}\sum_{j\in\mathbb{Z}}|\check{M}_{11}(t,x,y,j)|\leqslant\frac{C(\alpha)}{t^\alpha(1+t)^{1-\alpha}}.$$

Therefore, for any $1\leqslant q\leqslant p\leqslant\infty$, $\alpha\in(\frac{1}{2},1)$ and $\underline{\mathbf{W}}\in X_{\mathrm{ch}}^{\perp}$, there exists $C(\alpha)>0$ such that

$$\|\mathscr{M}_{11}(t)\underline{\mathbf{W}}\|_{X_p}\leqslant\frac{C(\alpha)}{(1+t)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}}\frac{1}{t^\alpha(1+t)^{1-\alpha}}\|\underline{\mathbf{W}}\|_{X_q}.$$

Moreover, we can improve the above estimate for t close to zero. Note that for the Laplacian operator, we have the general L^p-L^q estimate for all $t>0$. As a perturbation of the Laplacian operator, $\mathscr{M}_{11}$ has the same estimate for sufficiently small t, which can be seen by using the variation of constant formula as follows.

$$\|\mathscr{M}_{11}(t)\underline{\mathbf{W}}\|_{X_p} = \|(\,\mathrm{id}\, - \mathbf{E} * F)\mathrm{e}^{A_{\mathrm{ch}}t}\underline{\mathbf{W}}\|_{X_p} \leqslant C\|\mathrm{e}^{A_{\mathrm{ch}}t}\underline{\mathbf{W}}\|_{X_p} = C\|\mathrm{e}^{At}\mathbf{W}\|_{L^p},$$

where $\underline{\mathbf{W}} = \{\mathbf{W}_j(x)\}_{j\in\mathbb{Z}}$ and $\mathbf{W}(2\pi j + x) = \mathbf{W}_j(x)$ for all $j \in \mathbb{Z}$ and $x \in [-\pi, \pi]$. We now let $\mathbf{V}(t,x) = \mathrm{e}^{At}\mathbf{W}(x)$ and have

$$\mathbf{V}(t) = \mathrm{e}^{D\partial_{xx}t}\mathbf{W} + \int_0^t \mathrm{e}^{D\partial_{xx}(t-s)}\mathbf{f}'(\mathbf{u}_\star)\mathbf{V}(s)\mathrm{d}s.$$

from which we derive

$$\sup_{0<t\leqslant T} t^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|\mathbf{V}(t)\|_{L^p} \leqslant \|\mathbf{W}\|_{L^q} + CT^{1-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \sup_{0<t\leqslant T} t^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|\mathbf{V}(t)\|_{L^p}.$$

Taking T sufficiently small such that $CT^{1-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \leqslant \frac{1}{2}$, we obtain

$$\|\mathrm{e}^{At}\mathbf{W}\|_{L^p} \leqslant \frac{C}{t^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}}\|\mathbf{W}\|_{L^q},$$

which implies that

$$\|\mathscr{M}_{11}(t)\underline{\mathbf{W}}\|_{X_p} \leqslant \frac{C}{t^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}}\|\underline{\mathbf{W}}\|_{X_q}, \text{ for all } 0 < t \leqslant T.$$

Therefore, for any $1 \leqslant q \leqslant p \leqslant \infty$, there exists $C > 0$ such that

$$\|\mathscr{M}_{11}(t)\underline{\mathbf{W}}\|_{X_p} \leqslant \frac{C}{t^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}}\frac{1}{(1+t)}\|\underline{\mathbf{W}}\|_{X_q}.$$

■

Remark 3.3.3 *By* (3.3.1),(3.3.2) *and a similar argument as in Proposition 3.3.2, it is not hard to conclude that, for any $j \in \mathbb{Z}^+$, $1 \leqslant q \leqslant p \leqslant \infty$ and $(\underline{\theta}, \underline{\mathbf{W}}) \in \ell^1 \times X_{\mathrm{ch}}^\perp$, there exists a positive constant C such that, for all $t > 0$,*

$$\begin{pmatrix} \|\delta_+^j\mathscr{M}_{00}(t)\underline{\theta}\|_{\ell^p} & \|\delta_+^j\mathscr{M}_{01}(t)\underline{\mathbf{W}}\|_{\ell^p} \\ \|\delta_+^j\mathscr{M}_{10}(t)\underline{\theta}\|_{X_p} & \|\delta_+^j\mathscr{M}_{11}(t)\underline{\mathbf{W}}\|_{X_p} \end{pmatrix}$$
$$\leqslant C\begin{pmatrix} (1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p}+j)}\|\underline{\theta}\|_{\ell^q} & (1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p}+j+1)}\|\underline{\mathbf{W}}\|_{X_q} \\ (1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p}+j+1)}\|\underline{\theta}\|_{\ell^q} & t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}(1+t)^{-(1+\frac{j}{2})}\|\underline{\mathbf{W}}\|_{X_q} \end{pmatrix}. \tag{3.3.7}$$

3.4 Maximal regularity and nonlinear stability

In this section, we prove the main theorem–Theorem 2. To achieve this, we first introduce a Banach space that our argument will be based on. We then collect several maximal regularity results since the normal form system is quasilinear. Based on our normal form and the general $L^p - L^q$ linear estimates, we can apply a fixed point argument to the variation of constant formula, thus obtaining the nonlinear stability result.

We choose $r \in (4, +\infty)$ and define

$$Z = \{(\underline{\theta}, \underline{\mathbf{W}}) \in C((0,+\infty), \ell^1 \times (X_{\text{ch}} \cap \mathscr{T}_{\text{ch}}^{-1}(H^2))) \mid \|(\underline{\theta}, \underline{\mathbf{W}})\|_Z < \infty\},$$

where

$$\begin{aligned}\|(\underline{\theta}, \underline{\mathbf{W}})\|_Z = &\sup_{t>0} \|\underline{\theta}(t)\|_{\ell^1} + \sup_{t>0} (1+t)^{\frac{1}{2}} \|\underline{\theta}(t)\|_{\ell^\infty} + \sup_{t>0} (1+t)^{\frac{5}{4}} \|\delta^2 \underline{\theta}\|_{\ell^2} \\ &+ \sup_{t>0} (1+t)^{\frac{1}{2}} \|\underline{\mathbf{W}}\|_{X_1} + \sup_{t>0} (1+t) \|\underline{\mathbf{W}}\|_{X_\infty} + + \sup_{t>0} (1+t)^{\frac{5}{4}} \|\delta_+ \underline{\mathbf{W}}\|_{X_2} \\ &+ \left(\int_0^\infty (1+t)^r \|\delta_+ \partial_{xx} \underline{\mathbf{W}}(t)\|_{X_2}^r \right)^{1/r}.\end{aligned}$$

Here we have $\delta^2 := \delta_- \delta_+$, where $\delta_\pm$ is defined in (3.1.20).

Lemma 3.4.1 (**maximal regularity**) *For any given $T > 0$ and $r \in (1, +\infty)$, there exists a positive constant C such that the following holds. If $(\underline{\eta}, \underline{\mathbf{v}}) \in L^r((0,T), \ell^2 \times X_2)$ and if $(\underline{\theta}, \underline{\mathbf{w}})$ satisfies*

$$\begin{pmatrix} \underline{\theta}(t) \\ \underline{\mathbf{w}}(t) \end{pmatrix} = \int_0^t \mathscr{M}(t-s) \begin{pmatrix} \underline{\eta}(s) \\ \underline{\mathbf{v}}(s) \end{pmatrix} \mathrm{d}s, \quad t \in [0,T],$$

then

$$\int_0^T \|\partial_{xx} \underline{\mathbf{w}}(t)\|_{X_2}^r \mathrm{d}t \leqslant C \int_0^T \left(\|\underline{\eta}(t)\|_{\ell^2} + \|\underline{\mathbf{v}}(t)\|_{X_2} \right)^r \mathrm{d}t.$$

Proof. The result just follows from the standard maximal regularity results on the Laplacian operator and the robustness of maximal regularity with respect to lower order perturbations. To see that, we first recall $\mathscr{M}(t) = \mathrm{e}^{A_{\text{nf}} t}$, where A_{nf} is defined in (3.1.22). By [41], the maximal regularity result holds when we just replace A_{nf} by A_0, which is defined as

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & D\partial_{xx} \end{pmatrix}.$$

Viewing $A_{\rm nf}$ as a perturbation of A_0, we have

$$\begin{pmatrix}\underline{\theta}(t)\\ \underline{\mathbf{w}}(t)\end{pmatrix} = \int_0^t \mathscr{M}(t-s)\begin{pmatrix}\underline{\eta}(s)\\ \underline{\mathbf{v}}(s)\end{pmatrix}\mathrm{d}s = \int_0^t \mathrm{e}^{A_0(t-s)}\left((A_{\rm nf}-A_0)\begin{pmatrix}\underline{\theta}(s)\\ \underline{\mathbf{w}}(s)\end{pmatrix}+\begin{pmatrix}\underline{\eta}(s)\\ \underline{\mathbf{v}}(s)\end{pmatrix}\right)\mathrm{d}s.$$

Then by the maximal regularity property of A_0, we obtain

$$\int_0^T \|\partial_{xx}\underline{\mathbf{w}}(t)\|_{X_2}^r \mathrm{d}t \leqslant C\int_0^T\left(\|(A_{\rm nf}-A_0)\begin{pmatrix}\underline{\theta}(s)\\ \underline{\mathbf{w}}(s)\end{pmatrix}\|_{\ell^2\times X_2}+\|\begin{pmatrix}\underline{\eta}(s)\\ \underline{\mathbf{v}}(s)\end{pmatrix}\|_{\ell^2\times X_2}\right)^r\mathrm{d}s.$$

We observe that, for any $\epsilon>0$, there exists $K(\epsilon)>0$ such that

$$\|(A_{\rm nf}-A_0)\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{w}}\end{pmatrix}\|_{\ell^2\times X_2}\leqslant \epsilon\|A_0\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{w}}\end{pmatrix}\|_{\ell^2\times X_2}+K(\epsilon)\|\begin{pmatrix}\underline{\theta}\\ \underline{\mathbf{w}}\end{pmatrix}\|_{\ell^2\times X_2}.$$

In addition, it is straightforward to see that

$$\int_0^T\|\begin{pmatrix}\underline{\theta}(t)\\ \underline{\mathbf{w}}(t)\end{pmatrix}\|_{\ell^2\times X_2}^r\mathrm{d}t\leqslant C\int_0^T\|\begin{pmatrix}\underline{\eta}(t)\\ \underline{\mathbf{v}}(t)\end{pmatrix}\|_{\ell^2\times X_2}^r\mathrm{d}t.$$

The conclusion follows by combing the above three inequalities and taking ϵ sufficiently small. ■

We also prove a corollary which will be useful in the proof of nonlinear stability.

Corollary 3.4.2 *For given $\alpha\in\mathbb{R}$ and $r\in(1,\infty)$, there exists a positive constant C such that, if*

$$\begin{pmatrix}\underline{\theta}(t)\\ \underline{\mathbf{w}}(t)\end{pmatrix}=\int_{t-1}^t\mathscr{M}(t-s)\begin{pmatrix}\underline{\eta}(s)\\ \underline{\mathbf{v}}(s)\end{pmatrix}\mathrm{d}s,\quad t\geqslant 1,$$

then

$$\int_1^\infty(1+t)^\alpha\|\partial_{xx}\underline{\mathbf{w}}(t)\|_{X_2}^r\mathrm{d}t\leqslant C\int_0^\infty(1+t)^\alpha\left(\|\underline{\eta}(t)\|_{\ell^2}+\|\underline{\mathbf{v}}(t)\|_{X_2}\right)^r\mathrm{d}t.$$

Proof. We first note that, for $t\in[n,n+1)$, $n\in\mathbb{N}\backslash\{0\}$,

$$\begin{aligned}\begin{pmatrix}\underline{\theta}(t)\\ \underline{\mathbf{w}}(t)\end{pmatrix}&=\left(\int_{n-1}^t-\int_{n-1}^{t-1}\right)\mathscr{M}(t-s)\begin{pmatrix}\underline{\eta}(s)\\ \underline{\mathbf{v}}(s)\end{pmatrix}\mathrm{d}s\\ &=\left(\int_0^{t-n+1}-\int_0^{t-n}\right)\mathscr{M}(t-n+1-s)\begin{pmatrix}\underline{\eta}(n-1+s)\\ \underline{\mathbf{v}}(n-1+s)\end{pmatrix}\mathrm{d}s.\end{aligned}$$

Applying Lemma 3.4.1 to the above expression, we obtain

$$\int_n^{n+1} \|\partial_{xx}\underline{\mathbf{w}}(t)\|_{X_2}^r \mathrm{d}t \leqslant C\int_{n-1}^{n+1}\left(\|\underline{\eta}(t)\|_{\ell^2} + \|\underline{\mathbf{v}}(t)\|_{X_2}\right)^r \mathrm{d}t.$$

The conclusion follows from multiplying both sides with $n^\alpha \sim (1+t)^\alpha$ and summing over $n \in \mathbb{N}\backslash\{0\}$. ■

