Collective Fluctuations of Wilson Dirac Spectra and First Order Phase Transitions

Jacobus Verbaarschot

jacobus.verbaarschot@stonybrook.edu

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Stony Brook University

Acknowledgments

Collaborators: Gernot Akemann (Bielefeld)
Poul Damgaard (NBIA)
Mario Kieburg (Bielefeld)
Kim Splittorff (NBI)
Savvas Zafeiropoulos (Stony Brook)

Relevant Papers

P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Microscopic Spectrum of the Wilson Dirac Operator, Phys. Rev. Lett.

G. Akemann, P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Spectrum of the Wilson Dirac Operator at Finite Lattice Spacing, to be published.

P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Microscopic Spectrum of the Wilson Dirac Operator, arXiv:1001.2937 [hep-th].

K. Splittorff and J. J. M. Verbaarschot, The Wilson Dirac Spectrum for QCD with Dynamical Quarks, Phys. Rev. **D** 84 (2011) 065031 [arXiv:1104.6229 [hep-lat]].

K. Splittorff and J. J. M. Verbaarschot, The Microscopic Twisted Mass Dirac Spectrum, Phys. Rev. **D8** 5 (2012) 105008, arXiv:1201.1361 [hep-lat].

M. Kieburg, J. J. M. Verbaarschot, S. Zafeiropoulos Eigenvalue Density of the Non-Hermitian Wilson Dirac Operator,, Phys. Rev. Lett. **108** (2012) 022001 arXiv:1109.0656 [hep-lat].

M. Kieburg, K. Splittorff and J. J. M. Verbaarschot, The Realization of the Sharpe-Singleton Scenario, Phys. Rev. **D85**(2012) 094011 [arXiv:1202.0620 [hep-lat]].

M. Kieburg, K. Splittorff and J. J. M. Verbaarschot, to be published.

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I. Introduction

Wilson Dirac Operator

Spectra and Phases

First Order Scenario

Doubling Problem

- Because the Dirac operator in linear in the derivatives we have the dispersion relation

$$E = \sin pa.$$

This results in an unwanted low energy mode.

- Way out (Wilson 1974) : Add $a\nabla^2$ to the Dirac operator so that the modes at $p = \pi/a$ are lifted.

$$D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - \frac{1}{2}a\nabla_\mu\nabla_\mu^*.$$

- In agreement with the Nielsen-Ninomiya theorem this term violates the chiral symmetry of the Dirac operator.

Wilson Dirac Spectra

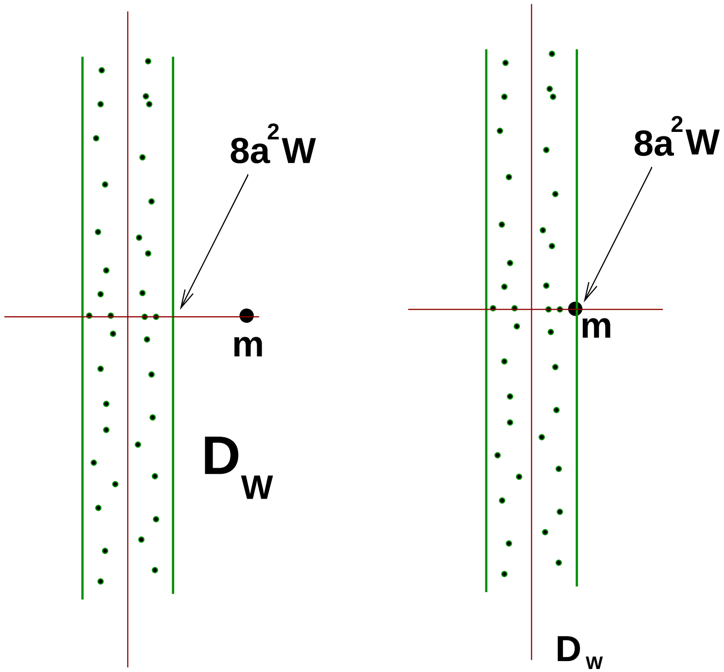
Block structure

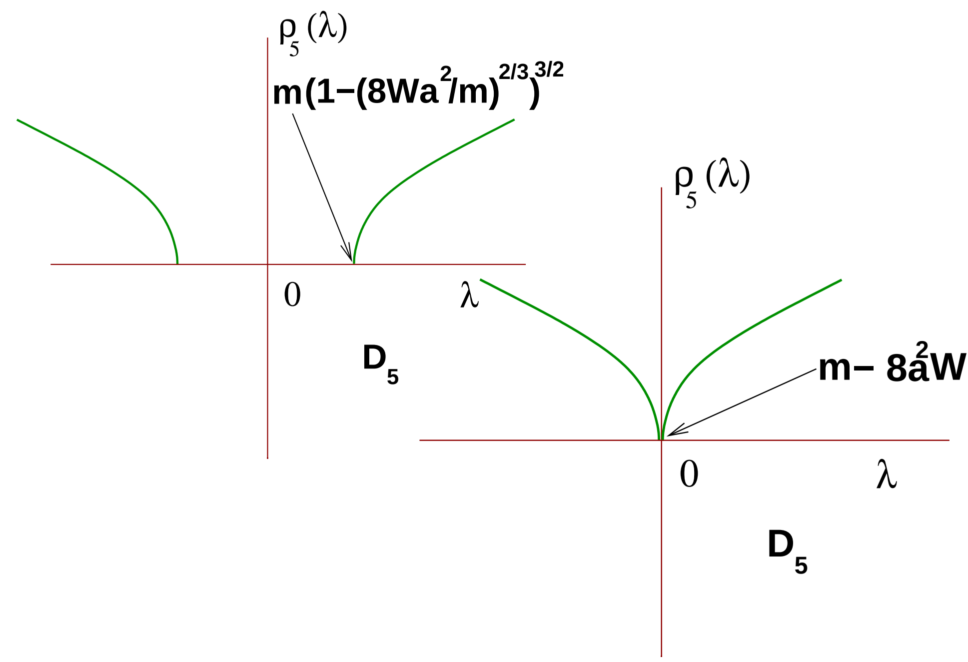
$$D_W = \begin{pmatrix} aA & id \\ id^\dagger & aB \end{pmatrix}$$

