Duality and gauge theories at finite density

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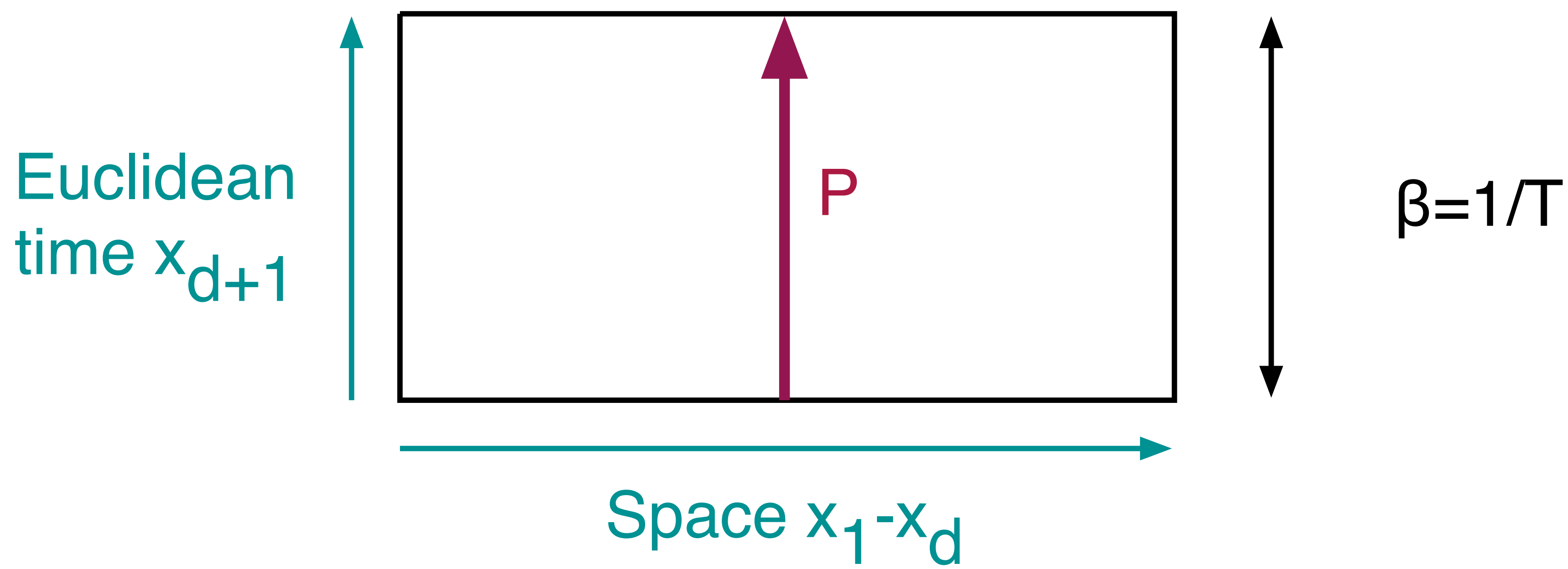
The sign problem

- The phase diagram of QCD in the T-μ plane is a key goal in the study of QCD.
- The partition function for gauge theories at finite density naturally appears as a sum over complex weights. This kind of problem occurs in many areas of physics and is known as the sign problem.
- Several lattice approaches (reweighting, imaginary chemical potential, complex Langevin) have been studied extensively, but are not fully satisfactory (de Forcrand, 2010; Gupta, 2011; Aarts, 2012)

The sign problem: a simple example

QCD with heavy quarks at finite density

Quarks are favored over antiquarks in the path integral



$$S_{eff} = \int d^{d+1}x \left[\frac{1}{4g^2} \left(F^a_{\mu\nu}\right)^2 - h_F \left(e^{\beta\mu} Tr_F(P) + e^{-\beta\mu} Tr_F(P^+) \right) \right]$$

The Polyakov loop operator represents the insertion of a static quark

$$P(\vec{x}) = \mathcal{P} \exp \left[i \int_0^\beta dt A_4(\vec{x}, t) \right]$$

$$h_F \sim \exp\left(-\beta M_q\right)$$

Finite density and CT symmetry 1

A key feature of finite density is CT symmetry

$$S_{eff} = \int d^{d+1}x \left[\frac{1}{4g^2} \left(F^a_{\mu\nu}\right)^2 - h_F \left(e^{\beta\mu} Tr_F(P) + e^{-\beta\mu} Tr_F(P^+) \right) \right]$$

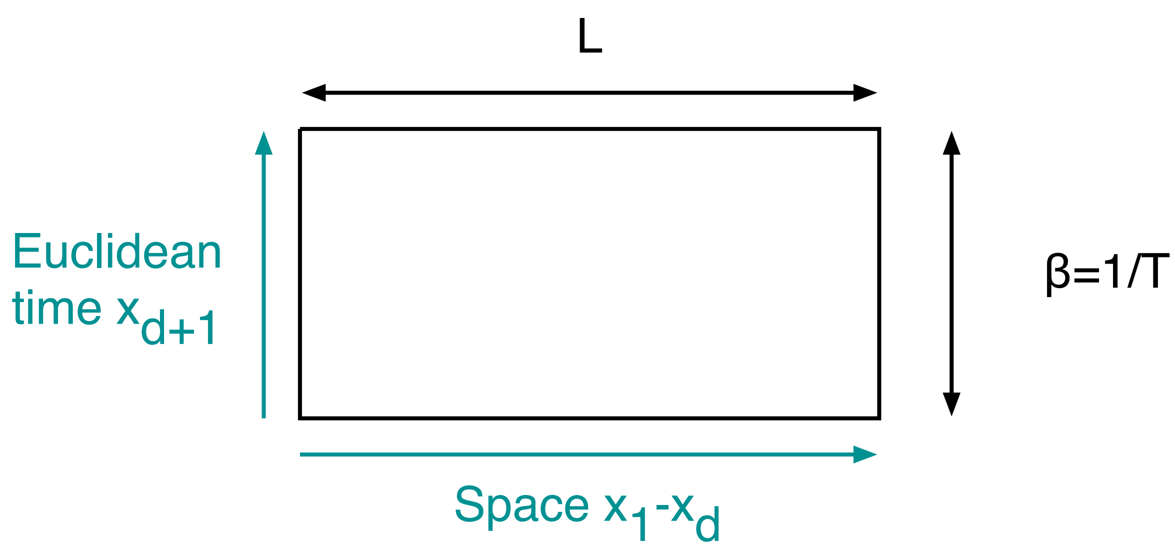
$$\mathcal{C} : A_\mu \to -A_\mu$$

$$\mathcal{T} : i \to -i$$

$$\mathcal{CT} : P \to P \qquad\qquad P(\vec{x}) = \mathcal{P} \exp\left[i \int_0^\beta dt A_4 (\vec{x}, t) \right]$$

Whenever a model has an antilinear symmetry like CT, we can say it has generalized PT symmetry.