Lemma 3.4.3 *If $(\underline{\theta}_0, \underline{\mathbf{W}}_0) \in \ell^1 \times (X_{\mathrm{ch}}^{\perp} \cap \mathscr{T}_{\mathrm{ch}}^{-1}(H^2))$, the solution of the linear system*

$$\begin{pmatrix} \underline{\theta}(t) \\ \underline{\mathbf{W}}(t) \end{pmatrix} = \mathscr{M}(t)\begin{pmatrix} \underline{\theta}_0 \\ \underline{\mathbf{W}}_0 \end{pmatrix}$$

belongs to Z and there exists a positive constant $C_1 > 0$ such that

$$\| \begin{pmatrix} \underline{\theta}(t) \\ \underline{\mathbf{W}}(t) \end{pmatrix} \|_Z \leqslant C_1 \| \begin{pmatrix} \underline{\theta}_0 \\ \underline{\mathbf{W}}_0 \end{pmatrix} \|_{\ell^1 \times (X_{\mathrm{ch}} \cap \mathscr{T}_{\mathrm{ch}}^{-1}(H^2))}. \tag{3.4.1}$$

Proof. By Proposition 3.3.2, it is straightforward to see that

$$\begin{aligned}
\|\mathscr{M}_{00}(t)\underline{\theta}_0\|_{\ell^1} &\leqslant C\|\underline{\theta}_0\|_{\ell^1}, & \|\mathscr{M}_{01}(t)\underline{\mathbf{W}}_0\|_{\ell^1} &\leqslant \frac{C}{(1+t)^{1/2}}\|\underline{\mathbf{W}}_0\|_{X_1}, \\
\|\mathscr{M}_{00}(t)\underline{\theta}_0\|_{\ell^\infty} &\leqslant \frac{C}{(1+t)^{1/2}}\|\underline{\theta}_0\|_{\ell^1}, & \|\mathscr{M}_{01}(t)\underline{\mathbf{W}}_0\|_{\ell^\infty} &\leqslant \frac{C}{1+t}\|\underline{\mathbf{W}}_0\|_{X_1}, \\
\|\delta^2\mathscr{M}_{00}(t)\underline{\theta}_0\|_{\ell^2} &\leqslant \frac{C}{(1+t)^{5/4}}\|\underline{\theta}_0\|_{\ell^1}, & \|\delta^2\mathscr{M}_{01}(t)\underline{\mathbf{W}}_0\|_{\ell^2} &\leqslant \frac{C}{(1+t)^{7/4}}\|\underline{\mathbf{W}}_0\|_{X_1}, \\
\|\mathscr{M}_{10}(t)\underline{\theta}_0\|_{X_1} &\leqslant \frac{C}{(1+t)^{1/2}}\|\underline{\theta}_0\|_{\ell^1}, & \|\mathscr{M}_{11}(t)\underline{\mathbf{W}}_0\|_{X_1} &\leqslant \frac{C}{1+t}\|\underline{\mathbf{W}}_0\|_{X_1}, \\
\|\mathscr{M}_{10}(t)\underline{\theta}_0\|_{X_\infty} &\leqslant \frac{C}{1+t}\|\underline{\theta}_0\|_{\ell^1}, & \|\mathscr{M}_{11}(t)\underline{\mathbf{W}}_0\|_{X_\infty} &\leqslant \frac{C}{1+t}\|\underline{\mathbf{W}}_0\|_{X_\infty}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\|\delta_+\partial_{xx}\underline{\mathbf{W}}(t)\|_{X_2} \leqslant C\|\delta_+ A_0 \mathscr{M}(t)\begin{pmatrix} \underline{\theta}_0 \\ \underline{\mathbf{W}}_0 \end{pmatrix}\|_{\ell^2\times X_2} \\
\leqslant& C(\|\delta_+ A_{\mathrm{nf}}\mathscr{M}(t)\begin{pmatrix} \underline{\theta}_0 \\ \underline{\mathbf{W}}_0 \end{pmatrix}\|_{\ell^2\times X_2} + \|\delta_+(A_{\mathrm{nf}} - A_0)\mathscr{M}(t)\begin{pmatrix} \underline{\theta}_0 \\ \underline{\mathbf{W}}_0 \end{pmatrix}\|_{\ell^2\times X_2}).
\end{aligned}$$

We need to show that the two terms on the right hand side of the above inequality decay sufficiently fast. On the one hand, we claim that

$$\|\delta_+ A_{\mathrm{nf}}\mathscr{M}(t)\begin{pmatrix} \underline{\theta}_0 \\ \underline{\mathbf{W}}_0 \end{pmatrix}\|_{\ell^2\times X_2} \leqslant \frac{C}{(1+t)^{\frac{3}{2}}}\|\begin{pmatrix} \underline{\theta}_0 \\ \underline{\mathbf{W}} \end{pmatrix}\|_{\ell^2\times\mathscr{T}_{\mathrm{ch}}^{-1}(H^2)}, \text{ for all } t \geqslant 0.$$

Actually, for $t \in [0,1]$, the above inequality is true since δ_+ is bounded and

$$\|A_{\mathrm{nf}}\mathscr{M}(t)\begin{pmatrix}\underline{\theta}_0\\ \underline{\mathbf{W}}_0\end{pmatrix}\|_{\ell^2\times X_2} \leqslant C\|A_{\mathrm{nf}}\begin{pmatrix}\underline{\theta}_0\\ \underline{\mathbf{W}}_0\end{pmatrix}\|_{\ell^2\times X_2} \leqslant C\|\begin{pmatrix}\underline{\theta}_0\\ \underline{\mathbf{W}}\end{pmatrix}\|_{\ell^2\times \mathscr{T}_{\mathrm{ch}}^{-1}(H^2)}.$$

For $t \in [1,\infty]$, we first point out that, to show $\|\|A_{\mathrm{nf}}\mathscr{M}(t) = A_{\mathrm{nf}}\mathrm{e}^{A_{\mathrm{nf}}t}\|\|_{\ell^2\times X_2}$ decays with rate t^{-1} as t goes to ∞, we only have to show that the supremum norm of its Fourier-Bloch counterpart $\widehat{A}_{\mathrm{nf}}(\sigma)M(t,\sigma)$ decays with rate t^{-1} as t goes to ∞, just as in the scalar heat equation case. This is true by applying the steps in Lemma 3.2.4 and Lemma 3.2.6 to $\widehat{A}_{\mathrm{nf}}(\sigma)M(t,\sigma)$. Second, it is straightforward to see that the discrete derivative operator δ_+ gives an extra $t^{-1/2}$ decay, which concludes our justification. On the other hand, we have the explicit expression, using that δ_+ and A_{nf}, A_0 commute,

$$\delta_+(A_{\mathrm{nf}}-A_0)\mathscr{M}(t)\begin{pmatrix}\underline{\theta}_0\\ \underline{\mathbf{W}}_0\end{pmatrix} = \begin{pmatrix}0 & \delta_+\Gamma\\ A_{\mathrm{ch}}\mathbf{E}* & \mathbf{f}'(\mathbf{u}_\star)-\mathbf{E}*\delta_+\Gamma\end{pmatrix}\begin{pmatrix}\delta_+\underline{\theta}(t)\\ \delta_+\underline{\mathbf{W}}(t)\end{pmatrix}.$$

We apply Proposition 3.3.2 again and obtain

$$\|A_{\mathrm{ch}}\mathbf{E}*(\delta_+\underline{\theta}(t))\|_{\ell^2} \leqslant C\|\delta^2\underline{\theta}(t)\|_{\ell^2} \leqslant \frac{C}{(1+t)^{\frac{5}{4}}}\left(\|\underline{\theta}_0\|_{\ell^1}+\|\underline{\mathbf{W}}_0\|_{X_2}\right),$$
$$\|\mathbf{f}'(\mathbf{u}_\star)(\delta_+\underline{\mathbf{W}}(t))\|_{X_2} \leqslant C\|\delta_+\underline{\mathbf{W}}(t)\|_{X_2} \leqslant \frac{C}{(1+t)^{\frac{5}{4}}}(\|\underline{\theta}_0\|_{\ell^1}+\|\underline{\mathbf{W}}_0\|_{X_2}).$$

In addition, recalling that Γ is defined in (3.1.20), we conclude that, for any $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that

$$\|\delta_+\Gamma(\delta_+\underline{\mathbf{W}}(t))\|_{X_2} \leqslant C\|\partial_x\delta_+\underline{\mathbf{W}}(t)\|_{X_2} \leqslant \epsilon\|\delta_+\partial_{xx}\underline{\mathbf{W}}(t)\|_{X_2} + K(\epsilon)\|\delta_+\underline{\mathbf{W}}(t)\|_{X_2}.$$

Therefore, by choosing ϵ sufficiently small, we conclude that

$$\|\delta_+\partial_{xx}\underline{\mathbf{W}}(t)\|_{X_2} \leqslant \frac{C}{(1+t)^{\frac{5}{4}}}\left(\|\underline{\theta}_0\|_{\ell^1}+\|\underline{\mathbf{W}}_0\|_{\mathscr{T}_{\mathrm{ch}}^{-1}(H^2)}\right),$$

which shows that

$$\left(\int_0^\infty (1+t)^r\|\delta_+\partial_{xx}\underline{\mathbf{W}}(t)\|_{X_2}^r\mathrm{d}t\right)^{1/r} \leqslant C\left(\|\underline{\theta}_0\|_{\ell^1}+\|\underline{\mathbf{W}}_0\|_{\mathscr{T}_{\mathrm{ch}}^{-1}(H^2)}\right).$$

This proves the lemma. ■

Lemma 3.4.4 *For $\|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z < \varepsilon$, where ε is sufficiently small ($0 < \varepsilon \leqslant \varepsilon_0$), there exists a positive constant $C_2 \geqslant 1$ such that*

$$\| \int_0^t \mathscr{M}(t-s) \begin{pmatrix} \mathbf{N}^\theta(\underline{\theta}(s), \underline{\mathbf{W}}(s)) \\ \mathbf{N}^{\mathbf{w}}(\underline{\theta}(s), \underline{\mathbf{W}}(s)) \end{pmatrix} \mathrm{d}s \|_Z \leqslant C_2 \|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z^2. \tag{3.4.2}$$

Moreover, for $(\underline{\theta}_1, \underline{\mathbf{W}}_1)$, $(\underline{\theta}_2, \underline{\mathbf{W}}_2)$ with their norms in Z smaller than ε, we have

$$\begin{aligned} &\| \int_0^t \mathscr{M}(t-s) \begin{pmatrix} \mathbf{N}^\theta(\underline{\theta}_1(s), \underline{\mathbf{W}}_1(s)) - \mathbf{N}^\theta(\underline{\theta}_2(s), \underline{\mathbf{W}}_2(s)) \\ \mathbf{N}^{\mathbf{w}}(\underline{\theta}_1(s), \underline{\mathbf{W}}_1(s)) - \mathbf{N}^{\mathbf{w}}(\underline{\theta}_2(s), \underline{\mathbf{W}}_2(s)) \end{pmatrix} \mathrm{d}s \|_Z \\ \leqslant & C_2 \left(\sum_{j=1}^2 \|(\underline{\theta}_j(t), \underline{\mathbf{W}}_j(t))\|_Z \right) \|(\underline{\theta}_1(t) - \underline{\theta}_2(t), \underline{\mathbf{W}}_1(t) - \underline{\mathbf{W}}_2(t))\|_Z. \end{aligned} \tag{3.4.3}$$

Proof. We start with proving the estimate (3.4.2). The proof is fairly straightforward. The strategy is to use estimates for the linear part $\mathscr{M}(t)$ in Proposition 3.3.2, the estimates for the nonlinear terms in Lemma B.1.2 from the appendix, and the maximal regularity estimates in Lemma 3.4.1, Corollary 3.4.2. For simplicity, we denote

$$\mathbf{N}^\theta(s) = \mathbf{N}^\theta(\underline{\theta}(s), \underline{\mathbf{W}}(s)), \quad \mathbf{N}^{\mathbf{w}}(s) = \mathbf{N}^{\mathbf{w}}(\underline{\theta}(s), \underline{\mathbf{W}}(s)).$$

By Lemma B.1.2, we have that

$$\begin{aligned} \|\mathbf{N}^\theta(s)\|_{\ell^1} &\leqslant \frac{C}{(1+s)^{\frac{3}{2}}} \|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z^2 + \frac{C}{(1+s)^{\frac{1}{4}}} \|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z \|\delta_+ \partial_{xx} \underline{\mathbf{W}}(s)\|_{X_2}, \\ \|\mathbf{N}^\theta(s)\|_{\ell^2} &\leqslant \frac{C}{(1+s)^{\frac{3}{2}}} \|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z^2 + \frac{C}{(1+s)^{\frac{1}{2}}} \|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z \|\delta_+ \partial_{xx} \underline{\mathbf{W}}(s)\|_{X_2}, \\ \|\mathbf{N}^{\mathbf{w}}(s)\|_{X_1} &\leqslant \frac{C}{1+s} \|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z^2 + \frac{C}{(1+s)^{\frac{1}{4}}} \|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z \|\delta_+ \partial_{xx} \underline{\mathbf{W}}(s)\|_{X_2}, \\ \|\mathbf{N}^{\mathbf{w}}(s)\|_{X_2} &\leqslant \frac{C}{(1+s)^{\frac{5}{4}}} \|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z^2 + \frac{C}{(1+s)^{\frac{1}{2}}} \|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z \|\delta_+ \partial_{xx} \underline{\mathbf{W}}(s)\|_{X_2}. \end{aligned} \tag{3.4.4}$$

