On manifolds with Ricci curvature lower bound and Kähler manifolds with nonpositive bisectional curvature

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Gang Liu

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
Doctor of Philosophy

Prof. Dr. Jiaping Wang

May, 2013

Acknowledgements

There are many people that have earned my gratitude for their contribution to my time in graduate school.

First and foremost, I would like to take this opportunity to express my deepest gratitude to my advisor Prof. Jiaping Wang, for introducing me to this vast subject including both Riemannian geometry and Kähler geometry. I also thank him for sharing with me the wonderful ideas and insights to various problems of mathematics, for his constant encouragement and support.

I would like to thank Professor Shing-Tung Yau, Professor Richard Schoen, Professor Peter Li, Professor Fangyang Zheng, Professor Lei Ni for their helpful conversations and comments on my work.

I also would like to thank my committee members, Prof. Gulliver, Prof. Marden and Prof. McGehee, for reviewing my thesis and serving on the committee of my thesis defense. I really appreciate their help.

I would like to thank some of my friends: Dr. Guoyi Xu, Dr. Bo Yang, Professor Yuanyuan for their suggestions and discussions.

Abstract

The relation between curvature and topology is a fundamental problem in differential geometry. For example, the Gauss-Bonnet theorem says the sign of curvature could determine the genus of the surface. Brendle and Schoen [8] proved that if a compact manifold has sectional curvature between $\frac{1}{4}$ and 1, then it is a space form.

In the thesis, first, we classify complete noncompact three dimensional manifold with nonnegative Ricci curvature. As a corollary, we confirms a conjecture of Milnor in dimension three. Note that in the compact case, the classification was done by R. Hamilton by using the Ricci flow. Also, previously, there are some partial classifications assuming additional conditions. Our proof will be based on the minimal surface theory developed by Schoen and Yau [74], Schoen and Fischer Colbrie [24].

Next we study compact Kähler manifolds with nonpositive bisectional curvature. In particular, we confirm a conjecture of Yau which states that for there is a canonical fibration structure for these manifolds. More relating results will be proved.

In the third part, we generalize the classical volume comparison theorem to the Kähler setting. We prove a few gap theorems which tells us some differences between Kähler geometry and Riemannian geometry. We also show that locally, the volume of a Kähler-Einstein manifold is no greater than that of the complex space forms. Note that when the bisectional curvature is bounded from below, the sharp volume comparison was obtained by Li and Wang.

Then we prove a rigidity result for volume entropy. This was first proved by Ledrapier and Wang. Our proof is much shorter and simpler.

Finally, we study complete manifolds with nonnegative Bakry-Emery Ricci curvature. It turns out that when the potential f is bounded, geometrically these manifolds will be very similar with manifolds of nonnegative Ricci curvature. In particular, we partially classify complete three dimensional manifold with nonnegative Bakry-Emery Ricci curvature.

Contents

Chapter 1

Introduction

In this chapter, we will recall some backgrounds and basic definitions necessary for each subsequent chapters.

1.1 Three manifolds with nonnegative Ricci curvature

Let M be a complete manifold with nonnegative Ricci curvature, then it is a fundamental question in geometry to find the restriction of the topology on M. Recall in 2-dimensional case, Ricci curvature is the same as Gaussian curvature K. It is a well known result that if $K \geq 0$, the universal cover is either conformal to $\mathbb{S}^2$ or $\mathbb{C}$.

Let us consider 3-manifolds with nonnegative Ricci curvature. By using the Ricci flow, Hamilton [29] classified all compact 3-manifolds with nonnegative Ricci curvature. He proved that the universal cover is either diffeomorphic to $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$. In the latter two cases, the metric is a product on each factor $\mathbb{R}$. For the noncompact case, there are some partial classification results. Anderson-Rodriguez [3] and Shi [67] classified these manifolds by assuming the upper bound of the sectional curvature. Zhu [87] proved that if the volume grows like r^3, then the manifold is contractible. Based on Schoen and Yau's work [74], Zhu [88] also proved that if the Ricci curvature is quasi-positive, then the manifold is diffeomorphic to $\mathbb{R}^3$.

In late 1970s, Yau initiated a program of using minimal surfaces to study 3-manifolds. It turns out that this method is very powerful. For example, Schoen and Yau proved the famous positive mass conjecture [75][76]. Meeks and Yau [58][59] proved the loop theorem, sphere theorem and Dehn lemma together with the equivariant forms. In [74], Schoen and Yau proved that a complete noncompact 3-manifold with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$, they also announced the classification of complete noncompact 3-manifolds with nonnegative Ricci curvature.

In section 2, we classify complete noncompact 3-manifolds with nonnegative Ricci curvature in full generality. The proof is based on the minimal surface theory developed by Schoen and Fischer-Colbrie [24], Schoen and Yau [74] , Schoen [64]. We will use the following theorem frequently.

Theorem 1.1.1. *(Schoen-Yau[74]) Let M^3 be a complete 3-manifold with nonnegative Ricci curvature. Let Σ be a complete oriented stable minimal surface in M, then Σ is totally geodesic, and the Ricci curvature of M normal to Σ vanishes at all points on Σ.*

Below is the main theorem in section 2:

Theorem 1.1.2. *Let M^3 be a complete noncompact 3-manifold with nonnegative Ricci curvature, then either M^3 is diffeomorphic to $\mathbb{R}^3$ or the universal cover of M^3 is isometric to a Riemann product $N^2 \times \mathbb{R}$ where N^2 is a complete 2-manifold with nonnegative sectional curvature.*

In [50], Milnor proposed the following conjecture:

Conjecture 1.1.1. *If a complete manifold has nonnegative Ricci curvature, then the fundamental group is finitely generated.*

Corollary 1.1.1. *Milnor's conjecture is true in dimension* 3*.*

So far Milnor's conjecture is still open in higher dimensions.

1.2 Compact Kähler manifolds with nonpositive bisectional curvature

The uniformization theorem of Riemann surfaces says the sign of curvature could determine the conformal structure in some sense. Explicitly, if the curvature is positive, it is covered by either $\mathbb{P}^1$ or $\mathbb{C}$. On the other hand, if the curvature is less than a negative constant, it is covered by the unit disk $\mathbb{D}^2$.

It is natural to wonder whether there are generalizations in higher dimensions. For the compact case, the famous Frankel conjecture says if a compact Kähler manifold has positive holomorphic bisectional curvature, then it is biholomorphic to $\mathbb{CP}^n$. This conjecture was solved by Mori [53] and Siu-Yau [69] independently. In fact Mori proved the stronger Hartshorne conjecture. Later, Mok [52] solved the generalized Frankel conjecture, the result says that, if a compact Kähler manifold has nonnegative holomorphic bisectional curvature, then the universal cover is isometric-biholomorphic to $(\mathbb{C}^k, g_0) \times (\mathbb{P}^{n_1}, \theta_1) \times \cdots \times (\mathbb{P}^{n_l}, \theta_l) \times (M_1, g_1) \times \cdots \times (M_i, g_i)$, where g_0 is flat; θ_k are metrics on $\mathbb{P}^{n_k}$ with nonnegative holomorphic bisectional curvature; (M_j, g_j) are compact irreducible Hermitian symmetric spaces.

If the curvature is negative, the current knowledge is much less satisfactory. For example, a famous conjecture of Yau says if a simply connected complete Kähler manifold has sectional curvature between two negative constants, then it is a bounded domain. So far, it is not even known whether there exists a nontrivial bounded holomorphic function on such manifolds.

As in the Riemannian case, it is often important to understand the difference between the negative curved case and nonpositive case. The former tends to be hyperbolic in some sense, while the latter usually possesses some rigidity properties. For compact Kähler manifolds with nonpositive holomorphic bisectional curvature, there is a conjecture of Yau:

Conjecture 1.2.1. *Let M^n be a compact Kähler manifold with nonpositive holomorphic bisectional curvature. Then there exists a finite cover M' of M such that M' is a holomorphic and metric fibre bundle over a compact Kähler manifold N with nonpositive bisectional curvature and $c_1(N) < 0$, and the fibre is a flat complex torus.*

Recall that the fiber bundle $M \to N$ is called a metric bundle, if for any $p \in N$, there is some neighborhood $p \in U \subset N$ such that the bundle over U is isometric to the product of the fiber and U. In [83], Yau proved the following

Theorem 1.2.1. *Let M be a compact complex submanifold of a complex torus T^n. Then M is a torus bundle over a complex submanifold N in T^n, such that the induced Kähler metric on N has negative definite Ricci tensor in an open dense set of N.*

Since complex submanifolds in T^n has nonpositive holomorphic bisectional curvature, Yau's theorem confirms the conjecture when M is a complex submanifold of T^n. Zheng [89] proved this conjecture under the extra assumption M has nonpositive sectional curvature and the metric is real analytic. In [81], Wu and Zheng proved this conjecture by only assuming that the metric is real analytic. They first proved a local splitting by a careful study of the foliation at the points where the ricci curvature has the maximal rank. By real analyticity, the foliation could be extended to the whole manifold. We confirm the conjecture in section 3.

Theorem 1.2.2. *Let (M^n, g) be a compact Kähler manifold with nonpositive holomorphic bisectional curvature. Then there exists a finite cover $\hat{M}$ of M such that $\hat{M}$ is a holomorphic and metric fibre bundle over a compact Kähler manifold N^k with nonpositive bisectional curvature and $c_1(N) < 0$, and the fibre is a flat complex torus T. Furthermore, $\hat{M}$ is diffeomorphic to $T \times N$. Finally, let r be the maximal rank of the Ricci curvature of g, then $r = k$.*

Remark 1.2.1. *$dim(N) = Kod(M)$, the Kodaira dimension of M.*

Remark 1.2.2. *In [28], Guler and Zheng proved a real version of the theorem above. They need to assume that the metric is real analytic. Interestingly, the real analytic assumption is necessary in their theorem. Indeed, there is an example of Gromov which says that the manifold does not split, but the maximal rank of Ricci curvature is strictly less than the dimension. The theorem above shows that Gromov's example cannot be extended to the Kähler case.*

See more applications to canonical metrics and Ricci flow in section 3.

1.3 Volume comparison for Kähler manifolds

Comparison theorems are fundamental tools in geometric analysis. They are vital in the estimates of the spectrums, heat kernels and the Sobolev constants. The classical Bishop-Gromov's relative volume comparison theorem [6][26][41] in Riemannian geometry is the following:

Theorem 1.3.1. *Let M^n be a complete Riemannian manifold such that $Ric \geq (n-1)K$. For any $p \in M$ and $0 < a < b$, the volume of geodesic balls satisfy*

$$\frac{Vol(B_p(b))}{Vol(B_p(a))} \leq \frac{Vol(B_{M_K}(b))}{Vol(B_{M_K}(a))},$$

where M_K is the simply connected real space form with sectional curvature K, $Vol(B_{M_K}(r))$ is the volume of the geodesic ball in M_K with radius r. The equality holds iff $B_p(b)$ is isometric to $B_{M_K}(b)$.

The key ingredient in theorem 1.3.1 is the Laplacian comparison theorem [13][77]:

Theorem 1.3.2. *Let M^n be a complete Riemannian manifold with $Ric \geq (n-1)K$. Let M_k be the simply connected real space form with sectional curvature K. Denote $r_M(x)$ to be distance function from p to x in M. Let r_{M_k} be the distance function on M_k. Then for any $x \in M$, $y \in M_k$ with $r_M(x) = r_{M_k}(y)$,*

$$\Delta r_M(x) \leq \Delta r_{M_k}(y).$$

The model spaces in above theorems are real space forms. In the Kähler category, it is a natural question whether we can replace the model spaces by Kähler models, i.e, complex space forms which are Kähler manifolds with constant holomorphic sectional curvature. In [46], Li and Wang showed that when the bisectional curvature has a lower bound, both theorems above hold with Kähler models. So the question left is: what can we get if we only assume the lower bound of the Ricci curvature? In [34], we prove the following

Theorem 1.3.3. *Let $M^n (n = dim_{\mathbb{C}} M)$ be a Kähler manifold with real analytic metric. Assume $Ric \geq K$ (K is any real number). Given any point $p \in M$, there exists $r = r(p, M) > 0$ such that for any $0 < a < b < r$, the volume of geodesic balls satisfy*

$$\frac{Vol(B_{M^n}(p,b))}{Vol(B_{M^n}(p,a))} \leq \frac{Vol(B_{N_K}(b))}{Vol(B_{N_K}(a))},$$

where N_K denotes the rescaled complex space form with $Ric = K$, $\Delta_{N_K} r$ is the Laplacian of distance function on N_K. The equality holds iff M is locally isometric to N_K.

Remark 1.3.1. *Theorem 1.3.3 is a local version of Bishop-Gromov's relative volume comparison theorem on Kähler manifolds. However, one cannot directly extend theorem*

3 to any radius. A simple example is the product of $\mathbb{P}^1$ with the standard product metric. Then the diameter is greater than that of the complex space form. This implies when r is large, the inequality in theorem 3 does not hold.

We can prove a result which is slightly stronger than theorem 1.3.3:

Theorem 1.3.4. *Under the same assumption as in theorem 1.3.3, there exists $r_0 = r_0(p, M) > 0$ such that for any $r < r_0$, the average Laplacian comparison holds:*

$$\frac{\int_{\partial B_p(r)} \Delta r}{A(\partial B_p(r))} \leq \Delta_{N_K} r(r),$$

where $\Delta_{N_K} r$ is the Laplacian of distance function on N_K. Moreover, the equality holds iff M is locally isometric to N_K.

Remark 1.3.2. *Theorem 1.3.4 is a local version of theorem 1.3.2 in the average sense. However, on Kähler manifolds with Ricci curvature lower bound, the pointwise Laplacian comparison does not hold even locally.*

Next we consider the comparison in general situation.

Theorem 1.3.5. *Let $M^m (m > 1)$ be a complete Kähler manifold with Ric $\geq (2m - 1)k (k \neq 0)$ and denote $B_x(r)$ to be the geodesic ball in M centered at x with radius r. Let N be the $2m$ dimensional simply connected real space form with sectional curvature k and denote $B_N(r)$ to be the geodesic ball in N with radius r. For any point $p \in M$ and constants $0 < c < a < b$, there exists a constant $\epsilon = \epsilon(b, a, m, k) > 0$ so that the area of the geodesic spheres satisfies*

$$\frac{A(\partial B_p(b))}{A(\partial B_p(a))} \leq \frac{A(\partial B_N(b))}{A(\partial B_N(a))}(1 - \epsilon).$$

Furthermore, if $k = -1$, then ϵ depends only on $c, b - a, m$.

Remark 1.3.3. *When the bisectional curvature is bounded from below, P. Li and J. Wang [44] proved the sharp version of theorem 1.3.1 comparing with the complex space forms . However, if we only assume the Ricci curvature has a positive lower bound, one cannot expect a sharp estimate of theorem 1.3.1 comparing with the complex space forms.*

Theorem 1.3.5 has several corollaries:

Corollary 1.3.1. *Using the same notation as in theorem 1.3.5, we have*

$$\frac{Vol(B_p(b))}{Vol(B_p(a))} \leq \frac{Vol(B_N(b))}{Vol(B_N(a))}(1-\epsilon)$$

where $\epsilon = \epsilon(b, a, m, k) > 0$. *If* $k = -1$, ϵ *depends only on* $b - a, c, m$.

Definition 1.3.1. *Let* (M^n, g) *be a complete Riemannian manifold. Choose a point* $p \in M$, *define the volume entropy of* M *to be* $h(M, g) = \overline{\lim_{r\to+\infty}} \frac{\ln Vol(B_p(r))}{r}$ *where* $B_p(r)$ *is the geodesic ball in* M *centered at* p *with radius* r.

Corollary 1.3.2. *Let* M^m *(*$m > 1$*) be a complete Kähler manifold with Ric* $\geq -(2m-1)$, *then the volume entropy* $h(M)$ *satisfies*

$$h(M) \leq 2m - 1 - \epsilon$$

where ϵ *is a positive constant depending only on* m.

Corollary 1.3.3. *Let* M^m *(*$m > 1$*) be a complete Kähler manifold with Ric* $\geq (2m-1)$, *then the diameter* $d(M)$ *satisfies*

$$d(M) \leq \pi - \epsilon$$

where ϵ *is a positive constant depending only on* m.

Corollary 1.3.4. *Under the same assumption as in theorem 1.3.5, let* λ_1 *be the first eigenvalue of the Laplacian with Dirichlet boundary condition, then we have*

$$\lambda_1(B_p(r)) \leq \lambda_1(B_N(r)) - \epsilon$$

where ϵ *is a positive constant depending only on* m, k *and* r.

Remark 1.3.4. *The corollaries above are counterparts of Bishop-Gromov volume comparison theorem [6], Bonnet-Meyers theorem [13], Cheng's spectrum estimate [10].*

Given a stronger condition in theorem 1.3.5, we can obtain a better result. Explicitly, we have the following:

Theorem 1.3.6. *Let $M^m(m>1)$ be a complete Kähler manifold with $Ric \geq (2m-1)k$, $k \neq 0$. Let N be the $2m$ dimensional simply connected real space form with sectional curvature k . For a point $p \in M$, denote $r_M(x)$ to be distance function from p to x in M. Let r_N be the distance function on N. If $r \leq \frac{i_0}{2}$ where i_0 is the injective radius at p, then*

$$\frac{1}{A(\partial B_p(r))}\int_{\partial B_p(r)} \Delta r_M \leq \Delta r_N(r) - \epsilon \tag{1.1}$$

where ϵ is a positive constant depending only on m, k and r. In particular, if p is a pole, then (3.5) holds for any $r > 0$. In this case, if $r \geq c > 0$, then there exists a constant $\delta > 0$ depending only on m, k, c such that $\epsilon > \delta > 0$.

When the metric is unitary invariant with respect to a point, we have the sharp Laplacian comparison.

Theorem 1.3.7. *Let $M^m(m > 1)$ be a complete Kähler manifold with $Ric \geq (m+1)k$ and suppose the metric is unitary invariant with respect to p in M. Let M_k be the complex space form with holomorphic bisectional curvature k. Denote $r_M(x)$ to be distance function from p to x in M. Let r_{M_k} be the distance function on M_k. Then for any $x \in M$, $y \in M_k$ with $r_M(x) = r_{M_k}(y)$,*

$$\Delta r_M(x) \leq \Delta r_{M_k}(y).$$

Remark 1.3.5. *It is shown in [35] that in general, the sharp Laplacian comparison does not hold comparing with the complex space forms.*

Finally, we have the counterpart of Yau's gradient estimate [82] on Kähler manifolds:

Theorem 1.3.8. *Let $M^m(m>1)$ be a complete Kähler manifold with $Ric \geq -(2m-1)$. If f is a positive harmonic function on M, then*

$$|\nabla \log f| \leq 2m - 1 - \epsilon \tag{1.2}$$

where ϵ is a positive constant depending only on m.

Remark 1.3.6. *Yau's gradient estimate is sharp in the Riemannian case, see [45].*

1.4 Rigidity of volume entropy

Definition 1.4.1. *For a complete Riemannian manifold M, define the volume entropy v of M as*

$$v(M) = \overline{\lim_{r\to\infty}} \frac{\ln volB_M(x,r)}{r}$$

where $B_M(x,r)$ is the geodesic ball of radius r centered at x in M.

The following theorem is due to F. Ledrappier and X. Wang in [47]:

Theorem 1.4.1. *Let (M^n, g) be a closed Riemannian manifold with $Ric \geq -(n-1)$. Let $\tilde{M}$ be its universal cover, then the volume entropy satisfies $v(\tilde{M}) \leq n-1$. Moreover, $v(\tilde{M}) = n-1$ iff $\tilde{M}$ is the standard hyperbolic space with constant curvature -1.*

In section 5 we will give a new proof using only the volume comparison theorem.

1.5 Manifolds with nonnegative Bakry-Emery Ricci curvature

A smooth metric measure space is a triple $(M, g, e^{-f}dvol)$, where M is a smooth manifold; g is the Riemannian metric on M; f is a smooth function and $dvol$ is the volume form induced by g. This object has been studied extensively in geometric analysis in recent years, e.g, [61][42][80][55][56][57]. Perelman [61] introduces a functional which involves an integral of the scalar curvature with respect to a weighted measure. The Ricci flow is thus a gradient flow of such a functional. Metric measure spaces also arise in smooth collapsed Gromov-Hausdorff limits. In the physics literature, f is refered to as the dilation field. On the smooth metric measure space, there is an important curvature quantity called the Bakry-Emery Ricci curvature, which is defined in [7] by

$$Ric_f = Ric + \nabla^2 f.$$

One observes that $Ric_f = \lambda g$ for some constant λ is exactly the gradient Ricci soliton equation, which plays an essential role in the analysis of the singularities of the Ricci flow.

A lower bound for Bakry-Emery curvature is a natural assumption to make and it has significant geometric consequences. More generally, Ric_f has a natural extension to metric measure spaces, see [49][65][66]. Recently, in [80], G. F. Wei and W. Wylie proved the weighted volume comparison theorems; O. Munteanu and J. Wang established the gradient estimate for positive weighted harmonic functions. It should be noted that a while back, Lichnerowicz [40] has generalized the classical Cheeger-Gromoll splitting theorem [14] to the metric measure spaces with $Ric_f \geq 0$ and f is bounded(See [23] for more generalizations).

In Riemannian geometry, minimal surfaces arise naturally in the variation of the area functional. A minimal surface is called stable if the second variation of the area is nonnegative for any compactly supported variations. Minimal surfaces have their own beauties, e.g, Bernstein's theorem. Moreover, they have important applications to the geometry and topology of manifolds. For example, more than 60 years ago, the Synge theorem and the Bonnet-Meyers theorem were proved by the variation of geodesics(one dimensional minimal surface). More recently, by using minimal surfaces, Schoen and Yau proved the famous positive mass conjecture [75][76] . Meeks and Yau [58][59] proved the loop theorem, sphere theorem and Dehn lemma together with the equivariant forms. In [74], Schoen and Yau proved that a complete noncompact 3-manifold with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$. Anderson [1] studied the restriction of the first betti number for manifolds with nonnegative Ricci curvature; the author [34] used the minimal surface theory to classify complete three dimensional manifolds with nonnegative Ricci curvature.

In the study of smooth metric measure spaces, it is natural to add a weight e^{-f} on the area functional of the surface. The critical points of the weighted area functional are called weighted minimal surfaces. A weighted minimal surface is called stable if the second variation of the weighted area is nonnegative.

Very recently, X. Cheng, T. Mejia and D. T. Zhou [17] studied the stability condition and compactness of f-minimal surfaces. They [18] also gave eigenvalue estimates for certain closed f-minimal surfaces.

In the last chapter, we will investigate some geometric and topological results for smooth metric measure spaces via analyzing stable weighted minimal surfaces. We shall assume that the Bakry-Emery Ricci curvature is nonnegative.

Chapter 2

Three manifolds with nonnegative Ricci curvature

This section is based on [34].

Proof of the corollary 1.1.1.

M is diffeomorphic to $\mathbb{R}^3$, then the conclusion is obvious. Otherwise by theorem 1.1.2, M has nonnegative sectional curvature. Hence the corollary follows from a result of Gromov [27].

Proof of Theorem 1.1.2

We assume M is not flat, otherwise the conclusion is obvious.

Let us review Schoen and Yau's argument in [74]. Assume M is simply connected, if $\pi_2(M) \neq 0$, according to Lemma 2 in [74], M must have at least two ends. From Cheeger-Gromoll splitting theorem [14], the universal cover splits. So we assume $\pi_2(M) = 0$. Therefore, the universal cover of M is contractible. If M is not simply connected, Schoen and Yau [74] proved that $\pi_1(M)$ must have no torsion elements. Thus, after replacing M by a suitable covering, we may assume that $\pi_1(M) = \mathbb{Z}$ and that M is orientable. Let γ be a Jordan curve representing the generator of the fundamental group of M. Consider an exhaustion of M by Ω_i, where $\partial\Omega_i$ is a disjoint union of smooth 2-manifolds. We may assume that γ lies in each Ω_i. By Poincare duality for manifolds with boundary, there exists a oriented surface $\Sigma_i \subset \Omega_i$ such that $\partial\Sigma_i \subset \partial\Omega_i$, moreover, the oriented intersection number of Σ_i with γ is 1. We would like to minimize the area among all surfaces which are in the same homology class as Σ_i and with the same boundary as Σ_i. We can perturb the metric near $\partial\Omega_i$ such that the mean curvature is positive with respect to the outer normal vector. So there exists a minimizing surface for each i, which we still call Σ_i. For each i, the intersection of Σ_i with γ is nonempty. Therefore, a subsequence of Σ_i converges to an oriented stable minimal surface Σ in M. If the Ricci curvature is strictly positive on M, then this contradicts theorem 1.1.1.

Let us deal with the case when the Ricci curvature is nonnegative. For a fixed point $p \in M$, we may assume that p does not lie on γ, otherwise we perturb γ a little bit such that p is not on γ. According to the result in [19] by Ehrlich, we can perturb the metric such that the Ricci curvature is strictly positive in a small annulus around p, while the metric remains the same outside the annulus(this means that inside the ball bounded by the annulus, the Ricci curvature might be negative). For reader's convenience, we give the details as follows: According to the well-known formula, if $g(t) = e^{2tf}g_0$ and $|\nu|_{g(0)} = 1$, then

$$Ric^t(v, v) = e^{-2tf}(Ric(v, v) - t(n-2)\nabla^2 f(v, v) - t\Delta f + t^2(n-2)(v(f)^2 - |\nabla f|^2))$$

where $n = dim(M) = 3$. Define r to be the distance function to p. For a very small $R > 0$, consider the function $\rho = R - r$ for $\frac{R}{2} < r < R$. Then we extend ρ to be a positive smooth function for $0 \le r < \frac{R}{2}$. Define $f = -\rho^5$, for $|v| = 1$,

$$Ric^t(v, v) = e^{2t\rho^5}(Ric(v, v) + t(n-2)\nabla^2(\rho^5)(v, v) + t\Delta(\rho^5) + t^2(n-2)(v(\rho^5)^2 - |\nabla\rho^5|^2)).$$

Now $\nabla^2(\rho^5)(v, v) = 20\rho^3 v(\rho)^2 + 5\rho^4\nabla^2(\rho)(v, v)$, therefore,

$$Ric^t(v, v) \ge e^{2t\rho^5}(Ric(v, v) + 20t\rho^3 + 5t\rho^4(\Delta\rho + (n-2)\nabla^2(\rho)(v, v)) - 25(n-2)t^2\rho^8). \quad (2.1)$$

From now on, we restrict r such that $\lambda R < r < R$, where $\lambda > \frac{1}{2}$ is to be determined. Using the fact that near p, the manifold is almost Euclidean, for small R, we have

$$|\Delta\rho + (n-2)\nabla^2\rho(v, v)| \le \frac{9(2n-3)}{8(R-\rho)}.$$

We plug this in (1). So for all small t, $g(t)$ have strictly positive Ricci curvature in an annulus $B_p(R)\backslash B_p(\lambda R)$ for $\lambda = \frac{7}{8}$. The metric remains the same outside $B_p(R)$. The deformation is C^4 continuous with respect to the metric and C^∞ with respect to t.

We apply this perturbation finitely many times so that the Ricci curvature is positive on γ(each time we perturb the metric a little bit around a point) and that the Ricci curvature is nonnegative except a small neighborhood of p. Then we can minimize the area as before. This will yield a complete stable minimal surface Σ. Now the claim is that Σ must pass through the small neighborhood of p. If this is not true, then on Σ, the Ricci curvature is nonnegative, the normal Ricci curvature is strictly positive somewhere on γ. This contradicts theorem 1.1.1.

Using t to denote the deformation parameter, we shrink the size of the neighborhood of p where the Ricci curvature might be negative. So we get a sequence of metrics on M and for each metric, a stable minimal surface passing through a small neighborhood of p. We may let $t \to 0$ sufficiently fast so that these metrics are converging to the initial metric in C^4 sense. Taking the limit for a subsequence of these complete minimal surfaces, we obtain a complete oriented stable minimal surface passing through p, with the initial metric. According to theorem 1.1.1, this surface is totally geodesic with vanishing normal Ricci curvature.

Since the manifold is not flat, there exists a neighborhood U such that the scalar curvature is strictly positive in U. Consider a point $p \in U$ and a sequence of points $p_i \to p$, where all $p_i \in U$. Through each p_i, there exists a complete totally geodesic surface H_i. So a subsequence of H_i converges to a complete totally geodesic surface H through p. We assume that the normal vector of H_i at p_i converges to the normal vector of H at p. We can choose p_j so that for any $j > i$, p_j does not lie on H_i. Therefore, for all large i, H_i does not coincide with H.

By the assumption of U, H_i and H are not flat. They have nonnegative sectional curvature, so they are conformal to $\mathbb{C}$. The normal bundle is trivial. We denote the unit normal vector of H by N. For any $x \in H$, when k is very large, we shall construct a piece $\Sigma_k \subset H_k$. For a shortest geodesic on H connecting p and x, we assume $x = exp_p(v)$ where $v \in T_pH$. If the geodesic is not unique, then we just choose one. We parallel transport the vector v along the shortest geodesic connecting p and p_k to obtain a tangent vector u_k at p_k. Then we project u_k to $T_{p_k}(H_k)$ to get $v_k \in T_{p_k}(H_k)$. Define a point $x_k = exp_{p_k} v_k$. Since we may have multiple choices of v, x_k may be different. However, when k is very large, these x_k are close to x, since $p_k \to p$ and the normal vector of H_k at p_k is converging to the normal vector of H at p. Moreover, these x_k

belong to the same piece of H_k, i.e, the H_k distances between them are very small, since H_k and H are simply connected. Let $r = \frac{1}{10} inj_M(x)$ where $inj_M(x)$ denotes the injective radius of M at x. Define $\Sigma_k = B_{H_k}(x_k, r)$. From the construction of x_k, for k large, the normal vector of H at x and the normal vector of H_k at x_k are close in the obvious sense, as the normal vectors of H and H_k are parallel along each surfaces. Since x_k is very close to x, $inj_M(x_k) \geq \frac{1}{2} inj_M(x) \geq r$. Therefore $dist_M(\partial B_{H_k}(x_k, r), x) \geq r - dist_M(x_k, x) > 5dist_M(x, x_k)$ for k large. Thus if l is the normalized shortest geodesic connecting x and Σ_k, l will intersect the inner part of Σ_k, say at the point $\overline{x}_k$. Triangle inequality implies that $dis_{H_k}(x_k, \overline{x}_k) \leq 2dis_M(x, x_k)$. Therefore, the unit normal vector of H at x and the unit normal vector of H_k at $\overline{x}_k$ are close in the obvious sense.