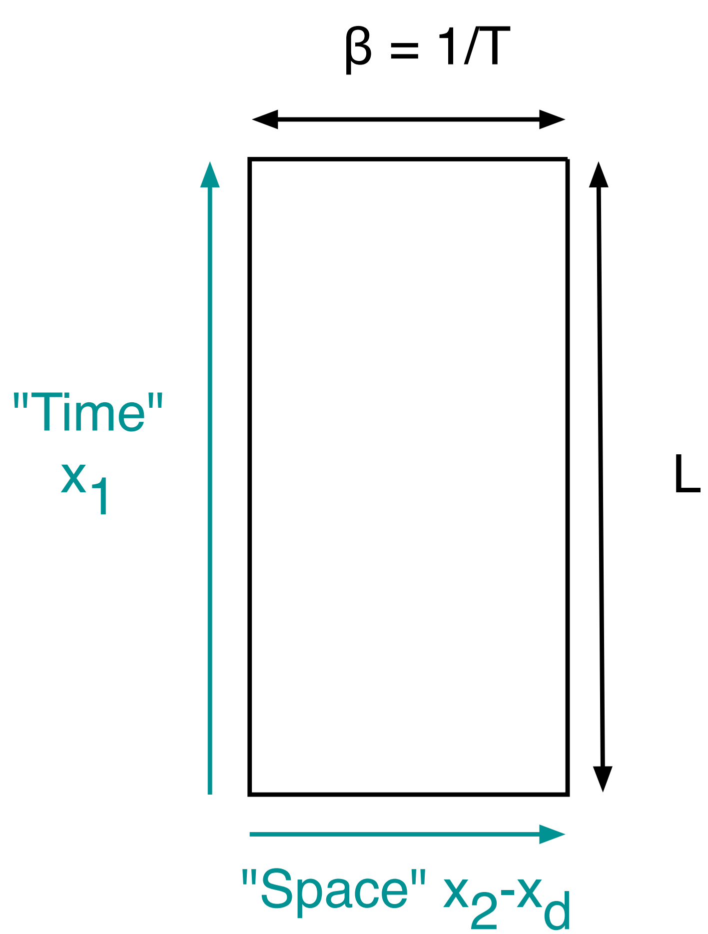
Finite density and CT symmetry 2



$$Z = Tr\left[\exp^{-\beta H_L}\right]$$

$$H_L = H - \mu \int d^d x\, j^0$$

H_L is Hermitian



$$Z = Tr\left[\exp^{-LH_\beta}\right]$$

$$H_\beta = H - i\mu \int d^d x\, j^d$$

H_β has generalized PT symmetry!

$$\mathcal{C} : j^d \to -j^d$$

$$\mathcal{T} : i \to -i$$

PT symmetry and non-Hermitian models

- Origin: ix^3 potential has only real energy eigenvalues (Bender and Boettcher, 1998). Model itself derived from Lee-Yang theory.

$$H = p^2 + ix^3$$

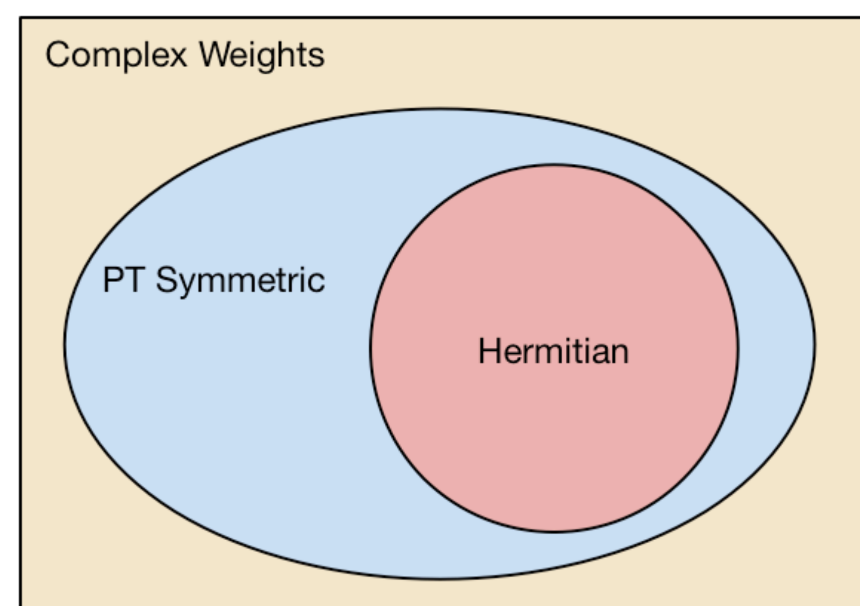
- Suppose an antilinear operator PT commutes with another operator H: then the eigenvalues of H are either real or half of a complex-conjugate pair.

$$H\left|\psi\right\rangle = E\left|\psi\right\rangle$$

$$H\mathcal{PT}\left|\psi\right\rangle = \mathcal{PT}\,H\left|\psi\right\rangle = \mathcal{PT}\,E\left|\psi\right\rangle = E^{*}\mathcal{PT}\left|\psi\right\rangle$$

- If H has such a symmetry, then Z is always real, but not necessarily positive.

$$Z = \sum_r e^{-\beta E_r} + \sum_p \left(e^{-\beta E_p} + e^{-\beta E_p^*}\right)$$

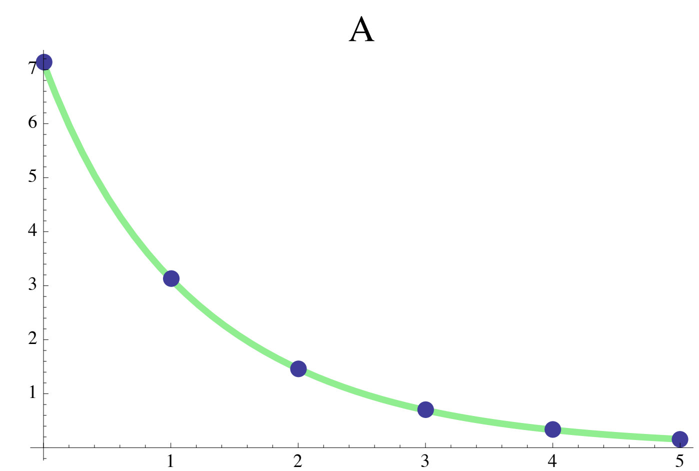


Classification of Phases via $\mathcal{PT}$ Symmetry

Example: d=1 Z(3) spin chain with complex action (Meisinger, mco & Wiser, 2010)

