Awakening of Geometrical Thought in Early Culture

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Awakening of Geometrical Thought in Early Culture

by Paulus Gerdes

With Foreword by Dirk J. Struik

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Foreword by Dirk J. Struik [1894-2000]

1

How did our mathematical concepts originate? And how did our science of mathematics come into being?

For many mathematicians the answer to the second question has been easy. Mathematics is a deductive science, and therefore originated with the Greeks, beginning with Thales and Pythagoras about 500 B.C. From them came many of our terms, even the term *mathematics* itself. The term *geometry* shows that the Greeks took many of their ideas from the Egyptians, because it referred to the annual surveying of the land after the floods of the Nile. Thus, according to this theory, Egyptians, as well as Babylonians, had mathematics, but mostly in an empirical way. The same held for China.

When, with the publication of such work as that of Neugebauer on Babylonia and Needham on China in the 1930s and later, it became clear that the mathematics of the Bronze Age empires was far more sophisticated than was believed, many mathematicians were willing to admit that the origin of mathematics as a science had to be traced back from the fourth century B.C. to the Sumerians and perhaps the Egyptians and Chinese as well.

This meant that mathematics began in the period when scribes of the Bronze Age states began to use symbols and special terms for mathematical concepts. But where did these concepts, and some of the terms already in existence, come from?

In years long past, there was a simple answer. God had bestowed on Adam in Paradise a lot of mathematical knowledge, which after his expulsion he bequeathed to his son Seth, the father of Enos. Enos, having a foreboding of the Flood, had his knowledge inscribed on two pillars, which survived the Flood. In the course of time they were seen and studied by many a traveler, among them the patriarch Abraham, who brought his knowledge to Egypt. And the Egyptians taught the Greeks.

We find such a story in Josephus, in the writings of the seventeenth-century mathematician Tacquet, and in other places. We present this story to our friends the Creationists, but prefer to search for the origin of mathematical concepts elsewhere.

We shall have to watch the gradual evolution of *Homo sapiens* all through the millennia of the prehistoric period for the earliest stages of tool-making, of fishing and hunting to agriculture, cattle raising, and trade—all through the Stone Age.

There has been much speculation on how the process of acquiring knowledge of mathematical concepts, of forms and number, has actually occurred.

One approach can be found in the words of one historian that "the first geometrical [and arithmetical] considerations of man . . . seem to have had their origin in simple observation, stemming from human ability to recognize physical form [and quantity], and compare shapes and sizes." For instance: the form of sun, moon, and certain flower heads led to the concept of a circle, the shape of ropes to line and curves, further spider webs and honeycombs to more intricate forms, to triangles, spirals, solids. Comparing heaps of objects to each other led to counting, first only one, two, many, etc. This approach stresses *onlooking*, *reflection*. It is a *static* point of view. We can call this the attitude of *homo observans*.

Another approach, presented by Seidenberg, looks at religious impulses like the building of altars. As explained in this book, this is not very satisfactory. What Gerdes stresses goes beyond this and also beyond observation, and is the approach through the effects of *labor*. Ever since the hominoids began to walk erectly, their hands became free to make tools in the production of their livelihood—first very primitive, but gradually evolving into well-constructed artifacts. Man discovers, improves, constructs, uses all kinds of forms. The number concept grows. Man builds tents, houses; makes baskets, bags, nets, pottery, and weapons. Through the millennia, first very slowly, then more rapidly, a great amount of knowledge of a mathematical art is obtained. This is a *dynamic* approach, the approach of *homo laborans*. It is implicit in the Marxian point of view, and we find it, for instance, in a note by Frederick Engels (1885), where he points out that the basic ideas of line, surface, angle, and number are all borrowed from reality in the interplay of Man and Nature. The objects seen in nature and in tools, in the villages and in the fields, are never exact straight lines, circles, triangles, squares. Only by activity throughout the centuries could Man be led from these forms to the abstract concepts of mathematics.

Man, changing Nature, changes himself.

We do not entirely deny the value of the other approaches; they stand in a dialectical relation to each other and the dynamic point of view. There are still other factors to take into account, for instance that of playing man, the man of games with a mathematical strand. The *homo ludens*.

During the many centuries the tools improved. For instance, arrowheads and hand axes become more efficient, well made; the same holds for baskets, pottery, nets. The tools became more symmetrical because of increased efficiency; and so we find, for instance, baskets taking the form of cylinders or prisms.

Incidentally, the symmetry and harmony of forms that turn out to be most efficient (many examples appear in this book) also strike us as more agreeable, *beautiful*. A source of the birth of aesthetics? We can refer to the book.

2

In order to obtain more factual information on Stone Age development, we can search for remnants of this age. There are some rods, wood or bone, found in Africa, perhaps 10,000 years old,* with carvings of parallel lines, perhaps the tally of hunting results. Then there are the famed cave paintings in Spain and

*Struik refers to a bone found at Ishango (Congo). Dating estimates of this bone now range from 8,000 to 20,000 B.C. A still older bone with twenty-nine clearly marked notches was found iSn a cave in the Lebombo Mountains on the border between South Africa and Swaziland. This bone has been dated at approximately 35,000 B.C. (see Gerdes 1994).

France, also very ancient, which show mathematical traces, if only by the fact that they are two-dimensional projections of solid bodies, hence exercises in mapping. We can also study arrowheads and other artifacts.

Much richer information can be obtained by studying the culture of present-day indigenous peoples still living in Stone Age conditions or at any rate retaining customs and memories of older times before Western influence set in. Their culture may contain many strains millennia old. Though we have some accounts of mathematical lore by travelers or missionaries, such as some reports on the counting of American Indians or the games of Polynesians dating to the nineteenth and early twentieth century, a systematic study of these cultures from a mathematical point of view only took place in the years after World War II, and has led to a novel field called *ethnomathematics.* This term was proposed by Professor Ubiratan D'Ambrosio of Brazil, who has studied, among other things, Latin American indigenous cultures.