with $\quad A^\dagger = A, \; B^\dagger = B.$

Hermitian Wilson Dirac operator

$$D_5 \equiv \gamma_5 (D_W + m) = D_5^\dagger$$

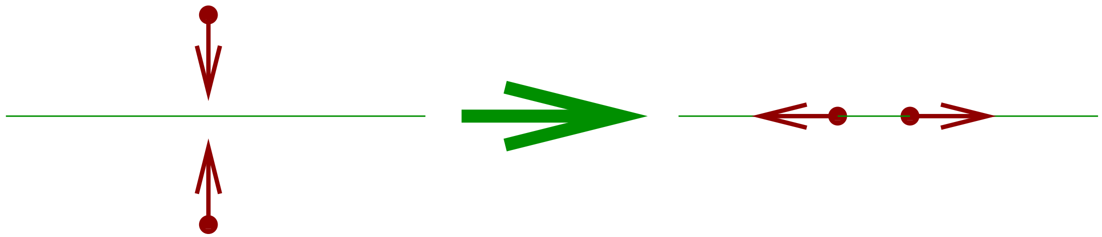




Aoki Phase

Pseudo Hermitian Operators

$$D_W^\dagger \neq D_W$$



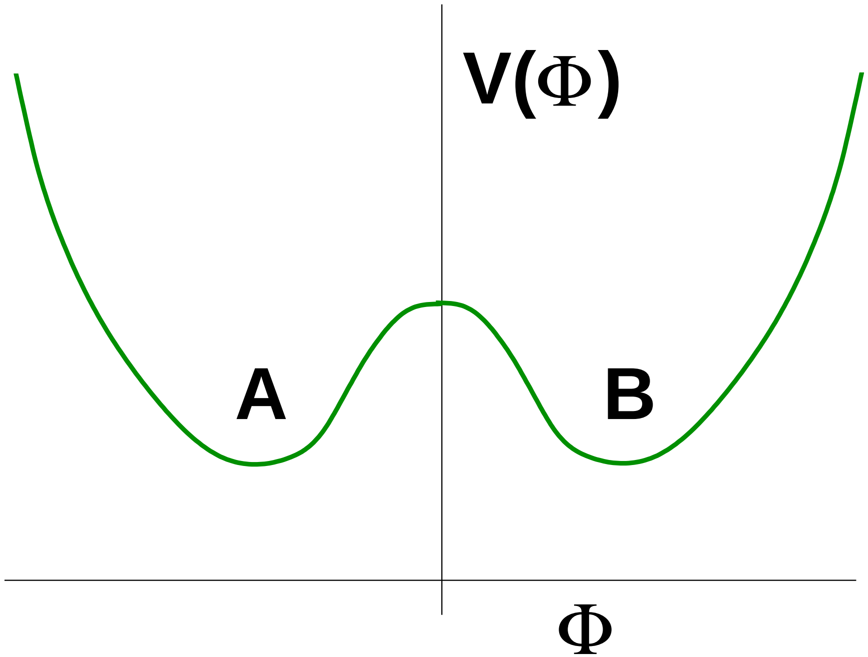
$$D_W^\dagger = \gamma_5 D_W \gamma_5$$

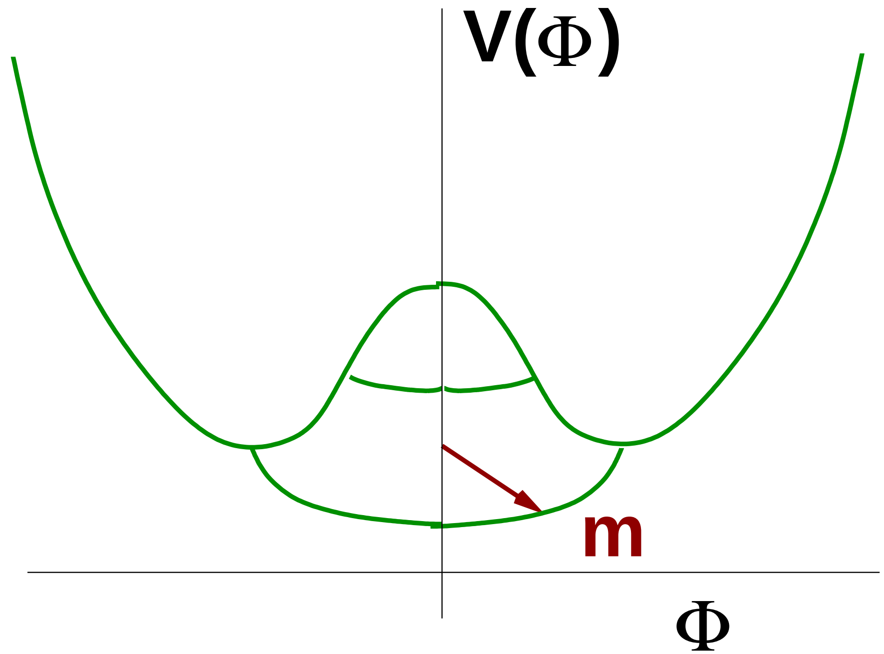
- Two complex eigenvalues can collide and turn into a pair of real eigenvalues with oposie chirality. The number of real eigenvalues does not change under small deformations of the operator.
- What is the distribution of the real pairs?

Spectra and Phases

- A phase transition takes place when the gap closes. The new phase is known as the Aoki phase.
- The transition to the Aoki phase is continuous.
- Sharpe and Singleton proposed another possibility that results in a first order phase transition.
- How can we understand the first order behavior in terms of collective fluctuations of Dirac eigenvalues?