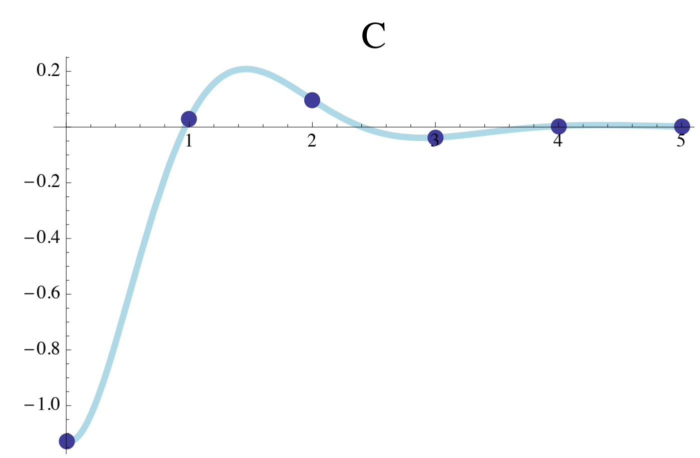
- Region I: PT symmetry is unbroken, and all eigenvalues are real. Behavior of correlation functions similar to a Hermitian system.

$$E_j \in R \;\; \forall j$$



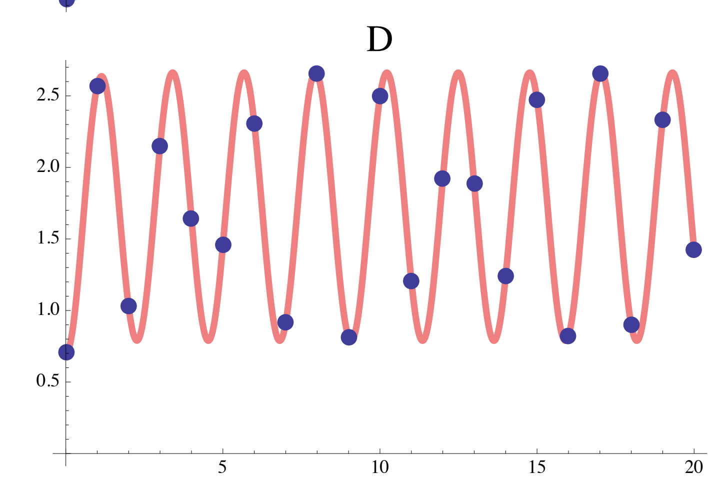
- Region II: PT symmetry is broken by a one or more pairs of excited states becoming complex. Thermodynamic properties are unaffected, but oscillatory behaviors appears in correlation functions.

$$E_j \neq E_j^* \quad j > 0$$

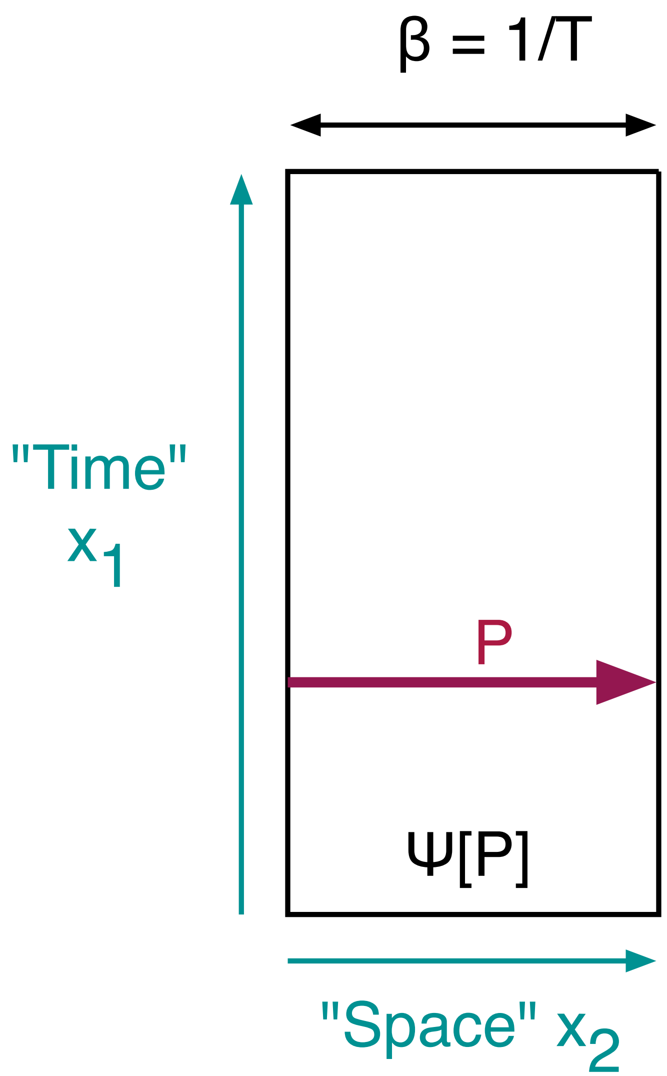


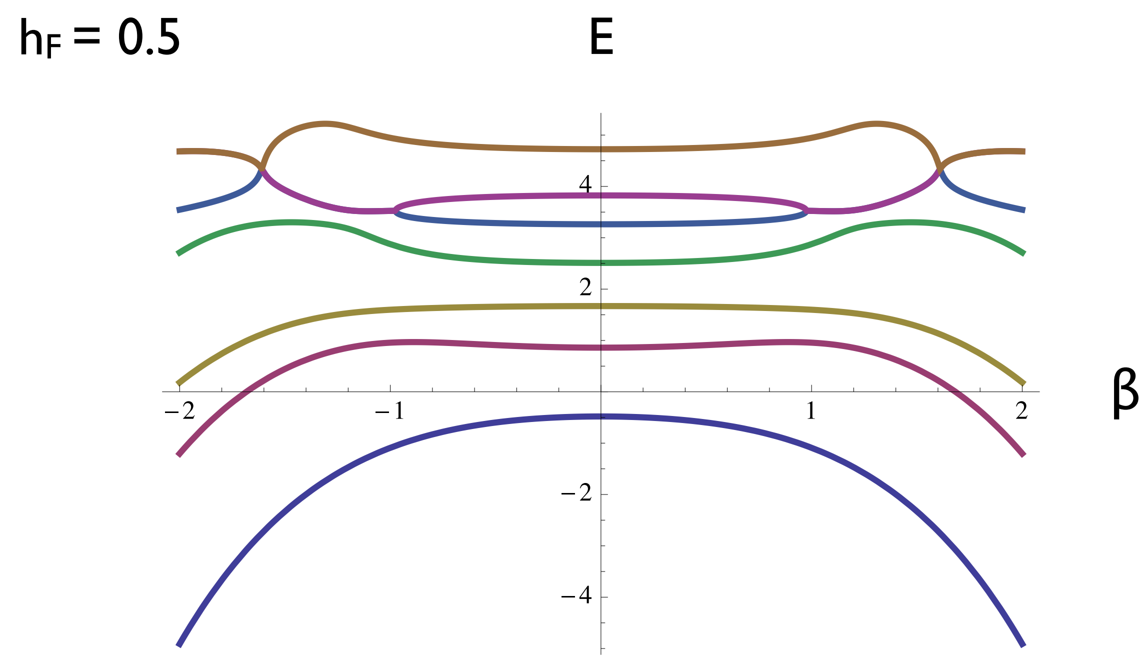
- Region III: PT symmetry is broken by the ground state becoming complex. The system is in a spatially modulated phase ("crystal").

