Seiberg's duality and $\mathcal{N} = 2$ Supersymmetric QCD

Mikhail Shifman and Alexei Yung

1 Introduction

Seiberg's duality

$$SU(N),\ N_f \text{ quarks} \qquad \Longleftrightarrow \qquad SU(\tilde{N}),\ N_f \text{ dual quarks},\ \ M$$

where N_f is the number of quark flavors, $\tilde{N} = (N_f - N)$.

Question:

Can Seiberg's duality be seen from $\mathcal{N} = 2$ QCD?

Argyres, Plesser and Seiberg 1996:
$SU(\tilde{N})$ at the root of the baryonic branch

Increase mass term for adjoint matter μ and decouple it

$$\mathcal{N} = 2 \text{ QCD} \qquad \Rightarrow \qquad \mathcal{N} = 1 \text{ QCD}$$

Shifman and Yung 2011: Baryonic branch corresponds to a different vacuum

$\mathcal{N}=2$ QCD: r-vacua where r quarks condense, $r \leq N$

Baryonic branch $\Longleftrightarrow$ $r=N$ vacuum

Seiberg's duality $\Longleftrightarrow$ $r=0$ monopole vacum

Our setup: $\mathcal{N}=2$ QCD with gauge group $U(N)=SU(N)\times U(1)$ and N_f flavors of quarks deformed by mass term for adjoint matter μ

We consider all r vacua.

Weak coupling description only in

- $r=N$ vacuum
- zero vacua at $r<\tilde{N}$

2 r Vacua

$\mathcal{N} = 2$ QCD with gauge group $U(N) = SU(N) \times U(1)$ and
N_f flavors of fundamental matter – quarks

$$N + 1 < N_f < \frac{3}{2} N$$

The field content:

U(1) gauge field A_μ

SU(N) gauge field A_μ^a, $a = 1, ..., N^2 - 1$

complex scalar fields a, and a^a

$+$ fermions

Complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) $+$ fermions

$k = 1, ..., N$ is the color index, A is the flavor index, $A = 1, ..., N_f$

Mass term for the adjoint chiral field

$$\mathcal{W}_{\rm br} = \mu \, {\rm Tr}\, \Phi^2, \qquad \Phi = \frac{1}{2} \mathcal{A} + T^a \mathcal{A}^a$$

$\boxed{r \text{ Vacuum at large } m_A}$

First r (s)quarks condense, $r \leq N$

F-terms in the potential

$$\left| \tilde{q}_A q^A + \sqrt{2}\, \frac{\partial \mathcal{W}_{\rm br}}{\partial \Phi} \right|^2 , \qquad \left| (\sqrt{2}\Phi + m_A) q^A \right|^2$$

Adjoint fields:

$$\langle \mathrm{diag}\Phi \rangle \approx -\frac{1}{\sqrt{2}} \left[m_1, ..., m_r, 0, ..., 0 \right] ,$$

For $r < N$ classically unbroken gauge group

$$U(N-r) \qquad \xrightarrow{\text{adjoints}} \qquad U(1)^{N-r} \qquad \xrightarrow{(N-r-1) \text{ monopoles}} \qquad U(1)$$

Number of isolated vacua with $r < N$

$$\mathcal{N}_{r<N} = \sum_{r=0}^{N-1} (N-r)\, C^r_{N_f} = \sum_{r=0}^{N-1} (N-r)\, \frac{N_f!}{r!(N_f-r)!}$$

Low energy theory at small $(m_A - m_B)$

$$U(r) \times U(1)^{(N-r)} \to U(1)^{\mathrm{unbr}}$$

r quarks + $(N-r-1)$ monopoles.

Quark VEV's

$$\langle q^{kA} \rangle \;=\; \langle \bar{\tilde{q}}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \sqrt{\xi_r} & 0 & \ldots & 0 \end{pmatrix},$$

$$k \;=\; 1, ..., r\,, \qquad A = 1, ..., N_f\,,$$

At large m_A $\qquad \xi_P \approx 2\, \mu m_P, \;\; P = 1, ..., r.$

We consider $r < \frac{N_f}{2}$

Then $U(r) \times U(1)^{(N-r)}$ is infrared-free and weakly coupled if

$$\sqrt{\xi_P} \ll \Lambda_{\mathcal{N}=2}$$

At small quark masses quark and monopole VEVs are determined by the roots of the Seiberg-Witten curve

$$y^2 = \prod_{P=1}^{N} (x - \phi_P)^2 - 4 \left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}} \right)^{2N-N_f} \prod_{A=1}^{N_f} \left(x + \frac{m_A}{\sqrt{2}} \right)$$