We also exploit the estimates from Proposition 3.3.2 and obtain the following estimates.

$$N_1 = \| \int_0^t \mathscr{M}_{00}(t-s) \mathbf{N}^\theta(s) \mathrm{d}s \|_{\ell^1} \leqslant C \int_0^t \|\mathbf{N}^\theta(s)\|_{\ell^1} \mathrm{d}s,$$

$$N_2 = \| \int_0^t \mathscr{M}_{01}(t-s)\mathbf{N}^{\mathbf{w}}(s)\mathrm{d}s\|_{\ell^1} \leqslant C\int_0^t \frac{\|\mathbf{N}^{\mathbf{w}}(s)\|_{X_1}}{(1+t-s)^{\frac{1}{2}}}\mathrm{d}s,$$

$$N_3 = (1+t)^{\frac{1}{2}}\| \int_0^t \mathscr{M}_{00}(t-s)\mathbf{N}^{\theta}(s)\mathrm{d}s\|_{\ell^\infty} \leqslant C(1+t)^{\frac{1}{2}}\int_0^t \frac{\|\mathbf{N}^{\theta}(s)\|_{\ell^1}}{(1+t-s)^{\frac{1}{2}}}\mathrm{d}s,$$

$$N_4 = (1+t)^{\frac{1}{2}}\| \int_0^t \mathscr{M}_{01}(t-s)\mathbf{N}^{\mathbf{w}}(s)\mathrm{d}s\|_{\ell^\infty} \leqslant C(1+t)^{\frac{1}{2}}\int_0^t \frac{\|\mathbf{N}^{\mathbf{w}}(s)\|_{X_1}}{1+t-s}\mathrm{d}s,$$

$$N_5 = (1+t)^{\frac{5}{4}}\|\delta^2 \int_0^t \mathscr{M}_{00}(t-s)\mathbf{N}^{\theta}(s)\mathrm{d}s\|_{\ell^2} \leqslant C(1+t)^{\frac{5}{4}}\int_0^t \frac{\|\mathbf{N}^{\theta}(s)\|_{\ell^1}}{(1+t-s)^{\frac{5}{4}}}\mathrm{d}s,$$

$$N_6 = (1+t)^{\frac{5}{4}}\|\delta^2 \int_0^t \mathscr{M}_{01}(t-s)\mathbf{N}^{\mathbf{w}}(s)\mathrm{d}s\|_{\ell^2} \leqslant C(1+t)^{\frac{5}{4}}\int_0^t \frac{\|\mathbf{N}^{\mathbf{w}}(s)\|_{X_2}}{(1+t-s)^{\frac{3}{2}}}\mathrm{d}s,$$

$$N_7 = (1+t)^{\frac{1}{2}}\| \int_0^t \mathscr{M}_{10}(t-s)\mathbf{N}^{\theta}(s)\mathrm{d}s\|_{X_1} \leqslant C(1+t)^{\frac{1}{2}}\int_0^t \frac{\|\mathbf{N}^{\theta}(s)\|_{\ell^1}}{(1+t-s)^{\frac{1}{2}}}\mathrm{d}s,$$

$$N_8 = (1+t)^{\frac{1}{2}}\| \int_0^t \mathscr{M}_{11}(t-s)\mathbf{N}^{\mathbf{w}}(s)\mathrm{d}s\|_{X_1} \leqslant C(1+t)^{\frac{1}{2}}\int_0^t \frac{\|\mathbf{N}^{\mathbf{w}}(s)\|_{X_1}}{1+t-s}\mathrm{d}s,$$

$$N_9 = (1+t)^{\frac{5}{4}}\|\delta_+ \int_0^t \mathscr{M}_{10}(t-s)\mathbf{N}^{\theta}(s)\mathrm{d}s\|_{X_2} \leqslant C(1+t)^{\frac{5}{4}}\int_0^t \frac{\|\mathbf{N}^{\theta}(s)\|_{\ell^1}}{(1+t-s)^{\frac{5}{4}}}\mathrm{d}s,$$

$$N_{10} = (1+t)^{\frac{5}{4}}\|\delta_+ \int_0^t \mathscr{M}_{11}(t-s)\mathbf{N}^{\mathbf{w}}(s)\mathrm{d}s\|_{X_2} \leqslant C(1+t)^{\frac{5}{4}}\int_0^t \frac{\|\mathbf{N}^{\mathbf{w}}(s)\|_{X_2}}{(1+t-s)^{\frac{3}{2}}}\mathrm{d}s,$$

$$N_{11} = (1+t)\| \int_0^t \mathscr{M}_{10}(t-s)\mathbf{N}^{\theta}(s)\mathrm{d}s\|_{X_\infty} \leqslant C(1+t)\int_0^t \frac{\|\mathbf{N}^{\theta}(s)\|_{\ell^1}}{1+t-s}\mathrm{d}s,$$

$$N_{12} = (1+t)\| \int_0^t \mathscr{M}_{11}(t-s)\mathbf{N}^{\mathbf{w}}(s)\mathrm{d}s\|_{X_\infty} \leqslant C(1+t)\int_0^t \frac{\|\mathbf{N}^{\mathbf{w}}(s)\|_{X_1}}{(1+t-s)(t-s)^{\frac{1}{2}}}\mathrm{d}s. \tag{3.4.5}$$

At this point, we substitute (3.4.4) into (3.4.5), estimate the resulting integrals, and find

$$N_j \leqslant C\|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z^2, \text{ for all } 1 \leqslant j \leqslant 12. \tag{3.4.6}$$

The calculations establishing the estimates for N_1, ..., N_{11} are based on the following elementary integral estimates.

$$\int_0^t \frac{1}{(1+t-s)^\alpha}\frac{1}{(1+s)^\beta}\mathrm{d}s \leqslant \frac{C}{(1+t)^\alpha}\int_0^{\frac{t}{2}}\frac{1}{(1+s)^\beta}\mathrm{d}s + \frac{C}{(1+t)^\beta}\int_0^{\frac{t}{2}}\frac{1}{(1+s)^\alpha}\mathrm{d}s.$$

For the estimate on N_{12}, we just need to show that the following integral expression

$$\begin{aligned} h(t) =&(1+t)\int_0^t \frac{1}{(1+t-s)(t-s)^{\frac{1}{2}}}\frac{1}{1+s}\mathrm{d}s+ \\ &(1+t)\left(\int_0^t \left(\frac{1}{(1+t-s)(t-s)^{\frac{1}{2}}}\frac{1}{(1+s)^{\frac{5}{4}}}\right)^{\frac{r}{r-1}}\mathrm{d}s\right)^{1-\frac{1}{r}} \end{aligned}$$

has a uniform upper bound for $t \in (0,\infty)$. First, for all $t \in (0,1]$, there exists $C > 0$ such that,

$$h(t) \leqslant 2\left(\int_0^1 (t-s)^{-\frac{1}{2}}\mathrm{d}s + \left(\int_0^1 (t-s)^{-\frac{r}{2(r-1)}}\mathrm{d}s\right)^{1-\frac{1}{r}}\right) \leqslant C$$

Second, for $t \in [1,\infty)$, we have

$$\begin{aligned} &(1+t)\int_0^t \frac{1}{(1+t-s)(t-s)^{\frac{1}{2}}}\frac{1}{1+s}\mathrm{d}s \\ \leqslant&\frac{C}{(1+t)^{\frac{1}{2}}}\int_0^{\frac{t}{2}}\frac{1}{1+s}\mathrm{d}s + C\int_{\frac{t}{2}}^t \frac{1}{(1+t-s)(t-s)^{\frac{1}{2}}}\mathrm{d}s \\ \leqslant&C\left(1+\int_0^\infty \frac{1}{(1+s)s^{\frac{1}{2}}}\mathrm{d}s\right) \leqslant C. \end{aligned}$$

Similar arguments show that the second part of $h(t)$ is also uniformly bounded on $[1,\infty)$.

The estimates on N_1, ..., N_{12} bound the Z-norm of the left-hand side of (3.4.2), except for the maximal regularity component. Thus it remains to show that

$$\left(\int_0^\infty (1+t)^r\|\delta_+\partial_{xx}\mathscr{W}(t)\|_{X_2}^r\mathrm{d}t\right)^{\frac{1}{r}} \leqslant C\|(\underline{\theta}(t), \underline{\mathbf{W}}(t))\|_Z^2, \tag{3.4.7}$$

where

$$\mathscr{W}(t)=\int_0^t \mathscr{M}_{10}(t-s)\mathbf{N}^\theta(s)+\mathscr{M}_{11}(t-s)\mathbf{N}^{\mathbf{w}}(s)\mathrm{d}s.$$

For $t\in[0,1]$, by maximal regularity in Lemma 3.4.1, we have

$$\begin{aligned}\int_0^1(1+t)^r\|\delta_+\partial_{xx}\mathscr{W}(t)\|_{X_2}^r\mathrm{d}t\leqslant&C\int_0^1\|\partial_{xx}\mathscr{W}(t)\|_{X_2}^r\mathrm{d}t\\ \leqslant&C\int_0^1\left(\|\mathbf{N}^\theta(t)\|_{\ell^2}+\|\mathbf{N}^{\mathbf{w}}(t)\|_{X_2}\right)^r\mathrm{d}t\\ \leqslant&C\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Z^{2r}.\end{aligned}\tag{3.4.8}$$

For $t\in[1,\infty)$, we split $\mathscr{W}$ into two parts, that is,

$$\mathscr{W}=\left(\int_0^{t-1}+\int_{t-1}^t\right)\mathscr{M}_{10}(t-s)\mathbf{N}^\theta(s)+\mathscr{M}_{11}(t-s)\mathbf{N}^{\mathbf{w}}(s)\mathrm{d}s=\mathscr{W}_1+\mathscr{W}_2.$$

By Corollary 3.4.2, we have

$$\begin{aligned}\int_1^\infty(1+t)^r\|\delta_+\partial_{xx}\mathscr{W}_2(t)\|_{X_2}^r\mathrm{d}t\leqslant&C\int_1^\infty(1+t)^r\|\partial_{xx}\mathscr{W}_2(t)\|_{X_2}^r\mathrm{d}t\\ \leqslant&C\int_0^\infty(1+t)^r\left(\|\mathbf{N}^\theta(t)\|_{\ell^2}+\|\mathbf{N}^{\mathbf{w}}(t)\|_{X_2}\right)^r\mathrm{d}t\\ \leqslant&C\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Z^{2r}.\end{aligned}$$

By similar arguments as in Lemma 3.4.3 and with the condition that $t-s>1$, we can show that

$$\begin{aligned}&\|\delta_+\partial_{xx}\left(\mathscr{M}_{10}(t-s)\mathbf{N}^\theta(s)+\mathscr{M}_{11}(t-s)\mathbf{N}^{\mathbf{w}}(s)\right)\|_{X_2}\\ \leqslant&\frac{C}{(1+t-s)^{\frac54}}\left(\|\mathbf{N}^\theta(s)\|_{\ell^1}+\|\mathbf{N}^{\mathbf{w}}(s)\|_{X_2}\right)\\ \leqslant&\frac{C}{(1+t-s)^{\frac54}(1+s)^{\frac54}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Z\Big(\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Z+(1+s)\|\delta_+\partial_{xx}\underline{\mathbf{W}}(s)\|_{X_2}\Big).\end{aligned}$$

As a result, we obtain

$$\begin{aligned}&(1+t)\|\delta_+\partial_{xx}\mathscr{W}_1(t)\|_{X_2}\\ \leqslant&(1+t)\int_0^{t-1}\|\delta_+\partial_{xx}\left(\mathscr{M}_{10}(t-s)\mathbf{N}^\theta(s)+\mathscr{M}_{11}(t-s)\mathbf{N}^{\mathbf{w}}(s)\right)\|_{X_2}\mathrm{d}s\\ \leqslant&(1+t)\int_0^{t-1}\frac{C}{(1+t-s)^{\frac54}(1+s)^{\frac54}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Z\cdot\\ &(\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Z+(1+s)\|\delta_+\partial_{xx}\underline{\mathbf{W}}(s)\|_{X_2})\,\mathrm{d}s\\ \leqslant&\frac{C}{(1+t)^{\frac14}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^{2r},\end{aligned}$$

which immediately implies that

$$\int_1^{\infty} (1+t)^r \|\delta_+ \partial_{xx} \mathscr{W}_1(t)\|_{X_2}^r \mathrm{d}t \leqslant C \| \left(\underline{\theta}(t), \underline{\mathbf{W}}(t)\right) \|_Z^{2r}.$$

Together with (3.4.8), this establishes (3.4.7) and concludes the proof.

In a completely analogous fashion, one establishes the Lipshitz estimates. ■

We now prove our main theorem.