First Order Scenario





In the first order scenario the effective potential for the chiral condensate has only two minima. A nonzero quark mass slightly tilts the potential and the condensate jumps from one minimum the the other. Sharpe-Singleton
There is no flat direction and the pion mass does not vanish for vanishing quark mass.

Effective potential for the chiral condensate in continuum limit. The chiral condensate rotates the direction of the quark mass.
Because of the flat direction the pion mass vanishes for vanishing quark mass

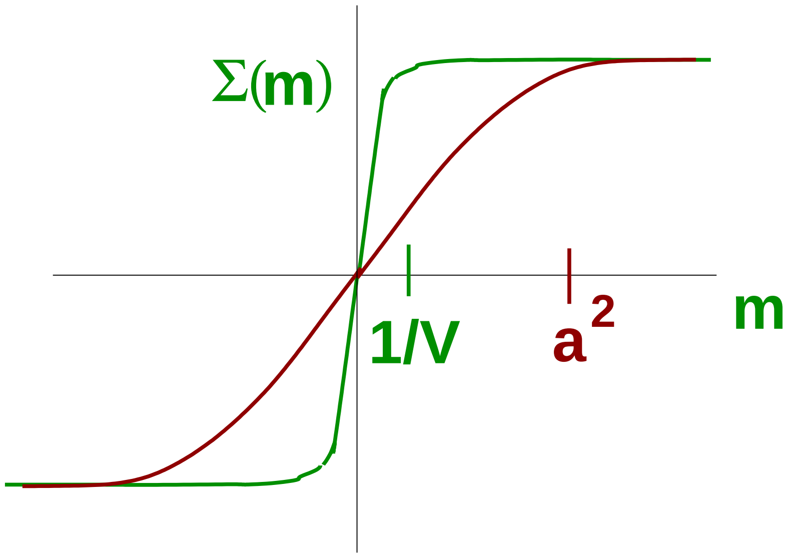
$$m_\pi \sim \sqrt{m}.$$

In the Aoki phase the direction of the chiral condensate is realigned by discretization effects.

First Order Scenario and Dirac Spectra

- How can we understand a first order scenario in terms of Dirac spectra?
- Wilson Dirac eigenvalues are in a strip of width $\sim a^2$.
- Banks-Casher Relation

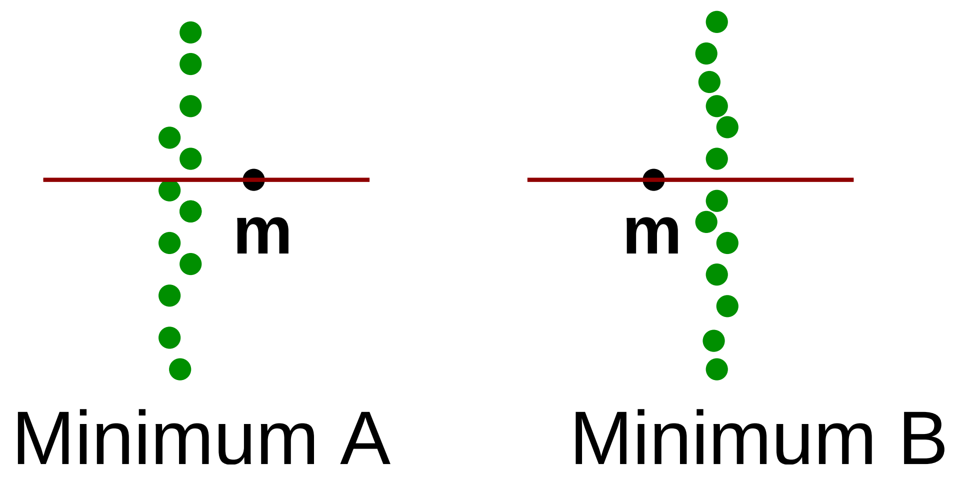
$$\Sigma(m) = \lim \frac{1}{V} \sum_k \frac{2m}{\lambda_k^2 + m^2}$$



Mass Dependence of the Chiral Condensate in a First Order Scenario (green) and for the Aoki phase (red).

First Order Scenario and Dirac Spectra

- Because the Wilson Dirac operator in neither Hermitian nor anti-Hermitian its eigenvalues can move.
- Because of the fermion determinant they will be repelled from the quark mass.
- The finite jump of the Dirac spectrum results in a first order phase transition.



The fuzzy string of eigenvalues is repelled from the mass, m , which results in a first order phase transition.

II. Collective Spectral Fluctuations

Collective Spectral Fluctuations

Chiral Lagrangian

Sign of Low Energy Constants

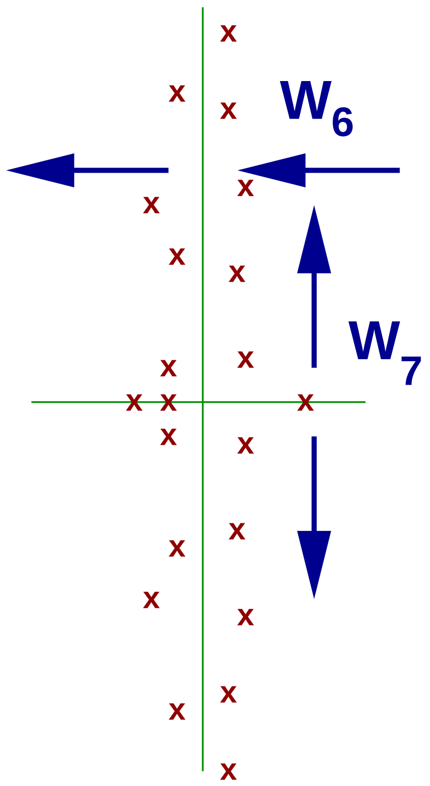
Microscopic versus Collective Spectral Fluctuations

- Microscopic fluctuations are fluctuations that occur on the scale of the average eigenvalue spacing.
- Collective fluctuations occur on the scale of many eigenvalue spacings.
- Collective fluctuations have to be consistent with the symmetries of the Dirac operator.

