Generalized complex structures on 4-manifolds

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Haojie Chen

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Tian-Jun Li

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Dedication

To my wife and collaborator Xiaolan.

Abstract

We study the existence of generalized complex structures on closed smooth four-manifolds in this thesis. We first prove an integrability theorem of generalized complex structures in terms of almost bihermitian structures. After reviewing the results of type zero and type two generalized complex structures, we prove the necessary and sufficient topological conditions for the existence of type one generalized complex structures on compact four-manifolds. We also obtain a nonexistence result of $\mathbb{T}^2$-invariant untwisted generalized complex structures with mixed types on the four torus $\mathbb{T}^4$. Finally, we prove a result about finite group actions on tamed almost complex four-manifolds.

Contents

Chapter 1

Introduction

Generalized complex geometry was introduced by Hitchin and developed in many aspects by Cavalcanti, Gualtieri and others. The notion of generalized complex structures naturally generalizes symplectic and complex structures and provides a wider background to study their common properties as well as the link between symplectic and complex geometries. Shortly after its introduction, generalized complex geometry was brought into physics and played an important role in the field of supersymmetric flux compactifications in string theory. The starting point lies in Gualtieri's discovery that the geometry of the generalized Kähler structure is equivalent to the geometry of a famous structure discovered by Gates, Hall and Roček in their study of supersymmetric sigma-models in 80s. In principle, zero flux compactification leads to Calabi-Yau structure while nonzero flux compactification leads to generalized Calabi-Yau structure on the space with extra dimension.

Generalized complex structure is defined to be an endomorphism $\mathcal{J}$ of the generalized tangent bundle $T \oplus T^*$ satisfying $\mathcal{J}^2 = -id$ such that it is H-integrable with respect to a real closed three form H. The integrability condition is defined in terms of either the Courant bracket or conditions of pure spinor forms (Chapter 2). Together with Xiaolan Nie, we prove the result which transforms the integrability condition into conditions of almost bihermitian structures (an almost bihermtian structure is a triple of two almost complex structures j_+, j_- compatible with a Riemannian metric g).

Theorem 1.1. *Given a generalized almost complex structure $\mathcal{J}$ on a smooth manifold M^{2n}, there exists an almost bihermitian structure (g, j_+, j_-) and a real two form b, such that*

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} j_+ + j_- & \omega_-^{-1} - \omega_+^{-1} \\ \omega_+ - \omega_- & -(j_+^* + j_-^*) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix},$$

where $\omega_\pm = gj_\pm$. Denote $T_+^{1,0}, T_-^{1,0} \subset T \otimes \mathbb{C}$ to be the corresponding $+i$-eigenbundles of j_+, j_- and H a closed real three form. Then $\mathcal{J}$ is H-integrable if and only if the following two conditions hold:

- $d\omega_+(X_+ + X_-, Y_+ + Y_-, Z_+) - (\omega_+ + \omega_-)(X_-, [Y_+ + Y_-, Z_+]) + (\omega_+ + \omega_-)(Y_-, [X_+ + X_-, Z_+]) - Z_+\omega_+(X_-, Y_-) = -i(H + db)(X_+ + X_-, Y_+ + Y_-, Z_+)$
- $d\omega_-(X_+ + X_-, Y_+ + Y_-, Z_-) - (\omega_- + \omega_+)(X_+, [Y_+ + Y_-, Z_-]) + (\omega_- + \omega_+)(Y_+, [X_+ + X_-, Z_-]) - Z_-\omega_-(X_+, Y_+) = i(H + db)(X_+ + X_-, Y_+ + Y_-, Z_-)$

where $X_\pm, Y_\pm, Z_\pm$ are sections of $T_\pm^{1,0}$.

Generalized complex structures in dimension two are B-field transform of complex or symplectic structures, so there is no obstruction to the existence. The first interesting dimension is dimension four. Generalized complex structures in dimension four on connected manifolds are divided into four cases: type zero, type two, type one and mixed types with both zero and two. While the existences of type zero and type two generalized complex structures are equivalent to existences of symplectic and complex structures, the existences of type one and mixed types generalized complex structures are new to the world of 4-manifolds. With Theorem 1.1 and the existence of Gauduchon metric in the conformal class of almost Hermitian metrics on almost complex manifolds, Xiaolan Nie and I obtain the necessary and sufficient topological condition for the existence of type one generalized complex structure on a closed 4-manifold.

Theorem 1.2. *A closed orientable 4-manifold admits type one generalized complex structures if and only if it has the structure of transversely holomorphic 2-foliations.*

Generalized complex structures with mixed types on 4-manifolds seem to be more mysterious. For example, Cavalcanti and Gualtieri proved that there exist generalized complex structures with mixed types on $(2k+1)\mathbb{CP}^2$, $k \geq 0$, in contrast with the fact that there is no symplectic or complex structure or type one generalized complex structure on $(2k+1)\mathbb{CP}^2$ for $k > 1$ (Chapter 4). It is natural to ask the following question:

Question 1: Does every almost complex 4-manifolds admit generalized complex structures with mixed types?

Using T-duality, we study the above question on $\mathbb{T}^4$ and prove the following.

Theorem 1.3. *There is no untwisted $\mathbb{T}^2$-invariant generalized complex structure of mixed types with at least one Lagrangian orbit on $\mathbb{T}^4$.*

Finally we study a different subject about the finite group actions on almost complex 4-manifolds. An almost complex manifold (M, j) is said to be tamed if there exists a symplectic structure ω such that $\omega(v, jv) > 0$ for $v \neq 0$. Let $\kappa_j^t = \{[\omega] \in H^2(M, \mathbb{R}) | j \textit{ tamed by } \omega\}$ and define $C_j = \min_{[\omega]\in\kappa_j^t \cap H^2(X,\mathbb{Z})}\{K_j \cdot [\omega]\}$ where K_j is canonical class. With these concepts, we obtain the following:

Theorem 1.4. *Let (M, j) be a minimal tamed almost complex 4-manifold with $b^+ > 1$. Assume either $\chi(M) \neq 0$ or the symplectic Kodaira dimension $\kappa^s = 2$, then there is a constant $C = m_1 C_j^2 + m_2$, where m_1, m_2 are positive constants related to b_1, b_2, such that for any nontrivial j-holomorphic $\mathbb{Z}_p$-action on M, $p \leq C$.*

The structures of the thesis are as follows. In Chapter 2, we review the fundamental concepts and results in generalized complex geometry. We define Courant bracket and generalized complex structures. Then we introduce the notion of type and discuss some classifications in terms of types. After that we give the notion of canonical bundle and state Gualteri's deformation theorem. To illustrate the application of T-duality to generalized complex geometry, we present Cavalcanti and Gualtieri's results on the transformation of generalized complex structures between T-dual pairs.

In Chapter 3, we introduce the concept of almost generalized Kähler structure. There exist almost generalized Kähler structures associated to any generalized almost complex structure. Using this, we give a description of generalized almost complex structure in terms of almost bihermitian structures and prove Theorem 1.1 subsequently. To better understand the integrability conditions, we give a reproof of Gualtieri's theorem on the equivalence of generalized Kähler structure and bihermitian structures with additional conditions. If the two almost complex structures of the almost bihermitian structures commute with each other, we derive a convenient criterion which is essentially used in Chapter 5.

In Chapter 4, we start the discussion of generalized complex geometry on 4-manifolds. These structures are divided into four cases and we describe the type two and type zero cases together. The existence of type two or zero structures is equivalent to existence of complex or symplectic structures which are classical topics on 4-manifolds. So we briefly review the theory of compact complex surfaces and symplectic 4-manifolds. Particularly, we discuss the existence of complex and symplectic structures on the 4-manifolds: $S^1 \times N^3$, surface bundle over surface, $n\mathbb{CP}^2$.

In Chapter 5, we focus on the type one generalized complex structure on 4-manifolds. We begin with generalized almost complex structure with type one and show that its existence only depends on the Euler characteristic and intersection form of 4-manifolds. To obtain integrable type one generalized complex structure, we introduce the concept of transversely holomorphic foliation. After proving the existence of Gauduchon metric on almost complex manifolds, we prove Theorem 1.2. As applications, we derive the existence of type one generalized complex structures on 4-manifolds of $S^1 \times N^3$, surface bundle over surface, etc. We also give nilpotent examples and compute its deformations.

In Chapter 6, we discuss the last case-generalized complex structure with mixed types on 4-manifolds. We review the holomorphic Poisson structure as well as C^∞ log transform approaches to construct generalized complex structures with mixed types. With respect to almost bihermitian structures, we obtain a noncommutative property. We proceed to study the existence of generalized complex structure with mixed types on $\mathbb{T}^4$. In particular, we prove Theorem 1.3.

In Chapter 7, we study the finite group actions on almost complex 4-manifolds. We review Weimin Chen's result of finite group actions on symplectic 4-manifolds which essentially applies the equivariant Seiberg-Witten theory. Combing it with new computations, we prove Theorem 1.4. If the almost complex manifold is not minimal, we obtain an even simpler result.

Chapter 2

Generalized complex geometry

After the built-up work of Hitchin in 2002 [32], generalized complex geometry was extensively studied in Gualtieri's thesis [29]. Among it, many fundamental concepts and results are founded, for example the generalized Darboux theorem, deformation theory of generalized complex structures and generalized Kähler structures. In a series of collaborate work (for example [9] [10]), Cavalcanti and Gualtieri were able to find more topological and geometric properties of generalized complex structures, such as the transformation of generalized complex structures under T-duality, construction of new generalized complex 4-manifolds, etc.

In this chapter, we give the preliminary definitions and results in generalized complex geometry. In section 1, we define the concept of generalized almost complex structure and H-integrable generalized complex structure. Then we give examples and define isomorphisms between generalized complex structures. There is a notion called the type of generalized complex structures. Points with locally constant types are called regular points. We then state the generalized Darboux theorem which gives a local classification of generalized complex structure around regular points.

In section 2, we define the canonical bundle associated to a generalized complex structure. In particular, generalized complex structure whose canonical bundle has a non-vanishing closed section is called a generalized Calabi-Yau structure. We define the Lie algebroid cohomology later and use it to state the deformation theory of generalized complex structures.

In section 3, we briefly introduce the notion of T-duality from [9]. The existence theorem of T-dual space is very useful. For a T-dual pair, T-duality turns out to induce an isomorphism between the invariant Courant algebroid structures which then induces transformation of invariant generalized complex structure from one space to its dual space.

2.1 Generalized complex structures

The idea of generalized geometry comes from replacing the tangent bundle T by the generalized tangent bundle $T \oplus T^*$. Recall the classical Newlander-Nirenberg theorem states that the existence of a complex structure on a manifold is equivalent to an endomorphism j of the tangent bundle satisfying $j^2 = -id$ (namely, almost complex structure) and whose $+i$ eigenbundle is closed under the Lie bracket. Generalized complex structure is a natural extension of the latter concept on the generalized tangent bundle. First, there is a symmetric paring on $T \oplus T^*$:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)),$$

where $X, Y \in T$ and $\xi, \eta \in T^*$.

Definition 2.1. A generalized almost complex structure on a manifold M is an endomorphism $\mathcal{J}$ of $T \oplus T^*$ preserving the natural paring such that $\mathcal{J}^2 = -id$.

Manifolds with almost complex structure are even dimensional. It follows that generalized almost complex structures guarantees the existence of almost complex structures (Chapter 3). Therefore, we assume the manifolds have dimension $2n$ from now on.

Given a closed real 3-form H, define the Courant bracket on $T \oplus T^*$ as:

$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi + i_Y i_X H,$$

where $X, Y \in T$ and $\xi, \eta \in T^*$ and i_X denoting the interior multiplication of X. It extends to $(T \oplus T^*) \otimes \mathbb{C}$ linearly. Courant bracket resembles many of the properties of Lie bracket on the tangent bundle except the Jacobi identity.

Definition 2.2. A generalized almost complex structure is said to be H-integrable if its $+i$ eigenspace, $L \subset (T \oplus T^*) \otimes \mathbb{C}$ is closed under the Courant bracket associated to H.

A generalized complex structure is an H-integrable generalized almost complex structure for some closed real 3-form H. If H is d-exact, we call that the generalized complex structure is untwisted.

Example 2.3. (Complex structures) Let j be an integrable almost complex structure, define

$$\mathcal{J}_j = \begin{pmatrix} -j & 0 \\ o & j^* \end{pmatrix}.$$

Then $\mathcal{J}_j$ gives an untwisted generalized complex structure $(H = 0)$ with $+i$ eigenbundle $L = T^{0,1} \oplus T^{*1,0}$.

Example 2.4. (Symplectic structures) Given a symplectic structure ω on a manifold, let

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Then $\mathcal{J}_\omega$ defines an untwisted generalized complex structure with $+i$ eigenspace $L = \{X - i\omega(X) : X \in T \otimes \mathbb{C}\}$

If two manifolds have generalized complex structures, their product has a natural product generalized complex structure. Therefore, the products of complex manifolds and symplectic manifolds have generalized complex structures. There is an additional construction.

Example 2.5. (B-field transform) A real 2-form B acts on T by interior multiplication. It exponentiates to an bundle automorphism e^B on $T \oplus T^*$ by $e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$. For $X+\xi, Y+\eta \in T \oplus T^*$, by computations of Courant bracket [29], we have

$$[e^{-B}(X+\xi), e^{-B}(Y+\eta)]_{H+dB} = e^{-B}([X+\xi, Y+\eta]_H).$$

If $\mathcal{J}$ is an H-integrable generalized complex structure, let $\mathcal{J}_B = e^{-B}\mathcal{J}e^B$. Denote L, L_B the $+i$-eigenbundles of $\mathcal{J}$ and $\mathcal{J}_B$ respectively. Then $L_B = e^{-B}L$. So $\mathcal{J}_B$ gives an $H+dB$-integrable generalized complex structure. We call such a construction the B-field transform.

Let $\phi : M \to N$ be a diffeomorphism. If $\mathcal{J}$ is a H-integrable generalized complex structure on N, then the pull back $\phi^*\mathcal{J}$ is naturally defined using the induced isomorphism between $T_M \oplus T_M^*$ and $T_N \oplus T_N^*$. In particular, $\phi^*\mathcal{J}$ is a ϕ^*H-integrable generalized complex structure on M. Now we can define the isomorphism between generalized complex structures.

Definition 2.6. Let $\mathcal{J}_1$ be an H_1-integrable generalized complex structure on M_1 and $\mathcal{J}_2$ an H_2-integrable generalized complex structure on M_2. $\mathcal{J}_1$ is said to be isomorphic to $\mathcal{J}_2$ if there is a diffeomorphism $\varphi : M_1 \to M_2$ and a real two form B on M_1 such that $\varphi^*H_2 = H_1 + dB$ and $\mathcal{J}_1 = e^{-B}\varphi^*\mathcal{J}_2 e^B$.

Associated to a generalized complex structure, there is an important notion called the type.

Definition 2.7. Given a generalized complex structure $\mathcal{J}$ on M^{2n}, the **type** of $\mathcal{J}$ at a point x is defined to be $n - \frac{dim \pi_x \mathcal{J}(T^*)}{2}$, where $\pi_x : T_x \oplus T_x^* \to T_x$ is the first projection.

It is a fact that the types are integers between n and 0 and keep the same parity (but may have jump phenomenons) within a connected component of M [29]. They are also invariant under the B-field transform as well as pull-backs. Generalized complex structures with constant types have good properties. First of all, the optimal cases type n and 0 can be characterized globally. It follows from Examples 2.3, 2.4 that generalized complex structures from complex structures have type n and generalized complex structures from symplectic structures have type 0. Gualtieri proves the converse:

Proposition 2.8. *(Gualtieri) A generalized complex structure on M^{2n} has constant type n (or 0) if and only if it a B-field transform of a complex structure (or a symplectic structure).*

Since generalized complex structures on a surface can only have constant type 1 or 0, we have:

Corollary 2.9. *On a surface, generalized complex structures are B-field transform of complex structures or symplectic structures.*

For generalized complex structures with constant types greater than 0 and less than n, there is a locally generalized Darboux theorem.

Theorem 2.10. *(Gualtieri) If a generalized complex structure $\mathcal{J}$ on M^{2n} has constant type k in a neighborhood of x, then there is a smaller neighbourhood of x such that in the neighborhood $\mathcal{J}$ is isomorphic to the product structure of an open set in $\mathbb{C}^k$ with an open set in the standard symplectic space $(\mathbb{R}^{2n-2k}, \omega_0)$.*

2.2 Canonical bundle and deformations of generalized complex structures

Generalized complex structure can be formulated in terms of pure spinors. Let T_x, T_x^* be the tangent and cotangent space at x on M^{2n}, $S_x = \oplus_{i=0}^{2n} \wedge^i T_x^*$ be the space of forms. Then there is a Clifford action of $CL(T_x \oplus T_x^*)$ on S_x which is induced by

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi,$$

where $X \in T_x, \xi \in T^*_x$ and $\varphi \in S_x$. For any nonzero element φ, its null space is $L_\varphi = \{v \in T_x \oplus T^*_x, v \cdot \varphi = 0\}$. The paring restricted to L_φ is actually zero. We call that φ is a pure spinor if $dim(L_\varphi) = 2n$, i.e. L_φ is a maximal isotropic subspace. Conversely, any maximal isotropic subspace corresponds to a unique pure spinor up to scaler [14].

Now given a generalized almost complex structure $\mathcal{J}$, its $+i$-eigenbundle L is a maximal isotropic subbundle of $(T \oplus T^*) \otimes \mathbb{C}$. By the above argument there is a unique line bundle U in $S = \oplus_{i=0}^{2n} \wedge^i T^* \otimes \mathbb{C}$ such that each fiber of U consists of the pure spinors corresponding to L_x.

Definition 2.11. The unique complex line bundle U is called the canonical bundle of $\mathcal{J}$.

