

# NLO BFKL and anomalous dimensions of light-ray operators

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JLAB & ODU

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- Regge limit in the coordinate space
- “BFKL” representation of 4-point correlation function in  $\mathcal{N} = 4$  SYM.
- Light-ray operators.
- “DGLAP” representation of 4-point correlation function.
- Anomalous dimensions from DGAP vs BFKL representations.

## Gluon operators of leading twist

$$\mathcal{O}_n^g \equiv F_{-i}^a \nabla_{-}^{n-2} F_{-}^{ai}$$

Anomalous dimension (in gluodynamics)

$$\gamma_n = N_c \left[ -\frac{4}{j-1} + \frac{4}{j} - \frac{4}{j+1} + \frac{4}{j+2} + 4[\psi(j+1) + C] - \frac{11}{3} \right] + \mathcal{O}(\alpha_s^2)$$

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BFKL gives  $\gamma(j\alpha_s)$  at the non-physical point  $j \rightarrow 1$

$$\gamma_j = \left[ A_n^{\text{LO BFKL}} + (j-1)B_n^{\text{NLO BFKL}} + \dots \right] \left( \frac{\alpha_s N_c}{\pi(j-1)} \right)^n$$

$\omega = j-1 \rightarrow 0$  corresponds to  $F_{-i}^a \nabla_-^{\omega-1} F_-^{ai}$  - some non-local operator.

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Q: which one?

A: gluon light-ray (LR) operator

## 4-point correlation function (CF)

$$A(x, y, x', y') = [(x - y)^2(x' - y')^2]^{2+\gamma_k} N_c^2 \langle \mathcal{O}(x) \mathcal{O}(y) \mathcal{O}(x') \mathcal{O}(y') \rangle$$

$\mathcal{O} = \phi_I^a \phi_I^a$  - Konishi operator ( $\gamma_k$  - anomalous dimension)

In a conformal theory the amplitude is a function of two conformal ratios

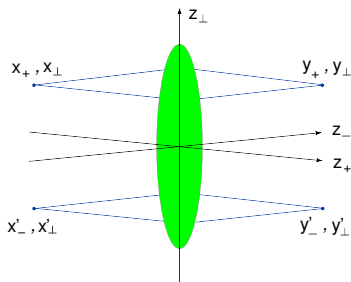
$$A = F(R, R')$$
$$R = \frac{(x - y)^2(x' - y')^2}{(x - x')^2(y - y')^2}, \quad R' = \frac{(x - y)^2(x' - y')^2}{(x - y')^2(x' - y)^2}$$

At large  $N_c$

$$A(x, y, x', y') = A(g^2 N_c) \quad g^2 N_c = \lambda \quad \text{‘t Hooft coupling}$$

# Regge limit in the coordinate space

Regge limit:  $x_+ \rightarrow \rho x_+$ ,  $x'_+ \rightarrow \rho x'_+$ ,  $y_- \rightarrow \rho' y_-$ ,  $y'_- \rightarrow \rho' y'_-$      $\rho, \rho' \rightarrow \infty$



Full 4-dim conformal group:  $A = F(R, r)$

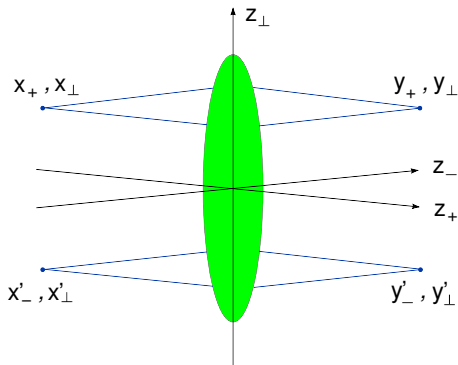
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \rightarrow \frac{\rho^2 \rho'^2 x_+ x'_+ y_- y'_-}{(x-x')_{\perp}^2 (y-y')_{\perp}^2} \rightarrow \infty$$

$$r = \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y)^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\rightarrow \frac{[(x'-y')_{\perp}^2 x_+ y_- + x'_+ y'_- (x-y)_{\perp}^2 + x_+ y'_- (x'-y)_{\perp}^2 + x'_+ y_- (x-y')_{\perp}^2]^2}{(x-x')_{\perp}^2 (y-y')_{\perp}^2 x_+ x'_+ y_- y'_-}$$

## 4-dim conformal group versus $SL(2, C)$

Regge limit:  $x_+ \rightarrow \rho x_+, x'_+ \rightarrow \rho x'_+, y_- \rightarrow \rho' y_-, y'_- \rightarrow \rho' y'_-$   
 $\rho, \rho' \rightarrow \infty$



Regge limit symmetry: 2-dim conformal group  $SL(2, C)$  formed from  $P_1, P_2, M^{12}, D, K_1$  and  $K_2$  which leave the plane  $(0, 0, z_\perp)$  invariant.