Denote the initial tangent vector of l at x by e. The oriented distance is defined by $d_k(x) = dist_M(x, \Sigma_k) Sign(\langle e, N \rangle)$ for $x \in H$. The function $Sign(t) = 1$ when $t > 0$; $Sign(t) = -1$ when $t < 0$; $Sign(t) = 0$ when $t = 0$. For any $x \in H$, $d_k(x)$ is well defined and smooth for k sufficiently large. Via the second variation of arc length, there is a nice pinching estimate for the Hessian of $d_k(x)$ when $d_k(x)$ is very small, namely,

$$-d_k(x)(R_{NijN} + Sign(d_k(x))\epsilon(k, x)) \leq (d_k(x))_{ij} \leq -d_k(x)(R_{NijN} - Sign(d_k(x))\epsilon(k, x))$$

where $\lim\limits_{k \to \infty} \epsilon(k, x) = 0$ and the convergence is uniform for any compact set of H. In the above estimate, we have used the fact that for k large, the normal direction of H_k at $\overline{x}_k$ and the normal direction of H at x are close in the obvious sense. Since d_k does not vanish identically, after a suitable rescaling, a subsequence converges to a nonzero function f when $k \to \infty$. Then f satisfies

$$f_{ij} + f R_{NijN} = 0 \tag{2.2}$$

where f_{ij} is the Hessian of f on H with the induced metric. Moreover, $\Delta f = 0$ since the normal Ricci curvature vanishes identically.

Remark 2.0.1. *We use the rescaled distance function to approximate the variational vector field on H. If the surfaces H_k and H are properly embedded, then we can simply define $d_k(x) = dist_M(x, H_k) Sign(\langle e, N \rangle)$. We define the function $d_k(x)$ as in last paragraph because in the final part of the paper, when we try to show that M is simply connected at infinity, we obtain stable minimal surfaces which could be immersed and improper.*

Lemma 2.0.1. *$f \equiv Constant$.*

Proof. First, H is conformal to $\mathbb{C}$, since it is not flat and the Gaussian curvature is nonnegative. We may assume f changes sign, otherwise from the Liouville property for positive harmonic functions on H, f is constant. We observe that the vanishing points of f consists of the geodesics on H, since ∇f is parallel along the vanishing points of f(the hessian of f vanishes when f vanishes, see (2.2)). Moreover, these geodesics do not intersect, otherwise $\nabla f = 0$ along one geodesic. Combining this with (2), we find $f \equiv 0$. This is a contradiction.

Now suppose the zero set of f contains at least 2 distinct geodesics. Let us call them L_1, L_2. We claim that L_1, L_2 are proper on H. The reason is this: we can write f as the real part of a holomorphic function $h = f + ig$, since f is harmonic. By Cauchy-Riemann relation, along the vanishing set of f, g is strictly monotonic, $|\nabla g|$ is constant along L_1 and L_2(since $|\nabla f|$ is constant on each of these two geodesics). But in a compact set of H, $|h|$ is bounded, therefore, L_1, L_2 are properly embedded on H. Consider the function $d(x) = dist_H(x, L_2)$ for $x \in L_1$. From the Hessian comparison, we can show that $d'' \le 0$. Since L_1 and L_2 never intersect, $d(x) \equiv d_0$. Using the Hessian comparison again, we find the metric to be flat in the domain Ω bounded by L_1 and L_2 on H. therefore the scalar curvature of the ambient space vanishes on Ω. Considering (2), we find that f is linear on Ω. However, the vanishing points of f have two components, this is a contradiction.

Thus the vanishing points of f consist of one geodesic. By the monotonicity of g, for any $t \in \mathbb{R}$, there exists exactly one solution to the equation $h(z) = (0, t) \in \mathbb{C}$. By big Picard theorem for entire functions, infinity can not be an essential singularity for the entire function h, since h can take each value $(0, t)$ only once. Therefore, h is a polynomial. Using again that there exists exactly one solution to the equation $h(z) = (0, t) \in \mathbb{C}$, we find h to be a linear function. After some conformal transformation, we may assume $f = x$ on the complex plane. Suppose the metric on H is given by $ds^2 = e^{2\rho}(dx^2 + dy^2)$ using Cartisian coordinate on $\mathbb{C}$.

Let $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}$, then

$$\langle \nabla_{e_1} e_1, e_1 \rangle = e^{2\rho}\rho_1, \langle \nabla_{e_1} e_1, e_2 \rangle = -\langle \nabla_{e_2} e_1, e_1 \rangle = -e^{2\rho}\rho_2.$$

Therefore

$$\nabla_{e_1} e_1 = \rho_1 e_1 - \rho_2 e_2.$$

Similarly

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \rho_2 e_1 + \rho_1 e_2, \nabla_{e_2} e_2 = \rho_2 e_2 - \rho_1 e_1.$$

So the Hessian of f is given by

$$f_{11} = 0 - (\nabla_{e_1} e_1) f = -\rho_1, f_{12} = 0 - (\nabla_{e_1} e_2) f = -\rho_2, f_{22} = 0 - (\nabla_{e_2} e_2) f = \rho_1.$$

Let us write (2) as $f_{ij} + f\tau_{ij} = 0$. Therefore, the norm of the tensor τ is

$$|\tau_{ij}| = \frac{\sqrt{2}|\nabla_E \rho|}{|x|e^{2\rho}}$$

(here ∇_E, Δ_E denotes the gradient and the Laplacian with respect to the standard metric on $\mathbb{C}$). Since the Ricci curvature of the ambient manifold is nonnegative and that the normal Ricci curvature vanishes, $|\tau_{ij}| \leq \sqrt{2}K$ where $K = -\frac{\Delta_E \rho}{e^{2\rho}}$ is the Gaussian curvature on the surface. Therefore

$$\frac{|\nabla_E \rho|}{|x|} \leq -\Delta_E \rho.$$

Let $h = -\rho$, so

$$\Delta_E h \geq \frac{|\nabla_E h|}{|x|} \geq \frac{|\nabla_E h|}{r}$$

where $r^2 = x^2 + y^2$. By Cohn-Vossen inequality, $\int K ds^2 \leq 2\pi$. Therefore,

$$\int \frac{|\nabla_E h|}{|x|} dxdy \leq \int \Delta_E h dxdy < \infty.$$

Define

$$g(t) = \int_{B(t)} \frac{|\nabla_E h|}{r} dxdy$$

where $B(t)$ is the Euclidean disk centered at the origin with radius t. We have

$$t \int_{\partial B(t)} \frac{|\nabla_E h|}{r} dl \geq \int_{B(t)} \Delta_E h dxdy \geq \int_{B(t)} \frac{|\nabla_E h|}{r} dxdy.$$

That is to say,

$$tg' \geq g.$$

Solving this inequality, combining with the condition that g is bounded, we find that

$$g \equiv 0.$$

Therefore H is flat. But this contradicts the assumption that H is not flat. Thus the lemma is proved.

□

We plug this result in (2). It turns out that $R_{iNNj} = 0$ on H. So in fact the rank of the Ricci curvature is 2 at p. Therefore, through each point close to p, there is a unique totally geodesic surface. From linear algebra, we see these surfaces vary smoothly. By the calculus of variation, the variational vector field of each surface satisfies equation (2.2). According to the lemma, after a reparametrization, we may assume the variational vector fields of these surfaces are given by $\nu = N$. We call these surfaces Σ_t, $-\epsilon < t < \epsilon$. Given a point $x \in \Sigma_t$, if $X \in T_x\Sigma_t$, then $\nabla_X N = 0$, as Σ_t is totally geodesic. Since $N = \nu$, we may extend X in a small neighborhood of x in M such that $X \in T\Sigma$ and $[X, N] = 0$. We have $< \nabla_N N, X >= - < \nabla_N X, N >= - < \nabla_X N, N >= 0$. Since $X \in T_x\Sigma_t$ is arbitrary, $\nabla_N N = 0$. Thus the unit normal vector of these surfaces is parallel and Σ_t are all isometric to Σ_0 via the integral curve of the variational vector field. Let I be the maximal connected interval of t such that there exists a local isometry $F : \Sigma \times I \to M$ with $F(\Sigma, 0) = \Sigma_0$. From the definition of I, it is easy to see that I is closed. Let $c(t)$ denote the integral curve of the normal vector field N such that $c(0) = p$. Then for any $t \in I$, the scalar curvature at $c(t)$ are the same, since F is a local isometry. I is open, since for any $t \in I$, the scalar curvature at $c(t)$ is positive, we can extend I a little bit more at the end points. Therefore we have a local isometry $F : \Sigma \times \mathbb{R} \to M$, which means that the universal cover of M splits.

Now assume that M is contractible. To prove that M is diffeomorphic to $\mathbb{R}^3$, from a topological result by Stallings [70], it suffices to prove that M is simply connected at infinity and irreducible. Suppose M is not simply connected at infinity, this means that there exists a sequence of closed curves σ_i tending to infinity such that for any immersed disk D_i with $\partial D_i = \sigma_i$, $D_i \cap K \neq \Phi$ where K is a fixed compact set of M. We may assume these disks are area minimizing, by the compactness and regularity result in Theorem 3 of [64], a subsequence of D_i converges to a complete stable minimal surface which could be immersed and improper.

We can apply the argument as before. For reader's convenience, we give some details here. Given a point $p \in M$, we perturb the metric such that $Ric > 0$ in $K \backslash B_p(r)$ and $Ric \geq 0$ in $M \backslash B_p(r)$. Then for the perturbed metric, we have a complete immersed(not necessarily proper) stable minimal surface Σ_i which intersects K, thus intersects $B_p(r)$ at some p_i. The surfaces (Σ_i, p_i) have uniform regularity in any compact set in M. When the perturbation is smaller and smaller, a subsequence of (Σ_i, p_i) converges to a stable minimal surface (Σ, p). According to theorem 1.1.1, Σ is totally geodesic and the normal Ricci curvature vanishes. Then we can use arguments before to show that M splits, which contradicts that M is not simply connected at infinity.

To prove that M is irreducible, we can invoke the solution of Poincare conjecture by Perelman [61][62][63]. Therefore M is diffeomorphic to $\mathbb{R}^3$. This completes the proof of theorem 1.1.2.

$\square$

Chapter 3

Compact Kähler manifolds with nonpositive bisectional curvature

This section is based on [36].

Theorem 3.0.1. *Let M^n be a compact Kähler manifold with nonpositive holomorphic bisectional curvature. Suppose $N^k \subset M$ is a complete(compact or noncompact) immersed complex submanifold of M which is flat and totally geodesic. If in addition, $Ric(M)|_{TN} = 0$, then M splits globally, i.e, the universal cover $\tilde{M}$ is isometric and biholomorphic to $\mathbb{C}^k \times T^{n-k}$ where T^{n-k} is a complete Kähler manifold of dimension $n-k$.*

Remark 3.0.2. *All conditions in theorem* 3.0.1 *are "local" around N, except that the holomorphic bisectional curvature on M is nonpositive, thus it might be interesting to see that local conditions imply global splitting. Theorem 3.0.1 also holds if we assume the manifold has nonnegative bisectional curvature. We can also weaken the condition by assuming that M is complete with bounded curvature. The condition that Ric vanishes along the tangent of N is necessary. For instance, if M is a compact locally symmetric Hermitian space with rank greater than* 1 *covered by an irreducible bounded symmetric domain, then there is a totally geodesic, flat complex submanifold immersed in M, however, M does not split. Finally, this theorem is not true in the Riemannian case.*

In [81], Wu and Zheng studied the foliation given by the kernel of the Ricci tensor at the points where Ric has the maximal rank. In [22], Ferus showed that the leaves are complete. The following corollary can be regarded as the converse in some sense.

Corollary 3.0.1. *Let M^n be a compact Kähler manifold with nonpositive bisectional curvature. Define $U(i) = \{x \in M | rank(Ric(x)) = i\}$. Let p be an interior point of $U(i)$, then there is a foliation near p by the kernel of the Ricci curvature. If the leaf through p extends to a complete leaf $L \subset U(i)$ which is the kernel of the Ricci curvature, then i is the maximal rank of the Ricci curvature.*

Next we discuss two applications of theorem 1.2.2. The existence of canonical metric is a central topic in Kähler geometry. Yau [82] solved the famous Calabi conjecture. He proved that any Kähler manifold with $c_1 < 0$ or $c_1 = 0$ admits a unique Kähler-Einstein metric. Aubin [4] also obtained the proof when $c_1 < 0$. If a Kähler manifold has nonpositive holomorphic bisectional curvature, it is natural to ask whether there exists canonical metrics.

Corollary 3.0.2. *Let (M^n, g) be a compact Kähler manifold with nonpositive holomorphic bisectional curvature. Then the manifold admits a canonical metric which is locally a product of a flat metric and a Kähler-Einstein metric with negative scalar curvature. More precisely, the manifold is locally biholomorphic and isometric to $(D^{n-k}, g_1) \times (U^k, g_2)$, where $k = Kod(M)$ and (D^{n-k}, g_1) is a flat complex Euclidean ball with a small radius and (U^k, g_2) is an small ball with Kähler-Einstein metric such that $Ric(g_2) = -g_2$.*

Proof. According to theorem 1.2.2, there exists a finite cover $\hat{M}$ of M such that there exists a flat fibration $T^{n-k} \to \hat{M} \to N$. The universal cover $\tilde{M}$ is biholomorphic to $\mathbb{C}^{n-k} \times \tilde{N}$ where $\tilde{N} \to N$ is the universal covering. Since $c_1(N) < 0$, N admits a unique Kähler-Einstein metric g_2, thus $\tilde{N}$ admits a complete Kähler-Einstein metric with negative scalar curvature. Any element $a \in \pi_1(M)$ induces a deck transformation f on $\tilde{M}$ which descends to a biholomorphism of $\tilde{N}$. By Yau's Schwartz lemma [85], the Kähler-Einstein metric on $\tilde{N}$ is unique. Thus f preserves the Kähler-Einstein metric g_2 on $\tilde{N}$. Therefore, the product metric $\mathbb{C}^{n-k} \times (\tilde{N}, g_2)$ descends to a metric to M which is canonical. $\square$

Remark 3.0.3. *We have to lift the metric to the universal cover. Since $\hat{M}$ is not necessarily a regular covering of M, there might be no deck transformation on $\hat{M}$.*

It is also interesting to analyze the long time behavior of the normalized Kähler-Ricci flow

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}} - g_{i\bar{j}} \tag{3.1}$$

on such manifolds. Cao [11] proved that if a manifold (M, ω) has $c_1 < 0$ or $c_1 = 0$ and $c_1 = \lambda[\omega]$, then the Kähler-Ricci flow will converge to the unique Kähler-Einstein metric. Tsuji [79] and Tian-Zhang [78] proved that if a Kähler manifold has $c_1 \leq 0$, then the normalized Kähler-Ricci flow has long time existence. In [71], Song and Tian considered the normalized Kähler-Ricci flow on an elliptic surface $f : X \to \Sigma$ where some of the fibers may be singular. It was shown that the solution of the flow converges to a generalized Kähler-Einstein metric. This result was generalized in [72] to the fibration $f : X \to X_{can}$ where X is a nonsingular algebraic variety with semi-ample canonical bundle and X_{can} is its canonical model . We have a result with the similar spirit below,

Corollary 3.0.3. *Let M^n be a compact Kähler manifold with nonpositive bisectional curvature, then the for any initial Kähler metric $g(0)$ on M, the normalized Kähler-Ricci flow will converge in C^∞ to the Kähler-Einstein metric which is a factor in the canonical metric of M^n in corollary 3.0.2.*

Proof. Taking $\hat{M}$ in theorem 1.2.2, we consider the normalized Kähler-Ricci flow on $\hat{M}$ which is diffeomorphic to $T \times N$. Recall a theorem of M. Gill [25] which generalizes a theorem in [73] by Song and Weinkove,

Theorem 3.0.2. *Let $X = Y \times T$ where Y is a Kähler manifold with negative first Chern class and T be a complex torus. Let $\omega(t)$ be the normalized Kähler-Ricci flow on X with any initial metric $\omega(0)$, then $\omega(t)$ converges to $\pi^*(\omega_Y)$ in $C^\infty(X, \omega_0)$ sense as $t \to \infty$ where $\pi : X \to Y$ is the projection and ω_Y is the Kähler-Einstein metric on Y.*

Note that $\hat{M}$ is not necessarily biholomorphic to $T \times N$. However, $\hat{M}$ is locally biholomorphic to $T \times U$ where U is an open set in N, thus there is a flat metric ω_T on the fibre independent of the projection to N. Then one can check that the proof of theorem 3.0.2 in [25] works for this case without any modification.

□

The proof of theorem 1.2.2 uses Hamilton's Ricci flow [29] and Hamilton's maximum principle for tensors([30][15][9]), together with some argument in [81] by Wu and Zheng.

We will use the invariant convex set constructed in [9] by Böhm and Wilking. The key point is to prove that there exists a small $\epsilon > 0$ such that the after the Ricci flow, $Ric(g_t) \leq 0$ for all $0 < t < \epsilon$(note that the holomorphic bisectional curvature is not necessarily nonpositive for small t). The final assertion $rank(Ric(g_0)) = k$ will follow from argument of Yu [86].

Remark 3.0.4. *There is a general philosophy that the Ricci flow makes the curvature towards positive, e.g, Hamilton-Ivey pinching estimate [31][33]. So it might be interesting to see that in our case, at least in a short time, the Ricci curvature remains nonpositive.*

3.1 The proof of theorem 1.2.2

Proof. Let $g(t)$ be the solution to the Ricci flow equation $\frac{\partial g(t)}{\partial t} = -2Ric(g(t))$ with $g(0) = g$. Following Böhm and Wilking in [9], we shall construct a family of convex sets V_t which are invariant under parallel transport such that the curvature tensor of $g(t)$ lies inside V_t for small t.

Proposition 3.1.1. *Let V_t be a family of Kähler algebraic curvature operators satisfying the following conditions:*

(1). $Ric(\alpha, \overline{\alpha}) \leq 0$ for any $\alpha \in T^{1,0}M$.

(2). $|R_{x\overline{x}u\overline{v}}|^2 \leq (1 + tK_1)Ric(u, \overline{u})Ric(v, \overline{v})$ for any $x, u, v \in T^{1,0}M$ and $|x|_{g(t)} = 1$.

(3).$||R|| \leq K_2 + tK_3$.

Then for suitable positive constants K_1, K_2, K_3, there exists a $\epsilon > 0$ such that the V_t is invariant under the Ricci flow for $0 \leq t < \epsilon$. Here R stands for the curvature operator.

Proof. First, we prove V_t is a convex for each t. It is easy to see that condition (1) and (3) defines a convex set. For condition (2), suppose R, S are two tensors satisfying (1), (2), then for any $0 \leq \lambda \leq 1$, define

$$T = \lambda R + (1 - \lambda)S.$$

$$
\begin{aligned}
|T_{x\overline{x}u\overline{v}}|^2 &= |\lambda R_{x\overline{x}u\overline{v}} + (1-\lambda)S_{x\overline{x}u\overline{v}}|^2 \\
&\leq (1+tK_1)|\lambda\sqrt{Ric_R(u,\overline{u})Ric_R(v,\overline{v})} + (1-\lambda)\sqrt{Ric_S(u,\overline{u})Ric_S(v,\overline{v})}|^2 \\
&\leq (1+tK_1)(\lambda Ric_R(u,\overline{u}) + (1-\lambda)Ric_S(u,\overline{u}))(\lambda Ric_R(v,\overline{v}) + (1-\lambda)Ric_S(v,\overline{v})) \\
&= (1+tK_1)Ric_T(u,\overline{u})Ric_T(v,\overline{v}).
\end{aligned} \tag{3.2}
$$

Therefore, V_t is convex.

Now let us check that when $t = 0$, the curvature tensor R_0 of (M^n, g) is in V_0. If we choose K_2 very large, then (1) and (3) hold. To check (2), we notice that for fixed x, $R_{x\overline{x}p\overline{q}}$ is a Hermitian form. Let e_i be the eigenvectors where $i = 1, 2, ..n$ and

$$R_{x\overline{x}e_i\overline{e_j}} = \delta_{ij}\lambda_i$$

where λ_i are all nonpositive. Suppose $u = \sum\limits_{i=1}^{n} u_i e_i, v = \sum\limits_{i=1}^{n} v_i e_i$, then

$$
\begin{aligned}
|R_{x\overline{x}u\overline{v}}|^2 &= |\sum_{i=1}^{n} u_i\overline{v_i}\lambda_i|^2 \\
&\leq (\sum_{i=1}^{n} |u_i\sqrt{-\lambda_i}|^2)(\sum_{i=1}^{n} |\overline{v_i}\sqrt{-\lambda_i}|^2) \\
&= R_{x\overline{x}u\overline{u}}R_{x\overline{x}v\overline{v}} \\
&\leq Ric(u,\overline{u})Ric(v,\overline{v}).
\end{aligned} \tag{3.3}
$$

Let us state Hamilton's maximum principle for tensors. Let M^n be a closed oriented manifold with a smooth family of Riemannian metric $g(t)$, $t \in [0, T]$. Let $V \to M$ be a real vector bundle with a time dependent metric h and $\Gamma(V)$ be the vector space of smooth sections on V. Let ∇_t^L denote the corresponding Levi-Civita connection on $(M, g(t))$. Furthermore, let ∇_t denote a time dependent metric connection on V. For a section $R \in \Gamma(V)$, define a new section $\Delta_t R \in \Gamma(V)$ as follows. For $p \in M$ choose an orthonormal basis of V_p(the fiber of V at p) and extend it along the radial geodesics in $(M, g(t))$ emanating from p by parallel transport of ∇_t to an orthonormal basis $X_1(q)$, ..., $X_d(q)$ of V_q for all q in a small neighborhood of p. If f_i satisfies $R = \sum\limits_{i=1}^{d} f_i X_i$, then

$$(\Delta_t R)(p) = \sum_{i=1}^{d} (\Delta_t f_i) X_i(p)$$

where Δ_t is the Beltrami Laplacian on functions.

Suppose that a time dependent section $R(\cdot, t) \in \Gamma(V)$ satisfies the parabolic equation

$$\frac{\partial R(p,t)}{\partial t} = (\Delta_t R)(p,t) + f(R(p,t)) \tag{3.4}$$

where $f : V \to V$ is a local Lipschitz map mapping each fibre V_q to itself. Roughly speaking, Hamilton's maximum principle says that the dynamics of the parabolic equation (3.4) is controlled by the ordinary differential equation

$$\frac{dR}{dt} = f(R(p,t)). \tag{3.5}$$

More precisely, we have the following version of Hamilton's maximum principle in [9] and [15]:

Theorem 3.1.1. *For $t \in [0, \delta]$, let $C(t) \subseteq V$ be a closed subset, depending continuously on t. Suppose that each of the sets $C(t)$ is invariant under parallel transport, fiberwise convex and that the family of $C(t)(0 \le t \le \delta)$ is invariant under the ordinary differential equation (3.5). Then for any solution $R(p,t) \in \Gamma(V)$ on $M \times [0, \delta]$ of parabolic equation (3.4) with $R(\cdot, 0) \in C(0)$, we have $R(\cdot, t) \in C(t)$ for all $t \in [0, \delta]$.*

Let us go back to the proof of proposition 3.1.1. In view of theorem 3.1.1, we just need to prove that $V(t)$ is invariant under the ODE equation of the curvature operator, i.e, we drop the Laplacian in the evolution equation of the curvature operator. For any $R(0) \in V_0$, we consider perturbation $R_\lambda(0) = R(0) - \lambda R'$ for the initial condition of the ODE, where λ is a small positive number and R' is the curvature tensor with holomorphic sectional curvature 1. For simplicity, when λ is fixed, we use R to denote the solution to the ODE with initial condition $R_\lambda(0)$.

Lemma 3.1.1. *There exist positive constants $\epsilon, A, K_1, K_2, K_3$ which are independent of λ such that $\epsilon K_1 \le 1$ and for any $t \in [0, \epsilon]$, the solution R satisfies*

(1'). $Ric(\alpha, \overline{\alpha}) \le -\frac{\lambda}{2} e^{-At}$ for any $e_\alpha \in T^{1,0}M$ and $|e_\alpha|_{g(t)} = 1$.

(2'). $|R_{x\overline{x}u\overline{v}}|^2 \le (1 + tK_1) Ric(u, \overline{u}) Ric(v, \overline{v})$ for any $x, u, v \in T^{1,0}M$ and $|x|_{g(t)} = 1$.

(3'). $||R|| \le K_2 + tK_3$.

Proof. We can find $B > 0$ such that $||R|| \le B$ for all small t and λ. Take $K_2 = B$. If K_3 is big enough, then (3') will be preserved for small t and λ.

Claim 3.1.1. *If R satisfies (1'), (2') and (3') of the Lemma at time t, then there exists $C > 0$ depending only on the bound of the curvature tensor such that $|R_{i\bar{j}k\bar{l}}| \leq C\sqrt{-Ric(i,\bar{i})}$ and $|R_{i\bar{j}k\bar{l}}| \leq C\sqrt{Ric(i,\bar{i})Ric(j,\bar{j})}$ at time t for any $e_i, e_j, e_k, e_j \in T^{1,0}M$ and that the length is 1 in $g(t)$.*

Proof. The proof follows if we polarize the curvature tensor. □

In the following, C will denote a positive constant which depends only on the bound of the curvature tensor. R satisfies the ODE

$$\frac{d}{dt}R_{i\bar{j}k\bar{l}} = \sum R_{i\bar{j}**}R_{****} + \sum R_{i***}R_{*\bar{j}**}$$

where $*$ are indices. By Claim 3.1.1, we have

$$|\frac{d}{dt}R_{i\bar{j}k\bar{l}}| \leq C\sqrt{Ric(i,\bar{i})Ric(j,\bar{j})}$$

It is easy to see that (1'), (2') and (3') in the Lemma hold for $t = 0$. If the Lemma is not true, let t_0 be the first time so that the Lemma fails. There are two possibilities:

(i) (1') does not hold in $[0, t_1)$ for any $t_1 > t_0$.

(ii) (2') does not hold in $[0, t_1)$ for any $t_1 > t_0$.

In case (i), after a slight computation, Claim 3.1.1 implies

$$\frac{d}{dt}(\frac{Ric(\alpha,\overline{\alpha})}{g(t)(\alpha,\overline{\alpha})}) \leq -CRic(\alpha,\overline{\alpha})$$

for $|\alpha|_{g(t)} = 1$. If $A > 2C$, this contradicts (i).

For case (ii), Claim 4.2.1 gives

$$\frac{d}{dt}((1+tK_1)Ric(u,\overline{u})Ric(v,\overline{v}) - \frac{|R_{x\overline{x}u\overline{v}}|^2}{g(t)(x,\overline{x})}) \geq (K_1 - C)Ric(u,\overline{u})Ric(v,\overline{v}) > 0 \quad (3.6)$$

if $|x|_{g(t)} = 1, K_1 > 2C + 10, t_0 < \epsilon < \frac{1}{2K_1}$. This contradicts (ii). The Lemma is thus proved. □

Proposition 3.1.1 follows if we let $\lambda \to 0$ in the Lemma. □

By theorem 3.1.1, $Ric(g(t)) \leq 0$ for small $t > 0$. If $Ric < 0$ for some small $t > 0$, then $c_1(M) < 0$. Otherwise, the rank of the Ricci curvature is less than n for some $t > 0$. We shall show that the rank of Ric_t is constant and the null space is parallel.

We use the arguments in [9](page 676-677). Consider

$$\frac{\partial Ric(v,\overline{v})}{\partial t} = \Delta_t Ric_{v\overline{v}} + \sum R_{v\overline{v}**}R_{****} + \sum R_{v****}R_{*\overline{v}**}.$$

Define $\tilde{Ric}_t = e^{Ht}Ric_t$. By Proposition 3.1.1, if H is large, then

$$\frac{\partial \tilde{Ric}_{v\overline{v}}}{\partial t} \leq \Delta_t \tilde{Ric}_{v\overline{v}}. \tag{3.7}$$

Now we show that the rank of $\tilde{Ric}$ is constant for small $t > 0$. Let $0 \geq \mu_1 \geq \mu_2 \geq .... \geq \mu_n$ denote the eigenvalues of $\tilde{Ric}$ and let

$$\sigma_l = \mu_1 + \mu_2 + ... + \mu_l.$$

Fix $p \in M$ and let $e_1(t_0), e_2(t_0), ..., e_l(t_0)$ be an orthogonal basis of $T_p^{1,0}(M)$ such that $\sigma_l(t_0) = \sum\limits_{i=1}^{l} \tilde{Ric}_{t_0}(e_i(t_0), \overline{e_i(t_0)})$. Now

$$\begin{aligned} \sigma_l'(t_0) :&= \limsup_{t \nearrow t_0} \frac{\sigma_l(t_0) - \sigma_l(t)}{t_0 - t} \\ &\leq \frac{d}{dt}|_{t=t_0} \sum_{i=1}^{l} \tilde{Ric}_t(e_i(t_0), \overline{e_i(t_0)}) \\ &\leq \sum_{i=1}^{l} \Delta \tilde{Ric}_{t_0}(e_i(t_0), \overline{e_i(t_0)}) \\ &\leq \Delta \sigma_l \end{aligned} \tag{3.8}$$

Thus

$$\frac{\partial \sigma_l}{\partial t} \leq \Delta \sigma_l$$

in the support function sense. By the strong maximum principle, either $\sigma_l < 0$ for all small $t > 0$ or $\sigma_l \equiv 0$. This proves that $\tilde{Ric}$ has constant rank for small $t > 0$.

Let $v(t) \in T^{1,0}M$ be a smooth vector field on M depending smoothly on t such that $\tilde{Ric}_t(v, \overline{v}) = 0$. Since $\tilde{Ric} \leq 0$, from (3.7),

$$0 = (\frac{\partial}{\partial t}\tilde{Ric})(v, \overline{v}) \leq \sum_{i=1}^{n} \tilde{Ric}(\nabla_{e_i} v, \overline{\nabla_{e_i} v})$$

where $e_i \in T^{1,0}M$ is a local unitary frame on M. This shows that the rank of Ric_t is constant and the null space of Ric_t is parallel. Therefore, $(M, g(t))$ splits locally for all small $t > 0$. Therefore, for metric $g(0)$, the universal cover $\tilde{M}$ is biholomorphic and isometric to $\mathbb{C}^k \times Y^{n-k}$ with the product metric. Note that the Ricci flow on M preserves the local product structure, and for $\epsilon > t > 0$, the Ricci curvature on Y is strictly negative.

The rest proof of Theorem 1.2.2 uses the argument of Wu and Zheng [81]. For reader's convenience, we recall some details here. Denote by Γ the deck transformation group. For each $0 \leq t < \epsilon$, denote by $I_1, I_2(t)$ the group of holomorphic isometries of $\mathbb{C}^k$ and Y^{n-k} at time t. Any $f \in \Gamma$ induces a biholomorphism and isometry on $\mathbb{C}^k \times Y^{n-k}$ for any $0 \leq t < \epsilon$. Therefore $f = (f_1, f_2)$, where $f_1 \in I_1, f_2 \in \cap_{0 \leq t < \epsilon} I_2(t)$. Denote by $p_i : \Gamma \to I_i$ the projection map, and by $\Gamma_i = p_i(\Gamma)$ the image groups for $i = 1, 2$. Below are two key claims in [81]:

Claim 3.1.2. *The group Γ_2 is discrete.*

Claim 3.1.3. *There exists a finite index subgroup of $\Gamma' \subseteq \Gamma$ such that Γ'_2 acts freely on Y, and Γ'_1 contains translation only. Here $\Gamma'_i = p_i(\Gamma'), i = 1, 2$.*

Wu and Zheng proved the two claims by using ideas in Eberlein [20][21] and Nadel [60]. For our case, Claim 3.1.2 follows by applying Wu and Zheng's argument to $g(t)$ for small $t > 0$(note that in this case $Ric(Y) < 0$). For Claim 3.1.3, Wu and Zheng's proof can be carried out without any modification.