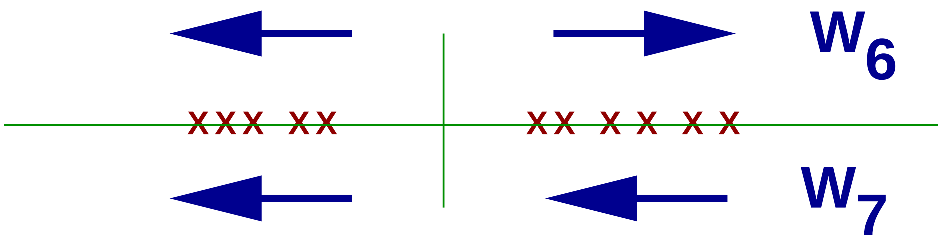
Collective Fluctuations due to Instantons

- The fluctuations of the number of quasi zero modes of the Dirac operator scales with $\sqrt{V}$ and induces fluctuation in the average spectral density of $\delta\rho/\langle\rho\rangle \sim 1/\sqrt{V}$. This gives rise to a Thouless energy with the correct scaling behavior.
- This implies that the microstructure in the spectrum of the Dirac operator gets wiped out at a scale of $\lambda \sim 1/\sqrt{V}$.
- This scale is known as the Thouless energy.

Possible Collective Spectral Fluctuations



Spectrum of D_W



Spectrum of D_5

Collective spectral fluctuations of the Wilson Dirac operator consistent with complex conjugation or reality.

What is the source of these fluctuations? Kieburg-Splittorff-JV-2013

Chiral Lagrangian

Chiral Lagrangian for Wilson Fermions in the microscopic domain

$$
\begin{aligned}
-\mathcal{L} \quad &= \quad \frac{1}{2} m V \Sigma \mathrm{Tr}(U + U^\dagger) + \\
&\quad a^2 V W_6 [\mathrm{Tr}(U + U^\dagger)]^2 + a^2 V W_7 [\mathrm{Tr}(U - U^\dagger)]^2 - a^2 V W_8 \mathrm{Tr}(U^2 + U^{-2})
\end{aligned}
$$

Sharpe-Singleton-1998, Rupak-Shoresh-2002, Bär-Rupak-Shoresh-2004, Damgaard-Splittorff-JV- 2011

Single trace terms are responsible for microscopic eigenvalue fluctuations.

Trace Squared Terms

- Trace squared terms can be linearized at the expense of a Gaussian integral and then can be added to the mass term.
- This random mass can interpreted in terms collective fluctuations of Dirac eigenvalues.

Collective Eigenvalue Fluctuations and W_6

For $W_6 < 0$ we have

$$e^{-a^2 V W_6 \mathrm{Tr}^2(U+U^{-1})} \sim \int dy e^{-y^2/(16V|W_6|a^2) - \frac{1}{2} y \mathrm{Tr}(U+U^{-1})}$$

The partition function can be written as

$$\begin{aligned} Z(m; W_6, W_8) &= \int dy e^{-y^2/16V|W_6|a^2} Z(m-y; W_6 = 0, W_8) \\ &= \int dy e^{-y^2/16V|W_6|a^2} \langle \prod_k (m - y - \lambda_k) \rangle. \end{aligned}$$

A negative W_6 therefore corresponds to collective fluctuations of the strip of eigenvalues.

Eigenvalue fluctuations cannot generate a positive value of W_6 .

Signs of W_6 , W_7 and W_8

- A positive W_6 would correspond to collective eigenvalues fluctuations in the imaginary direction which are forbidden because of the complex conjugation property. This implies that $W_6 < 0$.
  Akemann-Damgaard-Splittorff-JV-2010, Kieburg-Splittorff-JV-2012
- A similar argument can be made for W_7 . This results in a random mass term $\gamma_5 y$ and the requirement that $W_7 < 0$.
- The positivity of $W_8 - W_6 - W_7$ follows from pseudo-Hermiticity of the Wilson Dirac operator. Akemann-Damgaard-Splittorff-JV-2011, Hansen-Sharpe-2011
- $W_8 > 0$ follows from mass inequalities of the partial quenched theory. Hansen-Sharpe-2011

Predictions of First Order Scenario

- Since the quark mass remains at a finite distance from the eigenvalues, we have a minimum value for the pion mass.
- The first order scenario can only occur for dynamical quarks.
- In the quenched case only a transition to an Aoki phase can occur for a sufficiently small quark mass.
- This is in agreement with all lattice simulations with Wilson fermions.

III. Fluctuations of Wilson Dirac Spectra

Chiral Lagrangian

Random Matrix Theory

First Order Scenario

Dirac Spectra and Chiral Lagrangians

- The mass scale of Dirac spectra on the scale of the smallest is $\sim 1/V$. Therefore the Goldstone bosons corresponding to this mass scale have a mass $\sim 1/\sqrt{V}$.
- The Compton wavelength of these Goldstone bosons $\sim \sqrt{V}$ which for large volumes is much larger than the size of the box.
- Therefore when chiral symmetry is broken spontaneously the behavior of Dirac spectra on the scale of individual eigenvalue spacing can be obtained from the zero momentum part of a chiral Lagrangian.
- The zero momentum part is equivalent to a Random Matrix Theory with the same global symmetries.