One of the reasons for this interest has been political—anticolonialism. Starting with the impetus given by the Russian Revolution, the struggle against colonialism has led after the Second World War to the dissolution of the old colonial empires. The new politically independent states had to cope with the devastating influence of the colonial regime on the old native cultures, especially in Africa, Polynesia, and Micronesia. It has been a struggle to recoup native identities, if possible. The search for mathematical concepts inherent in these native cultures is part of this search for identity.

Pioneering here has been the work of Claudia Zaslavsky; in her book *Africa Counts* (1973), she surveys the mathematical (or "protomathematical," if you prefer) ideas in the cultures of peoples living south of the Sahara. She finds them in their counting, architecture, ornamentation, games, riddles, taboos, concepts of time, weights and measures, even magic squares.

Since the appearance of her book, many studies in this field have been published. We mention only Marcia Ascher's book *Ethnomathematics* (1991), which gives examples from many parts of the third world, including even kinship relations. As to Africa, here the main investigations have been led by Paulus

Gerdes and his collaborators. In this book he deals with the geometrical and ornamental aspect of native mathematics.

We learn in this book how mathematical concepts were involved in the construction of baskets, mats, bags, from reeds, leaves, and other parts of plants, as well as in the construction of homes and pyramids. In the course of the centuries, the artifacts and the methods of construction were improved, and so the concepts of triangle, hexagon, circle, and rectangle could be developed until they led to the abstractions of the science of mathematics.

Gerdes shows how, in the course of time, properties of these geometrical figures could be discovered, including the Theorem of Pythagoras. It has always been a mystery how knowledge of this theorem appears in Babylonia around 2000 B.C.—where did it come from? This look at the construction, use, and improvement of artifacts can also lead to other properties. Is it possible that Greek knowledge of the volume of a pyramid was developed out of the way fruit (say apples) is piled up in the markets and could this also have led to Pascal's triangle? Gerdes believes that the knowledge of the volume of the truncated pyramid could also have been the result of sophisticated methods born out of practices.

There is still another side of ethnomathematical study. It is its importance for education. If pupils from the villages (and ghettos) come to school and enter modern classrooms, will not the indigenous mathematics in their upbringing facilitate their acquisition of the modern mathematics of the classroom? This use of the "intuitive" native mathematics may well be of help in easing the mathematical angst we hear so much about.

This brings ethnomathematics in as a factor in the widespread discussion on the improvement of mathematical instruction in our schools. His ideas can have wide application. To the literature and the discussion of this subject, other writings of Professor Gerdes have also made their contribution.

Dirk J. Struik
Belmont, Massachusetts
March 1998

Preface

Most standard histories of mathematics ignore completely or pay little attention to the existence of mathematical traditions outside the so-called West. Geometry is presented as something very special, born among the "rational Greeks." Before them, at most some practical rules would have been known. Most standard textbooks ignore geometrical thinking in daily life, in particular in the daily life of the peoples of the "third world," of the "South."

Strong protests have arisen in recent decades against the ignorance of mathematics outside the "West" and "North," especially from the ethnomathematical movement. Claudia Zaslavsky's *Africa Counts: Number and Pattern in African Culture* (1999, first edition 1973), Ubiratan D'Ambrosio's *Sociocultural Bases for Mathematics Education* (1985) and *Etnomatemática* (1990), Alan Bishop's *Mathematical Enculturation* (1988), Marcia and Robert Ascher's *The Code of the Quipu: A Study in Media, Mathematics and Culture* (1981), Marcia Ascher's *Ethnomathematics: A Multicultural View of Mathematical Ideas* (1991), Michael Closs's *Native American Mathematics* (1986), George Gheverghese Joseph's *The Crest of the Peacock: Non-European Roots of Mathematics* (1991), and Arthur B. Powell and Marilyn Frankenstein's *Ethnomathematics*: *Challenging Eurocentrism in Mathematics Education* (1997) are extremely important in demystifying the dominant views about mathematics and in contributing to an alternative picture of mathematics as a panhuman activity.

In this perspective, *Awakening of Geometrical Thought in Early Culture* considers early geometrical thinking, both as embedded in various social activities surviving colonization in the life of the peoples of the “South,” and in early history. Chapter 1 discusses briefly some standard views of the origin of geometrical concepts. Chapter 2 analyzes alternative views of geometry stimulated by the philosophical reflections of Frederick Engels and presents a wholly unexplored field of research: geometrical thinking as embedded in mat- and basket-weaving. Chapter 3, constituting the principal part of the book, analyzes the emergence of a series of early geometrical concepts and relationships in socially important activities. Questions such as these are considered: *Where could the concept of a right angle have come from? Where did the idea of a regular hexagon arise? How is it possible to determine the rectangular base of a building?* Chapter 4 presents, on the basis of the ideas and the methodology developed in the previous chapter, a series of hypotheses on the possible role of social activity in the development of geometry in ancient Mesopotamia and Egypt. The last chapter offers some general ideas on the awakening of geometrical thought based on the analysis in this book.

In other work, I have tried to build upon ideas developed in *Awakening of Geometrical Thought in Early Culture* and, in particular, to give concrete examples of how (reconstructed) geometrical traditions may be incorporated into mathematics education. One of the objectives of ethnomathematical research is improving the teaching of mathematics by embedding it into the cultural context of students and teachers. Such mathematics education can heighten the appreciation of the scientific knowledge inherent in culture by using this knowledge to lay the foundations for providing quicker and better access to the scientific heritage of the whole of humanity.