Proof of Theorem 2. The proof is a fixed-point-theorem argument. We first recall the variation of constant formula,

$$\begin{pmatrix} \underline{\theta}(t) \\ \underline{\mathbf{W}}(t) \end{pmatrix} = \begin{pmatrix} \underline{\theta}_0 \\ \underline{\mathbf{W}}_0 \end{pmatrix} + \int_0^t \mathscr{M}(t-s) \begin{pmatrix} \mathbf{N}^{\theta}(\underline{\theta}(s), \underline{\mathbf{W}}(s)) \\ \mathbf{N}^{\mathbf{w}}(\underline{\theta}(s), \underline{\mathbf{W}}(s)) \end{pmatrix} \mathrm{d}s.$$

Let $\mathscr{P}$ be the right-hand side of the formula, that is

$$\mathscr{P} \begin{pmatrix} \underline{\theta}(t) \\ \underline{\mathbf{W}}(t) \end{pmatrix} = \begin{pmatrix} \underline{\theta}_0 \\ \underline{\mathbf{W}}_0 \end{pmatrix} + \int_0^t \mathscr{M}(t-s) \begin{pmatrix} \mathbf{N}^{\theta}(\underline{\theta}(s), \underline{\mathbf{W}}(s)) \\ \mathbf{N}^{\mathbf{w}}(\underline{\theta}(s), \underline{\mathbf{W}}(s)) \end{pmatrix} \mathrm{d}s.$$

Assume that now the initial value is sufficiently small, that is, for some small $\epsilon > 0$,

$$\|(\underline{\theta}_0, \underline{\mathbf{W}}_0)\|_{\ell^1 \times (Z \cap \mathscr{T}^{-1}(H^2))} \leqslant \epsilon.$$

If $(\underline{\theta}(t), \underline{\mathbf{W}}(t)) \in Z$ with norm smaller than ε, we know that

$$\mathscr{P} \begin{pmatrix} \underline{\theta}(t) \\ \underline{\mathbf{W}}(t) \end{pmatrix} \in Z.$$

By Lemma 3.4.3 and 3.4.4, we have that

$$\|\mathscr{P} \begin{pmatrix} \underline{\theta}(t) \\ \underline{\mathbf{W}}(t) \end{pmatrix} \|_Z \leqslant C_1 \epsilon + C_2 \| \begin{pmatrix} \underline{\theta}(t) \\ \underline{\mathbf{W}}(t) \end{pmatrix} \|_Z^2 \tag{3.4.9}$$

Moreover, we have

$$\begin{aligned} &\|\mathscr{P} \begin{pmatrix} \underline{\theta}_1(t) \\ \underline{\mathbf{W}}_1(t) \end{pmatrix} - \mathscr{P} \begin{pmatrix} \underline{\theta}_2(t) \\ \underline{\mathbf{W}}_2(t) \end{pmatrix} \|_Z \\ \leqslant & C_2 \left(\sum_{j=1}^{2} \|(\underline{\theta}_j(t), \underline{\mathbf{W}}_j(t))\|_Z \right) \|(\underline{\theta}_1(t) - \underline{\theta}_2(t), \underline{\mathbf{W}}_1(t) - \underline{\mathbf{W}}_2(t))\|_Z. \end{aligned} \tag{3.4.10}$$

We denote $B = \{\underline{\mathbf{V}} \in Z \mid \|\underline{\mathbf{V}}\|_Z \leqslant R\}$, where $R = \min(2C_1\epsilon, \varepsilon)$. We now take $\epsilon > 0$ small enough so that $2C_2R < 1$ and readily conclude, based on (3.4.9) and (3.4.10), that $\mathscr{P}(B) \subset B$ and that $\mathscr{P}$ is a strict contraction in B. By Banach's fixed point theorem, there is a unique fixed point of $\mathscr{P}$ in B, denoted as $(\underline{\theta}(t), \underline{\mathbf{W}}(t))$. Then $(\underline{\theta}(t), \underline{\mathbf{W}}(t))$ is a global solution of (3.1.22), and if we return to the original variables, we obtain a global solution of (1.3.2) which satisfies the decay estimate in Theorem 2. This concludes the proof. ■

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Appendix A

Grain boundary

Lemma A.0.5 *There exists a positive constant $N \geqslant 1$ such that*

$$\begin{aligned}
\|\partial_x(\widetilde{\mathcal{S}}(\mu,\delta))^{-1}f\|_{L^2_\eta} &\leqslant \frac{N}{\mu^{1/2}}\|f\|_{L^2_\eta}, \\
\|\partial_x^2(\widetilde{\mathcal{S}}(\mu,\delta))^{-1}f\|_{L^2_\eta} &\leqslant \frac{N}{\mu}\|f\|_{L^2_\eta}, \\
\|\partial_x^3(\widetilde{\mathcal{S}}(\mu,\delta))^{-1}f\|_{L^2_\eta} &\leqslant \frac{N}{\mu^{3/2}}\|f\|_{L^2_\eta}.
\end{aligned} \tag{A.0.1}$$

Proof. We define

$$\begin{aligned}
\mathcal{J} : L^2_\eta &\longrightarrow H^4_\eta \\
V &\longmapsto \int_{\mathbb{R}} G(x-y, g(x-y))V(y)dy,
\end{aligned}$$

where $G(x,a) = \int_{\mathbb{R}} \frac{e^{-itx}}{1/\mu(\mu t^2 - k^2_{x,\ell_*,\delta})^2 + a} dt, \quad g(x) = 6(C_+^{*2} + C_-^{*2}) - 1.$

By letting $b = \frac{k^2_{x,\ell_*,\delta}}{\mu^{1/2}}$, we have that

$$G(x,a) = \begin{cases} \frac{\pi}{2}\frac{1}{\mu^{1/4}a^{1/2}(b^2+a)^{1/4}} e^{(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x} \cdot \\ \Big((\frac{\sqrt{b^2+a}+b}{2\sqrt{b^2+a}})^{1/2} \cos((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x) - \\ (\frac{\sqrt{b^2+a}-b}{2\sqrt{b^2+a}})^{1/2} \sin((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)\Big), & x \leqslant 0, \\ \frac{\pi}{2}\frac{1}{\mu^{1/4}a^{1/2}(b^2+a)^{1/4}} e^{-(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x} \cdot \\ \Big((\frac{\sqrt{b^2+a}+b}{2\sqrt{b^2+a}})^{1/2} \cos((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x) + \\ (\frac{\sqrt{b^2+a}-b}{2\sqrt{b^2+a}})^{1/2} \sin((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)\Big), & x \geqslant 0, \end{cases}$$

$$G^{(1,0)}(x,a)=\begin{cases} -\frac{\pi}{2}\frac{1}{\mu^{1/2}a^{1/2}}e^{(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x}\cdot\\ \sin((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x), \quad x\leqslant 0,\\ -\frac{\pi}{2}\frac{1}{\mu^{1/2}a^{1/2}}e^{-(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x}\cdot\\ \sin((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x), \quad x\geqslant 0,\end{cases}$$

$$G^{(2,0)}(x,a)=\begin{cases} -\frac{\pi}{2}\frac{(b^2+a)^{1/4}}{\mu^{3/4}a^{1/2}}e^{(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x}\cdot\\ \Big((\frac{\sqrt{b^2+a}+b}{2\sqrt{b^2+a}})^{1/2}\cos((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)+\\ (\frac{\sqrt{b^2+a}-b}{2\sqrt{b^2+a}})^{1/2}\sin((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)\Big), \quad x\leqslant 0,\\ -\frac{\pi}{2}\frac{(b^2+a)^{1/4}}{\mu^{3/4}a^{1/2}}e^{-(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x}\cdot\\ \Big((\frac{\sqrt{b^2+a}+b}{2\sqrt{b^2+a}})^{1/2}\cos((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)-\\ (\frac{\sqrt{b^2+a}-b}{2\sqrt{b^2+a}})^{1/2}\sin((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)\Big), \quad x\geqslant 0,\end{cases} \tag{A.0.2}$$

$$G^{(3,0)}(x,a)=\begin{cases} -\frac{\pi}{2}\frac{(b^2+a)^{1/2}}{\mu a^{1/2}}e^{(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x}\cdot\\ \Big(\frac{\sqrt{a}}{\sqrt{b^2+a}}\cos((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)-\\ \frac{b}{\sqrt{b^2+a}}\sin((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)\Big), \quad x\leqslant 0,\\ \frac{\pi}{2}\frac{(b^2+a)^{1/2}}{\mu a^{1/2}}e^{-(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x}\cdot\\ \Big(\frac{\sqrt{a}}{\sqrt{b^2+a}}\cos((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)+\\ \frac{b}{\sqrt{b^2+a}}\sin((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x)\Big), \quad x>0,\end{cases}$$

It is then not hard to see from (A.0.2) that there exists $C_1\geqslant 1$ such that

$$\begin{aligned} \|G^{(0,j)}(x,g(x))\|_{L^1}&\leqslant C_1,\ \text{ for } j=0,1,2,3,4,\\ \|G^{(1,j)}(x,g(x))\|_{L^1}&\leqslant \frac{C_1}{\mu^{1/2}}(\mu+k^4_{x,\ell_*,\delta})^{1/4},\ \text{ for } j=0,1,2,3,\\ \|G^{(2,j)}(x,g(x))\|_{L^1}&\leqslant \frac{C_1}{\mu}(\mu+k^4_{x,\ell_*,\delta})^{1/2},\ \text{ for } j=0,1,2,\\ \|G^{(3,j)}(x,g(x))\|_{L^1}&\leqslant \frac{C_1}{\mu^{3/2}}(\mu+k^4_{x,\ell_*,\delta})^{3/4},\ \text{ for } j=0,1. \end{aligned} \tag{A.0.3}$$

Moreover, from estimates (A.0.3) and by Young's inequality, for $\eta>0$ sufficiently small, there exists a positive constant $C_2\geqslant 1$ such that

$$\|\partial_x^j\mathcal{J}(\mu,\delta)f\|_{L^2_\eta}\leqslant \frac{C_2}{\mu^{j/2}}\|f\|_{L^2_\eta}, \quad j=1,2,3. \tag{A.0.4}$$

To prove the lemma, we just need to show that $\widetilde{\mathcal{S}}\mathcal{J} : L^2_\eta \longrightarrow L^2_\eta$ is uniformly bounded and invertible for sufficiently small μ and δ. In fact,

$$\widetilde{\mathcal{S}}\mathcal{J} = \text{ id } + \mathcal{K}_1 + \mathcal{K}_2,$$

where

$$\begin{aligned}
(\mathcal{K}_1 V)(x) =& 2k^2_{x,\ell_*,\delta}\int_{\mathbb{R}} \Big(g''(x-y)G^{(0,1)}(x-y,g(x-y))+ \\
& g'(x-y)^2G^{(0,2)}(x-y,g(x-y))\Big)V(y)dy+ \\
& \mu\int_{\mathbb{R}}(g^{(4)}(x-y)G^{(0,1)}(x-y,g(x-y))+ \\
& 4g^{(3)}(x-y)G^{(1,1)}(x-y,g(x-y))+ \\
& 3g''(x-y)^2G^{(0,2)}(x-y,g(x-y))+g'(x-y)^4G^{(0,4)}(x-y,g(x-y))+ \\
& 4g'(x-y)^3G^{(1,3)}(x-y,g(x-y))+6g'(x-y)^2G^{(2,2)}(x-y,g(x-y))+ \\
& 4g^{(3)}(x-y)g'(x-y)G^{(0,2)}(x-y,g(x-y))+ \\
& 6g''(x-y)(g'(x-y)^2G^{(0,3)}(x-y,g(x-y))+ \\
& 2g'(x-y)G^{(1,2)}(x-y,g(x-y))+G^{(2,1)}(x-y,g(x-y))))V(y)dy. \\
(\mathcal{K}_2 V)(x) =& \int_{\mathbb{R}}(4g'(x-y)(k^2_{x,\ell_*,\delta}G^{(1,1)}+\mu G^{(3,1)})(x-y,g(x-y)))V(y)dy.
\end{aligned}$$

By Young's inequality and estimates (A.0.3), it is straightforward to conclude that the norm of $\mathcal{K}_1$ as an operator from L^2_η to L^2_η goes to 0 uniformly as $\mu > 0$ and δ go to zero. On the other hand, we denote $H(x,a) = k^2_{x,\ell_*,\delta}G^{(1,0)}(x,a) + \mu G^{(3,0)}(x,a)$ and have

$$H(x,a) = \begin{cases} -\frac{\pi}{2}e^{(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x}\cos((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x), & x \leqslant 0, \\ \frac{\pi}{2}e^{-(\frac{\sqrt{b^2+a}-b}{2})^{1/2}\mu^{-1/4}x}\cos((\frac{\sqrt{b^2+a}+b}{2})^{1/2}\mu^{-1/4}x), & x > 0, \end{cases}$$

Moreover, it is not hard to see that there exists some positive constant $C_3 \geqslant 1$ such that

$$\|H^{(0,1)}(x,a)\|_{L^1} \leqslant C_3(\mu + k^4_{x,\ell_*,\delta})^{1/4},$$

which, combined with Young's inequality, also shows that the norm of $\mathcal{K}_2$ as an operator from L^2_η to L^2_η goes to 0 uniformly as $\mu > 0$ and δ go to zero.

Thus, for sufficiently small μ and δ, we have $\widetilde{\mathcal{S}}^{-1} = \mathcal{J}(\widetilde{\mathcal{S}}\mathcal{J})^{-1}$, which, combined with estimate (A.0.4) and the fact that $(\widetilde{\mathcal{S}}\mathcal{J})^{-1}$ is uniformly bounded, shows that the lemma holds. ∎

Appendix B

Turing pattern

B.1 Estimates on nonlinear terms

In this section, we derive the estimates on the nonlinear terms $\mathbf{N}^\theta$ and $\mathbf{N}^\mathbf{w}$ in our normal form (3.1.21).