Example 2.12. The canonical bundle of a generalized complex structure from complex structure is $\wedge^n T^{*1,0}$ which coincides with the classical definition. The canonical bundle for a symplectic structure ω is trivialized by the section $e^{i\omega}$.

Remark 1: the type of $\mathcal{J}$ is the lowest degree of nonzero sections in U [10].

The above process can be reversed. By defining the null space of a pure spinor bundle U to be the $+i$-eigenbundle, a generalized almost complex structure $\mathcal{J}$ is uniquely defined by U. The integrability of $\mathcal{J}$ translates to differential conditions of U. Let $U_n = U$, $U_k = CL^{n-k}((T \oplus T^*) \otimes \mathbb{C}) \cdot U = \wedge^{n-k} \bar{L} \cdot U$, then Gualtieri proves the following equivalence.

Proposition 2.13. *A generalized almost complex structure is H-integrable if and only if $d_H(\mathcal{U}_n) \subset \mathcal{U}_{n-1}$, where $d_H = d + H\wedge$ and $\mathcal{U}_i$ are the space of sections of U_i.*

In particular, if the canonical bundle has a nowhere vanishing d_H closed section, the generalized almost complex structure $\mathcal{J}$ is H-integrable and is called a generalized Calabi-Yau structure. Any symplectic structure is a generalized Calabi-Yau structure.

The deformation of generalized complex structure is studied by Gualtieri. First, there is a one to one correspondence between generalized almost complex structures and maximal isotropic subbundles of $(T \oplus T^*) \otimes \mathbb{C}$, by relating $\mathcal{J}$ to its $+i$-eigenbundle L. If $\mathcal{J}$ is integrable, then L is closed under the Courant bracket, which make it to be a Lie algebroid. Therefore, by the theory of Lie algebroid there is a natural Lie algebroid derivative $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$ satisfying $d_L^2 = 0$. Gualtieri prove the following:

Proposition 2.14. *The differential complex $(C^\infty(\wedge^k L^*), d_L)$ is elliptic for an integrable generalized complex structure. Its cohomology groups $H_L^k(M)$ are finite dimensional vector space for a compact manifold M.*

Using techniques similar to the Kodaira-Spencer-Kuranishi theory of deformation of complex structures, Gualtieri obtains the deformation theorem for generalized complex structures.

Theorem 2.15. *(Gualtieri) The second cohomology group $H_L^2(M)$ locates the infinitesimal deformations of the generalized complex structure $\mathcal{J}$. Furthermore, there is a neighborhood $V \subset H_L^2(M)$ containing zero, a smooth family of generalized almost complex structures over V and an analytic obstruction map $\Psi : V \to H_L^3(M)$ such that $\Psi^{-1}(0) \subset V$ is precisely the subset which locates the integrable generalized almost complex structures. Any sufficiently deformation of $\mathcal{J}$ is isomorphic to one from the family over $\Psi^{-1}(0)$.*

2.3 T-duality and generalized complex structures

T-duality is a symmetry of quantum field theories with differing classical descriptions in physics. In [9], Cavalcanti and Gualtieri formulated and understood T-duality in terms of geometries of principle torus bundles with Neveu-Schwarz flux H. In particular, generalized complex structures can be transported from a principle bundle to its T-dual. We will apply this to study the existence of generalized complex structures on

4-manifolds (Chapter 6). In this section, we briefly introduce the definitions and results related to T-duality.

Definition 2.16. Let M_1 and M_2 be two principal T^k bundles over a common base manifold B and let $H_1 \in \Omega^3_{T^k}(M_1)$, $H_2 \in \Omega^3_{T^k}(M_2)$ be two invariant closed 3-forms. Consider the fiber product $M_1 \times_B M_2 = \{(x,y) \in M_1 \times M_2, p_1 x = p_2 y\}$ where p_1, p_2 are the two projections

$$p_1 : M_1 \times_B M_2 \to M_1, p_2 : M_1 \times_B M_2 \to M_2.$$

We call that (M_1, H_2) and (M_2, H_2) are T-dual if: $p_1^* H_1 - p_2^* H_2 = dF$, for some T^{2k}-invariant 2-form $F \in \Omega^2_{T^{2k}}(M_1 \times_B M_2)$ such that

$$F : t^k_{M_1} \otimes t^k_{M_2} \to \mathbb{R}$$

is nondegenerate, where $t^k_{M_i}$ are the vertical tangent vector space of the fibrations p_i.

For the existence of T-dual, the following theorem is very useful.

Theorem 2.17. *(Bouwknegt-Hannabuss-Mathai) Given a principle T^k bundle M_1 over a base manifold B and a closed 3-form $H_1 \in \Omega^3_{T^k}(M_1)$ representing a class in $H^3(M_1, \mathbb{Z})$ and satisfying $i_X i_Y H = 0$ for $X, Y \in t^k_{M_1}$, there exists another T^k bundle M_2 over B with 3-form H_2 such that (M_2, H_2) is T-dual to (M_1, H_1). If $H_1 = 0$, we can choose $H_2 = 0$ and M_2 be the product $T^k \times B$.*

As a consequence of T-duality, there is a Fourier-type isomorphism:

Theorem 2.18. *(Bouwknegt-Evslin-Mathai) If (M_1, H_2) and (M_2, H_2) are T-dual, with $p_1^* H_1 - p_2^* H_2 = dF$, then the map $\tau : \Omega^*_{T^k}(M_1) \to \Omega^*_{T^k}(M_2)$*

$$\tau(\rho) = \int_{T^k} e^F \wedge p_1^* \rho$$

induces an isomorphism between the differential complexes $(\Omega^*_{T^k}(M_1), d_{H_1})$ and $(\Omega^*_{T^k}(M_2), d_{H_2})$, where the integration is along the fibers of p_2 on $M_1 \times_B M_2$ and the result forms are interpreted as forms on M_2 accordingly.

The important discovery of Gualtieri and Cavalcanti is that there is an isomorphism of Courant algebroids with invariant sections between (M_1, H_2) and (M_2, H_2).

Theorem 2.19. *(Calvancanti-Gualtieri) Let (M_1, H_2) and (M_2, H_2) be T-dual space. Then there is an isomorphism of Courant algebroid $\varphi : (TM_1 \oplus T^*M_1)/T^k \to (TM_2 \oplus T^*M_2)/T^k$. Therefore, any invariant H_1-integrable generalized complex structures on M_1 can be transformed to an invariant H_2-integrable generalized complex structure on M_2. The transformation of canonical bundles coincides with the map τ locally.*

The following example from [9] is very illustrative and will be used in Chapter 6.

Example 2.20. Assume $\rho = e^{B+i\omega}\wedge\Omega$ be a local section of U_1, where U_1 is the canonical bundle of a generalized complex structure $\mathcal{J}_1$ on M_1 and $type(\mathcal{J}_1) = degree(\Omega) = i$. By the above theorem, the local section of the canonical bundle of the generalized complex structure $\mathcal{J}_2$ transformed from $\mathcal{J}_1$ is

$$\tau(\rho) = \int_{T^k} e^F \wedge \rho = \int_{T^k} e^{F+B+i\omega} \wedge \Omega.$$

As the type is the lowest degree of sections of the canonical bundle, let j be the smallest integer such that $\int_{T^k}(F + B + i\omega)^j \wedge \Omega \neq 0$, then the type of $\mathcal{J}_2$ is equal to $2j + deg(\Omega) - k = 2j + i - k$. Particularly, if $dimB = k$ we have the following phenomenons.

1. If $\mathcal{J}_1$ is a complex structure ($i = k$, $\rho = \Omega$) on a subset, and the fiber T^k is a complex submanifold. As F is nondegerate and the fiber is half dimensional, we have $\int_{T_k} F^{\frac{k}{2}} \wedge \Omega \neq 0$ and $j = \frac{k}{2}$. So $2j + i - k = k$ and $\mathcal{J}_2$ is also a complex structure.

2. If $\mathcal{J}_1$ is a complex structure with the fiber T^k a real manifold which means $\mathcal{J}_1(t^k)\cap t^k = 0$. Then $\int_{T_k} \Omega \neq 0$ and $j = 0$, $type(\mathcal{J}_2) = 0$. $\mathcal{J}_2$ is a symplectic structure.

3. If $\mathcal{J}_1$ is a symplectic structure ($i = 0, \rho = e^{i\omega}$) with T^k Lagrangian which means $\omega|_{T^k} = 0$. Then $\int_{T_k}(F + \omega)^k \neq 0$ and $j = k$, $type(\mathcal{J}_2) = k$. $\mathcal{J}_2$ is a complex structure.

4. If $\mathcal{J}_1$ is a symplectic structure with T^k symplectic which means $\omega|_{T^k}$ is a symplectic structure. Then $\int_{T_k}(F+\omega)^{\frac{k}{2}} \neq 0$ and $j = \frac{k}{2}$, $type(\mathcal{J}_2) = 0$. $\mathcal{J}_2$ is a symplectic structure.

Combing the four case, we have the following correspondent pairs for T-dual generalized complex structures.
(complex structure, complex fiber)$\leftrightarrow$ (complex structure, complex fiber)
(complex structure, real fiber)$\leftrightarrow$ (symplectic structure, Lagrangian fiber)
(symplectic structure, symplectic fiber)$\leftrightarrow$ (symplectic structure, symplectic fiber).

For T^k-invariant generalized complex structures, the orbit has the same types, so the correspondence can be applied even when the types are not constant throughout the whole manifold.

Chapter 3

Almost bihermitian structures and generalized complex structures

The integrability of generalized almost complex structure is defined in terms of involutive property of the maximal isotropic subbundle with respect to the Courant bracket. Practically this condition is not convenient to verify. With the concept of almost generalized Kähler structure, Xiaolan Nie and I are able to transform the condition into equations of classical geometric conditions, namely, forms and almost complex structures. With this transition, we reprove Gualtieri's celebrated theorem on the equivalence between generalized Kähler strucuture and target space geometry of $(2, 2)$ supersymmetric sigma model.

In section 1 we introduce the concept of almost generalized Kähler structure and find out the relation between almost generalized Kähler structure and generalized almost complex structure. Using it we find the explicit expressions for a generalized almost complex structure.

In section 2 we defined almost bihermitian structures. It is easy to find that the existence of almost bihermitian structure is the same with existence of almost complex structure, different from the integrable case. Using the almost bihermitian structure, we formulate the integrability of generalized almost complex structure in classical terms. When the two almost complex structures commute with each other, the conditions have much simpler forms which will be used in Chapter 5.

3.1 Almost generalized Kähler structures

Recall for a nondegenerate 2-form ω (or presymplectic structure) on M^{2n}, an almost complex structures j is said to be compatible with ω if $\omega(jX, jY) = \omega(X, Y)$ and $\omega(X, jX) > 0$ for any tangent vectors X, Y. By the standard obstruction and principle bundle theory, there always exist compatible almost complex structure for a nondegenerate 2-form (the whole space is contractible). On the other direction, given an almost complex structure j, the imaginary part of any Hermitian structure gives a nondegenerate 2-form compatible with j. These connections can be formulated in the setting of generalized complex geometry.

Definition 3.1. An almost generalized Kähler structure is a pair of generalized almost complex structure $(\mathcal{J}_1, \mathcal{J}_2)$ such that $\mathcal{J}_1\mathcal{J}_2 = \mathcal{J}_2\mathcal{J}_1 = \mathcal{G}$ and $(v, \mathcal{G}v) > 0$ for any nonzero $v \in T \oplus T^*$. A generalized Kähler structure is an almost generalized Kähler structure with both $\mathcal{J}_1$ and $\mathcal{J}_2$ being H-integrable for some closed 3-form H.

Nondegenerate 2-forms and almost complex structures induce generalized almost complex structures. The compatibility conditions between j and ω is equivalent to that $(\mathcal{J}_j, \mathcal{J}_\omega)$ forms an almost generalized Kähler structure.

The existence for compatible almost complex structures for nondegenerate 2-form can generalize to the following.

Proposition 3.2. *For any generalized almost complex structure $\mathcal{J}_1$, there exist another* *generalized almost complex structure $\mathcal{J}_2$, such that $(\mathcal{J}_1, \mathcal{J}_2)$ is an almost generalized Kähler structure.*

Proof. With the existence of generalized almost complex structure $\mathcal{J}_1$, the structure group of $T \oplus T^*$ can be reduced from $GL(4n)$ into $GL(n,n)$ whose maximal compact subgroup is $U(n,n)$. As $U(n,n)$ is homotopic equivalent to $U(n) \times U(n)$, the structure group can be then reduced to $U(n) \times U(n)$. With this reduction, there is a splitting of the associated generalized tangent bundle $T \oplus T^* = C_+ \oplus C_-$, where $C_\pm$ are postive/negative definite subbundles with respect to the inner product, and $\mathcal{J}(C_\pm) = C_\pm$. Define $\mathcal{J}_2 = -\mathcal{J}_1$ on C_+, and $\mathcal{J}_2 = \mathcal{J}_1$ on C_-. Then $(\mathcal{J}_1, \mathcal{J}_2)$ gives an almost generalized Kähler structure by definition. □

The remarkable property for an almost generalized Kähler structures is that it can be expressed by classical geometric structures explicitly.

Proposition 3.3. *For any almost generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$, there exists a real two form b and two almost complex structures j_1, j_2 compatible with a Riemannian metric g such that*

$$\mathcal{J}_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} j_+ + j_- & \omega_-^{-1} - \omega_+^{-1} \\ \omega_+ - \omega_- & -(j_+^* + j_-^*) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix},$$

$$\mathcal{J}_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} j_+ - j_- & -\omega_+^{-1} - \omega_-^{-1} \\ \omega_+ + \omega_- & -(j_+^* - j_-^*) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

Proof: Follow the argument in [29], since $\mathcal{G}$ is positive and $\mathcal{G}^2 = id$, the generalized tangent bundle splits into $2n$-dimensional, ± 1-eigenbundles of $\mathcal{G}$, $T \oplus T^* = C_+ \oplus C_-$. As $C_\pm$ are definite, they have zero intersection with T^*, so the projection $\pi : T \oplus T^* \to T$ induces isomorphisms between $C_\pm$ and T. Therefore $C_\pm$ are graphs of T for some homomorphism to T^*. Using the pair between T and T^*, each homomorphism splits into the sum of a symmetric and antisymmetric map. Notice C_+ is orthogonal to C_-, we get $C_\pm = T + (b \pm g)T$ for a 2-form b and a symmetric tensor g. The definiteness of C_+ insures that g is a Riemannian metric.

Now as $\mathcal{J}_1 C_\pm \subset C_\pm$, $-\mathcal{J}_1|_{C_\pm} = \pm\mathcal{J}_2|_{C_\pm}$, we have $\mathcal{J}_1 = \mathcal{J}_1|_{C_+} + \mathcal{J}_1|_{C_-}$ and $\mathcal{J}_2 = -\mathcal{J}_1|_{C_+} + \mathcal{J}_1|_{C_-}$. So once we know $\mathcal{J}_1|_{C_\pm}$, we can determine $\mathcal{J}_1, \mathcal{J}_2$. As $C_\pm$ are graphs of T, we transform $\mathcal{J}_1|_{C_\pm}$ to be two almost complex structures j_+, j_- on T. Now the final expression follows from linear algebra computations which we omit. □

Combing Proposition 3.2 and 3.3, we have the following corollary.

Corollary 3.4. *Let $\mathcal{J}$ be an almost generalized complex structure on M^{2n}, then there exists two almost complex structures j_+, j_- compatible with a Riemannian metric g and a real two form b, such that*

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} j_+ + j_- & \omega_-^{-1} - \omega_+^{-1} \\ \omega_+ - \omega_- & -(j_+^* + j_-^*) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix},$$

where $\omega_\pm = g j_\pm$.

3.2 Almost bihermitian structures and integrability of generalized almost complex structures

Recall an almost Hermitian structure is a pair of an almost complex structure with a compatible metric. Similarly, we can define:

Definition 3.5. An almost bihermitian structure is a triple of (g, j_1, j_2) where j_1, j_2 are two almost complex structures and g is a Rimannian metric such that both j_1 and j_2 are compatible with g. If $j_1 \neq \pm j_2$, we said the almost bihermtian structure is nontrivial.

Though almost bihermitian structure has more data than almost complex structure, the existence of nontrivial almost bihermitian structures is the same with existence of almost complex structure.

Proposition 3.6. *For any almost complex structure j, there exist $j' \neq \pm j$ and g, such that (j, j', g) is an almost bihermitian structure.*

Proof. Given the almost complex structure j, we can find a nondegenerate 2-form ω which is tamed by but not compatible with j, namely, $\omega(X, jX) > 0$, for any nonzero $X \in T$ and there exists $Y, Z \in T$ such that $\omega(Y, Z) \neq \omega(jY, jZ)$. For example, we can add a small real $(2,0) + (0,2)$ form to any compatible nondegenerate 2-form. The resulting 2-form is also nondegenerate and tamed but not compatible. As in [19], define $j' = -\omega^{-1} j^* \omega$, where j^* is the dual action of j on T^*, and ω is considered as a map from T to T^*. Let $g = j_+^* \omega - \omega j_+$. Then $g(u, v) = \omega(u, j_+ v) + \omega(v, j_+ u)$ and $g(v, v) = 2\omega(v, j_+ v) > 0$, so g is a Riemannian metric. Also it is easy to find $j'^* g j' = j^* g j = g$. As $j \neq \pm j'$, we proved that (j, j', g) is a nontrivial almost bihermitian structure. □

Now by Corollary 3.4 every generalized almost complex structure is induced by an almost bihermitian structure and 2-forms. Conversely, given an almost bihermitian structure and a 2-form, it is easy to verify that the expression in Corollary 3.4 gives a generalized almost complex structure. Therefore, up to B-field transform the study of generalized almost complex structure is equivalent to the study of geometry of almost bihermitian structures. We derive the integrability of $\mathcal{J}$ in terms of conditions of (g, j_1, j_2).