Awakening of Geometrical Thought in Early Culture is a briefer version in English of a book originally written in 1985 in German and Portuguese. A German-language version was published in 1990 under the title *Ethnogeometrie: Kulturanthropologische Beiträge zur Genese und Didaktik der Geometrie* (Bad Salzdethfurth: Verlag Franzbecker), with a preface by

Professor Peter Damerow (now at the Max Planck Institute for the History of Science, Berlin), and including chapters on the didactics of geometry in the context of an African country. Shorter Portuguese-language editions have been published by Universidade Pedagógica in Mozambique under the title *Cultura e o despertar do pensamento geométrico* (1991) and by the Universidade Federal do Paraná (Curitiba, Brazil, 1992) under the title *Sobre o despertar do pensamento geométrico*, with a preface by Professor Ubiratan D'Ambrosio (Universidade Estadual de Campinas). These three editions include a chapter on the artistic elaboration of symmetry ideas emerging from social activity that has not been included in the English version. Neither the Portuguese-language editions nor the English-language edition include the original introduction on mathematical under-development. The English edition includes a section on ancient Mesopotamian and Egyptian methods for the determination of the area of a circle that does not appear in the Portuguese-language editions. The German-language edition may be consulted for more notes and an extended bibliography,

Acknowledgments

I feel very honored that the late Dirk J. Struik, the "Nestor of the historians of mathematics" (and professor emeritus at the Massachusetts Institute of Technology), had been kind enough to provide the foreword to this book. His century-long work and active life have stimulated several generations of mathematicians and mathematics educators to reflect on the material and socio-cultural roots of mathematics, and to deepen understanding of the philosophy and history of mathematics. His letters to me for over two decades and our more recent conversations have encouraged and challenged me to pursue my research.

I thank Erwin and Doris Marquit for their hospitality when they received me in Minneapolis, and I am grateful for their able editing of my draft translation of the book into English.

MEP Publications released in 1985 Beatrice Lumpkin's translation of my book on the mathematical writings of Marx,

Karl Marx: Arrancar o véu misterioso à matemática (Eduardo Mondlane University, Maputo, 1983) under the title *Marx Demystifies Calculus.*

I am pleased that MEP Publications has once again been interested in publishing one of my books, making it available also to the North American public, and contributing in this way to the debate on the questions raised in this book.

Paulus Gerdes
Universidade Pedagógica
Maputo, Mozambique

Chapter One
Mathematicians on the Origin of Elementary Geometrical Concepts

In this chapter, I shall consider some widely held opinions of mathematicians about the origin and the early development of geometry; in chapters 4 and 5, I shall discuss other common ideas about early geometrical thought.

1. Did geometry have a beginning?

"Did geometry once have a beginning?" is a question that Julian Coolidge implicitly raises when he writes in his *History of Geometrical Methods* (1963), "Whatever be our definition of the *Homo sapiens*, he must be accorded some geometrical ideas; in fact, there would have been geometry if there had been no *Homines sapientes* at all" (1). Geometrical forms appear both in inanimate nature and also in organic life, and this phenomenon may be explained as a consequence of mechanical and physiological causes. Apart from this mechanical necessity—so asks Coolidge—what is the earliest example of an intentional geometrical construction? Maybe the making of a cell structure of the honey bee, "if we avoid metaphysical difficulties over the problem of the freedom of the will"? (1). No, the honeybee only optimizes, but "the ablest geometer among the animals is surely the spider" that weaves such beautiful (!) webs (2). According to Coolidge, geometry exists outside humans and their activities. Geometry is eternal. Coolidge's history of (human?) geometrical methods begins completely arbitrarily in Mesopotamia,[1] as he is lacking any criterion to find out when or which human beings became able to observe or perceive geometrical forms in nature.[2]

2. Does geometry equal deductive geometry?

Quite often it is said that geometry started in ancient Egypt.[3] Problems of field measurement led to a series of mostly only approximate formulas, but as Leonard Blumenthal asserts in his *Modern View of Geometry*, “the Egyptian surveyors were no more geometers than Adam was a zoologist when he gave names to the beasts of the field” (1961, 1). In his view, geometry emerged as a science as soon as it became *deductive* in ancient Greece. Even if one agrees to identify geometry with deductive geometry, another doubt arises: were not pre-Greek observations of, and reflections about, space rarely or never deductive? And does an induction not presuppose a deduction?

Also Herbert Meschkowski begins his well-known book *Evolution of Mathematical Thought* (1965) with Euclid’s *Elements*. He argues that the first childish steps were surpassed with the development of a rigorous system of mathematical proofs. Although it might be true that the ancient Egyptians and Babylonians had discovered quite a lot of theorems, nevertheless “these insights were acquired by *intuition* or by *direct observation*” (emphasis added). The transition from intuition and direct observation to the rigorous system of mathematical proofs remains without explanation and appears therefore absolute. And should not in particular this transition—if it had taken place in reality—have been one of the most important transformations in the evolution of mathematical thought? Now this transition seems to be a (nondialectical) leap. On the other hand, would, for example, the so-called Theorem of Pythagoras have been found through mere intuition? Or would it have been the result of pure direct observation?

3. Still in the dark: What is geometry?

Raymond Wilder, the late chairman of the American Mathematical Society (1955–1956) and of the Mathematical Association of America (1965–1966), stresses in the chapter on geometry in his book *Evolution of Mathematical Concepts* that "instead of looking for miracles or gods or superhuman individuals" in order to understand the level of Greek geometry, one should try to find the *continuous line* that leads from Egyptian and Babylonian geometry to Greek geometry (1968, 88). If one agrees, then one may still raise the question whether this line started in the ancient Orient or still earlier elsewhere. Wilder's answer remains in the dark: "There was a time" [where and when?] "when mathematics included nothing that one would place in a separate category and label geometry. . . . For at that time mathematics consisted solely of an arithmetic of whole numbers and fractions, together with an embryonic (albeit quite remarkable) algebra" (1968, 88). Would fractions have emerged earlier as the first geometrical concepts? If so, what then is geometry?