Lemma B.1.1 *For $\|\underline{\mathbf{W}}\|_{X_{\mathrm{ch}}}, \|\underline{\theta}\|_{\ell^1} < \varepsilon$, where ε is sufficiently small($0 < \varepsilon \leqslant \varepsilon_0$), there exists a nondecreasing function $C(\varepsilon) > 0$ such that, for all $1 \leqslant p \leqslant \infty$, the nonlinear terms in system (3.1.21) have the following estimates.*

$$
\begin{aligned}
|\mathbf{N}_j^\theta| \leqslant & C(\varepsilon)\Bigg[\sum_{k=-1}^{0} |(\delta_+\underline{\theta})_{j+k}|^2 + \left(\sum_{k=-1}^{1} |\theta_{j+k}|^3\right)\left(\sum_{k=-1}^{0} |(\delta_+\underline{\theta})_{j+k}|\right) + \\
& \left(\sum_{k=-1}^{1} |\theta_{j+k}|\right)\Big(|(\delta_+\underline{\mathbf{W}})_j(-\pi)| + |(\delta_+\partial_x\underline{\mathbf{W}})_j(-\pi)|\Big) + \\
& \|\mathbf{W}_j\|_{L^p}|(\delta_+\underline{\mathbf{W}})_j(-\pi)| + \|\mathbf{W}_j^2\|_{L^p}\Bigg], \\
\|\mathbf{N}_j^\mathbf{w}\|_{L^p} \leqslant & C(\varepsilon)\Bigg[\left(\sum_{k=-1}^{0} |(\delta_+\underline{\theta})_{j+k}|\right)\left(\sum_{k=-1}^{1} |\theta_{j+k}|\right) + |\theta_j|\|\mathbf{W}_j\|_{L^p} + \\
& \left(\sum_{k=-1}^{1} |\theta_{j+k}|\right)\left(\sum_{k=-1}^{1} |(\delta_+\underline{\mathbf{W}})_{j+k}(-\pi)| + |(\delta_+\partial_x\underline{\mathbf{W}})_{j+k}(-\pi)|\right) + \\
& \|\mathbf{W}_j\|_{L^p}|(\delta_+\underline{\mathbf{W}})_j(-\pi)| + \|\mathbf{W}_j^2\|_{L^p} + |\mathbf{N}_j^\theta| + |\mathbf{N}_{j+1}^\theta| + |\mathbf{N}_{j-1}^\theta|\Bigg].
\end{aligned}
\tag{B.1.1}
$$

Proof. We point out that throughout the proof, we repeatedly exploit the fact that the L^2 scalar product of an even function and an odd function are zero. We also recall that $\mathbf{u}_\star$ is even and $\mathbf{u}_{\rm ad}$ is odd. By equations (3.1.9),(3.1.10) and (3.1.21), we obtain

$$
\begin{aligned}
\mathbf{N}_j^\theta =& I_j + (II_j + III_j + IV_j + V_j)\,\mathscr{S}_j \text{ and}\\
\mathbf{N}_j^{\mathbf{w}} =& \left(\mathrm{id} - \frac{\partial \mathbf{G}_j}{\partial \mathbf{W}_j}\right)^{-1}\left(VI_j + VII_j + VIII_j + IX_j + \frac{\partial \mathbf{G}_j}{\partial \mathbf{W}_j} X_j\right), \text{ where}\\
\mathscr{S}_j =& \left(-1 + \langle \mathbf{W}_j(x) + \mathbf{H}_j(x), \mathbf{u}_{\rm ad}'(x-\theta_j)\rangle\right)^{-1};\\
\mathbf{G}_j =& \mathbf{G}(\theta_j, \mathbf{W}_j) = \langle \mathbf{W}_j(x), \mathbf{u}_{\rm ad}(x-\theta_j) - \mathbf{u}_{\rm ad}(x)\rangle \psi(x-\theta_j);\\
I_j =& (-\mathscr{S}_j - 1)(\delta_+ \Gamma W)_j;\\
II_j =& -\left(\mathbf{W}_{j+1}(-\pi) - \mathbf{W}_j(-\pi), D\left(\mathbf{u}_{\rm ad}'(\pi-\theta_j) - \mathbf{u}_{\rm ad}'(\pi)\right)\right);\\
III_j =& (\partial_x \mathbf{W}_{j+1}(-\pi) - \partial_x \mathbf{W}_j(-\pi), D\mathbf{u}_{\rm ad}(\pi-\theta_j));\\
IV_j =& (\partial_x \mathbf{H}_j(\pi) - \partial_x \mathbf{H}_j(-\pi), D\mathbf{u}_{\rm ad}(\pi-\theta_j)) - (\mathbf{H}_j(\pi) - \mathbf{H}_j(-\pi), D\mathbf{u}_{\rm ad}'(\pi-\theta_j));\\
V_j =& \langle \tilde g(\theta_j, \mathbf{W}_j + \mathbf{H}_j), \mathbf{u}_{\rm ad}(x-\theta_j)\rangle;\\
VI_j =& A(\mathbf{H}_j - (\underline{\mathbf{E}} * \underline{\theta})_j);\\
VII_j =& -\dot{\mathbf{H}}_j + \mathbf{u}_\star'(x-\theta_j)\dot\theta_j + (\underline{\mathbf{E}} * \underline{\dot\theta})_j - \langle \dot{\mathbf{W}}_j(x), \mathbf{u}_{\rm ad}(x-\theta_j) - \mathbf{u}_{\rm ad}(x)\rangle \psi(x-\theta_j);\\
VIII_j =& (\underline{\mathbf{E}} * (\delta_+ \Gamma \underline{\mathbf{W}} - \underline{\dot\theta}))_j;\\
IX_j =& \tilde g(\theta_j, \mathbf{W}_j + \mathbf{H}_j) + \left[\mathbf{f}'(\mathbf{u}_\star(x-\theta_j)) - \mathbf{f}'(\mathbf{u}_\star(x))\right](\mathbf{W}_j + \mathbf{H}_j);\\
X_j =& A(\underline{\mathbf{E}} * \underline{\theta})_j + A\mathbf{W}_j - (\underline{\mathbf{E}} * \delta_+ \Gamma \underline{\mathbf{W}})_j.
\end{aligned}
$$

We recall here that $\underline{\mathbf{E}}$ is defined in (3.1.18) and point out that the term in VII_j involving $\dot{\mathbf{W}}_j$ in fact cancels with a contribution from $\dot{\mathbf{H}}_j$. We now prove the estimate of $\mathbf{N}_j^\theta$.

Estimate on I_j: $|I_j| \leqslant C(\varepsilon)\Big(|(\delta_+\underline{\theta})_j| + |(\delta_-\underline{\theta})_j| + \|\mathbf{W}_j\|_{L^p}\Big)|(\delta_+\underline{\mathbf{W}})_j(-\pi)|$.

We first recall that $\mathbf{H}_j$ is defined in (3.1.10) and (3.1.13). We claim that the number c_j, appearing in the definition of $\mathbf{H}_j^2$ as in (3.1.15) and (3.1.16), can be estimated as

$$
|c_j| \leqslant C(\varepsilon)\left[|(\delta^2\underline{\theta})_j| + \Big(|(\delta_+\underline{\theta})_j| + |(\delta_-\underline{\theta})_j|\Big)\sum_{k=-1}^{1}\theta_{j+k} + |\theta_j|\|\mathbf{W}_j\|_{L^p}\right],
$$

where we use notation $\delta^2 = \delta_+\delta_-$. In fact, we have

$$
\begin{aligned}
&|\langle (\mathbf{u}_\star(x+\theta_j-\theta_{j+1}) + \mathbf{u}_\star(x+\theta_j-\theta_{j-1})), \mathbf{u}_{\rm ad}(x)\rangle| \leqslant C|(\delta^2\underline{\theta})_j|,\\
&|\langle \phi(x)(\mathbf{u}_\star(x+\theta_j-\theta_{j+1}) - \mathbf{u}_\star(x+\theta_j-\theta_{j-1})), \mathbf{u}_{\rm ad}(x)\rangle| \leqslant C(|(\delta_+\underline{\theta})_j|^2 + |(\delta_-\underline{\theta})_j|^2),
\end{aligned}
$$

$$\begin{aligned}&|\langle(\phi(x+\theta_j)-\phi(x))(\mathbf{u}_\star(x+\theta_j-\theta_{j+1})-\mathbf{u}_\star(x+\theta_j-\theta_{j-1})),\mathbf{u}_{\mathrm{ad}}(x)\rangle|\\ \leqslant& C|\theta_j|(|(\delta_+\underline{\theta})_j|+|(\delta_-\underline{\theta})_j|).\end{aligned}$$

We also have $|\mathbf{H}_j(x)|\leqslant C(\varepsilon)\left(|(\delta_+\underline{\theta})_j|+|(\delta_-\underline{\theta})_j|+|\theta_j|\|\mathbf{W}_j\|_{L^p}\right)$, from which we obtain the estimate.

Estimate on II_j: $|II_j|\leqslant C|\theta_j|^2|(\delta_+\underline{\mathbf{W}})_j(-\pi)|$.

This is straightforward.

Estimate on III_j: $|III_j|\leqslant C|\theta_j||(\delta_+\partial_x\underline{\mathbf{W}})_j(-\pi)|$.

This is straightforward.

Estimate on IV_j:

$$|IV_j|\leqslant C\left[|(\delta_+\underline{\theta})_j|^2+|(\delta_-\underline{\theta})_j|^2+|(\delta_+\underline{\theta})_j+(\delta_-\underline{\theta})_j|\left(|\theta_{j+1}|^3+|\theta_j|^3+|\theta_{j-1}|^3\right)\right].$$

We first simplify IV_j and obtain

$$\begin{aligned}IV_j=&\frac{1}{2}(\mathbf{u}_\star'(\pi-\theta_{j+1})-\mathbf{u}_\star'(\pi-\theta_{j-1}),D\mathbf{u}_{\mathrm{ad}}(\pi-\theta_j))-\\ &\frac{1}{2}(\mathbf{u}_\star(\pi-\theta_{j+1})-\mathbf{u}_\star(\pi-\theta_{j-1}),D\mathbf{u}_{\mathrm{ad}}'(\pi-\theta_j)).\end{aligned}$$

Then, it is not hard to see that

$$\begin{aligned}&\left|\frac{1}{2}\left(\mathbf{u}_\star'(\pi-\theta_{j+1})-\mathbf{u}_\star'(\pi-\theta_{j-1}),D\mathbf{u}_{\mathrm{ad}}(\pi-\theta_j)\right)-\right.\\ &\left.\frac{1}{2}\left(\mathbf{u}_{\star,\theta\theta}(\pi)(\theta_{j-1}-\theta_{j+1}),-D\mathbf{u}_{\mathrm{ad}}'(\pi)\theta_j\right)\right|\\ \leqslant& C\left(|\theta_j||\theta_{j+1}^3-\theta_{j-1}^3|+|\theta_j|^3|\theta_{j+1}-\theta_{j-1}|\right),\\ &\left|\frac{1}{2}\left(\mathbf{u}_\star(\pi-\theta_{j+1})-\mathbf{u}_\star(\pi-\theta_{j-1}),D\mathbf{u}_{\mathrm{ad}}'(\pi-\theta_j)\right)-\right.\\ &\left.\frac{1}{2}\left(\frac{1}{2}\mathbf{u}_{\star,\theta\theta}(\pi)(\theta_{j+1}^2-\theta_{j-1}^2),D\mathbf{u}_{\mathrm{ad}}'(\pi)\right)\right|\\ \leqslant& C\left(|\theta_{j+1}^4-\theta_{j-1}^4|+|\theta_j|^2|\theta_{j+1}^2-\theta_{j-1}^2|\right),\\ &\frac{1}{2}\left(\mathbf{u}_{\star,\theta\theta}(\pi)(\theta_{j-1}-\theta_{j+1}),-D\mathbf{u}_{\mathrm{ad}}'(\pi)\theta_j\right)-\frac{1}{2}\left(\frac{1}{2}\mathbf{u}_{\star,\theta\theta}(\pi)(\theta_{j+1}^2-\theta_{j-1}^2),D\mathbf{u}_{\mathrm{ad}}'(\pi)\right)\\ =&\frac{1}{4}\left(\mathbf{u}_{\star,\theta\theta}(\pi),D\mathbf{u}_{\mathrm{ad}}'(\pi)\right)\left[(\delta_-\underline{\theta})_j^2-(\delta_+\underline{\theta})_j^2\right],\end{aligned}$$

which establishes the estimate on IV_j as claimed.

Estimate on V_j: $|V_j| \leqslant C(\varepsilon)\left(|(\delta_+\underline{\theta})_j|^2 + |(\delta_-\underline{\theta})_j|^2 + \|\mathbf{W}_j^2\|_{L^p}\right)$.

Noting that $|V_j| \leqslant C(\varepsilon)\|(\mathbf{W}_j + \mathbf{H}_j)^2\|_{L^p}$ and applying the estimate of $\mathbf{H}_j$ into the inequality lead to the above estimate.

Estimate on $\mathscr{S}_j$: $|\mathscr{S}_j| \leqslant C(\varepsilon)$.

This is straightforward.