Random Matrix Theory for the Wilson Dirac Operator

Since the chiral Lagrangian is determined uniquely by symmetries, in the microscopic domain it also can be obtained from a random matrix theory with the same symmetries. In the sector of index ν the random matrix partition function is given by

$$Z^{\nu}_{N_f} = \int dA dB dW \det{}^{N_f}(D_W + m + z\gamma_5)\, P(D_W),$$

with

$$D_W = \begin{pmatrix} aA & C \\ -C^\dagger & aB \end{pmatrix} \qquad \text{and} \quad A^\dagger = A, \qquad B^\dagger = B.$$

where A is a square matrix of size $n \times n$, and B is a square matrix of size $(n+\nu) \times (n+\nu)$. The matrices C and D are complex $n \times (n+\nu)$ matrices. Damgaard-Splittorff-JV-2010

Comments

- The index of the D_W defined by spectral flow is equal to ν .
- Universality: the properties of this random matrix theory do not depend on the details of $P(D_W)$.
- In the microscopic domain, the Random Matrix Theory partition function reduces to the (twisted mass) chiral Lagrangian introduced before with $W_8 > 0$ and $W_6 = W_7 = 0$.
- The trace squared terms can be introduced via additional Gaussian integrals.
- The advantage of a random matrix formulations is that we can use powerful random matrix methods to calculate spectral observables.
  Kieburg-Zafeiropoulos-JV-2011, Akemann-Nagao-2011

Spectral Density of D_W

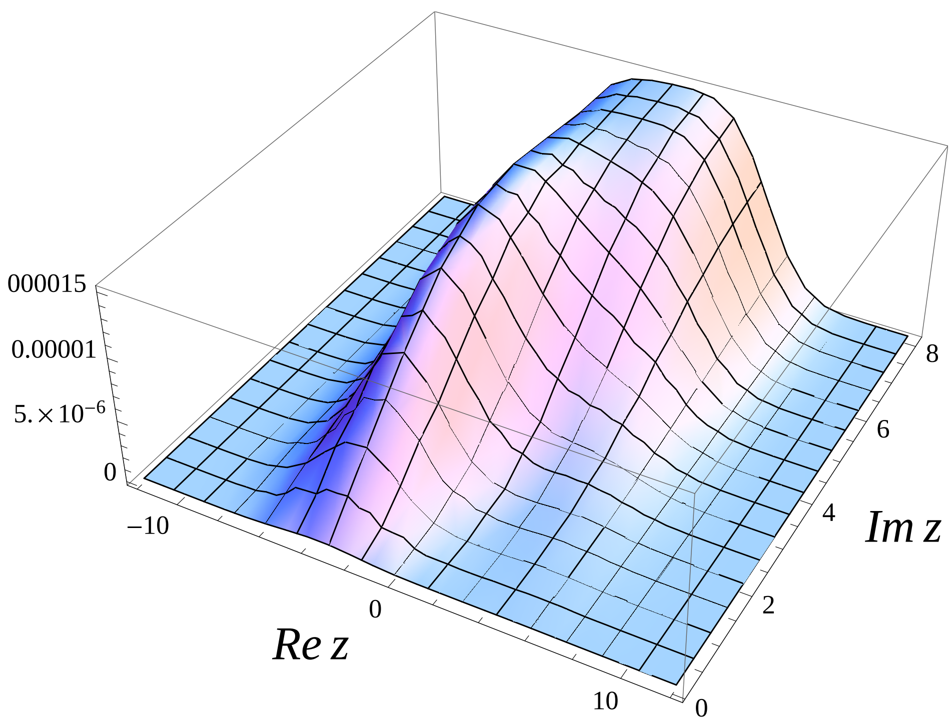
The spectral density of the complex eigenvalues of the Wilson Dirac operator is given by

Kieburg-JV-Zafeiropoulos-2011

$$\rho_c(z, z^*) = |z - z^*|^2 Z_{N_f=-2}(z, z^*; a_*) Z_{N_f=2}(z, z^*; a_8).$$

For two dynamical quarks with mass m the spectral density is given by

$$\rho_{N_f=2}(z, z^*) = \frac{|z - z^*|^2 (z - m^2)(z^* - m)^2 Z_{N_f=-2}(z, z^*; a_8) Z_{N_f=4}(m, m, z, z^*.; a_8)}{Z_{N_f=2}(m, m; a_8)}$$



Kieburg-Splittorf-JV-2012

Density of Complex Eigenvalues

$$\rho^{\nu}_{c,N_f}(z,z^*,m;a_6,a_8) = \frac{\int dy e^{-\frac{y^2 V}{16|W_6|a^2}} Z^{\nu}_{N_f}(m-y;0,a_8)\rho^{\nu}_{c,N_f=2}(z-y,z^*-y,m-y;0,a_8)}{Z^{\nu}_{N_f}(m;a_6,a_8)}$$

We work this out for $mV\Sigma \gg 1$ and $a^2 W_k V \gg 1$. Then we can use a mean field approximation. The Dirac spectrum is inside a strip

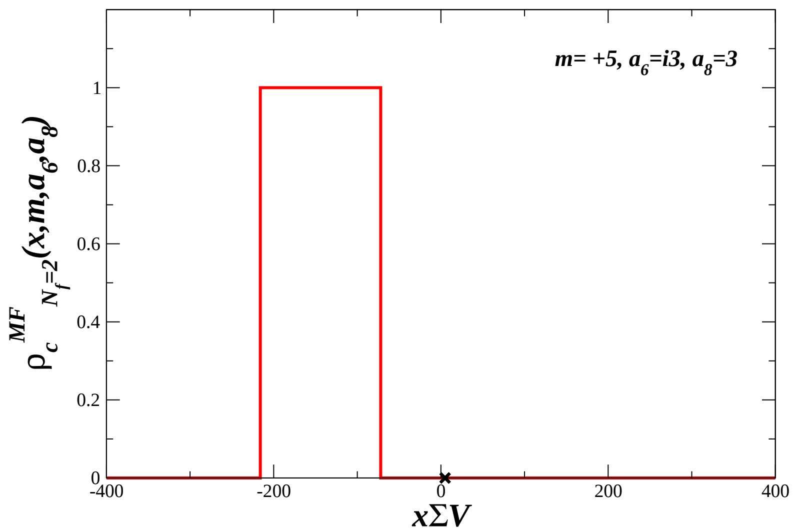
$$\rho^{\nu,\mathrm{MFT}}_{c,N_f=2}(z,z^*,m;0,a_8) = \theta(8a_8^2 - x\Sigma).$$