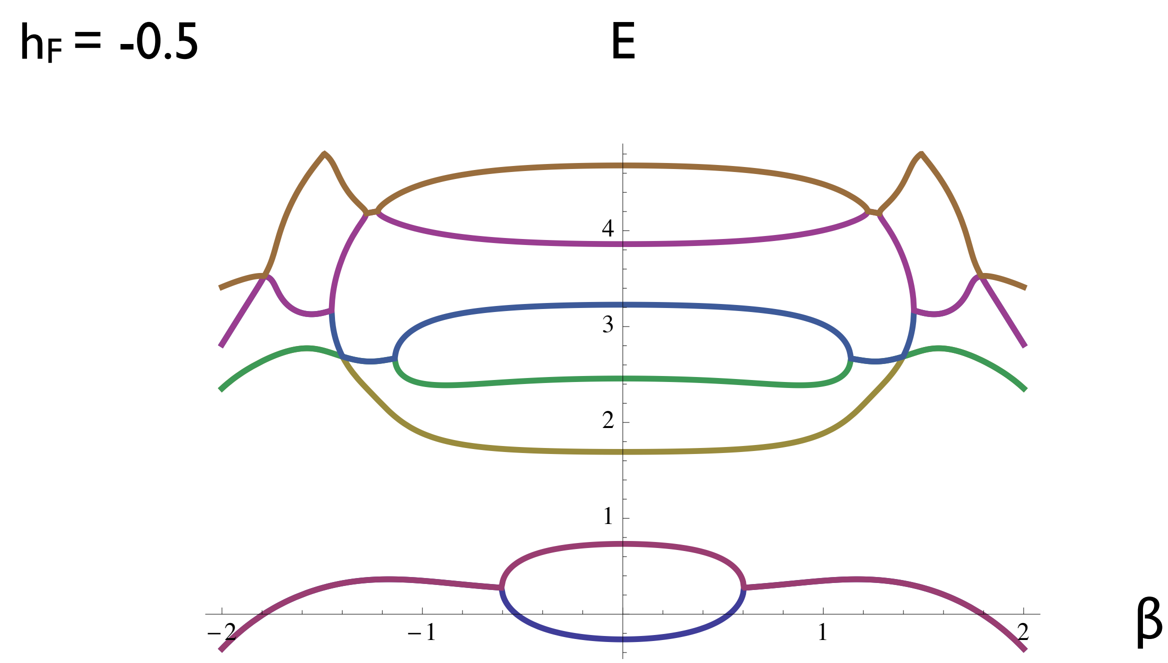
$$E_0 \neq E_0^*$$



Example: QCD$_2$ with heavy quarks







In 1+1 dimensions, problems reduces to PT-symmetric quantum mechanics on the gauge group.

Meisinger and mco, 2009

Abelian lattice models with CT symmetry

- Lattice duality maps CT-symmetric models with complex actions into dual models with real actions.
- Explicit duality relations are given for models for spin and gauge models based on Z(N) and U(1) symmetry groups.
- The dual forms are generalizations of the Z(N) chiral model and the lattice Frenkel-Kontorova model, respectively.
- For extended regions of parameter space, calculable for each model, duality resolves the sign problem for both analytic methods and computer simulations.
- From this equivalence, a rich set of spatially-modulated phases is found in the strong-coupling region of the original models.

d=2 U(1): derivation

Villain form of XY model with a chemical potential

Heat kernel action

$$Z = \int_{S^1} [d\theta] \sum_{n_\nu} \exp\left[-\frac{K}{2} \sum_{x,\nu} \left(\partial_\nu \theta\,(x) - i\mu\delta_{\nu 2} - 2\pi n_\nu\,(x)\right)^2\right]$$

Duality transform of action

$$Z = \int_{S^1} [d\theta] \prod_{x,\nu} \sum_{p_\nu(x)\in Z} \frac{1}{\sqrt{2\pi K}} e^{-p_\nu^2(x)/2K} e^{ip_\nu(x)(\nabla_\nu\theta(x) - i\delta_{\nu 2}\mu)}$$

Introduction of dual variables

$$p_\rho(x) = \epsilon_{\rho\nu}\nabla_\nu m(X)$$

$$Z = \sum_{\{m(X)\}\in Z} \frac{1}{\sqrt{2\pi K}} e^{K - \sum_{X,\nu}\left[(\nabla_\nu m(X))^2/2K + \mu\delta_{\nu 2}\epsilon_{\nu\rho}\nabla_\rho m(X)\right]}$$

