

**On fully nonlinear elliptic and parabolic equations in
domains with VMO coefficients**

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Dedication

To my parents, Yuhua Jiang and Guanxin Li.

Abstract

We prove the solvability in Sobolev spaces $W_p^{1,2}$, $p > d + 1$, of the terminal-boundary value problem for a class of fully nonlinear parabolic equations, including parabolic Bellman's equations, in bounded cylindrical domains with VMO "coefficients". The solvability in W_p^2 , $p > d$, of the corresponding elliptic boundary-value problem is also obtained. This research work has been published in [8].

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Chapter 1

Introduction

In this research work, we consider parabolic equations

$$\partial_t u(t, x) + F(D^2 u(t, x), t, x)$$

$$+ G(D^2 u(t, x), Du(t, x), u(t, x), t, x) = 0 \quad (1.1)$$

in subdomains of $\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$, where

$$\mathbb{R}^d = \{x = (x^1, \dots, x^d) : x^1, \dots, x^d \in \mathbb{R} = (-\infty, \infty)\}.$$

Here

$$D^2 u = (D_{ij} u), \quad Du = (D_i u), \quad D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j, \quad \partial_t = \frac{\partial}{\partial t}.$$

We introduce \mathcal{S} as the set of symmetric $d \times d$ matrices, fix some constants $\delta \in (0, 1)$ and $K \in \mathbb{R}_+ := (0, \infty)$, and throughout the article we assume that

(H₁) $F(u'', t, x)$ is convex and positive homogeneous of degree one with respect to $u'' \in \mathcal{S}$ and for all values of the arguments and $\xi \in \mathbb{R}^d$

$$\delta |\xi|^2 \leq F(u'' + \xi \xi^*, t, x) - F(u'', t, x) \leq \delta^{-1} |\xi|^2;$$

(H₂) $G(u'', u', u, t, x)$, $u'' \in \mathcal{S}$, $u' \in \mathbb{R}^d$, $u \in \mathbb{R}$, is nonincreasing in u and for all values of the arguments (notice u'' and not v'')

$$|G(u'', u', u, t, x) - G(u'', v', v, t, x)| \leq K(|u' - v'| + |u - v|),$$

$$|G(u'', u', u, t, x)| \leq \chi(|u''|)|u''| + K(|u'| + |u|) + \bar{G}(t, x),$$

where \bar{G} and χ are given functions such that $\chi : \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$ is bounded, decreasing, and $\chi(r) \rightarrow 0$ as $r \rightarrow \infty$;

(H₃) $F(u'', t, x) + G(u'', u', u, t, x)$ is convex with respect to $u'' \in \mathcal{S}$ and for all values of the arguments and $\xi \in \mathbb{R}^d$

$$\begin{aligned} \delta|\xi|^2 &\leq F(u'' + \xi\xi^*, t, x) + G(u'' + \xi\xi^*, u', u, t, x) \\ &\quad - F(u'', t, x) - G(u'', u', u, t, x) \leq \delta^{-1}|\xi|^2. \end{aligned}$$

We shall derive a priori estimates only using conditions (H₁) and (H₂). It is in the proofs of the solvability where condition (H₃) plays its role. However, we have the following

Conjecture. In (H₃) the convexity assumption on $F(u'', t, x) + G(u'', u', u, t, x)$ with respect to $u'' \in \mathcal{S}$ can be dropped.

To state our main results, we introduce a few notation. For $r > 0$, $x \in \mathbb{R}^d$, and $t \in \mathbb{R}$, we denote

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}, \quad Q_r(t, x) = (t, t + r^2) \times B_r(x).$$

If \mathcal{D} is a domain in \mathbb{R}^d and $-\infty \leq S < T < \infty$, we denote the parabolic boundary of the cylinder $(S, T) \times \mathcal{D}$ by

$$\partial'((S, T) \times \mathcal{D}) = (\{T\} \times \mathcal{D}) \cup ((S, T] \times \partial\mathcal{D}).$$

Finally, for any $T > 0$, we define $\mathcal{D}_T = (0, T) \times \mathcal{D}$.

The following VMO (vanishing mean oscillation) assumption is imposed on the leading term in (1.1) with a constant $\theta \in (0, 1]$ to be specified later.

Assumption 1.0.1. *There exists $R_0 \in (0, 1]$ such that for any $r \in (0, R_0]$, $\tau \in \mathbb{R}$, and $z \in \mathcal{D}$ one can find a function $\bar{F}(u'')$ (independent of (t, x)) satisfying condition (H₁) and such that for any $u'' \in \mathcal{S}$ with $|u''| = 1$ we have*

$$\int_{Q_r(\tau, z)} |F(u'', t, x) - \bar{F}(u'')| dx dt \leq \theta r^{d+2}. \quad (1.2)$$

The first main result of the article is about the terminal-boundary value problem for fully nonlinear parabolic equations with “VMO coefficients” in bounded cylinders.

Theorem 1.0.1. *Let $p > d + 1$ be a constant, $T \in \mathbb{R}_+$, and let \mathcal{D} be a bounded $C^{1,1}$ domain in \mathbb{R}^d . Assume that $\bar{G} \in L_p(\mathcal{D}_T)$. Then there exists a constant $\theta \in (0, 1]$ depending only on d, p, δ , and the $C^{1,1}$ norm of $\partial\mathcal{D}$ such that if Assumption 1.0.1 is satisfied with this θ , then the following assertions hold. For any $g \in W_p^{1,2}(\mathcal{D}_T)$, there is a unique solution $u \in W_p^{1,2}(\mathcal{D}_T)$ to (1.1) such that $u - g \in \dot{W}_p^{1,2}(\mathcal{D}_T)$. Moreover, we have*

$$\|u\|_{W_p^{1,2}(\mathcal{D}_T)} \leq N\|\bar{G}\|_{W_p^{1,2}(\mathcal{D}_T)} + N\|g\|_{W_p^{1,2}(\mathcal{D}_T)} + N_0, \quad (1.3)$$

where N depends only on d, p, δ, K, R_0 , the $C^{1,1}$ norm of $\partial\mathcal{D}$, and $\text{diam}(\mathcal{D})$ and N_0 depends only on the same objects, T , and χ . In particular, $N_0 = 0$ if $\chi \equiv 0$.

Here $W_p^{1,2}(\mathcal{D}_T)$ denotes the set of functions v defined on \mathcal{D}_T such that v, Dv, D^2v , and $\partial_t v$ are in $L_p(\mathcal{D}_T)$, and $\dot{W}_p^{1,2}(\mathcal{D}_T)$ is the set of all functions $v \in W_p^{1,2}(\mathcal{D}_T)$ such that if for the function u defined as $u(t, x) = v(t, x)$ for $(t, x) \in \mathcal{D}_T$, $u(t, x) = 0$ for $(t, x) \in (-\infty, T) \times \mathbb{R}^d \setminus \mathcal{D}_T$ we have $u \in W_p^{1,2}((-\infty, T) \times \mathbb{R}^d)$.

If F and G are independent of t , we also consider elliptic equations

$$F(D^2u(x), x) + G(D^2u(x), Du(x), u(x), x) = 0 \quad (1.4)$$

in subdomains of \mathbb{R}^d with Dirichlet boundary condition. In that case Assumption 1.0.1 becomes the following.

Assumption 1.0.2. *There exists $R_0 \in (0, 1]$ such that, for any $r \in (0, R_0]$ and $z \in \mathcal{D}$, one can find a function $\bar{F}(u'')$ (independent of x) satisfying condition (\mathbf{H}_1) and such that for any $u'' \in \mathcal{S}$ with $|u''| = 1$ we have*

$$\int_{B_r(z)} |F(u'', x) - \bar{F}(u'')| dx \leq \theta r^d. \quad (1.5)$$

Our next theorem is about the boundary value problem for elliptic equations with VMO coefficients in bounded domains.

Theorem 1.0.2. *Let $p > d$ be a constant and \mathcal{D} be a bounded $C^{1,1}$ domain in \mathbb{R}^d . Assume that \bar{G} is independent of t and $\bar{G} \in L_p(\mathcal{D})$. Then there exists a constant $\theta \in (0, 1]$, depending only on d, p, δ , and the $C^{1,1}$ norm of $\partial\mathcal{D}$, such that if Assumption 1.0.2 is satisfied with this θ , then the following assertions hold. For any $g \in W_p^2(\mathcal{D})$*

there is a unique solution $u \in W_p^2(\mathcal{D})$ to (1.4) such that $u - g \in \mathring{W}_p^2(\mathcal{D})$. Moreover, we have

$$\|u\|_{W_p^2(\mathcal{D})} \leq N\|\bar{G}\|_{W_p^2(\mathcal{D})} + N\|g\|_{W_p^2(\mathcal{D})} + N_0, \quad (1.6)$$

where N depends only on d, p, δ, R_0, K , the $C^{1,1}$ norm of $\partial\mathcal{D}$, and $\text{diam}(\mathcal{D})$ and N_0 depends only on the same objects and χ . In particular, $N_0 = 0$ if $\chi \equiv 0$.

Here $W_p^2(\mathcal{D})$ denotes the set of all functions v defined in \mathcal{D} such that v, Dv , and D^2v are in $L_p(\mathcal{D})$, and $\mathring{W}_p^2(\mathcal{D}) = \mathring{W}_p^1(\mathcal{D}) \cap W_p^2(\mathcal{D})$, where $\mathring{W}_p^1(\mathcal{D})$ is the subset of $W_p^1(\mathcal{D})$ consisting of functions for each of which there is a defining sequence vanishing on $\partial\mathcal{D}$.

In the literature, the interior $W_p^2, p > d$ estimates for a class of fully nonlinear uniformly elliptic equations of the form

$$F(D^2u, x) = f(x)$$

were first obtained by Caffarelli in [2] (see also [3]). His proof is geometric and is based on the Aleksandrov–Bakel’man–Pucci a priori estimate, the Krylov–Safonov Harnack inequality and a covering argument which can be found in [17] and [23]. Adapting this technique, similar interior estimates were proved by Wang [28] for parabolic equations. In the same paper, a boundary estimate is stated but without a proof; see Theorem 5.8 there. By exploiting a weak reverse Hölder’s inequality, the result of [2] was sharpened by Escauriaza in [9], who obtained the interior W_p^2 -estimate for the same equations allowing $p > d - \varepsilon$, with a small constant ε depending only on the ellipticity constant and d . Very recently, Winter [29] further extended this technique to establish the corresponding boundary estimate as well as the W_p^2 -solvability of the associated boundary-value problem. It is also worth noting that a solvability theorem in the space $W_{p,\text{loc}}^{1,2}(Q) \cap C(\bar{Q})$ can be found in [6] for the boundary-value problem for fully nonlinear parabolic equations. In these papers, a small oscillation assumption in the integral sense is imposed on the operators; see, for instance, [2, Theorem 1]. However, as pointed out in [29, Remark 2.3] and in [16] (see also [6, Example 8.3] for a relevant discussion), this assumption turns out to be equivalent to a small oscillation condition in the L_∞ sense, which, particularly in the *linear* case, is the same as what is required in the classical L_p theory based on the Calderón–Zygmund estimates. Thus, it seems to us that the results in [2, 28, 9, 6, 29] mentioned above are in general not formally applicable to the

operators under Assumption 1.0.1 or 1.0.2, in which local oscillations are measured in the average sense so that huge jumps in the L_∞ norm are allowed. It is still possible that the *methods* developed in the above cited articles can be used to obtain our results. In our opinion, our method is somewhat simpler and leads to the results faster.

The results obtained in this dissertation contain and generalize the Sobolev space theory of linear equations with VMO coefficients, which was developed about twenty years ago by Chiarenza, Frasca, and Longo in [4, 5] for non-divergence form elliptic equations, and later in [1] by Bramanti and Cerutti for parabolic equations. The proofs in these references are based on the Calderón–Zygmund theorem and the Coifman–Rochberg–Weiss commutator theorem. For further related results, we refer the reader to the book [22] and reference therein.

However, remarkably not all known results related to VMO coefficients and second-order elliptic and parabolic *linear* equations can be obtained from the results of the present dissertation.

The reader can find in [13, 14] a unified approach to investigating the L_p (and $L_q - L_p$) solvability of both divergence and non-divergence form parabolic and elliptic equations with leading coefficients that are in VMO in the spatial variables and only *measurable* in the time variable in the parabolic case. In the nonlinear setting, it is an extremely challenging problem whether or not one can treat F 's which are only measurable in t . The proofs in [13, 14] rely mainly on pointwise estimates of sharp functions of spatial derivatives of solutions, so that VMO coefficients are treated in a rather straightforward manner. This approach is rather flexible: it has been applied to both divergence and non-divergence form linear equations/systems with coefficients which are very irregular in some of the independent variables. For example, in [10, 11] Kim and Krylov established the solvability in Sobolev spaces of non-divergence elliptic and parabolic equations with leading coefficients measurable in a space variable and VMO in the other variables; in [7] Dong and Kim considered both divergence and non-divergence form higher-order elliptic and parabolic systems in the whole space, the half space and bounded domains with coefficients in the same class as in [13, 14]; see also the references in [7] for other results in this line of research.

Here we follow the general scheme in [13, 14] to study fully nonlinear elliptic and

parabolic equations in bounded domains or cylinders with VMO coefficients. This dissertation is a continuation of [16], in which interior estimates for elliptic Bellman's equations were obtained. The key ingredients in our proofs are the Evans–Krylov theorem applied to homogeneous equations with constant coefficients and a W_ε^2 estimate for elliptic equations with measurable coefficients, which is originally due to F.H. Lin [21] and extended to the parabolic case in [16]. We also remark that as in [9, 6, 29], by making use of a refined Aleksandrov–Bakel'man–Pucci estimate instead of the classical estimate, one can extend the range of p in our results to $p > d - \varepsilon$ in the elliptic case and to $p > d + 1 - \varepsilon$ in the parabolic case, where ε is a small constant depending only on d and δ . These ranges are sharp, as is seen from the example on nondivergence equations on page 18 in Section I.2 of [18] for the elliptic case, and one can show that for the parabolic case.

Remark 1.0.1. *A few comments on the structures of (1.1) and (1.4) are in order. Usually, the last two terms on the left-hand side of (1.1) are combined into one $H = F + G$. However, if we are given a function $H(u'', u', u, t, x)$, we can always represent it as $F + G$ with $F = H(u'', 0, 0, t, x) - H(0, 0, 0, t, x)$ and $G = H - F$. Then usual ellipticity, convexity in (u_{ij}) , Lipschitz continuity, and growth conditions with respect to (u'', u', u) from the theory of fully nonlinear equations will transform into our conditions even with $\chi \equiv 0$. Our form may look more attractive in the sense that no convexity condition with respect to u'' is imposed on G . The above decomposition of H lacks however the requirement that F be positive homogeneous of degree one. Then one defines*

$$\hat{F}(u'', t, x) = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} F(\lambda u'', t, x), \quad \hat{G} = F - \hat{F}$$

and combines \hat{G} with G . The fact that \hat{F} is well defined follows from the Lipschitz continuity and convexity of F in u'' . That \hat{F} is positive homogeneous of degree one is obvious. Furthermore, for each (t, x) , the functions $\frac{1}{\lambda} F(\lambda u'', t, x)$ are equicontinuous in u'' , and hence converge uniformly on compact sets which means exactly that

$$\chi(u'', t, x) := \frac{1}{|u''|} |\hat{F}(u'', t, x) - F(u'', t, x)| \rightarrow 0$$

as $|u''| \rightarrow \infty$.