Theorem 3.7. *Given a generalized almost complex structure $\mathcal{J}$ on a smooth manifold M^{2n}, let (g, j_+, j_-, b) be an almost bihermitian structure and a real two form associated to $\mathcal{J}$. Denote $T_+^{1,0}, T_-^{1,0} \subset T \otimes \mathbb{C}$ to be the corresponding $+i$-eigenbundles of j_+, j_- and H a closed real three form. Then $\mathcal{J}$ is H-integrable if and only if the following two conditions hold:*

- $d\omega_+(X_+ + X_-, Y_+ + Y_-, Z_+) - (\omega_+ + \omega_-)(X_-, [Y_+ + Y_-, Z_+]) + (\omega_+ + \omega_-)(Y_-, [X_+ + X_-, Z_+]) - Z_+ \omega_+(X_-, Y_-) = -i(H + db)(X_+ + X_-, Y_+ +$

$Y_-, Z_+)$ (1)

- $d\omega_-(X_+ + X_-, Y_+ + Y_-, Z_-) - (\omega_- + \omega_+)(X_+, [Y_+ + Y_-, Z_-]) + (\omega_- + \omega_+)(Y_+, [X_+ + X_-, Z_-]) - Z_-\omega_-(X_+, Y_+) = i(H + db)(X_+ + X_-, Y_+ + Y_-, Z_-)$ (2)

where $X_\pm, Y_\pm, Z_\pm$ *are sections of* $T_\pm^{1,0}$.

Proof. Given $\mathcal{J}$ as in Corollary 3.4 and H a closed real 3-form. First consider the $+i$ eigenbundle of

$$\mathcal{J}' = \frac{1}{2}\begin{pmatrix} j_+ + j_- & \omega_-^{-1} - \omega_+^{-1} \\ \omega_+ - \omega_- & -(j_+^* + j_-^*) \end{pmatrix}.$$

Then direct computation gives $\mathcal{J}'(X + \xi) = i(X + \xi)$ if and only if $j_+(X + g^{-1}\xi) = i(X + g^{-1}\xi)$ and $j_-(X - g^{-1}\xi) = i(X - g^{-1}\xi)$, i.e. $X + g^{-1}\xi \in T_+^{1,0}$ and $X - g^{-1}\xi \in T_-^{1,0}$. Note that a vector field $W \in T_+^{1,0}$ if and only if $g(W, Z_+) = 0$ for any $Z_+ \in T_+^{1,0}$, thus the above conditions give that $(X, \xi) \in +i$ eigenspace of $\mathcal{J}'$ is equivalent to:

$$g(X, Z_+) + \xi(Z_+) = g(X, Z_-) - \xi(Z_-) = 0 \qquad (3)$$

for any $Z_\pm \in T_\pm^{1,0}$.

Let L' be the $+i$ eigenspace of $\mathcal{J}'$, and we define the following injective maps

$$i_+ : T_+^{1,0} \longrightarrow L', \qquad i_- : T_-^{1,0} \longrightarrow L'$$

$$X_+ \longmapsto (X_+, gX_+) \qquad X_- \longmapsto (X_-, -gX_-).$$

then $i_+T_+^{1,0} \cap i_-T_-^{1,0} = 0$ and we have $L' = i_+T_+^{1,0} \oplus i_-T_-^{1,0}$. So for any $(X, \xi) \in L'$, there exists unique $X_+ \in T_+^{1,0}, X_- \in T_-^{1,0}$ such that $X + \xi = X_+ + X_- + g(X_+ - X_-)$. Now we can get that the $+i$ eigenspace of $\mathcal{J}$ is

$$L = e^b L' = \{X_+ + X_- + b(X_+ + X_-) + g(X_+ - X_-) | X_\pm \in T_\pm^{1,0}\},$$

then

$$[X_+ + X_- + b(X_+ + X_-) + g(X_+ - X_-), Y_+ + Y_- + b(Y_+ + Y_-) + g(Y_+ - Y_-)]_H$$

$$= e^b[X_+ + X_- + g(X_+ - X_-), Y_+ + Y_- + g(Y_+ - Y_-)]$$

$$+ i_{Y_++Y_-} i_{X_++X_-}(H + db).$$

Since $L' = e^{-b}L$, then

$$[L, L]_H \subset L \Leftrightarrow [L', L']_{H+db} = e^{-b}[L, L]_H \subset L'$$

i.e. L' is $H + db$ -involutive. So we only need to find the integrability condition for L'. Let $X + \xi = X_+ + X_- + g(X_+ - X_-)$, $Y + \eta = Y_+ + Y_- + g(Y_+ - Y_-) \in L'$, then

$$[X + \xi, Y + \eta]_{H+db} = [X_+ + X_-, Y_+ + Y_-] + \mathcal{L}_{X_++X_-} g(Y_+ - Y_-) -$$

$$i_{Y_++Y_-} d(g(X_+ - X_-)) + i_{Y_++Y_-} i_{X_++X_-}(H + db).$$

Use (3) we get that $[X + \xi, Y + \eta]_{H+db} \in L'$ if and only if

$$g([X_+ + X_-, Y_+ + Y_-], Z_+) + (\mathcal{L}_{X_++X_-} g(Y_+ - Y_-)$$

$$- i_{Y_++Y_-} d(g(X_+ - X_-))(Z_+) = -(H + db)(X_+ + X_-, Y_+ + Y_-, Z_+). \quad (4)$$

$$g([X_+ + X_-, Y_+ + Y_-], Z_-) - (\mathcal{L}_{X_++X_-} g(Y_+ - Y_-)$$

$$- i_{Y_++Y_-} d(g(X_+ - X_-))(Z_-) = (H + db)(X_+ + X_-, Y_+ + Y_-, Z_-). \quad (5)$$

We use $g = -\omega_\pm j_\pm$ and Cartan formula $\mathcal{L}_X = i_X d + d i_X$ for differential forms to get

$$\begin{aligned}
&\mathcal{L}_{X_++X_-} g(Y_+ - Y_-) - i_{Y_++Y_-} d(g(X_+ - X_-))(Z_+)\\
=&i(X_+ + X_-)\omega_-(Y_-, Z_+) + i\omega_+(Y_+, [X_+ + X_-, Z_+]) - i\omega_-(Y_-, [X_+ + X_-, Z_+])\\
&- i(Y_+ + Y_-)\omega_-(X_-, Z_+) - iZ_+(\omega_+(X_+, Y_-) - \omega_-(X_-, Y_+))\\
&- i\omega_+(X_+, [Y_+ + Y_-, Z_+] + i\omega_-(X_-, [Y_+ + Y_-, Z_+])\\
=&- i(X_+ + X_-)\omega_+(Y_-, Z_+) + i\omega_+(Y_+, [X_+ + X_-, Z_+]) - i\omega_-(Y_-, [X_+ + X_-, Z_+])\\
&+ i(Y_+ + Y_-)\omega_+(X_-, Z_+) - iZ_+(\omega_+(X_+, Y_-) + \omega_+(X_-, Y_+))\\
&- i\omega_+(X_+, [Y_+ + Y_-, Z_+]) + i\omega_-(X_-, [Y_+ + Y_-, Z_+]).
\end{aligned}$$

the last equality holds because $\omega_-(Y_-, Z_+) = ig(Y_-, Z_+) = i\omega_+(Y_-, J_+Z_+) = -\omega_+(Y_-, Z_+)$, Also $g([X_+ + X_-, Y_+ + Y_-], Z_+) = i\omega_+([X_+ + X_-, Y_+ + Y_-], Z_+)$, and $\omega_\pm$ are $(1,1)$ forms with respect to $j_\pm$, then we can rewrite (4) to be

$$\begin{aligned}&d\omega_+(X_+ + X_-, Y_+ + Y_-, Z_+) - (\omega_+ + \omega_-)(X_-, [Y_+ + Y_-, Z_+])\\ &+ (\omega_+ + \omega_-)(Y_-, [X_+ + X_-, Z_+]) - Z_+\omega_+(X_-, Y_-)\\ &= -i(H + db)(X_+ + X_-, Y_+ + Y_-, Z_+).\end{aligned}$$

Similarly (5) becomes

$$\begin{aligned}&d\omega_-(X_+ + X_-, Y_+ + Y_-, Z_-) - (\omega_- + \omega_+)(X_+, [Y_+ + Y_-, Z_-])\\ &+ (\omega_- + \omega_+)(Y_+, [X_+ + X_-, Z_-]) - Z_-\omega_-(X_+, Y_+)\\ &= i(H + db)(X_+ + X_-, Y_+ + Y_-, Z_-).\end{aligned}$$

The theorem is proved. □

Remark 2: The equations (4) and (5) directly indicate that $T_+^{1,0} + T_-^{1,0}$ is involutive. In the case $T_+^{1,0} \cap T_-^{1,0} = 0$, $T_+^{1,0} + T_-^{1,0} = T \otimes \mathbb{C}$, this follows automatically. Otherwise take $Z_+ = Z_- = Z \in T_+^{1,0} \cap T_-^{1,0}$ in (4), (5) and add them to get $g([X_+ + X_-, Y_+ + Y_-], Z) = 0$ for any $Z \in T_+^{1,0} \cap T_-^{1,0}$. Note that the orthogonal complement $(T_+^{1,0} \cap T_-^{1,0})^\perp$ is just $T_+^{1,0} + T_-^{1,0}$ as $T_+^{1,0} + T_-^{1,0} \subset (T_+^{1,0} \cap T_-^{1,0})^\perp$ and the two spaces have the same dimensions. Therefore $[X_+ + X_-, Y_+ + Y_-] \in T_+^{1,0} + T_-^{1,0}$ for any $X_\pm, Y_\pm \in T_\pm^{1,0}$.

Now we can reprove Gualtieri's theorem [29] on the equivalence between generalized Kähler strucuture and target space geometry of $(2,2)$ supersymmetric sigma model.

Proposition 3.8. *(Gualitieri) The almost generalized Kähler structure*

$$\mathcal{J}_{1,2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} j_+ \pm j_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(j_+^* \pm j_-^*) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix},$$

is integrable if and only if

- $T_{\pm}^{1,0}$ *are involutive.*
- $d_{+}^{c}\omega_{+} = -d_{-}^{c}\omega_{-} = -(H+db)$

Proof. For one direction, consider the generalized almost complex structures $\mathcal{J}_1, \mathcal{J}_2$. The equalities (4) and (5) give integrability of $\mathcal{J}_1$. Since $\mathcal{J}_2$ is obtained from $\mathcal{J}_1$ by replacing j_- by $-j_-$, then $\mathcal{J}_2$ is integrable if and only if the following are satisfied:

$$g([X_+ + \bar{X}_-, Y_+ + \bar{Y}_-], Z_+) + (\mathcal{L}_{X_+ + \bar{X}_-} g(Y_+ - \bar{Y}_-)$$
$$- i_{Y_+ + \bar{Y}_-} d(g(X_+ - \bar{X}_-))(Z_+) = -(H+db)(X_+ + \bar{X}_-, Y_+ + \bar{Y}_-, Z_+) \quad (6)$$

$$g([X_+ + \bar{X}_-, Y_+ + \bar{Y}_-], \bar{W}_-) - (\mathcal{L}_{X_+ + \bar{X}_-} g(Y_+ - \bar{Y}_-)$$
$$- i_{Y_+ + \bar{Y}_-} d(g(X_+ - \bar{X}_-))(\bar{W}_-) = (H+db)(X_+ + \bar{X}_-, Y_+ + \bar{Y}_-, \bar{W}_-). \quad (7)$$

where $X_\pm, Y_\pm, Z_\pm \in T_{\pm}^{1,0}$ and $\bar{X}_-, \bar{Y}_-, \bar{W}_- \in T_{-}^{0,1}$ denote the complex conjugate. Let $X_- = Y_- = 0$ in (5) and $\bar{X}_- = \bar{Y}_- = 0$ in (7), then add them to get:

$$g([X_+, Y_+], Z) - (\mathcal{L}_{X_+} g(Y_+) - i_{Y_+} d(g(X_+))(Z)$$
$$= (H+db)(X_+, Y_+, Z). \quad (8)$$

for any $Z = Z_- + \bar{W}_-$. Since $T_{-}^{1,0} \oplus T_{-}^{0,1} = T \otimes \mathbb{C}$, the equation (8) holds for any $Z \in T \otimes \mathbb{C}$. Let $Z = Z_+ \in T_{+}^{1,0}$ in (8) and $X_- = Y_- = 0$ in (4), adding them together gives $g([X_+, Y_+], Z_+) = 0$ for any $Z_+ \in T_{+}^{1,0}$, i.e. $[X_+, Y_+] \in T_{+}^{1,0}$ for any $X_+, Y_+ \in T_{+}^{1,0}$. Therefore $T_{+}^{1,0}$ is involutive. Now $g([X_+, Y_+]) = -\omega_+(j_+[X_+, Y_+]) = -i\omega_+([X_+, Y_+])$, we can rewrite the equality (8) to be:

$$i_{[X_+, Y_+]}\omega_+ - \mathcal{L}_{X_+} i_{Y_+}\omega_+ + i_{Y_+} d i_{X_+}\omega_+ = i_{Y_+} i_{X_+} i(H+db). \quad (9)$$

Note that $i_{[X_+, Y_+]} = [\mathcal{L}_{X_+}, i_{Y_+}] = \mathcal{L}_{X_+} i_{Y_+} - i_{Y_+}\mathcal{L}_{X_+}$. Use this relation to simplify (9) first and then apply $\mathcal{L}_{X_+} = i_{X_+} d + d i_{X_+}$ to get the following:

$$i_{Y_+} i_{X_+} d\omega_+ = -i_{Y_+} i_{X_+} i(H+db). \quad (10)$$

Similarly, let $X_+ = Y_+ = 0$ in (5) and take the complex conjugate to get

$$g([\bar{X}_-, \bar{Y}_-], \bar{Z}_+) - (\mathcal{L}_{\bar{X}_-} g(\bar{Y}_-) + i_{\bar{Y}_-} d(g(\bar{X}_-))(\bar{Z}_+)$$
$$= -(H + db)(\bar{X}_-, \bar{Y}_-, \bar{Z}_+). \tag{11}$$

Combine (6) and (11) and use the same argument as above: First we get for any $Z \in T \otimes \mathbb{C}$.

$$g([\bar{X}_-, \bar{Y}_-], Z) - (\mathcal{L}_{\bar{X}_-} g(\bar{Y}_-) - i_{\bar{Y}_-} d(g(\bar{X}_-))(Z)$$
$$= -(H + db)(\bar{X}_-, \bar{Y}_-, Z). \tag{12}$$

Then let $Z = \bar{W}_- \in T_-^{0,1}$ in (12) and $X_+ = Y_+ = 0$ in (7) to obtain $g([\bar{X}_-, \bar{Y}_-], \bar{W}_-) = 0$ for any $\bar{X}_-, \bar{Y}_- \in T_-^{0,1}$, i.e. $T_-^{0,1}$ is also integrable. Similarly we can use (12) to prove that:

$$i_{\bar{Y}_-} i_{\bar{X}_-} d\omega_- = -i_{\bar{Y}_-} i_{\bar{X}_-} i(H + db). \tag{13}$$

Note that $H + db$ is a $(1, 2) + (2, 1)$ form with respect to both $j_\pm$. Use the (p, q) decomposition of forms with respect to $j_\pm$ the equations (10) and (13) are equivalent to:

$$d_+^c \omega + = -d_-^c \omega_- = -(H + db)$$

where $d_\pm^c = i(\bar{\partial}_\pm - \partial\pm)$.

For the other direction. If $T_\pm^{1,0}$ are involutive and $d_+^c \omega_+ = -d_-^c \omega_- = -(H + db)$. Consider the integrability of $\mathcal{J}_1$ first. By the above theorem, we only need to show the equalities (1) (2) hold in three cases. If $X_- = Y_- = 0$, then (1) is true as $d\omega_+$ and $H + db$ are both $(1, 2) + (2, 1)$ forms for j_+. If $X_+ = Y_- = 0$, then the left hand side of (1) is equal to

$$d\omega_+(X_-, Y_+, Z_+) - (\omega_+ + \omega_-)(X_-, [Y_+, Z_+])$$
$$= d\omega_+(X_-, Y_+, Z_+)$$
$$= -i(H + db)(X_-, Y_+, Z_+).$$

The first equality holds because $T_+^{1,0}$ is involutive and $(\omega_+ + \omega_-)(X_-, W_+) = 0$. The second equality holds because $d_+^c \omega_+ = -(H + db)$. Now if $X_+ = Y_+ = 0$, as we calculate in Theorem 3.7, the left hand side of (1) becomes

$$
\begin{aligned}
& d\omega_+(X_-, Y_-, Z_+) - (\omega_+ + \omega_-)(X_-, [Y_-, Z_+]) + (\omega_- + \omega_+)(Y_-, [X_-, Z_+]) \\
& - Z_+\omega_-(X_-, Y_-) = ig([X_-, Y_-], Z_+) + i((\mathcal{L}_{X_-} g(-Y_-) - i_{Y_-} d(g(-X_-))(Z_+)) \\
& = -d\omega_-(X_-, Y_-, Z_+) = -i(H + db)(X_-, Y_-, Z_+)
\end{aligned}
$$

The third equality holds as $d^c_-\omega_- = (H + db)$. So we have proved (1), and similarly for (2) and the integrability of $\mathcal{J}_2$. $\square$

Almost bihermitian structure with two almost complex structures commuting with each other is interesting. With this assumption, we can simplify the equations (1), (2). To get the simplified version, we need the following observation:

Lemma 3.9. *Let M be a manifold with an almost bihermitian structure (g, j_+, j_-) and $j_+j_- = j_-j_+$. Then the two fundamental forms $\omega_\pm = gj_\pm$ satisfy the following relations:*

$$
\omega_-(X_+, Y_+) = \omega_+(X_-, Y_-) = 0, \;\; \omega_+(X, Y) + \omega_-(X, Y) = 0
$$

for any $X_\pm, Y_\pm \in T^{1,0}_\pm$ and $X, Y \in T^{1,0}_+ + T^{1,0}_-$.