4. Organizing spatial experiences

Contrary to Blumenthal and Meschkowski, the well-known geometer and didactician of mathematics Han Freudenthal evaluates in a completely different way the significance of the Greek deductive method when he notes forcefully: "Rather than as a positive element, I am inclined to view the Greek efforts to formulate and prove knowledge . . . by means of clumsy methods and governed by strict conventions, as a symptom of a terrifying *dogmatism*" that until today has retarded and sometimes endangered the spread and dissemination of mathematical knowledge (1982, 444). In Freudenthal's view, geometry did not begin late in history with the formulation of definitions and theorems, but as early as the organization of the spatial experiences that led to these definitions and theorems (1978, 278).

Why, when, and where did this organizing of spatial experiences begin? Or, which human beings are able to perceive geometric forms and relationships?

5. Who is able to perceive geometric forms and relationships?

Howard Eves, in his paper "The History of Geometry," answers the question, "Which human beings are able to perceive geometric forms and relationships?" in the same way as Coolidge: "All." However, he presents other reasons: "The first geometrical considerations of Man . . . seem to have had their origin in simple observations stemming from *human ability* to recognize physical form and to compare shapes and sizes" (1969, 165). Here he presupposes the ability to recognize and compare forms as a natural, a once-and-for-all given quality of human beings. Consequently, it turns out to be relatively easy to explain the origin of early geometrical concepts. For instance, the outline of the sun and the moon, the shape of the rainbow, and the seed heads of many flowers, etc. led to the conception of circles. A thrown stone describes a parabola; an unstretched cord hangs in a catenary curve; a wound-up cord lies in a spiral; spider webs illustrate regular polygons, etc. (168). So far, Eves's position may seem empiricist: the properties that are common to different objects are of an immediately visible and perceivable character. This perception remains mostly passive. Nevertheless he notes, "Physical forms that possess an *ordered* character, contrasting as they do with the haphazard and unorganized shapes of most bodies, *necessarily* attract the attention of a reflective mind—and some elementary geometric concepts are thereby brought to light," leading to a *"subconscious geometry"*[4] (166; emphasis added). But how do people know which forms possess an ordered character? Or better still, why and how did humans necessarily *learn* to discover order in nature? Why does the "subconscious geometry" transform itself in ancient Egypt and Mesopotamia, as Eves asserts (167), into "scientific geometry"?[5]

These questions indicate already how Eves's position may be dialectically sublated (*aufgehoben*): in order to geometrize, not only are geometrizable objects necessary, but also, to consider and perceive these objects, the ability to abstract all their other properties apart from their shape is also needed. This ability is the result of a long *historical development based on experience*, to paraphrase Frederick Engels.[6]

NOTES

1. Would it be by chance that for the same Coolidge the choice of geometrical axioms is completely arbitrary? See Coolidge, 1963, 423.

2. Cf. Simon: "Never and nowhere mathematics was invented. . . . Mathematical ideas are not at all restricted to Man. . . . When the spider produces its web, it uses its particularly built foot as a compass; the bees have solved a difficult maximum problem when they construct their hexagonal cells" (1973, xiii).

3. Cf. Ball: "Geometry is supposed to have had its origin in land surveying. . . . [S]ome methods of land-surveying must have been practiced from very early times, but the universal tradition of antiquity asserted that the origin of geometry was to be sought in Egypt" (1960, 5).

4. In what sense "subconscious"? "For want of a better name," Eves calls this knowledge of elementary geometric concepts "subconscious geometry." He notes, "This subconscious geometry was employed by very early man in the making of decorative ornaments and patterns, and it is probably quite correct to say that early art did much to prepare the way for later geometric development. The evolution of subconscious geometry in little children is well known and easily observed" (166).

5 Cf. Cantor: "Also geometrical concepts . . . must have emerged early in history. Objects and figures limited by straight lines and curves must have attracted the eye of Man, as soon as he started not only to see, but to look around himself" (1922, 1:15). What, however, could have caused this changeover from "seeing" to "looking around himself"?

6. "Counting requires not only objects that can be counted, but also the ability to exclude all properties of the objects considered except their number—and this ability is the product of a long historical development based on experience" (Engels 1987a, 36–37).

Chapter Two
How Did People Learn to Geometrize?

It may be said that geometry arose from the needs of human beings. The basic ideas of lines, surfaces, angles, polygons, cubes, spheres, etc., are all, in one way or another, "borrowed" from reality, observes Engels (1987a, 37). The important question is how were they borrowed from reality? In other words, how did the capacity to *geometrize* develop historically?

In his study *Dialectics of Nature*, Engels gave a hint about the direction in which we should look for an answer. As their intelligence grows in their creative interplay with nature, human beings develop their capacities of reflection, observation, and analysis. Human labor plays a fundamental role in this process. Geometrical ideas and relationships are *elaborated* by human beings (1987b, 476, 511).

1. The birth of geometry as a science

Broad outline. First approximation

Inspired by Engels's reflections, a series of historians, mathematicians, and philosophers stress, in broad outline, that geometry arose from practical life, from the effort to satisfy human needs. Its transformation into a mathematical theory required an immense period of time (see, e.g., Alexandrov 1977, 22).

Geometry emerged as an empirical, experimental science. In the interaction with their environment, the people of the Old Stone (Paleolithic) Age arrived at their first geometrical knowledge (see, e.g., Struik 1948 and 1967; Hauser 1955, 11; Wussing 1979, 31). The process of the elaboration of abstract representations of spatial relationships initially took place extremely slowly

(Molodschi 1977, 23). After having collected sufficient factual material with respect to the "simplest" spatial forms, it became possible—under special societal conditions, as, for example, in ancient Egypt, Mesopotamia, and China—to systematize the collected factual material (Ruzavin 1977, 39). With this systematization, geometry started its transformation from an empirical science into a mathematical science, achieving a first completion with Euclid's *Elements*: geometry as a "mathematical science with its logical structure—proving of affirmations—and the abstraction of the given object from its initial contents" (Alexandrov 1974, 47).