Remark 1.0.2. *There are natural and essentially unique candidates for the functions \bar{F} in Assumptions 1.0.1 and 1.0.2. To show them for a function f defined on a Borel set $\mathcal{U} \subset \mathbb{R}^{d+1}$, we set*

$$(f)_{\mathcal{U}} = \frac{1}{|\mathcal{U}|} \int_{\mathcal{U}} f(t, x) dx dt = \int_{\mathcal{U}} f(t, x) dx dt,$$

where $|\mathcal{U}|$ is the $d + 1$ -dimensional Lebesgue measure of \mathcal{U} . In case \mathcal{U} is a Borel subset of \mathbb{R}^d , we define $|\mathcal{U}|$ and $(f)_{\mathcal{U}}$ in a similar way. The reader understands that if f also depends on u'' : $f(u'', t, x)$, then after averaging with respect to (t, x) we will get the result depending on u'' as well, which we denote $(f)_{\mathcal{U}}(u'')$. Now it is easy to see that if (1.2) holds with an \bar{F} , then it also holds with

$$\bar{F}(u'') = (F)_{Q_r(\tau, z)}(u'')$$

provided that we multiply the right-hand side of (1.2) by a constant depending only on d . Thus defined $\bar{F}(u'')$ satisfies (H_1) as long as F does.

Remark 1.0.3. *A typical example when it is relatively easy to verify our hypotheses is given by the following Bellman's equation:*

$$\begin{aligned} \partial_t u(t, x) + \sup_{\omega \in \Omega} [a^{ij}(\omega, t, x) D_{ij} u(t, x) + b^i(\omega, t, x) D_i u(t, x) \\ - c(\omega, t, x) u(x) + f(\omega, t, x)] = 0, \end{aligned} \quad (1.7)$$

where the set Ω is a separable metric space, $a = (a^{ij})$, $b = (b^i)$, $c \geq 0$, and f are given functions which are measurable in (t, x) for each $\omega \in \Omega$ and continuous in ω for each (t, x) .

As usual, the summation convention is enforced throughout the article and the summation in (1.7) and in similar situations is performed before the supremum is taken. Equations of that type appear in many applications and, in particular, in the theory of optimal control of diffusion type processes they are the so-called Bellman's equations.

Introduce,

$$\begin{aligned} F(u'', t, x) &= \sup_{\omega \in \Omega} a^{ij}(\omega, t, x) u''_{ij}, \quad G(u'', u', u, t, x) \\ &= \sup_{\omega \in \Omega} [a^{ij}(\omega, t, x) u''_{ij} + b^i(\omega, t, x) u'_i - c(\omega, t, x) u + f(\omega, t, x)] - F(u'', t, x) \end{aligned}$$

and assume that for any ω the function $a^{ij}(\omega, t, x)u''_{ij}$ satisfies (\mathbf{H}_1) and the function $b^i(\omega, t, x)u'_i - c(\omega, t, x)u + f(\omega, t, x)$ satisfies (\mathbf{H}_2) . Then F and G satisfy (\mathbf{H}_1) - (\mathbf{H}_3) with $\chi \equiv 0$.

One can give several conditions in terms of a^{ij} , which are sufficient for (1.2) to hold. For instance, (1.2) is satisfied if for any $r \in (0, R_0]$, $t \in \mathbb{R}$, and $z \in \mathcal{D}$ one can find functions $\bar{a}^{ij}(\omega)$ such that the functions $\bar{a}^{ij}(\omega)u''_{ij}$ satisfy (\mathbf{H}_1) and for any $u'' \in \mathcal{S}$ with $|u''| = 1$

$$\int_{Q_r(\tau, z)} |\sup_{\omega} a^{ij}(\omega, t, x)u''_{ij} - \sup_{\omega} \bar{a}^{ij}(\omega)u''_{ij}| dx dt \leq \theta r^{d+2}$$

or, since the difference of supremums is less than the supremum of the absolute values of the differences, if for all i, j

$$\int_{Q_r(\tau, z)} \sup_{\omega} |a^{ij}(\omega, t, x) - \bar{a}^{ij}(\omega)| dx dt \leq \theta r^{d+2}. \quad (1.8)$$

In addition, if Ω is a finite set, then one can drop the last supremum and require the condition to hold for each ω . As in Remark 1.0.2, the latter condition holds with some \bar{a} if and only if it holds (with slightly modified right-hand side) with $\bar{a} = a_{Q_r(\tau, z)}$.

The remainder of the article is organized as follows. We consider elliptic equations in the half space with constant coefficients in Section 2.1 and with VMO coefficients in Section 2.2. With these preparations, the proof of Theorem 1.0.2 is given in Section 2.3. Then we turn to parabolic equations in the whole space with constant coefficients in Section 3.1 and with VMO coefficients in Section 3.2, as well as parabolic equations in the half space in Sections 3.3 and 3.4. Finally, the proof of Theorem 1.0.1 is presented at the end of Section 3.4. The reader may notice that we could have somewhat shortened the article by deriving some results for elliptic equations from their parabolic counterparts. We do not do that because it is much easier and shorter to explain the main ideas in the elliptic case.

A few times in the article we will be using known results from $C^{2+\alpha}$ theory of elliptic and parabolic fully nonlinear equations. Part of these results is proved for H concave in u'' and part for convex H . The reader understands that results for concave H are also applicable for equations with convex H since the transformation $H(u'') \rightarrow -H(-u'')$ changes the direction of convexity and does not affect the ellipticity condition.

Chapter 2

Solvability of elliptic equations

2.1 Elliptic equations with constant coefficients in \mathbb{R}_+^d

First we introduce a few more notation. Set $\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x^1 > 0\}$. For $r > 0$ and $x = (x^1, x') \in \mathbb{R}_+^d$, denote

$$B_r = B_r(0), \quad B_r(x^1) = B_r(x^1, 0),$$

$$B_r^+(x) = B_r(x) \cap \mathbb{R}_+^d, \quad B_r^+ = B_r^+(0), \quad B_r^+(x^1) = B_r^+(x^1, 0).$$

Recall that by Du and D^2u we denote the gradient and the Hessian of u , respectively.

In this section, we are interested in the equation

$$F(D^2u) = f(x), \tag{2.1}$$

in the half space \mathbb{R}_+^d with $F = F(u'', x)$ independent of x . Since F is convex and positive homogeneous of degree one, it has a representation as in Remark 1.0.3, so that we are dealing with Bellman's equations.

Lemma 2.1.1. *For any $u \in W_d^2(B_r^+)$ vanishing on $x^1 = 0$, we have*

$$\sup_{B_r^+} |u(x) - x^1(D_1u)_{B_r^+}|^d \leq Nr^{2d} \int_{B_r^+} |D^2u|^d dx,$$

where N depends only on d .

Proof. Let \tilde{u} be the odd extension of u with respect to x^1 , i.e., $\text{sgn}(x^1)$. By Lemma 8.2.1 in [15], $\tilde{u} \in W_d^2$. Note that

$$(\tilde{u})_{B_r} = 0, \quad (D_1 \tilde{u})_{B_r} = (D_1 u)_{B_r^+}, \quad (D_i \tilde{u})_{B_r} = 0 \quad \text{for } i \geq 2.$$

The lemma then follows from Lemma 2.1 of [16]. \square

Lemma 2.1.2. *Let $r \in (0, \infty)$, $\kappa \geq 2$ and let $v \in C(\bar{B}_{\kappa r}^+) \cap C_b^2(B_{\kappa \rho}^+)$ for any $\rho \in (0, r)$. Assume that v is a solution of (2.1) in $B_{\kappa r}^+$ with $f \equiv 0$ and $v = 0$ on $x^1 = 0$. Then there are constants $\alpha \in (0, 1)$ and N , depending only on d and δ , such that*

$$[D^2 v]_{C^\alpha(B_r^+)} \leq N(\kappa r)^{-2-\alpha} \sup_{\partial B_{\kappa r}^+} |v|.$$

Proof. Dilations show that it suffices to prove the inequality for $\kappa r = 1$. In this case, the result follows from Theorems 7.1 of [24] or of [25], which state that

$$[D^2 v]_{C^\alpha(B_{1/2}^+)} \leq N \sup_{B_1^+} |v|.$$

Due to the maximum principle, the lemma is proved. \square

Denote by \mathcal{S}_δ the set of symmetric $d \times d$ -matrices $\alpha = (\alpha^{ij})$ satisfying

$$\delta |\xi|^2 \leq \alpha^{ij} \xi_i \xi_j \leq \delta^{-1} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d.$$

Introduce \mathbb{L}_δ as the collection of operators $Lu = a^{ij} D_{ij} u$ with $a(x) = (a^{ij}(x)) \in \mathcal{S}_\delta$ for all $x \in \mathbb{R}^d$.

We need a slight generalization of the main result of [21] (stated as Lemma 2.3 in [16]) which can be proved in the same way as in [21] by using dilations and standard approximation arguments.

Lemma 2.1.3. *Let $r \in (0, \infty)$ and let $u \in C(\bar{B}_r) \cap W_d^2(B_\rho)$ for any $\rho \in (0, r)$. Then there are constants $\gamma \in (0, 1]$ and N , depending only on δ, d such that for any $L \in \mathbb{L}_\delta$ we have*

$$\int_{B_r} |D^2 u|^\gamma dx \leq N \left(\int_{B_r} |Lu|^d dx \right)^{\gamma/d} + Nr^{-2\gamma} \sup_{\partial B_r} |u|^\gamma.$$

Lemma 2.1.4. *Let $r \in (0, \infty)$ and let $w \in W_d^2(B_\rho^+) \cap C(\bar{B}_r^+)$ for any $\rho \in (0, r)$. Assume that $w = 0$ on ∂B_r^+ . Then there are constants $\gamma \in (0, 1]$ and N , depending only on δ and d , such that for any $L \in \mathbb{L}_\delta$,*

$$\int_{B_r^+} |D^2 w|^\gamma dx \leq N \left(\int_{B_r^+} |Lw|^d dx \right)^{\gamma/d}.$$

Proof. Denote $f = Lw$. Let \tilde{w} and \tilde{f} be the odd extension of w and f with respect to x^1 . Denote by $\tilde{L} \in \mathbb{L}_\delta$ the operator with coefficients

$$\tilde{a}^{ij}(x) = \text{sgn}(x^1) a^{ij}(|x^1|, x') \quad \text{for } i = 1, j \geq 2 \text{ or } j = 1, i \geq 2,$$

$$\tilde{a}^{ij}(x) = a^{ij}(|x^1|, x') \quad \text{otherwise.}$$

Clearly, $\tilde{w} \in C(\bar{B}_r) \cap W_d^2(B_\rho)$ for any $\rho < r$, $\tilde{w} = 0$ on ∂B_r , and $\tilde{L}\tilde{w} = \tilde{f}$ in B_r . Now Lemma 2.1.3 yields

$$\int_{B_r} |D^2 \tilde{w}|^\gamma dx \leq N \left(\int_{B_r} |\tilde{f}|^d dx \right)^{\gamma/d}.$$

To finish the proof of the lemma, it suffices to recall the definitions of \tilde{w} and \tilde{f} . \square

Everywhere below in this section α is the constant from Lemma 2.1.2 and γ is the one from Lemma 2.1.4.

Lemma 2.1.5. *Let $r \in (0, \infty)$, $\kappa \geq 16$, $x_0^1 \geq 0$. Let $u \in W_d^2(B_{\kappa r}^+(x_0^1))$ be a solution of (2.1) in $B_{\kappa r}^+(x_0^1)$ vanishing on $B_{\kappa r}(x_0^1) \cap \partial \mathbb{R}_+^d$. Then*

$$\begin{aligned} & \int_{B_r^+(x_0^1)} \int_{B_r^+(x_0^1)} |D^2 u(x) - D^2 u(y)|^\gamma dx dy \\ & \leq N \kappa^d \left(\int_{B_{\kappa r}^+(x_0^1)} |f|^d dx \right)^{\gamma/d} + N \kappa^{-\gamma\alpha} \left(\int_{B_{\kappa r}^+(x_0^1)} |D^2 u|^d dx \right)^{\gamma/d}, \end{aligned} \quad (2.2)$$

where the constant N depends only on d and δ .

Proof. Dilations show that it suffices to prove the lemma only for $\kappa r = 8$. We consider two cases.

Case 1: $x_0^1 > 1$. In this case, we have $B_{\kappa r/8}^+(x_0^1) = B_{r\kappa/8}(x_0^1) \subset \mathbb{R}_+^d$. Therefore, inequality (2.2) is an immediate consequence of Lemma 2.4 in [16] since $\kappa/8 \geq 2$ (cf. the comment at the beginning of the section).

Case 2: $x_0^1 \in [0, 1]$. Since $r = 8/\kappa \leq 1/2$, we have

$$B_r^+(x_0^1) \subset B_2^+ \subset B_4^+ \subset B_{\kappa r}^+(x_0^1).$$

By using a standard density argument, we may assume $u \in C_b^\infty(\bar{B}_{\kappa r}^+(x_0^1))$. Define $\hat{u}(x) := u(x) - x^1(D_1 u)_{B_4^+}$. Let v be a classical solution of (2.1) in B_4^+ with $f \equiv 0$ and boundary condition $v = \hat{u}$ on ∂B_4^+ . Such a solution exists due to Theorems 7.1 of [24] or of [25]. Then by Lemmas 2.1.2 and 2.1,

$$\begin{aligned} & \int_{B_r^+(x_0^1)} \int_{B_r^+(x_0^1)} |D^2 v(x) - D^2 v(y)| dx dy \leq N r^\alpha [D^2 v]_{C^\alpha(B_2^+)} \\ & \leq N r^\alpha \sup_{\partial B_4^+} |v| = N r^\alpha \sup_{\partial B_4^+} |\hat{u}| \leq N \kappa^{-\alpha} \left(\int_{B_4^+} |D^2 u|^d dx \right)^{1/d}. \end{aligned}$$

Recall that $\gamma \in (0, 1]$. By Hölder's inequality, we get

$$\begin{aligned} & \int_{B_r^+(x_0^1)} \int_{B_r^+(x_0^1)} |D^2 v(x) - D^2 v(y)|^\gamma dx dy \\ & \leq N \kappa^{-\gamma\alpha} \left(\int_{B_{\kappa r}^+(x_0^1)} |D^2 u|^d dx \right)^{\gamma/d}. \end{aligned} \quad (2.3)$$

Next we recall a simple and well-known fact that condition (H₁) implies that for any \mathcal{S} valued functions $u''(x)$ and $v''(x)$ there is an operator $L = a^{ij} D_{ij} \in \mathbb{L}_\delta$ such that $F(u''(x)) - F(v''(x)) = a^{ij} [u''_{ij} - v''_{ij}](x)$. Then set $w := \hat{u} - v$ in B_4^+ and notice that $w \in W_d^2(B_\rho^+) \cap C(\bar{B}_4^+)$ for any $\rho < 4$, $w = 0$ on ∂B_4^+ , and $F(D^2 \hat{u}) = f$.