$$A(x, y; x', y') \stackrel{s \rightarrow \infty}{\simeq} \frac{i}{2} \int d\nu f_+(\aleph(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\aleph(\lambda, \nu)/2}$$

L. Cornalba (2007)

$$f_+(\omega) = \frac{e^{i\pi\omega} - 1}{\sin \pi\omega} \text{ - signature factor}$$

$$\Omega(r, \nu) = \frac{\sin \nu \rho}{\sinh \rho}, \quad \cosh \rho = \frac{\sqrt{r}}{2}$$

- solution of the eqn  $(\square_{H_3} + \nu^2 + 1)\Omega(r, \nu) = 0$

The dynamics is described by:

$\aleph(\lambda, \nu)$  - pomeron intercept,

and

$F(\lambda, \nu)$  - “pomeron residue”.

$$A(x, y; x', y') \stackrel{s \rightarrow \infty}{\equiv} \frac{i}{2} \int d\nu f_+(\aleph(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\aleph(\lambda, \nu)/2}$$

Pomeron intercept:

$$\aleph(\nu, \lambda) = \frac{\lambda}{4\pi^2} \left[ \chi(\nu) + \frac{\lambda}{16\pi^2} \delta(\nu) \right] + \mathcal{O}(\lambda^3)$$

$$\chi(\nu) = 2\psi(1) - \psi\left(\frac{1}{2} + i\nu\right) - \psi\left(\frac{1}{2} - i\nu\right) = -S_1\left(-\frac{1}{2} + i\nu\right) + \text{c.c.} \quad - \text{BFKL intercept,}$$

$$\delta(\nu) = -\frac{3}{2}\zeta(3) + \pi^2 \ln 2 + \frac{\pi^2}{3} S_1\left(-\frac{1}{2} + i\nu\right) + 2S_3\left(-\frac{1}{2} + i\nu\right) + \pi^2 S_{-1}\left(-\frac{1}{2} + i\nu\right) - 4S_{-2,1}\left(-\frac{1}{2} + i\nu\right) + \text{c.c.}$$

- NLO BFKL intercept       $S_i(x)$  - harmonic sums      Lipatov, Kotikov (2000)

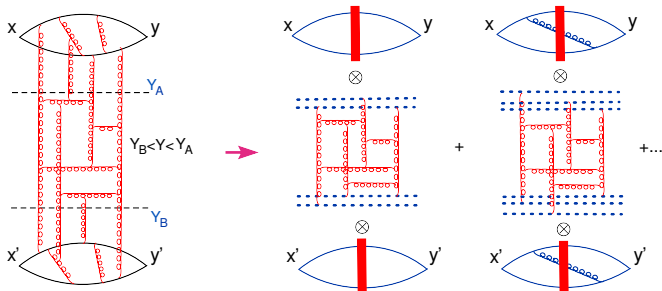
“Pomeron residue”  $F(\nu, \lambda)$ :

$$F(\nu, \lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + \dots$$

$$F_0(\nu) = \frac{\pi \sinh \pi\nu}{4\nu \cosh^3 \pi\nu} \quad \text{Cornalba, Costa, Penedones (2007)}$$

$$F_1(\nu) = \text{see below} \quad \text{G. Chirilli and I.B. (2009)}$$

# NLO Amplitude in $\mathcal{N}=4$ SYM theory: factorization in rapidity

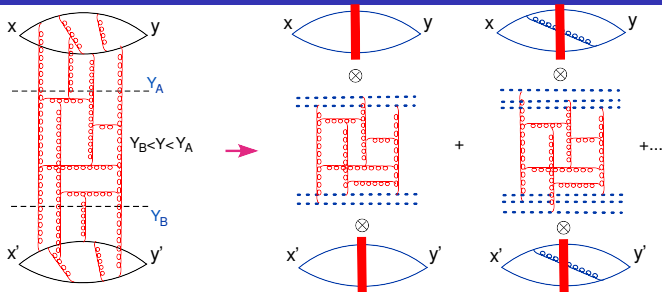


$$\begin{aligned}
 & [(x-y)^2(x'-y')^2]^{2+\gamma_k} \langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}(y')\} \rangle \\
 &= \int d^2z_{1\perp} d^2z_{2\perp} d^2z'_{1\perp} d^2z'_{2\perp} \text{IF}^{a_0}(x, y; z_1, z_2) [\text{DD}]^{a_0, b_0}(z_1, z_2; z'_1, z'_2) \text{IF}^{b_0}(x', y'; z'_1, z'_2)
 \end{aligned}$$

$$a_0 = \frac{x_+ y_+}{(x-y)^2}, \quad b_0 = \frac{x'_- y'_-}{(x'-y')^2} \Leftrightarrow \text{impact factors do not scale with energy}$$

$\Rightarrow$  all energy dependence is contained in  $[\text{DD}]^{a_0, b_0}$  ( $a_0 b_0 = R$ )

# NLO Amplitude in $\mathcal{N}=4$ SYM theory: factorization in rapidity



$$\begin{aligned}
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 \end{aligned}$$