In r vacuum the curve factorizes

$$y^2 = \prod_{P=1}^{r} (x - e_P)^2 \prod_{K=r+1}^{N-1} (x - e_K)^2 \, (x - e_N^+)(x - e_N^-)$$

quarks monopoles

$$e_N^2 = \frac{2S}{\mu}, \qquad S = \frac{1}{32\pi^2} \langle \mathrm{Tr}\, W_\alpha W^\alpha \rangle \qquad e_N^+ + e_N^- = 0$$

$\boxed{\text{Universal formula}}$ for VEVs of quarks and monopoles:

$$\xi_P = -2\sqrt{2}\,\mu\,\sqrt{(e_P - e_N^+)(e_P - e_N^-)}, \qquad P = 1, ..., N$$

Quarks:

$$\langle q^{kA} \rangle \;=\; \langle \bar{\tilde{q}}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \sqrt{\xi_r} & 0 & \ldots & 0 \end{pmatrix},$$

$$k \;=\; 1, ..., r\,, \qquad A = 1, ..., N_f\,,$$

Monopoles:

$$\langle M_{P(P+1)} \rangle = \langle \bar{\tilde{M}}_{P(P+1)} \rangle = \sqrt{\frac{\xi_P}{2}}, \qquad P = (r+1), ..., N$$

3 Λ vacua and zero vacua

Cachazo-Seiberg-Witten 2003:

$$M_A = \frac{\mu}{2}\left(m_A + \sqrt{m_A^2 - \frac{4S}{\mu}}\right), \qquad A = 1, ..., r$$
$$M_A = \frac{\mu}{2}\left(m_A - \sqrt{m_A^2 - \frac{4S}{\mu}}\right), \quad A = (r+1), ..., N_f$$

$$M_A = (\tilde{q}q)_A$$

Here gaugino condensate S is determined by matrix model superpotential, namely:

$$S^N = \mu^N \, \Lambda_{\mathcal{N}=2}^{N-\tilde{N}} \left(\frac{m}{2} - \frac{1}{2}\sqrt{m^2 - \frac{4S}{\mu}}\right)^r \left(\frac{m}{2} + \frac{1}{2}\sqrt{m^2 - \frac{4S}{\mu}}\right)^{N_f - r},$$

where we assume the equal-mass limit for simplicity.

This imply the following equation for quark condensate:

$$\frac{1}{\mu}\, M_A = m - \frac{1}{\mu^{\frac{N}{\tilde N}}\, \Lambda_{\mathcal{N}=2}^{\frac{N-\tilde N}{\tilde N}}}\, \frac{(\det M)^{\frac{1}{\tilde N}}}{M_A}\,.$$

Cachazo–Seiberg–Witten exact solution produces the same equations for the quark condensates as the continuation of the ADS superpotential to $N_f > N$.

Solution at large quark masses

$$\begin{aligned}
&M_A \approx \mu\, m, \qquad A = 1, ..., r\,;\\
&M_A \approx \mu\, \Lambda_{\mathcal{N}=2}^{\frac{N-\tilde N}{N-r}}\, m^{\frac{\tilde N - r}{N-r}}\, e^{\frac{2\pi k}{N-r}\, i}, \qquad A = (r+1), ..., N_f,\\
&k = 1, ..., (N-r)\,.
\end{aligned} \tag{1}$$

We have r large classical VEVs and $(N_f - r)$ small “quantum” VEVs.