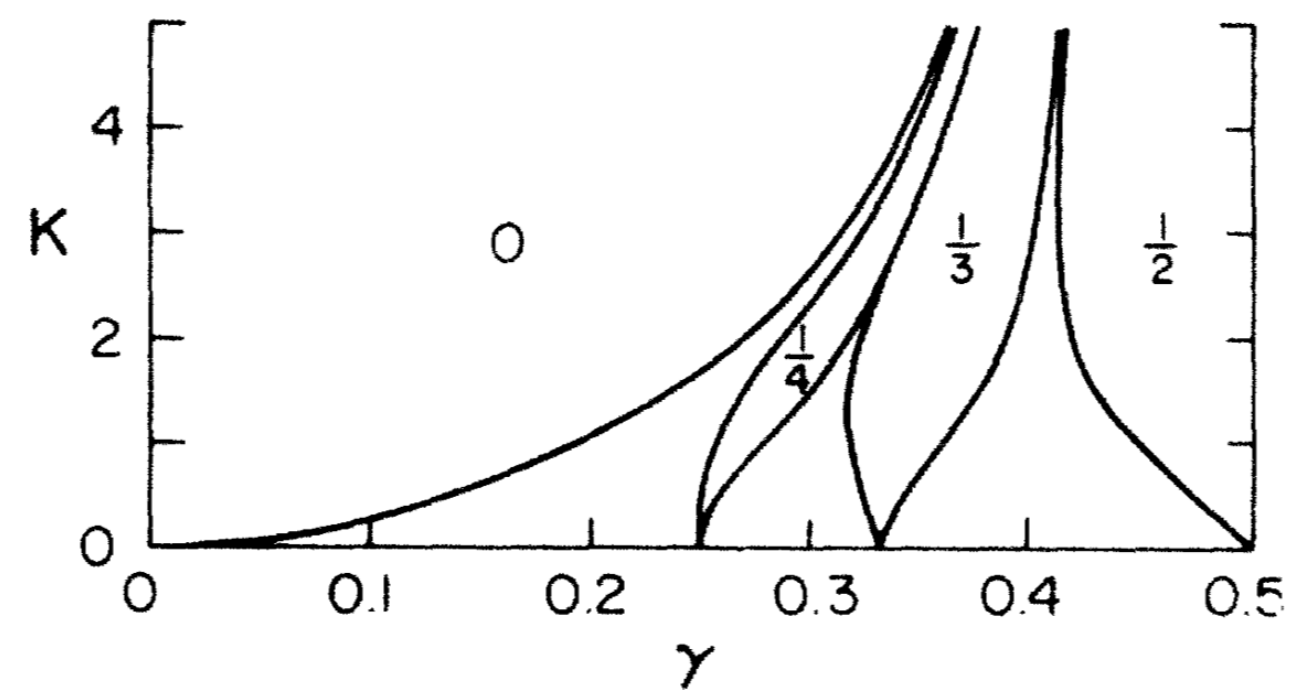
Introduction of dual scalar field

$$Z = \int_R [d\phi\,(X)]\, e^{-\sum_{X,\nu}\left[(\nabla_\nu\phi(X))^2/2K - \mu\nabla_1\phi(X)\right]} \sum_{\{m(X)\}\in Z} e^{2\pi i m(X)\phi(X)}$$

d=2 U(1): interpretation

m=1 only $$Z=\int_R [d\phi(X)] \exp\left[-\sum_{X,\mu}\frac{1}{2K}\left(\nabla_\mu\phi(X)\right)^2-\sum_X \mu\nabla_1\phi(X)+\sum_X 2y\cos\left(2\pi\phi(X)\right)\right]$$

- m=1 contributions only gives a lattice sine-Gordon model with an extra term: lattice form of Frenkel-Kontorova model.
- For fixed X2, derivative term counts the number of kinks on that slice.
- Continuum form equivalent to a massive Thirring model with a chemical potential.
- Frenkel-Kontorova model has rich modulated phase structure.



Chou and Griffiths, 1986

Duality: why it works

- Lattice duality for Abelian systems uses the Fourier transform on the group

Ising model:

$$e^{J\sigma\sigma'} = \cosh(J)\left[1 + \tanh(J)\sigma\sigma'\right]$$

- PT symmetry is analogous to reality of the Fourier transform

$$H(p,x) = p^2 + ix^3$$

$$H^*(p,x) = H(p,-x)$$

d=2 Z(N)

- Villain form again using methods of Elitzur *et al.* (1979)

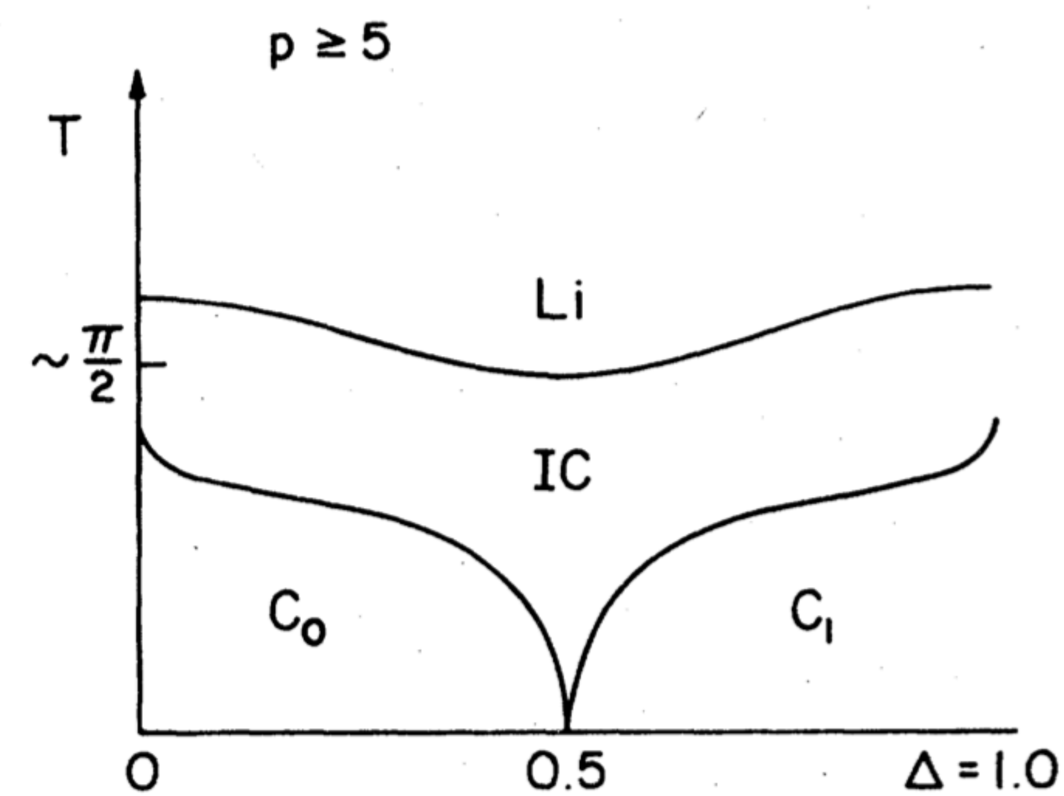
$$Z = \sum_m \sum_{n_\nu} \exp\left[-\frac{J}{2} \sum_{x,\nu} \left(\frac{2\pi}{N} \partial_\nu m\left(x\right) - i\mu\delta_{\nu 2} - 2\pi n_\nu\left(x\right) \right)^2 \right]$$

- Exact duality statements

$$J \leftrightarrow \tilde{J} = \frac{N^2}{4\pi^2 J}$$

$$\mu\delta_{\nu,2} \leftrightarrow \tilde{\mu} = -i\frac{2\pi J\mu}{N}\delta_{\nu,1}$$

- Incommensurate phase (IC) for J small (Ostlund, 1981) that extends the Coulomb phase for N>4. Li is the ordered phase. C0 is the dual ordered phase. C1 is a high-density phase.