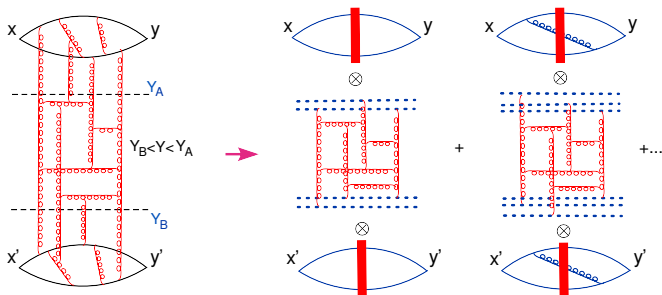
Dipole-dipole scattering

$$\chi(\gamma) \equiv 2C - \psi(\gamma) - \psi(1-\gamma)$$

$$[\mathbf{DD}] = \int d\nu \int dz_0 \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^{\frac{1}{2}+i\nu} \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^{\frac{1}{2}-i\nu} D\left(\frac{1}{2} + i\nu; \lambda\right) R^{\omega(\nu)/2}$$

$$D(\gamma; \lambda) = \frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)} \left\{ 1 - \frac{\lambda}{4\pi^2} \left[ \frac{\chi(\gamma)}{\gamma(1-\gamma)} - \frac{\pi^2}{3} \right] + \mathcal{O}(\lambda^2) \right\}$$

# NLO Amplitude in $\mathcal{N}=4$ SYM theory: factorization in rapidity



$$\begin{aligned}
 & [(x-y)^2(x'-y')^2]^{2+\gamma_k} \langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}(y')\} \rangle \\
 &= \int d^2z_{1\perp} d^2z_{2\perp} d^2z'_{1\perp} d^2z'_{2\perp} \mathbf{IF}^{a_0}(x, y; z_1, z_2) [\mathbf{DD}]^{a_0, b_0}(z_1, z_2; z'_1, z'_2) \mathbf{IF}^{b_0}(x', y'; z'_1, z'_2)
 \end{aligned}$$

Result :

(G.A. Chirilli and I.B.)

$$F(\nu) = \frac{N_c^2}{N_c^2 - 1} \frac{4\pi^4 \alpha_s^2}{\cosh^2 \pi\nu} \left\{ 1 + \frac{\alpha_s N_c}{\pi} \left[ \frac{\pi^2}{2} - \frac{2\pi^2}{\cosh^2 \pi\nu} - \frac{8}{1+4\nu^2} \right] + O(\alpha_s^2) \right\}$$

$$[x_{12}^2 x_{34}^2]^{2+\gamma_K} \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \frac{i}{2} \int d\nu \frac{\tanh \pi \nu}{\nu} F(\nu) \Omega(r, \nu) R^{\frac{1}{2}\aleph(\nu)} \tilde{f}_+(\aleph(\nu))$$

$\aleph(\nu)$  - pomeron intercept,  $\tilde{f}_+(\omega) = (e^{i\pi\omega} - 1) / \sin \pi\omega$  - signature factor

$$\Omega(r, \nu) = \frac{\nu}{2\pi^2} \frac{\sin 2\nu\rho}{\sinh \rho}, \quad \cosh \rho = \frac{\sqrt{r}}{2}$$

In the double limit (Regge) + ( $x_{12}^2 \rightarrow 0$ )

$$R = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \rightarrow \frac{x_{1+} x_{2+} x_{3-} x_{4-}}{x_{12\perp}^2 x_{34\perp}^2}$$

$$r \rightarrow \frac{x_{12+}^2 (x_{3-} x_{14\perp}^2 - x_{4-} x_{13\perp}^2)^2}{x_{1+} x_{2+} x_{3-} x_{4-} x_{12\perp}^2 x_{34\perp}^2}$$

$$\Omega(r, \nu) \rightarrow \frac{\nu}{2\pi^2 i} (r^{-\frac{1}{2}+i\nu} - r^{-\frac{1}{2}-i\nu})$$