By Claim 3.1.2 and Claim 3.1.3, we have a finite covering $M' = \tilde{M}/\Gamma'$ over M, and a holomorphic surjection $q : M' \to N$ induced by the projection from $\tilde{M}$ to Y. Here $N = Y/\Gamma'_2$ is a compact Kähler manifold. q makes M' a holomorphic fibre bundle over N with fibre being complex torus. M' is also isometric to a flat torus bundle over N. By using the same argument in [81], Theorem E, we can choose M' to be diffeomorphic to $T \times N$.

Finally, we will use the argument in [86] to show that the maximal rank of the Ricci curvature of g coincides with the dimension of N. Recall corollary C in [81]:

Theorem 3.1.2. *If M^n is a compact Kähler manifold with nonpositive bisectional curvature which has Ricci rank $r < n$, then the open set U in which the Ricci tensor has maximum rank r in the universal cover $\tilde{M}$ is locally holomorphically isometric to*

$L_a \times Y_a$, where L_a is a complete flat Kähler manifold, and Y_a is a Kähler manifold with nonpositive bisectional curvature and negative Ricci curvature.

Let f be the homomorphic embedding $L_a \to \tilde{M}$ given in the theorem above. By the evolution equation of the Kähler-Ricci flow,

$$\frac{\partial}{\partial t} Ric = \sqrt{-1}\partial\overline{\partial} R \tag{3.9}$$

where R is the scalar curvature and $Ric = R_{i\overline{j}} dz^i \wedge dz^{\overline{j}}$. Let p be any point in $f_* L_a$. Pulling back (3.9) to L_a by f and integrating on the interval $[0, \epsilon]$, we find that for $e_i, e_j \in T^{1,0} L_a$,

$$0 \geq f^* Ric_{i\overline{j}}(g(\epsilon)) - f^* Ric_{i\overline{j}}(g(0)) = \sqrt{-1}\partial_i \partial_{\overline{j}} \int_0^{\epsilon} R(p,t) dt, \tag{3.10}$$

since $Ric(g(\epsilon)) \leq 0$ and $Ric_{i\overline{j}}(g(0)) = 0$ for $e_i, e_j \in T^{1,0} L_a$. (3.10) implies that $-\int_0^{\epsilon} R(x) dt$ is a bounded plurisubharmonic function on L_a. Since L_a is flat, the function must be a constant. Therefore $R_{i\overline{j}}(g(\epsilon)) = 0$ for any $e_i, e_j \in T^{1,0} L_a$. This implies that $r = dim(N)$.

The proof of Theorem 1.2.2 is complete. □

Remark 3.1.1. *The analogous result of Proposition 3.1.1 is true for the Riemannian case, i.e, if a compact manifold has nonpositive sectional curvature, then after the Ricci flow, in a short time, the Ricci curvature will be nonpositive.*

3.2 The proof of theorem 3.0.1

Proof. First we run the Kähler-Ricci flow, then by the arguments above, the Ricci curvature will be nonpositive after a short time. Since N is an immersed totally geodesic flat complex submanifold of M and $Ric(M)|_{TN} = 0$, the last part of the proof in section 2 applies, e.g, equation (3.10). Therefore, $\tilde{M}$ has a flat factor $\mathbb{C}^k$. □

Proof of corollary 3.0.1: Let r be the maximal rank of the Ricci curvature of M. By using the same proof of theorem 3.1.2 in [81] , we can show that L is an immersed totally geodesic flat complex submanifold of M(just observe that near L, the rank of

Ricci curvature is locally maximal). By theorem 3.0.1. $i = rank(Ric(g(0), U(i))) \geq rank(Ric(g(\epsilon), U(i))) = rank(Ric(g(\epsilon))) = r$. The proof of corollary 3.0.1 is complete.

Chapter 4

Volume comparison theorems for Kähler manifolds

This chapter is based on [37] and [35]. Here are some notations in this chapter. We shall use Einstein summation in this paper. For a smooth function f on a manifold M, Δf denotes the standard Beltrami Laplacian if we use orthonormal frame; if we use unitary frame, then $\Delta f = f_{\alpha\overline{\beta}}g^{\alpha\overline{\beta}}$ which is one half of the Beltrami Laplacian. For $p \in M$, $B_p(r)$ denotes the geodesic ball in M centered at p with radius r. Vol denotes the volume and A denotes the area. Given a compact set $K \in M$, $⨍_K f$ is the average of the integral of f over K.

4.1 A Bochner type formula for functions on Kähler manifolds

Proposition 4.1.1. *Let $M^m (m > 1)$ be a complete Kähler manifold, $m = dim_{\mathbb{C}}(M)$. Let $f \in C^\infty(M)$ and assume that $\nabla f(p) \neq 0$ where $p \in M$. Choosing a unitary frame $e_\alpha \in T^{1,0}(M)(\alpha = 1, 2, ..m)$ near p so that $e_1 = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX)$ where $X = \frac{\nabla f}{|\nabla f|}$, we have*

$$\frac{1}{2}\langle \nabla f, \nabla (\sum_{\gamma \neq 1} f_{\gamma\overline{\gamma}})\rangle = f_{1\overline{1}}\Delta f - |f_{\alpha\overline{\beta}}|^2 + Re(divY) \tag{4.1}$$

where $Y = \sum_{\gamma\neq 1} f_{\overline{\alpha}} f_{\alpha\overline{\gamma}} e_\gamma$, $\Delta f = \sum_{\alpha} f_{\alpha\overline{\alpha}}$.

Proof. Recall the Bochner formula:

$$\frac{1}{2}\Delta(|\nabla f|^2) = |f_{\alpha\beta}|^2 + |f_{\alpha\overline{\beta}}|^2 + (\Delta f)_\alpha f_{\overline{\alpha}} + (\Delta f)_{\overline{\alpha}} f_\alpha + Ric_{\alpha\overline{\beta}} f_{\overline{\alpha}} f_\beta. \tag{4.2}$$

(4.2) can be decomposed into two parts, namely,

$$(f_{\overline{\alpha}} f_{\alpha\overline{\beta}})_\beta = |f_{\alpha\overline{\beta}}|^2 + (\Delta f)_\alpha f_{\overline{\alpha}}, \tag{4.3}$$

$$(f_\alpha f_{\overline{\alpha}\overline{\beta}})_\beta = |f_{\alpha\beta}|^2 + (\Delta f)_{\overline{\alpha}} f_\alpha + Ric_{\alpha\overline{\beta}} f_{\overline{\alpha}} f_\beta. \tag{4.4}$$

Define a vector field

$$Z = f_{\overline{\alpha}} f_{\alpha\overline{1}} e_1,$$

then (4.3) becomes

$$divY + divZ = |f_{\alpha\overline{\beta}}|^2 + (\Delta f)_\alpha f_{\overline{\alpha}}. \tag{4.5}$$

Now we compute

$$\begin{aligned} Re(divZ) &= Re(\sum_{\beta\neq 1} \langle \nabla_{e_\beta}(f_{\overline{\alpha}} f_{\alpha\overline{1}} e_1), e_{\overline{\beta}}\rangle + \langle \nabla_{e_1}(f_{\overline{\alpha}} f_{\alpha\overline{1}} e_1), e_{\overline{1}}\rangle) \\ &= Re(\sum_{\beta\neq 1} f_{\alpha\overline{1}} \langle \nabla_{e_\beta}(f_{\overline{\alpha}} e_1), e_{\overline{\beta}}\rangle + f_{\alpha\overline{1}} \langle \nabla_{e_1}(f_{\overline{\alpha}} e_1), e_{\overline{1}}\rangle + e_1(f_{\alpha\overline{1}})\langle f_{\overline{\alpha}} e_1, e_{\overline{1}}\rangle) \\ &= f_{1\overline{1}}\Delta f + \frac{1}{2}\langle \nabla f, \nabla(f_{1\overline{1}})\rangle. \end{aligned} \tag{4.6}$$

Plugging (4.6) in (4.5), we find

$$\frac{1}{2}\langle \nabla f, \nabla(\sum_{\gamma\neq 1} f_{\gamma\overline{\gamma}})\rangle = f_{1\overline{1}}\Delta f - |f_{\alpha\overline{\beta}}|^2 + Re(divY).$$

This completes the proof of proposition 4.1.1. □

Remark 4.1.1. *Note that in (4.152), it is assumed that $\nabla f \neq 0$ at p. In some applications, we will multiply (4.152) on both side by cut-off functions and do integration by parts. We can justify the integration by approximation of Morse functions, no matter whether ∇f is vanishing somewhere.*

4.2 Relative volume comparison

In this section we are going to prove theorem 1.3.5 and its corollaries, together with theorem 1.3.6. First we shall prove the corollaries in the introduction assuming theorem 1.3.5.

Proof of corollary 1.3.1: Suppose for sufficiently small ϵ,

$$\frac{Vol(B_p(b))}{Vol(B_p(a))} \geq \frac{Vol(B_N(b))}{Vol(B_N(a))}(1-\epsilon). \tag{4.7}$$

We have

$$\begin{aligned}\frac{Vol(B_p(b))}{Vol(B_p(a))} &= \frac{Vol(B_p(\frac{a+b}{2}))}{Vol(B_p(a))} + \frac{A(\partial(B_p(\frac{a+b}{2})))}{Vol(B_p(a))}\int_{\frac{a+b}{2}}^{b}\frac{A(\partial(B_p(r)))}{A(\partial(B_p(\frac{a+b}{2})))}dr \\ &\leq \frac{Vol(B_N(\frac{a+b}{2}))}{Vol(B_N(a))} + \frac{A(\partial(B_p(\frac{a+b}{2})))}{Vol(B_p(a))}\int_{\frac{a+b}{2}}^{b}\frac{A(\partial(B_N(r)))}{A(\partial(B_N(\frac{a+b}{2})))}dr.\end{aligned} \tag{4.8}$$

Putting (4.8), (4.7) together, after some manipulation, we find

$$\frac{A(\partial(B_p(\frac{a+b}{2})))}{Vol(B_p(a))} \geq \frac{A(\partial(B_N(\frac{a+b}{2})))}{Vol(B_N(a))}(1-\delta_1). \tag{4.9}$$

Also note that

$$\begin{aligned}\frac{Vol(B_p(a))}{A(\partial(B_p(\frac{a+b}{2})))} &= \frac{A(\partial(B_p(a)))}{A(\partial(B_p(\frac{a+b}{2})))}\int_0^a\frac{A(\partial(B_p(r)))}{A(\partial(B_p(a)))}dr \\ &\geq \frac{A(\partial(B_p(a)))}{A(\partial(B_p(\frac{a+b}{2})))}\int_0^a\frac{A(\partial(B_N(r)))}{A(\partial(B_N(a)))}dr.\end{aligned} \tag{4.10}$$

Combining (4.9), (4.10) together, we get

$$\frac{A(\partial(B_p(\frac{a+b}{2})))}{A(\partial(B_p(a)))} \geq \frac{A(\partial(B_N(\frac{a+b}{2})))}{A(\partial(B_N(a)))}(1-\delta_2). \tag{4.11}$$

In (4.9), (4.11), δ_1, δ_2 are positive constants depending only on ϵ, a, b, m, k. Moreover, $\lim_{\epsilon\to 0}\delta_i = 0$ for $i = 1, 2$. If $k = -1$, δ_i depends only on $\epsilon, b-a, c, m$.

If ϵ is sufficiently small, (4.11) contradicts theorem 1.3.5. $\square$

Proof of corollary 1.3.2: Let N be the $2m$ dimensional real space form with constant sectional curvature -1. Taking $a_i = i, b_i = i+1$ in corollary 1.3.1 for $i = 1, 2, ...,$ we have

$$\frac{Vol(B_p(i+1))}{Vol(B_p(i))} \leq (1-\epsilon_i)\frac{Vol(B_N(i+1))}{Vol(B_N(i))}. \tag{4.12}$$

According to corollary 1.3.1, there exists a positive constant δ such that $\epsilon_i > \delta$ for all $i \geq 1$. Therefore (4.12) becomes

$$\frac{Vol(B_p(i+1))}{Vol(B_p(i))} \leq (1-\delta)\frac{Vol(B_N(i+1))}{Vol(B_N(i))}. \tag{4.13}$$

By iteration of (4.13), it follows that

$$\frac{Vol(B_p(i))}{Vol(B_p(1))} \leq (1-\delta)^{i-1}\frac{Vol(B_N(i))}{Vol(B_N(1))}. \tag{4.14}$$

Thus

$$\begin{aligned} \frac{\ln Vol(B_p(i))}{i} \leq& \frac{i-1}{i}\ln(1-\delta) + \frac{\ln Vol(B_N(i))}{i} \\ &+ \frac{\ln Vol(B_p(1))}{i} - \frac{Vol(B_N(1))}{i}. \end{aligned} \tag{4.15}$$

When $i \to \infty$, the RHS of (4.15) is approaching $2m - 1 + \ln(1-\delta)$. This completes the proof of corollary 1.3.2. $\square$

Proof of corollary 1.3.3: Let S^{2m} be the $2m$ dimensional sphere with constant sectional curvature 1. Assuming $d(M) = d$, we pick two points $p, q \in M$ such that $dist(p,q) = d(M)$. According to corollary 1.3.1, there exists a positive constant ϵ such that

$$\frac{Vol(B_p(\frac{d}{2}))}{Vol(B_p(d))} \geq \frac{Vol(B_{S^{2m}}(\frac{d}{2}))}{Vol(B_{S^{2m}}(d))}(1+\epsilon),$$

$$\frac{Vol(B_q(\frac{d}{2}))}{Vol(B_q(d))} \geq \frac{Vol(B_{S^{2m}}(\frac{d}{2}))}{Vol(B_{S^{2m}}(d))}(1+\epsilon).$$

Therefore

$$\begin{aligned} 1 \geq& \frac{Vol(B_p(\frac{d}{2})) + Vol(B_q(\frac{d}{2}))}{Vol(M)} \\ \geq& 2(1+\epsilon)\frac{Vol(B_{S^{2m}}(\frac{d}{2}))}{Vol(B_{S^{2m}}(d))}. \end{aligned} \tag{4.16}$$

If d is sufficiently close to π, the right hand side of (4.16) is greater than 1. This is a contradiction. $\square$

Remark 4.2.1. *The counterexample in section 4 shows that when Ric $\geq 2m-1$, the diameter of the Kähler manifold could exceed that of $\mathbb{CP}^m$. The corollary says the diameter of the Kähler manifold can not be too close to that of S^{2m}.*

Proof of corollary 1.3.4:

We use the same notation as in theorem 1.3.5. Denote the area of the geodesic sphere $\partial B_p(r)$ by $A(r)$, the volume of the geodesic ball $B_p(r)$ by $V(r)$. Denote $\lambda_1(B_N(r))$ by λ_1 and let f be the nonnegative eigenfunction to the equation

$$\Delta f = -\lambda_1 f$$

on $B_N(r)$ with Dirichlet boundary condition. After normalization, we may assume $\int_{B_N(r)} f^2 = 1$. It is easy to see that f is a radical function. Pulling f back to the tangent space of p, via the exponential map, we may assume that f is defined on $B_p(r)$.

Suppose there is small constant ϵ such that

$$\lambda_1(B_p(r)) \geq \lambda_1 - \epsilon,$$

then we have the inequality

$$\lambda_1 - \epsilon \leq \frac{\int_{B_p(r)} |\nabla f|^2}{\int_{B_p(r)} f^2}. \tag{4.17}$$

Using integration by parts, we find

$$\frac{\int_{B(P,r)} (\lambda_1 f + \Delta f) f}{\int_{B_p(r)} f^2} \leq \epsilon. \tag{4.18}$$

By Cheng's argument in [10],

$$\lambda_1 f + \Delta f \geq 0$$

in $B_p(r)$. It is simple to see that f is strictly between two positive constants in $B_p(\frac{r}{2})$.

By (4.18), we have

$$(\min_{B_p(\frac{r}{2})} f)\frac{\int_{B_p(\frac{r}{2})}(\lambda_1 f+\Delta f)}{V(\frac{r}{2})} \le \frac{\int_{B_p(\frac{r}{2})} f(\lambda_1 f+\Delta f)}{V(\frac{r}{2})}$$
$$\le \frac{\int_{B_p(r)} f(\lambda_1 f+\Delta f)}{V(\frac{r}{2})}$$
$$\le \epsilon\frac{\int_{B_p(r)} f^2}{V(\frac{r}{2})} \tag{4.19}$$
$$\le \epsilon(\max_{B_p(r)} f^2)\frac{V(r)}{V(\frac{r}{2})}$$
$$\le C(r,k,m)\epsilon \max_{B_p(r)} f^2.$$

Therefore, we conclude

$$\fint_{B(P,\frac{r}{2})}(\lambda_1 f+\Delta f) \le C(r,k,m)\epsilon\frac{\max_{B_p(r)} f^2}{\min_{B_p(\frac{r}{2})} f} \tag{4.20}$$
$$= \delta(\epsilon,r,k,m).$$

Noting that f is a function of r and $f' \le 0$, we have

$$\fint_{B(P,\frac{r}{2})}\lambda_1 f = \lambda_1\frac{\int_0^{\frac{r}{2}} f(t)A(t)dt}{V(\frac{r}{2})}$$
$$= \lambda_1 f(\frac{r}{2}) + \lambda_1\int_0^{\frac{r}{2}}(-f'(t))\frac{V(t)}{V(\frac{r}{2})}dt \tag{4.21}$$
$$\ge C(r)$$

where

$$C(r) = \lambda_1 f(\frac{r}{2}) + \lambda_1\int_0^{\frac{r}{2}}(-f'(t))\frac{Vol_N(t)}{Vol_N(\frac{r}{2})}dt.$$

In the last inequality of (4.21), we have applied the Bishop-Gromov volume comparison. Using the divergence theorem, we have

$$\fint_{B(P,\frac{r}{2})}\Delta f = f'(\frac{r}{2})\frac{A(\frac{r}{2})}{V(\frac{r}{2})}. \tag{4.22}$$

Combining (4.20), (4.21), (4.22), we obtain

$$\frac{A(\frac{r}{2})}{V(\frac{r}{2})} \ge \frac{C(r)-\delta}{(-f'(\frac{r}{2}))} = \frac{A(B_N(\frac{r}{2}))}{Vol(B_N(\frac{r}{2}))} - \overline{\delta} \tag{4.23}$$

where $\delta, \overline{\delta}$ are small constants depending on ϵ, m, k, r.

If ϵ is very small, $\overline{\delta}$ is small. (4.23) contradicts theorem 1.3.5. $\square$

Proof of theorem 1.3.5:

We consider the case $Ric \geq -(2m-1)$ first.

Let $n = 2m$. For $x \in M$, define $r(x) = d(x, p)$. Choose an orthonormal frame h_i ($i = 1, 2, ..2m$) near x so that $h_1 = \nabla r$ and $Jh_{2\alpha-1} = h_{2\alpha}$ for $1 \leq \alpha \leq m$. Define a unitary frame $\{e_\alpha\}$ so that $e_\alpha = \frac{1}{\sqrt{2}}(h_{2\alpha-1} - \sqrt{-1}h_{2\alpha})$. Let ω^i be the dual 1-form of h_i. Define a tensor S near x such that

$$\begin{aligned} S &= S_{ij}\omega^i \otimes \omega^j \\ &= \coth r \sum_{i\neq 1} \omega^i \otimes \omega^i. \end{aligned} \tag{4.24}$$

It is simple to see that the tensor S is independent of the frame h_i, moreover, S is the Hessian of the distance function in real space form with sectional curvature -1. After the complexification, we find

$$S_{\alpha\overline{\beta}} = \begin{cases} 0 & \alpha \neq \beta \\ \coth r & \alpha = \beta, \alpha \neq 1 \\ \frac{1}{2}\coth r & \alpha = \beta = 1. \end{cases} \tag{4.25}$$

We introduce the proposition as follows:

Proposition 4.2.1. *Let M^n be a complete Riemannian manifold such that $Ric \geq -(n-1)$, $p \in M$ be a point. Define N to be the n dimensional real space form with constant sectional curvature -1. Given constants $b > a > c > 0$, $\epsilon > 0$, if the area of the geodesic spheres satisfies*

$$\frac{A(\partial B_p(b))}{A(\partial B_p(a))} \geq \frac{A(\partial B_N(b))}{A(\partial B_N(a))} - \epsilon, \tag{4.26}$$

there exists positive constants δ, C and a smooth function w defined in the annulus $T = \{x \in M | \frac{3a+2b}{5} \leq d(x,p) \leq \frac{2a+3b}{5}\}$ so that

$$\begin{aligned} \fint_T (|\nabla w - \nabla r|^2 + \sum_{i,j} |w_{ij} - S_{ij}|^2) &< \delta(b-a, c, n, \epsilon), \\ |\nabla w| &< C(b-a, c, n). \end{aligned} \tag{4.27}$$

Moreover,

$$\lim_{\epsilon \to 0} \delta(b-a, n, c, \epsilon) = 0.$$

Remark 4.2.2. *Proposition 4.2.1 originates from Cheeger and Colding's paper [12]. Their estimate depends on both the upper bound and lower bound of a and b which is not sufficient to prove corollary 1.3.2.*

Proof. For notational convenience, in the proof of proposition 4.2.1, δ denotes small positive constants depending only on $\epsilon, c, b-a, n$. C denotes positive constants depending only on $c, b-a, n$. Moreover, $\lim\limits_{\epsilon\to 0}\delta = 0$.

Define $\overline{\Delta}, \overline{\nabla}$ to be the Laplacian and the covariant derivatives in N. Pick a point $\overline{p}$ in N, define

$$A(a,b) = \{x \in M | a \leq d(x,p) \leq b\},$$

$$\overline{A}(a,b) = \{y \in N | a \leq d(y,\overline{p}) \leq b\}.$$

We solve the equation

$$\overline{\Delta}\overline{g} = 0 \tag{4.28}$$

in $\overline{A}(a,b)$ satisfying the boundary condition $\overline{g}(x) = 2$ on $\partial B(\overline{p}, a)$, $\overline{g}(x) = 1$ on $\partial B(\overline{P}, b)$. Then $\overline{g}$ is a radical function, say, $\overline{g} = \phi(r)$. Pulling $\overline{g}$ back to the tangent space of p, via the exponential map, we may assume that $\overline{g}$ is defined in $A(a,b)$. It is simple to check that $\overline{g}$ is strictly decreasing with respect to r, therefore $\overline{g}^{-1}$ exists. Moreover,

$$\overline{g}, |\overline{g}'|, |\overline{g}''|, |(\overline{g}^{-1})'|, |(\overline{g}^{-1})''| < C. \tag{4.29}$$

Similarly we solve the equation

$$\Delta g = 0 \tag{4.30}$$

in $A(a,b)$ satisfying the same boundary condition as $\overline{g}$.

Claim 4.2.1.

$$\begin{aligned} &|\frac{A(\partial B_p(a))}{Vol(A(a,b))} - \frac{A(\partial B_{\overline{p}}(a))}{Vol(\overline{A}(a,b))}| < \delta, \\ &|\frac{A(\partial B_p(b))}{Vol(A(a,b))} - \frac{A(\partial B_{\overline{p}}(b))}{Vol(\overline{A}(a,b))}| < \delta. \end{aligned} \tag{4.31}$$

Proof. Claim 4.2.1 follows from (4.26) and Bishop-Gromov volume comparison. □

The following claim is due to J. Cheeger and T. Colding [12]:

Claim 4.2.2.

$$\fint_{A(a,b)} |\nabla g - \nabla \overline{g}|^2 \leq \delta. \tag{4.32}$$

Proof. By maximum principle, $|g - \overline{g}| \leq 2$ in $A(a,b)$. By Laplacian comparison and (4.28), we have $\Delta \overline{g} \geq 0$, since $\overline{g}$ is decreasing with respect to r. Using integration by parts, we have

$$\begin{aligned}
\fint_{A(a,b)} |\nabla g - \nabla \overline{g}|^2 &= -\fint_{A(a,b)} (g - \overline{g})\Delta(g - \overline{g}) \\
&\leq 2\fint_{A(a,b)} \Delta \overline{g} \\
&\leq 2\frac{\int_{\partial A(a,b)} \frac{\partial \overline{g}}{\partial n} ds}{Vol(A(a,b))} \\
&= 2(\overline{g}'(b)\frac{A(\partial B_p(b))}{Vol(A(a,b))} - \overline{g}'(a)\frac{A(\partial B_p(a))}{Vol(A(a,b))}) \\
&\leq \delta.
\end{aligned}$$

In the last inequality, we have applied claim 4.2.1. □

The claim below comes from P. Li and R. Schoen in [43] and J. Cheeger and T. Colding [12]:

Claim 4.2.3. *For Dirichlet boundary condition on $A(a,b)$, the first eigenvalue $\lambda_1 \geq C$.*

Proof. Given a constant $\mu > 0$, define $f = (b + 1 - r)^\mu$ in $A(a,b)$. Then

$$\Delta f = \mu(\mu - 1)(b + 1 - r)^{\mu-2} - \mu(b + 1 - r)^{\mu-1}\Delta r,$$

$$|\nabla f| = \mu(b + 1 - r)^{\mu-1}.$$

Taking μ large, from the Laplacian comparison theorem, we may assume that

$$\Delta f \geq 1, |\nabla f| \leq C$$

in $A(a,b)$.

For any function $h \in C_c^\infty(A(a,b))$,

$$\begin{aligned}\int_{A(a,b)} h^2 &\leq \int_{A(a,b)} h^2 \Delta f \\ &= -2\int_{A(a,b)} h\langle \nabla f, \nabla h\rangle \\ &\leq 2C\int_{A(a,b)} |h||\nabla h| \\ &\leq 2C(\int_{A(a,b)} h^2)^{\frac{1}{2}}(\int_{A(a,b)} |\nabla h|^2)^{\frac{1}{2}}\end{aligned}$$

Therefore $\lambda_1 \geq \frac{1}{4C^2}$. This proves claim 4.2.3. □

Combining claim 4.2.2 and claim 4.2.3, one concludes that

$$⨍_{A(a,b)} |g - \overline{g}|^2 \leq \delta. \tag{4.33}$$

Let $a_1 = \frac{4a+b}{5}, a_2 = \frac{3a+2b}{5}, b_2 = \frac{2a+3b}{5}, b_1 = \frac{a+4b}{5}$, so $a < a_1 < a_2 < b_2 < b_1 < b$. Since g is a positive harmonic function in $A(a,b)$, by the gradient estimate of Cheng and Yau [16], we have

$$|\nabla g| \leq Cg \leq 2C \tag{4.34}$$

in $A(a_1, b_1)$. By a simple calculation, one can find a function $\psi(r)$ so that

$$\overline{\nabla}^2\psi(r) = \frac{1}{n}\overline{\Delta}(\psi(r)),$$

$$\psi(a) = 1, \psi'(a) = 1$$

in $\overline{A}(a,b)$. It is easy to see ψ is strictly increasing with respect to r, therefore ψ^{-1} exists. Moreover,

$$|\psi|, |\psi'|, |\psi''|, |(\psi^{-1})'|, |(\psi^{-1})''|, |(\psi^{-1})'''| < C. \tag{4.35}$$

For $x \in A_{a,b}$, define

$$u(x) = \psi \circ \overline{g}^{-1} \circ g(x), \overline{u}(x) = \psi \circ \overline{g}^{-1} \circ \overline{g}(x) = \psi(r(x)), \tag{4.36}$$

then by (4.29), (4.34), (4.35),

$$|\nabla u| \leq C \tag{4.37}$$

in $A(a_1, b_1)$.

The Bochner formula for $u(x)$ is

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta u\rangle + Ric(\nabla u, \nabla u).$$

Since $Ric \geq -(n-1)$, we can rewrite it as

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla \Delta u, \nabla u\rangle - \frac{1}{n}(\Delta u)^2 + (n-1)\langle \nabla u, \nabla u\rangle \geq |\nabla^2 u - \frac{1}{n}\Delta u|^2. \tag{4.38}$$

Now we want to get the estimate of the Hessian of u.

We will multiply both side of (4.38) by a cut-off function and do integration by parts.