It follows by the above that there exists an operator $L \in \mathbb{L}_\delta$ such that $Lw = f$ in B_4^+ . By Lemma 2.1.4 and the fact that $\kappa r = 8$, we get

$$\int_{B_r^+(x_0^1)} |D^2 w|^\gamma dx \leq N \kappa^d \int_{B_4^+} |D^2 w|^\gamma dx \leq N \kappa^d \left(\int_{B_{\kappa r}^+(x_0^1)} |f|^d dx \right)^{\gamma/d}$$

and

$$\begin{aligned} & \int_{B_r^+(x_0^1)} \int_{B_r^+(x_0^1)} |D^2 w(x) - D^2 w(y)|^\gamma dx dy \\ & \leq N \kappa^d \left(\int_{B_{\kappa r}^+(x_0^1)} |f|^d dx \right)^{\gamma/d}. \end{aligned}$$

Combining this with (2.3) and observing that $D^2 u = D^2 v + D^2 w$ yield (2.2). The lemma is proved. \square

If g is a measurable function in \mathbb{R}^d , define its maximal function by

$$\mathbb{M}(g)(x) = \sup_{B_r(y) \ni x} \int_{B_r(y)} |g(z)| dz.$$

It is easy to see that, for any $r > 0$ and $x \in \mathbb{R}_+^d$, we have

$$\int_{B_r^+(x)} |g(z)| dz \leq 2 \int_{B_r(x)} |g(z) I_{\mathbb{R}_+^d}(z)| dz \leq 2\mathbb{M}(g I_{\mathbb{R}_+^d})(x). \quad (2.4)$$

Next in the measure space \mathbb{R}_+^d endowed with the Borel σ -field and Lebesgue measure consider the filtration of dyadic cubes $\mathfrak{C} = \{\mathbb{C}_n, n \in \mathbb{Z}\}$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and \mathbb{C}_n is the collection of cubes

$$(i_1 2^{-n}, (i_1 + 1) 2^{-n}] \times \dots \times (i_d 2^{-n}, (i_d + 1) 2^{-n}], \quad i_1, \dots, i_d \in \mathbb{Z}, \quad i_1 \geq 0.$$

For $x \in \mathbb{R}_+^d$ introduce

$$g_\gamma^\#(x) = \sup_{C \in \mathfrak{C}: x \in C} \left(\int_C \int_C |g(y) - g(z)|^\gamma dy dz \right)^{1/\gamma}.$$

Notice that if $x \in C \in \mathfrak{C}$, then for the smallest $r > 0$ such that $C \subset B_r(x)$ we have

$$\int_C \int_C |g(y) - g(z)|^\gamma dy dz \leq N(d) \int_{B_r^+(x)} \int_{B_r^+(x)} |g(y) - g(z)|^\gamma dy dz.$$

This along with (2.4) and Lemma 2.1.5 lead to the following.

Corollary 2.1.6. *Let $u \in \dot{W}_d^2(\mathbb{R}_+^d)$ be a solution of (2.1) in \mathbb{R}_+^d . Then, for any $x \in \mathbb{R}_+^d$ and $\kappa \geq 16$, we have*

$$(D^2 u)_\gamma^\#(x) \leq N \kappa^{d/\gamma} \mathbb{M}^{1/d}(|f|^d I_{\mathbb{R}_+^d})(x) + N \kappa^{-\alpha} \mathbb{M}^{1/d}(|D^2 u|^d I_{\mathbb{R}_+^d})(x),$$

where the constant N depends only on d and δ .

Now we recall Theorem 5.3 of [16] which is a version of the Fefferman–Stein theorem: Let $p \in (1, \infty)$ and $\gamma \in (0, 1]$. Then for any $g \in L_p(\mathbb{R}_+^d)$, we have

$$\|g\|_{L_p(\mathbb{R}_+^d)} \leq N \|g_\gamma^\#\|_{L_p(\mathbb{R}_+^d)}, \quad (2.5)$$

where N depends on p, γ , and d only.

Theorem 2.1.7. *Let $p > d$. (i) Let $u \in \dot{W}_p^2(\mathbb{R}_+^d)$ satisfy (2.1). Then there exists $N = N(d, \delta, p)$ such that*

$$\|D^2u\|_{L_p(\mathbb{R}_+^d)} \leq N\|f\|_{L_p(\mathbb{R}_+^d)}.$$

(ii) *For any $\lambda > 0$ and $u \in \dot{W}_p^2(\mathbb{R}_+^d)$, we have*

$$\lambda\|u\|_{L_p(\mathbb{R}_+^d)} + \|D^2u\|_{L_p(\mathbb{R}_+^d)} \leq N\|F(D^2u) - \lambda u\|_{L_p(\mathbb{R}_+^d)}, \quad (2.6)$$

where N depends only on d, p , and δ .

(iii) *For any $f \in L_p(\mathbb{R}_+^d)$ and $\lambda > 0$, there is a unique solution $u \in \dot{W}_p^2(\mathbb{R}_+^d)$ of the equation*

$$F(D^2u) - \lambda u = f. \quad (2.7)$$

Proof. (i) First fix $\kappa \geq 16$. It follows from Corollary 2.1.6, (2.5), and the Hardy–Littlewood theorem on maximal functions that

$$\|D^2u\|_{L_p(\mathbb{R}_+^d)} \leq N\kappa^{d/\gamma}\|f\|_{L_p(\mathbb{R}_+^d)} + N\kappa^{-\alpha}\|D^2u\|_{L_p(\mathbb{R}_+^d)}, \quad (2.8)$$

where $N = N(d, \delta, p)$. Assertion (i) is proved once noting that the inequality holds for arbitrary $\kappa \geq 16$.

(ii) Assertion (i) implies that, to prove (2.6), it suffices to prove

$$\lambda\|u\|_{L_p(\mathbb{R}_+^d)} \leq N\|f\|_{L_p(\mathbb{R}_+^d)}, \quad (2.9)$$

where $f = F(D^2u) - \lambda u$.

We may assume that u is smooth in $\bar{\mathbb{R}}_+^d$ and vanishes for x large and for $x^1 = 0$. Take an operator $L \in \mathbb{L}_\delta$ such that $Lu - \lambda u = f$. Then we obtain (2.9) by Theorem 3.5.15 and the proof of Lemma 3.5.5 in [12] with N depending only on d, p , and δ .

(iii) The proof of the solvability of (2.7) relies on its solvability in $C^{2+\alpha}(\bar{\mathbb{R}}_+^d)$ with zero boundary condition ($\alpha \in (0, 1)$ is perhaps different from the one above). First we assume that $f \in C_0^\infty(\mathbb{R}_+^d)$ and by using classical results (see, for instance, [12] or [26]) find a function $u \in C^{2+\alpha}(\bar{\mathbb{R}}_+^d)$ with $u(0, \cdot) = 0$ satisfying (2.7). Simple barriers show that $u(x) \rightarrow 0$ exponentially fast as $|x| \rightarrow \infty, x^1 \geq 0$.

Furthermore, there is a well-known and standard procedure (see, for instance the proof of Lemma 2.4.4 in [15]) to derive from (2.6) that

$$\|u\|_{W_p^2(B_1^+(x))} \leq N\|u\|_{L_p(B_2^+(x))} + N\|f\|_{L_p(B_2^+(x))}, \quad x \in \mathbb{R}_+^d, \quad (2.10)$$

where N is independent of x . To start the procedure it suffices to notice that for nonnegative $\zeta \in C_0^\infty(\mathbb{R}^d)$ we have that

$$F(D^2(\zeta u)) - \lambda \zeta u = \zeta f + g,$$

where

$$g = F(D^2(\zeta u)) - \zeta F(D^2 u),$$

and by the homogeneity and Lipschitz continuity of F

$$|g| \leq N |D^2(\zeta u) - \zeta D^2 u| \leq N(|Du| + |u|).$$

Upon combining (2.10) and the fact that $u, f \in L_p(\mathbb{R}_+^d)$, we conclude that $u \in \dot{W}_p^2(\mathbb{R}_+^d)$, so that estimate (2.6) is applicable.

Having done this step we approximate the given $f \in L_p(\mathbb{R}_+^d)$ in the $L_p(\mathbb{R}_+^d)$ norm by functions $f_n \in C_0^\infty(\mathbb{R}_+^d)$, which would give us a sequence of $u_n \in \dot{W}_p^2(\mathbb{R}_+^d)$ with the $\dot{W}_p^2(\mathbb{R}_+^d)$ norms bounded such that $F(D^2 u_n) = f_n$. A subsequence $u_{n'}$ converges then uniformly on compact subsets of $\bar{\mathbb{R}}_+^d$ to a function $u \in \dot{W}_p^2(\mathbb{R}_+^d)$. That u satisfies (2.7) now follows from Theorems 3.5.15 and 3.5.6 (a) of [12]. This proves the existence of solutions.

As usual, uniqueness follows from the fact that $F(D^2 u) - F(D^2 v) = L(u - v)$, where $L \in \mathbb{L}_\delta$. The theorem is proved. \square

2.2 Elliptic equations with VMO coefficients in \mathbb{R}_+^d

We are about to deal with the equation

$$F(D^2 u, x) - \lambda u = f(x) \tag{2.11}$$

in the half space \mathbb{R}_+^d . Of course, Assumption 1.0.2 is supposed to hold with $\mathcal{D} = \mathbb{R}_+^d$.

Remark 2.2.1. *We are going to use the following fact: For any $\mu > 0$ there exists $\theta = \theta(\mu, d, \delta) > 0$ such that, if (1.5) holds with this θ for any $u'' \in \mathcal{S}$ with $|u''| = 1$, then*

$$\int_{B_r(z)} \sup_{u'' \in \mathcal{S}: |u''|=1} |F(u'', x) - \bar{F}(u'')| dx \leq \mu r^d. \tag{2.12}$$

To prove this observe that one can find $n = n(d, \delta, \mu)$ points u''_1, \dots, u''_n such that, for any x and any u'' with $|u''| = 1$, at least one of $|F(u'', x) - F(u''_i, x)| + |\bar{F}(u'') - \bar{F}(u''_i)|$ is less than $\mu/(4|B_1|)$. The latter is possible due to the Lipschitz continuity of F and \bar{F} in u'' (uniform with respect to x). After that it obviously suffices to choose $\theta = \mu/(4n)$.

We are also going to use the fact that the supremum in (2.12) is bounded by a constant depending only on δ and d .

Everywhere below in this section α is the constant from Lemma 2.1.2 and γ is the one from Lemma 2.1.4.

Lemma 2.2.1. *Let $\beta \in (1, \infty)$, $\lambda = 0$, $\kappa \geq 16$, $\mu, r > 0$, $x_0^1 \geq 0$, and $z \in \mathbb{R}_+^d$. Suppose that $\theta = \theta(\mu, d, \delta)$. Let $u \in \dot{W}_d^2(\mathbb{R}_+^d)$ be a solution of (2.11) vanishing outside $B_{R_0}^+(z)$. Then*

$$\begin{aligned} & \int_{B_r^+(x_0^1)} \int_{B_r^+(x_0^1)} |D^2 u(x) - D^2 u(y)|^\gamma dx dy \\ & \leq N\kappa^d \left(\int_{B_{\kappa r}^+(x_0^1)} |f|^d dx \right)^{\gamma/d} + N\kappa^d \left(\int_{B_{\kappa r}^+(x_0^1)} |D^2 u|^{\beta d} dx \right)^{\gamma/(\beta d)} \mu^{\gamma/(\beta' d)} \\ & \quad + N\kappa^{-\gamma\alpha} \left(\int_{B_{\kappa r}^+(x_0^1)} |D^2 u|^d dx \right)^{\gamma/d}, \end{aligned}$$

where $N = N(d, \delta, \beta)$ and $\beta' = \beta/(\beta - 1)$.

Proof. Introduce

$$\bar{F}(u'') = \begin{cases} (F)_{B_{R_0}^+(z)}(u''), & \text{if } \kappa r \geq R_0; \\ (F)_{B_{\kappa r}^+(x_0^1)}(u''), & \text{otherwise.} \end{cases}$$

Observe that

$$\bar{F}(D^2 u) = \hat{f}(x),$$

where

$$\hat{f}(x) = \bar{F}(D^2 u) - F(D^2 u, x) + f(x).$$

By Lemma 2.1.5,

$$\begin{aligned} & \int_{B_r^+(x_0^1)} \int_{B_r^+(x_0^1)} |D^2 u(x) - D^2 u(y)|^\gamma dx dy \\ & \leq N\kappa^d \left(\int_{B_{\kappa r}^+(x_0^1)} |\hat{f}|^d dx \right)^{\gamma/d} + N\kappa^{-\gamma\alpha} \left(\int_{B_{\kappa r}^+(x_0^1)} |D^2 u|^d dx \right)^{\gamma/d}. \end{aligned}$$

Notice that by the triangle inequality,

$$\int_{B_{\kappa r}^+(x_0^1)} |\hat{f}|^d dx \leq N \int_{B_{\kappa r}^+(x_0^1)} |f|^d dx + N \int_{B_{\kappa r}^+(x_0^1)} |\bar{F}(D^2u) - F(D^2u, x)|^d dx.$$

For any $x \in \mathbb{R}_+^d$, denote

$$h(x) = \sup_{u'' \in \mathcal{S}: |u''|=1} |F(u'', x) - \bar{F}(u'')|.$$

By Hölder's inequality,

$$\int_{B_{\kappa r}^+(x_0^1)} |\bar{F}(D^2u) - F(D^2u, x)|^d dx \leq \int_{B_{\kappa r}^+(x_0^1)} h^d(x) |D^2u|^d dx \leq J_1^{1/\beta} J_2^{1/\beta'},$$

where

$$J_1 = \int_{B_{\kappa r}^+(x_0^1)} |D^2u|^{\beta d} dx,$$

$$J_2 = \int_{B_{\kappa r}^+(x_0^1)} h^{\beta' d}(x) I_{B_{R_0}^+(z)} dx \leq N \int_{B_{\kappa r}^+(x_0^1)} h(x) I_{B_{R_0}^+(z)} dx.$$

If $\kappa r \geq R_0$, then by Remark 2.2.1

$$J_2 \leq N(\kappa r)^{-d} \int_{B_{R_0}^+(z)} h(x) dx \leq N(\kappa r)^{-d} R_0^d \int_{B_{R_0}^+(z)} h(x) dx \leq N\mu.$$

If $\kappa r < R_0$, then

$$J_2 \leq N \int_{B_{\kappa r}^+(x_0^1)} h(x) dx \leq N\mu.$$

Therefore,

$$\int_{B_{\kappa r}^+(x_0^1)} |\bar{F}(D^2u) - F(D^2u, x)|^d dx \leq N \left(\int_{B_{\kappa r}^+(x_0^1)} |D^2u(x)|^{\beta d} dx \right)^{1/\beta} \mu^{1/\beta'}$$

and this leads to the desired result. The lemma is proved. \square

Corollary 2.2.2. *Under the assumptions of Lemma 2.2.1, let $p > \beta d$. Then there is a constant N_0 , depending only on δ, β, d , and p , such that*

$$\|D^2u\|_{L_p(\mathbb{R}_+^d)} \leq N_0 \kappa^{d/\gamma} \|f\|_{L_p(\mathbb{R}_+^d)} + N_0 (\kappa^{d/\gamma} \mu^{1/(\beta' d)} + \kappa^{-\alpha}) \|D^2u\|_{L_p(\mathbb{R}_+^d)}.$$

Indeed it suffices to proceed as in the derivation of (2.8).