Small mass limit

Λ vacua

$$M_A \sim \mu \Lambda_{\mathcal{N}=2}, \qquad S \sim \mu \Lambda^2_{\mathcal{N}=2}$$

Zero vacua

$$M_A \approx \mu\, m, \qquad A = 1, ..., (N_f - r)$$

$$M_A \approx \mu\, \frac{m^{\frac{N-r}{\tilde{N}-r}}}{\Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{\tilde{N}-r}}}\, e^{\frac{2\pi k}{\tilde{N}-r}\, i}, \qquad A = (N_f - r + 1), ..., N_f,$$

$$k = 1, ..., (\tilde{N} - r)\,.$$

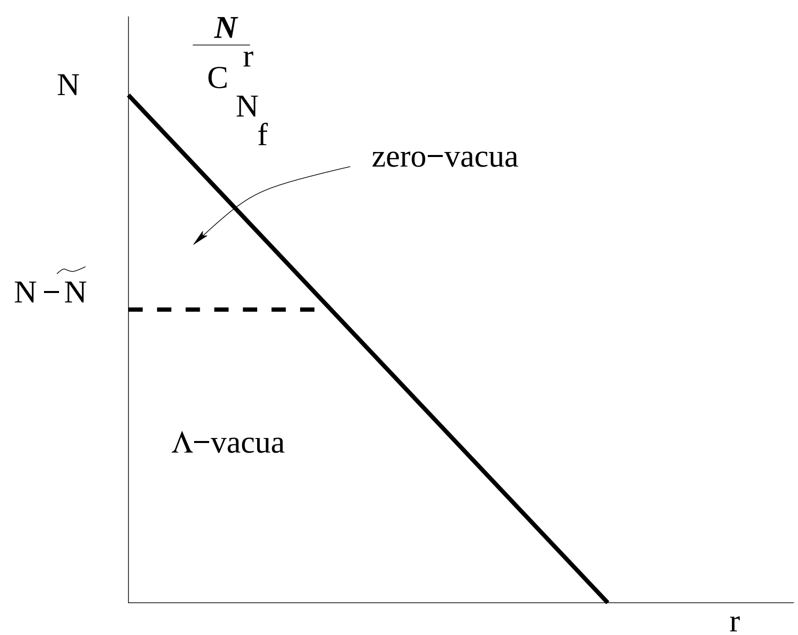
$$r < \tilde{N}$$

Gaugino condensate

$$S \approx \mu \, \frac{m^{\frac{N_f - 2r}{\tilde{N} - r}}}{\Lambda_{\mathcal{N}=2}^{\frac{N - \tilde{N}}{\tilde{N} - r}}} \, e^{\frac{2\pi k}{\tilde{N} - r} \, i} \, , \qquad k = 1, \ldots, (\tilde{N} - r) \, ,$$

Number of zero vacua

$$\mathcal{N}_{0-\mathrm{vac}} = \sum_{r=0}^{\tilde{N}-1} (\tilde{N} - r) \, C^r_{N_f} = \sum_{r=0}^{\tilde{N}-1} (\tilde{N} - r) \, \frac{N_f!}{r!(N_f - r)!}$$



4 Towards $\mathcal{N}=1$ QCD by increasing μ

Quark and monopole VEVs are determined by

$$\xi_P = -2\sqrt{2}\,\mu\sqrt{(e_P - e_N^+)(e_P - e_N^-)}, \qquad P = 1, ..., N$$

In r vacuum

$$\sqrt{2}\,e_P = -m_P, \qquad P = 1, ..., r$$

To ensure weak coupling we need

$$\sqrt{\xi_P} \ll \Lambda_{\mathcal{N}=2}$$

$$m_P = -\sqrt{2}\,e_P \to -\sqrt{2}\,e_N^{\pm}$$

Argyres-Douglas conformal regime. Strong coupling

Two exceptions: $r = N$ vacuum and zero vacua

5 μ-Duality

In zero vacuum r double roots

$$\sqrt{2}\, e_P = -m_P, \qquad P = 1, ..., r$$

$(N - \tilde{N})$ double roots

$$\sqrt{2}\, e_P = \Lambda_{\mathcal{N}=2}\, e^{\frac{2\pi i}{N-\tilde{N}}(P-\tilde{N})}, \qquad P = \tilde{N}, ..., (N-1),$$

while remaining $(\tilde{N} - r)$ double roots and unpaired roots are extremely small

$$e_P = 2\cos\frac{\pi(P-r)}{\tilde{N}-r}\,\frac{\Lambda_0}{\sqrt{2}}, \qquad P = (r+1), ..., (\tilde{N}-1)$$

$$e_N^{\pm} = \pm 2\frac{\Lambda_0}{\sqrt{2}}\,, \qquad \Lambda_0^{2(\tilde{N}-r)} = \frac{m^{N_f-2r}}{\Lambda_{\mathcal{N}=2}^{N-\tilde{N}}}\,, \qquad \Lambda_0 \ll m$$

Low energy theory: $\mathrm{U}(r)\times\mathrm{U}(1)^{\tilde{N}-r}$

r quarks and $(\tilde{N}-r)$ monopoles.