Ostlund, 1981

d=3 Z(N): duality

Heat kernel action for gauge and spins.

$$Z = \sum_{m,n_\nu,p_\nu,q_{\nu\rho}} \exp\left[-\frac{J}{2}\sum_{x,\nu}\left(\frac{2\pi}{N}\partial_\nu m\left(x\right) - \frac{2\pi}{N}p_\nu - i\mu_\nu - 2\pi n_\nu\left(x\right)\right)^2\right]$$

$$\times \exp\left[-\frac{K}{2}\sum_{x,\nu>\rho}\left(\frac{2\pi}{N}\left(\partial_\nu p_\rho - \partial_\rho p_\nu\right) - iG_{\nu\rho} - 2\pi q_{\nu\rho}\right)^2\right]$$

G is a background field, but corresponds to an electric field in Minkowski space. This is again a sign problem (Shintani *et al.* 2006; Alexandru 2008).

$$J \to \tilde{J} = \frac{N^2}{4\pi^2 K}$$

$$K \to \tilde{K} = \frac{N^2}{4\pi^2 J}$$

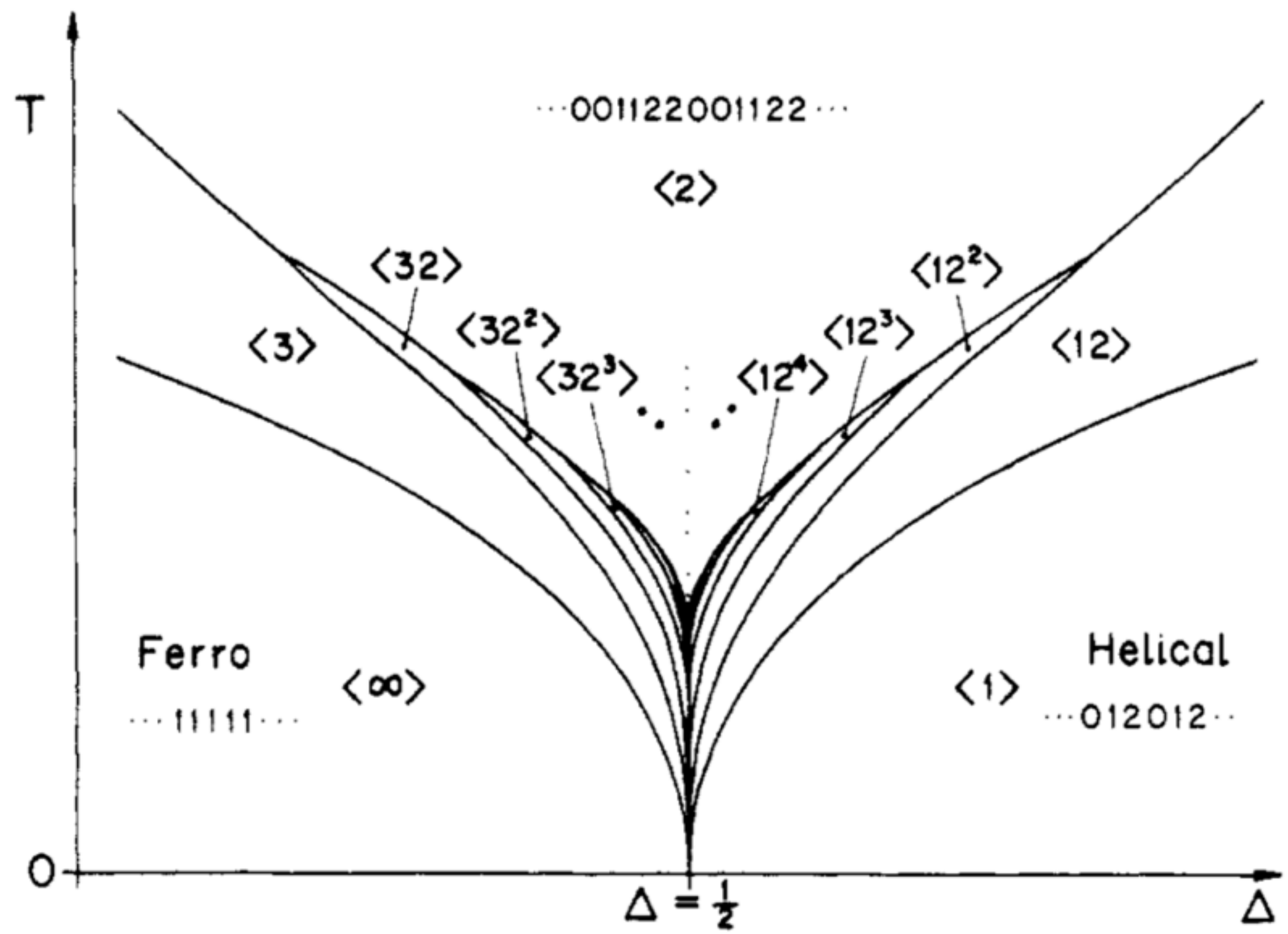
Duality swaps between gauge and spin degrees of freedom.

$$\mu_\nu \to \tilde{\mu}_\nu = -i\frac{2\pi K}{N}\epsilon_{\nu\rho\sigma}G_{\rho\sigma}$$

$$G_{\nu\rho} \to \tilde{G}_{\nu\rho} = -i\frac{2\pi J}{N}\epsilon_{\nu\rho\sigma}\mu_\sigma$$

d=3 Z(N): gauge theory

Duality in d=3 maps Z(N) gauge theory to the chiral Z(N) spin model



The dual model has a rich set of commensurate modulated phases.

$$\Delta = KG_{12}$$

Figure 1. Schematic representation (not to scale) of the (T, Δ) phase diagram of the chiral Potts or asymmetric clock model illustrating the unbounded sequences of commensurate phases of character $\langle 32^k \rangle$ and $\langle 12^k \rangle$ for $k = 0, 1, 2, \ldots$.