By taking $2\beta = 1 + p/d$ and choosing κ and θ in such a way that

$$N_0(\kappa^{d/\gamma} \mu^{1/(\beta'd)} + \kappa^{-\alpha}) \leq \frac{1}{2},$$

we arrive at the following corollary.

Corollary 2.2.3. *Let $p > d$, $u \in \dot{W}_d^2(\mathbb{R}_+^d)$ be a solution of (2.11) with $\lambda = 0$ vanishing outside $B_{R_0}^+(z)$, where $z \in \mathbb{R}_+^d$. Then there exist $\theta = \theta(d, p, \delta) \in (0, 1]$ and $N = N(d, p, \delta)$ such that if Assumption 1.0.2 is satisfied with this θ , then $\|D^2 u\|_{L_p(\mathbb{R}_+^d)} \leq N \|f\|_{L_p(\mathbb{R}_+^d)}$.*

The next theorem is the main result of this section.

Theorem 2.2.4. *Let $p > d$. Then there exist constants $\theta \in (0, 1]$ depending only on d, p, δ and λ_0 depending only on d, p, δ , and R_0 such that if Assumption 1.0.2 holds with this θ , then*

(i) *For any $\lambda \geq \lambda_0$ and any $u \in \dot{W}_p^2(\mathbb{R}_+^d)$ satisfying (2.11), we have*

$$\lambda \|u\|_{L_p(\mathbb{R}_+^d)} + \|D^2 u\|_{L_p(\mathbb{R}_+^d)} \leq N \|f\|_{L_p(\mathbb{R}_+^d)}, \quad (2.13)$$

where $N = N(d, p, \delta)$;

(ii) *For any $\lambda > 0$, there exists a constant $N = N(d, p, \delta, R_0, \lambda)$ such that if $u, v \in \dot{W}_p^2(\mathbb{R}_+^d)$ and $u - v \in \dot{W}_p^2(\mathbb{R}_+^d)$, then*

$$\|u\|_{W_p^2(\mathbb{R}_+^d)} \leq N \|F(D^2 u, \cdot) - \lambda u\|_{L_p(\mathbb{R}_+^d)} + N \|v\|_{W_p^2(\mathbb{R}_+^d)}, \quad (2.14)$$

Proof. We suppose that Assumption 1.0.2 holds with θ from Corollary 2.2.3.

(i) Take a nonnegative $\zeta \in C_0^\infty$ which has support in $B_{R_0}^+$ and is such that ζ^p integrates to one. For the parameter $z \in \mathbb{R}_+^d$ define

$$u_z(x) = u(x)\zeta(z - x)$$

and observe that for any $x \in \mathbb{R}_+^d$ we have

$$\int_{\mathbb{R}_+^d} \zeta^p(z - x) dz = 1. \quad (2.15)$$

Then notice that, by the homogeneity of F , for any $z \in \mathbb{R}_+^d$

$$F(D^2 u_z(x), x) = f_z(x) + \lambda u_z(x),$$

where

$$f_z(x) = f(x)\zeta(z-x) + F(D^2u_z(x), x) - F(\zeta(z-x)D^2u, x)$$

By Corollary 2.2.3 and the Lipschitz continuity of F in u'' ,

$$\begin{aligned} & \|\zeta(z-\cdot)|D^2u\|_{L_p(\mathbb{R}_+^d)}^p \leq N\|\zeta(z-\cdot)f\|_{L_p(\mathbb{R}_+^d)}^p \\ & + N\|D\zeta(z-\cdot)\|_{L_p(\mathbb{R}_+^d)}\|Du\|_{L_p(\mathbb{R}_+^d)}^p + N(\|D^2\zeta(z-\cdot)\|_{L_p(\mathbb{R}_+^d)} + \lambda\zeta(z-\cdot))\|u\|_{L_p(\mathbb{R}_+^d)}^p. \end{aligned}$$

Upon integrating through this estimate with respect to $z \in \mathbb{R}_+^d$ and using (2.15) we get

$$\begin{aligned} \|D^2u\|_{L_p(\mathbb{R}_+^d)}^p & \leq N_1(\|f\|_{L_p(\mathbb{R}_+^d)}^p + \lambda^p\|u\|_{L_p(\mathbb{R}_+^d)}^p) \\ & + N_2(\|Du\|_{L_p(\mathbb{R}_+^d)}^p + \|u\|_{L_p(\mathbb{R}_+^d)}^p), \end{aligned}$$

where $N_1 = N_1(d, \delta, p)$ and $N_2 = N_2(d, \delta, p, R_0)$. Furthermore, as in the proof of Theorem 2.1.7, by analyzing the proof of Lemma 3.5.5 of [12], we have for any $\lambda > 0$

$$\lambda\|u\|_{L_p(\mathbb{R}_+^d)} \leq N\|f\|_{L_p(\mathbb{R}_+^d)},$$

where N depends only on d, p , and δ . Hence

$$\lambda^p\|u\|_{L_p(\mathbb{R}_+^d)}^p + \|D^2u\|_{L_p(\mathbb{R}_+^d)}^p \leq N_1\|f\|_{L_p(\mathbb{R}_+^d)}^p + N_2(\|Du\|_{L_p(\mathbb{R}_+^d)}^p + \|u\|_{L_p(\mathbb{R}_+^d)}^p),$$

and to finish proving (2.13) with $N = 2N_1$ it only remains to use the multiplicative inequalities and choose $\lambda_0(d, \delta, p, R_0)$ sufficiently large.

(ii) Set $w = u - v$ and $f = F(D^2u, x) - \lambda u$. Observe that

$$F(D^2w, x) - \lambda w = f + [F(D^2w, x) - F(D^2w + D^2v, x)] + \lambda v$$

and $|F(D^2w, x) - F(D^2w + D^2v, x)| \leq N|D^2v|$. Then we see that (2.14) follows from the proof of Assertion (i). The theorem is proved. \square

The following solvability theorem is a standard result, which however will not be used later in the paper. The main emphasis here is on the method of proof.

Theorem 2.2.5. *Let $p > d$. Then there exist constants $\theta \in (0, 1]$ depending only on d, p, δ and λ_0 depending only on d, p, δ , and R_0 such that if Assumption 1.0.2 holds with this θ , then for any $v \in W_p^2(\mathbb{R}_+^d)$, $\lambda > 0$, and $f \in L_p(\mathbb{R}_+^d)$, there exists a unique $u \in W_p^2(\mathbb{R}_+^d)$ satisfying (2.11) and such that $u - v \in \mathring{W}_p^2(\mathbb{R}_+^d)$.*

Proof. We take θ from Theorem 2.2.4 and first we assume that $F(u'', x)$ is infinitely differentiable with respect to $x \in \mathbb{R}^d$ and each of its derivatives is continuous in (u'', x) and Lipschitz continuous in u'' uniformly with respect to x (in particular, if in addition F_{x^k} are differentiable with respect to u'' for $u'' \neq 0$, then $F_{x^k u_{ij}}$ are bounded for $u'' \neq 0$). By mollifying $F(u'', x)$ with respect to u'' and using its properties listed above we obtain a sequence $F^n(u'', x)$ of functions infinitely differentiable in (u'', x) , converging to F as $n \rightarrow \infty$, convex in u'' and such that, for all n and all values of the arguments and $\xi \in \mathbb{R}^d$ and $v'' \in \mathcal{S}$, we have

$$\begin{aligned} \delta|\xi|^2 &\leq F_{u_{ij}}^n \xi^i \xi^j \leq \delta^{-1}|\xi|^2, \quad |F^n - F_{u_{ij}}^n u_{ij}| \leq 1, \\ -F_{u_{ij} x^k}^n v_{ij}'' \xi^k - F_{x^k x^r}^n \xi^k \xi^r &\leq N(|v''| + |\xi|)|\xi|, \end{aligned}$$

where N is independent of n . It follows that the function $-F(-u'', x) - \lambda u$ is of class $\bar{\mathcal{F}}$ introduced in [12, §6.1]. We also take $f \in C_0^\infty(\mathbb{R}_+^d)$ and $v \in C^\infty(\mathbb{R}^d)$ with compact support. Then by classical results (see, for instance, [12] or [26]) equation (2.11) with boundary condition $u = v$ on $\partial\mathbb{R}_+^d$ has a unique solution u , which is twice continuously differentiable in $\bar{\mathbb{R}}_+^d$. As in the proof of Theorem 2.1.7 we have that $u \in \mathring{W}_p^2(\mathbb{R}_+^d)$ and estimate (2.14) holds (with $F(D^2 u, x) - \lambda u = f$).

Next, we consider the general situation. Take the function ζ from the proof of assertion (i) of Theorem 2.2.4 but such that (2.15) holds with $p = 1$. For $n = 1, 2, \dots$ define

$$F_n(u'', x) = \int_{\mathbb{R}_+^d} F(u'', x + y/n) \zeta(y) dy.$$

Obviously, these infinitely differentiable functions of x are positive homogeneous of degree one and satisfy (H₁) and Assumption 1.0.2 with the same parameters as F does.

Then we approximate f and v in appropriate norms with functions f_n and v_n possessing the properties described above. This yields a sequence of $u_n \in W_p^2(\mathbb{R}_+^d)$ with uniformly bounded $W_p^2(\mathbb{R}_+^d)$ norms and such that $u_n - v_n \in \mathring{W}_p^2(\mathbb{R}_+^d)$ and

$$F_n(D^2 u_n, x) - \lambda u_n(x) = f_n(x).$$

By embedding theorems there is a $u \in W_p^2(\mathbb{R}_+^d)$ and a subsequence, which is still denoted by $\{u_n\}$, such that $u_n \rightarrow u$ uniformly on compact subsets of $\bar{\mathbb{R}}_+^d$. In particular $u = v$ on $\partial\mathbb{R}_+^d$, so that $u - v \in \mathring{W}_p^2(\mathbb{R}_+^d)$.

Since $f_n \rightarrow f$ in $L_p(\mathbb{R}_+^d)$, we may assume that $f_n \rightarrow f$ (a.e.). Therefore, if we define

$$\bar{F}_{n_0}(u'', x) = \sup_{n \geq n_0} F_n(u'', x), \quad \underline{F}_{n_0}(u'', x) = \inf_{n \geq n_0} F_n(u'', x),$$

then, for any n_0 , (a.e.)

$$\begin{aligned} \liminf_{n \rightarrow \infty} \bar{F}_{n_0}(D^2 u_n, x) &\geq \liminf_{n \rightarrow \infty} F_n(D^2 u_n, x) = f(x) + \lambda u(x), \\ \limsup_{n \rightarrow \infty} \underline{F}_{n_0}(D^2 u_n, x) &\leq \limsup_{n \rightarrow \infty} F_n(D^2 u_n, x) = f(x) + \lambda u(x). \end{aligned}$$

It follows by Theorems 3.5.15 and 3.5.6 of [12] that for any n_0 (a.e.)

$$\bar{F}_{n_0}(D^2 u, x) \geq f(x) + \lambda u(x) \geq \underline{F}_{n_0}(D^2 u, x). \quad (2.16)$$

Now observe that, for each $u'' \in \mathcal{S}$, $F_n(u'', x) \rightarrow F(u'', x)$ (a.e.). Since both parts are positive homogeneous and Lipschitz continuous in u'' with constant independent of n we also have

$$\bar{F}_{n_0}(u'', x) - \underline{F}_{n_0}(u'', x) \leq \varepsilon_{n_0}(x)|u''|,$$

where $\varepsilon_{n_0}(x) \rightarrow 0$ (a.e.) as $n_0 \rightarrow \infty$. After that to finish proving the existence it only remains to pass to the limit in (2.16).

Uniqueness is proved in the same way as in Theorem 2.1.7. The theorem is proved. \square

2.3 Proof of Theorem 1.0.2

First we state a generalization of a result of [16]. The point is that in that paper the counterpart of our Assumption 1.0.2 is formulated as (1.8) (in its elliptic version).

Theorem 2.3.1. *Let $p > d$. Then there exist constants $\theta \in (0, 1]$ depending only on d, p, δ and λ_0 depending only on d, p, δ , and R_0 such that if Assumption 1.0.2 holds with this θ and $\mathcal{D} = \mathbb{R}^d$, then*

(i) *For any $u \in W_p^2(\mathbb{R}^d)$ satisfying (2.11), we have*

$$\lambda \|u\|_{L_p(\mathbb{R}^d)} + \|D^2 u\|_{L_p(\mathbb{R}^d)} \leq N \|f\|_{L_p(\mathbb{R}^d)}, \quad (2.17)$$

if $\lambda \geq \lambda_0$, where $N = N(d, p, \delta)$, and we have

$$\|u\|_{W_p^2(\mathbb{R}_+^d)} \leq N \|f\|_{W_p^2(\mathbb{R}_+^d)} \quad (2.18)$$

if $\lambda > 0$ with $N = N(d, p, \delta, R_0, \lambda)$.

(iii) For any $\lambda > 0$ and $f \in L_p(\mathbb{R}^d)$, there exists a unique $u \in W_p^2(\mathbb{R}^d)$ satisfying (2.11).

We only give a few comments on the proof of this theorem. In case F is independent of x the a priori estimate (2.17) is obtained in [16] for all $\lambda > 0$. When F depends on x one obtains the estimate (2.17) for $\lambda \geq \lambda_0$ and (2.18) for $\lambda > 0$ as in the proof of Theorem 2.2.4. After the necessary a priori estimates are obtained, it is stated in [16] that the solvability theorems are derived in a standard way without giving any specific details. This standard way is presented in the proof of Theorem 2.2.5.

Proof of Theorem 1.0.2. As in the proof of Theorem 2.2.4, we first establish (1.6) as an a priori estimate and the case of general g is reduced to the case $g \equiv 0$ by replacing the unknown function u with $u - g$. We will see that to obtain the a priori estimate we do not need condition (H₃).