Emergence of geometry as the perception of spatial forms. Second approximation

The development that led to the transformation of geometry from an empirical science to a theoretical science was, according to Alexandrov (1974, 47) and Molodschi (1977, 23), long and complex. Material objects and their relationships existed already much earlier than *Homo sapiens*. The circular appearance of the sun and moon, the smooth surface of a lake, the straightness of a beam of light, etc., were always present and gave people the possibility of observing them. But exact circles, straight lines, or triangles never exist in nature. The chief reason, in Alexandrov's view, that people gradually became capable of working out geometric concepts lies in the fact that human observation of nature was not a passive but an active one in the sense that, to meet their practical needs, human beings made objects more and more regular in shape. When they built their dwellings, enclosed their plots of land, stretched bowstrings in their bows, modeled their clay pots, etc., they discovered that a pot is curved, but a stretched bowstring is straight. In short, stresses Alexandrov, human beings "first gave form to their material and only then recognized form as that which is impressed on *material* and can therefore be considered in itself as an abstraction from the material" (1977, 10; emphasis added). As human beings made more and more regular shapes and compared them with one another, they learned to perceive "form unattached from the qualitative particularity of the compared objects” (Molodschi 1977, 23). Once capable of recognizing the form of the objects as such, people could make products of better quality, which, once again, contributed to a more precise elaboration of the abstract concept of form. The dialectical interplay between active life and abstract thinking constitutes the motor of the development of geometry.

2. An example of the influence of labor on the emergence of early geometrical notions

In his study “Numbers in Paleolithic Graphic Art and the Initial Stages in the Development of Mathematics” (1977), Frolov analyzes important aspects of the emergence of the earliest geometrical notions in history. Archaeological and paleoneurological research shows that not only *Homo sapiens* in the Upper Paleolithic, but already their precursors of the Mousterian, possessed well-developed speech and quite a high level of abstract concepts. Already before labor had had a considerable influence on the development of thinking, hand axes became smaller and more elegant, taking on a geometrically regular and symmetrical shape. To produce them, a sequence of multiple and varied work operations were necessary, which led to a change in the higher mental functions, like attention, memory, and language. It was not accidental that gradually a symmetrical shape was chosen: symmetry of the cutting edge reduces the resistance of a hard body, diminishes friction, requires less muscular effort, etc.; a *symmetrical* shape was, therefore, the most rational. In other words, the first stages of tool-making activity show that a symmetrical shape is not an *imitation* of symmetrical forms in nature, but rather that it was attained in the course of the production traditions of thousands of generations. The formation of the concept of symmetry was dialectical. A significant step took place: the most rational form became what was considered beautiful; the symmetrical shape increasingly acquired an *independent*, *technical*, and *aesthetic* significance (Frolov 1977/78, 148–52; see also Breuil and Lantier 1959, 215 ff).

The Mousterians already fabricated more than sixty types of tools. They also knew how to build dwellings for long occupancy, and made the first attempts at depiction. In particular, a piece of bone, more than 50,000 years old, found at La Ferrassie in France, was covered with groups of fine parallel notches, provoking various speculations. Okladnikov interprets these as the "first ornamental compositions on our planet," as a decisive step in the development of art, and the logic of abstract concepts. He writes that the creator of these notches

> was capable of overcoming the inertia of long-term mental stagnation and the chaos of associations. He brought order into the stormy chaos of impressions. From them he selected what was significant for him, and expressed it in the abstract form of symmetrically arranged geometrical lines. Clarity in place of the unclear and diffuse, order instead of disorder, logic in place of cloudy sensations and flashes: here is the objective meaning of this most ancient specimen of ornamentation. (cited in Frolov 1977, 155)

Frolov regards this composition of groups of parallel notches as a first *"mathematical structure,"* which emerged after many hundreds of thousands of years of practical application of identical groups of rhythmic blows to obtain symmetrical tool shapes from stone, and after numerous experiments in working bone with cutting tools that left incisions. This is a possible interpretation, but it does not clarify why the notches were carved exactly parallel to one another. Would the thinking of their creator already have been sufficiently independent, sufficiently freed from matter to have been able to conceive such a pattern of parallel notches? Or did the Mousterians perhaps have other working experiences in which they found parallel lines? The search for other possible contexts is further stimulated when Frolov observes about the paintings in the caves of the Mousterians:

> The use of the time factor in the "development" of rock compositions in the depths of caves and the "winding" of scenes on many places on the cylindrical surfaces of mobile objects is of particular interest. . . . The genesis of

> rectangular figures in Paleolithic art reflected, in…particular, the existence at that time of concepts about the areas of objects. (1978, 75)

But whatever could have been the reason for the development of the idea of the area of an object?

Parallel lines, spirals, right angles—in what other contexts could these concepts have emerged?

3. An unexplored field: Geometrical concepts in weaving

In his famous study *Science in History*, Bernal suggests where we might look for an answer:

> The idea of a right angle existed certainly before building and, probably, even before textile weaving. Among the mural paintings in the caves of Lascaux one encounters rectangular figures divided as a little bit irregular chess board, in which the squares are painted alternately in different colours. The most probable origin of these drawings may be found in the art of interlacing, that as we know was already really practiced during the Paleolithic. (1971, 251)

Not only the idea of a right angle, but also the notions of parallel lines and of spirals that develop with time might have been formed in mat- and basket-weaving activities. Basketry was already known during the Paleolithic and was, probably, a prior stage to weaving. Both techniques are based on regularity and perhaps led people, as Bernal supposes, “to distinguish patterns and to use them in art and later in geometrical figures and in mathematical analysis” (1971, 51). An attempt to analyze this hypothesis immediately confronts one with some difficulties.