In [12], Cheeger and Colding choose the cut-off function to be a function of g. To make that work, the upper bound of a is needed. To avoid this problem, we define the cut-off function $\overline{\varphi}$ to be a function of r, explicitly,

$$\overline{\varphi}(r) = \begin{cases} 0 & a \leq r < a_1 \\ \frac{r-a_1}{a_2-a_1} & a_1 \leq r \leq a_2 \\ 1 & a_2 \leq r \leq b_2 \\ \frac{b_1-r}{b_1-b_2} & b_2 < r < b_1 \\ 0 & b_1 \leq r \leq b. \end{cases} \tag{4.39}$$

Define

$$\varphi(x) = \overline{\varphi} \circ \overline{g}^{-1} \circ g \tag{4.40}$$

in $A(a, b)$. From claim 4.2.2, (4.29), (4.33), (4.34), it is easy to see that

$$⨍_{A(a,b)} |\nabla\overline{\varphi} - \nabla\varphi|^2 + |\varphi - \overline{\varphi}|^2 \leq \delta. \tag{4.41}$$

Multiplying (4.38) on both side by $\overline{\varphi}^2$ and using integration by parts, we find

$$\begin{aligned} \frac{1}{Vol(A(a_1,b_1))}\int_{A(a_1,b_1)} & -\frac{1}{2}\langle \nabla\overline{\varphi}^2, \nabla|\nabla u|^2\rangle + \overline{\varphi}^2(\Delta u)^2 + 2\overline{\varphi}\Delta u\langle\nabla\overline{\varphi}, \nabla u\rangle - \frac{1}{n}\overline{\varphi}^2(\Delta u)^2 \\ & + (n-1)\overline{\varphi}^2|\nabla u|^2 \geq \frac{1}{Vol(A(a_1,b_1))}\int_{A(a_1,b_1)} \overline{\varphi}^2|\nabla^2 u - \frac{1}{n}\Delta u|^2. \end{aligned} \tag{4.42}$$

Let us write the first term of (4.42) as

$$-\frac{1}{Vol(A(a_1,b_1))}\int_{A(a_1,b_1)}\frac{1}{2}\langle\nabla\overline{\varphi}^2,\nabla|\nabla u|^2\rangle = -\frac{1}{2}⨍_{A(a_1,b_1)}(\overline{\varphi}^2)_i(u_j^2)_i$$
$$= -2⨍_{A(a_1,b_1)}\overline{\varphi}\overline{\varphi}_i u_j u_{ji}$$
$$= -2(⨍_{A(a_1,b_1)}\overline{\varphi}(\overline{\varphi}_i-\varphi_i)u_j u_{ji} + ⨍_{A(a_1,b_1)}\overline{\varphi}\varphi_i u_j u_{ji}). \tag{4.43}$$

We will estimate the two terms in the RHS of (4.43) separately. Using (4.37), (4.41), we find

$$\begin{aligned}
&|-2⨍_{A(a_1,b_1)}\overline{\varphi}(\overline{\varphi}_i-\varphi_i)u_j u_{ji}|\\
&\leq 2⨍_{A(a_1,b_1)}|\overline{\varphi}||\nabla\overline{\varphi}-\nabla\varphi||\nabla u||u_{ji}|\\
&\leq C⨍_{A(a_1,b_1)}\overline{\varphi}|\nabla\varphi-\nabla\overline{\varphi}||\nabla^2 u|\\
&\leq C(⨍_{A(a_1,b_1)}|\nabla\overline{\varphi}-\nabla\varphi|^2)^{\frac{1}{2}}(⨍_{A(a_1,b_1)}\overline{\varphi}^2|\nabla^2 u|^2)^{\frac{1}{2}}\\
&\leq \delta(⨍_{A(a_1,b_1)}\overline{\varphi}^2|\nabla^2 u|^2)^{\frac{1}{2}}\\
&\leq \frac{1}{2}(\delta+\delta⨍_{A(a_1,b_1)}\overline{\varphi}^2|\nabla^2 u|^2).
\end{aligned} \tag{4.44}$$

In (4.44) we may assume that ϵ is so small that $\delta \leq \frac{1}{2}$. For the other term on the RHS of (4.43), integration by parts gives

$$\begin{aligned}
-2⨍_{A(a_1,b_1)}\overline{\varphi}\varphi_i u_j u_{ji} &= -⨍_{A(a_1,b_1)}\overline{\varphi}\varphi_i(u_j^2)_i\\
&= ⨍_{A(a_1,b_1)}\overline{\varphi}_i\varphi_i u_j^2 + ⨍_{A(a_1,b_1)}\overline{\varphi}\varphi_{ii}u_j^2\\
&= ⨍_{A(a_1,b_1)}\langle\nabla\varphi,\nabla\overline{\varphi}\rangle|\nabla u|^2+\overline{\varphi}\Delta\varphi|\nabla u|^2.
\end{aligned} \tag{4.45}$$

Plugging (4.44) and (4.45) in (4.43), we find that

$$\begin{aligned}
&⨍_{A(a_1,b_1)}\{\langle\nabla\overline{\varphi},\nabla\varphi\rangle|\nabla u|^2+\overline{\varphi}\Delta\varphi|\nabla u|^2+\frac{1}{2}\delta+\frac{1}{2}\delta\overline{\varphi}^2|\nabla^2 u|^2+\overline{\varphi}^2(\Delta u)^2\\
&+2\overline{\varphi}\Delta u\langle\nabla\overline{\varphi},\nabla u\rangle-\frac{1}{n}\overline{\varphi}^2(\Delta u)^2+(n-1)\overline{\varphi}^2|\nabla u|^2\}\\
&\geq ⨍_{A(a_1,b_1)}\overline{\varphi}^2|\nabla^2 u-\frac{1}{n}\Delta u|^2.
\end{aligned} \tag{4.46}$$

We can rewrite (4.46) as

$$\begin{aligned}
&\fint_{A(a_1,b_1)} \{\langle\nabla\varphi,\nabla\overline{\varphi}\rangle|\nabla u|^2+\overline{\varphi}\Delta\varphi|\nabla u|^2+\frac{1}{2}\delta+\frac{\delta}{2n}\overline{\varphi}^2(\Delta u)^2+\overline{\varphi}^2(\Delta u)^2 \\
&+2\overline{\varphi}\Delta u\langle\nabla\overline{\varphi},\nabla u\rangle-\frac{1}{n}\overline{\varphi}^2(\Delta u)^2+(n-1)\overline{\varphi}^2|\nabla u|^2\} \\
&\geq(1-\frac{\delta}{2})\fint_{A(a_1,b_1)}\overline{\varphi}^2|\nabla^2 u-\frac{1}{n}\Delta u|^2 \\
&\geq(1-\frac{\delta}{2})\frac{Vol(A(a_2,b_2))}{Vol(A(a_1,b_1))}\fint_{A(a_2,b_2)}|\nabla^2 u-\frac{1}{n}\Delta u|^2 \\
&\geq C\fint_{A(a_2,b_2)}|\nabla^2 u-\frac{1}{n}\Delta u|^2.
\end{aligned} \tag{4.47}$$

We claim that up to a negligible error, we can replace all functions on the LHS of (4.47) by the corresponding radical functions, namely,

$$\begin{aligned}
\nabla\varphi &\longrightarrow \nabla\overline{\varphi}, \\
\nabla u &\longrightarrow \nabla\overline{u}, \\
\Delta\varphi &\longrightarrow (\overline{\varphi}\circ\overline{g}^{-1})''|\nabla\overline{g}|^2, \\
\Delta u &\longrightarrow (\overline{\psi}\circ\overline{g}^{-1})''|\nabla\overline{g}|^2.
\end{aligned}$$

To justify the above substitution we use the following standard lemma:

Lemma 4.2.1. *Let $k_1,...,k_N,\overline{k}_1,..,\overline{k}_N$ be functions on a measure space U such that for all i,*

$$Sup(|k_i|+|\overline{k}_i|)\leq C,$$
$$\int_U|k_i-\overline{k}_i|\leq\epsilon,$$

then

$$\int_U|k_1\cdots k_N-\overline{k}_1\cdots\overline{k}_N|\leq NC^{N-1}\epsilon.$$

Proof. If we write

$$\begin{aligned}
k_1\cdots k_N-\overline{k}_1\cdots\overline{k}_N &= (k_1-\overline{k}_1)k_2\cdots k_N \\
&+\sum_{i=1}^{N-2}\overline{k}_1\cdots\overline{k}_i(k_{i+1}-\overline{k}_{i+1})k_{i+2}\cdots k_N \\
&+\overline{k}_1\cdots\overline{k}_{N-1}(k_N-\overline{k}_N),
\end{aligned}$$

then the conclusion is obvious. $\square$

It is simply to see that the area of the geodesic sphere in $A(a,b)$ satisfies a pinching estimate:

$$\frac{A(\partial B_N(b))}{A(\partial B_N(a))} \geq \frac{A(\partial B_p(b))}{A(\partial B_p(a))} \geq \frac{A(\partial B_N(b))}{A(\partial B_N(a))} - \delta. \tag{4.48}$$

Therefore, after the replacement of functions, the LHS of (4.47) is very small. Thus we have

$$⨍_{A(a_2,b_2)} |\nabla^2 u - \frac{1}{n}\Delta u|^2 \leq \delta. \tag{4.49}$$

Since $\Delta g = 0$, it follows from (4.29), (4.33), (4.34), (4.35), (4.36) and claim 4.2.2 that

$$⨍_{A(a_1,b_1)} |\Delta u - (\overline{\psi} \circ \overline{g}^{-1})''|\nabla \overline{g}|^2|^2 \leq \delta. \tag{4.50}$$

Also note

$$(\overline{\psi} \circ \overline{g}^{-1})''|\nabla \overline{g}|^2 = \overline{\Delta}\overline{u}$$

and

$$\overline{\nabla}^2 \overline{u} = \frac{1}{n}\overline{\Delta}\overline{u},$$

therefore

$$⨍_{A(a_2,b_2)} |\nabla^2 u - \overline{\nabla}^2 \overline{u}|^2 \leq \delta. \tag{4.51}$$

Let us define

$$w = \psi^{-1} \circ u = \overline{g}^{-1} \circ g, \tag{4.52}$$

Putting claim 4.2.2, (4.33), (4.35), (4.37), and (4.51) together, we find

$$⨍_{A(a_2,b_2)} \sum_{i,j} |w_{ij} - S_{ij}|^2 \leq \delta \tag{4.53}$$

where we have used the fact that S is the pull back of the Hessian of the distance function from N.

It follows from (4.35) and (4.37) that

$$|\nabla w| \leq C. \tag{4.54}$$

in $A(a_1,b_1)$.

Putting (4.29) and claim 4.2.2 together, we find

$$⨍_{A(a_1,b_1)} |\nabla w - \nabla r|^2 \leq \delta. \tag{4.55}$$

Combining (4.53), (4.54) and (4.55), we complete the proof of proposition 4.2.1. □

Now we are ready to prove theorem 1.3.5. If there exists a small constant $\epsilon > 0$ such that

$$\frac{A(\partial B_P(b))}{A(\partial B_P(a))} \geq \frac{A(\partial B_N(b))}{A(\partial B_N(a))}(1-\epsilon), \tag{4.56}$$

according to proposition 4.2.1, there exists a smooth function w in $T = A(\frac{3a+2b}{5}, \frac{2a+3b}{5})$ such that (4.27) holds. For simplicity, we write $a_1 = \frac{3a+2b}{5}, b_1 = \frac{2a+3b}{5}, a_2 = \frac{2a_1+b_1}{3}, b_2 = \frac{a_1+2b_1}{3}$(note that our notation is a little bit different from proposition 4.2.1).

Applying (4.152) to w, we find

$$\frac{1}{2}\langle \nabla w, \nabla(\sum_{\gamma\neq 1} w_{\gamma\bar{\gamma}})\rangle - (w_{1\bar{1}}\Delta w - |w_{\alpha\bar{\beta}}|^2 + Re(divY)) = 0 \tag{4.57}$$

where $Y = \sum_{\gamma\neq 1} w_{\bar{\alpha}} w_{\alpha\bar{\gamma}} e_\gamma$. Let $\varphi(r)$ to be the Lipschitz cut-off function in the annulus T depending only on the distance to p, given by

$$\varphi(r) = \begin{cases} \frac{r-a_1}{a_2-a_1} & a_1 \leq r \leq a_2 \\ 1 & a_2 < r < b_2 \\ \frac{b_1-r}{b_1-b_2} & b_2 \leq r \leq b_1. \end{cases} \tag{4.58}$$

Multiplying φ on both sides of (4.57), we integrate in the annulus T and take the average. It follows that

$$\frac{1}{2}\fint_T \varphi\langle \nabla w, \nabla(\sum_{\gamma\neq 1} w_{\gamma\bar{\gamma}})\rangle - \fint_T \{\varphi w_{1\bar{1}}\Delta w - |w_{\alpha\bar{\beta}}|^2 + \varphi Re(divY)\} = 0. \tag{4.59}$$

Using integration by parts, we find

$$\frac{1}{2}\fint_T \varphi\langle \nabla w, \nabla(\sum_{\gamma\neq 1} w_{\gamma\bar{\gamma}})\rangle = -\fint_T \varphi\Delta w\sum_{\gamma\neq 1} w_{\gamma\bar{\gamma}} - \frac{1}{2}\fint_T \sum_{\gamma\neq 1} w_{\gamma\bar{\gamma}}\langle \nabla w, \nabla\varphi\rangle, \tag{4.60}$$

$$\begin{aligned}\fint_T \varphi(w_{1\bar{1}}\Delta w - |w_{\alpha\bar{\beta}}|^2) + \varphi Re(divY) = &\fint_T \varphi(w_{1\bar{1}}\Delta w - |w_{\alpha\bar{\beta}}|^2) \\ &- Re\fint_T \sum_{\gamma\neq 1} \varphi_\gamma w_{\bar{\alpha}} w_{\alpha\bar{\gamma}}.\end{aligned} \tag{4.61}$$

Note that in (4.60) and (4.61), Δw is one half of the real Laplacian of w.

Therefore

$$\begin{aligned}&-\fint_T \varphi\Delta w\sum_{\gamma\neq 1} w_{\gamma\bar{\gamma}} - \frac{1}{2}\fint_T \sum_{\gamma\neq 1} w_{\gamma\bar{\gamma}}\langle \nabla w, \nabla\varphi\rangle - \\ &\fint_T \varphi(w_{1\bar{1}}\Delta w - |w_{\alpha\bar{\beta}}|^2) + Re\fint_T \sum_{\gamma\neq 1} \varphi_\gamma w_{\bar{\alpha}} w_{\alpha\bar{\gamma}} = 0\end{aligned} \tag{4.62}$$

Following (4.27), we see that up to a negligible error, we can replace the functions in (4.62) by the corresponding functions of r. Explicitly,

$$w_{ij} \to S_{ij}, \nabla w \to \nabla r.$$

By (4.56), we can also replace the volume element by that of the real space form N.

In order to derive a contradiction to (4.56), we just need to prove that after the replacement, there is an explicit gap between the LHS and the RHS of (4.62). To prove this, it suffices to find a gap between the LHS and RHS of (4.152) if we do the replacement:

$$f_{\alpha\overline{\beta}} \to S_{\alpha\overline{\beta}}, \nabla f \to \nabla r. \tag{4.63}$$

Using (4.25), we find the gap between the LHS and RHS in (4.152) is

$$\begin{aligned}
&\frac{1}{2}\langle \nabla r, \nabla \sum_{\gamma\neq 1} S_{\gamma\overline{\gamma}}\rangle - (S_{1\overline{1}}S_{\alpha\overline{\alpha}} - |S_{\alpha\overline{\beta}}|^2 + RedivY)\\
&= -\frac{m-1}{2}(\coth r)' - \frac{1}{2}\coth r\frac{2m-1}{2}\coth r + (\frac{1}{2}\coth r)^2 + (m-1)(\coth r)^2\\
&= -\frac{m-1}{2}((\coth r)' - (\coth r)^2)\\
&= \frac{m-1}{2} > 0.
\end{aligned} \tag{4.64}$$

This proves theorem 1.3.5 for $Ric \geq -(2m-1)$. The proof for other cases are similar. We complete the proof of theorem 1.3.5. $\square$

Proof of theorem 1.3.6: We first prove theorem 1.3.6 assuming P is a pole, $Ric \geq -(2m-1)$ and $r > 5$. Let $n = 2m$.

For notational convenience, in the proof of theorem 1.3.6, δ denotes small positive constants depending only on ϵ, n. C denotes constants depending only on n. Moreover, $\lim_{\epsilon\to 0}\delta = 0$. We use $\overline{r}_{ij}$ to denote the Hessian of the distance function on N.

Claim 4.2.4. $1 - n \leq \Delta r \leq 100(n-1)$ *for* $r > 1$.

Proof. By the Laplacian comparison theorem,

$$\Delta r \leq (n-1)\coth r \leq 100(n-1)$$

if $r > 1$. The Bochner formula gives

$$\frac{\partial \Delta r}{\partial r} + \frac{1}{n-1}(\Delta r)^2 - n + 1 \leq 0. \tag{4.65}$$

If $\Delta r < 1 - n$, then after a simple ODE analysis, Δr will blow up when r is large. This is a contradiction. □

Choose an orthonormal frame h_i $(i = 1, 2, ..2m)$ near $x \in M$ such that $h_1 = \nabla r$ and $Jh_{2\alpha-1} = h_{2\alpha}$ for $1 \leq \alpha \leq m$. Define a unitary frame $e_\alpha(\alpha = 1, ...m)$ so that $e_\alpha = \frac{1}{\sqrt{2}}(h_{2\alpha-1} - \sqrt{-1}h_{2\alpha})$ for all α.

Claim 4.2.5. *Along the geodesic emanating from p, we have $\int_r^{r+1} |r_{ij}|^2 < C$ for any $r \geq 1$.*

Proof. According to the Bochner formula,

$$\frac{\partial \Delta r}{\partial r} + |r_{ij}|^2 - (n-1) \leq 0. \tag{4.66}$$

The result follows from claim 4.2.4 after we integrate (4.66) along the geodesic from r to $r+1$. □

Now we argue by contradiction for theorem 1.3.6. Given any $r_0 > 5$, assume the average of the Laplacian satisfies

$$\fint_{\partial B_p(r_0)} \Delta r \geq \Delta r_N(r_0) - \epsilon \tag{4.67}$$

where ϵ is a small positive constant, then $\partial B_p(r)$ can be decomposed into two parts, namely,

$$\begin{aligned} E_1 &= \{x \in \partial B_p(r_0) | \Delta r \geq \Delta r_N(r_0) - \sqrt{\epsilon}\}, \\ E_2 &= \{x \in \partial B_p(r_0) | \Delta r < \Delta r_N(r_0) - \sqrt{\epsilon}\}. \end{aligned} \tag{4.68}$$

For $i = 1, 2$, define the cone as follows:

$$F_i = \{\theta \in UT_p(M) | exp_p(r_0\theta) \in E_i\}. \tag{4.69}$$

We also define

$$E_i(r) = \{x \in M | x \in exp_p(rF_i)\}. \tag{4.70}$$

Claim 4.2.6.

$$\frac{A(E_1)}{A(\partial B_p(r_0))} \geq 1 - \delta. \tag{4.71}$$

Proof. From (4.67), (4.68), we have

$$\begin{aligned}\Delta r_N(r_0) - \epsilon &\leq \fint_{\partial B_p(r_0)} \Delta r \\ &= \frac{\int_{E_1} \Delta r}{A(\partial(B_p(r_0)))} + \frac{\int_{E_2} \Delta r}{A(\partial(B_p(r_0)))} \\ &\leq \frac{A(E_1)}{A(\partial(B_p(r_0)))}\Delta r_N(r_0) + (1 - \frac{A(E_1)}{A(\partial(B_p(r_0)))})(\Delta r_N(r_0) - \sqrt{\epsilon}\}).\end{aligned} \tag{4.72}$$

After a simple manipulation of (4.72), claim 4.2.6 follows. □

Claim 4.2.7. *If $r_0 - 2 \leq r \leq r_0$, then $\frac{A(E_1 r)}{A(\partial B_p(r))} \geq 1 - \delta$.*

Proof. Since we have two bounds for Δr,

$$\begin{aligned}\frac{A(E_1(r))}{A(E_1(r_0))} &\geq \frac{1}{C}, \\ \frac{A(E_2(r))}{A(E_2(r_0))} &\leq C\end{aligned} \tag{4.73}$$

for $r_0 - 2 \leq r \leq r_0$. Claim 4.2.7 follows from (4.71) and (4.73). □

At the point $q = exp_p(r\theta)$, choose an orthonormal frame $\{d_1, ...d_n\}$ near q such that $d_1 = \nabla r$; $Jd_{2\alpha-1} = d_{2\alpha}$ for $1 \leq \alpha \leq m$. Define a unitary frame $\{e_\alpha\}(\alpha = 1, ...m)$ so that $e_\alpha = \frac{1}{\sqrt{2}}(d_{2\alpha-1} - \sqrt{-1}d_{2\alpha})$ for all α.

Claim 4.2.8. *Along the geodesic from p satisfying $\theta \in F_1$, $\int\limits_{r_0-2}^{r_0} |r_{ij} - \overline{r}_{ij}|^2 \leq \delta$.*

Proof. We write the Bochner formula as

$$\frac{\partial \Delta r}{\partial r} + \frac{1}{n-1}(\Delta r)^2 + \sum_{i\neq j} r_{ij}^2 + \sum_{i\neq 1}(\frac{\Delta r}{n-1} - r_{ii})^2 - n + 1 \leq 0. \tag{4.74}$$

According to (4.68), we have

$$\Delta r(r_0, \theta) \geq \Delta r_N(r_0) - \delta \tag{4.75}$$

for $\theta \in F_1$. After a simple analysis of (4.65), it follows that

$$|\Delta r_N(r) - \Delta r(r, \theta)| < \delta \tag{4.76}$$

for $r_0 - 2 \le r \le r_0$. Integrating (4.74) along the geodesic from $r_0 - 2$ to r_0, in view of (4.76), we find that

$$\begin{aligned}\int_{r_0-2}^{r_0} \Big(\sum_{i\neq j} r_{ij}^2 + \sum_{i\neq 1}\big(\frac{\Delta r}{n-1} - r_{ii}\big)^2\Big) &\le \int_{r_0-2}^{r_0} \Big(-\frac{\partial \Delta r}{\partial r} + n - 1 - \frac{1}{n-1}(\Delta r)^2 \\ &= \Delta r(r_0-2,\theta) - \Delta r(r_0,\theta) + 2(n-1) - \frac{1}{n-1}\int_{r_0-2}^{r_0}(\Delta r)^2 \\ &\le \delta.\end{aligned} \tag{4.77}$$

Combining (4.76) and (4.77), claim 4.2.8 follows. $\square$

Applying (4.152) to the distance function to p, we find

$$\frac{1}{2}\langle \nabla r, \nabla(\sum_{\gamma\neq 1} r_{\gamma\overline{\gamma}})\rangle = r_{1\overline{1}}\Delta r - |r_{\alpha\overline{\beta}}|^2 + Re(divY). \tag{4.78}$$

where $Y = \sum_{\gamma\neq 1} r_{\overline{\alpha}} r_{\alpha\overline{\gamma}} e_\gamma$. It is simple to see

$$div|_M Y = div|_{\partial B_p(r)} Y.$$

Thus after the integration of (4.78) on the geodesic sphere $\partial B_p(r)$, we find

$$\int_{\partial B_p(r)} \frac{\partial \sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}}{\partial r} = \int_{\partial B_p(r)} -2\sum_{\alpha,\beta} |r_{\alpha\overline{\beta}}|^2 + 2 r_{\alpha\overline{\alpha}} r_{1\overline{1}}. \tag{4.79}$$

For notational simplicity, we use $\int$ to denote $\int_{\partial B_p(r)}$, $⨍$ to denote the average of $\int_{\partial B_p(r)}$. Taking the average of (4.79) on $\partial B_p(r)$, we get

$$\frac{\partial ⨍ \sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}}{\partial r} = -2⨍ \sum_{\alpha,\beta} |r_{\alpha\overline{\beta}}|^2 + 2⨍ r_{1\overline{1}}\Delta r + 2⨍ \sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}\Delta r - 2⨍ \sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}} ⨍ \Delta r. \tag{4.80}$$

Integrating (4.80) from $r-1$ to r, we find

$$\begin{aligned}
&⨍_{\partial B_p(r)}\sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dA - ⨍_{\partial B_P(r-1)}\sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dA \\
&= \int\limits_{r-1}^{r}(-2⨍\sum_{\alpha,\beta}|r_{\alpha\overline{\beta}}|^2 dA_t + 2⨍ r_{1\overline{1}}\Delta r dA_t \\
&+ 2⨍\sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}\Delta r dA_t - 2⨍\sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dA_t ⨍\Delta r dA_t)dt.
\end{aligned} \tag{4.81}$$

Integration of (4.81) from r_0-1 to r_0 yields

$$\begin{aligned}
&\int_{r_0-1}^{r_0}⨍_{\partial B_p(r)}\sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dAdr - \int_{r_0-1}^{r_0}⨍_{\partial B_p(r-1)}\sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dAdr \\
&= \int_{r_0-1}^{r_0}\int\limits_{r-1}^{r}(-2⨍\sum_{\alpha,\beta}|r_{\alpha\overline{\beta}}|^2 dA_t + 2⨍ r_{1\overline{1}}\Delta r dA_t \\
&+ 2⨍\sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}\Delta r dA_t - 2⨍\sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dA_t ⨍\Delta r dA_t)dtdr.
\end{aligned} \tag{4.82}$$

We come to the estimate of the first term in the LHS of (4.82).

$$\begin{aligned}
&\int_{r_0-1}^{r_0}⨍_{\partial B_p(r)}\sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dAdr \\
&= \int_{r_0-1}^{r_0}\frac{\int_{E_1(r)}\sum\limits_{\alpha\neq 1}(r_{\alpha\overline{\alpha}} - \overline{r}_{\alpha\overline{\alpha}})dA}{A(\partial B_p(r))}dr + \int_{r_0-1}^{r_0}\frac{\int_{E_2(r)}\sum\limits_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dA}{A(\partial B_p(r))}dr \\
&+ \int_{r_0-1}^{r_0}\frac{\int_{E_1(r)}\sum\limits_{\alpha\neq 1}\overline{r}_{\alpha\overline{\alpha}}dA}{A(\partial B_p(r))}dr.
\end{aligned} \tag{4.83}$$

Claim 4.2.9.

$$\begin{aligned}
&|\int_{r_0-1}^{r_0}\frac{\int_{E_1(r)}\sum\limits_{\alpha\neq 1}(r_{\alpha\overline{\alpha}} - \overline{r}_{\alpha\overline{\alpha}})dA}{A(\partial B_p(r))}dr| \leq \delta, \\
&|\int_{r_0-1}^{r_0}\frac{\int_{E_2(r)}\sum\limits_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dA}{A(\partial B_p(r))}dr| \leq \delta, \\
&|\int_{r_0-1}^{r_0}\frac{\int_{E_1(r)}\sum\limits_{\alpha\neq 1}\overline{r}_{\alpha\overline{\alpha}}dA}{A(\partial B_p(r))}dr - \int_{r_0-1}^{r_0}\overline{r}_{\alpha\overline{\alpha}}dr| \leq \delta.
\end{aligned} \tag{4.84}$$

Proof. By claim 4.2.4, for $r_0 - 2 \le r \le r_0$, we have the relation

$$\frac{1}{C} \le \frac{dA(\partial B_p(r))(\theta)}{dA(\partial B_p(r_0))(\theta)} \le C \tag{4.85}$$

where $dA(\partial B_p(r))(\theta)$ is the area element of $\partial B_p(r)$. Therefore we get

$$\begin{aligned}
\int_{r_0-1}^{r_0} \frac{\int_{E_1(r)} \sum\limits_{\alpha\neq 1}(r_{\alpha\overline{\alpha}} - \overline{r}_{\alpha\overline{\alpha}})dA}{A(\partial B_p(r))dA} dr &\le \int_{r_0-1}^{r_0} \frac{\int_{E_1(r)} \sum\limits_{\alpha\neq 1}|r_{\alpha\overline{\alpha}} - \overline{r}_{\alpha\overline{\alpha}}|dA}{A(\partial B_p(r))} dr \\
&\le C\frac{\int_{r_0-1}^{r_0}\int_{E_1(r)}|r_{\alpha\overline{\alpha}} - \overline{r}_{\alpha\overline{\alpha}}|dA}{A(\partial B_p(r_0))} dr \\
&\le C\frac{\int_{F_1}\int_{r_0-1}^{r_0}|r_{\alpha\overline{\alpha}} - \overline{r}_{\alpha\overline{\alpha}}|drdA_{r_0}}{A(\partial B_p(r_0))} \\
&\le C\frac{\int_{F_1(r)}(\int_{r_0-1}^{r_0}\sum\limits_{\alpha\neq 1}|r_{\alpha\overline{\alpha}} - \overline{r}_{\alpha\overline{\alpha}}|^2 dr)^{\frac{1}{2}}dA_{r_0}}{A(\partial B_p(r_0))} \\
&\le \delta.
\end{aligned} \tag{4.86}$$

In the last inequality, we have used claim 4.2.6 and claim 4.2.8. Similarly

$$\begin{aligned}
\int_{r_0-1}^{r_0} \frac{\int_{E_2(r)} \sum\limits_{\alpha\neq 1} r_{\alpha\overline{\alpha}}dA}{A(\partial B_p(r))} dr &\le \int_{r_0-1}^{r_0} \frac{\int_{E_2(r)} \sum\limits_{\alpha\neq 1}|r_{\alpha\overline{\alpha}}|dA}{A(\partial B_p(r))} dr \\
&\le C\frac{\int_{r_0-1}^{r_0}\int_{E_2(r)}\sum\limits_{\alpha\neq 1}|r_{\alpha\overline{\alpha}}|dA}{A(\partial B_p(r_0))} dr \\
&\le C\frac{\int_{F_2(r)}\int_{r_0-1}^{r_0}\sum\limits_{\alpha\neq 1}|r_{\alpha\overline{\alpha}}|drdA_{r_0}}{A(\partial B_p(r_0))} \\
&\le C\frac{\int_{F_2(r)}(\int_{r_0-1}^{r_0}\sum\limits_{\alpha\neq 1}|r_{\alpha\overline{\alpha}}|^2 dr)^{\frac{1}{2}}dA_{r_0}}{A(\partial B_p(r_0))} \\
&\le \delta.
\end{aligned} \tag{4.87}$$

$$\begin{aligned}
\left|\int_{r_0-1}^{r_0} \frac{\int_{E_1(r)}\sum\limits_{\alpha\neq 1}\overline{r}_{\alpha\overline{\alpha}}dA}{A(\partial B_p(r))}dr - \int_{r_0-1}^{r_0}\sum_{\alpha\neq 1}\overline{r}_{\alpha\overline{\alpha}}dr\right| &= \int_{r_0-1}^{r_0}(1 - \frac{A(E_1(r))}{A(\partial B_p(r))})\sum_{\alpha\neq 1}\overline{r}_{\alpha\overline{\alpha}}dr \\
&\le \delta.
\end{aligned} \tag{4.88}$$

This completes the proof of claim 4.2.9. $\square$

Claim 4.2.9 says up to a negligible error, we can replace $r_{\alpha\overline{\beta}}$ by $\overline{r}_{\alpha\overline{\beta}}$ in the first term of (4.82) , where $\overline{r}_{\alpha\overline{\beta}}$ is the complexification of $\overline{r}_{ij}$. Similarly, we can apply the replacement to all other terms in (4.82) with a negligible error. In order to get a contradiction to (4.67), we just need to prove that after the replacement, there is an explicit gap between the LHS and the RHS of (4.82). It suffices to find a gap between of LHS and RHS of (4.152) after the replacement $f_{\alpha\overline{\beta}} \to \overline{r}_{\alpha\overline{\beta}}, \nabla f \to \nabla r$. The computation of the gap is the same as (4.64).

We have thus proved theorem 1.3.6 when p is a pole, $Ric \geq -(2m-1)$, $r > 5$. The proof of the general case is similar. $\square$

4.3 Average Laplacian comparison for some special cases

Proof of theorem 1.3.7:

For simplicity, we write $r_M(x)$ as r. Near a point $q \in M$, choose a unitary frame $\{e_\alpha\} \in T^{1,0}(M)$ such that $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1}J\nabla r)$. Since the metric is unitary invariant, it is simple to see $r_{\alpha\overline{\beta}} = 0$ if $\alpha \neq \beta$; $r_{\alpha\overline{\alpha}} = r_{\beta\overline{\beta}}$ if $\alpha \neq 1$ and $\beta \neq 1$; $r_{\alpha\beta} = 0$ unless $\alpha = \beta = 1$; $r_{11} = -r_{1\overline{1}}$.

Using $r_{1\overline{1}} = \Delta r - (m-1)r_{2\overline{2}}$ and (4.2), we find

$$\frac{1}{2}R_{1\overline{1}} + \frac{\partial \Delta r}{\partial r} + (m-1)|r_{2\overline{2}}|^2 + 2(\Delta r - (m-1)r_{2\overline{2}})^2 = 0. \tag{4.89}$$

Applying (4.152) to r, after a simplification, we have

$$\frac{\partial r_{2\overline{2}}}{\partial r} = 2r_{2\overline{2}}(\Delta r - mr_{2\overline{2}}). \tag{4.90}$$

We use $\overline{\Delta}r$ and $\overline{r}_{\alpha\overline{\beta}}$ to denote the Laplacian and complex Hessian of the distance function in M_k. Let us write down the equations for M_k analogue to (4.89) and (4.90). Explicitly,

$$\frac{1}{2}(m+1)k + \frac{\partial \overline{\Delta}r}{\partial r} + (m-1)|\overline{r}_{2\overline{2}}|^2 + 2(\overline{\Delta}r - (m-1)\overline{r}_{2\overline{2}})^2 = 0, \tag{4.91}$$

$$\frac{\partial \overline{r}_{2\overline{2}}}{\partial r} = 2\overline{r}_{2\overline{2}}(\overline{\Delta}r - m\overline{r}_{2\overline{2}}). \tag{4.92}$$

We shall regard $r_{\alpha\overline{\beta}}$ and $\overline{r}_{\alpha\overline{\beta}}$ as functions of r. (4.89), (4.91) give

$$\begin{aligned}(\overline{\Delta}r-\Delta r)' &\geq \frac{1}{2}(R_{1\overline{1}}-(m+1)k)-f(r)|\overline{\Delta}r-\Delta r|-f(r)|\overline{r}_{2\overline{2}}-r_{2\overline{2}}| \\ &\geq -f(r)|\overline{\Delta}r-\Delta r|-f(r)|\overline{r}_{2\overline{2}}-r_{2\overline{2}}|.\end{aligned} \tag{4.93}$$

Similarly (4.90), (4.92) give

$$(\overline{r}_{2\overline{2}}-r_{2\overline{2}})' \geq -f(r)|\overline{\Delta}r-\Delta r|-f(r)|\overline{r}_{2\overline{2}}-r_{2\overline{2}}|. \tag{4.94}$$

In (4.93) and (4.94), $f(r)$ is a suitable positive function depending only on the metric g of M. (4.93) and (4.94) yield

$$(\overline{\Delta}r-\Delta r+(\overline{r}_{2\overline{2}}-r_{2\overline{2}}))' \geq -2f(r)(|\overline{\Delta}r-\Delta r|+|\overline{r}_{2\overline{2}}-r_{2\overline{2}}|). \tag{4.95}$$

We divide the proof of theorem 1.3.7 into two cases. Namely, $k=-1$ and $k=1$. Note that the case $k=0$ is included in the real Laplacian comparison theorem.