Cornalba's formula as  $x_{12\perp}^2 \rightarrow 0$

$$\begin{aligned}
 & [x_{12\perp}^2 x_{34\perp}^2]^{2+\gamma_K} \int dx_{2+} dx_{3-} \langle \mathcal{O}(L_1 + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_2 + x_{3-}, x_{2\perp}) \mathcal{O}(x_{3-}, x_{3\perp}) \rangle \\
 &= \frac{i\alpha_s^2}{8} \pi^2 L_1 L_2 \int_0^1 du dv \int d\nu \frac{\tanh \pi\nu}{\nu \cosh^2 \pi\nu} \left( \frac{x_{12\perp}^2 x_{34\perp}^2 \bar{u}u \bar{v}v}{[x_{13\perp}^2 \nu + x_{14\perp}^2 \bar{v}]^2} \right)^{\frac{1}{2}+i\nu} \left( \frac{L_1^2 L_2^2 \bar{u}u \bar{v}v}{x_{12\perp}^2 x_{34\perp}^2} \right)^{\aleph(\nu)/2} f_+
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{i\alpha_s^2}{8} \pi^2 \int_0^1 dv \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\xi f(\aleph(\xi)) \quad \xi \equiv \frac{1}{2} + i\nu \\
 &\times \frac{(\bar{v}v)^{1-\xi+\frac{\aleph(\xi)}{2}} \cos \pi\xi}{(\xi - \frac{1}{2}) \sin^3 \pi\xi} \frac{B(2 - \xi + \frac{\aleph(\xi)}{2})}{[x_{13\perp}^2 \nu + x_{14\perp}^2 \bar{v}]^{2+\aleph(\xi)}} \left( \frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 \nu + x_{14\perp}^2 \bar{v}]^2} \right)^{-\xi - \frac{\aleph(\xi)}{2}} (L_1 L_2)^{1+\aleph(\xi)}
 \end{aligned}$$

= "BFKL" representation of the amplitude in the (Regge) + ( $x_{12}^2 \rightarrow 0$ ) limit

## Gluon light-ray (LR) operator of twist 2

$$F_{-i}^a(x'_+ + x_\perp)[x'_+, x_+]^{ab} F_{-i}^{b\ i}(x_+ + x_\perp)$$

Forward matrix element - gluon parton density

$$z^\mu z^\nu \langle p | F_{\mu\xi}^a(z)[z, 0]^{ab} F_\nu^{b\xi}(0) | p \rangle^\mu \stackrel{z^2=0}{=} 2(pz)^2 \int_0^1 dx_B x_B D_g(x_B, \mu) \cos(pz) x_B$$

Evolution equation (in gluodynamics)

$$\begin{aligned} & \mu^2 \frac{d}{d\mu^2} F_{-i}^a(x'_+ + x_\perp)[x'_+, x_+]^{ab} F_{-i}^{b\ i}(x_+ + x_\perp) \\ &= \int_{x_+}^{x'_+} dz'_+ \int_{x_+}^{z'_+} dz_+ K(x'_+, x_+; z'_+, z_+; \alpha_s) F_{-i}^a(z'_+ + x_\perp)[z'_+, z_+]^{ab} F_{-i}^{b\ i}(z_+ + x_\perp) \end{aligned}$$

“Forward” LR operator

$$F(L_+, x_\perp) = \int dx_+ F_{-i}^a(L_+ + x_+ + x_\perp)[L_+ + x_+, x_+]^{ab} F_{-i}^{b\ i}(x_+ + x_\perp)$$



## Expansion in local operators

$$F(L_+, x_\perp) = \sum_{n=2}^{\infty} \frac{L_+^{n-2}}{(n-2)!} \mathcal{O}_n^g(x_+, x_\perp), \quad \mathcal{O}_n^g \equiv F_{-i}^a \nabla_-^{n-2} F_-^{ai}$$

## Evolution equation for $F(L_+, x_\perp)$

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} F(L_+, x_\perp) &= \int_0^1 du K_{gg}(u, \alpha_s) F(uL_+, x_\perp) \\ \Rightarrow \gamma_n(\alpha_s) &= -\frac{1}{2} \int_0^1 du u^{n-2} K_{gg}(u, \alpha_s) \quad \mu \frac{d}{d\mu} \mathcal{O}_n^g = -\gamma_n(\alpha_s) \mathcal{O}_n^g \end{aligned}$$

## $uK_{gg}$ - DGLAP kernel

$$uK_{gg}(u) = \frac{\alpha_s N_c}{\pi} \left( \bar{u}u + \left[ \frac{1}{\bar{u}u} \right]_+ - \frac{1}{12c} \delta(\bar{u}) \right) + \text{higher orders in } \alpha_s$$

Conformal LR operator ( $j = \frac{1}{2} + i\nu$ )

$$F^\mu(L_+, x_\perp) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} (L_+)^{-\frac{3}{2}+i\nu} \mathcal{F}_{\frac{1}{2}+i\nu}^\mu(x_\perp)$$

$$\mathcal{F}_j^\mu(x_\perp) = \int_0^\infty dL_+ L_+^{1-j} F^\mu(L_+, x_\perp)$$

Evolution equation for “forward” conformal light-ray operators

$$\Rightarrow \mu^2 \frac{d}{d\mu^2} \mathcal{F}_j(z_\perp) = \int_0^1 du K_{gg}(u, \alpha_s) u^{j-2} \mathcal{F}_j(z_\perp)$$

$\Rightarrow \gamma_j(\alpha_s)$  is an analytical continuation of  $\gamma_n(\alpha_s)$

# Supermultiplet of twist-2 operators

$SU_4$  singlet operators.