Fisher and Yeomans, 1981

General result for Z(N)

Arbitrary PT-symmetric Z(N) potential has N real parameters

$$V(z) = \sum_{j=0}^{N-1} v_j z^j \qquad V_j = V(\omega^j) \qquad \omega = \exp(2\pi i/N)$$

CT invariance $\quad V_{N-j} = V_j^*$

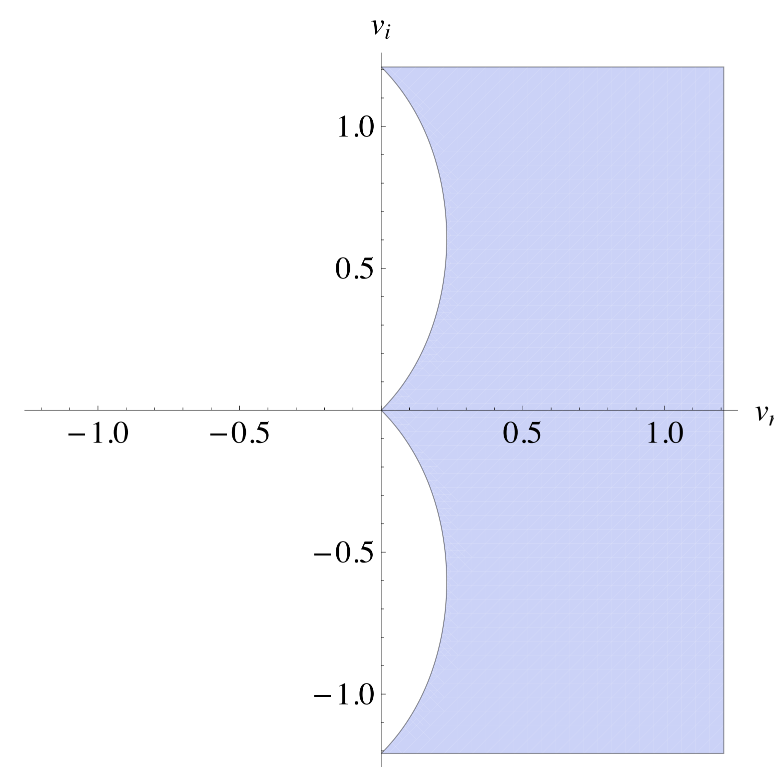
Duality is a Z(N) Fourier transform:

$$\exp\left(-\tilde{V}_j\right) = \sum_{k=0}^{N-1} \omega^{jk} \exp(-V_k)$$

Dual positive weight region for Z(3)

$$v_r = v_1 + v_2$$

$$v_i = v_1 - v_2$$



In the dual positive weight region

$$\tilde{V}(w) = \sum_{j=0}^{N-1} \tilde{v}_j w^j$$

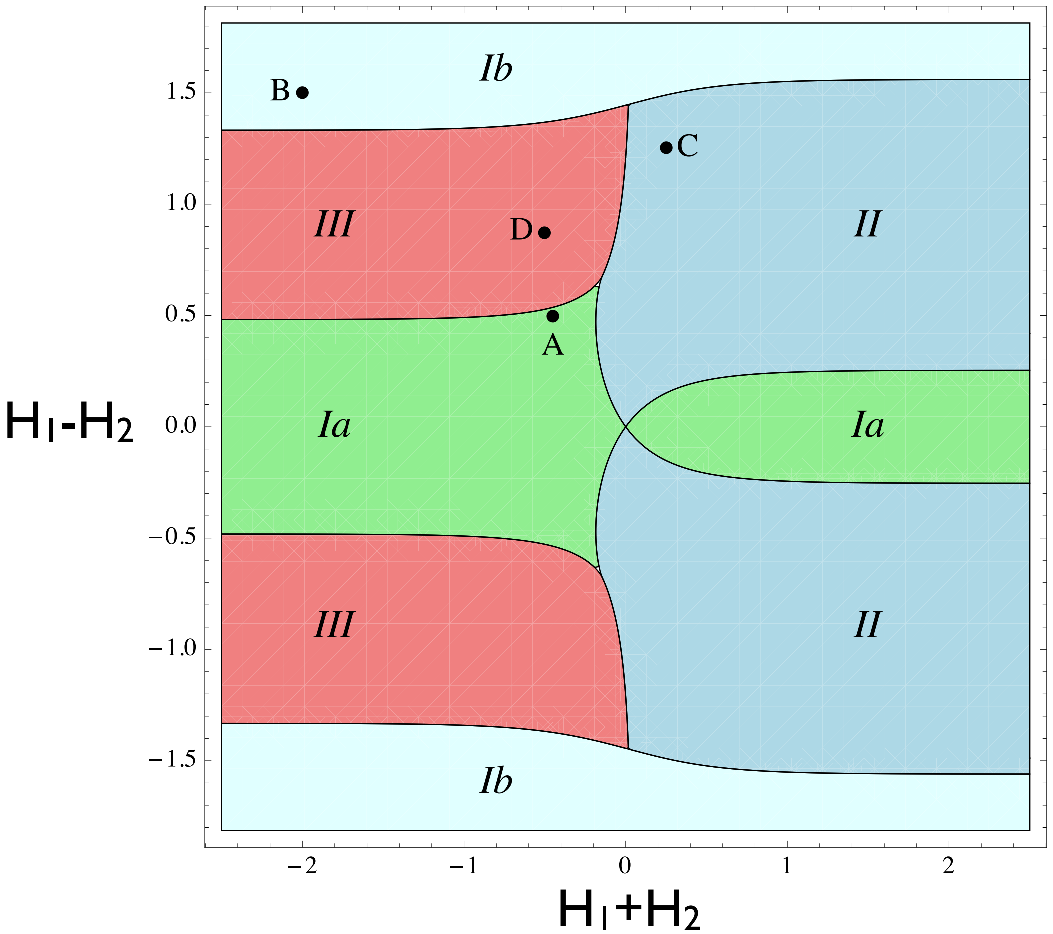
dual potential is real with chiral phases;
N real parameters