First consider the case $k=-1$. It suffices to prove the claim below:

Claim 4.3.1. *$\overline{\Delta}r-\Delta r$ and $\overline{r}_{2\overline{2}}-r_{2\overline{2}}$ are always nonnegative.*

Proof. First we prove the claim under the assumption as follows:

$$\Delta r<\overline{\Delta}r \text{ and } r_{2\overline{2}}<\overline{r}_{2\overline{2}} \text{ when } r \text{ is small.} \tag{4.96}$$

If the claim is not true, there are three possibilities.

1. When r is increasing, $\overline{\Delta}r-\Delta r$ is becoming negative before $\overline{r}_{2\overline{2}}-r_{2\overline{2}}$ does.
2. $\overline{r}_{2\overline{2}}-r_{2\overline{2}}$ is becoming negative before $\overline{\Delta}r-\Delta r$ does.
3. There exists a constant $r_0>0$ such that $\overline{r}_{2\overline{2}}|_{r=r_0}=r_{2\overline{2}}|_{r=r_0}$, $\overline{\Delta}r|_{r=r_0}=\Delta r|_{r=r_0}$. $\Delta r<\overline{\Delta}r$ and $r_{2\overline{2}}<\overline{r}_{2\overline{2}}$ for $r<r_0$.

For case 1, let $r=r_0>0$ be the first radius such that $\overline{\Delta}r-\Delta r$ is becoming negative while $\overline{r}_{2\overline{2}}(r_0)-r_{2\overline{2}}(r_0)>0$.

We are going to prove

$$(\overline{\Delta}r-\Delta r)'|_{r=r_0}>0. \tag{4.97}$$

(4.89) and (4.91) give

$$\begin{aligned}(\overline{\Delta}r-\Delta r)' \geq\ & 2(\Delta r-(m-1)r_{2\overline{2}})^2+(m-1)r_{2\overline{2}}^2 \\ &-(2(\overline{\Delta}r-(m-1)\overline{r}_{2\overline{2}})^2+(m-1)\overline{r}_{2\overline{2}}^2).\end{aligned} \tag{4.98}$$

To prove (4.97), it suffices to prove

$$2(\Delta r-(m-1)r_{2\bar{2}})^2+(m-1)r_{2\bar{2}}^2-2(\overline{\Delta}r-(m-1)\overline{r}_{2\bar{2}})^2-(m-1)\overline{r}_{2\bar{2}}^2>0. \quad (4.99)$$

Since $(\Delta r-\overline{\Delta}r)|_{r=r_0}=0$, after a simplification, (4.99) is equivalent to

$$(\overline{r}_{2\bar{2}}-r_{2\bar{2}})(4\Delta r-(2m-1)(r_{2\bar{2}}+\overline{r}_{2\bar{2}}))>0. \quad (4.100)$$

According to the assumption in case 1, we have

$$(\overline{r}_{2\bar{2}}-r_{2\bar{2}})|_{r=r_0}>0. \quad (4.101)$$

Using $k=-1$ and the assumption in case 1, we find

$$\Delta r|_{r=r_0}=\overline{\Delta}r|_{r=r_0}>\frac{2m-1}{2}\overline{r}_{\bar{2}2}|_{r=r_0}. \quad (4.102)$$

Therefore

$$(4\Delta r-(2m-1)(r_{2\bar{2}}+\overline{r}_{2\bar{2}}))|_{r=r_0}>0. \quad (4.103)$$

Putting (4.101) and (4.103) together, we obtain the proof of (4.100) and (4.97). However, (4.97) contradicts the assumption that $\overline{\Delta}r-\Delta r$ is becoming negative when $r=r_0$. Therefore case 1 can not happen.

Now consider case 2. Let $r=r_0>0$ be the first radius such that $\overline{r}_{2\bar{2}}-r_{2\bar{2}}$ is becoming negative while $\overline{\Delta}r(r_0)-\Delta r(r_0)>0$.

Using (4.90), (4.92), $r_{2\bar{2}}|_{r=r_0}=\overline{r}_{2\bar{2}}|_{r=r_0}$ and $\overline{r}_{2\bar{2}}>0$, we find

$$(\overline{r}_{2\bar{2}}-r_{2\bar{2}})'|_{r=r_0}=(2\overline{r}_{2\bar{2}}(\overline{\Delta}r-\Delta r))|_{r=r_0}>0 \quad (4.104)$$

(4.104) contradicts the assumption that $\overline{r}_{2\bar{2}}-r_{2\bar{2}}$ is becoming negative at $r=r_0$. Therefore case 2 can not happen.

Consider case 3 now. Using the assumption that $\Delta r<\overline{\Delta}r$ and $r_{2\bar{2}}<\overline{r}_{2\bar{2}}$ for $r<r_0$, we integrate (4.95) from $\frac{r_0}{2}$ to r_0. It follows

$$(\overline{\Delta}r-\Delta r+\overline{r}_{2\bar{2}}-r_{2\bar{2}})|_{r=r_0}>0. \quad (4.105)$$

This contradicts the assumption of case 3.

So far we have proved claim 4.3.1 under the condition (4.96). For general case, let $\tilde{g} = (1+\epsilon)g$ where ϵ is a small positive constant. It is simple to check that $\tilde{g}$ satisfies $Ric(\tilde{g}) \geq -(m+1)\tilde{g}$. After a simple computation of the asymptotic expansion of $\Delta_{\tilde{g}} r$ and $(r_{\tilde{g}})_{2\bar{2}}$ for small r, $\tilde{g}$ satisfies (4.96). The proof of claim 4.3.1 is complete if we let ϵ approach 0.

□

The proof of the case $k = -1$ is complete.

Now we turn to the case $k = 1$.

Claim 4.3.2. *For any $r_0 > 0$, if*

$$\overline{r}_{2\bar{2}} - r_{2\bar{2}} \geq 0, \overline{\Delta} r - \Delta r - (m-1)(\overline{r}_{2\bar{2}} - r_{2\bar{2}}) \geq 0 \tag{4.106}$$

for any $r \in (0, r_0)$, there exists a function $g(r)$ such that

$$(\overline{\Delta} r - \Delta r - (m-1)(\overline{r}_{2\bar{2}} - r_{2\bar{2}}))' \geq g(r)(\overline{\Delta} r - \Delta r - (m-1)(\overline{r}_{2\bar{2}} - r_{2\bar{2}})) \tag{4.107}$$

for $r \in (0, r_0)$.

Proof. By (4.89), (4.90), (4.91) and (4.92), we have

$$\begin{aligned}
&(\overline{\Delta} r - \Delta r - (m-1)(\overline{r}_{2\bar{2}} - r_{2\bar{2}}))' \\
&= \frac{1}{2}(R_{1\bar{1}} - (m+1)) + (m-1)r_{2\bar{2}}^2 + 2(\Delta r - (m-1)r_{2\bar{2}})^2 - (m-1)\overline{r}_{2\bar{2}}^2 \\
&- 2(\overline{\Delta} r - (m-1)\overline{r}_{2\bar{2}})^2 + (m-1)(2r_{2\bar{2}}\Delta r - 2mr_{2\bar{2}}^2 - 2\overline{r}_{2\bar{2}}\overline{\Delta} r + 2m\overline{r}_{2\bar{2}}^2) \\
&\geq (m-1+2(m-1)^2 - 2m(m-1))r_{2\bar{2}}^2 + 2(\Delta r)^2 + (-4(m-1) + 2(m-1))r_{2\bar{2}}\Delta r \\
&- ((m-1+2(m-1)^2 - 2m(m-1))\overline{r}_{2\bar{2}}^2 + 2(\overline{\Delta} r)^2 + (-4(m-1) + 2(m-1))\overline{r}_{2\bar{2}}\overline{\Delta} r) \\
&= -(m-1)r_{2\bar{2}}^2 + 2(\Delta r)^2 - 2(m-1)r_{2\bar{2}}\Delta r - (-(m-1)\overline{r}_{2\bar{2}}^2 + 2(\overline{\Delta} r)^2 - 2(m-1)\overline{r}_{2\bar{2}}\overline{\Delta} r) \\
&= (m-1)(\overline{r}_{2\bar{2}} - r_{2\bar{2}})(\overline{r}_{2\bar{2}} + r_{2\bar{2}}) - 2(\overline{\Delta} r - \Delta r)(\overline{\Delta} r + \Delta r) \\
&+ 2(m-1)\overline{\Delta} r(\overline{r}_{2\bar{2}} - r_{2\bar{2}}) + 2(m-1)r_{2\bar{2}}(\overline{\Delta} r - \Delta r) \\
&= (\overline{\Delta} r - \Delta r)(2(m-1)r_{2\bar{2}} - 2(\overline{\Delta} r + \Delta r)) + (m-1)(\overline{r}_{2\bar{2}} - r_{2\bar{2}})(\overline{r}_{2\bar{2}} + r_{2\bar{2}} + 2\overline{\Delta} r).
\end{aligned} \tag{4.108}$$

Note that $\overline{\Delta} r - \Delta r \geq 0$ when $r < r_0$. To prove claim 4.3.2, by (4.107) and (4.108), it suffices to find a function $g(r)$ satisfying the two inequalities below:

$$2(m-1)r_{2\bar{2}} - 2(\overline{\Delta} r + \Delta r) \geq g, \tag{4.109}$$

$$(m-1)(\overline{r}_{2\overline{2}}+r_{2\overline{2}}+2\overline{\Delta}r)\geq -(m-1)g. \tag{4.110}$$

We take

$$g=-2\overline{\Delta}r-(\overline{r}_{2\overline{2}}+r_{2\overline{2}}), \tag{4.111}$$

then g satisfies (4.110). Plugging g in (4.109), after a slight simplification, it suffices to prove

$$2(m-1)r_{2\overline{2}}+\overline{r}_{2\overline{2}}+r_{2\overline{2}}\geq 2\Delta r. \tag{4.112}$$

Since $k=1$, a simple computation gives

$$\overline{\Delta}r<\frac{2m-1}{2}\overline{r}_{2\overline{2}}. \tag{4.113}$$

Putting (4.106) and (4.113) together, we get

$$\Delta r-(m-1)r_{2\overline{2}}\leq \overline{\Delta}r-(m-1)\overline{r}_{2\overline{2}}\leq \frac{\overline{r}_{2\overline{2}}}{2}. \tag{4.114}$$

An observation of (4.90) gives

$$r_{2\overline{2}}>0. \tag{4.115}$$

(4.114) and (4.115) imply (4.112). The proof of claim 4.3.2 is complete. $\square$

Since $\Delta r=(m-1)r_{2\overline{2}}+(\Delta r-(m-1)r_{2\overline{2}})$, to prove theorem 1.3.7 for the case $k=1$, it suffices to prove the claim as follows:

Claim 4.3.3. $\overline{r}_{2\overline{2}}-r_{2\overline{2}}\geq 0$, $\overline{\Delta}r-\Delta r-(m-1)(\overline{r}_{2\overline{2}}-r_{2\overline{2}})\geq 0$ *for all* r.

Proof. We first prove claim 4.3.3 under the assumption that $\overline{r}_{2\overline{2}}-r_{2\overline{2}}$ and $\overline{\Delta}r-\Delta r-(m-1)(\overline{r}_{2\overline{2}}-r_{2\overline{2}})$ are positive for small r. If claim 4.3.3 does not hold, there are three possibilities.

1. When r is increasing, $\overline{\Delta}r-\Delta r-(m-1)(\overline{r}_{2\overline{2}}-r_{2\overline{2}})$ is becoming negative before $\overline{r}_{2\overline{2}}-r_{2\overline{2}}$ does.

2. $\overline{r}_{2\overline{2}}-r_{2\overline{2}}$ is becoming negative before $\overline{\Delta}r-\Delta r-(m-1)(\overline{r}_{2\overline{2}}-r_{2\overline{2}})$ does.

3. There exists a constant $r_0>0$ such that $\overline{r}_{2\overline{2}}|_{r=r_0}=r_{2\overline{2}}|_{r=r_0}$, $\overline{\Delta}r-\Delta r-(m-1)(\overline{r}_{2\overline{2}}-r_{2\overline{2}})|_{r=r_0}=0$. $\overline{\Delta}r-\Delta r-(m-1)(\overline{r}_{2\overline{2}}-r_{2\overline{2}})>0$ and $\overline{r}_{2\overline{2}}-r_{2\overline{2}}>0$ for $r<r_0$.

For case 1, we apply claim 4.3.2. Let $r=r_0>0$ be the first radius such that $\overline{\Delta}r-\Delta r-(m-1)(\overline{r}_{2\overline{2}}-r_{2\overline{2}})$ is becoming negative while $\overline{r}_{2\overline{2}}(r_0)-r_{2\overline{2}}(r_0)>0$. Integrating

(4.107) from $\frac{r_0}{2}$ to r_0, we find $\overline{\Delta}r - \Delta r - (m-1)(\overline{r}_{2\overline{2}} - r_{2\overline{2}})|_{r=r_0} > 0$. This contradicts the assumption of case 1.

For case 2 and case 3, we can get the same contradiction as in the proof of claim 4.3.1.

Thus we have completed the proof of claim 4.3.3 under the condition that $\overline{r}_{2\overline{2}} - r_{2\overline{2}}$ and $\overline{\Delta}r - \Delta r - (m-1)(\overline{r}_{2\overline{2}} - r_{2\overline{2}})$ are positive for small r. To remove the condition, we can use the same strategy as in the proof of claim 4.3.1. □

This proves the case when $k = 1$. The proof of theorem 1.3.7 is complete.

□

Using $⨍$ to denote the average on the geodesic sphere $\partial B_p(r)$, we are going to prove the following theorem:

Theorem 4.3.1. *Let M^m be a complete Kähler manifold such that $Ric \geq -(m+1)$. Consider a point $p \in M$, define r to be the distance function to p on M. Near a point $q \in M$, choose a unitary frame $\{e_\alpha\} \in T^{1,0}(M)$ such that $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1}J\nabla r)$. Let $\overline{M}^m$ be the simply connected complex space form with constant bisectional curvature -1. Use $\overline{r}_{\alpha\overline{\beta}}$ and $\overline{\Delta}r$ to denote the complex hessian and Laplacian of the distance function on $\overline{M}$. If either Δr or $\sum\limits_{\alpha\neq 1} r_{\alpha\overline{\alpha}}$ is a function of r, then inside the injective radius of p, the following average Laplacian comparison holds,*

$$⨍ \Delta r \leq \overline{\Delta}r(r). \tag{4.116}$$

Proof. To prove theorem 4.3.1, it suffices to prove the following claim:

Claim 4.3.4.

$$⨍ \Delta r \leq \overline{\Delta}r, ⨍ \sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}} \leq \sum_{\alpha\neq 1} \overline{r}_{\alpha\overline{\alpha}}. \tag{4.117}$$

Proof. Let $u(r) = ⨍ \Delta r, v(r) = ⨍ \sum\limits_{\alpha\neq 1} r_{\alpha\overline{\alpha}}$. Integrating the Bochner formula (4.2) on the geodesic sphere, we find

$$\int_{\partial B_p(r)} \frac{\partial \Delta r}{\partial r} \leq \int_{\partial B(P,r)} -\frac{1}{2}Ric_{1\overline{1}} - 2(\Delta r - \sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}})^2 - \sum_{\alpha\neq 1} r_{\alpha\overline{\alpha}}^2.$$

Taking the average on the geodesic sphere $\partial B_p(r)$, we get

$$\frac{\partial ⨍ \Delta r}{\partial r} \leq -\frac{1}{2}⨍ Ric_{1\bar{1}}+2⨍(\Delta r)^2-2(⨍ \Delta r)^2-2⨍(\Delta r-\sum_{\alpha\neq 1} r_{\alpha\bar{\alpha}})^2-\sum_{\alpha\neq 1}⨍ |r_{\alpha\bar{\alpha}}|^2. \quad (4.118)$$

If Δr is a function of r, then (4.118) becomes

$$\frac{\partial ⨍ \Delta r}{\partial r} \leq -\frac{1}{2}⨍ Ric_{1\bar{1}} - 2⨍(\Delta r - \sum_{\alpha\neq 1} r_{\alpha\bar{\alpha}})^2 - \sum_{\alpha\neq 1}⨍ |r_{\alpha\bar{\alpha}}|^2. \quad (4.119)$$

After a slight simplification, we obtain

$$u' \leq -\frac{1}{2}(⨍ R_{1\bar{1}}) - 2u^2 + 4uv - \frac{2m-1}{m-1}v^2. \quad (4.120)$$

(4.80) becomes

$$\frac{\partial ⨍ \sum\limits_{\alpha\neq 1} r_{\alpha\bar{\alpha}}}{\partial r} = -2⨍ \sum_{\alpha,\beta} |r_{\alpha\bar{\beta}}|^2 + 2⨍ r_{1\bar{1}}\Delta r. \quad (4.121)$$

Further simplification gives

$$v' \leq 2uv - \frac{2m}{m-1}v^2. \quad (4.122)$$

If $\sum\limits_{\alpha\neq 1} r_{\alpha\bar{\alpha}}$ is a function of r, then (4.118) becomes

$$\frac{\partial ⨍ \Delta r}{\partial r} \leq -\frac{1}{2}⨍ Ric_{1\bar{1}} - 2(⨍ \Delta r)^2 - 2⨍ |\sum_{\alpha\neq 1} r_{\alpha\bar{\alpha}}|^2 - \sum_{\alpha\neq 1}⨍ |r_{\alpha\bar{\alpha}}|^2 + 4⨍(\Delta r \sum_{\alpha\neq 1} r_{\alpha\bar{\alpha}}) \quad (4.123)$$

where we expanded the term $2⨍(\Delta r - \sum\limits_{\alpha\neq 1} r_{\alpha\bar{\alpha}})^2$ in (4.118). (4.123) is equivalent to (4.120).

For (4.80), we write it as

$$\frac{\partial ⨍ \sum\limits_{\alpha\neq 1} r_{\alpha\bar{\alpha}}}{\partial r} = -2⨍ \sum_{\alpha\neq 1,\beta\neq 1} |r_{\alpha\bar{\beta}}|^2 + 2⨍(\Delta r - \sum_{\alpha\neq 1} r_{\alpha\bar{\alpha}})\sum_{\alpha\neq 1} r_{\alpha\bar{\alpha}} \quad (4.124)$$

which could be written as (4.122).

Combining (4.120), (4.122), the proof of claim 4.3.4 is almost the same as the proof of claim 4.3.1, so we skip the proof here. $\square$

We complete the proof of theorem 4.3.1. $\square$

Remark 4.3.1. *Note that in the proof of theorem 4.3.1, one just needs to assume* $\fint Ric_{1\bar{1}}$ *to be bounded from below by a negative constant.*

4.4 An example

In this section, we give a simple example showing that when the Ricci curvature is bounded from below by a positive constant, the diameter of the Kähler manifold could exceed the diameter of the complex space forms. This implies that in general situation, the sharp version of theorem 1.3.5 is not true comparing with the complex space forms.

Let $N^m = \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$ to be the Kähler manifold equipped with the product metric, each $\mathbb{CP}^1$ has the Fubini-Study metric. We can rescale N^m so that $Ric = g$. It is simple to see

$$diam(N^m) = \sqrt{m}\pi.$$

After a rescaling, $\mathbb{CP}^m$ inherits a Kähler-Einsten metric with $Ric = g$. Given a unit vector $X \in T(\mathbb{CP}^m)$, one can see that

$$R_{XJXJXX} = \frac{2}{m+1},$$

therefore

$$diam(\mathbb{CP}^m) = \frac{\pi}{\sqrt{\frac{2}{m+1}}}.$$

If $m > 1$, one sees that

$$diam(N^m) > diam(\mathbb{CP}^m).$$

One can compare this example with the result of Li and Wang in [44]. Their theorem says that for a complete Kähler manifold, if the bisectional curvature is bounded from below by a positive constant, then $\mathbb{CP}^m$ has the maximal diameter.

If we apply theorem 1.3.3 to the example, then for small r,

$$A(\partial B_{N^m}(r)) \leq A(\partial B_{\mathbb{CP}^m}(r)).$$

However, if r lies between $diam(\mathbb{CP}^m)$ and $diam(N^m)$, then the inequality does not hold. It is not clear to the author whether the sharp version of theorem 1.3.5 is true when the Ricci curvature is bounded from below by a negative constant. We can show that along the diagonal of $\mathbb{CP}^1 \times \mathbb{CP}^1$, the Laplacian of the distance function is greater than that of $\mathbb{CP}^2$. However, the Laplacian of the distance function in $\mathbb{CH}^1 \times \mathbb{CH}^1$ along the diagonal is smaller than that of $\mathbb{CH}^2$.

4.5 Gradient estimate

Proof of theorem 1.3.8:

Let us recall the following theorem due to Yau [82]:

Theorem 4.5.1. *Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below by -(n-1). If f is a positive harmonic function on M, then*

$$|\nabla \log f| \leq n-1. \tag{4.125}$$

Set $n = 2m$, $h = \log f$. By direct computation, we find

$$\Delta h = -|\nabla h|^2. \tag{4.126}$$

At a point $p \in M$ such that $\nabla h \neq 0$, choose an orthonormal frame $\{d_1, ...d_n\}$ near p such that $d_1 = \frac{\nabla h}{|\nabla h|}$; $Jd_{2\alpha-1} = d_{2\alpha}$ for $1 \leq \alpha \leq m$. Define a unitary frame $\{e_\alpha\}(\alpha = 1, ...m)$ so that $e_\alpha = \frac{1}{\sqrt{2}}(d_{2\alpha-1} - \sqrt{-1}d_{2\alpha})$ for all α.

Using the Bochner formula, we compute

$$\begin{aligned}
\Delta|\nabla h|^2 &= 2h_{ij}^2 + 2Ric(\nabla h, \nabla h) + 2\langle \nabla h, \nabla \Delta h\rangle \\
&\geq 2\sum_{i\neq j} h_{ij}^2 + 2h_{11}^2 + \frac{2(\Delta h - h_{11})^2}{n-1} + 2\sum_{i\neq 1}(\frac{\Delta h - h_{11}}{n-1} - h_{ii})^2 \\
&- 2(n-1)|\nabla h|^2 - 2\langle \nabla h, \nabla |\nabla h|^2\rangle \\
&= 2\sum_{i\neq j} h_{ij}^2 + 2h_{11}^2 + 2\sum_{i\neq 1}(\frac{\Delta h - h_{11}}{n-1} - h_{ii})^2 \\
&+ \frac{2}{n-1}(|\nabla h|^4 + 2|\nabla h|^2 h_{11} + h_{11}^2) - 2(n-1)|\nabla h|^2 - 2\langle \nabla h, \nabla |\nabla h|^2\rangle \\
&= 2\sum_{i\neq j} h_{ij}^2 + \frac{2n}{n-1}h_{11}^2 + 2\sum_{i\neq 1}(\frac{\Delta h - h_{11}}{n-1} - h_{ii})^2 \\
&+ \frac{2}{n-1}|\nabla h|^4 - 2(n-1)|\nabla h|^2 - \frac{2n-4}{n-1}\langle \nabla h, \nabla |\nabla h|^2\rangle.
\end{aligned} \tag{4.127}$$

In the computation above, we have used the fact

$$\langle \nabla h, \nabla |\nabla h|^2 \rangle = h_i (h_j^2)_i = 2|\nabla h|^2 h_{11}. \tag{4.128}$$

Now we define

$$\begin{aligned} u &= 2\sum_{i\neq j} h_{ij}^2 + \frac{2n}{n-1} h_{11}^2 + 2\sum_{i\neq 1} (\frac{\Delta h - h_{11}}{n-1} - h_{ii})^2 \geq 0, \\ g &= |\nabla h|^2. \end{aligned} \tag{4.129}$$

Theorem 4.5.1 says that

$$0 \leq g \leq (n-1)^2. \tag{4.130}$$

We may write (4.127) as

$$\begin{aligned} \Delta g &\geq u + \frac{2}{n-1} g^2 - 2(n-1)g - \frac{2n-4}{n-1}\langle \nabla h, \nabla g \rangle \\ &= u + \frac{2}{n-1} g(g-(n-1)^2) - \frac{2n-4}{n-1}\langle \nabla h, \nabla g \rangle \\ &\geq u + 2(n-1)(g-(n-1)^2) - \frac{2n-4}{n-1}\langle \nabla h, \nabla g \rangle. \end{aligned} \tag{4.131}$$

In the second inequality we have used (4.130). Define a new function

$$w = (n-1)^2 - g, \tag{4.132}$$

then

$$0 \leq w \leq (n-1)^2. \tag{4.133}$$

Moreover, w satisfies the inequality

$$-\Delta w \geq u - 2(n-1)w + \frac{2n-4}{n-1}\langle \nabla h, \nabla w \rangle,$$

that is,

$$\Delta w + \frac{2n-4}{n-1}\langle \nabla h, \nabla w \rangle + u \leq 2(n-1)w. \tag{4.134}$$

Let us invoke a theorem in [41], page 76, which is proved by the standard Di Georgi-Nash-Moser iteration:

Theorem 4.5.2. *Let M^n be a complete Riemannian manifold with $Ric \geq k$. Let p be a point in M. If f is a nonnegative function on M satisfying the inequality*

$$\Delta f \leq Af$$

for some constant $A \geq 0$, then there exist positive constants λ, C depending only on r, A, k, n such that

$$(\fint_{B_p(r)} f^\lambda)^{\frac{1}{\lambda}} \leq C \inf_{B_p(\frac{r}{16})} f.$$

We would like to apply theorem 4.5.2 to the function w in (4.134). The situation is a little bit different: there is a first order term in (4.134). However, the coefficient of the first order term in (4.134) is bounded, theorem 4.5.2 works for our case. Therefore we have

$$(\fint_{B_p(r)} w^\lambda)^{\frac{1}{\lambda}} \leq C \inf_{B_p(\frac{r}{16})} w. \tag{4.135}$$

Define a cut-off function φ depending only on the distance to p, given by

$$\varphi(r) = \begin{cases} 1 & 0 \leq r \leq 1 \\ 2 - r & 1 < r < 2 \\ 0 & r \geq 2. \end{cases} \tag{4.136}$$

Multiplying (4.134) on both side by $\varphi^2 w^{-\frac{1}{3}}$, after the integration, we get

$$\int \varphi^2 w^{-\frac{1}{3}} \Delta w + \frac{2n-4}{n-1} \langle \nabla h, \nabla w \rangle w^{-\frac{1}{3}} \varphi^2 + u w^{-\frac{1}{3}} \varphi^2 \leq 2(n-1) \int w^{\frac{2}{3}} \varphi^2.$$

Integration by parts gives

$$\begin{aligned} 2(n-1) \int w^{\frac{2}{3}} \varphi^2 \geq & \int u w^{-\frac{1}{3}} \varphi^2 - \int \langle \nabla(\varphi^2 w^{-\frac{1}{3}}), \nabla w \rangle + \frac{3(2n-4)}{n-1} \int \langle \nabla h, \nabla w^{\frac{1}{3}} \rangle w^{\frac{1}{3}} \varphi^2 \\ = & \int u w^{-\frac{1}{3}} \varphi^2 - 6 \int \varphi w^{\frac{1}{3}} \langle \nabla \varphi, \nabla w^{\frac{1}{3}} \rangle + 3 \int \varphi^2 |\nabla w^{\frac{1}{3}}|^2 \\ & + \frac{3(2n-4)}{n-1} \int \langle \nabla h, \nabla w^{\frac{1}{3}} \rangle w^{\frac{1}{3}} \varphi^2. \end{aligned} \tag{4.137}$$

Using Schwartz inequality, we find

$$\begin{aligned} & -6 \int \varphi w^{\frac{1}{3}} \langle \nabla \varphi, \nabla w^{\frac{1}{3}} \rangle \geq -\delta \int \varphi^2 |\nabla w^{\frac{1}{3}}|^2 - \frac{9}{\delta} \int |\nabla \varphi|^2 w^{\frac{2}{3}}, \\ & \frac{3(2n-4)}{n-1} \int \langle \nabla h, \nabla w^{\frac{1}{3}} \rangle w^{\frac{1}{3}} \varphi^2 \geq -\delta \int \varphi^2 |\nabla w^{\frac{1}{3}}|^2 - \frac{C_1}{\delta} \int |\nabla h|^2 \varphi^2 w^{\frac{2}{3}}. \end{aligned} \tag{4.138}$$

where C_1 is a constant depending only on n.

We take $\delta = 1$. Noting that $|\nabla h| \leq n-1, |\nabla \varphi| \leq 2$, we yield from (4.137) and (4.138) that

$$\begin{aligned} C_2 \int_{B_p(2)} w^{\frac{2}{3}} &\geq \int_{B_p(1)} u w^{-\frac{1}{3}} \\ &\geq (n-1)^{-\frac{2}{3}} \int_{B_p(1)} u. \end{aligned} \tag{4.139}$$

where C_2 is a positive constant depending only on n. Using (4.133), (4.135), (4.139) and the relative volume comparison theorem, we find

$$⨍_{B_p(1)} u \leq C_3 (w(p))^{\alpha} \tag{4.140}$$

where C_3, α are positive constants depending only on n. Following (4.129), (4.140), we obtain

$$⨍_{B_p(1)} 2 \sum_{i \neq j} h_{ij}^2 + \frac{2n}{n-1} h_{11}^2 + 2 \sum_{i \neq 1} (\frac{\Delta h - h_{11}}{n-1} - h_{ii})^2 \leq C(n)(w(p))^{\alpha}. \tag{4.141}$$

(4.126), (4.132), (4.133), (4.135) imply

$$⨍_{B_p(1)} (\Delta h + (n-1)^2)^2 \leq C(n)(w(p))^{\beta}. \tag{4.142}$$

where β is a positive constant depending only on n. (4.141) and (4.142) imply

$$⨍_{B_p(1)} 2 \sum_{i \neq j} h_{ij}^2 + \frac{2n}{n-1} h_{11}^2 + 2 \sum_{i \neq 1} (1 - n - h_{ii})^2 \leq C(n)(w(p))^{\gamma}. \tag{4.143}$$

where $\gamma = \gamma(n) > 0$. Now we would like to use the Kähler structure of M. Applying (4.152) to h, we find

$$\frac{1}{2} \langle \nabla h, \nabla (\sum_{\gamma \neq 1} h_{\gamma \overline{\gamma}}) \rangle = h_{1\overline{1}} \Delta h - |h_{\alpha \overline{\beta}}|^2 + Re(divY) \tag{4.144}$$

where $Y = \sum\limits_{\gamma \neq 1} h_{\overline{\alpha}} h_{\alpha \overline{\gamma}} e_{\gamma}$.