Observe that Theorems 2.2.4 and 2.3.1 with $\lambda = \lambda_0$ imply that

$$\|D^2u\|_{L_p(\mathbb{R}_+^d)} \leq N(\|F(D^2u)\|_{L_p(\mathbb{R}_+^d)} + \|u\|_{L_p(\mathbb{R}_+^d)}), \quad \forall u \in \dot{W}_p^2(\mathbb{R}_+^d),$$

$$\|D^2v\|_{L_p(\mathbb{R}^d)} \leq N(\|F(D^2v)\|_{L_p(\mathbb{R}^d)} + \|v\|_{L_p(\mathbb{R}^d)}), \quad \forall v \in W_p^2(\mathbb{R}^d), \quad (2.19)$$

where $N = N(d, p, \delta, R_0)$ (provided that $\theta = \theta(d, p, \delta)$ is chosen appropriately).

Now assume that $u \in \dot{W}_p^2(\mathcal{D})$ satisfies

$$F(D^2u(x), x) + G(D^2u(x), Du(x), u(x), x) = 0. \quad (2.20)$$

We move the term G to the right-hand side and define

$$f(x) := -G(D^2u(x), Du(x), u(x), x).$$

After that by using the technique based on flattening the boundary, partitions of unity, and interpolation inequalities allowing one to estimate Du through D^2u and u and also using (2.19) we obtain that

$$\|D^2u\|_{L_p(\mathcal{D})} \leq N_1(\|f\|_{L_p(\mathcal{D})} + \|u\|_{L_p(\mathcal{D})}), \quad (2.21)$$

provided that θ is sufficiently small depending only on d, p, δ , and the $C^{1,1}$ norm of ∂D . Here N_1 depends only on d, p, δ, R_0 , and the $C^{1,1}$ norm of $\partial \mathcal{D}$. Below by N_i we denote the same type of constants as N_1 . It follows from the definition of f and (H₂) that, for any $s > 0$,

$$\begin{aligned} \|f\|_{L_p(\mathcal{D})} &\leq \|\chi(|D^2u|)D^2u\|_{L_p(\mathcal{D})} + K(\|Du\|_{L_p(\mathcal{D})} + \|u\|_{L_p(\mathcal{D})}) \\ &\quad + \|\bar{G}\|_{L_p(\mathcal{D})} \leq \chi(s)\|D^2u\|_{L_p(\mathcal{D})} + \|\chi\|_{L_\infty} s |\mathcal{D}|^{1/p} \\ &\quad + K(\|Du\|_{L_p(\mathcal{D})} + \|u\|_{L_p(\mathcal{D})}) + \|\bar{G}\|_{L_p(\mathcal{D})}. \end{aligned} \quad (2.22)$$

Upon taking s large so that $N_1\chi(s) \leq 1/2$, we get from (2.21), (2.22), and the interpolation inequality that

$$\|u\|_{W_p^2(\mathcal{D})} \leq N_2(\|u\|_{L_p(\mathcal{D})} + \|\bar{G}\|_{L_p(\mathcal{D})} + \|\chi\|_{L_\infty} s |\mathcal{D}|^{1/p}). \quad (2.23)$$

To estimate the term $\|u\|_{L_p(\mathcal{D})}$ on the right-hand side of (2.23), we rewrite (2.20) as

$$\begin{aligned} &F(D^2u(x), x) + G(D^2u(x), Du(x), u(x), x) \\ &\quad - G(D^2u(x), 0, 0, x) = -G(D^2u(x), 0, 0, x). \end{aligned} \quad (2.24)$$

Note that, by conditions (H₁) and (H₂), there exist $L \in \mathbb{L}_\delta$ and bounded measurable functions $b = (b^1, \dots, b^d)$ and c such that the left-hand side of (2.24) can be represented as $Lu + b^i D_i u - cu$. Since G is nonincreasing in u , we have $c \geq 0$. Therefore, by Alexandrov's estimate

$$\sup_{\mathcal{D}} |u|, \quad \|u\|_{L_p(\mathcal{D})} \leq N \|G(D^2u(\cdot), 0, 0, \cdot)\|_{L_p(\mathcal{D})},$$

where $N = N(d, p, \delta, \text{diam}(\mathcal{D}))$. Again by condition (H₂), for any $s_1 > 0$,

$$\begin{aligned} \|u\|_{L_p(\mathcal{D})} &\leq N \|\chi(|D^2u|)D^2u\|_{L_p(\mathcal{D})} + N \|\bar{G}\|_{L_p(\mathcal{D})} \\ &\leq N_3(\chi(s_1)\|D^2u\|_{L_p(\mathcal{D})} + \|\chi\|_{L_\infty} s_1 |\mathcal{D}|^{1/p} + \|\bar{G}\|_{L_p(\mathcal{D})}). \end{aligned} \quad (2.25)$$

Combining (2.23) with (2.25) and taking s_1 sufficiently large so that $N_2 N_3 \chi(s_1) \leq 1/2$, we arrive at

$$\|u\|_{W_p^2(\mathcal{D})} \leq N_4 \|\tilde{G}\|_{L_p(\mathcal{D})} + N_4(s + s_1) \|\chi\|_{L_\infty} |\mathcal{D}|^{1/p},$$

which is (1.6) in the case that $g = 0$.

To prove the existence and uniqueness of solutions, we first assume that $H := F + G$ and g are smooth in x and the domain is of class $C^{2+\alpha}$. Then, under conditions (H₂) and (H₃), it is known (cf. the proof of Theorem 2.2.5) that there is a unique $C^2(\mathcal{D})$ solution u with a given smooth boundary data. This solution is certainly in $W_p^2(\mathcal{D})$ and we have an estimate of its $W_p^2(\mathcal{D})$ norm. After that we mollify the original F and G in x , mollify g , and approximate \mathcal{D} by a sequence of increasing smooth domains \mathcal{D}_n with the $C^{1,1}$ norm under control. We take these domains because otherwise after mollifications F may fail to satisfy (1.5) for all $x \in \mathcal{D}$. After that it suffices to repeat the last part of the proof of Theorem 2.2.5. To see that the limiting function u satisfies $u - g \in \dot{W}_p^2(\mathcal{D})$, we use the fact that $u_n, g_n \in W_p^2(\mathcal{D}_n)$ with uniformly bounded norms and the fact that $(u_n - g_n)I_{\mathcal{D}_n} \in \dot{W}_p^1(\mathcal{D})$ with uniformly bounded norms. Of course, while passing to the limits and proving uniqueness we use that for any $u, v \in W_p^2(\mathcal{D})$ there is an operator $L \in \mathbb{L}_\delta$ and bounded measurable functions $b = (b^1, \dots, b^d)$ and c satisfying $|b| \leq K$, $0 \leq c \leq K$ such that

$$\begin{aligned} & H(D^2u, Du, u, x) - H(D^2v, Dv, v, x) \\ &= H(D^2u, Du, u, x) - H(D^2v, Du, u, x) \\ &+ H(D^2v, Du, u, x) - H(D^2v, Dv, v, x) \\ &= L(u - v) + b^i D_i(u - v) - c(u - v). \end{aligned}$$

The theorem is proved. □

Chapter 3

Solvability of Parabolic Bellman's Equation

3.1 Parabolic Bellman's equations in \mathbb{R}^{d+1} with constant coefficients

In this section we consider the equation

$$\partial_t u(t, x) + F(D^2 u) = f(t, x), \quad (3.1)$$

in the whole space. Due to the same reasons as in Section 2.1, equation (3.1) can be written as a parabolic Bellman's equation

$$\partial_t u(t, x) + \sup_{\omega \in \Omega} [a^{ij}(\omega) D_{ij} u(t, x)] = f(t, x).$$

For $r > 0$, introduce $Q_r := Q_r(0, 0)$. The following is Lemma 4.2.2 in [15].

Lemma 3.1.1. *Let $p \in [1, \infty)$. Then there is a constant $N = N(d, p)$ such that for any $r \in (0, \infty)$ and $u \in C_{loc}^\infty(\mathbb{R}^{d+1})$ we have*

$$\begin{aligned} \int_{Q_r} |Du - (Du)_{Q_r}|^p dx dt &\leq Nr^p \int_{Q_r} (|D^2 u| + |\partial_t u|)^p dx dt, \\ \int_{Q_r} |u(t, x) - (u)_{Q_r} - x^i (D_i u)_{Q_r}|^p dx dt \\ &\leq Nr^{2p} \int_{Q_r} (|D^2 u| + |\partial_t u|)^p dx dt. \end{aligned}$$

The second lemma is a parabolic embedding theorem proved as Lemma II.3.3 in [19].

Lemma 3.1.2. *Let $p > (d + 2)/2$ and $u \in W_p^{1,2}(Q_1)$. Then for any $(t, x) \in Q_1$,*

$$|u(t, x)| \leq N \|u\|_{W_p^{1,2}(Q_1)},$$

where $N = N(d, p)$.

Let $v = u(t, x) - (u)_{Q_r} - x^i (D_i u)_{Q_r}$, then v belongs to $W_p^{1,2}(Q_r)$ whenever u does. Noting that

$$D_{ij}v = D_{ij}u, \quad \partial_t v = \partial_t u, \quad D_i v = D_i u - (D_i u)_{Q_r},$$

we get the following corollary by dilations and combining Lemmas 3.1.1 and 3.1.2.

Corollary 3.1.3. *Let $p > (d + 2)/2$ and $r \in (0, \infty)$. Then for any $u \in W_p^{1,2}(Q_r)$, we have*

$$\begin{aligned} & \sup_{(t,x) \in Q_r} |u(t, x) - (u)_{Q_r} - x^i (D_i u)_{Q_r}|^p \\ & \leq N r^{2p} \int_{Q_r} (|D^2 u| + |\partial_t u|)^p dx dt, \end{aligned}$$

where $N = N(d, p)$.

Lemma 3.1.4. *Let $r \in (0, \infty)$ and $\kappa \geq 2$. Let $v \in C_b^{1,2}(Q_{\kappa r})$ be a solution of (3.1) with $f \equiv 0$. Then there are constants $\alpha \in (0, 1)$ and N depending only on d and δ such that*

$$\int_{Q_r} \int_{Q_r} |D^2 v(t, x) - D^2 v(s, y)| dx dt dy ds \leq N \kappa^{-2-\alpha} r^{-2} \sup_{\partial' Q_{\kappa r}} |v|.$$

Proof. Dilations show that we may concentrate on the case when $r = 1/\kappa$. In this case one routinely derives from Theorem 5.5.2 in [12] that there exist α, N depending only on δ, d such that for any $(t, x), (s, y) \in Q_{1/2}$, we have

$$|D^2 v(t, x) - D^2 v(s, y)| \leq N (|x - y|^\alpha + |t - s|^{\alpha/2}) \sup_{Q_1} |v| \quad (3.2)$$

Thanks to the maximum principle, the lemma is proved. \square

Remark 3.1.1. By “routinely derives” we mean the following. First observe that we may assume that

$$\sup_{Q_1} |v| = 1.$$

Indeed, if the sup is zero, we have nothing to prove. However if the sup is different from zero we can replace v with the ratio of v and the sup.

Then we approximate $F(u'')$ by smooth convex functions $F^n(u'')$ so that $F^n \rightarrow F$ as $n \rightarrow \infty$ uniformly on compact sets and for all values of variables

$$\delta|\xi|^2 \leq F_{u_{ij}}^n \xi^i \xi^j \leq \delta^{-1}|\xi|^2, \quad |F^n - F_{u_{ij}}^n u_{ij}| \leq 1.$$

To do that it suffices to mollify $F(u'')$ with respect to u'' . Then we approximate v on $\partial'Q_1$ uniformly by infinitely differentiable functions ϕ^n such that $|\phi^n| \leq 1$. Next, we apply Theorem 6.2.5 of [12] to find a unique $u^n \in C^{1,2}(Q_1) \cap C(\bar{Q}_1)$ such that

$$\partial_t u^n + F^n(D^2 u^n) - \frac{1}{n} u^n = 0 \quad \text{in } Q_1$$

and $u^n = \phi^n$ on $\partial'Q_1$.

This theorem also guarantees that

$$u^n, Du^n, D^2 u^n, \partial_t u^n \in C^{1,2}([\varepsilon, 1 - \varepsilon] \times \bar{B}_\varepsilon)$$

for any $\varepsilon \in (0, 1/2)$. By the maximum principle u^n are uniformly bounded in Q_1 . Since

$$\partial_t v + F^n(D^2 v) - \frac{1}{n} v = F^n(D^2 v) - F(D^2 v) - \frac{1}{n} v$$

and the latter tends to zero uniformly in Q_1 , by the maximum principle $u^n \rightarrow v$ uniformly in Q_1 .

Now we can formally apply Theorem 5.5.2 of [12] and get that the $C^{1+\alpha/2, 2+\alpha}(Q_{1/2})$ norms of u^n are uniformly bounded and, in particular, for any $(t, x), (s, y) \in Q_{1/2}$, we have

$$|D^2 u^n(t, x) - D^2 u^n(s, y)| \leq N(|x - y|^\alpha + |t - s|^{\alpha/2}),$$

where N depends only on d and δ . Since $u^n \rightarrow v$ uniformly and $D^2 u^n$ are uniformly equicontinuous in $Q_{1/2}$, we have that $D^2 u^n \rightarrow D^2 v$ in $Q_{1/2}$, which yields

$$|D^2 v(t, x) - D^2 v(s, y)| \leq N(|x - y|^\alpha + |t - s|^{\alpha/2})$$

and this coincides with (3.2).

Introduce \mathbb{L}_δ as before Lemma 2.1.4 but allow the dependence of the coefficients on (t, x) rather than on x only.

Lemma 3.1.5. *Let $r \in (0, \infty)$ and let $u \in C(\bar{Q}_r) \cap W_{d+1}^{1,2}(Q_\rho)$ for any $\rho \in (0, r)$. Then there are constants $\gamma \in (0, 1]$ and N , depending only on δ, d , such that for any $L \in \mathbb{L}_\delta$ we have*

$$\begin{aligned} \int_{Q_r} |D^2 u|^\gamma dx dt &\leq N r^{-2\gamma} \sup_{\partial' Q_r} |u|^\gamma \\ &+ N \left(\int_{Q_r} |\partial_t u + Lu|^{d+1} dx dt \right)^{\gamma/(d+1)}. \end{aligned} \quad (3.3)$$

Proof. If we prove (3.3) with ρ in place of r for any $\rho \in (0, r)$, then by passing to the limit we will obtain (3.3) as is. Hence, we may assume that $u \in W_{d+1}^{1,2}(Q_r)$. As usual, we may also assume that $r = 1$. Then we may also assume that the coefficients $a^{ij}(t, x)$ of L are infinitely differentiable in \mathbb{R}^{d+1} . Now set $f = \partial_t u + Lu$ in Q_1 and extend $f(t, x)$ for $t \leq 0$ as zero. Also set $u(t, x) = u(-t, x)$ for $t \leq 0$. Observe that the new u belongs to $W_{d+1}^{1,2}((-1, 1) \times B_1)$. After that define $v(t, x)$ as a unique $W_{d+1, \text{loc}}^{1,2}((-1, 1) \times B_1) \cap C([-1, 1] \times \bar{B}_1)$ solution of $\partial_t v + Lv = f$ with terminal and lateral conditions being u . The existence and uniqueness of such a solution is a classical result (see, for instance, Theorem IV.9.1 of [19] or Theorem 7.17 of [20]). By uniqueness $v = u$ in Q_1 , so that owing to Corollary 4.2 of [16],

$$\begin{aligned} \int_{Q_1} |D^2 u|^\gamma dx dt &= \int_{Q_1} |D^2 v|^\gamma dx dt \leq N \left(\int_{(-1, 1) \times B_1} |f|^{d+1} dx dt \right)^{\gamma/(d+1)} \\ &+ N \sup_{\partial'(-1, 1) \times B_1} |v|^\gamma = N \left(\int_{Q_1} |f|^{d+1} dx dt \right)^{\gamma/(d+1)} + N \sup_{\partial' Q_1} |u|^\gamma. \end{aligned}$$

The lemma is proved. \square

We note that a slightly weaker statement than Lemma 3.1.5 can be found in [27], where for the proof the reader is referred to [28].