The folding of a leaf already leads to a straight line (see fig. 2.1). In a few minutes, one may produce a simple basket out of palm leaves and use it for carrying fish, as may be illustrated by the basket in fig. 2.2, coming from the Mozambican province of Nampula. After having been used once or a few times, it is thrown away. The ephemeral character of the materials that were used makes it very difficult to reconstruct the history of mat- and basket-weaving. It is not accidental, therefore, that books on the

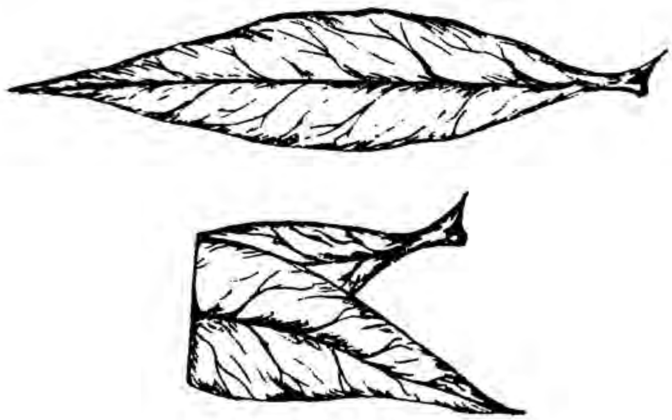


Fig. 2.1. The folding of a leaf

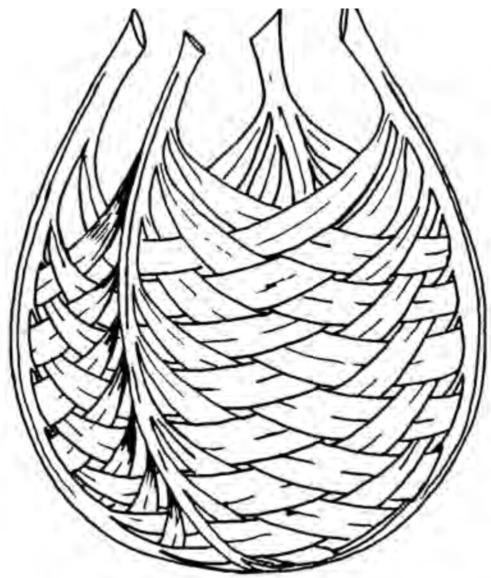


Fig. 2.2. Simple disposable basket made out of palm leaves

history of technology normally dedicate no space or only a few pages to the history of mat- and basket-weaving (see, e.g., Jonas et al. 1969 and Sworykin et al. 1964). Existing and surviving techniques may be analyzed for a better understanding of the development of interlacing. Ethnographic data may be helpful in attempts to reconstruct some fragments of the emergence of geometrical concepts in weaving.[1]

NOTES

1. Also, D. Smith supposes in his *History of Mathematics* that such connections exist, but he does not advance an analysis of them: "A . . . prehistoric stage of mathematical development is seen in the use of simple geometric forms as *were suggested by the plaiting of rushes,* the first step in textile art" (1958 1:15; emphasis added). See also Lietzman 1940, 9.

Chapter Three
Early Geometrical Concepts and Relationships in Societal Activities

1. The concept of a right angle

Already during the Lower Paleolithic Period, the hominids had developed in their labor activities a first feeling for angle amplitudes—for example, *in what direction* does one have to hew to obtain sharper hand axes (fig. 3.1)?[1] To fabricate more effective harpoons (fig. 3.2)? They discovered the optimal direction for throwing their assagais (fig. 3.3).

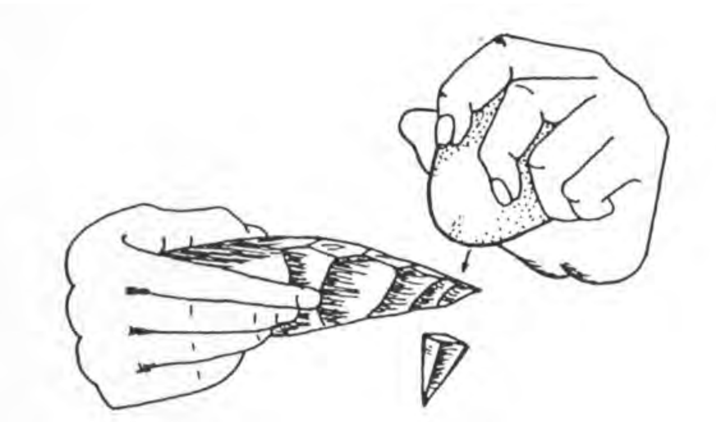


Fig. 3.1. Production of hand axes

To avoid the overturn of their windscreens, the Australian aborigines were forced to put the upper sticks *perpendicularly* to the supporting sticks (fig. 3.4). To avoid their dams being swept away by the water, the Wagheni of Congo, the Lamuts of the