Suppose at a point $p \in M$,

$$|\nabla h(p)| > n - 1 - \epsilon \tag{4.145}$$

where ϵ is a very small constant. Then

$$w(p) \leq C(n)\epsilon. \tag{4.146}$$

Integrating (4.144) on the geodesic ball $B_p(r)$, we get

$$\begin{aligned} &-\int_{B_p(r)} \Delta h \sum_{\alpha\neq 1}(h_{\alpha\overline{\alpha}}+2m-1)+\frac{1}{2}\int_{\partial B_p(r)}\sum_{\alpha\neq 1}(h_{\alpha\overline{\alpha}}+2m-1)\langle\nabla h,\nabla r\rangle \\ &=\int_{B_p(r)} h_{1\overline{1}}\Delta h-|h_{\alpha\overline{\beta}}|^2+\frac{1}{2}Re\int_{\partial B_p(r)}\sum_{\alpha\neq 1}|\nabla h|h_{1\overline{\alpha}}\langle e_\alpha,\nabla r\rangle. \end{aligned} \tag{4.147}$$

Define the annulus $A=\{x\in M\|\frac{1}{2}\leq d(x,p)\leq 1\}$. Integrating (4.147) with respect to r from $\frac{1}{2}$ to 1, dividing both side by $Vol(B_p(1))$, we find

$$\begin{aligned} &-\int_{\frac{1}{2}}^{1}\frac{\int_{B_p(r)}\Delta h\sum\limits_{\alpha\neq 1}(h_{\alpha\overline{\alpha}}+2m-1)}{Vol(B_p(1))}dr+\frac{1}{2}\frac{\int_A\sum\limits_{\alpha\neq 1}(h_{\alpha\overline{\alpha}}+2m-1)\langle\nabla h,\nabla r\rangle}{Vol(B_p(1))} \\ &=\int_{\frac{1}{2}}^{1}\frac{\int_{B(P,r)}h_{1\overline{1}}\Delta h-|h_{\alpha\overline{\beta}}|^2}{Vol(B_p(1))}dr+\frac{1}{2}\frac{Re\int_A\sum\limits_{\alpha\neq 1}|\nabla h|h_{1\overline{\alpha}}\langle e_\alpha,\nabla r\rangle}{Vol(B_p(1))}. \end{aligned} \tag{4.148}$$

In view of (4.143), after the complexification, we obtain

$$⨍_{B_p(1)}\sum_{\alpha\neq\beta}|h_{\alpha\overline{\beta}}|^2+(h_{1\overline{1}}+\frac{2m-1}{2})^2+\sum_{\alpha\neq 1}(1-2m-h_{\alpha\overline{\alpha}})^2\leq C(n)\epsilon^\gamma. \tag{4.149}$$

Following (4.149) and the relative volume comparison, we see that up to a negligible error, we can replace the complex hessian of h in (4.148) by the corresponding constants in (4.149). Explicitly,

$$h_{\alpha\overline{\beta}}\to\begin{cases} 0 & \alpha\neq\beta \\ 1-2m & \alpha=\beta,\alpha\neq 1 \\ \frac{1-2m}{2} & \alpha=\beta=1 \end{cases} \tag{4.150}$$

In order to get a contradiction to (4.145), it suffices to find a gap between the LHS and the RHS of (4.148) if we replace $h_{\alpha\overline{\beta}}$ by (4.150). Plugging (4.150) in (4.148), the

LHS is 0, the RHS is

$$\begin{aligned}
&(\frac{1-2m}{2}(\frac{1-2m}{2}+(m-1)(1-2m))-(\frac{1-2m}{2})^2 \\
&-(m-1)(1-2m)^2)\int_{\frac{1}{2}}^{1}\frac{Vol(B(P,r))}{Vol(B(P,1))}dr \\
&=-\frac{(2m-1)^2(m-1)}{2}\int_{\frac{1}{2}}^{1}\frac{Vol(B(P,r))}{Vol(B(P,1))}dr \\
&\leq -\frac{(2m-1)^2(m-1)}{2}C(n)
\end{aligned} \tag{4.151}$$

where $C(n)$ is a positive constant depending only on n.

The proof of theorem 1.3.8 is complete. □

In the rest of this chapter we will prove theorem 1.3.3.

First, we state two propositions which demonstrate the relation between the derivatives of $A(\partial B_p(r))$ and covariant derivatives of the curvature tensor at p. Section 7 is the first part of the proof of proposition 4.6.1. We shall estimate the derivatives of $A(\partial B_p(r))$ up to order 4. In the estimate of the 4th derivative, the Kähler condition is employed. The most important part is section 8. We use an induction to prove proposition 1. Besides the routine computation, there are two technical lemmas which simplify the computation of higher order covariant derivatives of the curvature tensor significantly. One should note that the Kähler condition is essential in these two lemmas. The last section is devoted to giving an example showing that the pointwise Laplacian comparing with the complex space form does not necessarily hold if the complex dimension is greater or equal to 2.

4.6 Basic set up

Throughout the rest of this chapter, for derivatives of functions of r, we are always evaluating at $r = 0$. Given a point p on a Kähler manifold M^n, fix a unit vector $e_0 \in T_pM$. Along the geodesic l from p with initial direction e_0, consider the Jacobian equation $J'' = R(e_0, J)e_0$. Set up an orthonormal frame $\{e_k\}$ at p such that $Je_{2i} =$

$e_{2i+1}, Je_{2i+1} = -e_{2i}$ for $0 \le i \le n-1$. Parallel transport the frame along the geodesic l. Consider the Jacobian field J_u with initial value $J_u(0) = 0, J'_u(0) = e_u$.

We may write

$$J_u = J_u(r, e_0) = \sum_{i=1}^{\infty} \sum_{v=0}^{2n-1} r^i C_{u,i}^v e_v \tag{4.152}$$

where $C_{u,i}^v$ are constants independent of r. Denote $R_{e_0 e_u e_0 e_v}$ by R_{uv} when e_0 is fixed. Plugging (2.1) in the Jacobian equation, we get

$$\sum_i \sum_v i(i-1) r^{i-2} C_{u,i}^v e_v = \sum_k \sum_w r^k C_{u,k}^w R(e_0, e_w) e_0. \tag{4.153}$$

Along the geodesic l,

$$R(e_0, e_w)e_0 = \sum_{s=0}^{2n-1} \sum_{j=0}^{\infty} \frac{R_{sw}^{(j)}}{j!} e_s r^j$$

where $R_{sw}^{(j)}$ denotes the jth covariant derivative of R_{sw} along e_0 at p. Inserting it in (2.2), we get

$$\sum_{i,v} i(i-1) r^{i-2} C_{u,i}^v e_v = \sum_{k,j,w,s} r^{k+j} C_{u,k}^w \frac{R_{sw}^{(j)}}{j!} e_s.$$

Comparing coefficients, we obtain

$$C_{u,i}^v = \sum_{k+j=i-2,w} C_{u,k}^w \frac{R_{vw}^{(j)}}{j! i(i-1)}. \tag{4.154}$$

A simple iteration gives

$$C_{u,1}^v = \delta_u^v; C_{u,2}^w = 0; C_{u,3}^v = \sum_w C_{u,1}^w \frac{R_{vw}}{6} = \frac{R_{uv}}{6};$$

$$C_{u,4}^v = \sum_w C_{u,1}^w \frac{R'_{vw}}{12} = \frac{R'_{vu}}{12};$$

$$C_{u,5}^v = \sum_w (C_{u,1}^w \frac{R''_{vw}}{40} + C_{u,3}^w \frac{R_{vw}}{20}) = \frac{1}{120} (\sum_s R_{us} R_{sv} + 3R''_{uv}).$$

$$J_u = re_u + \frac{r^3}{6} R_{uv} e_v + \frac{r^4}{12} R'_{uv} e_v + \frac{r^5}{120} (\sum_s R_{us} R_{sv} + 3R''_{uv}) e_v + O(r^6). \tag{4.155}$$

Using dA to denote the standard measure of the unit tangent bundle $UT_p(M)$ at p, via exponential map, we write $\int_{\partial B(p,r)} dA$ as $\int$. Defining

$$W = \frac{\int \sqrt{det < J_u, J_v >}}{r^{2n-1}},$$

we introduce two propositions as follows:

Proposition 4.6.1. *Under the same condition as in theorem 4, if the derivatives of* W *with order from* 1 *to* $(2m-1)(m \geq 1)$ *are the same as that of the complex space form, we have*

Conclusion 1 :

If $m = 2$, $Ric = K$ *at* p.

If $m \geq 3$, *then* $R_{i\bar{j}k\bar{l}} = \frac{K}{n+1}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$ *at* p. *Moreover, for any unit vectors* $u, v, e_0 \in UT_p(M)$, $R_{uv}^{(\lambda)} = 0$ *for* $1 \leq \lambda \leq m-3$ *and* $Ric^{(l)}(e_0, e_0) = 0$ *for* $1 \leq l \leq 2m-4$. *The superscripts are orders of covariant derivatives along direction* e_0.

Conclusion 2 : $W^{(2m)}$ *is less than or equal to that of the complex space form.*

Proposition 4.6.2. *Under the same condition as in theorem 4, if the derivatives of* W *with order from* 1 *to* $(2m)(m \geq 1)$ *are the same as the complex space form,* $W^{(2m+1)} = 0$.

We divide the proof of proposition 4.6.1 into two parts: $m = 1, 2$ and $m \geq 3$.

4.7 The proof of proposition 4.6.1: Part I

This section treats the case $m = 1, 2$. By (2.1), we have

$$\frac{< J_u, J_v >}{r^2} = \sum_{i,j,w} r^{i+j-2} C_{u,i}^w C_{v,j}^w. \tag{4.156}$$

By (2.4),

$$\frac{< J_u, J_u >}{r^2} = 1 + \frac{R_{uu}}{3} r^2 + \frac{R'_{uu}}{6} r^3 + (\frac{2}{45} \sum_s R_{us}^2 + \frac{1}{20} R''_{uu}) r^4 + O(r^5).$$

If $u \neq v$,

$$\frac{< J_u, J_v >}{r^2} = \frac{1}{3} R_{uv} r^2 + \frac{R'_{uv}}{6} r^3 + (\frac{2}{45} \sum_s R_{us} R_{vs} + \frac{1}{20} R''_{uv}) r^4 + O(r^5).$$

Now use the above two expressions to see that

$$\frac{det < J_u, J_v >}{r^{4n-2}} = 1 + \frac{1}{3}\sum_u R_{uu}r^2 + \frac{1}{6}\sum_u R'_{uu}r^3 + (\frac{2}{45}\sum_{u,s} R^2_{us} + \frac{1}{20}\sum_u R''_{uu} + \frac{1}{9}\sum_{u<v} R_{uu}R_{vv} - \frac{1}{9}\sum_{u<v} R^2_{uv})r^4 + O(r^5). \tag{4.157}$$

Considering the identity $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$, we get

$$\frac{\sqrt{det < J_u, J_v >}}{r^{2n-1}} = 1 + \frac{1}{6}\sum_u R_{uu}r^2 + \frac{1}{12}\sum_u R'_{uu}r^3 + (\frac{1}{45}\sum_{u,s} R^2_{us} + \frac{1}{40}\sum_u R''_{uu} + \frac{1}{18}\sum_{u<v} R_{uu}R_{vv} - \frac{1}{18}\sum_{u<v} R^2_{uv} - \frac{1}{72}(\sum_u R_{uu})^2)r^4 + O(r^5). \tag{4.158}$$

Since $W = \frac{\int \sqrt{det<J_u,J_v>}}{r^{2n-1}}$, we find

$$W'(0) = 0, W''(0) = -cs$$

where c is a positive constant depending only on n, s is the scalar curvature at p. Therefore $W''(0)$ is less than or equal to that of the complex space form. This proves proposition 1 for $m = 1$.

Now we consider $m = 2$. According to the assumption of proposition 1, W'' is the same as that of the complex space form. Therefore $s = nK$ at p. Since the Ricci curvature is bounded from below by K, $Ric = Kg$ at p. By (3.3), it is simple to see that the r^3 coefficient of W is 0 by symmetry. Thus to complete the proof for $m = 2$, we just need to show that the 4th derivative of W is less than or equal to that of the complex space form.

We keep in mind that $Ric = Kg$ at p. The r^4 coefficient of W is

$$
\begin{aligned}
c_4 &= \int (\frac{1}{45}\sum_{u,s} R_{us}^2 + \frac{1}{40}\sum_u R''_{uu} + \frac{1}{18}\sum_{u<v} R_{uu}R_{vv} - \frac{1}{18}\sum_{u<v} R_{uv}^2 - \frac{1}{72}(\sum_u R_{uu})^2) \\
&= \frac{1}{360}\int (8\sum_u R_{uu}^2 + 16\sum_{u<v} R_{uv}^2 + 9\sum_u R''_{uu} + 20\sum_{u<v} R_{uu}R_{vv} \\
&- 20\sum_{u<v} R_{uv}^2 - 5(\sum_u R_{uu})^2) \\
&= \frac{1}{360}\int (-2\sum_u R_{uu}^2 + 10(\sum_u R_{uu})^2 - 4\sum_{u<v} R_{uv}^2 + 9\sum_u R''_{uu} - 5(\sum_u R_{uu})^2) \\
&= \frac{1}{360}\int (9\sum_u R''_{uu} - 4\sum_{u<v} R_{uv}^2 - 2\sum_u R_{uu}^2 + 5(\sum_u R_{uu})^2).
\end{aligned}
$$

Note that the Ricci curvature attains the minimum K at p, so

$$
\sum_u R''_{uu} = -Ric''(e_0, e_0) \le 0.
$$

Therefore we have

$$
\begin{aligned}
c_4 &= \frac{1}{360}\int (9\sum_u R''_{uu} - 4\sum_{u<v} R_{uv}^2 - 2\sum_u R_{uu}^2 + 5K^2) \\
&\le -\frac{1}{360}\int (2\sum_u R_{uu}^2 - 5K^2) \\
&= -\frac{1}{360}\int (2\sum_{u\ne 1} R_{uu}^2 + 2R_{11}^2 - 5K^2) \\
&\le -\frac{1}{360}\int (\frac{1}{n-1}(\sum_{u\ne 1} R_{uu})^2 + 2R_{11}^2 - 5K^2) \\
&= -\frac{1}{360}\int (\frac{1}{n-1}(Ric(e_0, e_0) + R_{11})^2 + 2R_{11}^2 - 5K^2) \\
&= -\frac{1}{360}\int (\frac{1}{n-1}K^2 + \frac{2}{n-1}KR_{11} + (\frac{1}{n-1} + 2)R_{11}^2 - 5K^2) \\
&\le -\frac{1}{360}(\int \frac{1}{n-1}K^2 + \frac{2}{n-1}K\int R_{11} + C_1(\int R_{11})^2 - \int 5K^2) \\
&= C_2K^2.
\end{aligned}
\tag{4.159}
$$

In the inequalities above, C_1, C_2 are constants depending only on n.

We explain the inequalities above. In the first inequality, we drop the two terms $\sum_{u<v} R_{uv}^2$ and $\sum_u R''_{uu}$. In the second inequality, we apply Schwartz inequality for directions

e_u that are perpendicular to e_1, e_0. In the third inequality we use Schwartz inequality $\int R_{11}^2 \geq C(\int R_{11})^2$. We make use of the Kähler condition to obtain $\int R_{11} = C_3 s = nC_3 K$, where C_3 is a constant depending only on n. This explains the last equality.

The right hand side of (4.159) is exactly the case of the complex space form. Therefore when W', W'' are the same as the complex space form, $W^{(3)} = 0$ and $W^{(4)}$ is less than or equal to that of the complex space form. (4.159) becomes an equality if and only if the holomorphic sectional curvature is constant at p and $Ric''(e_0, e_0) = 0$ for any $e_0 \in UT_pM$. This completes the proof for $m = 2$.

4.8 The proof of proposition 4.6.1: Part II

This section deals with the case $m \geq 3$. Denote $Ric^{(l)}(e_0, e_0)$ by $Ric^{(l)}$. According to the assumption of proposition 4.6.1, the derivatives of W with order from 1 to $(2m-1)$ are the same as the complex space form. Follow results in the last section, the holomorphic sectional curvature is constant at p and $Ric'' = 0$ for any e_0. That is to say, at p,

$$R_{i\bar{j}k\bar{l}} = \frac{K}{n+1}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}), Ric'' = 0.$$

Therefore, we proved conclusion 1 of proposition 4.6.1 for $m = 3$.

Now we use induction. Assuming conclusion 1 of proposition 4.6.1 holds for $k = m$, we shall prove that for $k = m + 1$.

Claim 4.8.1. *Under the hypothesis of the induction above, $C^v_{u,i}(i \leq m)$ are constants independent of the direction e_0. In fact, they are the same as that of the complex space form.*

Proof. Insert the induction hypothesis in (4.154).

□

Let us write

$$\frac{det < J_u, J_v >}{r^{4n-2}} = 1 + \sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j + O(r^{2m+1}). \tag{4.160}$$

Combining claim 4.8.1 with (4.152), we find that a_i are constants independent of the direction e_0. (4.152) also yields $C^v_{u,m+1} = C^u_{v,m+1}$ for all u, v. Direct expansion of the determinant via (4.152) gives

$$\begin{aligned} b_{2m} = & \sum_{u,v}(C^v_{u,m+1})^2 + 4\sum_{u<v} C^u_{u,m+1}C^v_{v.m+1} + 2\sum_u C^u_{u,2m+1} - 4\sum_{u<v} C^v_{u,m+1}C^u_{v,m+1} \\ & + \sum_{i=1}^m C^v_{u,m+i}C_{i,m,u,v} + C_{0,m} \end{aligned} \tag{4.161}$$

where $C_{i,m,u,v}$ and $C_{0,m}$ are all constants independent of the direction e_0.

Note also

$$b_m = 2\sum_u C^u_{u,m+1} + Constant. \tag{4.162}$$

Applying $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \sum\limits_{k=3}^{\infty} \lambda_k x^k (|x| < 1)$, we obtain

$$\begin{aligned} \frac{\sqrt{det < J_u, J_v >}}{r^{2n-1}} = & 1 + \frac{1}{2}(\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j) - \frac{1}{8}(\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j)^2 \\ & + \sum_{k=3}^{\infty} \lambda_k (\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j)^k + O(r^{2m+1}). \end{aligned} \tag{4.163}$$

Lemma 4.8.1. *the $2m$th order coefficient of the expansion of W is*

$$\begin{aligned} c_{2m} = & \int (\frac{1}{2}\sum_{u,v}(C^v_{u,m+1})^2 + 2\sum_{u<v} C^u_{u,m+1}C^v_{v,m+1} + \sum_u C^u_{u,2m+1} \\ & - 2\sum_{u<v} C^v_{u,m+1}C^u_{v,m+1} - \frac{1}{2}(\sum_u C^u_{u,m+1})^2 + \sum_{i=1}^m C^v_{u,m+i}\widetilde{C}_{i,m,u,v}) + \widetilde{C}_{0,m} \end{aligned} \tag{4.164}$$

where $\widetilde{C}_{i,m,u,v}$ and $\widetilde{C}_{0,m}$ are constants independent of the direction e_0.

Proof. It suffices to find out the contribution of each term in (4.164) to c_{2m}. We keep in mind that coefficients a_i in (4.160) are independent of e_0.

By (4.161), the contribution of term $1 + \frac{1}{2}(\sum\limits_{i=1}^{m-1} a_i r^i + \sum\limits_{j=m}^{2m} b_j r^j)$ to c_{2m} is

$$\begin{aligned} & \int \frac{1}{2}\sum_{u,v}(C^v_{u,m+1})^2 + 2\sum_{u<v} C^u_{u,m+1}C^v_{v,m+1} + \sum_u C^u_{u,2m+1} - 2\sum_{u<v} C^v_{u,m+1}C^u_{v,m+1} \\ & + \frac{1}{2}(\sum_{i=1}^m C^v_{u,m+i}C_{i,m,u,v} + C_{0,m}). \end{aligned} \tag{4.165}$$

The contribution of the term $-\frac{1}{8}(\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j)^2$ to c_{2m} is

$$-\int (\frac{1}{8}b_m^2 + \sum_{i=1}^{m} C_{u,m+i}^{v} p_{i,m,u,v}) + p_{0,m}. \tag{4.166}$$

By (4.162), it could be written as

$$-\int (\frac{1}{2}(\sum_u C_{u,m+1}^{u})^2 + \sum_{i=1}^{m} C_{u,m+i}^{v} p_{i,m,u,v}) + p_{0,m}. \tag{4.167}$$

The contribution of $\sum_{k=3}^{\infty} \lambda_k (\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j)^k$ to c_{2m} is

$$\int \sum_{i=1}^{m} C_{u,m+i}^{v} q_{i,m,u,v} + q_{0,m}. \tag{4.168}$$

Here, $p_{i,m,u,v}$, $q_{i,m,u,v}$, $p_{0,m}$ and $q_{0,m}$ are all constants independent of the direction e_0. The proof is complete if we combine these equalities. □

Lemma 4.8.2.

$$c_{2m} = \int Q(R_{uv}^{(m-2)}) + \sum_{i=-2}^{m-4} h_{m,i} \int R_{11}^{(m+i)} + C_m \int Ric^{(2m-2)} + Constant \tag{4.169}$$

where Q is a negative definite quadratic form, $h_{m,i}$ are constants and C_m is a negative constant.

Proof. By the induction hypothesis and (4.154), we have

$$\begin{aligned} C_{u,2m+1}^{u} &= \sum_{k+j=2m-1,w} \frac{C_{u,k}^{w} R_{uw}^{(j)}}{j!(2m+1)2m} \\ &= \frac{1}{(2m+1)2m}(\sum_w (\frac{R_{uw}^{(m-2)} C_{u,m+1}^{w}}{(m-2)!} \\ &\quad + \sum_{j=m-1}^{2m-2} B_{j,m,w,u} R_{uw}^{(j)}) + R_{uu} C_{u,2m-1}^{u}) \end{aligned} \tag{4.170}$$

where $B_{j,m,w,u}$ are constants. For $i \le m$, we have

$$C_{u,m+i}^{v} = \sum_{j=m-2}^{m+i-3} d_{m,i,j,w,u} R_{uw}^{(j)} + Constant \tag{4.171}$$

where $d_{m,i,j,w,u}$ are constants. In particular, we have

$$C^v_{u,m+1} = \sum_{k+j=m-1,w} C^w_{u,k}\frac{R^{(j)}_{vw}}{j!m(m+1)} = \frac{1}{m(m+1)}(\frac{R^{(m-2)}_{vu}}{(m-2)!} + C^v_{u,m-1}R_{vv}). \quad (4.172)$$

By the induction hypothesis,

$$\sum_u R^{(m-2)}_{uu} = -Ric^{(m-2)} = 0. \quad (4.173)$$

Therefore

$$\begin{aligned}\sum_u (R^{(m-2)}_{uu})^2 &= (\sum_u R^{(m-2)}_{uu})^2 - 2\sum_{u<v} R^{(m-2)}_{uu}R^{(m-2)}_{vv} \\ &= -2\sum_{u<v} R^{(m-2)}_{uu}R^{(m-2)}_{vv}.\end{aligned} \quad (4.174)$$

Inserting (4.170), (4.171), (4.172) in (4.164), we find

$$c_{2m} = \int Q(R^{(m-2)}_{uv}) + \sum_{i=-2}^{m-2}\int\sum_{u,v} h_{m,i,u,v}R^{(m+i)}_{uv} + Constant. \quad (4.175)$$

Now we prove that Q is negative definite. Let us check each term in (4.164). By (4.172) ,the term $\frac{1}{2}\sum_{u,v}(C^v_{u,m+1})^2$ in (4.164) contributes to the quadratic term

$$\sum_{u,v}\frac{1}{2m^2(m+1)^2((m-2)!)^2}(R^{(m-2)}_{uv})^2. \quad (4.176)$$

The term $2\sum_{u<v} C^u_{u,m+1}C^v_{v,m+1}$ contributes to the quadratic term

$$\sum_{u<v}\frac{2}{m^2(m+1)^2((m-2)!)^2}R^{(m-2)}_{uu}R^{(m-2)}_{vv}. \quad (4.177)$$

By (4.174), it could be written as

$$-\frac{1}{m^2(m+1)^2((m-2)!)^2}\sum_u (R^{(m-2)}_{uu})^2. \quad (4.178)$$

By (4.170) and (4.172), the term $\sum_u C^u_{u,2m+1}$ contributes to the quadratic term

$$\sum_{u,v}\frac{1}{2m^2(m+1)(2m+1)((m-2)!)^2}(R^{(m-2)}_{uv})^2. \quad (4.179)$$

The term $-2\sum_{u<v} C^v_{u,m+1}C^u_{v,m+1}$ contributes to the quadratic term

$$-\sum_{u<v}\frac{2}{m^2(m+1)^2((m-2)!)^2}(R^{(m-2)}_{uv})^2. \tag{4.180}$$

The term $-\frac{1}{2}(\sum_u C^u_{u,m+1})^2$ is obvious semi-negative definite.

Combine (4.176), (4.177), (4.178), (4.179) and (4.180), it follows that the quadratic form in (4.169) is negative definite.

Consider the linear terms in (4.175). By the induction hypothesis, the coefficients $h_{m,i,u,v}$ are unchanged if we take a unitary transformation keeping the direction e_0 fixed. Comparing the coefficients of the linear order terms, we see $h_{m,i,u,v} = 0$ if $u \neq v$; $h_{m,i,u,u} = h_{m,i,v,v}$ if $u \neq e_1$ and $v \neq e_1$. Therefore, the linear terms $h_{m,i,u,u}R^{(m+i)}_{uu}$ could be absorbed in $Ric^{(m+i)}$ with the terms $-h_{m,i}R^{(m+i)}_{11}$ left. Also note that by induction hypothesis, $Ric^{(l)} = 0$ for $0 < l \leq 2m-3(Ric^{(2m-3)}$ vanishes as the Ricci curvature attains its minimum at p). Finally, one verifies that $\sum_u C^u_{u,2m+1}$ is the only term in (4.164) that has contribution to $R^{(2m-2)}_{uv}$. Therefore the linear terms in (4.175) could be written as $\sum_{i=-2}^{m-4} h_{m,i}\int R^{(m+i)}_{11} + C_m\int Ric^{(2m-2)}$. From (4.170), it is simple to check that C_m is negative.

□

By the induction hypothesis and that the Ricci curvature attains its minimum at p, we have $Ric^{(2m-2)} \geq 0$. It follows from lemma above

$$c_{2m} \leq \sum_{i=-2}^{m-4} h_{m,i}\int R^{(m+i)}_{11} + Constant. \tag{4.181}$$

We would like to prove that the linear terms $\int R^{(m+i)}_{11}$ vanish for $-2 \leq i \leq m-4$. Note that by symmetry, if $m+i$ is odd, the integral equals 0. Let us deal with case when $m+i$ is even. We shall check when $i = m-4$. Other cases are similar. Let

$$A = -\frac{1}{4}\int R^{(2m-4)}_{11}. \tag{4.182}$$

Set up an orthonormal frame $\{f_i\}$ at p such that $Jf_{2j} = f_{2j+1}, Jf_{2j+1} = -f_{2j}$ for $0 \leq j \leq n-1$. Letting $\beta_j = \frac{1}{2}(f_{2j} - \sqrt{-1}f_{2j+1})$, in a small neighborhood of p, we parallel transport the frame along each geodesic through p. Suppose

$$e_0 = \sum_{j=0}^{n-1}(z_j\beta_j + \overline{z_j}\overline{\beta_j}). \tag{4.183}$$

Lemma 4.8.3. *Under the assumption of the induction in proposition 1, $Rm^{(\lambda)} = 0$ at p for $1 \le \lambda \le m-3$, where $Rm^{(\lambda)}$ denotes any covariant derivative of the curvature tensor with order λ at p.*

Proof. We use induction. If $\lambda = 0$, lemma 4.8.3 automatically holds since there is nothing to prove. Suppose lemma 3 holds for $k < \lambda$. For $k = \lambda$, we plug (4.183) in $R^{(\lambda)}_{uv}$.

Claim 4.8.2. *We can commute the covariant derivatives for $R^{(\lambda)}_{uv}$.*

Proof. We only need to consider the case $\lambda \ge 2$. By the induction hypothesis of lemma 3, the covariant derivatives of the curvature tensor vanish up to order $\lambda - 1$ at p. If $\lambda > 3$, claim 2 follows from the ricci identity. Now suppose $\lambda = 2$. By ricci identity, the difference of commuting the covariant derivatives is a function of the curvature tensor. Note that the curvature tensor at p is the same as the complex space form. We complete the proof for $\lambda = 2$. □

We insert (4.183) in $R^{(\lambda)}_{Je_0Je_0}$. By the claim and Bianchi identities, $R^{(\lambda)}_{Je_0Je_0}$ becomes a polynomial with variables $z_j, \overline{z_j}$. The coefficients of the polynomial are exactly all the covariant derivatives of Rm at p with order λ. According to the assumption of lemma, $R^{(\lambda)}_{Je_0Je_0}$ is identically 0 for all e_0. Therefore, the coefficients of the polynomial are all 0. This completes the induction of lemma. □

Lemma 4.8.4. *Under the assumption of the induction in proposition 4.6.1, A could be written as $\sum_{i=1}^{m-2} g_{i,m}\Delta^i s$ where s denotes the scalar curvature, $g_{i,m}$ are constants depending only on n, m, i.*

Proof. Define $X = \frac{1}{2}(e_0 - \sqrt{-1}Je_0)$, then $A = \int R_{X\overline{X}X\overline{X},e_0e_0...e_0}$ where the number of e_0 is $2m-4$. Plugging (4.183) in it, after the integration, we find

$$A = \sum_{\alpha_1\alpha_2...\alpha_{2m}} (\int \alpha_1\alpha_2...\alpha_{2m})R_{\alpha_1\alpha_2\alpha_3\alpha_4,\alpha_5....\alpha_{2m}} \tag{4.184}$$

where α_i is $\{z_j\}$ or $\{\overline{z_k}\}$ for $0 \le j, k \le n-1$, $\alpha_1, \alpha_3 \in \{z_j\}$, $\alpha_2, \alpha_4 \in \{\overline{z_k}\}$. Under the subscript of R, z_j stands for β_j, $\overline{z_k}$ stands for $\overline{\beta_k}$.