(Korchemsky et al)

$$S_{1n}(z) = \mathcal{O}_g^n(z) + \frac{n-1}{24}\mathcal{O}_\lambda^n(z) + \frac{n(n-1)}{24}\mathcal{O}_\phi^n(z)$$

$$S_{2n}(z) = \mathcal{O}_g^n(z) - \frac{1}{24}\mathcal{O}_\lambda^n(z) - \frac{n(n+1)}{72}\mathcal{O}_\phi^n(z)$$

$$S_{3n}(z) = \mathcal{O}_g^n(z) - \frac{n+2}{12}\mathcal{O}_\lambda^n(z) + \frac{(n+1)(n+2)}{24}\mathcal{O}_\phi^n(z)$$

$$\mathcal{O}_\lambda^n(z) = \bar{\lambda}^a \nabla^{n-1} \lambda^a(z) + \text{descendants}$$

$$\mathcal{O}_\phi^n(z) = \bar{\phi}^{a,I} \nabla^n \phi^{a,I}(z) + \text{descendants}$$

All operators have the same anomalous dimension

$$\gamma_n(\alpha_s) = \frac{2\alpha_s}{\pi} N_c [\psi(n-1) + C] + \dots$$

## Glينو and scalar LR operators

$$\Lambda(L_+, x_\perp) = \int dx'_+ \bar{\lambda}^a(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab} \lambda^b(x_+ + x_\perp)$$

$$\Lambda_j(x_\perp) = \int_0^\infty dL_+ L_+^{-j} \Lambda(L_+, x_\perp)$$

$$\Phi(L_+, x_\perp) = \int dx'_+ \phi^{a,I}(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab} \phi^{b,I}(x_+ + x_\perp)$$

$$\Phi_j(x_\perp) = \int_0^\infty dL_+ L_+^{-j-1} \Phi(L_+, x_\perp)$$

$SU_4$  singlet LR operators.

$$S_{1j}(x_\perp) = F_j(x_\perp) + \frac{j-1}{24} \Lambda_j(x_\perp) + \frac{j(j-1)}{24} \Phi_j(x_\perp)$$

$$S_{2j}(x_\perp) = F_j(x_\perp) - \frac{1}{24} \Lambda_j(x_\perp) - \frac{j(j+1)}{72} \Phi_j(x_\perp)$$

$$S_{3j}(x_\perp) = F_j(x_\perp) - \frac{j+2}{12} \Lambda_j(x_\perp) + \frac{(j+1)(j+2)}{24} \Phi_j(x_\perp)$$

All operators have the same anomalous dimension

$$\gamma_j(\alpha_s) = \frac{2\alpha_s}{\pi} N_c [\psi(j-1) + C] + \dots$$

## Three-point CF of LR operator and two locals

Three-point correlation function of local operators ( $\mathcal{O} = \phi_I^a \phi_I^a$  - Konishi operator)

$$(\mu^2 z_{23}^2)^{-\gamma_K} \langle S_{1n}(z_1) \mathcal{O}(z_2) \mathcal{O}(z_3) \rangle = \sum c_n \frac{[1 + (-1)^n]}{z_{12}^2 z_{13}^2 z_{23}^2} \left( \frac{z_{12-}}{z_{12}^2} - \frac{z_{13-}}{z_{13}^2} \right)^n \left( \frac{\mu^{-2} z_{23}^2}{z_{12}^2 z_{13}^2} \right)^{\frac{1}{2} \gamma(n, \alpha_s)}$$

$\Rightarrow$

$$\begin{aligned} & x_{23\perp}^2 (\mu^2 x_{23\perp}^2)^{\gamma_K} \int dx_{1+} \langle S_{1n}(x_{1+}, x_{1\perp}) \mathcal{O}(x_{2-}, x_{2\perp}) \mathcal{O}(x_{3-}, x_{3\perp}) \rangle \\ &= -2\pi i \frac{c_n [1 + (-1)^n]}{x_{23\perp}^4} \frac{\Gamma(1 + 2n + \gamma_n)}{\Gamma^2(1 + n + \frac{1}{2} \gamma_n)} \left( \frac{x_{23\perp}^2}{x_{2-} x_{3-}} \right)^{1 + \frac{\gamma_n}{2}} \left( \frac{x_{12\perp}^2}{x_{2-}} + \frac{x_{13\perp}^2}{x_{3-}} \right)^{-1 - n - \gamma_n} \end{aligned}$$

CF of light-ray operator and two local operators

$$\begin{aligned} & x_{23\perp}^2 (\mu^2 x_{23\perp}^2)^{\gamma_K} \langle S_{1j}(x_{1\perp}) \mathcal{O}(x_{2-}, x_{2\perp}) \mathcal{O}(x_{3-}, x_{3\perp}) \rangle \\ &= -2\pi i \frac{c_j [1 + e^{i\pi j}]}{x_{23\perp}^4} \frac{\Gamma(1 + 2j + \gamma_j)}{\Gamma^2(1 + j + \frac{1}{2} \gamma_j)} \left( \frac{x_{23\perp}^2}{x_{2-} x_{3-}} \right)^{1 + \frac{\gamma_j}{2}} \left( \frac{x_{12\perp}^2}{x_{2-}} + \frac{x_{13\perp}^2}{x_{3-}} \right)^{-1 - j - \gamma_j} \end{aligned}$$