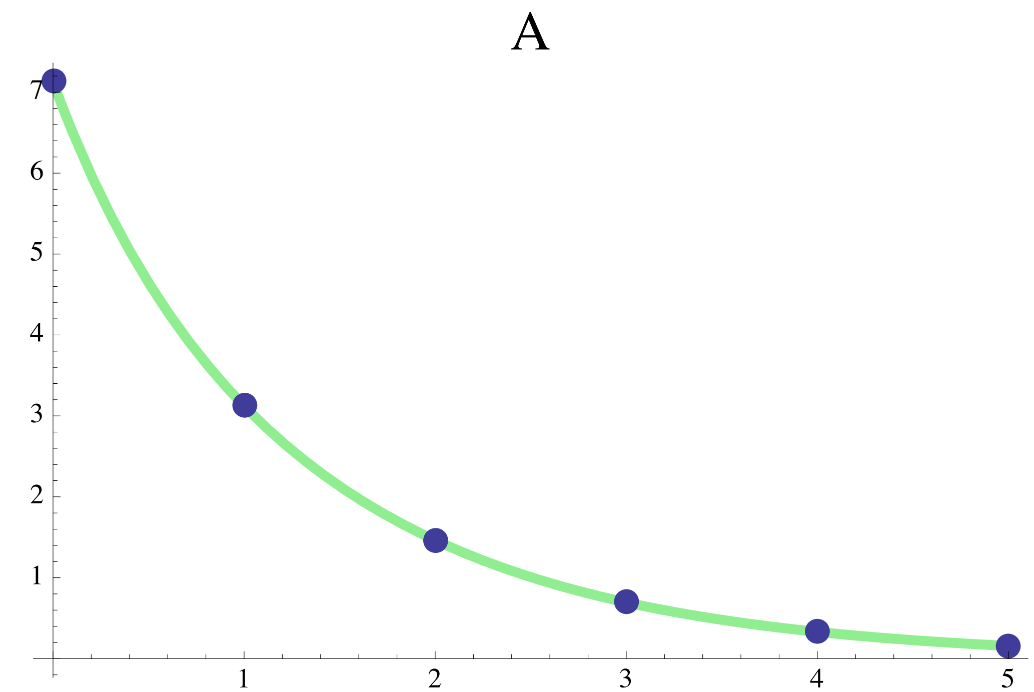
$$\tilde{v}_j = \tilde{v}_{N-j}^*$$

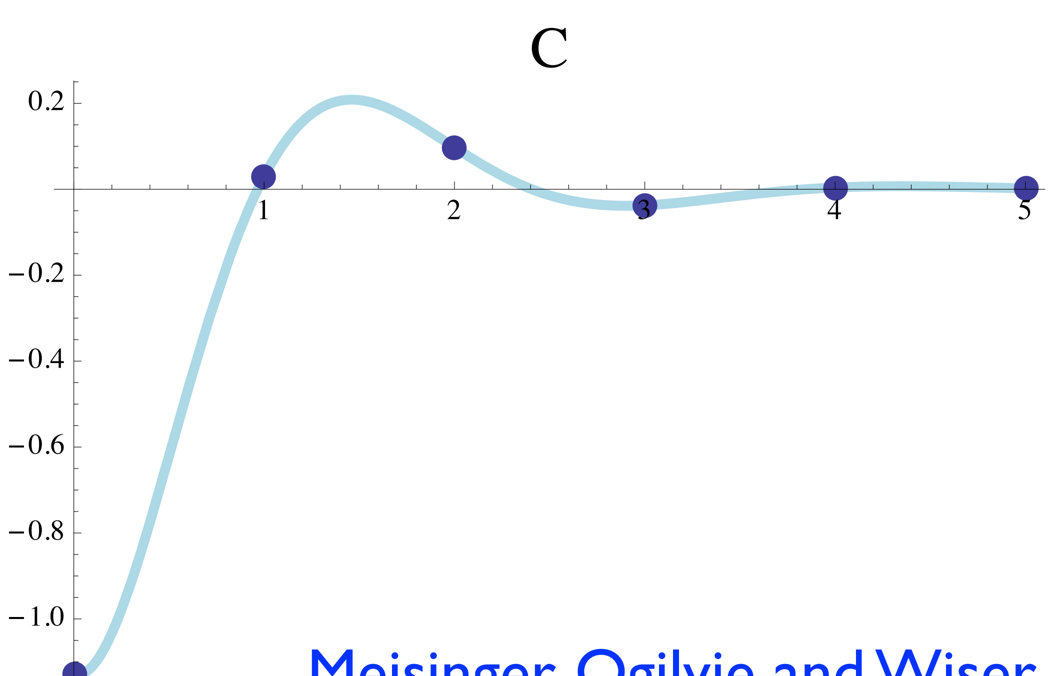
Conclusion and Prospects

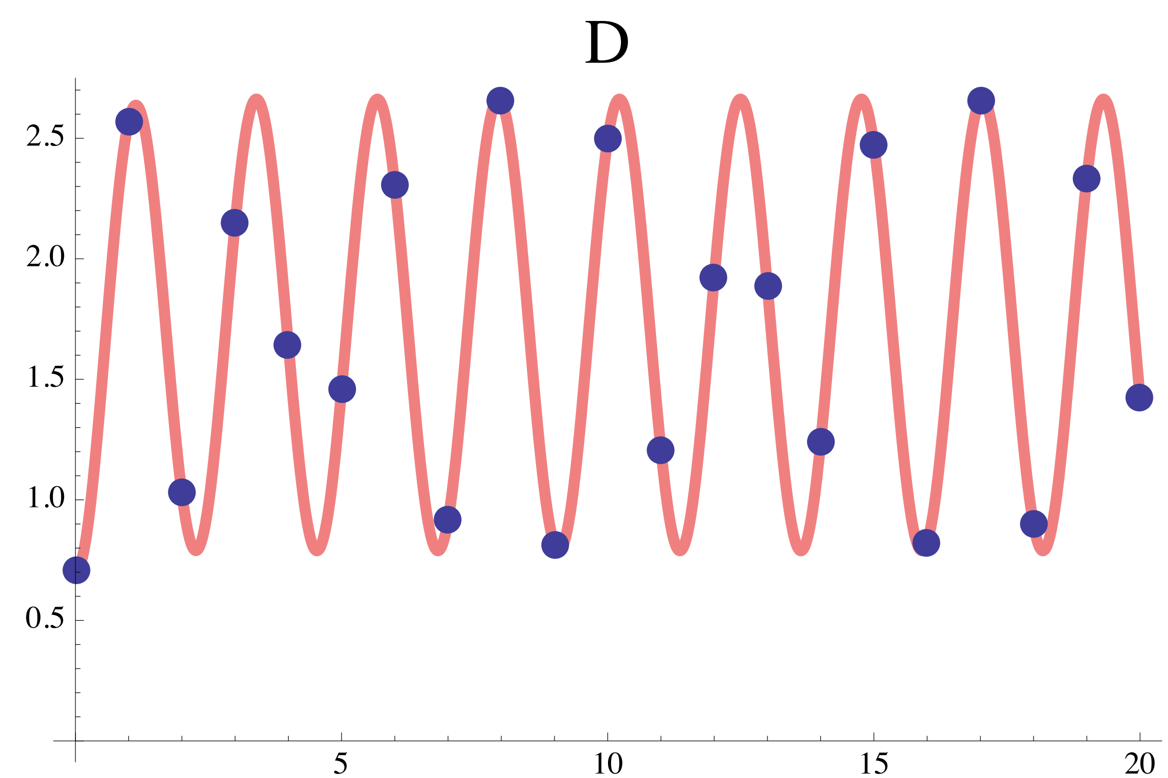
- Solution of the sign problem: both analytical and simulation methods can be applied to a large class of Abelian models.
- SU(N) deformed to $U(1)^{N-1}$ can be treated.
- d=4 (Gattringer *et al.*)
- finite temperature
- Connection to condensed matter physics, *e.g.*, bosonic quantum Hall effect

Solution of the $\mathcal{PT}$-symmetric Z(3) model in d=1









Meisinger, Ogilvie and Wiser, 2010