Combining our estimates of $I_j - V_j$ and $\mathscr{S}_j$, we obtain the first inequality in (B.1.1).

Now, we have to show that the estimate of $\mathbf{N}_j^{\mathbf{w}}$ in (B.1.1) is true.

Estimate on VI_j:

$$|VI_j| \leqslant C(\varepsilon)\left[\left(|(\delta_+\underline{\theta})_j| + |(\delta_-\underline{\theta})_j|\right)\sum_{k=-1}^{1}|\theta_{j+k}| + |\theta_j|\|\mathbf{W}_j\|_{L^p}\right].$$

First, for f 2π-periodic and smooth, we have

$$|f(x-\theta_1) - f(x-\theta_2) - f'(x)(\theta_2-\theta_1)| \leqslant C(|\theta_2-\theta_1|^2 + |\theta_2||\theta_2-\theta_1|).$$

If in addition, f is odd, we have

$$|f(\theta_1) - f(\theta_2) - f'(0)(\theta_1-\theta_2)| \leqslant C|\theta_2^3 - \theta_1^3|.$$

The latter implies that

$$|c_j - \frac{1}{4}(\delta^2\underline{\theta})_j| \leqslant C(\varepsilon)\left(|(\delta_+\underline{\theta})_j|^2 + |(\delta_-\underline{\theta})_j|^2 + |\theta_j|\|\mathbf{W}_j\|_{L^p}\right).$$

Moreover, by the former inequality, we have

$$\begin{aligned}|VI_j| \leqslant & C\left(|(\delta_+\underline{\theta})_j|^2 + |(\delta_-\underline{\theta})_j|^2 + |\theta_j||(\delta_+\underline{\theta})_j| + |\theta_j||(\delta_-\underline{\theta})_j|\right) + \\ & \left|c_j A\psi(x-\theta_j) - \frac{1}{4}(\delta^2\underline{\theta})_j A\psi(x)\right| \\ \leqslant & C(\varepsilon)\left[\left(|(\delta_+\underline{\theta})_j| + |(\delta_-\underline{\theta})_j|\right)\sum_{k=-1}^{1}|\theta_{j+k}| + |\theta_j|\|\mathbf{W}_j\|_{L^p}\right].\end{aligned}$$

Estimate on VII_j:

$$|VII_j| \leqslant C\left[\left(\sum_{k=-1}^{1}|\theta_{j+k}|\right)\left(\sum_{k=-1}^{1}|\dot{\theta}_{j+k}|\right) + |\dot{\theta}_j|\|\mathbf{W}_j\|_{L^p}\right]$$

Noting that $(\underline{\mathbf{E}} * \underline{\theta})_j$ is the linear part of $\mathbf{H}_j + \mathbf{u}_\star(x-\theta_j) - \mathbf{u}_\star(x)$ and there is no term invovling $\dot{\mathbf{W}}_j$ in VII_j, we have

$$\begin{aligned}|VII_j| \leqslant & C\Big(|\theta_{j+1}||\dot{\theta}_{j+1}| + |\theta_{j-1}||\dot{\theta}_{j-1}| + |\theta_j||\dot{\theta}_j|\Big) + \\ & \Big|c_j\psi'(x-\theta_j)\dot{\theta}_j\Big| + \left|\frac{1}{4}(\delta^2\underline{\dot{\theta}})_j\psi(x) - \tilde{\dot{c}}_j\psi(x-\theta_j)\right|,\end{aligned}$$

where $\tilde{\dot{c}}_j = \dot{c}_j + \langle \dot{\mathbf{W}}_j(x), \mathbf{u}_{\mathrm{ad}}(x-\theta_j) - \mathbf{u}_{\mathrm{ad}}(x)\rangle$.
First, we note that

$$\Big|c_j\psi'(x-\theta_j)\dot{\theta}_j\Big| \leqslant C|\dot{\theta}_j|\left[|(\delta^2\underline{\theta})_j| + \Big(|(\delta_+\underline{\theta})_j| + |(\delta_-\underline{\theta})_j|\Big)\sum_{k=-1}^{1}\theta_{j+k} + |\theta_j|\|\mathbf{W}_j\|_{L^p}\right].$$

Moreover, we claim that

$$|\tilde{\dot{c}}_j| \leqslant C\left[|(\delta_+\underline{\dot{\theta}})_j| + |(\delta_-\underline{\dot{\theta}})_j| + \left(\sum_{k=-1}^{1}|\theta_{j+k}|\right)\left(\sum_{k=-1}^{1}|\dot{\theta}_{j+k}|\right) + |\dot{\theta}_j|\|\mathbf{W}_j\|_{L^p}\right],$$

$$|\tilde{\dot{c}}_j - \frac{1}{4}(\delta^2\underline{\dot{\theta}})_j| \leqslant C\left[\left(\sum_{k=-1}^{1}|\theta_{j+k}|\right)\left(\sum_{k=-1}^{1}|\dot{\theta}_{j+k}|\right) + |\dot{\theta}_j|\|\mathbf{W}_j\|_{L^p}\right].$$

In fact, we have

$$\begin{aligned}
&|\langle \phi(x)(\dot{\mathbf{u}}_\star(x+\theta_j-\theta_{j+1}) - \dot{\mathbf{u}}_\star(x+\theta_j-\theta_{j-1})), \mathbf{u}_{\mathrm{ad}}(x)\rangle| \\
\leqslant & C\Big(|(\delta_+\underline{\theta})_j||(\delta_+\underline{\dot{\theta}})_j| + |(\delta_-\underline{\theta})_j||(\delta_-\underline{\dot{\theta}})_j|\Big), \\
&|\langle \phi'(x+\theta_j)\dot{\theta}_j(\mathbf{u}_\star(x+\theta_j-\theta_{j+1}) - \mathbf{u}_\star(x+\theta_j-\theta_{j-1})), \mathbf{u}_{\mathrm{ad}}(x)\rangle| \\
\leqslant & C|\dot{\theta}_j|\Big(|(\delta_+\underline{\theta})_j| + |(\delta_-\underline{\theta})_j|\Big), \\
&|\langle (\phi(x+\theta_j) - \phi(x))(\dot{\mathbf{u}}_\star(x+\theta_j-\theta_{j+1}) - \dot{\mathbf{u}}_\star(x+\theta_j-\theta_{j-1})), \mathbf{u}_{\mathrm{ad}}(x)\rangle| \\
\leqslant & C|\theta_j|\Big(|(\delta_+\underline{\dot{\theta}})_j| + |(\delta_-\underline{\dot{\theta}})_j|\Big), \\
&|\langle (\dot{\mathbf{u}}_\star(x+\theta_j-\theta_{j+1}) + \dot{\mathbf{u}}_\star(x+\theta_j-\theta_{j-1})), \mathbf{u}_{\mathrm{ad}}(x)\rangle| \\
\leqslant & C\Big(|(\delta_+\underline{\dot{\theta}})_j| + |(\delta_-\underline{\dot{\theta}})_j|\Big), \\
&|\langle (\dot{\mathbf{u}}_\star(x+\theta_j-\theta_{j+1}) + \dot{\mathbf{u}}_\star(x+\theta_j-\theta_{j-1})), \mathbf{u}_{\mathrm{ad}}(x)\rangle + \delta^2\underline{\dot{\theta}}_j| \\
\leqslant & C\Big(|(\delta_+\underline{\theta})_j||(\delta_+\underline{\dot{\theta}})_j| + |(\delta_-\underline{\theta})_j||(\delta_-\underline{\dot{\theta}})_j|\Big),
\end{aligned}$$

which establishes the claim and thus the estimate on VII_j.

Estimate on $VIII_j$:

$$|VIII_j| \leqslant C\Big(|\mathbf{N}^\theta_j| + |\mathbf{N}^\theta_{j+1}| + |\mathbf{N}^\theta_{j-1}|\Big)$$

The calculation is straightforward using the expressions for $\mathbf{K}_j$ and $\dot{\theta}_j$.

Estimate on IX_j:

$$|IX_j| \leqslant C(\varepsilon)\left[\left(\sum_{k=0}^{1}|(\delta_-\underline{\theta})_{j+k}|\right)\left(\sum_{k=-1}^{1}|\theta_{j+k}\right) + +|\theta_j||\mathbf{W}_j| + |\theta_j|^2\|\mathbf{W}_j\|_{L^p} + |\mathbf{W}_j|^2\right]$$

The calculation is straightforward using the estimate on $\mathbf{H}_j$.

Estimate on $\frac{\partial \mathbf{G}_j}{\partial \mathbf{W}_j}X_j$:

$$\begin{aligned}|\frac{\partial \mathbf{G}_j}{\partial \mathbf{W}_j}X_j| \leqslant & C(\varepsilon)\left(|\theta_j|\sum_{k=-1}^{1}\Big(|(\delta_+\underline{\theta})_{j+k}| + |(\delta_+\underline{\mathbf{W}})_{j+k}(-\pi)|\Big)+\right.\\ & \left.|\langle A\mathbf{W}_j(x), \mathbf{u}_{\mathrm{ad}}(x-\theta_j) - \mathbf{u}_{\mathrm{ad}}(x)\rangle|\right).\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}&\langle A\mathbf{W}_j(x), \mathbf{u}_{\mathrm{ad}}(x-\theta_j) - \mathbf{u}_{\mathrm{ad}}(x)\rangle\\ =&\langle \mathbf{W}_j(x), A^*\left(\mathbf{u}_{\mathrm{ad}}(x-\theta_j) - \mathbf{u}_{\mathrm{ad}}(x)\right)\rangle +\\ &(\partial_x\mathbf{W}_{j+1}(-\pi) - \partial_x\mathbf{W}_j(-\pi), D(\mathbf{u}_{\mathrm{ad}}(\pi-\theta_j) - \mathbf{u}_{\mathrm{ad}}(\pi))) -\\ &(\mathbf{W}_{j+1}(-\pi) - \mathbf{W}_j(-\pi), D(\mathbf{u}'_{\mathrm{ad}}(\pi-\theta_j) - \mathbf{u}'_{\mathrm{ad}}(\pi))).\end{aligned}$$

Thereofore, we have

$$\begin{aligned}|\frac{\partial \mathbf{G}_j}{\partial \mathbf{W}_j}X_j| \leqslant & C(\varepsilon)|\theta_j|\Bigg(\sum_{k=-1}^{1}\Big(|(\delta_+\underline{\theta})_{j+k}| + |(\delta_+\underline{\mathbf{W}})_{j+k}(-\pi)|\Big)+\\ & \|\mathbf{W}_j\|_{L^p} + |(\delta_+\partial_x\underline{\mathbf{W}})_j(-\pi)|\Bigg).\end{aligned}$$

Estimate on $(\,\mathrm{id} - \frac{\partial \mathbf{G}_j}{\partial \mathbf{W}_j})^{-1}$: For any $\underline{\theta} \in \ell^\infty$ and $p \in [1, \infty]$, there exists a constant $C > 0$ such that

$$|||(\,\mathrm{id} - \frac{\partial \mathbf{G}_j}{\partial \mathbf{W}_j})^{-1}|||_{L^p} \leqslant C.$$

Combining estimates on VI_j to IX_j, $\frac{\partial \mathbf{G}_j}{\partial \mathbf{W}_j}$ and $(\,\mathrm{id} - \frac{\partial \mathbf{G}_j}{\partial \mathbf{W}_j})^{-1}$, we obtain the second inequality in (B.1.1). ■

Moreover, we have the following lemma.

Lemma B.1.2 *There exist $C>0$ and $\eta>0$ such that, for all $(\underline{\theta},\underline{\mathbf{W}})\in Y$ with its Y-norm smaller than η, we have*

$$\|\mathbf{N}^{\theta}(s)\|_{\ell^1}\leqslant\frac{C}{(1+s)^{\frac{3}{2}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^2+\frac{C}{(1+s)^{\frac{1}{4}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y\|\delta_+\partial_{xx}\underline{\mathbf{W}}(s)\|_{X_2},$$

$$\|\mathbf{N}^{\theta}(s)\|_{\ell^2}\leqslant\frac{C}{(1+s)^{\frac{3}{2}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^2+\frac{C}{(1+s)^{\frac{1}{2}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y\|\delta_+\partial_{xx}\underline{\mathbf{W}}(s)\|_{X_2},$$

$$\|\mathbf{N}^{\mathbf{w}}(s)\|_{X_1}\leqslant\frac{C}{1+s}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^2+\frac{C}{(1+s)^{\frac{1}{4}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y\|\delta_+\partial_{xx}\underline{\mathbf{W}}(s)\|_{X_2},$$

$$\|\mathbf{N}^{\mathbf{w}}(s)\|_{X_2}\leqslant\frac{C}{(1+s)^{\frac{5}{4}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^2+\frac{C}{(1+s)^{\frac{1}{2}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y\|\delta_+\partial_{xx}\underline{\mathbf{W}}(s)\|_{X_2}.$$

Proof. The estimates are obtained through a direct calculation from the estimates in Lemma B.1.1. We sketch the computation for $\|\mathbf{N}^{\theta}(s)\|_{\ell^1}$, and the others follow similarly.