# Three-point CF of LR operator and two locals

Proof: compare conformal Ward identities for  $\int dx_+ S_{1n}(x_+, x_\perp)$  and  $S_{1j}(x_\perp)$

$$i[D, \int dx_+ S_{1n}(x_+, x_\perp)] = \int dx_+ \left( x_+ \frac{\partial}{\partial x_+} + n + 2 + x_i^\perp \frac{\partial}{\partial x_i^\perp} \right) S_{1n} = (n + 1) \int dx_+ S_{1n}(x_+, x_\perp)$$

vs

$$i[D, \Phi(x_+, x_\perp) \Phi(x'_+, x_\perp)] = \left[ 2 + x_+ \frac{\partial}{\partial x_+} + x'_+ \frac{\partial}{\partial x'_+} + x_i^\perp \frac{\partial}{\partial x_i^\perp} \right] \Phi(x_+, x_\perp) \Phi(x'_+, x_\perp)$$
$$\Rightarrow i[D, \Phi_j(x_\perp)] = \left[ j + 1 + (x - x_0)_i^\perp \frac{\partial}{\partial x_i^\perp} \right] \Phi_j(x_\perp)$$

Similarly

$$\Rightarrow i[K, \int dx_+ \Phi(x_\perp) \nabla_-^n \Phi(x_\perp)] = 4 \left[ n + \frac{1}{2} (x - x_0)_i^\perp \frac{\partial}{\partial x_i^\perp} \right] \int dx_+ x_* \Phi(x_\perp) \nabla_-^n \Phi(x_\perp)$$

vs

$$i[K, \int dx_+ \int_0^\infty ds_+ (s_+)^{-j-1} \Phi(x_+ + \frac{s_+}{2} + x_\perp) \Phi(x_+ - \frac{s_+}{2} + x_\perp)]$$
$$= 4 \left[ j + \frac{1}{2} (x - x_0)_i^\perp \frac{\partial}{\partial x_i^\perp} \right] \int dx_+ \int_0^\infty ds_+ (s_+)^{-j-1} \Phi(x_+ + \frac{s_+}{2} + x_\perp) \Phi(x_+ - \frac{s_+}{2} + x_\perp)$$

CF of light-ray operator and two local operators “in forward kinematics”

$$\begin{aligned}
 & x_{23\perp}^2 (\mu^2 x_{23\perp}^2)^{\gamma_K} \int dx_{3-} \langle S_j^\mu(x_{1\perp}) \mathcal{O}(L_- + x_{3-}, x_{2\perp}) \mathcal{O}(x_{3-}, x_{3\perp}) \rangle \\
 &= \int_0^1 du \frac{c(j, \alpha_s) [1 + e^{i\pi j}] L_-^j (\bar{u}u)^j}{[x_{12\perp}^2 u + x_{13\perp}^2 \bar{u}]^{1+j}} \left( \frac{\bar{u}u \mu^{-2} x_{23\perp}^2}{[x_{12\perp}^2 u + x_{13\perp}^2 \bar{u}]^2} \right)^{\frac{1}{2}\gamma(j, \alpha_s)}
 \end{aligned}$$

CF of light-ray operator and two local operators “in forward kinematics”

$$\begin{aligned}
 & x_{23\perp}^2 (\mu^2 x_{23\perp}^2)^{\gamma_K} \int dx_{3-} \langle S_j^\mu(x_{1\perp}) \mathcal{O}(L_- + x_{3-}, x_{2\perp}) \mathcal{O}(x_{3-}, x_{3\perp}) \rangle \\
 &= \int_0^1 du \frac{c(j, \alpha_s) [1 + e^{i\pi j}] L_-^j (\bar{u}u)^j}{[x_{12\perp}^2 u + x_{13\perp}^2 \bar{u}]^{1+j}} \left( \frac{\bar{u}u \mu^{-2} x_{23\perp}^2}{[x_{12\perp}^2 u + x_{13\perp}^2 \bar{u}]^2} \right)^{\frac{1}{2}\gamma(j, \alpha_s)}
 \end{aligned}$$

Limit  $x_{23\perp} \equiv \Delta \rightarrow 0$

$$\begin{aligned}
 & \Delta^2 (\mu^2 \Delta^2)^{\gamma_K} \int dx_{3-} \langle S_j^+(x_{1\perp}) \mathcal{O}(L_- + x_{3-}, x_{2\perp}) \mathcal{O}(x_{3-}, x_{3\perp}) \rangle \\
 &= \frac{\Gamma^2(1 + j + \frac{\gamma}{2})}{\Gamma(2 + 2j + \gamma)} \frac{c(j, \alpha_s) [1 + e^{i\pi j}] L_-^j}{(x_{12\perp}^2)^{1+j} (\mu^2 x_{12\perp}^2)^{-\gamma(j, \alpha_s)}} (\mu \Delta)^{\gamma(j, \alpha_s)}
 \end{aligned}$$