The mean field limit of the partition function is given by

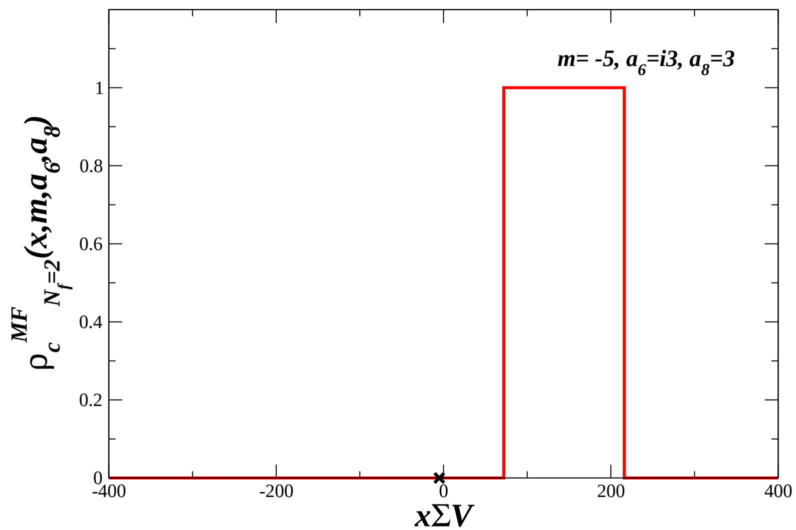
$$Z_2^{\mathrm{MF}}(m;0,a_8) = e^{2mV\Sigma - 4Va_8^2} + e^{-2mV\Sigma-4Va_8^2} + \theta(8a_8^2 - |m\Sigma|)e^{Vm^2\Sigma^2/8a_8^2 + 4Va_8^2}.$$

Kieburg-Splittorff-JV-2012

The First Order Scenario at Work



Spectral density of Wilson Dirac operator for $m = 5$, $a_6^2 V = -9$, $a_8^2 V = 9$.



Spectral density of Wilson Dirac operator for $m = -5$, $a_6^2 V = -9$, $a_8^2 V = 9$.

The strip of eigenvalues is repelled from the quark mass. The distance between the quark mass and the strip of eigenvalues is given by

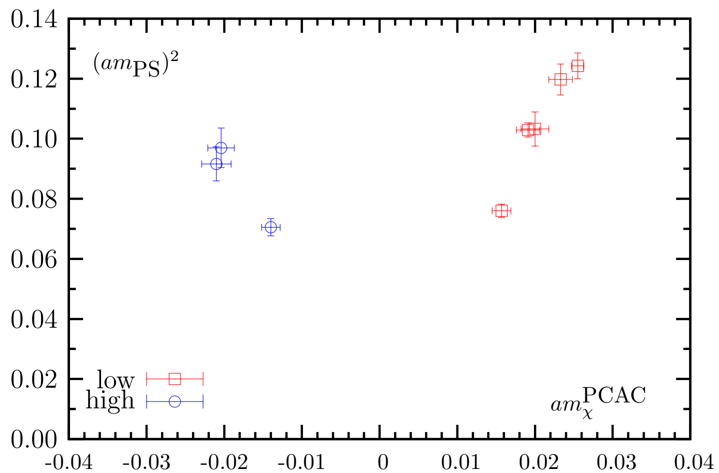
$$|m|V - 8(a_8 + 2a_6)V/\Sigma.$$

Minimum Pion Mass

- Minimum Pion mass Sharpe-Singleton-2004, Münster-2004

$$m_\pi^2 = \frac{2|m|\Sigma - 16(W_8 + 2W_6)a^2}{F_\pi^2}.$$

When $W_8 + 2W_6 < 0$ we have a minimum pion mass. This has been observed in lattice simulations with twisted mass fermions. Jansen-etal-2005



The minimum pion mass is $O(a)$.

IV. Determining the Low Energy Constants

Exact Results

Comparison to Lattice Data

Simple Relations for the Small a -Limit

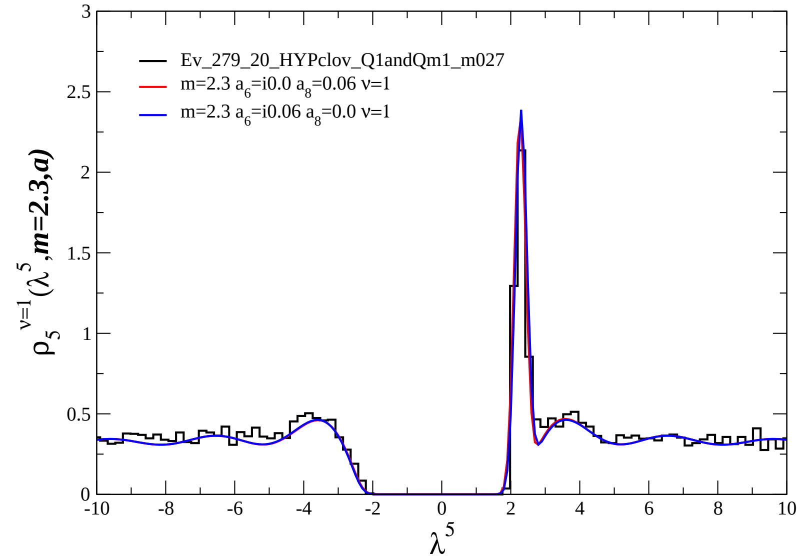
Determining the Low Energy Constants from Dirac Spectra

- Low Energy Constants can be obtained by fitting analytical results to lattice data

- We will use exact results for Dirac spectral to do so:
  - ⋆ Density of complex eigenvalues: $\rho_c(z)$
  - ⋆ Density of the right handed modes: $\rho_r(x)$
  - ⋆ Density of the left handed modes: $\rho_l(x)$
  - ⋆ The chirality distribution: $\rho_\chi(x) \equiv \rho_r(x) - \rho_l(x)$
  - ⋆ The eigenvalue density of $\gamma_5 D_W$: $\rho_5(x)$

Damgaard-Splittorff-JV-2010,Akemann-Nagao-2011,Larssen-2012,Kieburg-JV-Zafeiropoulos-2011,Splittorff-JV-2011,Kieburg-Splittorff-JV-2012

Eigenvalue Density of $\gamma_5(D_W + m)$



The microscopic spectrum of $\gamma_5(D_W + m)$ for $\nu = 1$ Damgaard-Heller-Splittorff-2012.