Proof. Since j_+ and j_- are both compatible with g, also $j_+j_- = j_-j+$, we have $\omega_-(X_+, Y_+) = g(j_-X_+, Y_+) = g(j_+j_-X_+, j_+Y_+) = g(j_-j_+X_+, j_+Y_+) =$ $- g(j_-X_+, Y_+) = -\omega_-(X_+, Y_+)$. Thus $\omega_-(X_+, Y_+) = 0$. Similarly $\omega_+(X_-, Y_-)$ $= 0$. Now we can show the third equality. Given any $X, Y \in T^{1,0}_+ + T^{1,0}_-$, there exist $X_\pm, Y_\pm \in T^{1,0}_\pm$ such that $X = X_+ + X_-, Y = Y_+ + Y_-$. Note that $\omega_+(X_+, Y_+) = \omega_-(X_-, Y_-) = 0$. Also $\omega_+(X_+, Y_-) = ig(X_+, Y_-) = i\omega_-(X_+, j_-Y_-) = -\omega_-(X_+, Y_-)$. Therefore $\omega_+(X, Y) + \omega_-(X, Y) = \omega_+(X_-, Y_-) + \omega_-(X_+, Y_+) = 0$. $\square$

Corollary 3.10. *Given a generalized almost complex structure $\mathcal{J}$ on a smooth manifold M^{2n}, let (g, j_+, j_-, b) be an almost bihermitian structure and a real two form associated* *to $\mathcal{J}$. Use the same notation as above. If j_+ and j_- commute, then $\mathcal{J}$ is H-integrable if and only if $T_+^{1,0}+T_-^{1,0}$ is involutive and the forms $\omega_\pm, b$ and H satisfy the conditions:*

$$d\omega_+|_{T_+^{1,0}+T_-^{1,0}} = -d\omega_-|_{T_+^{1,0}+T_-^{1,0}} = -i(H+db)|_{T_+^{1,0}+T_-^{1,0}}$$

Proof. If $\mathcal{J}$ is H-integrable, then $T_+^{1,0}+T_-^{1,0}$ is involutive as noted in Remark 2. Use the formula $d\omega_\pm(X,Y,Z) = X\omega(Y,Z)+\omega(X,[Y,Z])+c.p.$ (where $c.p.$ indicates cyclic permutation) to expand $d\omega_\pm(X_++X_-,Y_++Y_-,Z_+)$. Since $j_+j_- = j_-j_+$, applying the above lemma, $\omega_+(X,Y)+\omega_-(X,Y)=0$ for any $X,Y\in T_+^{1,0}+T_-^{1,0}$, we obtain that

$$d\omega_+(X_++X_-,Y_++Y_-,Z_+) = -d\omega_-(X_++X_-,Y_++Y_-,Z_+). \tag{14}$$

Similarly we have

$$d\omega_+(X_++X_-,Y_++Y_-,Z_-) = -d\omega_-(X_++X_-,Y_++Y_-,Z_-) \tag{15}$$

Use Lemma 3.9 again to reduce the integrability conditions in Theorem 3.7 to:

$$d\omega_+(X_++X_-,Y_++Y_-,Z_+) = -i(H+db)(X_++X_-,Y_++Y_-,Z_+)$$

$$d\omega_-(X_++X_-,Y_++Y_-,Z_-) = i(H+db)(X_++X_-,Y_++Y_-,Z_-)$$

Hence we get that

$$d\omega_+(X_++X_-,Y_++Y_-,Z_++Z_-) = -d\omega_-(X_++X_-,Y_++Y_-,Z_++Z_-)$$
$$= -i(H+db)(X_++X_-,Y_++Y_-,Z_++Z_-).$$

That is,

$$d\omega_+|_{T_+^{1,0}+T_-^{1,0}} = -d\omega_-|_{T_+^{1,0}+T_-^{1,0}} = -i(H+db)|_{T_+^{1,0}+T_-^{1,0}}.$$

Conversely, if $T_+^{1,0}+T_-^{1,0}$ is involutive, the above equations and Lemma 3.9 give (1), (2) in Theorem 3.7, which imply that $\mathcal{J}$ is H-integrable. $\square$

Remark 3: When $j_+ = j_-$, the above conditions indicate $T_+^{1,0}$ is integrable; and when $j_+ = -j_-$, the conditions indicate that $d\omega_+ = d\omega_- = 0$. In these special cases we see that the generalized complex structures are b-transforms of complex or symplectic structures.

Chapter 4

Generalized complex structures with constant type 0 or 2 on 4-manifolds

Generalized complex structures on 4-manifolds have types 0, 1 ,2. As the types keep the same parity within a connected component, on connected 4-manifolds they are divided into four cases:

- Constant type 0. By Proposition 2.8, these structures are B-field transform of symplectic structures.
- Constant type 1.
- Constant type 2. By Proposition 2.8, there structures are B-field transform of complex structures.
- Mixed types of 0 and 2.

Therefore the existence of generalized complex structure with constant type 0 or 2 on 4-manifolds is equivalent to the existence of symplectic or complex structures on

4-manifolds. Recall 4-manifolds with complex structures are called complex surfaces, which was studied by algebraic geometers in as early as late 19th century. With the extensive work of Kodaira, the theory of compact complex surfaces are well-developed nowadays. On the other side, the existence of symplectic structures on 4-manifolds is still one of the important problems in symplectic topology, where remarkable and fascinate results were proved. In this chapter, we give a brief review of the known results about complex surfaces and symplecit 4-manifolds. We expect that some of the results have generalization to the generalized complex geometry for our purpose.

In section 1, we discuss minimal model of compact complex surfaces and state the Enriques-Kodaira classification of minimal complex surfaces in terms of Kodaira dimension. With the classification, we discuss existence of complex structure on three particular cases: $S^1 \times N^3$, surface bundle over surface, $n\mathbb{CP}^2 = \mathbb{CP}^2 \# \mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2$.

In section 2, we discribe the minimal model of symplectic 4-manifolds. We give Tian-Jun Li's classification of symplectic 4-manifolds using the symplectic Kodaira dimension. Then we discuss the existence of symplectic structures on $S^1 \times N^3$, surface bundle over surface similarly. To illustrate the nonexistence on $n\mathbb{CP}^2$, we briefly introduce results of Seiberg-Witten theory on 4-manifolds. Finally, we state Akhmedov's results on existence of symplectic structures on some exotic 4-manifolds.

4.1 Compact complex surfaces

The classification of compact complex surfaces is based on the minimal models of complex surfaces. A complex surface M is said to be minimal if it contains no rational curves C (a complex submanifold biholomorphic to $\mathbb{CP}^1$) with $C^2 = -1$. If M is not minimal, then Grauert's criterion ensures that there is a surjective bimeromorphism (blow down) of M to a complex surface M' which contracts one of the exceptional curves. The process can be repeated on M' and after finite times, the resulting surface will be minimal which is called minimal model of M. Except for certain rational or geometrically ruled surfaces, the minimal model for a complex surface is unique.

Now for a complex surface, denote K its canonical line bundle. Let $P_n(M) = dim_{\mathbb{C}} H^0(M, K^{\otimes n})$ be the plurigenera. It is easy to find that $P_n(M) = P_n M'$ if M' is the blow down of M.

Definition 4.1. The Kodaira dimension κ of M is defined in the following way:

1. $\kappa = -\infty$ if $P_n(M) = 0$ for all n.

2. $\kappa = 0$ if some $P_n(M)$ is nonzero and $\{P_n(M)\}$ is a bounded sequence.

3. $\kappa = 1$ if $\{P_n(M)\}$ is unbounded but $\{P_n(M)/n\}$ is bounded.

4. $\kappa = 2$ if $\{P_n(M)/n\}$ is unbounded.

Using the notion of Kodaira dimension, the Enriques-Kodaira classification of minimal complex surfaces lists:

Theorem 4.2. *Let M be a minimal compact complex surface,*
1. If $\kappa = -\infty$, then M is either geometrically ruled, or surface of class VII with $b_1 = 1$ or biholomorhpic to $\mathbb{CP}^2$.
2. If $\kappa = 0$, then M is one of the following: complex torus, K3 surface, hyperelliptic surface, Enriques surface, Kodaira surface.
3. If $\kappa = 1$, then M is a properly elliptic surface.
4. If $\kappa = 2$, then M is called a surface of general type.

The Kodaira dimension is actually a differential invariant. This is based on the stronger version-Van De Ven's conjecture which is now a theorem [5].

Theorem 4.3. *For compact complex surfaces M and N, if M is diffeomorphic to N, then $P_n(M) = P_n(N)$.*

The proof of the theorem is nontrivial which use the classifications as well as the Seiberg-Witten theory.

Now having the classifications, we can discuss the existence of complex structures on particular 4-manifolds. We restrict to three cases: product of S^1 and three manifolds, surface fiber bundles over surfaces, and $\mathbb{CP}^2\#\mathbb{CP}^2\cdots\#\mathbb{CP}^2$.

Proposition 4.4. *Let $M = S^1 \times N^3$ be a compact smooth 4-manifold, where N^3 is an oriented closed 3-manifold. If N^3 is Seifert fibered, then there is complex structures on M. Conversely, if there is a complex structure j on M, then (M, j) is either a Hopf surface, or a ruled surface diffeomorphic to $S^2 \times T^2$, or elliptic surfaces which have no singular fibers.*

Proof. It is a classical theorem that $S^1 \times N^3$ is an elliptic surface if N^3 is Seifert fibered, for example see the proof in [44]. Conversely, assuming M is a complex surface, as any compact oriented 3-manifold is spin, M is spin with $W_2(M) = 0$. So the intersection form is even and M is minimal. $\chi(M) = \chi(S^1) \times \chi(N^3) = 0$. Minimal surface of general type has $\chi > 0$, so $\kappa(M) \leq 1$. From the classification, if $\kappa(M) = -\infty$, M is either class VII with $b_2 = 0$ or geometrically ruled. Minimal VII surface with $b_2 = 0$ are classified as either Hopf surface or Inoue surface. It's showed in [23] that Inoue surface can not be a product. With the fact that the only ruled spin minimal surface of $\chi = 0$ is $S^2 \times T^2$, M is either a Hopf surface or diffeomorphic to $S^2 \times T^2$. If $\kappa(M) = 0, 1$, M has elliptic fibration over Riemann surface. The vanishing of $\chi(M)$ excludes any singular fibers due to the formula of characteristic. Yet there may be multiple fibers of M. □

Next, we study the complex structure on fiber bundles. We have:

Proposition 4.5. *Any smooth 4-manifolds diffeomorphic to an oriented S^2 bundle over a compact surface have complex structures which are geometrically ruled.*

Proof. Any oriented S^2 bundle has transition groups in $Diff^+(S^2)$. As $Diff^+(S^2)$

is homotopy equivalent to $SO(3)$ by Smale's theorem, we can reduce the transition group into $SO(3)$ by the classical fiber bundle theory. Therefore any oriented S^2 bundle is the unit sphere bundle of an orientable $\mathbb{R}^3$ vector bundle. $\mathbb{R}^3$-bundle over surface Σ are classified by the second Whitney class of Σ. So there are only two isomorphic classes of oriented $\mathbb{R}^3$ vector bundles hence S^2 bundles over a compact surface. Actually the diffeomorphism type of the total space are also distinct. One is the trivial bundle $S^2 \times \Sigma$, another is the twisted product denoted by $S^2 \hat{\times} \Sigma$ distinguished by the fact that it is not spin.

There is automatically complex structure on the product. For the twisted product, consider the holomorphic fiber bundle of $\mathbb{P}(L \oplus \mathbb{C}) \to \Sigma$, where L is a holomorphic line bundle over Σ with $c_1(L) \neq 0$. $\square$

For fiber bundle with fiber genus $g \geq 1$, we do not know the explicit classification of bundle structures, therefore there is not an existence result in general. If the fibration is holomorphic, there is a canonical complex structure on the total space. However, holomorphic fibrations are rather restricted.

Last, we discuss the well known nonexistence result on connected sums of $\mathbb{CP}^2$. Denote $n\mathbb{CP}^2$ the connected sum of n copies of $\mathbb{CP}^2$. For the almost complex structure, by the criterion of Wu and Hirzebruch-Hopf, we have:

Proposition 4.6. *There is almost complex structures on $n\mathbb{CP}^2$ if and only if $n = 2k+1$, $k \geq 0$.*

Actually for simply connected smooth 4-manifolds, there exist almost complex structure if and only if b^+ is odd [27].

Recall the Bogomolov-Miyaoka-Yau inequality says that: for compact complex surface of general type, $C_1^2 \leq 3C_2$. With this we have:

Proposition 4.7. *There is no complex structure on $(2k+1)\mathbb{CP}^2$ for $k \geq 1$.*

Proof. If there is complex structure on $(2k+1)\mathbb{CP}^2$ for $k \geq 1$. Then we have $C_1^2 = 2\chi + 3\sigma = 10k - 1$, and $C_2 = 2k + 1$. As $C_1^2 > 0$, the complex structure is projective. By Noether's formula, we have $p_g = k > 0$. So due to the Enriques-Kodaira classification, the complex surface will be surface of general type. But this contradicts with Bogomolov-Miyaoka-Yau inequality when $k > 1$.

4.2 Symplectic 4-manifolds

The classification of compact symplectic 4-manifolds is far from complete compared to the compact complex surface. However, some concepts and results about complex surfaces have analogues on symplectic 4-manifolds. There is a coarse classification in terms of the symplectic Kodaira dimension defined by Tian-Jun Li.

A symplectic 4-manifold (M, ω) is said to be symplectically minimal if there is no embedded symplectic sphere with self-intersection -1. By the work in [37], symplectically minimality is equivalent to smoothly minimality which means there is no embedded sphere with square -1. So we are safe to say (M, ω) is minimal. If (M, ω) is not minimal, one can blow down the -1 spheres similarly to obtain a symplectic minimal model. If (M, ω) is not topologically rational or ruled, namely, M is not diffeomorphic to connected sums of $\overline{\mathbb{CP}}^2$ with $\mathbb{CP}^2$ or S^2 bundles, then there is a unique symplectic minimal model up to diffeomorphism [35]. For a symplectic 4-manifold, Li defines the following.

Definition 4.8. The symplectic Kodaira dimension κ^s of a minimal symplectic 4-manifold (M, ω) is defined in the following way:

1. $\kappa^s = -\infty$ if $K_\omega \cdot [\omega] < 0$ or $K_\omega \cdot K_\omega < 0$

2. $\kappa^s = 0$ if $K_\omega \cdot [\omega] = 0$ and $K_\omega \cdot K_\omega = 0$

3. $\kappa^s = 1$ if $K_\omega \cdot [\omega] > 0$ and $K_\omega \cdot K_\omega = 0$

4. $\kappa^s = 2$ if $K_\omega \cdot [\omega] > 0$ or $K_\omega \cdot K_\omega > 0$

For a general symplectic 4-manifold, the symplectic Kodaira dimension is defined to be symplectic Kodaira dimension of one of its minimal models.

A basic theorem in [35] says (also see [16]):

Theorem 4.9. *The symplectic Kodaira dimension κ^s is well defined. It only depends on the smooth structure of M and coincides with the holomorphic Kodaira dimension if M have complex structure.*

In terms of the symplectic Kodaira dimension, Li proves:

Theorem 4.10. *Let (M, ω) be a minimal symplectic 4-manifold,*
1. $\kappa^s = -\infty$ if and only if M is diffeomorphic to $\mathbb{CP}^2$ or a S^2-bundle.
2. $\kappa^s = 0$ if and only if K_ω is a torsion class. The homology of M must be isomorphic to the following: K3 surface, hyperelliptic surface, Enriques surface, T^2-bundle over T^2.
3. $\kappa^s = 2$ is called a symplectic 4-manifold of general type.

Beside the symplectic Kodaira dimension, there are other characterizations of symplectic 4-manifolds. Recall a Lefschetz pencil on a compact 4-manifold X consists of a nonempty finite set $B \subset X$ and a smooth map $\pi : X - B \to \mathbb{CP}^1$ such that around each point of B, π is the projectivization $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$ and near each critical point there is coordinate (z_1, z_2) such that $\pi(z_1, z_2) = z_1 z_2$. Lefschetz pencils can be always blowed up to eliminate the base locus and become a Lefschetz fibration. Based on Donaldson's work on Lefschetz fibration, Gompf got the following.

Theorem 4.11. *(Gompf) A closed 4-manifold has symplectic structure if and only if it has Lefschetz pencils.*

With the above results, Auroux obtain another characterization:

Theorem 4.12. *(Auroux) A closed 4-manifold has symplectic structures if and only if it is a symplectic branch cover over $\mathbb{CP}^2$.*

Now we divide our discussion of existence of symplectic structure explicitly into three cases: $S^1 \times N^3$, fiber bundles and connected sum of $\mathbb{CP}^2$.

Theorem 4.13. *(Friedl-Vidussi) Let $M = S^1 \times N^3$ be a compact smooth 4-manifold, where N^3 is an oriented closed 3-manifold. Then M has symplectic structure if and only if N^3 is a fiber bundle over S^1.*

The proof uses a good combination of the twisted Alexander polynomial of 3-manifold and Gauge theory on 4-manifolds which we refer to [21].