From the expression of (4.184), we see that $z_i, \overline{z_i}$ must all go in pairs in the sequence $\alpha_1\alpha_2..\alpha_{2m}$, otherwise the integral $\int \alpha_1\alpha_2...\alpha_{2m}$ equals 0. Switching the covariant derivatives in (4.25), using Kähler identities, we can rearrange (4.25) as

$$A = \sum_{I_1, I_2, ... I_n} C_{I_1 I_2 .. I_n} R_{I_1 I_2 ... I_n} + B \tag{4.185}$$

where the symbol I_j denotes $z_j\overline{z_j}..z_j\overline{z_j}$; subscripts after the fourth subscript of R denote the covariant derivatives; $C_{I_1 I_2 .. I_n}$ are the coefficients; $\sum\limits_j |I_j| = 2m$; B is the combination of covariant derivatives of Rm with lower order. From (4.23), we see that the coefficients $C_{I_1 I_2 .. I_n}$ in (4.185) are unitary invariants. For fixed $I_3, I_4, ..I_n$, let $d = |I_1| + |I_2|$. Denote $C_{I_1 I_2 .. I_n}$ by C_p where $0 \le |I_1| = p \le d$. We want to find the relations of $\{C_p\}$. Take a unitary transformation:

$\widetilde{\beta_i} = \beta_i$ for $i \ne 1, 2$; $\beta_1 = \cos\theta \widetilde{\beta_1} + \sin\theta \widetilde{\beta_2}$; $\beta_2 = -\sin\theta \widetilde{\beta_1} + \cos\theta \widetilde{\beta_2}$.

Insert the unitary transformation above in (4.185), the new coefficient $\tilde{C_d}$ becomes $\sum\limits_{p=0}^{d} C_p \cos^{2p}\theta \sin^{2(d-p)}\theta$. Therefore we have:

$$\sum_{p=0}^{d} C_p \cos^{2p}\theta \sin^{2(d-p)}\theta = C_d = C_d(\cos^2\theta + \sin^2\theta)^d. \tag{4.186}$$

Claim 4.8.3. $C_p = C_d \binom{d}{p}$

Proof. Divide by $cos^{2d}\theta$ on both sides, (4.186) becomes

$$\sum_{p=0}^{d} C_p \tan^{2(d-p)}\theta = C_d = C_d(1 + \tan^2\theta)^d.$$

Since θ is arbitrary, the claim follows. □

$\frac{C_p}{C_d} = \binom{d}{p}$. Since we can substitute any index u, v for $1, 2$, the ratio of all coefficients in (4.185) are determined. Note that to get the relations between C_p, we only use the condition that the form (4.182) is unitary invariant. Since $\Delta^{m-2}s$ is also unitary invariant with respect to the frame, we can write it in the form as (4.185). By the same argument, the ratio of all coefficients of $\Delta^{m-2}s$ are the same as (4.185). It follows that the term $\sum_{I_1,I_2,...I_n} C_{I_1I_2..I_n} R_{I_1I_2...I_n}$ in (4.185) equals $C(m,n)\Delta^{(m-2)}s$ modulo lower order covariant derivatives, where $C(m,n)$ is a constant depending only on m,n.

Now we make an important observation. From the Ricci identity, $R_{i_1\overline{i_2}....i_p\alpha\beta i_{p+3}..i_{2m}} - R_{i_1\overline{i_2}...i_p\beta\alpha i_{p+3}..i_{2m}}$ is the sum of $(RmRm^{(p-4)})_{,i_{p+3}..i_{2m}}$. By lemma 3, $Rm^{(\lambda)} = 0$ for $1 \leq \lambda \leq m-3$. It follows that $(RmRm_{,i_5...ip})_{,i_{p+3}..i_{2m}}$ can be expanded as a linear combination of the covariant derivatives of curvature tensor. Therefore $A - C(m,n)\Delta^{(m-2)}s$ can be written as a linear combination of the covariant derivatives of the curvature tensor with the highest order $2m-6$. Furthermore it is unitary invariant since the curvature tensor is unitary invariant at p. By recursive arguments, we complete the proof of the lemma. □

From the induction in proposition 4.6.1, $Ric^{(l)} = 0$ for $1 \leq l \leq 2m-4$. Integrating with respect to the unit sphere in T_pM, by similar arguments as in the proof of lemma 4.8.4, we find that for l even,

$$0 = \int Ric_{e_0e_0,e_0e_0...e_0} = \sum_{k=1}^{\frac{l}{2}} C_{l,k}\Delta^k s \tag{4.187}$$

where the order of the covariant derivative above is l. It is straightforward to check that the highest order coefficient $C_{l,\frac{l}{2}}$ is not equal to 0. Then by a recursive argument, $\Delta^k s = 0$ at p for $1 \leq k \leq m-2$. Combine this with lemma 4, it follows that $A = 0$. Similarly all linear terms in (4.169) vanish. Therefore, under the induction hypothesis in proposition 4.6.1, in order that c_{2m} in (4.169) achieves the maximum, $Ric^{(2m-2)} = 0$ and $R_{uv}^{(\lambda)} = 0$ for $1 \leq \lambda \leq m-2$. This is exactly the case of the complex space form. Therefore we complete the induction in proposition 4.6.1. As a byproduct, we proved conclusion 2 in proposition 4.6.1. The proof of proposition 4.6.1 is complete. □

4.9 The proof of theorem 1.3.5

Under the assumption of proposition 4.6.2, using the same argument as in the last section, we find that $W^{(2m+1)}$ is a linear combination of $\int R_{11}^{(m+i)}(1 \leq i \leq m-3)$(the terms with order greater than $2m-3$ could be absorbed in $Ric^{(m+i)}$ to vanish). Similar as the proof of lemma 4.8.4, $W^{(2m+1)}$ is equal to 0. This completes the proof of proposition 4.6.2.

Consider two cases below:

1. All coefficients of the power series of W are equal to that of the complex space form. Follow proposition 4.6.1, all covariant derivatives of the curvature tensor at p are the same as the complex space form. Since the metric is real analytic, we conclude that near p, the manifold is isometric to the complex space form.

2. There is a $i_0 \geq 1$ such that for all $i < i_0$, the coefficients of the power series of W are equal to that of the complex space form, but the i_0th coefficient is less than that of the complex space form. Checking the power series of $\frac{W'}{W}$ at p, we find that for sufficiently small r, $\frac{W'}{W}$ is less than that of the complex space form. Follow the definition of W, for small r,

$$\frac{\int_{\partial B_p(r)} \Delta r}{A(\partial B_p(r))} = \frac{\int (\sqrt{det < J_u, J_v >})'}{\int \sqrt{det < J_u, J_v >}} < \Delta_{N_K} r(r).$$

The proof of theorem 1.3.5 is complete. □

4.10 An example

In this section we give an example showing that the analogous Laplacian comparison theorem is not true on Kähler manifolds when the Ricci curvature is bounded from below by a nonzero constant. The example is in dimension 2. For higher dimensions, the construction is similar.

Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ in the usual way. The corresponding almost complex structure J is given by $J\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2}, J\frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x_1}, J\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4}, J\frac{\partial}{\partial x_4} = -\frac{\partial}{\partial x_3}$.

Given a small ball near the origin of $\mathbb{C}^2$, define the function f to be

$$f = |z_1|^2 + |z_2|^2 + a|z_1|^4 + 8a|z_1|^2|z_2|^2 + a|z_2|^4 + \frac{8}{3}a^2|z_1|^6 +$$
$$28a^2|z_1|^4|z_2|^2 + 28a^2|z_1|^2|z_2|^4 + \frac{8}{3}a^2|z_2|^6 + p(|z_1|, |z_2|)$$

where a is a nonzero constant and p is a homogeneous polynomial of degree 8 which will be determined later.

We define

$$\omega = \frac{\sqrt{-1}}{2}\partial\overline{\partial} f = \frac{\sqrt{-1}}{2}\sum_{i,j} g_{i\overline{j}} dz_i \wedge d\overline{z_j}.$$

It is straightforward to check that ω defines a Kähler metric g if the ball is sufficiently small (note that the metric is not complete).

Direct computation gives

$$g_{1\bar{1}} = 1 + 4a|z_1|^2 + 8a|z_2|^2 + 24a^2|z_1|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_2|^4 \\ + O((|z_1| + |z_2|)^6));$$

$$g_{2\bar{2}} = 1 + 4a|z_2|^2 + 8a|z_1|^2 + 24a^2|z_2|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_1|^4 \\ + O((|z_1| + |z_2|)^6));$$

$$g_{1\bar{2}} = 8a\overline{z_1}z_2 + 56a^2z_1\overline{z_1}^2z_2 + 56a^2\overline{z_1}z_2^2\overline{z_2} + O((|z_1| + |z_2|)^6)).$$

Therefore

$$\begin{aligned} det(g_{i\bar{j}}) &= g_{1\bar{1}}g_{2\bar{2}} - |g_{1\bar{2}}|^2 \\ &= (1 + 4a|z_1|^2 + 8a|z_2|^2 + 24a^2|z_1|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_2|^4) \\ &\quad (1 + 4a|z_2|^2 + 8a|z_1|^2 + 24a^2|z_2|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_1|^4) \\ &\quad - |8a\overline{z_1}z_2 + 56a^2z_1\overline{z_1}^2z_2 + 56a^2\overline{z_1}z_2^2\overline{z_2}|^2 + O((|z_1| + |z_2|)^6) \\ &= 1 + 12a(|z_1|^2 + |z_2|^2) + 84a^2(|z_1|^4 + |z_2|^4) + 240a^2|z_1|^2|z_2|^2 \\ &\quad + O((|z_1| + |z_2|)^6)). \end{aligned}$$

Using $log(1 + x) = x - \frac{1}{2}x^2 + O(x^3)$, we have

$$Ric + 12ag = \partial\overline{\partial}(-log(detg_{i\bar{j}}) + 12af) = \partial\overline{\partial}(O((|z_1| + |z_2|)^6)).$$

Therefore $Ric + 12ag$ vanishes up to order 3 at the origin. Moreover, if we choose the function p to be $-\lambda(|z_1|^8 + |z_2|^8 + 8(|z_1|^6|z_2|^2 + |z_1|^2|z_2|^6))$, after a direct computation,

$$Ric + 12ag = \partial\overline{\partial}(24\lambda(|z_1|^2 + |z_2|^2)^3 + O((|z_1| + |z_2|)^6)$$

where the term $O((|z_1| + |z_2|)^6)$ does not depend on λ. If λ is sufficiently large, $Ric + 12ag \geq 0$ near the origin. Set $K = -12a$. Thus, near the origin, $Ric \geq K$. By direct computation, at the origin, $R_{1212} = R_{1313} = R_{1414} = 4a$; $R_{1u1v} = 0$ if $u \neq v$. Combining this with the fact that the second derivatives of the Ricci tensor vanish at the origin, after a slight computation, we find that the fourth order term of (4.158) is greater than that of the complex space form if $e_0 = \frac{\partial}{\partial x_1}$. So when r is very small, along the geodesic with initial direction $\frac{\partial}{\partial x_1}$ at the origin, $\sqrt{det < J_u, J_v >}$ is greater than that of the complex space form. Since $\Delta r = \frac{\partial \log \sqrt{det < J_u, J_v >}}{\partial r}$, it follows that the pointwise Laplacian comparing with the complex space forms is not true for Kähler manifolds.

Chapter 5

Rigidity of volume entropy

This section is based on [38]. We present the proof of 1.4.1.

Proof. The inequality $v(\tilde{M}) \leq n-1$ directly follows from the volume comparison. We have to deal with the equality case. We shall construct a Busemann function u on $\tilde{M}$ such that $\Delta u = n-1$ in the distribution sense. By the result of Li-Wang in [48], we know $\tilde{M}$ is the hyperbolic space form since $\tilde{M}$ has bounded curvature. Now take a fixed R such that $R > 50 diam(M)$. Pick a point $O \in \tilde{M}$ and define $r(x) = d(O, x)$.

Claim 5.0.1. There exists a sequence $r_i \to \infty$ so that the area of the geodesic spheres satisfy

$$\frac{A(\partial(B(O, r_i + 50R)))}{A(\partial(B(O, r_i - 50R)))} \to e^{100(n-1)R}.$$

We prove the claim by contradiction. Suppose there exist $r_0 > 100R > 0$ and $\epsilon > 0$ such that for any $r > r_0$,

$$\frac{A(\partial(B(O, r + 50R)))}{A(\partial(B(O, r - 50R)))} \leq e^{100(n-1)R}(1-\epsilon).$$

By an iteration argument we find that for sufficiently large r,

$$A(\partial(B(O, r))) \leq C(1-\epsilon)^{\frac{r}{100R}} e^{(n-1)r}$$

where C is a constant independent of r. After the integration, we find that the volume entropy is smaller than $n-1$. This is a contradiction.

We take the sequence r_i in claim 1 and define

$$A_i = \{x \in \tilde{M} | r_i - 50R \leq d(x, O) \leq r_i + 50R\}.$$

Claim 5.0.2. $⨍_{A_i} \Delta r \geq n - 1 - \epsilon(i, R)$ where $\epsilon(i, R) \to 0$ when $i \to \infty$. The symbol $⨍$ means the average.

After integration by parts, the claim follows from Bishop-Gromov's volume comparison.

Given a point $P \in M$, for all preimages of P in $\tilde{M}$, consider the subset $P_j(i)$ such that $B(P_j(i), R) \subseteq A_i$. Let E_i be the maximal set of $P_j(i)$ such that for $j_1 \neq j_2$ in E_i, $B(P_{j_1}(i), R) \bigcap B(P_{j_2}(i), R) = \Phi$. Take $F_i = \bigcup_{j \in E_i} B(P_j(i), R), G_i = \bigcup_{j \in E_i} B(P_j(i), 5R)$.

By Bishop-Gromov's volume comparison, we have

$$\frac{vol(F_i)}{vol(G_i)} \geq g(R, n).$$

Now by a standard covering technique, we find that

$$G_i = \bigcup_{j \in E_i} B(P_j(i), 5R) \supseteq B(O, r_i + 10R) \backslash B(O, r_i - 10R).$$

we have

$$\frac{vol(G_i)}{vol(A_i)} \geq h(R, n)$$

where $g(R, n), h(R, n)$ are positive functions independent of i. Therefore, we have

$$\frac{vol(F_i)}{vol(A_i)} \geq g(R, n)h(R, n).$$

By the Laplacian comparison, we find that for each i,

$$⨍_{F_i} \Delta r \geq n - 1 - \frac{\epsilon(R, i)}{g(R, n)h(R, n)} - \delta(i, n)$$

where $\delta(i, n) \to 0$ when $i \to \infty$.

Therefore there exists at least one j in E_i such that

$$⨍_{B(P_j(i), R)} \Delta r \geq n - 1 - \frac{\epsilon(R, i)}{g(R, n)h(R, n)} - \delta(i, n). \tag{5.1}$$

Note that $B(P_j(i), R)$ is isometric to $B(P_0, R)$ where P_0 is a fixed preimage of P in $\tilde{M}$. Consider the function $u_i(x) = r(x) - d(O, P_j(i))$ in $B(P_j(i), R)$, we pull u_i back to $B(P_0, R)$. Note that $u_i(P_0) = 0$. Since u_i is a uniformly Lipschitz sequence, we can extract a subsequence so that $u_i \to u_R$ in $B(P_0, R)$. By the Laplacian comparison, we can easily get

$$\int_{B(P_0(i),R)} u_R \Delta\varphi \geq (n-1) \int_{B(P_0(i),R)} \varphi \tag{5.2}$$

for any $\varphi \in C_0^\infty(B(P_0, R)), \varphi \geq 0$.

One the other hand, since u_R is a limit of the distance function, the standard Laplacian comparison implies

$$\int_{B(P_0(i),R)} u_R \Delta\varphi \leq (n-1) \int_{B(P_0(i),R)} \varphi \tag{5.3}$$

for any $\varphi \in C_0^\infty(B(P_0(i), R)), \varphi \geq 0$.

Therefore $\Delta u_R = n-1$ in the distribution sense. Furthermore, since u_R is a limit of the distance function, $|\nabla u_R| = 1$. Let $R \to \infty$, we can extract a subsequence of u_R so that $u_R \to u$. Then u is defined on $\tilde{M}$. It satisfies $|\nabla u| = 1$ and $\Delta u = n-1$. According to the argument at the beginning of the proof, $\tilde{M}$ is the hyperbolic space form. $\square$

Using the same proof, we can prove the following theorem which is also due to F. Ledrappier and X. Wang [47]:

Theorem 5.0.1. *Let M be a compact Kähler manifold with $dim_{\mathbb{C}} M = m$ and $\tilde{M}$ be its universal cover. If the bisectional curvature $K_{\mathbb{C}} \geq -2$, then the volume entropy satisfies $v \leq 2m$. Moreover, if the equality holds iff $\tilde{M}$ is the complex hyperbolic space form.*

Remark 5.0.1. *It is not clear to the author whether theorem 2 still holds if we relax the condition to $Ric \geq -2(m+1)$.*

Chapter 6

Manifolds with nonnegative Bakry-Emery Ricci curvature

This chapter is based on [39]. Below is the organization of this chapter. First, we will derive the second variation formula for the weighted area(see also [17] and [5] for the derivation). Then we give an application to compact stable f-minimal surfaces. This generalizes some previous works of Heintze and Karcher [32]. An example is given to show that a result of Schoen and Fischer-Colbrie [24] cannot be extended to the case when Bakry-Emery Ricci curvature is nonnegative. Then we give an application of the stability inequality to noncompact case. Finally we study the topology of complete 3-manifolds with nonnegative Bakry-Emery Ricci curvature.

6.1 Second variation formula

Definition 6.1.1. *Let $(M^m, g, e^{-f}dv)$ be a complete smooth metric measure space and Σ be a complete submanifold in M. We say Σ is f-minimal in M, if the first variation of the e^{-f} weighted area functional vanishes at Σ. Σ is called stable f-minimal if the second variation of the e^{-f} weighted area functional is nonnegative along any compactly supported variational normal vector field.*

Proposition 6.1.1. *Let $(M^m, g, e^{-f}dv)$ be a complete smooth metric measure space and Σ^n be a complete f-minimal submanifold in M. Let $e_i (0 \leq i \leq n)$ be an orthonormal* *frame in an open set of Σ. Define ∇^T and $\nabla^\perp$ to be the connections projected to the tangential and normal spaces on Σ. Then*

$$H = \nabla^\perp f$$

where $H = -\sum_i \nabla^\perp_{e_i} e_i$ is the mean curvature vector. If $\Sigma_t(-\epsilon < t < \epsilon)$ is a smooth family of the submanifolds such that $\Sigma_0 = \Sigma$ and the variational normal vector field ν is compactly supported on Σ_t, then at $t = 0$,

$$\frac{d^2 \int_{\Sigma_t} e^{-f}}{dt^2} = \int_\Sigma e^{-f}(-\sum_{i=1}^n R_{i\nu\nu i} - \frac{1}{2}\Delta_\Sigma(|\nu|^2) + |\nabla_\Sigma \nu|^2 - 2|A^\nu|^2 - f_{\nu\nu} + \frac{1}{2}\langle \nabla^T f, \nabla^T(|\nu|^2)\rangle)$$

where $A^\nu_{ij} = -\langle \nabla_{e_i} e_j, \nu\rangle$.

Proof. For any point $p \in \Sigma_0$, consider a local frame $e_i(1 \le i \le n)$ near p such that they are tangential to Σ_t and $[e_i, \nu] = 0$ for all small t. We can also assume that at p, e_i is an orthonormal frame and $\nabla^T_{e_i} e_j = 0$. Let $g_{ij} = \langle e_i, e_j\rangle$ and g^{ij} be the inverse matrix of g_{ij}. We have

$$\frac{d \int_{\Sigma_t} e^{-f}}{dt} = \int_{\Sigma_t} e^{-f}\langle H - \nabla^\perp f, \nu\rangle$$

where

$$H = -(\nabla_{e_i} e_j)^\perp g^{ij}.$$

Thus if Σ_0 is e^{-f} minimal,

$$H = \nabla^\perp f.$$

At p, we have

$$\begin{aligned}
\frac{d\langle H, \nu\rangle}{dt} &= -(\langle \nabla_\nu \nabla_{e_i} e_j, \nu\rangle g^{ij} + \langle \nabla_{e_i} e_j, \nabla_\nu \nu\rangle g^{ij} + \langle \nabla_{e_i} e_j, \nu\rangle \nu(g^{ij})) \\
&= -(\sum_{i=1}^n R_{\nu ii\nu} + \langle \nabla_{e_i}\nabla_\nu e_i, \nu\rangle - \langle H, \nabla_\nu \nu\rangle - \sum_{i,j=1}^n \langle \nabla_{e_i} e_j, \nu\rangle(\langle \nabla_\nu e_i, e_j\rangle + \langle \nabla_\nu e_j, e_i\rangle)) \\
&= -(\sum_{i=1}^n R_{\nu ii\nu} + \frac{1}{2}\Delta_\Sigma(|\nu|^2) - \sum_{i=1}^n |\nabla_{e_i}\nu|^2 + 2\sum_{i,j=1}^n |\langle \nabla_{e_i} e_j, \nu\rangle|^2 - \langle H, \nabla_\nu \nu\rangle).
\end{aligned} \tag{6.1}$$

$$
\begin{aligned}
\frac{d\langle \nabla^{\perp} f, \nu\rangle}{dt} &= \nu\nu(f) \\
&= f_{\nu\nu} + \langle \nabla^T f, \nabla_\nu \nu\rangle + \langle \nabla^{\perp} f, \nabla_\nu \nu\rangle \\
&= f_{\nu\nu} + \sum_{i=1}^{n} e_i(f)\langle e_i, \nabla_\nu \nu\rangle + \langle \nabla^{\perp} f, \nabla_\nu \nu\rangle \\
&= f_{\nu\nu} - \frac{1}{2}\langle \nabla^T f, \nabla^T(|\nu|^2)\rangle + \langle \nabla^{\perp} f, \nabla_\nu \nu\rangle.
\end{aligned}
\tag{6.2}
$$

Since Σ_0 is f minimal, by the two equalities above, we have

$$
\begin{aligned}
\frac{d^2 \int_{\Sigma_t} e^{-f}}{dt^2} &= \frac{d \int_{\Sigma_t} e^{-f}\langle H - \nabla^{\perp} f, \nu\rangle}{dt} \\
&= \int_{\Sigma} e^{-f}(-\sum_{i=1}^{n} R_{i\nu\nu i} - \frac{1}{2}\Delta_{\Sigma}(|\nu|^2) + |\nabla_{\Sigma}\nu|^2 - 2|A^{\nu}|^2 - f_{\nu\nu} + \frac{1}{2}\langle \nabla^T f, \nabla^T(|\nu|^2)\rangle).
\end{aligned}
\tag{6.3}
$$

□

Corollary 6.1.1. *Let $(M^m, g, e^{-f}dv)$ be a complete oriented Riemannian manifold and Σ_t be a smooth family of oriented hypersurfaces in M. Let N be the unit normal vector field on Σ_t. Suppose the variational vector field for Σ_t is given by λN where λ is smooth function with compact support on Σ_t. If Σ_0 is e^{-f} minimal, then the mean curvature of Σ_0 satisfies*

$$H = f_n.$$

where f_n is the normal derivative of f. Moreover,

$$\frac{d^2 \int_{\Sigma_t} e^{-f}}{dt^2}|_{t=0} = \int_{\Sigma_0} (|\nabla\lambda|^2 - \lambda^2(Ric_f(n,n) + |A|^2))e^{-f}$$

where $Ric_f = Ric + \nabla^2 f$, A is the second fundamental form. Therefore, the stability inequality is

$$\int_{\Sigma_0} (|\nabla\lambda|^2 - \lambda^2(Ric_f(n,n) + |A|^2))e^{-f} \geq 0$$

for any compactly supported function λ on Σ_0.

Proof. Since Σ_0 is weighted minimal, according to Proposition 6.1.1,

$$H = \langle \nabla^{\perp} f, N\rangle = f_n.$$

Let $\nu = \lambda N$. For an orthonormal frame e_i at a point on Σ_0,

$$\begin{aligned} |\nabla_\Sigma \nu|^2 &= |\langle \nabla_{e_i}(\lambda N), \nabla_{e_i}(\lambda N)\rangle|^2 \\ &= |\nabla \lambda|^2 + \sum_{i,j} |\langle \nabla_{e_i}(\lambda N), e_j\rangle|^2 \\ &= |\nabla \lambda|^2 + \lambda^2 |A|^2. \end{aligned} \tag{6.4}$$

Therefore

$$\begin{aligned} \frac{d^2 \int_{\Sigma_t} e^{-f}}{dt^2} &= \int_{\Sigma_0} e^{-f}(-\sum_{i=1}^{n} R_{i\nu\nu i} - \frac{1}{2}\Delta_\Sigma(|\nu|^2) + |\nabla_\Sigma \nu|^2 - 2|A^\nu|^2 - f_{\nu\nu} + \frac{1}{2}\langle \nabla^T f, \nabla^T(|\nu|^2)\rangle) \\ &= \int_{\Sigma_0} e^{-f}(-\lambda^2 Ric_f(n,n) - \lambda\Delta\lambda - \lambda^2|A|^2 + \langle \nabla f, \nabla\lambda\rangle\lambda) \\ &= \int_{\Sigma_0} (|\nabla\lambda|^2 - \lambda^2(Ric_f(n,n) + |A|^2))e^{-f}. \end{aligned} \tag{6.5}$$

In the last step, we have used the integration by parts. □

6.2 An application to the compact case

In [68], Simons observed that there are no closed, stable minimal 2-sided hypersurfaces in a manifold with positive Ricci curvature. Later Heintze and Karcher [32] proved that the exponential map of the normal bundle of a hypersurface $\Sigma \in M$ is area decreasing, if Σ is stable, minimal and M has nonnegative Ricci curvature. Anderson extended this result, he also proved that a version of the Cheeger-Gromoll splitting theorem in the compact case, see [2]. More recently, F. Morgan [54] obtained the upper bound of weighted volume of one side of a hypersurface which generalizes some works in [32]. See also chapeter 18 in [51] for more discussion.

In this section, we shall prove the following:

Theorem 6.2.1. *Let $(M^m, g, e^{-f}dv)$ be an oriented complete Riemannian manifold and Σ be a closed oriented stable f-minimal hypersurface in M. If $Ric_f \geq 0$, then Σ is totally geodesic and $Ric_f(n,n) = 0$. If Σ is weighted f-area-minimizing in its homology class, then M^m is isometric to a quotient of $\Sigma \times \mathbb{R}$. In this case, if $m = 3$, then topologically Σ is either a sphere or a torus. In the torus case, M^3 is flat.*

Proof. The first conclusion follows if we take $\lambda = 1$ in corollary 6.1.1. Let N be the unit normal vector field on Σ. For x close to Σ in M, consider the oriended distance function $d(x) = Sign(x)dist(x, \Sigma)$, where $Sign(x)$ is 1 if x is on one side of Σ; $Sign(x) = -1$ if x is on the other side of Σ. Then $d(x)$ is smooth near Σ and let Σ_t be the level set of $d(x)$. Then for t small, Σ_t is a smooth family of hypersurfaces on M and we have

$$\frac{d(H - f_n)}{dt} = -Ric(n, n) - |A|^2 - f_{nn} = -Ric_f(n, n) - |A|^2 \leq 0.$$

Note that Σ_0 is totally geodesic and $f_n = H = 0$ at $t = 0$. Therefore

$$H - f_n \leq 0$$

for all t and

$$\frac{d \int_{\Sigma_t} e^{-f}}{dt} = \int_{\Sigma_t} (H - f_n)e^{-f} \leq 0.$$

Since Σ_0 is area-minimizing in its homology class, Σ_t are all totally geodesic. By induction, one can easily show that M is isometric to the quotient of $\Sigma_0 \times \mathbb{R}$. Therefore

$$f_n = H = 0, f_{nn} = \frac{\partial f_n}{\partial t} = 0, Ric_{nn} = 0$$

for all t.

Now consider the case when $m = 3$. Let e_1, e_2 be a local orthonormal frame on Σ_0. Let S be the scalar curvature on M; $S_f = S + \Delta f$; K_Σ be the Gaussian curvature on Σ. Since Σ_0 is totally geodesic,

$$2K_{\Sigma_0} = 2R_{1221} = S - 2Ric_{nn} = S_f - f_{11} - f_{22} = S_f - \Delta_{\Sigma_0} f.$$

In the above equality, we have used the fact that $f_{nn} = 0$. Since $S_f \geq 0$, the Gauss-Bonnet theorem says that Σ_0 is either a sphere or a torus. In the torus case, $S_f = 0$ everywhere, thus on Σ, $Ric + \nabla^2 f = 0$. So Σ is a 2 dimensional steady soliton. Thus the Gaussian curvature on Σ is nonnegative. This means that Σ and M are flat. $\square$

6.3 An example

In [24], R. Schoen and D. Fischer-Colbrie proved the following theorem:

Theorem 6.3.1 (R. Schoen and D. Fischer-Colbrie)**.** *Let M be a complete oriented 3-manifold with nonnegative scalar curvature. Let Σ be an oriented complete stable minimal surface in M, then if Σ is compact, then it is conformal to $\mathbb{S}^2$ or a torus $\mathbb{T}^2$; if Σ is not compact, it is conformally covered by $\mathbb{C}$.*

In view of Theorem 6.2.1, it is natural to ask whether we can weaken the condition in theorem 1 when $dim(M) = 3$. We will show that at least locally, even if the Bakry-Emery Ricci curvature is nonnegative, the stability of a weighted stable minimal surface Σ does not provide any information on the conformal structure on Σ.

Let M^3 be an oriented manifold with nonnegative Bakry-Emery Ricci curvature and Σ be an oriented stable f-minimal surface in M. In this section we will give an explicit example so that Σ is hyperbolic.

Let (Σ, ds^2_Σ) be a complete surface with curvature -1. Let $M = (-\frac{1}{2}, \frac{1}{2}) \times \Sigma$ and define metric on M by

$$ds^2 = dt^2 + g(t)ds^2_\Sigma.$$

Note that the metric on M is not complete. Let $p \in M$ and consider a product chart $U \ni p$ such that $e_1 = \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial x_2}$ are tangential to Σ_t and $\frac{\partial}{\partial t} = e_3$ on U. We may assume that e_1, e_2, e_3 is an orthogonal frame in U and $ds^2_\Sigma(e_1, e_1) = ds^2_\Sigma(e_2, e_2) = 1$. Then

$$\langle \nabla_{e_1} e_3, e_1 \rangle = \langle \nabla_{e_2} e_3, e_2 \rangle = \frac{1}{2} g'(t),$$

$$\langle \nabla_{e_1} e_3, e_2 \rangle = \langle \nabla_{e_2} e_3, e_1 \rangle = 0.$$

Therefore, $\nabla^{\Sigma_t} A = 0$ for all t. By Gauss equation,

$$K_{\Sigma_t} - \frac{R_{1221}}{g^2} = \frac{A_{11}A_{22}}{g^2} = \frac{\langle \nabla_{e_1} e_3, e_1 \rangle \langle \nabla_{e_2} e_3, e_2 \rangle}{g^2}.$$

Since the Gaussian curvature $K_{\Sigma_t} = -\frac{1}{g}$,

$$R_{1221} = -g - \frac{1}{4} g'^2.$$

It is easy to see that $\nabla_{e_3} e_3 \equiv 0$, thus

$$\begin{aligned} R_{1331} &= \langle \nabla_{e_1} \nabla_{e_3} e_3, e_1 \rangle - \langle \nabla_{e_3} \nabla_{e_1} e_3, e_1 \rangle \\ &= -\langle \nabla_{e_3} \nabla_{e_1} e_3, e_1 \rangle \\ &= -(e_3(\langle \nabla_{e_1} e_3, e_1 \rangle) - |\nabla_{e_3} e_1|^2) \\ &= -\frac{1}{2} g'' + \frac{1}{4} \frac{g'^2}{g}. \end{aligned} \tag{6.6}$$

From the same computation, we see that $R_{1332} = 0$. By Codazzi equation,

$$R_{1223} = (\nabla^{\Sigma_t}_{e_1} A)(e_2, e_2) - (\nabla^{\Sigma_t}_{e_2} A)(e_1, e_2) = 0.$$

Therefore

$$Ric_{11} = \frac{R_{1221}}{g} + R_{1331} = -1 - \frac{1}{2} g'' = Ric_{22},$$

$$Ric_{33} = -\frac{g''}{g} + \frac{1}{2}(\frac{g'}{g})^2,$$

$$Ric_{12} = Ric_{13} = Ric_{23} = 0.$$

Let $f = f(t)$ be a function of M, then

$$f_{11} = -\langle \nabla f, \nabla_{e_1} e_1 \rangle = \frac{g' f'}{2} = f_{22},$$

$$f_{12} = f_{13} = f_{23} = 0, f_{33} = f''.$$

Therefore

$$Ric_f(e_1, e_1) = -1 - \frac{g''}{2} + \frac{f' g'}{2}, Ric_f(e_3, e_3) = \frac{-2g'' g + g'^2 + 2g^2 f''}{2g^2}$$

If $f = -2t^2, g = 1 - 2t^2$, then one gets that

$$Ric_f(e_2, e_2) = Ric_f(e_1, e_1) = 1 + 8t^2 \geq 0, Ric_f(e_3, e_3) = 4(\frac{1}{(1-2t^2)^2} - 1) \geq 0.$$

Therefore, M has nonnegative Bakry-Emery Ricci curvature. Moreover, the second fundamental form and $Ric_f(e_3, e_3)$ vanish at $t = 0$. According to corollary 1, Σ_0 is a stable f-minimal surface in M. However, Σ is hyperbolic.