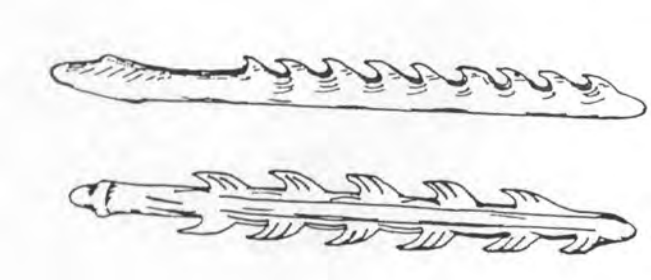


Fig. 3.2. Harpoon points

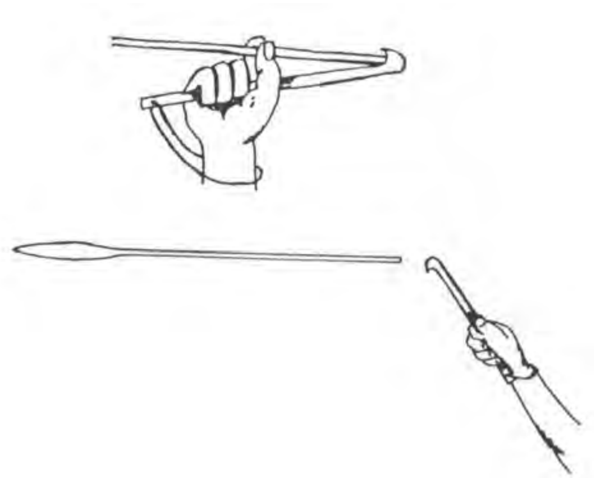


Fig. 3.3. Throwing of an assagai

Camchatca peninsula, and the Camaiura Indians of Brazil saw themselves forced to fasten the barrier sticks perpendicularly to the supporting sticks.[2] To make a fire as quickly as possible, the hardwood fire drill has to be rotated perpendicularly to the softwood (see the example of Australian aborigines in fig. 3.5).[3]

Many hunting communities discovered that their arrows flew easier and more forcefully when they were released perpendicularly to the bow (fig. 3.6). Mozambican fishermen learned to fasten the floaters perpendicularly to their *mitumbui* and *cangaia* boats to maintain their equilibrium (fig. 3.7).

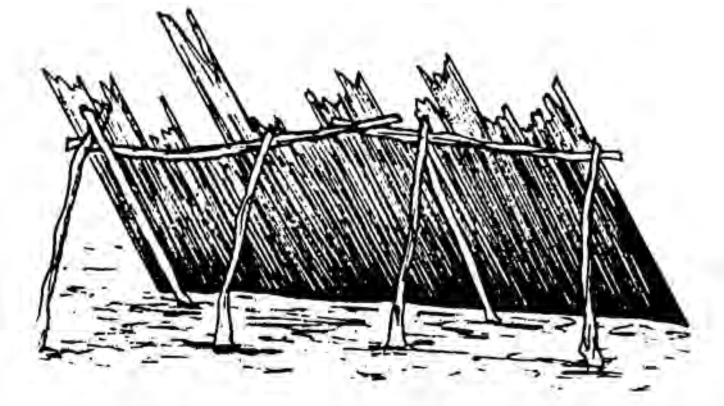


Fig. 3.4. Windscreen

These are only a few examples of situations where people—to satisfy best their needs—felt themselves forced through their labor and the characteristics of the material with which they were working to prefer mutually perpendicular directions.

The most widespread, and probably one of the oldest, activitity encountered daily that required perpendicular orientations was the binding of objects. A problem that occurs frequently, for example, when weaving baskets and mats, constructing floats or boats, building shelters or houses, is how to bind fast two or more sticks, stems, or branches with the help of strands or thinner ropes. If one chooses an arbitrary folding angle, as in fig. 3.8, then the sticks can easily loosen and become undone (fig. 3.9). Through experience, one learns only one position is suitable for

Fig. 3.5. Fire drill from Australia

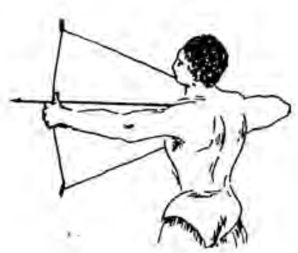


Fig. 3.6. Shooting an arrow

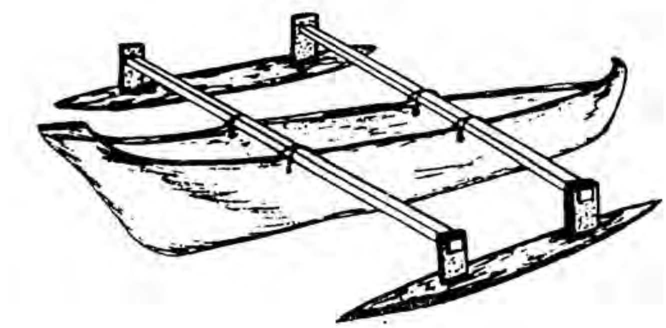


Fig. 3.7. *Cangaia* boat

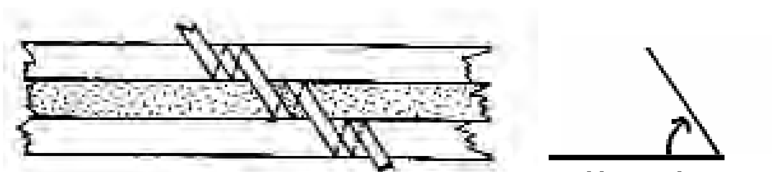
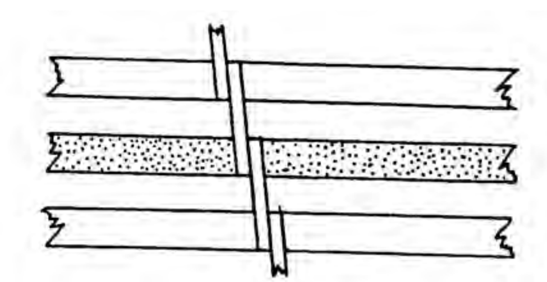


Fig. 3.8. Arbitrary folding angle



Fi.g. 3.9. Loosening of the bond

the fastening of two sticks (fig. 3.10). To bind together three or more sticks with the same thread, the perpendicular position is better approximated when the thread is thinner (fig. 3.11).