Light-ray operators in (+) and (-) directions:

$$\Phi^+(L_+, x_\perp) = \int dx'_+ \phi^{a,I}(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab} \phi^{b,I}(x_+ + x_\perp)$$

$$\Phi_j^+(x_\perp) = \int_0^\infty dL_+ L_+^{-j-1} \Phi(L_+, x_\perp)$$

$$\Phi^-(L_-, x_\perp) = \int dx'_- \phi^{a,I}(L_- + x_- + x_\perp)[x'_- + x_-, x_-]^{ab} \phi^{b,I}(x_- + x_\perp)$$

$$\Phi_j^-(x_\perp) = \int_0^\infty dL_- L_-^{-j-1} \Phi(L_-, x_\perp)$$

CF of two LRs :  $\langle S_{j=\frac{3}{2}+i\nu}^+(x_{1\perp}) S_{j'=\frac{3}{2}+i\nu'}^-(x_{3\perp}) \rangle = \frac{\delta(\nu - \nu') a(j, \alpha_s)}{(x_{13\perp}^2)^{j+1} (x_{13\perp}^2 \mu^2)^{\gamma(j, \alpha_s)}}$

# “DGLAP” representation of 4-point CFs

⇒ Light-cone OPE

$$\begin{aligned} & x_{12\perp}^2 (\mu^2 x_{12\perp}^2)^{\gamma_K} \int dx_{1+} \mathcal{O}(L_1 + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \\ &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj c(j, \alpha_s) [1 + e^{i\pi j}] L_1^j (\mu^2 x_{12\perp}^2)^{\gamma(j, \alpha_s)} S_j^+(x_{1\perp}) \end{aligned}$$

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⇒ “DGLAP” result (at  $x_{12}^2 \rightarrow 0$ )

$$\mu^{-4} (\mu^4 x_{12}^2 x_{34}^2)^{1+\gamma_k} \int dx_{2+} dx_{3-} \langle \mathcal{O}(L_1 + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_2 + x_{4-}, x_{3\perp}) \mathcal{O}(x_{4-}, x_{4\perp}) \rangle$$

# “DGLAP” representation of 4-point CFs

⇒ Light-cone OPE

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# “DGLAP” representation of 4-point CFs

⇒ Light-cone OPE

$$\begin{aligned} & x_{12\perp}^2 (\mu^2 x_{12\perp}^2)^{\gamma_K} \int dx_{1+} \mathcal{O}(L_1 + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \\ &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj c(j, \alpha_s) [1 + e^{i\pi j}] L_1^j (\mu^2 x_{12\perp}^2)^{\gamma(j, \alpha_s)} S_j^+(x_{1\perp}) \end{aligned}$$

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“DGLAP” result (leading twist)

$$\begin{aligned} & \mu^{-4} (\mu^4 x_{12\perp}^2 x_{34\perp}^2)^{1+\gamma_k} \int dx_{2+} dx_{3-} \langle \mathcal{O}(L_1 + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_2 + x_{4-}, x_{3\perp}) \mathcal{O}(x_{4-}, x_{4\perp}) \rangle \\ &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj C(j, \alpha_s) \int_0^1 dv \frac{[1 + e^{i\pi j}](L_1 L_2)^j}{[x_{12\perp}^2 v + x_{13\perp}^2 \bar{v}]^{1+j}} \left( \frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^2} \right)^{\frac{1}{2}\gamma(j, \alpha_s)} (\bar{v}v)^{j+\frac{1}{2}\gamma(j, \alpha_s)} \end{aligned}$$

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$$\begin{aligned} & \mu^{-4} (\mu^4 x_{12\perp}^2 x_{34\perp}^2)^{1+\gamma_k} \int dx_{2+} dx_{3-} \langle \mathcal{O}(L_1 + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_2 + x_{4-}, x_{3\perp}) \mathcal{O}(x_{4-}, x_{4\perp}) \rangle \\ &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj C(j, \alpha_s) \int_0^1 dv \frac{[1 + e^{i\pi j}](L_1 L_2)^j}{[x_{12\perp}^2 v + x_{13\perp}^2 \bar{v}]^{1+j}} \left( \frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^2} \right)^{\frac{1}{2}\gamma(j, \alpha_s)} (\bar{v}v)^{j+\frac{1}{2}\gamma(j, \alpha_s)} \end{aligned}$$