Next, we review the results of symplectic structure on fiber bundles.,

Theorem 4.14. *Let M be the total space of an oriented Σ_h fiber bundle over Σ_g. Then M has symplectic structure if and only if it is in the following case:*

1. $h \neq 1$

2. $h = 1, g = 0$ *and the bundle structure is trivial.*

3. $h = 1, g = 1$

4. $h = 1, g > 1$, *and also* $[F] \neq 0$ *in* $H_2(M, \mathbb{R})$

Proof. By Thurston's results on symplectic fibration, there are symplectic structures on symplectic fiber bundles if the fiber is homologically nontrivial in $H_2(M, \mathbb{R})$ which is true if $h \neq 1$. As any surface bundle over surface can be deformed to be s symplectic fiber bundle, case 1 follows. When $h = 1$, the situation depends on the genus of the base space. If $g = 0$ and M is a nontrivial T^2-bundle over S^2, then $b_1(M) = 1$, $b^2(M) = 0$ and M does not admit symplectic structures. For torus bundle with $g = 1$, if the fiber is not homologically essential, it is argued in [24] that M is also a principal S^1-bundle over a principal S^1-bundle over T^2 and has symplectic structures. If $g > 1$, Rafal Walczak [43] proved that the fiber bundle admits symplectic structures only if the fiber is homologically essential. Together with Thurston's result, this proves case 4. $\square$

To state the nonexistence results of symplectic structure, we introduce the Seiberg-Witten theory on 4-manifold briefly. Recall that the $Spin^c$ group is the group generated by $Spin$ and S^1 in the complexified Clifford algebra $CL \otimes \mathbb{C}$. A $Spin^c$ structure on a manifold is a lifting of the frame bundle to a principle $Spin^c$ bundle. There always exist $Spin^c$ structures on an oriented 4-manifold. The set of $Spin^c$ structure SP is an affine space modelled on $H^2(M, \mathbb{Z})$. The Seiberg-Witten invariant for a smooth 4-manifold with $b^+ > 1$ is a defined to a map $SW : SP \to \mathbb{Z}$. If $b^+ = 1$, fixing a connected component of set of real second cohomology classes with positive square (forward cone), two Seiberg-Witten invariants are defined, which are: $SW_+ : SP \to \mathbb{Z}$ and $SW_- : SP \to \mathbb{Z}$. For the Seiberg-Witten invariants, we have the following well known results.

Theorem 4.15. *1. The Seiberg-Witten invariant is a diffeomorphism invariant of smooth 4-manifold.*
2. If $b^+(M) > 1$ and M admits a metric with positive scaler curvature, then $SW_M = 0$.
3. If $M = M_1 \# M_2$ with $b^+(M_1) > 1$ and $b^+(M_2) > 1$, then $SW_M = 0$.
4. Let (M, ω) be a symplectic 4-manifold. The canonical class K of ω determines a unique $Spin^c$ structure denoting σ_{can}. If $b^+ = 1$, choose the component which contains $[\omega]$ as the forward cone. Then $SW(\sigma_{can}) = 1$ if $b^+ > 1$, or $SW_-(\sigma_{can}) = 1$ if $b^+ = 1$.
5. Let (M, ω) be a symplectic 4-manifold with $b^+ = 1$. Then there are the following relations:

$$SW_+(\sigma_{can} \otimes e) = (-1)^{(2-b_1)/2} SW_-(\sigma_{can} \otimes (K - e)),$$

If M is simply connected,

$$SW_-(\sigma_{can} \otimes (K - e)) - (-1)^{(2-b_1)/2} SW_-(\sigma_{can} \otimes e) = 1$$

Using these results, the nonexistence of symplectic structure on $(2k+1)\mathbb{CP}^2$ can be proved.

Theorem 4.16. *There is no sympletic structure on $(2k+1)\mathbb{CP}^2$ for $k \geq 1$.*

Proof. If there exist symplectic structures, then by Theorem 4.15, $SW(\sigma_{can}) = 1$. But as $\mathbb{CP}^2$ admits metrics with positive scaler curvature, and the connected sum of two manifold admits positive scaler curvature metric if they both admit [28]. The existence of symplectic structure contradict with 2 in Theorem 4.15. □

On the other side, using delicate and powerful topological constructions, Akhmedov [1] [2] proves the following:

Theorem 4.17. *There exist symplectic structure on some smooth four manfolds which are homeomorphic to* $3\mathbb{CP}^2\#k(\overline{\mathbb{CP}^2})$ *for* $k = 4, 5, 6, 8, 10$ *or* $(2n-1)\mathbb{CP}^2\#2n(\overline{\mathbb{CP}^2})$ *for* $n > 3$.

It follows from the nonexistence theorem and the blow up operation that the smooth structures on these manifolds are exotic.

Chapter 5

Generalized complex structures with type one on 4-manifolds

With respect to the types, the situations for existence of generalized complex structures on 4-manifolds are very different. For complex structures, there is not a general criterion but the classification theory is usually used to study the existence. For symplectic structures, Gompf proves that a compact 4-manifold has a symplectic structure if and only if it has Lefschetz pencils. In this chapter, we establish a similar result for generalized complex structures with type one on 4-manifolds. First, by projecting the maximal isotropic bundles to the complexied tangent bundle, we find regular 2-foliations with transversely holomorphic structure on every type one generalized complex 4-manifold. With the existence of Gauduchon metric in the conformal class of almost Hermitian metrics on almost complex manifolds and the integrability conditions in Chapter 3, we prove that this is sufficient: a compact orientable 4-manifold has type one generalized complex structure if and only if it admits transversely holomorphic 2-foliations.

In section 1, we study generalized almost complex structure with type one on 4-manifolds. It turns out that the existence is a purely topological requirement in terms of the intersection form and Euler characteristic. This is based on Matsushita's criterion for the existence of 2-distributions on 4-manifolds.

In section 2, we give two definitions of transversely holomorphic foliation and prove that they are equivalent. For a generalized complex structures with type one, we construct a transversely holomorphic 2-foliation.

In section 3, we introduce the Gauduchon metric on almost complex structure and show its existence in each conformal class of almost Hermitian metric. Combing it with Corollary 3.10, we give the proof of our main theorem. We apply the theorem to study the existence of type one generalized complex structure on $S^1 \times N^3$, surface bundle over surface. We also give examples on $\mathbb{T}^4$ and nilmanifolds and study their deformations.

5.1 Generalized almost complex structure with type one on 4-manifolds

With generalized almost complex structure expressed in terms of almost bihermitian structures, the type requirement transforms to conditions for the almost bihermtian structure.

Lemma 5.1. *Given a generalized almost complex structure $\mathcal{J}$ on a smooth 4-manifold M^4, let (g, j_+, j_-) be an associated almost bihermitian structure. Then $\mathcal{J}$ has type one if and only if j_+, j_- induce different orientations. Also, we have $j_+j_- = j_-j_+$ in this case.*

Proof. We follow the argument in [29]. Let $T_+^{1,0}$ and $T_-^{1,0}$ be the i-eigenbundles of j_+, j_-. By the computations in Theorem 3.7, $T_+^{1,0} + T_-^{1,0} = \pi(L)$ where L is i-eigenbundle of $\mathcal{J}$ and $\pi : T \oplus T^* \to T$ is the first projection. As type of $\mathcal{J}$ is equal to $2 - \frac{dim_{\mathbb{R}}\pi(\mathcal{J}T^*)}{2} = \frac{dim_{\mathbb{C}}\pi(L)\cap\pi(\bar{L})}{2}$. For type one $\mathcal{J}$, $T_+^{1,0} + T_-^{1,0}$ is a 3-dimensional complex vector bundle. So $T_+^{1,0} \cap T_-^{1,0} = B$ is a complex line bundle. Let $K = B \oplus \bar{B}$ and $L = K^\perp$ its g-orthogonal complement. We have $j_+|_K = j_-|_K$, $j_\pm(L) \subset L$. Since there are only two orthogonal complex structures on L, $j_+|_L = -j_-|_L$ because $j_+ \neq j_-$. Therefore j_+, j_- induce different orientations and also $j_+j_- = j_-j_+$.

Conversely, assume j_+, j_- induce different orientations. Let ω_+, ω_- be their fundamental forms. Denote $*$ to be the Hodge operator with respect to g and orientation of j_+, then $*\omega_+ = \omega_+$ and $*\omega_- = -\omega_-$. As any anti-self-dual 2-form is invariant under actions of j_+, we have $\omega_-(j_+X, j_+Y) = \omega(X, Y)$. Therefore $j_+j_- = j_-j_+$. Let $K' = ker(j_+ - j_-) \in (T \oplus T^*) \otimes \mathbb{C}$, as $(j_+ - j_-)(j_+ + j_-) = 0$, by the opposite orientations of j_+, j_-, we have $dim_{\mathbb{C}} K' = 2$ and $dim_{\mathbb{C}}(T_+^{1,0} + T_-^{1,0}) = 3$. So type of $\mathcal{J}$ is one. □

Having this, we get the topological conditions of existence of type one generalized almost complex structures.

Proposition 5.2. *A 4-manifold admits type one generalized almost complex structure if and only if it has an orientable 2-distribution.*

Proof. By the above lemma, if (g, j_+, j_-) is an associated almost bihermitian structure, then j_+, j_- induce different orientations and $j_+j_- = j_-j_+$. Therefore $ker(j_+ \pm j_-)$ will be even dimensional as they are invariant under $j_\pm$. Since j_+, j_- induce different orientations, $ker(j_+ \pm j_-)$ each gives an orientable 2-distribution. On the other side, if E is an orientable 2-distribution, choose a Riemannian metric g such that $T = E \oplus N$, where N is orthogonal to E. Denote ρ_E to be the rotation of E by $+\pi/2$, and similarly ρ_N be the rotation of N by $+\pi/2$. Define $j_+ = \rho_E + \rho_N, j_- = \rho_E - \rho_N$, then (g, j_+, j_-) is an almost bihermitian structure with j_+, j_- opposite orentations. By Lemma 5.1 it induces almost generalized complex structure of type one. □

In [39], the topological requirements for the existence of orientable 2-distribution on a compact oriented 4-manifold are given explicitly. For the reader's convenience, we state it here: let σ and χ be the signature and Euler characteristic of the manifold, then it admits an orientable 2-distribution if and only if: (1) if the intersection form is indefinite, then $\sigma + \chi \equiv 0 \ mod\ 4$, and $\sigma - \chi \equiv 0 \ mod\ 4$; (2) if the intersection form is definite, then $\sigma + \chi \equiv 0 \ mod\ 4$, $\sigma - \chi \equiv 0 \ mod\ 4$ and $|\sigma| + \chi \geq 0$. Using the criterions, we can get:

Corollary 5.3. *There is no type one generalized almost complex structure on $n\mathbb{CP}^2$, $n \geq 1$.*

5.2 Transversely holomorphic foliations

We start with the standard definition of C^∞ foliations.

Definition 5.4. A C^∞ codimension p foliation $\mathcal{F}$ on a smooth manifold M^n is given by an open covering $\{U_i\}_{i\in I}$ of M together with homeomorphisms $f_i : U_i \to \mathbb{R}^n$ such that: if $U_i \cap U_j \neq \emptyset$, the transition functions $f_{ij} = f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ are smooth and given by: $f_{ij}(x, y) = (f_{ij}^1(x, y), f_{ij}^2(y))$, where $(x, y) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$, and f_{ij}^2 are diffeomorphisms between open sets in $\mathbb{R}^p$.

The subsets $f_i^{-1}(\mathbb{R}^{n-p} \times \{c\})$ in U_i will glue together to give connected immersed submanifolds in M, which are called the leaves of M. Use F to denote the tangent distribution of $\mathcal{F}$, whose fiber at each point x is the tangent space to the leaf through x. A foliation $\mathcal{F}$ is called **orientable** if F is an orientable vector bundle. From now on we assume that all the foliations in the rest context are orientable without notification.

Definition 5.5. A transversely holomorphic foliation with complex codimension m is a foliation such that $p = 2m$ and f_{ij}^2 are biholomorphisms between open sets in $\mathbb{C}^m$.

Like the complex structure and almost complex structure, we can characterize the transversely holomorphic foliation in terms of transversely almost complex structure.

Let $N = T/F$ be the normal bundle of a foliation $\mathcal{F}$.

Definition 5.6. A transversely almost complex structure of a foliation $\mathcal{F}$ is an endomorphism

$$j : N \to N$$

with $j^2 = -id$. (F, j) is said to be integrable if $[\phi^{-1}(N^{0,1}), \phi^{-1}(N^{0,1})] \subset \phi^{-1}(N^{0,1})$, where $\phi : T \otimes \mathbb{C} \to N \otimes \mathbb{C}$ is the projection and $N^{0,1} \subset N \otimes \mathbb{C}$ is the $-i$-eigenbundle of j.

Proposition 5.7. *[18] There is one to one correspondence between transversely holomorphic foliations and foliations with integrable transversely almost complex structure.*

Proof: Use the notations as above. For one direction, given a transversely holomorphic foliation $\mathcal{F}$, write the diffemorphisms f_i in Definition 5.5 to be (f_i^1, f_i^2), where $f_i^2 : U_i \longrightarrow \mathbb{C}^m$ are submersions. Then $N|_{U_i}$ can be identified with $f_i^{2*}T\mathbb{C}^m$. If the transition functions f_{ij}^2 are holomorphic, then the almost complex structure on $N|_{U_i}$ induced by the holomorphic structure of $T\mathbb{C}^m$ does not depend on U_i and match together to give a transversely almost complex structure. As $\phi^{-1}(N^{0,1}) = f_{i*}^{-1}(T\mathbb{R}^{n-2m} \otimes \mathbb{C} \times T^{0,1}\mathbb{C}^m)$ is involutive in each U_i, the transversely almost complex structure is integrable.

For the other direction, if $\phi^{-1}(N^{0,1})$ is involutive, since $\phi^{-1}(N^{0,1}) + \overline{\phi^{-1}(N^{0,1})}$ $= T \otimes \mathbb{C}$, the existence of transversely holomorphic structure follows from the complex Frobenious theorem [40]. $\square$

Now we can relate transversely holomorphic foliations to generalized complex structures.

Proposition 5.8. *A generalized complex structure with type one on 4-manifolds uniquely determines a transversely holomorphic 2-foliations structure.*

Proof. If $\mathcal{J}$ is a type one generalized complex structure on M^4, let (g, j_+, j_-) be an associated almost bihermitian structure. From Lemma 5.1, j_+, j_- induce different orientations and j_+, j_- commute. Therefore $ker(j_+ \pm j_-)$ are 2-distributions and we have the decompositions $T_+^{1,0} = A \oplus B$, $T_-^{1,0} = \bar{A} \oplus B$, where:

$$A = T_+^{1,0} \cap T_-^{0,1}, \qquad B = T_+^{1,0} \cap T_-^{1,0}$$

are complex line bundles. From Corollary 3.10, $T_+^{1,0} + T_-^{1,0} = A \oplus \bar{A} \oplus B$ is involutive. Taking conjugate gives that $A \oplus \bar{A} \oplus \overline{B}$ is also involutive. Therefore their intersection

$A \oplus \bar{A} = F \otimes \mathbb{C}$ is involutive, where $F = ker(j_+ + j_-)$ is the 2-distribution. Moreover, let $N = T/F$, then the projection $\pi : T \otimes \mathbb{C} \to N \otimes \mathbb{C}$ induces an isomorphism between $B \oplus \bar{B}$ and $N \otimes \mathbb{C}$. Let $N^{0,1} = \pi(B)$, then it defines a transversely almost complex structure $j : N \to N$. As

$$\phi^{-1}(N^{0,1}) = A \oplus \bar{A} \oplus B = T_+^{1,0} + T_-^{1,0}$$

is involutive, j is integrable. Thus it gives a transversely holomorphic foliation by Proposition 5.7. Moreover, since $\pi^{-1}(N^{0,1}) = T_+^{1,0} + T_-^{1,0}$ is just the projection of the $+i$ eigenbundle of $\mathcal{J}$ to $T \otimes \mathbb{C}$, the transversely holomorphic 2-foliation is uniquely determined by $\mathcal{J}$.

5.3 Integrability

To construct integrable type one generalized complex structure, we first introduce the Gauduchon metric on almost complex manifolds.

Definition 5.9. Given an almost complex manifold (M^{2n}, j), let $T^{*(p,q)} = \wedge^p T^{*(1,0)} \otimes \wedge^q T^{*(0,1)}$ be the bundle with sections (p, q) forms. Define $\overline{\partial} = \pi_{p,q+1} \circ d : T^{*(p,q)} \to T^{*(p,q+1)}$, $\partial = \pi_{p+1,q} \circ d : T^{*(p,q)} \to T^{*(p+1,q)}$ to be the formal differential operators. Then an almost Hermitian metric g is said to be a Gauduchon metric if $\partial\overline{\partial}\omega^{n-1} = 0$ where $\omega = gj$ is the fundamental form.

The existence of Gauduchon metric is well known on compact complex manifolds in any conformal class of Hermitian metrics [22]. The proof actually does not require integrability of almost complex structures and the existence is also true for almost complex manifolds. We thank Tedi Draghici for providing us the following simple proof.

Proposition 5.10. *In any conformal class of almost Hermitian metrics on a compact connected almost complex manifold, there exists a unique Gauduchon metric up to scalars.*

Proof. Given a Hermitian metric g, let ω be the fundamental $(1,1)$-form. Then ω^{n-1} is a $(n-1,n-1)$-form, and $d\omega^{n-1} = (\partial + \overline{\partial})\omega^{n-1}$, $\partial\overline{\partial}\omega^{n-1} = -\overline{\partial}\partial\omega^{n-1}$. So we have

$$\partial\overline{\partial}\omega^{n-1} = -id^c d\omega^{n-1} = idd^c\omega^{n-1}$$

where $d^c = i(\overline{\partial} - \partial)$. Let $*$ be the Hodge star operator associated to g, then $*j*d\omega^{n-1} = -i(\overline{\partial} - \partial)\omega^{n-1} = -d^c\omega^{n-1}$. As $*\omega = \frac{\omega^{n-1}}{n-1}$, $\partial\overline{\partial}\omega^{n-1} = 0$ if and only if $dd^c\omega^{n-1} = 0$, if and only if $d*j*d\omega^{n-1} = 0$. Composing the above equation with $*$, we get g is Gauduchon if and only if $\delta j \delta \omega = 0$, where $\delta = -*d*$ is the codifferential.