The red and blue curves represents the analytical result for the resolvent Splittorff-JV-2011

$$G^\nu(m, z; a) = \frac{1}{16a^2\pi} \int \frac{dsdt}{t - is} e^{-[(s+iz)^2+(t-z')^2]/16a^2} \frac{(m - is)^\nu}{(m - t)^\nu} \tilde{Z}^\nu_{1|1}(\sqrt{m^2 + s^2}, \sqrt{m^2 - t^2}; a = 0$$

where

$$\tilde{Z}^\nu_{1|1}(x, y; a = 0) = \frac{y^\nu}{x^\nu}[yK_{\nu+1}(y)I_\nu(x) + xK\nu(y)I_{\nu+1}(x)]$$

The width of the topological peak behaves as $\sim 1/\sqrt{V}$.

Observables in the small a Limit

- The density of the projection of the eigenvalues of D_W on the imaginary axis. According to the Banks-Casher formula we have

$$\Delta = \frac{\pi}{\Sigma V}.$$

- The average number of the additional real modes for $\nu = 0$:

$$N_{\rm add}^{\nu=0} \quad \overset{a \ll 1}{=} \quad 2Va^2(W_8 - 2W_7).$$

- The width of the Gaussian shaped strip of complex eigenvalues:

$$\frac{\sigma^2}{(\Delta)^2} \quad \overset{a \ll 1}{=} \quad \frac{4}{\pi^2}a^2V(W_8 - 2W_6).$$

Observables in the small a Limit

- The variance of the distribution of chirality over the real eigenvalues:

$$\frac{\langle x^2 \rangle_{\rho_\chi^\nu}}{\Delta^2} \overset{a\ll 1}{=} \frac{8}{\pi^2} V a^2 (\nu W_8 - W_6 - W_7), \; \nu > 0.$$

There are linear dependencies between the relations. This results in the consistency condition

$$\frac{\langle x^2 \rangle_{\rho_\chi}^{\nu=1}}{\Delta^2} = \frac{\sigma^2}{\Delta^2} + \frac{2}{\pi^2} N_{\rm add}^{\nu=0}.$$

Kieburg-JV-Zafeiropoulos-2013

V. Conclusions

- Collective spectral fluctuations limit the domain of validity of random matrix theory.

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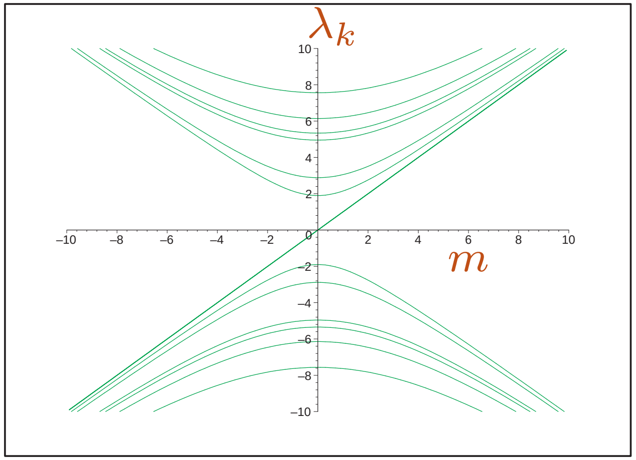
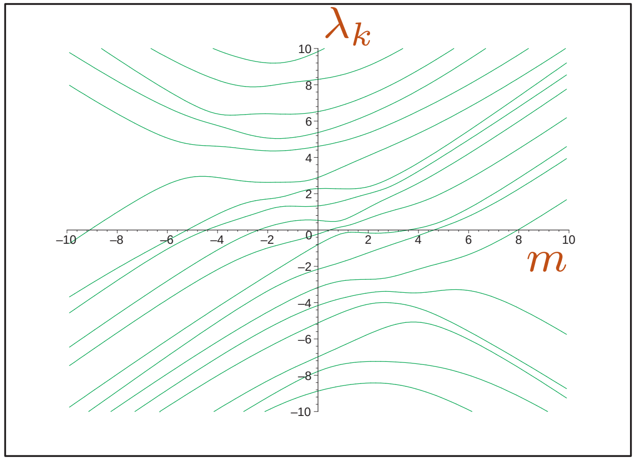
- Collective spectral fluctuations drive the first order phase transition for Wilson fermions.
- Eigenvalue fluctuations constrain the signs of the low energy constants in the chiral Lagrangian.
- The value of the low-energy constants can determined by fitting analytical results for Dirac spectra to lattice data.

Spectral Flow at Nonzero Topology





The topological charge is given by the difference of the number of positive eigenvalues for large positive charge and the number of positive eigenvalues for large negative mass.

If ϕ is an eigenfunction of a topological zero mode,then

$$a = 0: \qquad D_W\phi = 0, \qquad \gamma_5\phi = \phi, \Longrightarrow \gamma_5(D_W + m)\phi = m\phi.$$

$$a \neq 0: \qquad \gamma_5(D_W + m_k)\phi_k = 0 \Longrightarrow D_W\phi_k = -m_k\phi_k.$$

For $a \neq 0$ the flow-line may cross the x -axis more than once so that the number is real modes is larger than ν . This generically does not happen in the ϵ domain.