Everywhere below in this section α is the constant from Lemma 3.1.4 and γ is the one from Lemma 3.1.5.

Lemma 3.1.6. *Let $r \in (0, \infty)$ and $\kappa \geq 2$. Let $u \in W_{d+1}^{1,2}(Q_{\kappa r})$ be a solution to (3.1).*

Then

$$\begin{aligned} & \int_{Q_r} \int_{Q_r} |D^2 u(t, x) - D^2 u(s, y)|^\gamma dx dt dy ds \\ & \leq N \kappa^{d+2} (|f|^{d+1})_{Q_{\kappa r}}^{\gamma/(d+1)} + N \kappa^{-\alpha\gamma} (|D^2 u|^{d+1})_{Q_{\kappa r}}^{\gamma/(d+1)}, \end{aligned}$$

where N depends only on d and δ .

Proof. As usual, it suffices to prove the lemma for $r = 1$. We follow the proof of Lemma 2.4 in [16] and, as there, without trouble reduce the general case to the one that $u \in C_b^\infty(\bar{Q}_\kappa)$. Define $\hat{u} := u - (u)_{Q_\kappa} - x^i (D_i u)_{Q_\kappa}$ and let $v \in C_b^{1,2}(Q_\kappa) \cap C(\bar{Q}_\kappa)$ be a solution of (3.1) in Q_κ with $f \equiv 0$ and $v = \hat{u}$ on $\partial' Q_\kappa$. Such a solution v exists by Theorem 6.4.1 of [12]. By Lemma 3.1.4, Hölder's inequality, and Corollary 3.1.3, we have

$$\begin{aligned} & \int_{Q_1} \int_{Q_1} |D^2 v(t, x) - D^2 v(s, y)|^\gamma dx dt dy ds \\ & \leq N \kappa^{-\gamma(2+\alpha)} \sup_{\partial' Q_\kappa} |v|^\gamma \leq N \kappa^{-\alpha\gamma} (|D^2 u|^{d+1} + |\partial_t u|^{d+1})_{Q_\kappa}^{\gamma/(d+1)}. \end{aligned} \quad (3.4)$$

Let $w := \hat{u} - v$ in \bar{Q}_κ . Then by the same argument as in the proof of Lemma 2.4 in [16] or our Lemma 2.1.5, we obtain that there exists an operator $L \in \mathbb{L}_\delta$, such that $\partial_t w + Lw = f$. Then by Lemma 3.1.5,

$$\begin{aligned} & \int_{Q_1} |D^2 w|^\gamma dx dt \leq N \kappa^{d+2} \int_{Q_\kappa} |D^2 w|^\gamma dx dt \\ & \leq N \kappa^{d+2} \left(\int_{Q_\kappa} |f|^{d+1} dx dt \right)^{\gamma/(d+1)} \end{aligned}$$

and

$$\int_{Q_1} \int_{Q_1} |D^2 w(t, x) - D^2 w(s, y)|^\gamma \leq N \kappa^{d+2} \left(\int_{Q_\kappa} |f|^{d+1} dx dt \right)^{\gamma/(d+1)}.$$

By combining this inequality and (3.4) and observing that $D^2 u = D^2 v + D^2 w$ and

$$|\partial_t u| = |f - F(D^2 u)| \leq |f| + N |D^2 u|, \quad (3.5)$$

we get the desired result. The lemma is proved. \square

The next theorem is the main result of this section. For simplicity of notation set

$$L_p = L_p(\mathbb{R}^{d+1}), \quad W_p^{1,2} = W_p^{1,2}(\mathbb{R}^{d+1}). \quad (3.6)$$

Theorem 3.1.7. *Let $p > d + 1$. (i) Let $u \in W_p^{1,2}$ be a solution to (3.1). Then*

$$\|D^2u\|_{L_p} + \|\partial_t u\|_{L_p} \leq N\|f\|_{L_p}, \quad (3.7)$$

where N depends only on p , d , and δ .

(ii) For any $\lambda > 0$ and $f \in L_p$, there exists a unique solution $u \in W_p^{1,2}$ of the equation

$$\partial_t u + F(D^2u) - \lambda u = f. \quad (3.8)$$

Furthermore,

$$\lambda\|u\|_{L_p} + \|D^2u\|_{L_p} + \|\partial_t u\|_{L_p} \leq N\|f\|_{L_p}, \quad (3.9)$$

where N depends only on p , d , δ , and λ .

Proof. (i) The estimate of the D^2u term on the left-hand side of (3.7) is derived from Theorem 5.3 of [16] and Lemma 3.1.6 in the same way as Theorem 2.5 (i) of [16] or Theorem 2.1.7 (i). Of course this time we use the filtration of parabolic dyadic cubes. The estimate of $\partial_t u$ follows from that of D^2u and (3.5).

(ii) To prove the a priori estimate (3.9) we replace f with $-\lambda u + f$ in the above estimates and get

$$\|\partial_t u\|_{L_p} + \|D^2u\|_{L_p} \leq \lambda\|u\|_{L_p} + \|f\|_{L_p}.$$

Hence it suffices to prove that

$$\lambda\|u\|_{L_p} \leq N\|f\|_{L_p},$$

which is done in the same way as in the elliptic case. After that, the solvability of (3.8) is proved in the same way as in Theorem 2.1.7. The theorem is proved. \square

3.2 Parabolic equations in \mathbb{R}^{d+1} with VMO coefficients

In this section, we consider the parabolic equation

$$\partial_t u(t, x) + F(D^2u(t, x), t, x) - \lambda u(t, x) = f(t, x). \quad (3.10)$$

Everywhere below in this section, Assumption 1.0.1 is supposed to hold with $\mathcal{D} = \mathbb{R}^{d+1}$, α is the constant from Lemma 3.1.4, and γ is the one from Lemma 3.1.5. We use notation (3.6) and recall that $\theta(\mu, d, \delta)$ is introduced in Remark 2.2.1.

Lemma 3.2.1. *Let $\beta \in (1, \infty)$, $\lambda = 0$, $\mu, r \in (0, \infty)$, $\kappa \geq 2$, and $(t_0, x_0) \in \mathbb{R}^{d+1}$. Suppose that $\theta = \theta(\mu, d, \delta)$. Let $u \in W_{d+1}^{1,2}$ be a solution of (3.10) vanishing outside $Q_{R_0}(t_0, x_0)$. Then,*

$$\begin{aligned} \int_{Q_r} \int_{Q_r} |D^2 u(t, x) - D^2 u(s, y)|^\gamma dx dt dy ds &\leq N \kappa^{d+2} \left(|f|^d \right)_{Q_{\kappa r}}^{\gamma/(d+1)} \\ &\quad + N \kappa^{d+2} \left(|D^2 u|^{\beta(d+1)} \right)_{Q_{\kappa r}}^{\gamma/(\beta d + \beta)} \mu^{\gamma/(\beta' d + \beta')} \\ &\quad + N \kappa^{-\alpha \gamma} \left(|D^2 u|^{d+1} \right)_{Q_{\kappa r}}^{\gamma/(d+1)}, \end{aligned} \quad (3.11)$$

where $N = N(d, \delta, \beta)$ and $\beta' = \beta/(\beta - 1)$.

Proof. We will basically repeat the proof of Lemma 2.2.1 adapting it to the parabolic case and the whole space. Introduce

$$\bar{F}(u'') = \begin{cases} (F)_{Q_{R_0}(t_0, x_0)}(u''), & \text{if } \kappa r \geq R_0; \\ (F)_{Q_{\kappa r}}(u''), & \text{otherwise,} \end{cases}$$

and

$$h(t, x) = \sup_{u'' \in \mathcal{S}: |u''|=1} |F(u'', t, x) - \bar{F}(u'')|.$$

Note that

$$\partial_t u + \bar{F}(D^2 u) = \tilde{f},$$

where

$$\tilde{f}(t, x) = f(t, x) + \bar{F}(D^2 u) - F(D^2 u, t, x).$$

By Lemma 3.1.6 and the triangle inequality,

$$\begin{aligned} \int_{Q_r} \int_{Q_r} |D^2 u(t, x) - D^2 u(s, y)|^\gamma dx dt dy ds \\ \leq N \kappa^{d+2} \left((|f|^{d+1})_{Q_{\kappa r}}^{\gamma/(d+1)} + J^{\gamma/(d+1)} \right) \end{aligned}$$

$$+N\kappa^{-\alpha\gamma} \left(|D^2u|^{d+1} \right)_{Q_{\kappa r}}^{\gamma/(d+1)}, \quad (3.12)$$

where $N = N(d, \delta)$ and

$$J = \int_{Q_{\kappa r}} |\bar{F}(D^2u) - F(D^2u, t, x)|^{d+1} I_{Q_{R_0}}(t_0, x_0) dx dt \leq J_1^{1/\beta} J_2^{1/\beta'},$$

with

$$J_1 = \int_{Q_{\kappa r}} |D^2u|^{\beta(d+1)} dx dt,$$

$$J_2 = \int_{Q_{\kappa r}} h^{\beta'(d+1)} I_{Q_{R_0}}(t_0, x_0) dx dt \leq N \int_{Q_{\kappa r}} h I_{Q_{R_0}}(t_0, x_0) dx dt.$$

If $\kappa r < R_0$, we have

$$J_2 \leq N \int_{Q_{\kappa r}} h dx dt \leq N\theta.$$

If $\kappa r \geq R_0$, we have

$$J_2 \leq N(\kappa r)^{-d-2} \int_{Q_{R_0}(t_0, x_0)} h dx dt$$

$$\leq N(\kappa r)^{-d-2} R_0^{d+2} \int_{Q_{R_0}(t_0, x_0)} h dx dt \leq N\mu.$$

Therefore, in any case,

$$J \leq N \left(\int_{Q_{\kappa r}} |D^2u(x)|^{\beta(d+1)} dx dt \right)^{1/\beta} \mu^{1/\beta'}.$$

Substituting the above inequality back into (3.12), we get (3.11). The lemma is proved. \square

From Lemma 3.2.1, by a standard argument using Theorem 5.3 of [16] and the Hardy–Littlewood theorem, we arrive at the following corollary.

Corollary 3.2.2. *Let $p > d + 1$, and $u \in W_{d+1}^{1,2}$ be a solution of (3.10) with $\lambda = 0$ vanishing outside Q_{R_0} . Then there exist constants N and θ depending only on p , d , and δ , such that if Assumption 1.0.1 is satisfied with this θ , then*

$$\|D^2u\|_{L_p} + \|\partial_t u\|_{L_p} \leq N\|f\|_{L_p}.$$

For any $T \in [-\infty, \infty)$, we denote

$$\mathbb{R}_T^{d+1} = \{(t, x) \in \mathbb{R}^{d+1} : t > T\}.$$

The main result of this section is the following theorem.

Theorem 3.2.3. *Let $p > d + 1$ and $T \in [-\infty, \infty)$. Then there exist $\theta \in (0, 1]$, depending only on d, δ, p and a constant λ_0 , depending only on d, δ, p , and R_0 , such that if Assumption 1.0.1 is satisfied with this θ , then*

(i) *For any $\lambda \geq \lambda_0$ and any $u \in W_{d+1}^{1,2}(\mathbb{R}_T^{d+1})$ satisfying (3.10), we have*

$$\lambda \|u\|_{L_p(\mathbb{R}_T^{d+1})} + \|\partial_t u\|_{L_p(\mathbb{R}_T^{d+1})} + \|D^2 u\|_{L_p(\mathbb{R}_T^{d+1})} \leq N \|f\|_{L_p(\mathbb{R}_T^{d+1})}, \quad (3.13)$$

where $N = N(d, \delta, p)$

(ii) *For any $\lambda > 0$, there exists a constant $N = N(d, p, \delta, R_0, \lambda)$ such that for any $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ satisfying (3.10) we have*

$$\|u\|_{W_p^{1,2}(\mathbb{R}_T^{d+1})} \leq N \|f\|_{L_p(\mathbb{R}_T^{d+1})}. \quad (3.14)$$

(iii) *For any $\lambda > 0$ and $f \in L_p(\mathbb{R}_T^{d+1})$, there exists a unique solution of (3.10) in $W_p^{1,2}(\mathbb{R}_T^{d+1})$.*

Proof. First we assume $T = -\infty$. The proof of Theorem 2.2.5 shows that assertion (iii) follows from (i) and (ii). We suppose that Assumption 1.0.1 holds with θ from Corollary 3.2.2.

Take a nonnegative function $\zeta \in C^\infty$ which has support in $-Q_{R_0}$ and is such that ζ^p integrates to one. Fix $(s, y) \in \mathbb{R}^{d+1}$, and define

$$u_{(s,y)}(t, x) = u(t, x)\zeta(s - t, y - x),$$

Then $u_{(s,y)}(t, x)$ is supported in $Q_{R_0}(s, y)$, and

$$\partial_t u_{(s,y)} + F(u_{(s,y)}, t, x) = f_{(s,y)},$$

where

$$\begin{aligned} f_{(s,y)}(t, x) &= f(t, x)\zeta(s - t, y - x) + F(u_{(s,y)}, t, x) \\ &\quad - F(\zeta(s - t, y - x)D^2 u, t, x) - (\partial_t \zeta)(s - t, y - x)u + \lambda u_{(s,y)}. \end{aligned}$$

By Corollary 3.2.2 and condition (H₁),

$$\begin{aligned} & \|\zeta(s - \cdot, y - \cdot)\partial_t u\|_{L^p}^p + \|\zeta(s - \cdot, y - \cdot)D^2 u\|_{L^p}^p \\ & \leq N\|\zeta(s - \cdot, y - \cdot)f\|_{L^p}^p + N\|D\zeta(s - \cdot, y - \cdot)Du\|_{L^p}^p \\ & \quad + \|(|\partial_t \zeta| + |D^2 \zeta| + \lambda|\zeta|)(s - \cdot, y - \cdot)u\|_{L^p}^p. \end{aligned}$$

Integrating the above inequality over $(s, y) \in \mathbb{R}^{d+1}$ we get

$$\begin{aligned} \|\partial_t u\|_{L^p}^p + \|D^2 u\|_{L^p}^p & \leq N_1(\|f\|_{L^p}^p + \lambda^p \|u\|_{L^p}^p) \\ & \quad + N_2(\|Du\|_{L^p}^p + \|u\|_{L^p}^p), \end{aligned}$$

where $N_1 = N_1(d, \delta, p)$ and $N_2 = N_2(d, \delta, p, R_0)$. Now to conclude (3.13) and (3.14), it suffices to use again the proof of Lemma 3.5.5 of [12] as in Theorem 2.2.4. This completes the proof of the theorem in the special case when $T = -\infty$.