On the other side, as $\omega^{n-1}\wedge : \Omega^1(M) \to \Omega^{2n-1}(M)$ is bijective, there is a 1-form θ such that $d\omega^{n-1} = \theta \wedge \omega^{n-1}$. We call it the Lee form of g. By direct calculations in terms of $(1,0),(0,1)$ components, we have $\theta = j\delta\omega$. With above arguments we get g is Gauduchon is and only if θ is co-closed. Now a conformal change of g is $\bar{g} = e^f g$ for some real function f. As $d(e^f\omega^{n-1}) = (df + \theta) \wedge e^f\omega^{n-1}$, the Lee form $\bar{\theta}$ of $\bar{g}$ is $\theta + df$. Assume $\theta = \theta_h + d\alpha + \delta\beta$ is the Hodge decomposition of θ, where θ_h is the harmonic part. Let $f = -\alpha$, we have $\delta\bar{\theta} = \delta(\theta + df) = \delta(\theta_h + \delta\beta) = 0$. The uniqueness of f up to scaler gives the uniqueness of the metric. □

Now we are ready to prove the following existence theorem.

Theorem 5.11. *A compact orientable 4-manifold admits type one generalized complex structures if and only if it has the structure of transversely holomorphic 2-foliations.*

Proof: We only need to prove the existence. Given a transversely holomorphic 2-foliation (F, j) on a compact 4-manifold M, we want to construct an almost bihermitian structure (g, j_+, j_-) such that j_+, j_- induce different orientations and the conditions in Corollary 2.6 are satisfied. First embed the bundle $N = T/F$ into T to get a splitting $T = N \oplus F$. Choose a Riemannian metric g on M such that g is compatible with j on N and N is orthogonal to F. Define two almost complex structures on M by $j_+ = -j$ on N and j_+ to be the rotation by $\pi/2$ on F and

$$j_- = -j_+|_F + j_+|_N.$$

Then (g, j_+, j_-) is an almost bihermitian structure on M^4. By definition, j_+, j_- induce different orientations and $j_+j_- = j_-j_+$. Also we have $T_+^{1,0} + T_-^{1,0} = N^{0,1} \oplus F \otimes \mathbb{C}$. The integrability of j ensures that $T_+^{1,0} + T_-^{1,0}$ is involutive. By Proposition 5.10, we can assume g is a Gauduchon metric with respect to j_+ (otherwise, replace it with the Gauduchon metric in its conformal class and it's still compatible with j_+, j_-). Now by Corollary 3.10, (g, j_+, j_-) gives an H-integrable generalized complex structure if

$$d\omega_+|_{T_+^{1,0}+T_-^{1,0}} = -iH|_{T_+^{1,0}+T_-^{1,0}} = -d\omega_-|_{T_+^{1,0}+T_-^{1,0}}.$$

As $T_+^{1,0} + T_-^{1,0}$ is involutive and $j_+j_- = j_-j_+$, from the equalities (14) and (15) in Corollary 3.10, we know that $d\omega_+|_{T_+^{1,0}+T_-^{1,0}} = -d\omega_-|_{T_+^{1,0}+T_-^{1,0}}$. Then the theorem follows from the next lemma. □

Lemma 5.12. *Given (g, j_+, j_-) an almost bihermitian structure on a four manifold M^4 with j_+, j_- inducing opposite orientations , use the notations as above. If $\partial_+\bar{\partial}_+\omega_+ = 0$, then there exists a closed real three form H such that*

$$d\omega_+|_{T_+^{1,0}+T_-^{1,0}} = -iH|_{T_+^{1,0}+T_-^{1,0}}.$$

Proof. Use the same notions as above. Let $A = T_+^{1,0} \cap T_-^{0,1}$ and $B = T_+^{1,0} \cap T_-^{1,0}$ be complex line bundles. As $T \otimes \mathbb{C} = F \otimes \mathbb{C} \oplus B \oplus \bar{B}$, we have the splitting $\wedge^3 T^* \otimes \mathbb{C} = \oplus_{k+m+n=3} \wedge^k \mathbb{F}^* \otimes \wedge^m B^* \otimes \wedge^n \bar{B}^*$, with $\mathbb{F} = F \otimes \mathbb{C}$. For any three form $\phi \in \Omega^3(M, \mathbb{C})$, we denote $\phi^{k,m,n}$ to be its components corresponding to the splitting, where $k \leq 2, m, n \leq 1$ because of dimension reason. Now to find a close real three form H such that $d\omega_+|_{T_+^{1,0}+T_-^{1,0}} = -iH|_{T_+^{1,0}+T_-^{1,0}}$, write $H = H^{2,1,0} + H^{2,0,1} + H^{1,1,1}$. Then the equality in the lemma is equivalent to

$H^{2,1,0} = i(d\omega_+)^{2,1,0}$, since $T_+^{1,0} + T_-^{1,0} = F \otimes \mathbb{C} \oplus B$. As H is real, we get that

$$H^{2,0,1} = \overline{H^{2,1,0}} = -i(d\omega_+)^{2,0,1}.$$

So only $H^{1,1,1}$ is not determined, and we need to find a three form $H^{1,1,1}$ such that $dH = 0$. Write

$$\bar{\partial}_+\omega_+ = (\bar{\partial}_+\omega_+)^{2,1,0} + (\bar{\partial}_+\omega_+)^{2,0,1} + (\bar{\partial}_+\omega_+)^{1,1,1}.$$

For any $X \in A \subset T_+^{1,0}, Y \in \bar{A} \subset T_+^{0,1}, Z \in B \subset T_+^{1,0}$, we have

$$(\bar{\partial}_+\omega_+)^{2,1,0}(X,Y,Z) = \bar{\partial}_+\omega_+(X,Y,Z) = 0$$

The second equality holds as $\bar{\partial}_+\omega_+$ is a $(1,2)$ form with respect to j_+. Thus $(\bar{\partial}_+\omega_+)^{2,1,0}$ must be 0 since it is a section of $\wedge^2(A^* \oplus \bar{A}^*) \otimes B^*$. Using the fact $d\omega_+ = \partial_+\omega_+ + \bar{\partial}_+\omega_+$ on a 4-manifold, we obtain that

$$(d\omega_+)^{2,1,0} = (\partial_+\omega_+)^{2,1,0} + (\bar{\partial}_+\omega_+)^{2,1,0} = (\partial_+\omega_+)^{2,1,0},$$

$$\text{thus} \quad (d\omega_+)^{2,0,1} = \overline{(d\omega_+)^{2,1,0}} = \overline{(\partial_+\omega_+)^{2,1,0}} = (\bar{\partial}_+\omega_+)^{2,0,1}. \tag{16}$$

As $\bar{\partial}_+\omega_+ = (\bar{\partial}_+\omega_+)^{2,0,1} + (\bar{\partial}_+\omega_+)^{1,1,1}$, $\partial_+\bar{\partial}_+\omega_+ = 0$ gives that

$$\partial_+(\bar{\partial}_+\omega_+)^{2,0,1} = -\partial_+(\bar{\partial}_+\omega_+)^{1,1,1}$$

Similarly $\bar{\partial}_+(\bar{\partial}_+\omega_+)^{2,0,1} = -\bar{\partial}_+(\bar{\partial}_+\omega_+)^{1,1,1}$ as $\bar{\partial}_+\bar{\partial}_+\omega_+ = 0$.

Note that $d\varphi = \partial\varphi + \bar{\partial}\varphi$ for any three form on a 4-manifold, we actually get

$$d(\bar{\partial}_+\omega_+)^{1,1,1} = -d(\bar{\partial}_+\omega_+)^{2,0,1}$$

Hence by (16),

$$d(\bar{\partial}_+\omega_+)^{1,1,1} = -d(d\omega_+)^{2,0,1}$$

Now that $dH = 0$ if and only if

$$\begin{aligned} dH^{1,1,1} &= -dH^{2,1,0} - dH^{2,0,1} \\ &= -id\overline{(d\omega_+)^{2,0,1}} + id(d\omega_+)^{2,0,1} \\ &= -2\mathrm{Im}d(d\omega_+)^{2,0,1} \\ &= 2\mathrm{Im}d(\bar{\partial}_+\omega_+)^{1,1,1}. \end{aligned}$$

where Im denotes the imaginary part of the form. Let $H^{1,1,1} = 2\mathrm{Im}(\bar{\partial}_+\omega_+)^{1,1,1}$, as $H^{2,0,1} + H^{2,1,0} = -i(d\omega_+)^{2,0,1} + i\overline{(d\omega_+)^{2,0,1}} = 2\mathrm{Im}(d\omega_+)^{2,0,1}$,

$$H = 2\mathrm{Im}(\bar{\partial}_+\omega_+)^{1,1,1} + 2\mathrm{Im}(d\omega_+)^{2,0,1}$$

is a closed real three form such that $d\omega_+|_{T_+^{1,0}+T_-^{1,0}} = -iH|_{T_+^{1,0}+T_-^{1,0}}$. $\square$

In [4], Bailey defines the obstruction forms for a regular Poisson structure with transverse complex structure. Our lemma lead to the statement that for a transversely holomorphic 2-foliation on a 4-manifold, the Gauduchon metric g induces a Poisson structure such that the obstruction forms vanish.

With the above theorem, we can construct interesting examples of type one generalized complex structures on 4-manifold. The following example is type one generalized complex structures with noncompact symplectic leaves such that they are everywhere dense.

Example 5.13. Consider the linear foliation on $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$. Let $X = (a, b, 0, 0)$, $Y = (0, 0, c, d)$ be two constant vector fields on $\mathbb{R}^4$, with $a/b, c/d$ irrational numbers, and E be the constant 2-plane field perpendicular to X, Y. Then X, Y passes to a 2-foliation $\mathcal{F}$ on T^4, and E passes to a two distribution $\tilde{E}$ on T^4 which is transversal to $\mathcal{F}$. Any linear transformation $L : E \to E$ with $L^2 = -id$ induces an integrable transversely almost complex structure of $\mathcal{F}$. The leaves of $\mathcal{F}$ are immersed $\mathbb{R}^2$ and are each everywhere dense in $\mathbb{T}^4$. By Theorem 5.11, there are generalized complex structures of type one associated to the linear foliation.

Next we can discuss the existence of type one generalized complex structures on particular 4-manifolds. We first consider the existence of type one generalized complex structure on surface bundle over surface. There is transversely holomorphic foliation structures on any surface bundle over a Riemann surface, where the foliation comes from the fibering and the holomorphic structure is pulled back from the base. As a consequence, we reprove the result in [4].

Corollary 5.14. *A compact orientable 4-manifold which is diffeomorphic to a surface bundle over Riemann surface has type one generalized complex structures.*

We obtain the complete topological characterization of generalized Calabi-Yau 4-manifolds.

Corollary 5.15. *A compact 4-manifold has generalized Calabi-Yau structure if and only if it is diffeomorphic to one of the following: a symplectic 4-manifold, $\mathbb{T}^4$, $K3$ surfaces, Kodaira surfaces, surface bundle over T^2.*

Proof. Generalized Calabi-Yau structure has nonvanishing section of canonical bundle, so the types keep the same on connected component. Type zero generalized Calabi-Yau structures are B-field transform of symplectic structures. Type two generalized Calabi-Yau structures are B-field transform of Calabi-Yau structure, which only exist on $\mathbb{T}^4$, $K3$ surfaces and Kodaira surfaces by the Enriques-Kodaira classification. For a type one generalized Calabi-Yau 4-manifold M, in [33] Hitchin proves that M must fiber over T^2. So we only need to show any surface bundle over T^2 has type one generalized Calabi-Yau structures.

By Corollary 5.14, there is type one generalized complex structure $\mathcal{J}$ on M associated to any complex structure j on T^2. From Chapter 3 we have the i-eigenbundle for $\mathcal{J}$ is

$$L = \{X_+ + X_- + b(X_+ + X_-) + g(X_+ - X_-)|X_\pm \in T_\pm^{1,0}\}.$$

Abusing of notations, let $B = T_+^{1,0} \cap T_-^{1,0}$ and $\bar{B}^* \subset T^* \times \mathbb{C}$, with the Clifford action, the canonical bundle of $\mathcal{J}$ is $e^b \bar{B}^*$. Let $\Omega^{0,1}$ be the bundle of $(0,1)$ forms on T^2 and $p : M \to T^2$ the fibering, then $\bar{B}^* = p^*(\Omega^{0,1})$. If $\rho \in C^\infty(\Omega^{0,1})$ is a nonvanishing closed form on T^2, $p^*(\rho)$ gives a nonvanishing closed section of $\bar{B}^*$ and $e^b p^*(\rho)$ is a d_{db}-closed nonvanishing section of the canonical bundle. Therefore, $\mathcal{J}$ is an untwisted generalized Calabi-Yau structure. $\square$

Then we consider the existence on $S^1 \times N^3$ for N^3 a compact 3-manifold. Recall transversely holomorphic 1-foliation on 3-manifold is called transversely holomorphic flow, we have the following.

Corollary 5.16. *Let N^3 be a compact orientable 3-manifold with transversely holomorphic flows. Then there is type one generalized complex structures on $S^1 \times N^3$, or more generally on a circle bundle over N^3. In particular, it is true for N^3 being any Seifert fibered 3-manifolds.*

Proof. The transversely holomorphic structure on N^3 descends to the product directly. Since circle bundles are locally diffeomorphic to the product, it is also true. Seifert fibered 3-manifolds all have transversely holomorphic flows [6]. □

Example 5.17. Especially there are type one generalized complex structures on $S^1 \times L(p,q)$ for any (p,q), where $L(p,q)$ are Lens spaces. In [42], it is proved that there is also even type generalized complex structures on $S^1 \times L(p,1)$ with arbitrarily many type change loci.

Three-manifolds with orientable transversely holomorphic flows are classified in [6],[25]. There are six classes of foliations among them, which are: Seifert fibrations, linear foliations on T^3, strong stable foliations of T^2, foliations from holomorphic suspension of S^2, affine foliations on $S^2 \times S^1$ and Poincaré foliations on S^3. Analogue to Theorem 4.13, we propose the following question:

Question 2: For a closed 4-manifold $S^1 \times N^3$ with type one generalized complex structure, does N^3 has transversely holomorphic flows or not?

At the end, we study the deformation of type one generalized complex structure on 4-manifolds. For a transversely holomorphic 2-foliation F, let $S = \phi^{-1}(N^{0,1})$ be the integrable subbundle, where $\phi : T \otimes \mathbb{C} \to N \otimes \mathbb{C}$. As S is a Lie algebroid, there is a Lie algebroid differential d_S such that $(C^\infty(\wedge^k S^*), d_S)$ is an elliptic complex. In [18], it is proved that the infinitesimal deformations of F is the second cohomology $H^2_S(M)$, and the related Kurunishi theory of deformations of transversely holomorphic foliations are studied in details. Now the generalized complex structure associated to F has i-eigenbundle L such that $\pi(L) = S$, where $\pi : T \oplus T^* \to T$. Denote the dual map by $\pi^* : S^* \to L^*$. It induces a homomorphism between the complex $(C^\infty(\wedge^k S^*), d_S) \to (C^\infty(\wedge^k L^*, d_L)$ and then induces to a homomorphism $\bar{\pi}^* : H_S^k(M) \to H_L^k(M)$. Recall the infinitesimal deformations of generalized complex structure is in $H_L^2(M)$. So the two deformations are related by the map $\bar{\pi}^*$. We analyze this by the following three examples.

Example 5.18. Let $X_1 = (\sqrt{2}, 1, 0, 0), X_2 = (0, 0, 1, \sqrt{2}), X_3 = (0, 1, 0, 0), X_4 = (0, 0, 0, 1)$ be four vector fields on $\mathbb{T}^4$. Denote the corresponding dual forms by $X_1^*, X_2^*, X_3^*, X_4^*$. Let

$$L =< X_1 - iX_2^*, X_2 + iX_1^*, X_3 - iX_4, X_4^* + iX_3^* >\subset (T \oplus T^*) \otimes \mathbb{C}.$$

Then $L + \bar{L} = (T \oplus T^*) \otimes \mathbb{C}$. Also L is isotropic and closed under the trivial Courant bracket. Therefore L determines a generalized complex structure $\mathcal{J}_L$. As $\pi(L) =< X_1, X_2, X_3 - iX_4 >$ is 3-dimensional, $\mathcal{J}_L$ is type one generalized complex structure on $\mathbb{T}^4$. The invariant Lie algbroid cohomologies of $\mathcal{J}_L$ are given by:

$$IH_L^1(T^4) = \mathbb{C}^4 =< X_1^* + iX_2, X_2^* - iX_1^*, X_3^* + iX_4^*, X_4 - iX_3 >,$$

similarly, $IH_L^2(T^4) = \mathbb{C}^6$.

The 2-foliation determined by $\mathcal{J}_L$ is $F =< X_1, X_2 >$ and $S = \pi(L) =< X_1, X_2, X_3 - iX_4 >$. The invariant Lie algebroid cohomologies of S are:

$$IH_S^1(T^4) = \mathbb{C}^3 =< X_1^*, X_2^*, X_3^* + iX_4^* >,$$

$$IH_S^2(T^4) = \mathbb{C}^3.$$

The map $\bar{\pi}^*$ is the natural injection.

A Lie algebra g is called nilpotent if $g^i = 0$ for some $i > 1$, where $g^k = [g^{k-1}, g]$ is defined inductively with $g^1 = g$. Given a nilpotent Lie algebra g, the exponential map to its simply connected Lie group is bijective. So we identify g with its simply connected Lie group. A classical theorem of Mel'cev [38] states that g has a discrete lattice such that the quotient is compact if and only if the structure constants with respect to a basis are rational. Up to isomorphism, there are two nonabelian classes of 4-dimensional nilpotent algebra: $g_{4,1}, g_{4,2}$.