First, for terms only involving $\underline{\theta}$, we notice that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}|(\delta_+\underline{\theta})_j|^2=-\sum_{j\in\mathbb{Z}}\theta_j(\delta^2\underline{\theta})_j\leqslant\|\underline{\theta}\|_{\ell^2}\|\delta^2\underline{\theta}\|_{\ell^2}\leqslant\frac{1}{(1+s)^{\frac{3}{2}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^2,\\
&\sum_{j\in\mathbb{Z}}|\theta_j|^3|(\delta_+\underline{\theta})_j|\leqslant\|\underline{\theta}\|_{\ell^\infty}^2\|\underline{\theta}\|_{\ell^2}\|\delta_+\underline{\theta}\|_{\ell^2}\leqslant\frac{1}{(1+s)^2}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^4.
\end{aligned}$$

Second, for terms involving $\underline{\mathbf{W}}$, we observe that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}|\theta_j||(\delta_+\partial_x\underline{\mathbf{W}})_j(-\pi)|&\leqslant\|\underline{\theta}\|_{\ell^2}\left(\sum_{j\in\mathbb{Z}}\left(\int_{-\pi}^{\pi}(\partial_{xx}\mathbf{W}_{j+1}(x)-\partial_{xx}\mathbf{W}_j(x))\,\mathrm{d}x\right)^2\right)^{\frac{1}{2}}\\
&\leqslant\sqrt{2\pi}\|\underline{\theta}\|_{\ell^2}\|\delta_+\partial_{xx}\underline{\mathbf{W}}\|_{X_2}\\
&\leqslant\frac{\sqrt{2\pi}}{(1+s)^{\frac{5}{4}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y(1+s)\|\delta_+\partial_{xx}\underline{\mathbf{W}}(s)\|_{X_2}.
\end{aligned}$$

Similarly, for $\sum_{j\in\mathbb{Z}}|\theta_j||(\delta_+\underline{\mathbf{W}})_j(-\pi)|$, we have

$$\sum_{j\in\mathbb{Z}}|\theta_j||(\delta_+\underline{\mathbf{W}})_j(-\pi)|\leqslant\sqrt{2\pi}\|\underline{\theta}\|_{\ell^2}\|\delta_+\partial_x\underline{\mathbf{W}}\|_{X_2}.$$

Using the "homogeneous matching boundary conditions " (3.1.11), we have

$$\begin{aligned}\|\delta_+\partial_x\underline{\mathbf{W}}\|_{X_2} &= \left(-\sum_{j\in\mathbb{Z}}\int_{-\pi}^{\pi}\left(\delta_+\underline{\mathbf{W}}\right)_j(x)\left(\delta_+\partial_{xx}\underline{\mathbf{W}}\right)_j(x)\mathrm{d}x\right)^{\frac{1}{2}}\\ &\leqslant \|\delta_+\underline{\mathbf{W}}\|_{X_2}^{\frac{1}{2}}\|\delta_+\partial_{xx}\underline{\mathbf{W}}\|_{X_2}^{\frac{1}{2}}\\ &\leqslant \|\delta_+\underline{\mathbf{W}}\|_{X_2}+\|\delta_+\partial_{xx}\underline{\mathbf{W}}\|_{X_2}.\end{aligned}$$

We plug the latter estimate into the former one and obtain that

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}|\theta_j||(\delta_+\underline{\mathbf{W}})_j(-\pi)|\\ \leqslant&\sqrt{2\pi}\|\underline{\theta}\|_{\ell^2}\left(\|\delta_+\underline{\mathbf{W}}\|_{X_2}+\|\delta_+\partial_{xx}\underline{\mathbf{W}}\|_{X_2}\right)\\ \leqslant&\frac{\sqrt{2\pi}}{(1+s)^{\frac{3}{2}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^2+\frac{\sqrt{2\pi}}{(1+s)^{\frac{1}{4}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y\|\delta_+\partial_{xx}\underline{\mathbf{W}}(s)\|_{X_2}.\end{aligned}$$

For $\sum_{j\in\mathbb{Z}}\|\mathbf{W}_j\|_{L^p}|(\delta_+\underline{\mathbf{W}})_j(-\pi)|$, we take $p=2$ and follow steps as above, obtaining the following estimate.

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\|\mathbf{W}_j\|_{L^2}|(\delta_+\underline{\mathbf{W}})_j(-\pi)|\\ \leqslant&\frac{\sqrt{2\pi}}{(1+s)^2}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^2+\frac{\sqrt{2\pi}}{(1+s)^{\frac{3}{4}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y\|\delta_+\partial_{xx}\underline{\mathbf{W}}(s)\|_{X_2}.\end{aligned}$$

For $\sum_{j\in\mathbb{Z}}\|\mathbf{W}_j^2\|_{L^p}|$, we take $p=1$ and obtain that

$$\sum_{j\in\mathbb{Z}}\|\mathbf{W}_j^2\|_{L^1}\leqslant\|\underline{\mathbf{W}}\|_{X_2}^2\leqslant\frac{1}{(1+s)^{\frac{3}{2}}}\|(\underline{\theta}(t),\underline{\mathbf{W}}(t))\|_Y^2.$$

Combining the above estimate, we establish the first inequality in the lemma. ■

B.2 Bloch wave decomposition

In this section, we present the Bloch wave decomposition of the linear operator $\widetilde{A}$. We first recall that $\widetilde{A}$, as in (3.2.1), is defined as

$$\begin{aligned}\widetilde{A}:\quad (H^2(\mathbb{R}))^n &\longrightarrow (L^2(\mathbb{R}))^n\\ \mathbf{v} &\longmapsto D\partial_{xx}\mathbf{v}-\mathbf{f}'(\mathbf{u}_\star)\mathbf{v}.\end{aligned}$$

We introduce the direct integral [30, XIII.16.]

$$\begin{array}{rccc} \mathscr{B}: & L^2(\mathbb{T}_1,(L^2(\mathbb{T}_{2\pi}))^n) & \longrightarrow & (L^2(\mathbb{R}))^n \\ & \mathbf{U}(\sigma,x) & \longmapsto & \int_{\sigma\in\mathbb{T}_1} \mathrm{e}^{\mathrm{i}\sigma\cdot x}\mathbf{U}(\sigma,\cdot)\mathrm{d}\sigma \end{array}. \tag{B.2.1}$$

The direct interal is an isometric isomorphism with inverse

$$\begin{array}{rccc} \mathscr{B}^{-1}: & (L^2(\mathbb{R}))^n & \longrightarrow & L^2(\mathbb{T}_1,(L^2(\mathbb{T}_{2\pi}))^n) \\ & \mathbf{u}(x) & \longmapsto & \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}^m} \mathrm{e}^{\mathrm{i}\ell\cdot x}\widehat{\mathbf{u}}(\sigma+\ell). \end{array}$$

The following result from [42, 34] characterizes the Bloch wave decomposition of $\widetilde{A}$.

Theorem 5 (**Bloch wave decomposition**) *The linear operator $\widetilde{A}$ is diagonal in Bloch wave space. To be precise,*

$$\mathscr{B}^{-1}\widetilde{A}\mathscr{B} = \widehat{A} = \int_{-\frac{1}{2}}^{\frac{1}{2}} B(\sigma)\mathrm{d}\sigma, \tag{B.2.2}$$

where by $\widehat{A} = \int_{-\frac{1}{2}}^{\frac{1}{2}} B(\sigma)\mathrm{d}\sigma$, we mean that, given any $\mathbf{u}\in L^2(\mathbb{T}_1,(L^2(\mathbb{T}_{2\pi}))^n)$,

$$(\widehat{A}\mathbf{u})(\sigma) = B(\sigma)\mathbf{u}(\sigma),\ \ a.e.\ \ \sigma\in[-\frac{1}{2},\frac{1}{2}].$$

Moreover, we have the following spectral mapping property.

$$\mathrm{spec}(\widetilde{A}) = \mathrm{spec}(\widehat{A}) = \bigcup_{\sigma\in[-\frac{1}{2},\frac{1}{2}]} \mathrm{spec}(B(\sigma)). \tag{B.2.3}$$

B.3 Spectral properties of $\{\widehat{A}_{\mathrm{ch}}(\sigma)\}_{\sigma\in[-\frac{1}{2},\frac{1}{2}]}$

We recall that $\widehat{A}_{\mathrm{ch}}(\sigma)$ is defined in (3.2.9) as $\widehat{A}_{\mathrm{ch}}(\sigma) = \mathscr{F}_n B(\sigma)\mathscr{F}_n^{-1}$ and Y_q in (3.2.15) for $1\leqslant q\leqslant\infty$. We are concerned with their spectral properties as unbounded operators in Y_q, which is useful for the derivation of the estimates for $M(t,\sigma)$ as defined in (3.2.23).

We first show the well-definedness of $\widehat{A}_{\mathrm{ch}}(\sigma)$ in Y_q in the following lemma.

Lemma B.3.1 *For any given $\sigma\in[-\frac{1}{2},\frac{1}{2}]$, $\widehat{A}_{\mathrm{ch}}(\sigma)$ is an unbounded closed operator in Y_2, that is,*

$$\begin{array}{rccc} \widehat{A}_{\mathrm{ch}}(\sigma): & \mathscr{D}_2(\widehat{A}_{\mathrm{ch}}(\sigma))\subset Y_2 & \longrightarrow & Y_2 \\ & \underline{\mathbf{w}} & \longmapsto & \{-(\sigma+\ell)^2 D\mathbf{w}_\ell + \sum_{k\in\mathbb{Z}} \mathbf{h}_{\ell-k}\mathbf{w}_k\}_{\ell\in\mathbb{Z}}, \end{array} \tag{B.3.1}$$

where

$$\mathscr{D}_2(\widehat{A}_{\rm ch}(\sigma)) = \{\mathbf{w} \in Y_2 \mid \{(1+m^2)\mathbf{w}_m\}_{m\in\mathbb{Z}} \in Y_2\},$$
$$\underline{\mathbf{h}} = \{\mathbf{h}_\ell\}_{\ell\in\mathbb{Z}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{f}'(\mathbf{u}_\star(x))\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x.$$

Moreover, $\widehat{A}_{\rm ch}(\sigma)$ can naturally be considered as an unbounded closed operator in Y_q, with $\mathscr{D}_q(\widehat{A}_{\rm ch}(\sigma)) = \{\mathbf{w} \in Y_q \mid \{(1+m^2)\mathbf{w}_m\}_{m\in\mathbb{Z}} \in Y_q\}$, for all $1 \leqslant q \leqslant \infty$.

Proof. The expression for $\widehat{A}_{\rm ch}(\sigma)$ in Y_2 follows from a direct calculation. The extension to Y_q follows from the fact that the set $\{\underline{\mathbf{w}} \in Y_\infty \mid \underline{\mathbf{w}}$ has finitely many nonzero entries$\}$ is dense in Y_q and $\mathscr{D}_q(\widehat{A}_{\rm ch}(\sigma))$, for all $q \in [1,\infty]$. ■

We then have the following proposition.

Proposition B.3.2 *For any fixed $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$ and $p \in [1,\infty]$, $\widehat{A}_{\rm ch}(\sigma)$ defined in Y_q is sectorial and has compact resolvent. In fact, there exist $C > 0$, $\omega \in (\pi/2, \pi)$ and $\lambda_0 \in \mathbb{R}$, independent of σ and q, such that the sector $S(\lambda_0,\omega) = \{\lambda \in \mathbb{C} \mid 0 \leqslant |\arg(\lambda-\lambda_0)| \leqslant \omega, \lambda \neq \lambda_0\} \subseteq \rho(\widehat{A}_{\rm ch}(\sigma))$ and*

$$|||(\widehat{A}_{\rm ch}(\sigma)-\lambda)^{-1}|||_{Y_q} \leqslant C|\lambda-\lambda_0|^{-1}, \text{ for all } \lambda \in S(\lambda_0,\omega), \sigma \in [-\frac{1}{2}, \frac{1}{2}] \text{ and } q \in [1,\infty]. \tag{B.3.2}$$

Moreover, for any fixed $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, the spectrum of $\widehat{A}_{\rm ch}(\sigma)$ is independent of the choice of its underlying space Y_q and thus denoted as $\mathrm{spec}(\widehat{A}_{\rm ch}(\sigma))$, *for any $q \in [1,\infty]$, with* $\mathrm{spec}(\widehat{A}_{\rm ch}(\sigma)) = \mathrm{spec}(B(\sigma))$ *consisting only of isolated eigenvalues with finite multiplicity.*

Proof. We view $\widehat{A}_{\rm ch}(\sigma)$ as a perturbation of the Laplacian in the discrete Fourier space, that is,

$$\widehat{A}_{\rm ch}(\sigma) = L(\sigma) + H,$$

where $L(\sigma)\underline{\mathbf{w}} = \{-(\sigma+\ell)^2 D\mathbf{w}_\ell\}_{\ell\in\mathbb{Z}}$ and $H\underline{\mathbf{w}} = \{\sum_{k\in\mathbb{Z}} \mathbf{h}_{\ell-k}\mathbf{w}_k\}_{\ell\in\mathbb{Z}}$. It is straightforward to verify that the proposition holds for the Laplacian $L(\sigma)$. We only have to show that the perturbation H is good enough to preserve these properties. Noting that $H \in \mathscr{L}((\ell^p)^n)$ for any $p \in [1,\infty]$ with its norm uniformly bounded, we have, for $\lambda \in \rho(L(\sigma))$, $|\lambda|$ sufficiently large,

$$(\widehat{A}_{\rm ch}(\sigma)-\lambda)^{-1} = (L(\sigma)+H-\lambda)^{-1} = (L(\sigma)-\lambda)^{-1}(\text{ id } + H(L(\sigma)-\lambda)^{-1})^{-1}. \tag{B.3.3}$$

All assertions in the proposition easily follows from this expression (B.3.3), except for the fact that the spectrum of $\widehat{A}_{\text{ch}}(\sigma)$ is independent of q.