6.4 Applications to the noncompact case

Now consider the case when Σ is noncompact. The following proposition follows from a simple cut-off argument:

Proposition 6.4.1. *Let $(M^m, g, e^{-f}dv)$ be an oriented complete Riemannian manifold and Σ be a complete noncompact oriented stable f-minimal hypersurface in M. If $Ric_f \geq 0$ on Σ and that the weighted volume growth of Σ with respect to its intrinsic distance to a point $p \in \Sigma$ satisfy*

$$V_{\Sigma,f}(B_p(r)) \leq Cr^2$$

for all large r , then Σ is totally geodesic and $Ric_f(n,n) = 0$.

Proof. Let r be a distance function to $p \in M$. Given any $a > 1$, consider the cut-off function

$$\lambda(r) = \begin{cases} 1 & 0 \leq r \leq a \\ \frac{2\log a - \log r}{\log a} & a < r < a^2 \\ 0 & r \geq a^2. \end{cases} \tag{6.7}$$

Define $V(r) = \int_{B_\Sigma(p,r)} e^{-f}$. Plugging this in the stability inequality in corollary 1, we find that

$$\begin{aligned} &\int_{B_\Sigma(a)} (Ric_f(n,n) + |A|^2)|\lambda|^2 e^{-f} \\ &\leq \int_{B_\Sigma(a^2)} |\nabla\lambda|^2 e^{-f} \\ &= \int_a^{a^2} \frac{V'(r)}{r^2 \log^2 a} dr \\ &= \frac{V(r)}{r^2 \log^2 a}\Big|_{r=a}^{r=a^2} - \int_a^{a^2} V(r)\left(\frac{1}{r^2\log^2 a}\right)' dr \\ &\leq \frac{C}{\log^2 a} + C\frac{1}{\log^2 a}\int_a^{a^2} \frac{dr}{r} \\ &= O\left(\frac{1}{\log a}\right). \end{aligned} \tag{6.8}$$

The proposition follows by taking $a \to \infty$. $\square$

Now recall the following theorem in [80][55]

Lemma 6.4.1. *Let $(M^m, e^{-f}dv)$ be a smooth metric measure space with $Ric_f \geq 0$, then along any miminizing geodesic starting from $x \in B_p(R)$ we have*

$$\frac{J_f(x, r_2, \xi)}{J_f(x, r_1, \xi)} \leq e^{4A(R)}(\frac{r_2}{r_1})^{m-1}$$

for $0 < r_1 < r_2 < R$. In particular, for $0 < r_1 < r_2 < R$, the weighted area of the geodesic spheres satisfy

$$\frac{A_f(\partial B_x(r_2))}{A_f(\partial B_x(r_1))} \leq e^{4A(R)}\frac{r_2^{m-1}}{r_1^{m-1}}.$$

Here $A(R) = Sup_{x \in B_x(3R)}|f|(x)$ and $J_f(x, r, \xi) = e^{-f}J(x, r, \xi)$ is the e^{-f} weighted volume in geodesic polar coordinates.

If f is bounded, $Vol_f(B_x(r))$ has polynomial growth of order at most m.

Proposition 6.4.2. *Let $(M^3, e^{-f}dv)$ be a smooth metric measure space with $Ric_f \geq 0$ and f is bounded. If Σ is a complete weighted area-minizing hypersurface which is the boundary of least weighted area in M, then Σ is totally geodesic and $Ric_f(n, n) = 0$.*

Proof. According to lemma 6.4.1, the weighted volume of the geodesic sphere has at most quadratic growth. Since Σ is weighted area minimizing and is a boundary of least weighted area in M, $vol_f(\Sigma \vdash B_x(r)) \leq A_f(\partial B_x(r)) \leq Cr^2$. Proposition 6.4.2 follows from Proposition 6.4.1. □

6.5 Application to complete 3-manifolds with nonnegative Bakry-Emery Ricci curvature

The classification of complete 3-manifolds with nonnegative Ricci curvature has been complete by various authors' works. By using the Ricci flow, Hamilton [29][30] classified all compact 3-manifolds with nonnegative Ricci curvature. He proved that the universal cover is either diffeomorphic to $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$. In the latter two cases, the manifold splits.

In [74], Schoen-Yau proved that a complete noncompact 3-manifold with positive Ricci curvature is diffeomorphic to the Euclidean space. Anderson-Rodriguez [3] and Shi [67] classified complete noncompact 3-manifolds with nonnegative Ricci curvature by assuming an upper bound of the sectional curvature. Very recently, the author [34] classified all complete noncompact 3-manifolds with nonnegative Ricci curvature.

In view of the results above, it is natural to ask what happens to a 3-manifold when the Bakry-Emery Ricci curvature is nonnegative. Below is a partial classification when f is bounded.

Theorem 6.5.1. *Let $(M^3, g, e^{-f}dv)$ be a complete 3-manifold with bounded f and $Ric_f \geq 0$.*

- *If M is noncompact, then either M is contractible or through each point in M, there exists a totally geodesic surface with $Ric_f(n,n) = 0$. If in addition the rank of Ric_f is at least 2 everywhere, then the universal of M splits as a Riemann product as $\Sigma \times \mathbb{R}$. In particular, if the Bakry-Emery Ricci curvature is positive, then M is contractible.*
- *If M is compact, then either it is a quotient of $\mathbb{S}^3$ or the universal cover splits as a product $\Sigma \times \mathbb{R}$.*

In each splitting case, Σ is conformal to $\mathbb{S}^2$ or $\mathbb{C}$ and f is constant along the $\mathbb{R}$ factor.

Proof. First we consider the case when M is noncompact. The argument is similar to [74][34]. Assume M is simply connected, if $\pi_2(M) \neq 0$, according to Lemma 2 in [74], M must have at least two ends. From Lichnerowicz's extension of the Cheeger-Gromoll splitting theorem [40], the universal cover splits. So we assume $\pi_2(M) = 0$. Therefore, the universal cover of M is contractible. If M is not simply connected, Schoen and Yau [74] proved that $\pi_1(M)$ must have no torsion elements. Thus, after replacing M by a suitable covering, we may assume that $\pi_1(M) = \mathbb{Z}$ and that M is orientable.

Recall lemma 2.2 in [1] by Anderson:

Lemma 6.5.1. *(Anderson) Let M be a complete Riemannian manifold with finitely generated homology $H_1(M, \mathbb{Z})$. Then any non-zero line $\mathbb{R} \cdot \alpha, \alpha \in H_1(M, \mathbb{Z})$ gives rise to a complete homologically area-minimizing hypersurface Σ_α, which is the boundary of least area in a cover $\mathbb{Z} \to \overline{M} \to M$. Moreover, the volume growth of Σ_α satisfies $vol(\Sigma \vdash B^{\overline{M}}(r)) \leq vol(\partial B^{\overline{M}}(r))$ and the intersection number $I(\Sigma, \alpha) \neq 0$.*

The proof of the above lemma in [1] can be carried out without any modification to weighted volume case. Taking α to be the generator of $H_1(M, \mathbb{Z})$, we can find a complete oriented boundary Σ of least weighted area in the universal cover $\tilde{M}$. By proposition 3, Σ is totally geodesic and $Ric_f(n,n) = 0$. If $Ric_f > 0$ on M, then this is a contradiction.

Now consider the case when $Ric_f \geq 0$. We shall use a perturbation argument in [19][34]. For any point $p \in M$, consider a family of metric $g(t) = e^{2t\lambda} g_0$, where $\lambda = \lambda(x)$ is a function on M. Let (U, g_{ij}, x_i) be a normal coordinate for g_0 at p such that $\frac{\partial}{\partial x_i} = e_i$. We have

$$f_{ij}^t = e_j e_i(f) - (\nabla_{e_j}^t e_i) f,$$

$$\Gamma_{ij}^s(g(t)) = \frac{1}{2} g^{sl}(t)\left(\frac{\partial g_{il}(t)}{\partial x_j} + \frac{g_{jl}(t)}{\partial x_i} - \frac{\partial g_{ij}(t)}{\partial x_l}\right).$$

Then at p,

$$\Gamma_{ij}^s(g(t)) = t(\lambda_j \delta_{is} + \lambda_i \delta_{js} - \lambda_s \delta_{ij}).$$

Therefore,

$$\begin{aligned} f_{ij}^t - f_{ij} &= -\Gamma_{ij}^s(g(t)) f_s \\ &= t(f_s \lambda_s \delta_{ij} - \lambda_i f_j - \lambda_j f_i) \\ &\geq -3t|\nabla f||\nabla \lambda|. \end{aligned} \tag{6.9}$$

Let $m = dim(M) = 3$. Recall that

$$Ric^t(v,v) = (Ric(v,v) - t(m-2)\lambda_{vv} - t\Delta\lambda + t^2(m-2)(v(\lambda)^2 - |\nabla\lambda|^2))$$

for $|\nu|_{g_0} = 1$. Let $r(x) = dist(x,p)$ on M. For a very small $R > 0$, consider the function $\rho = R - r$ for $\frac{R}{2} < r < R$. Then we extend ρ to be a positive smooth function for $0 \leq r < \frac{R}{2}$. Define $\lambda = -\rho^5$.

Now

$$\nabla^2(\rho^5)(v,v) = 20\rho^3 v(\rho)^2 + 5\rho^4 \nabla^2(\rho)(v,v).$$

For $aR < r < R$, we have

$$\begin{aligned} Ric^t(v,v) + f_{vv}^t \geq{} & Ric^0(v,v) + f_{vv}^0 + 20t\rho^3 + 5t\rho^4(\Delta\rho + \\ & (m-2)\nabla^2(\rho)(v,v)) - 25(m-2)t^2\rho^8 - 15t\rho^4|\nabla f|. \end{aligned} \tag{6.10}$$

Using the fact that the manifold is almost Euclidean near p, for small R, we have

$$|\Delta\rho + (m-2)\nabla^2\rho(v,v)| \leq \frac{9(2m-3)}{8(R-\rho)}.$$

Therefore, there exists small $R > 0$ such that for all small t, $Ric^t_f(v,v) > 0$ in an annulus $B_p(R)\backslash B_p(aR)$ for $a = \frac{7}{8}$. The metric remains the same outside $B_p(R)$. The deformation is C^4 continuous with respect to the metric and C^∞ with respect to t.

Let γ be a closed curve in M which represents the generator of $\pi_1(M)$. We can apply the perturbation finitely many times such that $Ric_f > 0$ on γ and Ric_f is nonnegative on M except a small neighborhood U of p. Then for the perturbed metric g_t, we can apply lemma 2 to obtain a complete oriented boundary Σ of least weighted area in the universal cover $\tilde{M}$. Since g_t is uniformly equivalent to g_0, we can show Σ_t has quadratic weighted volume growth. Let $q \in \Sigma_t$, then for any $r > 0$,

$$\begin{aligned} vol_{g(t)}(\Sigma_t \vdash B_{g(t)}(q,\tilde{M})(r)) &\leq vol_{g(t)}(\Sigma_t \vdash B_{g(0)}(q,\tilde{M})(Cr)) \\ &\leq vol_{g(t)}(\partial B_{g(0)}(q,\tilde{M})(Cr)) \\ &\leq C vol_{g(0)}(\partial B_{g(0)}(q,\tilde{M})(Cr)) \\ &\leq C_1 r^2. \end{aligned} \tag{6.11}$$

If Σ_t does not intersect the preimage of U in $\tilde{M}$, then on Σ_t, $Ric_f \geq 0$ and $Ric_f > 0$ at $\Sigma_t \cap \gamma$. This contradicts proposition 6.4.2.

For each Σ_t, we can find deck transformation l_t on $\tilde{M}$ such that $l_t(\Sigma_t)$ intersects the preimage of U at some fixed compact set in $\tilde{M}$. Therefore, if we shrink the size of the neighborhood of p and let $t \to 0$ sufficiently fast, a subsequence of Σ_t will converge to a weighted area minimizing surface Σ satisfying

$$vol_{g(0)}(\Sigma \vdash B_{g(0)}(q,\tilde{M})(r)) \leq Cr^2.$$

Thus, by proposition 6.4.2, Σ is totally geodesic and $Ric_f(n,n) = 0$. Since p is arbitrary, though each point there exists a totally geodesic surface with $Ric_f(n,n) = 0$.

Now we use the assumption that the rank of Ric_f is at least 2 everywhere. Then through each point $p \in \tilde{M}$, there exists a unique totally geodesic surface. Therefore we have a foliation on $\tilde{M}$. We can parametrize the surfaces as Σ_t.

Let N be the unit normal vector and λN be the variational vector field of Σ_t. Since the smooth family of surfaces Σ_t never intersect with each other, λ is nonnegative. A simple computation shows that the variational vector field of these totally geodesic surfaces satisfies

$$\Delta\lambda + \lambda Ric(n,n) = 0.$$

Since

$$H = f_n = 0,$$

$$\begin{aligned}
0 &= \frac{df_n}{dt} \\
&= \lambda f_{nn} + \langle \nabla f, \nabla_{\lambda N} N \rangle \\
&= \lambda f_{nn} + \sum_{i=1}^{2} \langle \nabla f, e_i \rangle \langle e_i, \nabla_{\lambda N} N \rangle \\
&= \lambda f_{nn} - \langle \nabla f, \nabla \lambda \rangle.
\end{aligned} \tag{6.12}$$

In the above computation, e_i is an orthonormal frame on an open set of Σ.

But

$$0 = Ric_f(n, n) = Ric(n, n) + f_{nn},$$

thus we have

$$\Delta_f \lambda = \Delta \lambda - \langle \nabla \lambda, \nabla f \rangle = 0$$

on Σ.

The lemma below is close to corollary 1 in [16].

Lemma 6.5.2. *For a smooth metric measured space $(M, g, e^{-f} dv)$ with quadratic weighted volume growth, if λ is a positive function which satisfies $\Delta_f \lambda = 0$, then λ is a constant.*

Proof. Let $\lambda = e^h$, then

$$\Delta h + |\nabla h|^2 - \langle \nabla h, \nabla f \rangle = 0.$$

Let φ be a cut-off function, we find

$$\int \varphi^2 \Delta h e^{-f} + \int \varphi^2 |\nabla h|^2 e^{-f} - \int \varphi^2 \langle \nabla h, \nabla f \rangle e^{-f} = 0.$$

By integration by parts,

$$\int \varphi^2 (\Delta h) e^{-f} = -\int h_i 2 \varphi \varphi_i e^{-f} + \int h_i \varphi^2 f_i e^{-f}.$$

Therefore

$$\int \varphi^2 |\nabla h|^2 e^{-f} = 2 \int \varphi_i h_i \varphi e^{-f} \leq 2 (\int \varphi^2 |\nabla h|^2 e^{-f})^{\frac{1}{2}} (\int |\nabla \varphi|^2 e^{-f})^{\frac{1}{2}}.$$

Thus

$$\int \varphi^2|\nabla h|^2 e^{-f} \leq 4\int |\nabla\varphi|^2 e^{-f}.$$

Now we can use the same cut-off function in proposition 2 to show that $\nabla h \equiv 0$. Thus λ is a constant. □

Since λ is nonnegative, by the lemma, λ is constant. After a reparametrization of Σ_t, we may assume $\lambda = 1$. Now for $X \in T\Sigma_t$, $\nabla_X N = 0$, since Σ_t is totally geodesic. Since λ is a constant, we may assume $[X, N] = 0$. $\langle \nabla_N N, X\rangle = -\langle N, \nabla_N X\rangle = -\langle N, \nabla_X N\rangle = 0$. Thus $\nabla N \equiv 0$. Therefore M is locally isometric to $\Sigma \times \mathbb{R}$. f is constant along the $\mathbb{R}$ factor, since $f_n = 0$.

Now consider the case when M is compact. If the universal cover is compact, then according to Perelman's solution to the Poincare conjecture, M is covered by $\mathbb{S}^3$. If the universal cover $\tilde{M}$ is noncompact, then according to Theorem 6.6 in [80], $\tilde{M}$ splits as a product $\Sigma \times \mathbb{R}$.

Finally, we show that in the splitting case, Σ is conformal to $\mathbb{C}$ or $\mathbb{S}^2$. There are two methods to do this. Note that on Σ,

$$Ric_\Sigma + \nabla^2 f \geq 0.$$

Consider the conformal change of the metric $\tilde{g} = e^{-f}g$ on Σ, then the tensor

$$Ric_\Sigma(\tilde{g}) = Ric_\Sigma(g) + \frac{1}{2}(\Delta_\Sigma f)g \geq 0.$$

As f is bounded, $\tilde{g}$ is complete. Since Σ is simply connected, Σ is conformal to $\mathbb{C}$ or $\mathbb{S}^2$.

The second way is this: By lemma 6.4.1, the weighted volume growth of Σ is at most quadratic. Since f is bounded, the volume growth of Σ is at most quadratic. If Σ is conformal to the Poincare disk, then there exists a nontrivial bounded harmonic function on Σ. But according to corollary 1 in [16], the function is a constant. This is a contradiction.

□

Remark 6.5.1. *The bounded condition of f cannot be dropped in the above theorem. For example, consider the warped product metric $ds^2 = dt^2 + g(t)ds^2_\Sigma$ on $M = \mathbb{S}^2 \times \mathbb{R}$.*

Here ds^2_Σ *is the standard metric on* $\mathbb{S}^2$ *with curvature* 1*. Consider an orthogonal frame* e_1, e_2, e_3 *on* M *such that* $ds^2_\Sigma(e_1, e_1) = ds^2_\Sigma(e_2, e_2) = 1$ *and* $\frac{\partial}{\partial t} = e_3$*.*

If we take f *as a function of* t *on* M*, then by similar computations in section* 4*, we see*

$$Ric_f(e_1, e_1) = 1 - \frac{g''}{2} + \frac{f'g'}{2}, Ric_f(e_3, e_3) = \frac{-2g''g + g'^2 + 2g^2 f''}{2g^2}.$$

If $f(t) = t^2$*,* $g(t) = e^t$*, then one can check that* $Ric_f > 0$*, however,* M *is not a Riemann product or a contractible manifold.*

References

[1] M. Anderson, *On the topology of complete manifolds of nonnegative Ricci curvature*, Topology, Vol.29, No.1, pp 41-55, 1990.

[2] M. Anderson, *On Area-Minimizing Hypersurfaces in manifolds with nonnegative Ricci curvature*, Indiana Univ. Math. J. 32(1983), no. 5, 745-760.

[3] M. Anderson and L. Rodriguez, *Minimal surfaces and 3-manifolds with nonnegative Ricci curvature*, Math. Ann. 284, 461-475(1989).

[4] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) 102 (1978), no. 1, 63-95.

[5] V. Bayle, *Propriétés de concavité du profil isopérimétrique et applicationes*, graduate thesis, Institut Fourier, Univ. Joseph-Fourier, Grenoble I, 2004.

[6] Bishop. R.L, Crittenden. R.J.: *Geometry of Manifolds*, Pure and Applied Math., Vol. XV. New York-London: Academic Press 1964.

[7] D. Bakry and M. Emery, *Diffusions hypercontractives*, In Seminaire de probabilities, XIX, 1983/1984, 177-206, Lecture Notes in Math. 1123, Springer, Berlin, 1985.

[8] S. Brendle and R. Schoen, *Manifolds with 1/4-pinched curvature are space forms.* J. Amer. Math. Soc. 22 (2009), no. 1, 287307.

[9] C. Böhm and B. Wilking, *Nonnegative curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature*, Geom. Funct. Anal. 17(2007), 665-681.

[10] S. Y. Cheng, *Eigenvalue comparison theorems and geometric applications*, Math. Z. 143 (1975), 289-297.

[11] H.-D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), 359-372.

[12] J. Cheeger, T. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*. Ann. of Math. (2) 144 (1996), no. 1, 189-237.

[13] J. Cheeger, D. Ebin.: *Comparison Theorems in Riemannian Geometry*, 2000 Mathematics Subject Classification.Primary 53C20; Secondary 58E10.

[14] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom. 6, 119-128(1971).

[15] B. Chow and P. Lu, *The maximum principle for systems of parabolic equations subject to an avoidance set*, Pacific. J. Math. 214(2004), 201-222.

[16] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. 28 (1975), 333-354.

[17] X. Cheng, T. Mejia and D. T. Zhou, *Stability and compactness for complete f-minimal surfaces*, arxiv: 1210.8076.

[18] X. Cheng, T. Mejia and D. T. Zhou, *Eigenvalue estimate and compactness for closed f-minimal surfaces*, arxiv: 1210.8448.

[19] P. Ehrlich, *Metric deformation of curvature*, Geometriae Dedicata 5 (1976), no. 1, 1-23.

[20] P. Eberlein, *A canonical form for compact nonpositively curved manifolds whose fundamental group has nontrivial center*, Math. Ann. 260(1982), 23-29.

[21] P. Eberlein, *Euclidean de Rham factor of a lattice of nonpositive curvature*, J. Diff. Geom. 18(1983), 209-220.

[22] D. Ferus, *On the completeness of nullity foliations*. Michigan Math. J. 18 (1971) 61-64.

[23] F. Fang, X. D. Li and Z. L. Zhang, *Two generalizations of Cheeger-Gromoll splitting theorem via Bakry-Emery Ricci curvature*, Ann. Inst. Fourier(Grenoble)59, 2009, no.2, 563-573.

[24] R. Schoen and D. Fischer-Colbrie, *The structure of stable minimal surfaces in 3-manifolds with nonnegative scalar curvature*, Comm. Pure. Appl. Math., 33, 199-211, 1980.

[25] M. Gill, *Collapsing of Products Along the Kähler-Ricci Flow*, arxiv: 1203.3781v1.

[26] Gromov. M, Lafontaine, J., Pansu, P.: *Structure metrique pour les varietes Riamanniennes*, Nathen: Cedic/Fernand 1981.

[27] M. Gromov, *Curvature, diameter and Betti numbers*, Comment. Math. Helv. 56 (1981), no. 2, 179-195.

[28] D. Guler and F. Zheng, *On nonpositively curved compact Riemannian manifolds with degenerate Ricci tensor*. Trans. Amer. Math. Soc. 363 (2011), no. 3, 1265-1285.

[29] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. 17(1982), 255-306.

[30] R. Hamilton, *Four-manifolds with positive curvature operator*, J. Diff. Geom. 24, 153-179(1986)

[31] R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Diff. Geom. International Press 2 (1995), 7-136.

[32] E. Heintze and H. Karcher, *A general comparison theorem with applications to volume estimates for submanifolds*, Ann. Sci. Ecole Norm. Sup.11(1978), no. 4, 451-470.

[33] T. Ivey,*Ricci solitons on compact three-manifolds*, Diff. Geom. Appl. 3(1993), 301-307.

[34] G. Liu, *3-manifolds with nonnegative Ricci curvature*, to appear in Invent. Math.

[35] G. Liu, *Local volume comparison for Kähler manifolds*, to appear in Pacific. J. Math.

[36] G. Liu, *Compact Kähler manifold with nonpositive bisectional curvature*, arxiv: 1112.1479.

[37] G. Liu, *Kähler manifolds with Ricci curvature lower bound*, to appear in Asian. J. Math.

[38] G. Liu, *A short proof to the rigidity of volume entropy*, Math. Res. Lett. 18(2011), no.01, 151-153.

[39] G. Liu, *Stable weighted minimal surfaces in manifolds with nonnegative Bakry-Emery Ricci tensor*, arxiv: 1211.3770.

[40] A. Lichnerowicz, *Varietes riemanniennes a tensor C non negatif*, C. R. Acad. Sci. Paris Sr. A, 271(1970)650-653.

[41] P. Li, *Lecture notes on geometric analysis*, Lecture Notes Series, 6, Research Institute of Mathematics and Global Analysis Research Center, Seoul National University, Seoul, 1993.

[42] J. Lott, *Some geometric properties of the Bakry-Émery-Ricci tensor*, Comment. Math. Helv. 78(2003), 865-883.

[43] P. Li and R. Schoen, *L^p and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math. 153 (1984), 279-301.

[44] P. Li and J. Wang, *Comparison theorem for Kähler manifolds and positivity of spectrum*, J. Diff. Geom. 69 (2005), 43-74.

[45] P. Li and J. Wang, *Complete manifolds with positive spectrum, II*, J. Diff. Geom, 62(2002), 143-162.

[46] P. Li and J. Wang: *Comparison theorem for Kähler manifolds and positivity of spectrum*, J. Diff. Geom. 69 (2005), 43-74.

[47] F. Ledrappier and X. Wang, *A integral formula for the volume entropy and an application to rigidity*, to appear in J. Diff. Geom.

[48] P. Li and J. Wang, *Complete manifolds with positive spectrum II*, J. Diff. Geom. 62(2002), 143-162.

[49] J. Lott and C. Villani, *Ricci curvature for metric-measure space via optimal transport*, Annals of Math. 169(2009), 903-991.

[50] J. Milnor, *A note on curvature and fundamental group*, J. Diff. Geom. 2(1968), 1-7.

[51] F. Morgan, *Geometric measure theory, a beginner's guide*. Fourth Edition, Elsevier/Academic Press, Amsterdam, 2009.

[52] N. M. Mok, *The uniformization theorem for compact Kähler manifolds with nonnegative holomorphic bisectional curvature*, J. Diff. Geom. 27 (1988), 179-214.

[53] S. Mori, *Projective manifolds with ample tangent bundle*, Ann. of Math. 110(1979), 593-606.

[54] F. Morgan, *Manifolds with density*, Notices of the Amer. Math. Soc., 52(2005), no. 8, 853-858.

[55] O. Munteanu and J. Wang, *Smooth metric measure spaces with non-negative curvature*, Communications in analysis and geometry, Volume 19, Number 3, 451-486, 2011.

[56] O. Munteanu and J. Wang, *Analysis of weighted Laplacian and applications to Ricci solitons*, Comm. Anal. Geom. 20(2012), no.1, 55-94.

[57] O. Munteanu and J. Wang, *Geometry of manifolds with densities*, arxiv: 1211.3996.

[58] W. H. Meeks III and S. T. Yau, *Topology of three manifolds and the embedding problems in minimal surface theory*, Ann. of Math, 112(1980), 441-485.

[59] W. H. Meeks III and S. T. Yau, *The equivariant Dehn's lemma and loop theorem*, Comment. Math. Helv. 56 (1981), no. 2, 225-239.

[60] A. Nadel, *Semisimplicity of the group of biholomorphisms of the universal covering of a compact complex manifold with ample canonical line bundle*, Ann. of Math. 132(1990), 193-211.

[61] Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math 0211159.

[62] Grisha Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math 0303109.

[63] Grisha Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math 0307245.

[64] R. Schoen, *Estimates for stable minimal surfaces in three-dimensional manifolds*, Seminar on minimal submanifolds, 111-126, Ann. of Math. Stud., 103, 1983.

[65] K. Sturm, *On the geometry of metric measured spaces*, I. Acta. Math, 196(2006), 65-131.

[66] K. Sturm, *On the geometry of metric measured spaces*, II. Acta. Math, 196(2006), 133-177.

[67] W. X. Shi, *Complete noncompact three-manifolds with nonnegative Ricci curvature*, J. Diff. Geom. 29 (1989), no. 2, 353-360.

[68] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math.(2), 88(July 1968), no. 1, 62-105.

[69] Y. T. Siu and S. T. Yau, *Compact Kähler manifolds with positive bisectional curvature*, Invent. Math. 59(1980), 189-204.

[70] J. Stallings, *Group theory and three dimensional manifolds*, Yale Univ. Press, 1971.

[71] J. Song, G. Tian, *The Kähler-Ricci flow on minimal surfaces of positive Kodaria dimension*, Invent. Math. 170 (2007),no. 3, 609653.

[72] J. Song, G. Tian, *Canonical measures and Kähler-Ricci flow*, J. Amer. Math. 25(2012), 303-353.

[73] J. Song and B. Weinkove, *Lecture notes on the Kähler-Ricci flow*, arxiv: 1212.3653.

[74] R. Schoen and S. T. Yau, *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature*, Seminar on Differential Geometry, pp. 209-228, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

[75] R. Schoen and S. T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. 65 (1979), no. 1, 45-76.

[76] R. Schoen and S. T. Yau, *Proof of the positive mass theorem. II*, Comm. Math. Phys. 79 (1981), no. 2, 231-260.

[77] R. Schoen, S. T. Yau: *Lecture notes on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, Boston 1994.

[78] G. Tian, Z. Zhang, *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B 27 (2006), no. 2, 179-192.

[79] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988). no. 1, 123-133.

[80] G. Wei and W. Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. 83 (2009), no. 2, 377-405.

[81] H. Wu and F. Zheng, *Compact Kähler manifolds with nonpositive bisectional curvature*, J. Diff. Geom. 61(2002), 263-287.

[82] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28(1975), 201-228.

[83] S. T. Yau, *On the Curvature of Compact Hermitian Manifolds*, Invent. Math. 25, 213-239(1974).

[84] S. T. Yau, *A general Schwarz lemma for Khler manifolds.* Amer. J. Math. 100 (1978), no. 1, 197-203.

[85] S.-T. Yau. *On the Ricci curvature of a compact Kahler manifold and the complex Monge- Ampère equation, I*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.

[86] C. J. Yu, *A note on Wu-Zheng's splitting conjecture*, Proc. Amer. Math. Soc. 141 (2013), no. 5, 1791-1793.

[87] S. Zhu, *A finiteness theorem for Ricci curvature in dimension three*, J. Diff. Geom., (1993), 711-727.

[88] S. Zhu, *On open three manifolds of quasi-positive Ricci curvature*, Proc. Amer. Math. Soc. 120(1994), no. 2, 569-572.

[89] F. Zheng, *Kodaira dimensions and hyperbolicity for nonpositively curved Kahler manifolds*, Comment. Math. Helv. 77(2) (2002) 221-234.