Monopole VEVs are determined by small ξ_P, $P=(r+1),...,(\tilde{N}-1)$

$$\xi_P = -2\sqrt{2}\,\mu\,\sqrt{(e_P-e_N^+)(e_P-e_N^-)} = -4i\,\mu\Lambda_0\,\sin\frac{\pi(P-r)}{\tilde{N}-r}$$

As we increase μ monopole VEVs

become of order of $\Lambda_0 \Rightarrow$ strong coupling

Crossover to μ-dual theory

$$\langle\Phi\rangle \approx -\frac{1}{\sqrt{2}}\,\mathrm{diag}\left[m_1,...,m_r,0,...,0,c_1\Lambda_{\mathcal{N}=2},...,c_{N-\tilde{N}}\Lambda_{\mathcal{N}=2}\right]$$

Restoration of $U(\tilde{N})$ gauge group

$$y^2 = \left(x + \frac{m}{\sqrt{2}}\right)^{2r} \left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{2(N-\tilde{N})} \left\{ \prod_{P=r+1}^{\tilde{N}} (x-\phi_P)^2 \; - 4 \frac{\left(x + \frac{m}{\sqrt{2}}\right)^{N_f - 2r}}{\left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{N-\tilde{N}}} \right\}$$

This is a curve for $U(\tilde{N})$ gauge theory with N_f flavors

In r-vacuum quark VEVs

$$\langle q^{lA} \rangle \;=\; \langle \bar{\tilde{q}}^{lA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \sqrt{\xi_r} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$l \;=\; 1, ..., \tilde{N}\,, \qquad A = 1, ..., N_f\,,$$

At large μ we integrate out adjoint matter

Superpotential

$$\mathcal{W}_{\mu-\mathrm{dual}} = -\frac{1}{2\mu}\,(\tilde{q}_A q^B)(\tilde{q}_B q^A) + m_A \tilde{q}_A q^A$$

μ-dual $U(\tilde{N})$ theory is at weak coupling if

$$\xi \sim \mu m \ll \tilde{\Lambda}^2_{\mathcal{N}=1},$$

where

$$\tilde{\Lambda}^{N_f-3\tilde{N}}_{\mathcal{N}=1} = \frac{\Lambda^{N_f-2\tilde{N}}_{\mathcal{N}=2}}{\mu^{\tilde{N}}}$$

6 Generalized Seiberg's duality

Seiberg's duality is formulated for $r = 0$ (monopole) vacua. All other $r \neq 0$ vacua are runaway vacua at $\mu = \infty$

Original theory: integrate adjoint fields at large μ

$$-\frac{1}{2\mu}\,(\tilde{q}_A q^B)(\tilde{q}_B q^A) + m_A\,(\tilde{q}_A q^A)$$

Carlino, Konishi, Murayama, 2000

Generalized Seiberg's dual: $\mathrm{U}(\tilde{N})$ gauge theory with superpotential

$$\mathcal{W}_S = -\frac{1}{2\mu}\,\mathrm{Tr}\,(M^2) + m_A\,M_A^A + \frac{1}{\kappa}\,\tilde{h}_{Al}h^{lB}\,M_B^A, \qquad l = 1, ..., \tilde{N}$$

M_A^B is the Seiberg neutral mesonic M field

Giveon and Kutasov, 2008 Problem with vacuum counting

Integrate out "dual quarks" h^{lA} $\Rightarrow$ Affleck-Dine-Seiberg superpotential

$$\mathcal{W}_{\rm ADS} = -\frac{1}{2\mu}\,{\rm Tr}\,M^2 + m_A\,{\rm Tr}\,M + (N - N_f)\,\frac{(\det M)^{\frac{1}{N_f - N}}}{\Lambda^{\frac{3N-N_f}{N_f-N}}}$$