For $T > -\infty$, we extend f to be zero for $t \leq T$, and then find a unique solution $\tilde{u} \in W_p^{1,2}(\mathbb{R}^{d+1})$ of (3.10) in \mathbb{R}^{d+1} , the existence of which is guaranteed by the argument above. This in turn also yields the existence of a solution of (3.10) in \mathbb{R}_T^{d+1} satisfying (3.13) or (3.14) as appropriate. Its uniqueness in $W_p^{1,2}(\mathbb{R}_T^{d+1})$ follows as usual from the uniqueness for linear equations (with measurable coefficients) and parabolic Alexandrov's estimates. The theorem is proved. \square

We finish the section by proving the following result about the Cauchy problem. Denote by $\mathring{W}_p^{1,2}((0, T) \times \mathbb{R}^d)$ the set of functions of class $W_p^{1,2}((0, T) \times \mathbb{R}^d)$ having zero trace on the plane $\{(T, x) : x \in \mathbb{R}^d\}$.

Theorem 3.2.4. *Let $p > d + 1$ and $T > 0$. Then there exists $\theta \in (0, 1]$ depending only on d, δ, p , such that if Assumption 1.0.1 is satisfied with this θ , the following assertions hold:*

(i) *For any $v \in W_p^{1,2}((0, T) \times \mathbb{R}^d)$ and $f \in L_p((0, T) \times \mathbb{R}^d)$, there exists a unique solution $u \in W_p^{1,2}((0, T) \times \mathbb{R}^d)$ of (3.10) in $(0, T) \times \mathbb{R}^d$ with $\lambda = 0$ satisfying $u - v \in \mathring{W}_p^{1,2}((0, T) \times \mathbb{R}^d)$.*

(ii) *Moreover,*

$$\|u\|_{W_p^{1,2}((0, T) \times \mathbb{R}^d)} \leq N\|v\|_{W_p^{1,2}((0, T) \times \mathbb{R}^d)} + N\|f\|_{L_p((0, T) \times \mathbb{R}^d)}, \quad (3.15)$$

where $N = N(d, \delta, p, T, R_0)$.

Proof. As in the proof of Theorem 2.2.5, it suffices to prove (3.15) as an a priori estimate. By considering $u - v$ instead of the unknown function u , without loss of generality we may assume that $v \equiv 0$. Furthermore, having in mind the possibility of substitution $\hat{u} = e^t u$, we see that it suffices to consider equation (1.0.1) with $\lambda = 1$. We extend u to be zero for $t > T$. It is easily seen that the extended $u \in W_p^{1,2}(\mathbb{R}_0^{d+1})$ satisfies (3.10) in \mathbb{R}_0^{d+1} with $f(t, x) = 0$ for $t \geq T$. Estimate (3.15) then follows from Theorem 3.2.3 (ii). \square

3.3 Parabolic Bellman's equations in \mathbb{R}_+^{d+1} with constant coefficients

In this section, we consider equation (3.1) in the half space

$$\mathbb{R}_+^{d+1} := \mathbb{R} \times \mathbb{R}_+^d.$$

For $r > 0$, $t \in \mathbb{R}$ and $x = (x^1, x')$ in \mathbb{R}_+^d , denote

$$Q_r^+(t, x) = Q_r(t, x) \cap \mathbb{R}_+^{d+1}, \quad Q_r^+ = Q_r^+(0, 0), \quad Q_r^+(x^1) = Q_r^+(0, x^1, 0).$$

The following lemma can be deduced from Corollary 3.1.3 in the same way as Lemma 2.1.1 is proved.

Lemma 3.3.1. *Let $p > (d+2)/2$ and $r \in (0, \infty)$. Then for any $u \in W_p^{1,2}(Q_r^+)$ vanishing on $x^1 = 0$, we have*

$$\sup_{(t,x) \in Q_r^+} |u - x^1(D_1 u)_{Q_r^+}|^p \leq N r^{2p} \int_{Q_r^+} (|D^2 u| + |\partial_t u|)^p dx dt.$$

where N depends only on d and p .

Lemma 3.3.2. *Let $r \in (0, \infty)$, $\kappa \geq 2$, and $v \in C(\bar{Q}_{\kappa r}^+) \cap C_b^{1,2}(Q_{\kappa \rho}^+)$ for any $\rho \in (0, r)$. Assume that v is a solution of (3.1) with $f \equiv 0$ and $v = 0$ on $x^1 = 0$. Then there are constants $\alpha \in (0, 1)$ and N , depending only on d and δ , such that*

$$[D^2 v]_{C^\alpha(Q_r^+)} \leq N(\kappa r)^{-2-\alpha} \sup_{\partial' Q_{\kappa r}^+} |v|.$$

Proof. Dilations show that it suffices to prove the inequality for $\kappa r = 1$. We take a smooth domain \mathcal{D}_1 such that $B_{3/4}^+ \subset \mathcal{D}_1 \subset B_1^+$. As in Lemma 3.1.4, it then follows from Theorem 5.5.2 in [12] that

$$[D^2v]_{C^\alpha(Q_{1/2}^+)} \leq N \sup_{(0,3/4) \times \mathcal{D}_1} |v|.$$

Owing to the maximum principle, the lemma is proved. \square

The next lemma is a consequence of Lemma 3.1.5 and can be proved in the same way as Lemma 2.1.4 is proved.

Lemma 3.3.3. *Let $r \in (0, \infty)$ and let a function $u \in C(\bar{Q}_r^+) \cap W_{d+1}^{1,2}(Q_\rho^+)$ for any $\rho \in (0, r)$ and satisfy $u = 0$ on $\partial'Q_r^+$. Then there are constants $\gamma \in (0, 1]$ and N , depending only on δ and d , such that for any $L \in \mathbb{L}_\delta$ we have*

$$\int_{Q_r^+} |D^2u|^\gamma dx dt \leq N \left(\int_{Q_r^+} |\partial_t u + Lu|^{d+1} dx dt \right)^{\gamma/(d+1)}.$$

Everywhere below in this section α is the smallest of the constants called α in Lemmas 3.1.4 and 3.3.2 and γ is the smallest of the ones from Lemmas 3.1.5 and 3.3.3.

Lemma 3.3.4. *Let $r \in (0, \infty)$, $\kappa \geq 16$, and $x_0^1 \geq 0$. Let $u \in W_{d+1}^{1,2}(Q_{\kappa r}^+(x_0^1))$ be a solution to (3.1) in $Q_{\kappa r}^+(x_0^1)$ vanishing on $Q_{\kappa r}^+(x_0^1) \cap \partial\mathbb{R}^{d+1}$. Then*

$$\begin{aligned} & \int_{Q_r^+(x_0^1)} \int_{Q_r^+(x_0^1)} |D^2u(t, x) - D^2u(s, y)|^\gamma dx dt dy ds \\ & \leq N \kappa^{d+2} \left(\int_{Q_{\kappa r}^+(x_0^1)} |f|^{d+1} dx dt \right)^{\gamma/(d+1)} \\ & + N \kappa^{-\alpha\gamma} \left(\int_{Q_{\kappa r}^+(x_0^1)} |D^2u|^{d+1} dx dt \right)^{\gamma/(d+1)}, \end{aligned} \quad (3.16)$$

where the constant N depends only on d and δ .

Proof. As in the proof of Lemma 2.1.5 due to dilations, we only need to consider the case $\kappa r = 8$. Again, we consider the following two cases.

Case 1: $x_0^1 > 1$. In this case, we have $Q_{r\kappa/8}(x_0^1) \subset \mathbb{R}_+^{d+1}$ and inequality (3.16) is an immediate consequence of Lemma 3.1.6 since $\kappa/8 \geq 2$.

Case 2: $x_0^1 \in [0, 1]$. Since $r = 8/\kappa \leq 1/2$, we have

$$Q_r^+(x_0^1) \subset Q_{3/2}^+ \subset Q_4^+ \subset Q_{\kappa r}^+(x_0^1).$$

By using a standard approximating argument, we may assume that $u \in C_b^\infty(\bar{Q}_{\kappa r}^+(x_0^1))$.

Define $\hat{u} := u - x^1(D_1 u)_{Q_4^+}$. We claim that there exists a function v such that

- (i) $v \in C(\bar{Q}_4)$, $v = \hat{u}$ on $\partial' Q_4^+$;
- (ii) $v \in C_b^{1,2}(\bar{Q}_\rho^+)$ for any $\rho < 4$;
- (iii) v satisfies (3.1) in Q_4^+ with $f \equiv 0$.

The proof of this claim is obtained as follows. First we take smooth domains \mathcal{D}_n such that $B_{4-1/n}^+ \subset \mathcal{D}_n \subset B_4^+$ set $\mathcal{Q}_n = (0, 16) \times \mathcal{D}_n$ and by applying Theorem 6.4.1 of [12] find unique $v_n \in C_b^{1,2}(\mathcal{Q}_n) \cap C(\bar{\mathcal{Q}}_n)$ satisfying (3.1) with $f \equiv 0$, and boundary condition $v_n = \hat{u}$ on $\partial' \mathcal{Q}_n$. Then one routinely derives from Theorem 5.5.2 in [12] (cf. Remark 3.1.1) that there exists $\beta \in (0, 1)$ such that for any $\rho < 4$ the $C^{1+\beta/2, 2+\beta}(Q_\rho^+)$ norms of v_n are bounded for all large n . After that one takes a subsequence of v_n , if necessary, and finds a function v possessing the above properties (ii) and (iii). That v also satisfies (i) is proved in the same way as a similar statement is proved in Theorem 6.3.1 of [12].

Now Lemmas 3.3.2 and 3.3.1 and the maximum principle easily yield that

$$\begin{aligned} & \int_{Q_r^+(x_0^1)} \int_{Q_r^+(x_0^1)} |D^2 v(t, x) - D^2 v(s, y)| \, dx \, dt \, dy \, ds \\ & \leq N r^\alpha [D^2 v]_{C^\alpha(Q_{3/2}^+)} \leq N r^\alpha \sup_{\partial' Q_4^+} |v| \\ & \leq N \kappa^{-\alpha} \left(\int_{Q_4^+} (|D^2 u| + |\partial_t u|)^{d+1} \, dx \, dt \right)^{1/(d+1)}. \end{aligned}$$

Recall that $\gamma \in (0, 1]$. By Hölder's inequality,

$$\begin{aligned} & \int_{Q_r^+(x_0^1)} \int_{Q_r^+(x_0^1)} |D^2 v(t, x) - D^2 v(s, y)|^\gamma \, dx \, dt \, dy \, ds \\ & \leq N \kappa^{-\alpha \gamma} \left(\int_{Q_{\kappa r}^+(x_0^1)} (|D^2 u| + |\partial_t u|)^{d+1} \, dx \, dt \right)^{\gamma/(d+1)}. \end{aligned} \tag{3.17}$$

Next for $w := \hat{u} - v$ in Q_4^+ , we have $w \in W_{d+1}^{1,2}(Q_\rho^+)$ for any $\rho < 4$. By the same argument as in the proof of Lemma 2.1.5, we know that there exists an operator $L \in \mathbb{L}_\delta$ such that $\partial_t w + Lw = f$ in Q_4^+ . By Lemma 3.3.3 and the fact that $\kappa r = 8$, we get

$$\begin{aligned} \int_{Q_r^+(x_0^1)} |D^2 w|^\gamma dx dt &\leq N \kappa^{d+2} \int_{Q_4^+} |D^2 w|^\gamma dx dt \\ &\leq N \kappa^{d+2} \left(\int_{Q_4^+} |f|^{d+1} dx dt \right)^{\gamma/(d+1)} \\ &\leq N \kappa^{d+2} \left(\int_{Q_{\kappa r}^+(x_0^1)} |f|^{d+1} dx dt \right)^{\gamma/(d+1)} \end{aligned}$$

and

$$\begin{aligned} \int_{Q_r^+(x_0^1)} \int_{Q_r^+(x_0^1)} |D^2 w(t, x) - D^2 w(s, y)|^\gamma dx dt dy ds \\ \leq N \kappa^{d+2} \left(\int_{Q_{\kappa r}^+(x_0^1)} |f|^{d+1} dx dt \right)^{\gamma/d+1}. \end{aligned}$$

Upon combining this inequality with (3.17), observing that $D^2 u = D^2 v + D^2 w$, and using (3.5) we get (3.16). The lemma is proved. \square

As in the proof of Theorem 2.1.7, one derives the following theorem from Lemma 3.3.4, the Hardy–Littlewood theorem, and Theorem 5.3 of [16], which we apply to the filtration of dyadic parabolic cubes belonging to \mathbb{R}_+^{d+1} . Denote by $\mathring{W}_p^{1,2}(\mathbb{R}_+^{d+1})$ the set of functions from $W_p^{1,2}(\mathbb{R}_+^{d+1})$ with zero trace at $x^1 = 0$.

Theorem 3.3.5. *Let $p > d + 1$. (i) If $u \in \mathring{W}_p^{1,2}(\mathbb{R}_+^{d+1})$ satisfies (3.1) in \mathbb{R}_+^{d+1} , then*

$$\|D^2 u\|_{L_p(\mathbb{R}_+^{d+1})} + \|\partial_t u\|_{L_p(\mathbb{R}_+^{d+1})} \leq N \|f\|_{L_p(\mathbb{R}_+^{d+1})},$$

where N depends only on d , δ , and p .

(ii) For any $f \in L_p(\mathbb{R}_+^{d+1})$ and $\lambda > 0$, there exists a unique solution $u \in \mathring{W}_p^{1,2}(\mathbb{R}_+^{d+1})$ of the equation

$$\partial_t u(t, x) + F(D^2 u(t, x)) - \lambda u(t, x) = f(t, x).$$

Furthermore,

$$\lambda \|u\|_{L_p(\mathbb{R}_+^{d+1})} + \|D^2 u\|_{L_p(\mathbb{R}_+^{d+1})} + \|\partial_t u\|_{L_p(\mathbb{R}_+^{d+1})} \leq N \|f\|_{L_p(\mathbb{R}_+^{d+1})},$$

where N depends only on d , δ , and p .