Example 5.19. Consider $g_{4,1} =< x_1, x_2, x_3, x_4 | [x_1, x_2] = x_3 >$. It is the product of 3-dimensional Heisenberg algebra with $\mathbb{R}$. The compact quotient of $g_{4,1}$ is the Kodaira-Thurston manifold. Let $L =< x_1 + ix_2, x_1^* + ix_2^*, x_3 - ix_4^*, x_3^* - ix_4 >$. Then $L + \bar{L} = (g \oplus g^*) \otimes \mathbb{C}$, and L is maximal isotropic. The nonzero parts of Courant bracket of $g \oplus g^*$ are:

$$[x_1, x_2] = [x_3], [x_1, x_3^*] = -x_2^*, [x_2, x_3^*] = x_1^*.$$

So L is closed under the Courant bracket and defines an invariant type one generalized complex structure on M. For the invariant Lie algebroid cohomology, we have

$$IH^1_L(g) = \mathbb{C}^3 =< A, C, D >,$$

$$IH^2_L(g) = \mathbb{C}^4 =< A \wedge B, A \wedge C, C \wedge D, B \wedge D >$$

where $A = x_1^* - ix_2^*, B = x_1 - ix_2, C = x_3^* + ix_4, D = x_3 + ix_4^*$. Now $S =< x_1 + ix_2, x_3, x_4 >$, the induced Lie bracket is trivial on S. So the invariant cohomology of S are:

$$IH^1_S(g) = \mathbb{C}^3 =< x_1^* - ix_2^*, x_3^*, x_4^* >,$$

$$IH^2_S(g) = \mathbb{C}^3 =< (x_1^* - ix_2^*) \wedge x_3^*, (x_1^* - ix_2^*) \wedge x_4^*, x_3^* \wedge x_4^* > .$$

The map $\bar{\pi}^*$ is not injective, since $\bar{\pi}^*([(x_1^* - ix_2^*) \wedge x_4^*]) = [A \wedge D] = 0$ in $IH^2_L(g)$

Example 5.20. Consider the irreducible nilpotent algebra $g_{4,2} =< x_1, x_2, x_3, x_4 | [x_1, x_2] = x_3, [x_1, x_3] = x_4 >$. The structure constants are integers, so there is compact quotient N of $g_{4,2}$. Let $L =< x_1 + ix_2, x_1^* + ix_2^*, x_3 - ix_4^*, x_3^* - ix_4 >$ which is the same as the last problem. Then L defines a generalized almost complex structure. The nonzero Courant brackets are:

$[x_1, x_2] = [x_3], [x_1, x_3] = x_4,$

$[x_1, x_3^*] = -x_2^*, [x_2, x_3^*] = x_1^*, [x_1, x_4^*] = -x_3^*, [x_3, x_4^*] = x_1^*.$

So L is closed under the Courant bracket and induces an integrable type one generalized complex structure on N. The invariant Lie algebroid cohomology of L are:

$$IH_L^1(g) = \mathbb{C}^2 =< A, C >,$$

$$IH_L^2(g) = \mathbb{C}^2 =< A \wedge B, C \wedge D >$$

where A, B, C, D are the same as above. The invariant cohomology of S is:

$$IH_S^1(g) = \mathbb{C}^2 =< x_1^* - ix_2^*, x_3^* >,$$

$$IH_S^2(g) = \mathbb{C}^2 =< (x_1^* - ix_2^*) \wedge x_4^*, x_3^* \wedge x_4^* > .$$

The map $\bar{\pi}^*$ is not injective as $\bar{\pi}^*([(x_1^* - ix_2^*) \wedge x_4^*]) = [A \wedge D] = 0$ in $IH_L^2(g)$.

Chapter 6

Generalized complex structures with mixed types on 4-manifolds

Generalized complex structure with mixed types is the last class among the four classes of generalized complex structures on 4-manifolds and is also the most mysterious case. There are two known ways to construct generalized complex structures with mixed types. One is by deforming complex surfaces using holomorphic Poisson structures which was introduced in Gualtieri's thesis. It produces generalized complex structure with mixed types on $\mathbb{CP}^2$ for example. Another is using the C^∞ log transform surgery to symplectic 4-manifolds introduced by Cavalcanti and Gualtieri in [8].

Recall the Luttinger surgery in symplectic geometry is a $(p, q, 1)$ torus surgery around Lagrangian torus with Lagrangian framing such that the resulting manifold has symplectic structures. For a $(p, q, 0)$ torus surgery, the manifold obtained is not symplectic in general. The work in [8] shows that if the surged manifold is symplectic and the torus is symplectic with trivial normal bundle, then the resulting manifold of a $(p, q, 0)$ torus surgery has generalized complex structure of mixed types. Using this surgery, they construct generalized complex structures with mixed types on $3\mathbb{CP}^2\#19\overline{\mathbb{CP}^2}$ which does not have symplectic and complex structures. Based on the surgery, Torres proves that any finitely generated groups can be the fundamental groups of 4-manifolds with generalized complex structures of mixed types.

The above results demonstrates that the world of 4-manifolds with generalized complex structure of mixed tyeps may be much broader than that of complex surfaces or symplectic 4-manifolds. Therefore it is natural to ask the following question:

Question 3: Does every almost complex 4-manifolds admit generalized complex structures with mixed types?

There is no known obstruction for generalized complex structure with mixed types. As a trial, we study this question on the four torus $\mathbb{T}^4$. We assume there is a free $\mathbb{T}^2$ action on $\mathbb{T}^4$ and consider the existence of invariant generalized complex structure of mixed types on $\mathbb{T}^4$. Applying the T-duality results in Chapter 2, we obtain a nonexistence result under certain conditions.

In section 1, we give local examples of generalized complex structure with mixed types. Then we introduce holomorphic Poisson structures and its applications. We state Bailey's local classification of generalized complex structure with complex type. After it, we introduce Cavalcanti and Gualtieri's surgery on symplectic 2-torus as well as Torres's work. With respect to almost bihermitian structure, we obtain a noncommutative property of the two almost complex structures.

In section 2, we define $\mathbb{T}^2$ action on $\mathbb{T}^4$ and study the invariant complex as well as generalized complex structures with mixed types on $\mathbb{T}^4$. Use the T-duality we prove that there is no invariant untwisted generalized complex structure of mixed types with at least a Lagrangian orbit on $\mathbb{T}^4$.

6.1 Examples of generalized complex structure with mixed types

For a generalized complex structure with mixed types on 4-manifolds, we call the set of points with type 2 the complex locus. The rest set is called the symplectic locus. The most illustrative example of generalized complex structure with mixed types is the following:

Example 6.1. On $\mathbb{C}^2$, let $\rho = z_1 + dz_1 dz_2$. Then ρ is a pure spinor form with $d\rho = dz_1 = \partial_{z_2} \cdot \rho$. By the equivalent definition in Chapter 2, ρ defines a generalized complex structure $\mathcal{J}$. The type of $\mathcal{J}$ equals the lowest degree of ρ which is 0 when $z_1 \neq 0$ and 2 when $z_1 = 0$. So ρ gives an example of generalized complex structure with mixed types 0 and 2 on $\mathbb{C}^2$.

The above example is a special case of holomorphic Poisson deformation of complex structures.

Definition 6.2. A holomorphic Poisson structure on a complex manifold is a section $\Lambda \in H^0(M, T^{1,0} \wedge T^{1,0})$ such that $[\Lambda, \Lambda] = 0$ for the natural Schouten bracket on $T^{1,0} \wedge T^{1,0}$.

In the above example, $\Lambda = z_1 \partial_{z_1} \wedge \partial_{z_2}$ and $\rho = e^{\Lambda} \cdot dz_1 dz_2$ where $\cdot$ is the interior multiplication. In [29], Gualtieri obtains the following general result.

Theorem 6.3. *(Gualtieri) Let Λ be a nonzero holomorphic Poisson structure of a complex manifold. If K is the canonical bundle, then $e^{\Lambda} \cdot K$ is the canonical bundle of a generalized complex structure $\mathcal{J}_\Lambda$. In particular, if the zero set of Λ is nonempty, the types of the generalized complex structure $\mathcal{J}_\Lambda$ are changing and the complex locus is given by zero set.*

Example 6.4. On $\mathbb{CP}^2$, $T^{1,0} \wedge T^{1,0} = 3H$ where H is the hyperplane bundle. Any cubic polynomial gives a holomorphic Poisson bivector. Therefore we get generalized complex structure with mixed types on $\mathbb{CP}^2$ whose complex locus is the cubics.

With the above theorem, it is natural to ask if any complex locus is given by zero set of holomorphic Poisson structures. Using a Moser-Nash type argument, Bailey obtains the local characterization for complex locus.

Theorem 6.5. *(Bailey) Suppose $\mathcal{J}$ is a generalized complex structure with complex type at a point p. Then, in a neighbourhood of p, $\mathcal{J}$ is equivalent to a generalized complex structure induced by a holomorphic Poisson structure for some complex structure near p. In particular, the complex locus of a generalized complex structure locally admits the structure of an analytic subvariety.*

Generalized complex structure with mixed types can be also obtained from symplectic structures.

Definition 6.6. Let (M, ω) be a 4-manifold and $\Sigma \hookrightarrow M$ be a 2-torus with trivial normal bundle. Choosing an identification $\mathcal{N}\Sigma \cong \mathbb{T}^2 \times D^2$ such that $\pi_1(\partial\mathcal{N}\Sigma) = \pi_1(T^3) = \mathbb{Z}^3 =< [m], [l], [\mu] >$, where $[m], [l]$ are two generators of $\pi_1(\Sigma)$. Let $\phi_{p,q,k} : T^3 \to T^3$ be a diffeomorphism such that $\phi_{p,q,k}([\mu]) = p[m] + q[l] + k[\mu]$. Then the (p, q, k) torus surgery of M along Σ is $\tilde{M} = (M - \mathcal{N}\Sigma) \cup_{\phi_{p,q,k}} \mathbb{T}^2 \times D^2$.

The following result is due to Cavalcanti and Gualtieri.

Theorem 6.7. *(Cavalcanti-Gualtieri) Let M be a symplectic 4-manifold, with $\Sigma \subset M$ a symplectic 2-torus with self intersection 0. Then the result manifold $\tilde{M}$ of a $(p, q, 0)$ surgery along Σ admits a generalized complex structure with mixed types whose complex locus is an 2-torus.*

Using this, they proved the following.

Theorem 6.8. *There are generalized complex structures with mixed types on $3\mathbb{CP}^2\#19\overline{\mathbb{CP}^2}$.*

Proof. Consider an elliptically fibered $K3$ surface. Any smooth elliptic fiber is a symplectic 2-torus with trivial normal bundle. Then the manifold after the $(0, 1, 0)$ torus surgery is diffeomorphic to $3\mathbb{CP}^2\#19\overline{\mathbb{CP}^2}$, which admits generalized complex structure by the above result.

Using the $(p, q, 0)$ torus surgery, Torres proves the following.

Theorem 6.9. *(Torres) For any finitely generated group G, there exists 4-manifolds with generalized complex structure of mixed types, such that the fundamental group is isomorphic to G.*

For generalized complex structures with mixed types, the almost bihermitian structure have the following property.

Proposition 6.10. *Let $\mathcal{J}$ be a generalized complex structure with mixed types on a connected 4-manifold M, if (g, j_+, j_-) is any almost bihermitian structure associated to $\mathcal{J}$, then $j_+j_- \neq j_-j_+$.*

Proof. We prove it by contradiction. If $j_+j_- = j_-j_+$, then $(j_++j_-)(j_+-j_-) = 0$. At any point x, $T_x = ker(j_++j_-) \oplus ker(j_+-j_-)$. Let $H_x = ker(j_+-j_-)$. It is $j_\pm$-invariant then is even dimensional. If $dim(H_x) = 2$, by the argument in Lemma 5.1, $\mathcal{J}$ has type one on x which is impossible. So $dim(H_x) = 0$ *or* 4. Let $M_1 = \{x \in M | dim(H_x) = 0\}$, $M_2 = \{x \in M | dim(H_x) = 4\}$, then $M = M_1 \cup M_2$. Also on M_1, $j_+ = -j_-$, and on M_2, $j_+ = j_-$ by the above decomposition of T_x. If $j_+ \neq \pm j_-$, then M_1, M_2 are not empty set. But they are both closed subset which contradicts the connectedness of M. If $j_+ = j_-$ or $j_= - j_-$, then $\mathcal{J}$ is a complex structure or a symplectic structure by the Remark 3 after Corollary 3.10 which gives the contradiction. □

Remark 4: For a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$, if $j_+j_- \neq j_-j_+$ for the bihermitian structure, Hitchin [33] proves that $j_+j_- - j_-j_+$ induces a nonzero holomorphic Poisson structure to both j_1 and j_2. This gives strong constraints on the topology of underlying complex manifolds.

6.2 $\mathbb{T}^4$

Denote $\mathbb{T}^4$ the four dimensional torus $\mathbb{R}^4/\mathbb{Z}^4$. With addition, it is a compact abelian Lie group. For any complex structure on $\mathbb{T}^4$, the Albanese map induces a biholomorphism of it to $\mathbb{C}^2/\mathbb{Z}^4$. So the complex structure is biholomorphic to an invariant one, the holomorphic tangent bundle is trivial and any holomorphic Poisson structure of $\mathbb{T}^4$ is a constant section. Therefore we cannot obtain generalized complex structure with mixed types on $\mathbb{T}^4$ by deformations. On the other side, it is still not known if the C^∞ log transform can produce generalized complex structure with mixed types on $\mathbb{T}^4$ or not. Motivated by this, we propose the following question:

Question 4: Does there exists generalized complex structure with mixed types on $\mathbb{T}^4$.

Given a free action of $\mathbb{T}^2$ on $\mathbb{T}^4$, the quotient map induces a fibration of $\mathbb{T}^4$ over Riemann surface with fiber $\mathbb{T}^2$. By computing the Betti numbers, the bundle structure is the product structure $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$. So we can assume $\mathbb{T}^2$ is the first factor of $\mathbb{T}^4$ and the action is addition on the first factor and constant on the second. We say a generalized complex structure $\mathcal{J}$ is invariant under the $\mathbb{T}^2$ action if $p^*\mathcal{J} = \mathcal{J}$ for any $p \in \mathbb{T}^2$, where p^* is the pull back. For generalized complex structure with mixed types on 4-manifolds, generic points are symplectic type, so we can talk about Lagrangian submanifold which means it is within the symplectic points and Lagrangian with respect to the symplectic structure. We obtain the following result related to Question 4.

Theorem 6.11. *There is no $\mathbb{T}^2$-invariant untwisted generalized complex structure of mixed types with at least one Lagrangian orbit on $\mathbb{T}^4$.*

Proof. We prove this by contradiction. Assume there is an untwisted generalized complex structure $\mathcal{J}$ with mixed types with at least one Lagrangian orbit. Let Γ be the complex locus. By Bailey's theorem, it is locally the zero set of holomorphic Poisson bivector in $\mathbb{C}^2$ which is an analytic hypersurface with real dimension 2.

More explicitly, let p be a complex point, choose a neighborhood U around p in the sense of Bailey such that if (z_1, z_2) are the complex coordinates, the local pure spinor ρ of $\mathcal{J}$ is $\rho = f(z_1, z_2) + dz_1 \wedge dz_2$, where $f(z_1, z_2)$ is a holomorphic function of (z_1, z_2) and $\Gamma|_U = \{(z_1, z_2), f(z_1, z_2) = 0\}$. Therefore on $\Gamma|_U$, $\rho = dz_1 \wedge dz_2$ and $\mathcal{J}$ is the standard complex structure and Γ is a complex submanifold (Chapter 2). Since the $\mathbb{T}^2$ action preserve types of $\mathcal{J}$, Γ is invariant and consists of union of orbits which are copies of $\mathbb{T}^2$. Also $\mathbb{T}^2$ is connected, so each connected component of Γ maps to itself and equals an orbit.

Now applying the T-duality process in Chapter 2. As $\mathcal{J}$ is untwisted, we can assume $H = 0$, otherwise we replace $\mathcal{J}$ by a B-field transform. Take $\mathbb{T}^4$ as the trivial principle bundle of $\mathbb{T}^2$ over $\mathbb{T}^2$, by Theorem 2.17, $(\mathbb{T}^4, 0)$ is a T-dual to itself. Applying Theorem 2.19, $\mathcal{J}$ is transformed to another untwisted generalized complex structure $\mathcal{J}_2$ on $\mathbb{T}^4$. The T-dual of complex type generalized complex structure is also with complex type, so $\mathcal{J}_2$ is either a complex structure or a generalized complex structure with mixed types.

Actually it has to be a complex structure. If not, assuming $\mathcal{J}_2$ is a generalized complex structure with mixed types, by the arguments in Example 2.20, the dual of Γ is complex submanifolds of the complex locus of $\mathcal{J}_2$. By the assumption that there is at least a Lagrangian orbit of $\mathcal{J}$ and the T-dual of (symplectic, Lagrangian) pair is (complex, real), $\mathcal{J}_2$ has at least a fiber which is real inside the complex locus of $\mathcal{J}_2$. However, this violates Bailey's theorem which says that the complex locus is complex submanifolds instead of being real subset. So $\mathcal{J}_2$ is a complex structure with both real orbit and complex orbit. Now the theorem will follow from the following lemma.