Describes all r vacua but it is not a gauge theory

From Seiberg's dual theory at small quark masses

$$\begin{aligned} &-\frac{1}{\mu}\,M_A + \kappa\, m_A + \frac{1}{\kappa}\,\tilde h_{Al} h^{lA} = 0,\\ &M_A\, h^{lA} = \tilde h_{Al}\, M_A = 0, \end{aligned} \tag{2}$$

$$M_A = \mu\, m_A, \qquad (\tilde h h)_A = 0, \qquad A = 1, ..., p\,,$$

$$(\tilde h h)_A = -\kappa\, m_A, \qquad M_A = 0, \qquad A = (p+1), ..., N_f\,,$$

$$p > N$$

Number of these vacua

$$\mathcal{N}_{0-\mathrm{vac}} = \sum_{p=N+1}^{N_f} (p-N)\, C^{p}_{N_f} = \sum_{r=0}^{\tilde{N}-1} (\tilde{N}-r)\, C^{r}_{N_f}$$

$$p = N_f - r$$

Vacua seen in the Seiberg's dual theory at the classical level are zero vacua

Interpretation

In the zero vacua $\mathrm{U}(\tilde{N})$ is the true low-energy gauge group and dual quarks h are the correct low-energy degrees of freedom. Since the Seiberg dual theory is infrared-free it is weakly coupled in the small-m limit, provided the classical vacua exist, i.e. in the zero vacua.

Instead, in the Λ vacua, the dual quarks h are not the low-energy degrees of freedom. This explains why the Λ vacua are not seen quasiclassically. In fact, Seiberg's dual $\mathrm{U}(\tilde{N})$ theory is strongly coupled in the Λ vacua. Nevertheless, integrating out dual quarks leads to the correct ADS superpotential, which can be used only to determine chiral condensates from the chiral rings, à la Veneziano-Yankielowicz.

$\boxed{\text{Main question:}}$ Are μ-dual and Seiberg's dual match?

μ-dual	Seiberg's dual
$U(\tilde{N})$ gauge group	$U(\tilde{N})$ gauge group
N_f flavors of quarks q^{lA}	N_f flavors of "dual quarks" h^{lA}

The answer is YES

$$q^{lA} = \sqrt{-\frac{\mu}{\kappa}}\, h^{lA}\,, \quad N_A^B \equiv -\frac{1}{\mu}\, M_A^B\,, \quad l = 1, ..., \tilde{N}, \quad A = 1, ..., N_f$$

$$\mathcal{W}_S = -\frac{\mu}{2}\,\mathrm{Tr}\,(N^2) - \mu m_A\, N_A^A + \tilde{q}_{Al} q^{lB}\, N_B^A$$