Dirac Spectrum for $a = 0$

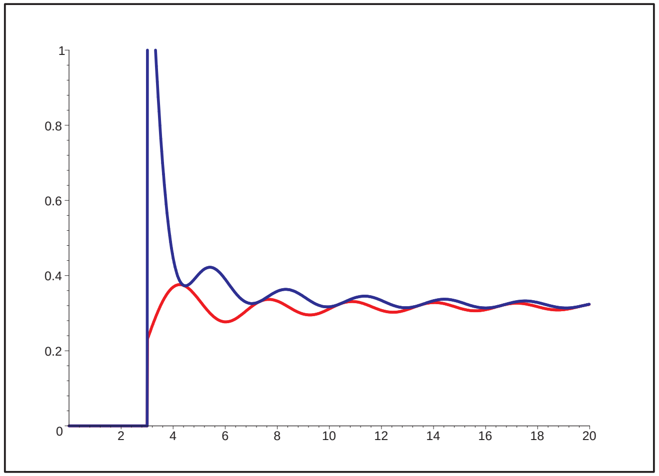
For $a = 0$ we have that

$$\gamma_5(D_W + m) = \begin{pmatrix} m & C \\ C^\dagger & -m \end{pmatrix}.$$

C can be brought to a diagonal form by a unitary transformation. This results in the spectrum of D_5

$$(\lambda_k - m)^2 - |c_k|^2 = 0 \quad \Longrightarrow \quad \lambda_k = \pm\sqrt{m^2 + |c_k|^2}$$

That is why $\gamma_5(D_W + m)$ has a gap $[-m, m]$ for $a = 0$. For $a \neq 0$ states intrude inside the gap.



The microscopic spectral density of $\gamma_5(D_W + m)$ for $m = 3$, $\nu = 0$ and $a = 0$ for different number of flavors. Results are shown for $N_f = 0$ (blue) and $N_f = 2$ (red).

Mean Field Result for the Gap

$W_8 > 0$

The spectral density of $\gamma_5(D_W + m)$ vanishes if the correspond resolvent is real. For large $mV\Sigma$ and a^2VW_8 the resolvent can be calculated by a saddle point approximation. The result for the spectral gap $[-z_c, z_c]$ is given by (in units where $\Sigma = 1$ and $W_8 = 1$)

$$z_c = m[1 - (8a^2/m)^{2/3}]^{3/2}.$$

The gap closes when $8a^2 = m$. This is the onset of the Aoki phase.

$W_8 < 0$

The spectrum of $D_5 = \gamma_5(D_W + m)$ is complex and this chiral Lagrangian cannot be used to calculate the spectrum of D_5 . To calculate the spectrum one has to introduce a partition function with quarks and conjugate anti-quarks as is the case for QCD at nonzero chemical potential (see Stephanov-1996).

Microscopic Spectral Spectral Density

The resolvent is given by

$$
\begin{aligned}
G^\nu(z,m;a) \quad &= \quad \int_{-\infty}^{\infty} ds \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, \frac{i}{2} \cos(\theta) e^{S_f+S_b} e^{(i\theta-s)\nu} \\
&\times \left(-m\sin(\theta) + im\sinh(s) + iz\cos(\theta) + iz\cosh(s)\right. \\
&\quad \left. +4a^2[\cos(2\theta) + \cosh(2s) + (e^{i\theta+s} + e^{-i\theta-s})] + 1\right).
\end{aligned}
$$

Here,

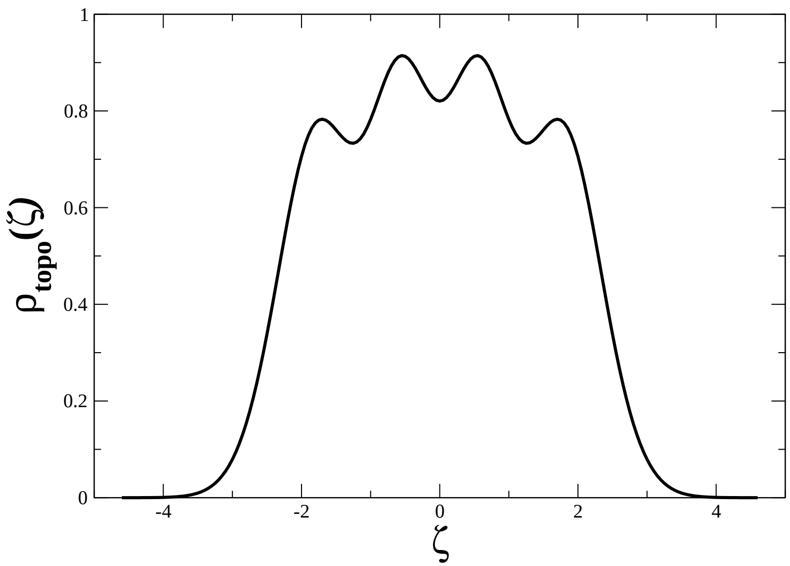
$$
\begin{aligned}
S_f \quad &= \quad -m\sin(\theta) + iz\cos(\theta) + 2a^2\cos(2\theta), \\
S_b \quad &= \quad -im\sinh(s) - iz\cosh(s) - 2a^2\cosh(2s).
\end{aligned}
$$

The microscopic quenched spectral density is equal to

$$
\rho_5^\nu(x,m;a) \quad = \quad \frac{1}{\pi}\mathrm{Im}[G^\nu(x,m;a))].
$$

Damgaard-Splittorff-JV-2010

Density of Real Eigenvalues



The quenched spectral density of the topological real eigenvalues of D_W for $\nu = 4$ and $a = 0.25$.

Damgaard-Splittorff-JV-2010

The real eigenvalues of D_W give rise to a cut on top of the cloud of complex eigenvalues. This implies that the spectral density of the real modes is given by the discontinuity of the chiral condensate. Because $\Sigma(m)$ is real for real m the discontinuity is given by the imaginary part.

$$\rho^{\text{topological}}(m) = \frac{1}{\pi}\text{Im}\,\Sigma(m,a).$$

For large ν the distribution of real eigenvalues approaches a semicircle. This may be the first time that a semi-circular distribution of eigenvalues has been in a physical system. Note that since this result is derived from a chiral Lagrangian, it is universal.