Lemma 6.12. *There is no $\mathbb{T}^2$-invariant complex structure on $\mathbb{T}^4$ with both complex orbit and real orbit.*

Proof. Let j be a $\mathbb{T}^2$-invariant complex structure on $\mathbb{T}^4$. As before, we assume the $\mathbb{T}^2$ action is given by addition in the first factor of a product, namely, if (x, y, z, w) is a point on $\mathbb{T}^4$ for $x, y, z, w \in S^1$, we assume the $\mathbb{T}^2$ action is given by

$$(x_0, y_0) \cdot (x, y, z, w) = (x + x_0, y + y_0, z, w)$$

for $(x_0, y_0) \in \mathbb{T}^2$. The canonical bundle K of j is holomorphic trivial, so $H^0(\mathbb{T}^4, K) = \mathbb{C}$. Let ϕ be a nowhere zero holomorphic section, as j is $\mathbb{T}^2$-invariant, $p^*\phi = k\phi$ for some constant k where $p \in \mathbb{T}^2$. As $H^0(\mathbb{T}^4, K)$ injects into the DeRham cohomology $H^2(\mathbb{T}^4, \mathbb{C})$ and $\mathbb{T}^2$ acts trivially on $H^2(\mathbb{T}^4, \mathbb{C})$, we have $k \equiv 1$ and ϕ is $\mathbb{T}^2$-invariant. Assume

$$\phi = f_1 dxdy + f_2 dxdz + f_3 dxdw + f_4 dydz + f_5 dydw + f_6 dzdw,$$

where $f_i, i = 1, \cdots 6$ are functions of (z, w) because of the invariant of ϕ. Now $d\phi = 0$ gives that

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_1}{\partial w} = 0.$$

Therefore, f_1 is a constant function. From Example 2.20 we get that an orbit Σ is complex if and only $\phi|_\Sigma = 0$, and is real if and only $\phi|_\Sigma \neq 0$. The coordinates for Σ is (x, y), so $\phi|_\Sigma = f_1 dxdy$. As f_1 is constant, the orbits are either all complex or all real. We complete the proof. $\square$

Chapter 7

Finite group actions

In this chapter, we turn to study a different topic about finite group actions on almost complex 4-manifolds. Ever since the beginning of the field of transformation groups, the study of finite group actions has been an important part. For example, the Smith theory about the homology of fixed points deals with the action of finite groups with prime order. In [13], Weimin Chen initiates the investigation of the finite symplectomorphism group of symplectic manifolds. He proved that for a large class of symplectic 4-manifolds, the order of any prime order symplectic transformation is bounded.

Inspired by the above result, we study the symmetry of almost complex 4-manifolds in this chapter. Recall for an almost complex manifold (M, j), j is said to be tamed by a symplectic form ω if $\omega(v, jv) > 0$ for any nonzero $v \in T$. Let $\kappa_j^t = \{[\omega] \in H^2(M, \mathbb{R}) | j \textit{ tamed by } \omega\}$. A tamed almost complex 4-manifold is an almost complex 4-manifold with $\kappa_j^t \neq \emptyset$. Assume (M, j) is j-holomorphic minimal, namely, there is no embedded j-holomorphic sphere with self intersection -1. Let K be the canonical class of (M, j). Assume $b^+(M) > 1$, if $K \neq 0$, by Taubes's work on the Seiberg-Witten theory of symplectic manifold [41], K is represented by symplectic surfaces. So $[\omega] \cdot K \geq 0$. Define $C_j = \min_{[\omega] \in \kappa_j^t \cap H^2(X, \mathbb{Z})} \{K \cdot [\omega]\}$. Our main theorem is:

Theorem 7.1. *Let (M, j) be a minimal tamed almost complex 4-manifold with $b^+ > 1$. Assume either $\chi(M) \neq 0$ or the symplectic Kodaira dimension $\kappa^s = 2$, then there is a constant $C = m_1 C_j^2 + m_2$, where m_1, m_2 are positive constants related to b_1, b_2, such that for any nontrivial j-holomorphic $\mathbb{Z}_p$-action on M, $p \leq C$.*

If the manifold is not j-holomorphic minimal, we have a nicer result:

Theorem 7.2. *Let (M, j) be a non-minimal tamed almost complex 4-manifold with $b^+ > 1$, then for any nontrivial j-equivariant $\mathbb{Z}_p$-action on M, $p \leq C$, where $C = m_1 C_j^2 + m_2$ and m_1, m_2, C_j are defined similarly.*

Remark 5. The conditions in our results are optimal. Otherwise, as there are nontrivial circle actions on the rational surfaces ($b^+ = 1$), and on the surfaces $T^2 \times \Sigma$ (which have zero Euler characteristic and symplectic Kodaira dimension $\kappa^s < 2$), no such bound exists on these manifolds. Theorem 7.1 contains the most possible cases in this sense.

7.1 Proof of the Theorems

We have the following lemma whose proof uses the representations of $\mathbb{Z}_p$.

Lemma 7.3. *If an oriented closed 4-manifold M admits a $\mathbb{Z}_p$-action, then the induced action on the integral homology is trivial when $p > 2 + b_1 + b_2$.*

Proof. We strictly follow [13]. Since p is odd, the $\mathbb{Z}_p$-action is orientation preserving. From the decompostion of $\mathbb{Z}_p$ representations [15], the induced integral representation of $\mathbb{Z}_p$ on $H^1(M, \mathbb{Z})$ is equivalent to the direct sum representation

$$\mathbb{Z}[\mathbb{Z}_p]^r \oplus \mathbb{Z}[\mu_p]^s \oplus Y$$

for some integers $r, s \geq 0$, where $\mathbb{Z}[\mathbb{Z}_p]$ is the regular representation of rank p, $\mathbb{Z}[\mu_p]$ is the cyclotomic representation of rank $p - 1$ and Y is the trivial representation with free part rank t. This gives $b_1 = rp+s(p-1)+t$. If $p > b_1+1$, we have $p-1 > rp+s(p-1)+t$, which gives $r = s = 0$, and the representation on $H^1(M,\mathbb{Z})$is trivial.

Similarly, if $p > b_2+1$, the induced action on $H^2(M,\mathbb{Z})$ is also trivial. By Poincaré duality we get that the $\mathbb{Z}_p$ action induces trivial action on homology. $\square$

About the symplectic group actions, Weimin Chen proves the following theorem.

Theorem 7.4. *(Weimin Chen) Let (M,ω) be a symplecticly minimal 4-manifold with $b^+ > 1$. If*

1. $[\omega] \in H^2(M;\mathbb{Q})$
2. *Either $\chi(M) \neq 0$ or $\kappa^s = 2$,*

then there exists a constant $C = m_1 C_\omega^2 + m_2 > 0$, where C_ω is the least integer among the integral multiple of $K \cdot [\omega]$, and m_1, m_2 are positive constants depending on the betti numbers, such that there are no nontrivial symplectic $\mathbb{Z}_p$-action of prime order on M provided that $p > C$.

Proof of Theorem 7.1. From Lemma 7.3, assuming $p > 2 + b_1 + b_2$, we have the $\mathbb{Z}_p$-action is homologically trivial. Since κ_j^t is an open cone in $H^2(M,\mathbb{R})$ and $H^2(M,\mathbb{Q})$ is dense in $H^2(M,\mathbb{R})$, we can choose a j-tamed symplectic form ω, such that $[\omega]$ is in $H^2(M,\mathbb{Z})$. Fix ω, and define $\omega_G = \frac{1}{p}\sum_{g\in G} g^*\omega$. Since G is j-equivarient, ω_G is also a j-tamed symplectic form as

$$\begin{aligned}\omega_G(v, Jv) &= \frac{1}{p}\sum_{g\in G} g^*\omega(v, Jv)\\ &= \frac{1}{p}\sum_{g\in G} \omega(gv, gJv)\\ &= \frac{1}{p}\sum_{g\in G} \omega(gv, Jgv) > 0.\end{aligned}$$

for any nonzero vector v in T. Furthermore, $[\omega] = [\omega_G]$ in $H^2(M,\mathbb{Z})$ because G acts homologically trivially. With respect to the symplectic structure (M, ω_G), the $\mathbb{Z}_p$-action is symplectic by

$$\begin{aligned} g_1^*\omega_G &= \frac{1}{p}\sum_{g\in G} g_1^* g^*\omega \\ &= \frac{1}{p}\sum_{g\in G}(g_1 g)^*\omega \\ &= \omega_G \end{aligned}$$

for any $g_1 \in G$. As (M, j) is j-holomorphic minimal and M is not rational or ruled, (M, ω_G) is also symplecticly minimal by [37]. Applying Theorem 7.4, we get $p \leq C'$, where

$$C' = m_1(K \cdot [\omega_G])^2 + m_2.$$

We can assume $m_2 > 2 + b_1 + b_2$ above to get the homologically trivial action. Since $[\omega] = [\omega_G]$, and ω is a fixed j-tamed symplectic form, we have the bound C' is applicable for any nontrivial j-equivariant $\mathbb{Z}_p$-action on M.

Define

$$C_j = \min_{[\omega]\in\kappa_j^t\cap H^2(X,\mathbb{Z})}\{K \cdot [\omega]\}.$$

Let $C = m_1 C_j^2 + m_2$. Then we have for any nontrivial j-equivariant $\mathbb{Z}_p$-action on M, $p \leq C$, which proves the theorem. $\square$

Proof of theorem 7.2. For the case when M is not minimal, we use the following lemma from [13].

Lemma 7.5. *Let G be a finite group acting on a symplectic 4-manifold (M, ω), where $[\omega] \in H^2(M, \mathbb{Z})$. If G preserves ω and induces a trivial action on $H^2(M, \mathbb{Z})$, then there exists a symplectic 4-manifold (M', ω') which is a symplectic blowdown of (M, ω), with an induced G-action preserving ω' and inducing trivial action on $H^*(M', \mathbb{Z})$ such that for any G-equivariant, ω'-compatible almost complex structure j, M' is j-holomorphic minimal. Furthermore, $[\omega'] \in H^2(M', \mathbb{Z})$ and $K' \cdot [\omega'] \leq K \cdot [\omega]$.*

To prove Theorem 7.2, using the same method of Theorem 7.1, we fix an $\omega \in \kappa_j^t$ with $[\omega] \in H^2(M, \mathbb{Z})$. For a nontrivial j-equivariant $\mathbb{Z}_p$ action on (M, j), define $\omega_G = \frac{1}{p}\sum_{g \in G} g^*\omega$. Then we have $\omega_G \in \kappa_J^t$, $[\omega] = [\omega_G]$, and G is symplectic with respect to (M, ω_G). Using Lemma 7.5, we get a (M', ω'_G) with a G-action such that $[\omega'_G] \in H^2(M', \mathbb{Z})$ and for any G-equivariant, ω'_G-compatible almost complex structure j', M' is j'-holomorphic minimal. Since M' is not rational or ruled, it's also symplecticly minimal. Now the images of the exceptional divisors in M' become fixed points for the induced G action by the definition of equivariantlly blow down. Ensuring the fixed points, Theorem 7.4 works for $(M'\omega'_G)$ even when $\chi(M') = 0$. Then we get $p \leq C'$, where $C' = m'_1(K' \cdot [\omega'_G])^2 + m'_2$. Since

$$K' \cdot [\omega'_G] \leq K \cdot [\omega_G],$$

$$b'_1 = b_1, b'_2 < b_2,$$

and $m'_1 < m_1,\ m'_2 < m_2$, we get

$$p \leq m_1(K \cdot [\omega_G])^2 + m_2 = m_1(K \cdot [\omega])^2 + m_2.$$

Sum all $[\omega] \in \kappa_j^t$ up, we get $p \leq m_1 C_j^2 + m_2$, where $C_j = \min_{[\omega] \in \kappa_j^t \cap H^2(X, \mathbb{Z})}\{K \cdot [\omega]\}$ $\square$

References

[1] Akhmedov, Anar, Park, B. Doug. Exotic smooth structures on small 4-manifolds with odd signatures. Invent. Math. 181 (2010), no. 3, 577-603

[2] Akhmedov, Anar. Small exotic 4-manifolds. Algebr. Geom. Topol. 8 (2008), no. 3, 1781–1794.

[3] Apostolov, Vestislav, Gualtieri, Marco. Generalized Kähler manifolds, commuting complex structures, and split tangent bundles. Comm. Math. Phys. 271 (2007), no. 2, 561-575.

[4] Bailey, Michael. On the local and global classification of generalized complex structures, D.Phil. thesis. (2012) arXiv:1201.0791.

[5] Barth, W; Peters, C; Van de Ven, A. Compact complex surfaces. Erg. Math. Grenzgebiete. Springer-Verlag, 1984.

[6] Brunella, Marco. On transversely holomorphic flows. I. Invent. Math. 126 (1996), no. 2, 265-279.

[7] Cavalcanti, Gil R. New aspects of the ddc-lemma. Oxford D.Phil. thesis. (2005) arXiv:math/0501406

[8] Cavalcanti, Gil R.; Gualtieri, Marco. A surgery for generalized complex structures on 4-manifolds. J. Differential Geom. 76 (2007), no. 1, 35–43

[9] Cavalcanti, Gil R.; Gualtieri, Marco. Generalized complex geometry and T-duality. A celebration of the mathematical legacy of Raoul Bott, (2010) 341–365

[10] Cavalcanti, G.R and Gualtieri.M, Blow-up of generalized complex 4-manifolds J. Topology 2 (2009) 840-864

[11] Chen, Haojie; Nie, Xiaolan. Generalized complex structures and almost bihermitian structures, arXiv:1210.0953, submitted.

[12] Chen, Haojie. Cyclic Symmetry of Tamed Almost Complex 4-manifolds, preprint.

[13] Chen, Weimin. On the orders of periodic diffeomorphisms of 4-manifolds. Duke Math. J. 156 (2011), no. 2, 273-310.

[14] Chevalley, Claude. The Algebraic Theory of Spinors and Clifford Algebras. Collected Works., volume 2. Springer Verlag, 1996.

[15] Curtis, C.W.; Reiner, I. Representation theory of finite groups and associative algebras, Wiley, New York, 1962.

[16] Dorfmeister, Josef; Zhang, Weiyi The Kodaira dimension of Lefschetz fibrations. Asian J. Math. 13 (2009), no. 3, 341–357.

[17] Draghici, Tedi; Li, Tian-Jun; Zhang, Weiyi, On the J-anti-invariant cohomology of almost complex 4-manifolds. Quart. J. Math. 00(2011), 1-29.

[18] Duchamp, T.; Kalka, M. Deformation theory for holomorphic foliations. J. Differential Geom. 14 (1979), no. 3, 317-337 (1980).

[19] Enrietti, Nicola; Fino, Anna; Grantcharov, Gueo. Tamed Symplectic forms and Generalized Geometry. arXiv:1112.2592

[20] Etgü, Tolga. Lefschetz fibrations, complex structures and Seifert fibrations on $S^1 M^3$. Algebr. Geom. Topol. 1 (2001), 469–489

[21] Friedl, Stefan; Vidussi, Stefano. Twisted Alexander polynomials detect fibered 3-manifolds. Ann. of Math. (2) 173 (2011), no. 3, 1587-1643.

[22] Gauduchon, Paul, Le thorme de l'excentricit nulle. (French) C. R. Acad. Sci. Paris Sr. A-B 285 (1977), no. 5, A387-A390

[23] Geiges, Hansjörg; Gonzalo, Jesús. Contact geometry and complex surfaces. Invent. Math. 121 (1995), no. 1, 147–209.

[24] Geiges, Hansjörg. Symplectic structures on T^2-bundles over T^2. Duke Math. J. 67 (1992), no. 3, 539–555.

[25] Ghys, Ëtienne. On transversely holomorphic flows. II. Invent. Math. 126 (1996), no. 2, 281–286.

[26] Gómez-Mont, Xavier. Transversal holomorphic structures. J. Differential Geom. 15 (1980), no. 2, 161-185 (1981).

[27] Gompf, Robert E; Stipsicz, András I. 4-manifolds and Kirby calculus. Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999.

[28] Gromov, Mikhael; Lawson, H. Blaine, Jr. The classification of simply connected manifolds of positive scalar curvature. Ann. of Math. (2) 111 (1980), no. 3, 423–434.

[29] Gualtieri, Marco. Generalized complex geometry, Oxford D.Phil. thesis. (2004)math.DG/0401221.

[30] Gualtieri, Marco. Generalized complex geometry. Ann. of Math. (2) 174 (2011), no. 1, 75-123

[31] Gualtieri, Marco. Generalized Kähler geometry, arXiv:1007.3485.

[32] Hitchin, Nigel. Generalized Calabi-Yau manifolds. Q. J. Math., 54:281308, 2003.

[33] Hitchin, Nigel. Instantons, Poisson structures and generalized Kähler geometry, Comm. Math. Phys. 265 (2006), no. 1, 131-164,

[34] Kobayashi, Shoshichi. Natural connections in almost complex manifolds. Explorations in complex and Riemannian geometry, 153-169, Contemp. Math., 332, Amer. Math. Soc., Providence, RI, 2003.

[35] Li, Tian-Jun. Symplectic 4-manifolds with Kodaira dimension zero. J. Differential Geom. 74 (2006), no. 2, 321–352.

[36] Li, Tian-Jun; Usher, Michael. Symplectic forms and surfaces of negative square. J. Symplectic Geom. 4 (2006), no. 1, 71-91.

[37] Li, Tian-Jun. Smoothly embedded spheres in symplectic 4-manifolds. Proc. Amer. Math. Soc. 127 (1999), no. 2, 609-613.

[38] Malćev, A. I. On a class of homogeneous spaces. Amer. Math. Soc. Translation 9 (1951).

[39] Matsushita, Yasuo. Fields of 2-planes and two kinds of almost complex structures on compact 4-dimensional manifolds. Math. Z. 207 (1991), no. 2, 281-291.

[40] Nirenberg, L. A complex Frobenius theorem, Seminar on Analytic Functions, Institute for Advanced Study, Princeton, 1957, 172-189.

[41] Taubes, Clifford Henry. The Seiberg-Witten invariants and symplectic forms. Math. Res. Lett. 1 (1994), no. 6, 809-822.

[42] Torres, Rafael. Constructions of Generalized Complex Structures in Dimension Four. Comm. Math. Phys. 314 (2012), no. 2, 351-371.

[43] Walczak, Rafal. Existence of symplectic structures on torus bundles over surfaces. Ann. Global Anal. Geom. 28 (2005), no. 3, 211-231

[44] Wood, John C. Harmonic morphisms, conformal foliations and Seifert fibre spaces. Geometry of low-dimensional manifolds, 1 (Durham, 1989), 247-259