To do that, we denote the spectrum of $\widehat{A}_{\text{ch}}(\sigma)$ defined on Y_q as $\text{spec}(\widehat{A}_{\text{ch}}(\sigma), q)$, which consists of eigenvalues with finite multiplicity, accumulating at infinity, only. Given any eigenfunction $\underline{\mathbf{v}} = \{\mathbf{v}_j\}_{j\in\mathbb{Z}}$, $\underline{\mathbf{v}}$ belongs to $\bigcap_{q\in[1,\infty]} Y_q$ since $\underline{\mathbf{v}}$ are smooth, that is, $\mathbf{v}_j$ decays algebraically with any rate. This establishes $\text{spec}(\widehat{A}(\sigma), q) = \text{spec}(\widehat{A}(\sigma), p)$, for any $p, q \in [1, \infty]$. ■

B.4 Perturbation results

We apply perturbation theory to the Bloch wave operator $B(\sigma)$ for σ near 0 and obtain more detailed spectral information, including the Taylor expansion of d in Hypotheses 1.3.2.

To this end, we define

$$\begin{array}{rccc} F: & [-\frac{1}{2}, \frac{1}{2}] \times \mathbb{C} \times H^2_\perp & \longrightarrow & L^2 \\ & (\sigma, \lambda, \mathbf{w}) & \longmapsto & (B(\sigma) - \lambda)(\mathbf{w} + \mathbf{u}'_\star), \end{array}$$

where $H^2_\perp = \{\mathbf{w} \in (H^2(\mathbb{T}_{2\pi}))^n \mid \langle \mathbf{w}, \mathbf{u}'_\star\rangle = 0\}$. A standard implicit-function-theorem argument shows that there are a small neighborhood of σ at the origin and a smooth function $(\lambda(\sigma), \mathbf{w}(\sigma))$ with $(\lambda(0), \mathbf{w}(0)) = 0$ defined in this neighborhood such that $F(\sigma, \lambda(\sigma), \mathbf{w}(\sigma)) = 0$. We denote $\mathbf{e}(\sigma) = \mathbf{u}'_\star + \mathbf{w}(\sigma)$. Similarly, replacing $B(\sigma)$ with its adjoint $B^*(\sigma)$, we obtain a smooth continuation of $\mathbf{u}_{\text{ad}}$, denoted as $\mathbf{e}^*(\sigma)$. Without loss of generality, we can assume that $\langle \mathbf{e}(\sigma), \mathbf{e}^*(\sigma)\rangle = 1$. Moreover, we have the following proposition.

Proposition B.4.1 *There exist positive numbers γ_0 and γ_1 such that for any $|\sigma| \leqslant \gamma_0$ in $\mathbb{R}$, $B(\sigma)$ has only one simple eigenvalue within the strip $|\operatorname{Re}\lambda| \leqslant \gamma_1$ in $\mathbb{C}$, which is exactly the continuation $\lambda(\sigma)$ of the eigenvalue $\lambda(0) = 0$. Moreover, $\lambda(\sigma)$ has the Taylor expansion,*

$$\lambda(\sigma) = -d\sigma^2 + \mathrm{O}(|\sigma|^3),$$

where $-\gamma_1/4 \leqslant -2d\sigma^2 < Re\lambda(\sigma) < -\frac{d}{2}\sigma^2$, for all $\sigma \in [-\gamma_0, \gamma_0]$ and

$$d = -\langle 2\mathrm{i}\frac{\partial^2 \mathbf{e}(0,x)}{\partial x \partial \sigma} - \mathbf{u}'_\star(x), D\mathbf{u}_{\text{ad}}(x)\rangle.$$

Proof. We first derive the explicit expression of d. To do that, taking first and second derivative with respect to σ of $F(\sigma, \lambda(\sigma), \mathbf{w}(\sigma)) = 0$, taking the inner product of the derivatives with $\mathbf{u}_{\mathrm{ad}}$ and letting $\sigma = 0$, we have

$$\lambda'(0) = \langle B(0)\partial_\sigma \mathbf{e}(0,x) + 2\mathrm{i}D\mathbf{u}''_\star(x), \mathbf{u}_{\mathrm{ad}}(x)\rangle,$$
$$\lambda''(0) = \langle B(0)\partial^2_\sigma \mathbf{e}(0,x) + (4\mathrm{i}D\partial_x - 2\lambda'(0))\partial_\sigma \mathbf{e}(0,x) - 2D\mathbf{u}'_\star(x), \mathbf{u}_{\mathrm{ad}}(x)\rangle.$$

Noting that $\mathrm{span}\{\mathbf{u}_{\mathrm{ad}}\} \perp Rg(B(0))$ and the inner product of an even function and an odd function is always 0, we have

$$\lambda'(0) = 0, \quad \lambda''(0) = 2\langle 2\mathrm{i}\frac{\partial^2 \mathbf{e}(0,x)}{\partial x \partial \sigma} - \mathbf{u}'_\star(x), D\mathbf{u}_{\mathrm{ad}}(x)\rangle.$$

It remains to prove the uniqueness of the eigenvalue of $B(\sigma)$ in a vertical strip centered at the origin for sufficiently small σ. First, there is no eigenvalue within the strip far away from the origin due to the fact that, by Proposition B.3.2, $\mathrm{spec}(B(\sigma))$ is in the same sector for every $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$. Secondly, the uniqueness within a small neighborhood of the origin follows from the above perturbation results. For the region inbetween, compactness and the local robustness of resolvent guarantee the absence of eigenvalues within this area. ■

Remark B.4.2 *(i) We stress that we may choose γ_0 as small as desired.*

(ii) The uniqueness implies that, for $|\sigma|$ sufficiently small, $\lambda(\sigma)$ is a real number since its complex conjugate is also an eigenvalue.

B.5 Properties of analytic semigroups $\{e^{\widehat{A}_{\mathrm{ch}}(\sigma)t}\}_{\sigma\in[-\frac{1}{2},\frac{1}{2}]}$

In this section, we will derive various estimates on $e^{\widehat{A}_{\mathrm{ch}}(\sigma)t}$. We first note that, by [10, 1.4], the interpolation space $\mathscr{D}_q(\widehat{A}_{\mathrm{ch}}(\sigma)^\alpha)$ is independent of σ,

$$\mathscr{D}_q(\widehat{A}_{\mathrm{ch}}(\sigma)^\alpha) = \{\underline{\mathbf{w}} \in Y_q \mid \{(1+m^2)^\alpha \mathbf{w}_m\}_{m\in\mathbb{Z}} \in Y_q\} =: Y^\alpha_q,$$

where $\|\underline{\mathbf{w}}\|_{Y^\alpha_q} = \|\{(1+m^2)^\alpha \mathbf{w}_m\}_{m\in\mathbb{Z}}\|_{Y_q}$. We then recall the definitions of $Y_{q,\mathrm{c}}(\sigma)$, $Y_{q,\mathrm{s}}(\sigma)$, $\widehat{A}_{\mathrm{c}}(\sigma)$ and $\widehat{A}_{\mathrm{s}}(\sigma)$ from (3.2.18). We now have the following proposition.

Proposition B.5.1 *For every $q \in [1, +\infty]$ and $\alpha > 0$, there exist positive constants $\epsilon \in (0,1)$, γ_2, $C(q)$, $C(\alpha)$ and $C(\alpha, q)$ such that*

$$
\begin{aligned}
|||e^{\widehat{A}_{\mathrm{c}}(\sigma)t}|||_{Y_{q,\mathrm{c}(\sigma)}} &\leqslant e^{-\frac{d}{2}\sigma^2 t}, \text{ for all } |\sigma| \leqslant \gamma_0, t \geqslant 0,\\
|||e^{\widehat{A}_{\mathrm{c}}(\sigma)t}|||_{Y_{q,\mathrm{c}(\sigma)} \to Y_q^\alpha} &\leqslant C(\alpha) e^{-\frac{d}{2}\sigma^2 t}, \text{ for all } |\sigma| \leqslant \gamma_0, t \geqslant 0,\\
|||e^{\widehat{A}_{\mathrm{s}}(\sigma)t}|||_{Y_{q,\mathrm{s}(\sigma)}} &\leqslant C(q) e^{-\frac{\gamma_1}{2}t}, \text{ for all } |\sigma| \leqslant \gamma_0, t \geqslant 0,\\
|||e^{\widehat{A}_{\mathrm{s}}(\sigma)t}|||_{Y_{q,\mathrm{s}(\sigma)} \to Y_q^\alpha} &\leqslant C(\alpha, q) t^{-\alpha} e^{-\gamma_1 t/2}, \text{ for all } |\sigma| \leqslant \gamma_0, t > 0,\\
|||e^{\widehat{A}_{\mathrm{ch}}(\sigma)t}|||_{Y_q} &\leqslant C(q) e^{-\epsilon d \sigma^2 t}, \text{ for all } |\sigma| \leqslant \gamma_0, t \geqslant 0,\\
|||e^{\widehat{A}_{\mathrm{ch}}(\sigma)t}|||_{Y_q} &\leqslant C(q) e^{-\gamma_2 t}, \text{ for all } \gamma_0 \leqslant |\sigma| \leqslant \frac{1}{2}, t \geqslant 0,\\
|||e^{\widehat{A}_{\mathrm{ch}}(\sigma)t}|||_{Y_q \to Y_q^\alpha} &\leqslant C(\alpha, p) t^{-\alpha} e^{-\gamma_2 t}, \text{ for all } \gamma_0 \leqslant |\sigma| \leqslant \frac{1}{2}, t > 0.
\end{aligned}
$$

Proof. We first derive estimates for the case $|\sigma| \leqslant \gamma_0$. For $\widehat{A}_{\mathrm{c}}(\sigma)$, we have $e^{\widehat{A}_{\mathrm{c}}(\sigma)t} = e^{\lambda(\sigma)t}$. The first two inequalities follow directly from the fact that $\mathrm{Re}\,\lambda(\sigma) < -\frac{d}{2}\sigma^2$ and $\mathbf{e}(\sigma)$ is smooth, by Proposition B.4.1, for $|\sigma| \leqslant \gamma_0$.

For $\widehat{A}_{\mathrm{s}}(\sigma)$, by Proposition B.3.2 and B.4.1, for any $\sigma \in (-\gamma_0, \gamma_0)$ and $q \in [1, \infty]$,

$$
\mathrm{spec}(\widehat{A}_{\mathrm{s}}(\sigma), q) \subset \mathbb{C} \backslash S(-\frac{\gamma_1}{2}, \tilde{\omega}), \text{ where } \tilde{\omega} \in (\frac{\pi}{2}, \pi).
$$

Moreover, for every $q \in [1, +\infty]$, there exists a positive constant $C(q)$ such that

$$
|||(\widehat{A}_{\mathrm{s}}(\sigma) - \lambda)^{-1}|||_{Y_{q,\mathrm{s}(\sigma)}} \leqslant C(q)|\lambda + \frac{\gamma_1}{2}|^{-1}, \text{ for all } |\sigma| \leqslant \gamma_0 \text{ and } \lambda \in S(-\frac{\gamma_1}{2}, \tilde{\omega}).
$$

Thus, by [10, Thm.1.3.4, 1.4.3], we immediately obtain the two inequalities for $\widehat{A}_{\mathrm{s}}(\sigma)$. The first inequality on $\widehat{A}_{\mathrm{ch}}(\sigma)$ follows directly by combining the first inequality for $\widehat{A}_{\mathrm{c}}(\sigma)$ and the first inequality for $\widehat{A}_{\mathrm{s}}(\sigma)$.

We now derive the estimates for the case $\gamma_0 < |\sigma| \leqslant \frac{1}{2}$. By a similar analysis as in Proposition B.4.1, there exists a positive constant γ_2 such that

$$
\mathrm{Re}(\mathrm{spec}\,\widehat{A}_{\mathrm{ch}}(\sigma)) < -2\gamma_2, \text{ for all } \gamma_0 < |\sigma| \leqslant \frac{1}{2}.
$$

It is then not hard to conclude that

$$
\mathrm{spec}(\widehat{A}_{\mathrm{ch}}(\sigma)) \subset \mathbb{C} \backslash S(-\gamma_2, \tilde{\omega}_1), \text{ where } \tilde{\omega}_1 \in (\frac{\pi}{2}, \pi).
$$

Moreover, for every $q \in [1, +\infty]$, there exists a positive constant $C(q)$ such that

$$|||(\widehat{A}_{\mathrm{ch}}(\sigma) - \lambda)^{-1}|||_{Y_q} \leqslant C(q)|\lambda + \gamma_2|^{-1}, \text{ for all } \gamma_0 < |\sigma| \leqslant \frac{1}{2} \text{ and } \lambda \in S(-\gamma_2, \tilde{\omega}_1).$$

Therefore, again by [10, Thm.1.3.4, 1.4.3], we immediately obtain the last two inequalities for $\widehat{A}_{\mathrm{ch}}(\sigma)$, which concludes the proof. ∎