3.4 Parabolic equations in \mathbb{R}_+^{d+1} with VMO coefficients

In this section, we consider parabolic equations in \mathbb{R}_+^{d+1} with variable coefficients

$$\partial_t u(t, x) + F(D^2 u(t, x), t, x) - \lambda u(t, x) = f(t, x). \quad (3.18)$$

In the sequel, Assumption 1.0.1 is supposed to hold with $\mathcal{D} = \mathbb{R}_+^{d+1}$, and the constants α and γ in Lemma 3.4.1 are taken from Section 3.3. Recall that $\theta(\mu, d, \delta)$ is introduced in Remark 2.2.1.

Lemma 3.4.1. *Let $\beta \in (1, \infty)$, $\lambda = 0$, $\mu, r > 0$, $\kappa \geq 16$, $x_0^1 \geq 0$, and $(\tau, z) \in \mathbb{R}_+^{d+1}$. Suppose that $\theta = \theta(\mu, d, \delta)$. Let $u \in \dot{W}_{d+1}^{1,2}(\mathbb{R}_+^{d+1})$ be a solution of (3.18) vanishing outside $Q_{R_0}^+(\tau, z)$. Then*

$$\begin{aligned} & \int_{Q_r^+(x_0^1)} \int_{Q_r^+(x_0^1)} |D^2 u(t, x) - D^2 u(s, y)|^\gamma dx dt dy ds \\ & \leq N \kappa^{d+2} \left(\int_{Q_{\kappa r}^+(x_0^1)} |f|^d dx dt \right)^{\gamma/(d+1)} \\ & + N \kappa^{d+2} \left(\int_{Q_{\kappa r}^+(x_0^1)} |D^2 u|^{\beta(d+1)} dx dt \right)^{\gamma/(\beta d + \beta)} \mu^{\gamma/(\beta' d + \beta')} \\ & + N \kappa^{-\alpha \gamma} \left(\int_{Q_{\kappa r}^+(x_0^1)} |D^2 u|^{d+1} dx dt \right)^{\gamma/(d+1)}, \end{aligned} \quad (3.19)$$

where $N = N(\delta, d, \beta)$ and $\beta' = \beta/(\beta - 1)$.

Proof. Introduce

$$\bar{F}(u'') = \begin{cases} (F)_{Q_{R_0}^+(\tau, z)}(u''), & \text{if } \kappa r \geq R_0; \\ (F)_{Q_{\kappa r}^+(x_0^1)}(u''), & \text{otherwise,} \end{cases}$$

and

$$h(t, x) = \sup_{u'' \in \mathcal{S}: |u''|=1} |F(u'', t, x) - \bar{F}(u'')|.$$

Note that

$$\partial_t u(t, x) + \bar{F}(D^2 u) = \tilde{f},$$

where

$$\tilde{f}(t, x) = f(t, x) + \bar{F}(D^2 u) - F(D^2 u, t, x).$$

By Lemma 3.3.4 and the triangle inequality,

$$\begin{aligned}
& \int_{Q_r^+(x_0^1)} \int_{Q_r^+(x_0^1)} |D^2 u(t, x) - D^2 u(s, y)|^\gamma dx dt dy ds \\
& \leq N \kappa^{d+2} \left(\int_{Q_{\kappa r}^+(x_0^1)} |f|^{d+1} dx dt \right)^{\gamma/(d+1)} + N \kappa^{d+2} J^{\gamma/(d+1)} \\
& \quad + N \kappa^{-\alpha\gamma} \left(\int_{Q_{\kappa r}^+(x_0^1)} |D^2 u|^{d+1} dx dt \right)^{\gamma/(d+1)}, \tag{3.20}
\end{aligned}$$

where $N = N(d, \delta)$,

$$J = \int_{Q_{\kappa r}^+(x_0^1)} |\bar{F}(D^2 u) - F(D^2 u, t, x)|^{d+1} I_{Q_{R_0}^+(\tau, z)} dx dt \leq J_1^{1/\beta} J_2^{1/\beta'},$$

Here

$$\begin{aligned}
J_1 &= \int_{Q_{\kappa r}^+(x_0^1)} |D^2 u|^{\beta(d+1)} dx dt, \\
J_2 &= \int_{Q_{\kappa r}^+(x_0^1)} h^{\beta'(d+1)} I_{Q_{R_0}^+(\tau, z)} dx dt \leq N \int_{Q_{\kappa r}^+(x_0^1)} h I_{Q_{R_0}^+(\tau, z)} dx dt.
\end{aligned}$$

If $\kappa r < R_0$, we have

$$J_2 \leq N \int_{Q_{\kappa r}^+(x_0^1)} h(t, x) dx dt \leq N \mu.$$

If $\kappa r \geq R_0$, we have

$$\begin{aligned}
J_2 &\leq N (\kappa r)^{-d-2} \int_{Q_{R_0}^+(\tau, z)} h(t, x) dx dt \\
&\leq N (\kappa r)^{-d-2} R_0^{d+2} \int_{Q_{R_0}^+(\tau, z)} h(t, x) dx dt \leq N \mu.
\end{aligned}$$

Therefore, in any case,

$$J \leq N \left(\int_{Q_{\kappa r}^+(x_0^1)} |D^2 u(x)|^{\beta(d+1)} dx dt \right)^{1/\beta} \mu^{1/\beta'}.$$

Substituting the above inequality back into (3.20) yields (3.19). The lemma is proved. \square

The proof of Lemma 3.4.1 is just a rather dull repetition of already given proofs of similar facts. The following corollary is obtained in the same way as similar assertions were obtained before.

Corollary 3.4.2. *Let $p > d + 1$ and $u \in \mathring{W}_{d+1}^{1,2}(\mathbb{R}_+^{d+1})$ be a solution to (3.10) with $\lambda = 0$ vanishing outside $Q_{R_0}^+(\tau, z)$, where $(\tau, z) \in \mathbb{R}_+^{d+1}$. Then there exist constants $\theta \in (0, 1]$ and N depending only on p, d , and δ , such that if Assumption 1.0.1 is satisfied with this θ , then*

$$\|D^2u\|_{L_p} + \|\partial_t u\|_{L_p} \leq N\|f\|_{L_p}.$$

Next we state the main result of this section, which is deduced from Corollary 3.4.2 by modifying the proof of Theorem 2.2.5. By $\mathring{W}_p^{1,2}((T, \infty) \times \mathbb{R}_+^d)$ we denote the set of functions of class $W_p^{1,2}((T, \infty) \times \mathbb{R}_+^d)$ with zero trace on $x^1 = 0$.

Theorem 3.4.3. *Let $p > d + 1$ and $T \in [-\infty, \infty)$. There exist constants $\theta = \theta(d, \delta, p) \in (0, 1]$, and λ_0 depending only on d, p, δ and R_0 , such that if Assumption 1.0.1 holds with this θ , then*

(i) *For any $\lambda \geq \lambda_0$ and $u \in \mathring{W}_p^{1,2}((T, \infty) \times \mathbb{R}_+^d)$ satisfying (3.18), we have*

$$\begin{aligned} \lambda\|u\|_{L_p((T, \infty) \times \mathbb{R}_+^d)} + \|\partial_t u\|_{L_p((T, \infty) \times \mathbb{R}_+^d)} + \|D^2u\|_{L_p((T, \infty) \times \mathbb{R}_+^d)} \\ \leq N\|f\|_{L_p((T, \infty) \times \mathbb{R}_+^d)}, \end{aligned}$$

where $N = N(d, \delta, p)$.

(ii) *For any $\lambda > 0$, there exists a constant $N = N(d, p, \delta, R_0, \lambda)$ such that for any $u \in \mathring{W}_p^{1,2}((T, \infty) \times \mathbb{R}_+^d)$ satisfying (3.18), we have*

$$\|u\|_{W_p^{1,2}((T, \infty) \times \mathbb{R}_+^d)} \leq N\|f\|_{L_p((T, \infty) \times \mathbb{R}_+^d)}.$$

(iii) *For any $\lambda > 0$ and $f \in L_p((T, \infty) \times \mathbb{R}_+^d)$, there exists a unique solution $u \in \mathring{W}_p^{1,2}((T, \infty) \times \mathbb{R}_+^d)$ of (3.18).*

We are now ready to prove Theorem 1.0.1.

3.5 Proof of Theorem 1.0.1

Proof. The proof is similar to that of Theorem 1.0.2 in Section 2.3 with some minor modifications. As before, we first establish (1.3) as an a priori estimate and we may assume that $g \equiv 0$.

We will see again that to obtain the a priori estimate we do not need condition (H₃).

Observe that Theorems 3.2.3 and 3.4.3 with $\lambda = \lambda_0$ imply that

$$\begin{aligned} \|\partial_t u\|_{L_p(\mathbb{R}_0^{d+1})} + \|D^2 u\|_{L_p(\mathbb{R}_0^{d+1})} &\leq N(\|\partial_t u + F(D^2 u)\|_{L_p(\mathbb{R}_0^{d+1})} \\ &\quad + \|u\|_{L_p(\mathbb{R}_0^{d+1})}), \quad \forall u \in W_p^{1,2}(\mathbb{R}_0^{d+1}), \\ \|\partial_t v\|_{L_p(\mathbb{R}_+ \times \mathbb{R}_+^d)} + \|D^2 v\|_{L_p(\mathbb{R}_+ \times \mathbb{R}_+^d)} &\leq N(\|\partial_t v + F(D^2 v)\|_{L_p(\mathbb{R}_+ \times \mathbb{R}_+^d)} \\ &\quad + \|v\|_{L_p(\mathbb{R}_+ \times \mathbb{R}_+^d)}), \quad \forall v \in \dot{W}_p^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+^d), \end{aligned} \quad (3.21)$$

where $N = N(d, p, \delta, R_0)$ (provided that $\theta = \theta(d, p, \delta)$ is chosen appropriately).

Now suppose that $u \in \dot{W}_p^{1,2}(\mathcal{D}_T)$ satisfies

$$\partial_t u + F(D^2 u, t, x) + G(D^2 u, Du, u(t, x), t, x) = 0 \quad (3.22)$$

in \mathcal{D}_T . We extend u and G to be zero for $t > T$. It is easily seen that the extended $u \in \dot{W}_p^{1,2}(\mathcal{D}_\infty)$ satisfies (3.22) in \mathcal{D}_∞ . Define

$$f(t, x) = -G(D^2 u(t, x), Du(t, x), u(t, x), t, x).$$

After that by using the technique based on flattening the boundary, partitions of unity, and interpolation inequalities allowing one to estimate Du through $D^2 u$ and u and also using (3.21) we obtain that

$$\|\partial_t u\|_{L_p(\mathcal{D}_\infty)} + \|D^2 u\|_{L_p(\mathcal{D}_\infty)} \leq N_1(\|f\|_{L_p(\mathcal{D}_\infty)} + \|u\|_{L_p(\mathcal{D}_\infty)}),$$

which is the same as

$$\|\partial_t u\|_{L_p(\mathcal{D}_T)} + \|D^2 u\|_{L_p(\mathcal{D}_T)} \leq N_1(\|f\|_{L_p(\mathcal{D}_T)} + \|u\|_{L_p(\mathcal{D}_T)}), \quad (3.23)$$

provided that θ is sufficiently small depending only on d, p, δ , and the $C^{1,1}$ norm of ∂D . Here N_1 depends only on d, p, δ, R_0 , and the $C^{1,1}$ norm of $\partial \mathcal{D}$.

It follows from the definition of f and (H₂) that, for any $s > 0$,

$$\begin{aligned} \|f\|_{L_p(\mathcal{D}_T)} &\leq \chi(s) \|D^2 u\|_{L_p(\mathcal{D}_T)} + \|\chi\|_{L_\infty} s T^{1/p} |\mathcal{D}|^{1/p} \\ &\quad + K(\|Du\|_{L_p(\mathcal{D}_T)} + \|u\|_{L_p(\mathcal{D}_T)}) + \|\bar{G}\|_{L_p(\mathcal{D}_T)}. \end{aligned} \quad (3.24)$$

Upon taking s large such that $N_1\chi(s) \leq 1/2$, we get from (3.23), (3.24) and the interpolation inequality that

$$\|u\|_{W_p^{1,2}(\mathcal{D}_T)} \leq N_2(\|u\|_{L_p(\mathcal{D}_T)} + \|\bar{G}\|_{L_p(\mathcal{D}_T)} + \|\chi\|_{L_\infty} s T^{1/p} |\mathcal{D}|^{1/p}), \quad (3.25)$$

where N_2 is the same type of constant as N_1 .

Next, one can estimate the $L_p(\mathcal{D}_T)$ norm of u by rewriting (3.22) similarly to (2.24) as

$$\partial_t u + Lu + b^i D_i u - cu = -G(D^2 u, 0, 0, t, x)$$

and using the parabolic Alexandrov estimates. This will lead to an a priori estimate (1.3) as in the proof of Theorem 1.0.2 with N depending also on T . To see that N can be chosen to be independent of T , we suppose without loss of generality that $\mathcal{D} \subset B_{R/2}$, where $R = 4\text{diam}(\mathcal{D})$, and take the barrier function v_0 defined on \mathbb{R}^d from Lemma 11.1.2 of [15], which satisfies in B_R ,

$$v_0 > 0, \quad Lv_0 + b^i D_i v_0 - cv_0 \leq -1.$$

Denote $v = u/v_0$. Then $v \in \dot{W}_p^{1,2}(\mathcal{D}_T)$ satisfies

$$\partial_t v + Lv + \tilde{b}^i D_i v - \tilde{c}v = -v_0^{-1} G(D^2(v_0 v), 0, 0, t, x)$$

in \mathcal{D}_T , where

$$\tilde{b}^i = b^i + 2a^{ij} v_0^{-1} D_j v_0, \quad \tilde{c} = -v_0^{-1} (Lv_0 + b^i D_i v_0 - cv_0).$$

It is easily seen that we can find constants $\tilde{K} > 0$ and $\nu > 0$ depending only on d, δ, K , and R , such that

$$|\tilde{b}| \leq \tilde{K}, \quad \nu \leq \tilde{c} \leq \tilde{K}.$$

We then write $\tilde{c} = \hat{c} + \nu$ so that $\hat{c} \geq 0$. As in the proof of Theorem 2.1.7 (ii), it holds that

$$\nu \|v\|_{L_p(\mathcal{D}_T)} \leq N(d, \delta, p) \|v_0^{-1} G(D^2(v_0 v), 0, 0, t, x)\|_{L_p(\mathcal{D}_T)},$$

which gives

$$\|u\|_{L_p(\mathcal{D}_T)} \leq N(d, \delta, p, R) \|G(D^2 u, 0, 0, t, x)\|_{L_p(\mathcal{D}_T)}, \quad (3.26)$$

owing to the properties of v_0 . Combining (3.26) and (3.25), we finish proving the a priori estimate as in the proof of Theorem 1.0.2.

With the a priori estimate (1.3) in hand, the existence and uniqueness are obtained by the same argument as at the end of Section 2.3 relying on condition (H_3) . The theorem is proved. \square

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