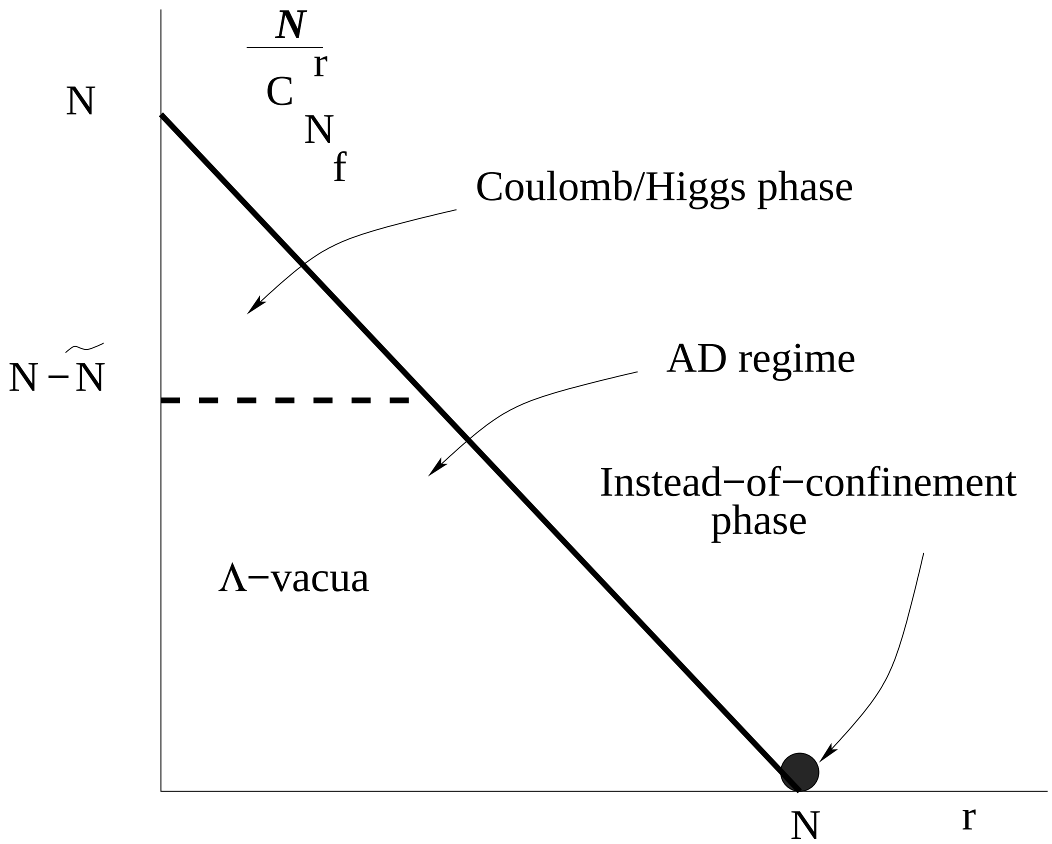
Integrate out N-fields

$$\mathcal{W}_S = \frac{1}{2\mu}\,(\tilde{q}_A q^B)(\tilde{q}_B q^A) - m_A \tilde{q}_A q^A$$

"Dual quarks" $\equiv$ quarks

7 Phases of $\mathcal{N} = 1$ QCD

$$N+1 < N_f < \frac{3}{2}N, \qquad \mu \gg \sqrt{\xi}, \qquad \sqrt{\xi} \ll \tilde{\Lambda}_{\mathcal{N}=1}$$



- Zero vacua. $U(\tilde{N})$ gauge group with N_f flavors of quarks r quarks condense. Higgs/Coulomb phase
- Λ vacua

Continuation of the Argyres-Douglas conformal strongly coupled regime to large μ

- $r = N$ Vacuum. *Shifman and Yung, 2009-2011*

Instead-of-confinement phase

$$S = 0, \qquad \xi_P = -2\sqrt{2}\,\mu\, e_P = 2\mu m_P, \qquad P = 1, ..., \tilde{N}$$

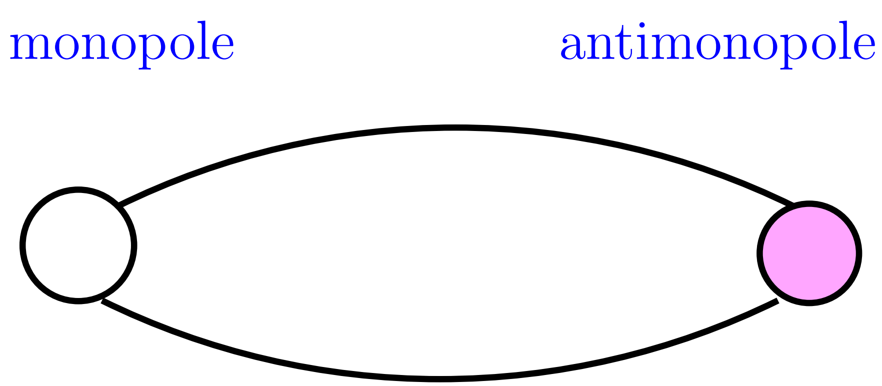
Instead-of-confinement phase

The small-mass limit can be described by weakly coupled infrared-free dual theory U($\tilde{N}$) gauge group with N_f flavors of quark-like dyons.

The quark-like dyons condense leading to the formation of non-Abelian strings which confine monopoles.

Higgs-screened quarks and gauge bosons decay into monopole-antimonopole pairs at CMS.

At $\xi \neq 0$ monopoles are confined and cannot move apart



In the region of small ξ Higgs-screened quarks and gauge bosons become stringy monopole-antimonopole mesons

8 Conclusions

- μ-Dual theory matches with Seiberg's dual in zero vacua
  $U(\tilde{N})$ gauge theory with N_f quarks
- In zero r-vacua we have Higgs/Coulomb phase.
  r quarks condense, $r < \tilde{N}$
- There is no quark confinement phase in $\mathcal{N} = 1$ SQCD in the domain of small ξ.
- The phase most close to what we observe in the real-world QCD is the "instead-of-confinement" phase present in the $r = N$ vacuum.