BFKL result ( $\aleph(\xi, \alpha_s) = \frac{\alpha_s N_c}{\pi} \chi(\xi) + \dots$  - pomeron intercept)

$$\begin{aligned} & [x_{12\perp}^2 x_{34\perp}^2]^{2+\gamma_k} \int dx_{2+} dx_{3-} \langle \mathcal{O}(L_1 + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_2 + x_{3-}, x_{2\perp}) \mathcal{O}(x_{3-}, x_{3\perp}) \rangle \\ &= \frac{i\alpha_s^2}{8} \pi^2 \int_0^1 dv \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\xi f(\aleph(\xi)) \quad \xi \equiv \frac{1}{2} + iv \\ &\times \frac{(\bar{v}v)^{1-\xi+\frac{\aleph(\xi)}{2}} \cos \pi \xi}{(\xi - \frac{1}{2}) \sin^3 \pi \xi} \frac{B(2 - \xi + \frac{\aleph(\xi)}{2})}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^{2+\aleph(\xi)}} \left( \frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^2} \right)^{-\xi - \frac{\aleph(\xi)}{2}} (L_1 L_2)^{1+\aleph(\xi)} \end{aligned}$$

# DGLAP vs BFKL for 4-point CFs

“DGLAP” result (leading twist)

$$\begin{aligned} & \mu^{-4} (\mu^4 x_{12}^2 x_{34}^2)^{1+\gamma_k} \int dx_{2+} dx_{3-} \langle \mathcal{O}(L_1 + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_2 + x_{4-}, x_{3\perp}) \mathcal{O}(x_{4-}, x_{4\perp}) \rangle \\ &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj C(j, \alpha_s) \int_0^1 dv \frac{[1 + e^{i\pi j}](L_1 L_2)^j}{[x_{12\perp}^2 v + x_{13\perp}^2 \bar{v}]^{1+j}} \left( \frac{x_{12}^2 x_{34}^2}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^2} \right)^{\frac{1}{2} \gamma(j, \alpha_s)} (\bar{v} v)^{j + \frac{1}{2} \gamma(j, \alpha_s)} \end{aligned}$$

BFKL result ( $\aleph(\xi, \alpha_s) = \frac{\alpha_s N_c}{\pi} \chi(\xi) + \dots$  - pomeron intercept)

$$\begin{aligned} & [x_{12}^2 x_{34}^2]^{2+\gamma_k} \int dx_{2+} dx_{3-} \langle \mathcal{O}(L_1 + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_2 + x_{3-}, x_{2\perp}) \mathcal{O}(x_{3-}, x_{3\perp}) \rangle \\ &= \frac{i\alpha_s^2}{8} \pi^2 \int_0^1 dv \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\xi f(\aleph(\xi)) \quad \xi \equiv \frac{1}{2} + i\nu \\ &\times \frac{(\bar{v} v)^{1-\xi + \frac{\aleph(\xi)}{2}} \cos \pi \xi}{(\xi - \frac{1}{2}) \sin^3 \pi \xi} \frac{B(2 - \xi + \frac{\aleph(\xi)}{2})}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^{2+\aleph(\xi)}} \left( \frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^2} \right)^{-\xi - \frac{\aleph(\xi)}{2}} (L_1 L_2)^{1+\aleph(\xi)} \end{aligned}$$

We compare these results around  $\xi \sim j - 1 \sim 0$

$$\Rightarrow 1 + \aleph(\xi, \alpha_s) = j \text{ and } \gamma(j, \alpha_s) = -2\xi - \aleph(\xi)$$

$$\aleph(\xi) - 2\aleph(\xi)\aleph'(\xi) \simeq \frac{\alpha_s N_c}{\pi \xi} + \frac{\zeta(3)\alpha_s^2}{\xi} + \dots \rightarrow \gamma_j = -2 \frac{\alpha_s N_c}{\pi(j-1)} + [0 + \zeta(3)(j-1)] \left( \frac{\alpha_s N_c}{\pi(j-1)} \right)^3 + \dots$$



$$\gamma_j = -2 \frac{\alpha_s N_c}{\pi(j-1)} + [0 + \zeta(3)(j-1)] \left( \frac{\alpha_s N_c}{\pi(j-1)} \right)^3 + \dots$$

is an anomalous dimension of light-ray operator  $F \nabla^{j-2} F(x_\perp)$

## 1 Conclusions:

- NLO BFKL gives anomalous dimensions of light-ray operators  $F_{+i} \nabla^{\omega-1} F_+^i$  at  $\omega \rightarrow 0$  (in all orders in  $\alpha_s$ )

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- NLO BFKL gives anomalous dimensions of light-ray operators  $F_{+i} \nabla^{\omega-1} F_{+}^i$  at  $\omega \rightarrow 0$  (in all orders in  $\alpha_s$ )

## 2 Outlook (joint project with V. Kazakov and E. Sobko):

- 3-point CF of  $\langle F \nabla^{\omega_1-1} F(x_{1\perp}) F \nabla^{\omega_2-1} F(x_{2\perp}) F \nabla^{\omega_3-1} F(x_{3\perp}) \rangle$  at  $\omega_1, \omega_2, \omega_3 \rightarrow 0$