The same perpendicular position *necessarily* also emerges when one sews reeds together to make a mat. The easiest way to

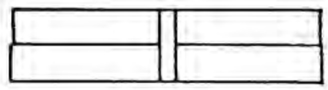


Fig. 3.10. Proper position

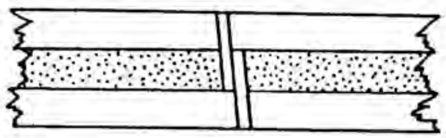


Fig. 3.11. Approximating a perpendicular position

bore through a reed with a needle is in a perpendicular direction (fig. 3.12), as this offers less resistance.

When one draws the thread tighter, it *automatically* assumes—independently of human will—a perpendicular position in relation to the reeds (fig. 3.13). On the basis of this experience, the other threads are sewn in the same way (fig. 3.14). Where should the last threads pass through the reeds? One discovers that a thread that does not pass through all reeds, as in the case of reed 1 in fig. 3.14, is not desirable. Reeds that are (much) longer than the others, like reed 2, make rolling up the mat difficult.

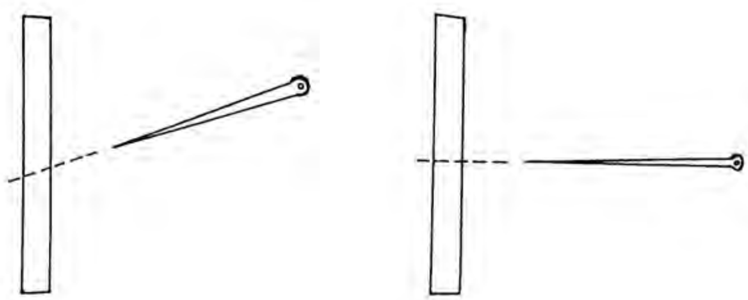


Fig. 3.12. Boring through a reed with a needle

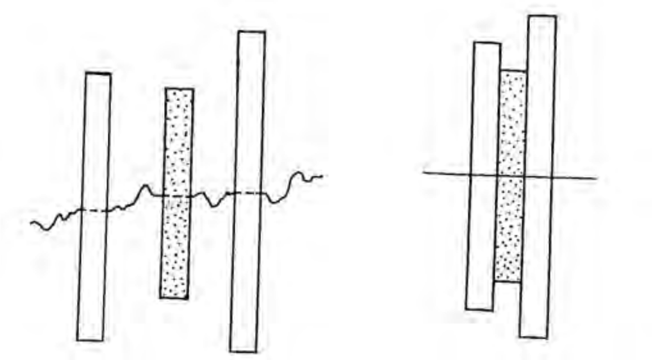
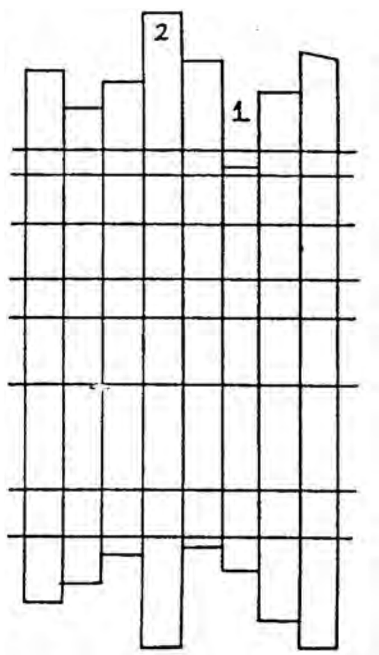


Fig. 3.13. Tightening the thread



Fi.g. 3.14. One of the threads does not pass through all reeds

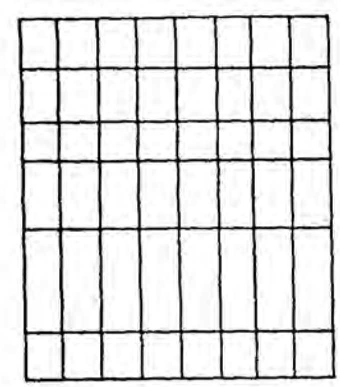


Fig. 3. 15. Rectangular mat

This process not only leads to the formation of the concept of a right angle,[4] but gives rise also to a first *conceptualization of a rectangle*. The almost necessary rectangularity of the mat (fig. 3.15) facilitates, in turn, the fabrication of other similar mats; as raw material, one needs reeds with the same length. At the time of the Paleolithic, there were already needles in use that, apart from having served for the processing of hides and, perhaps, the fabrication of collars, may have also been used for mat-making.

One may also arrive at the same rectangular form in other ways, such as in the case of Chinese mats or the hammocks of the Yanomama Indians in northern Brazil (Biocca 1980, 152), where two threads are simultaneously interlaced up and down in such a

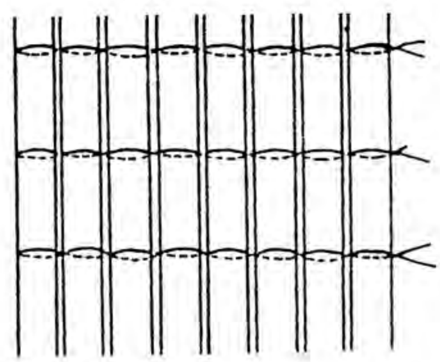


Fig. 3.16. Pairs of threads are simultaneously interlaced

way that when one thread goes over the reed, the other passes under it (fig. 3.16).

The concepts of right angle and rectangle were elaborated through the practical activity of human beings. Once discovered and "anchored," they could be applied to other situations where no immediate material necessity existed to favor these forms, as, for example, in the rectangular weaving of strands of (approximately) the same width (fig. 3.17), where other amplitudes of angle are possible and indeed are sometimes chosen (fig. 3.18).[5]

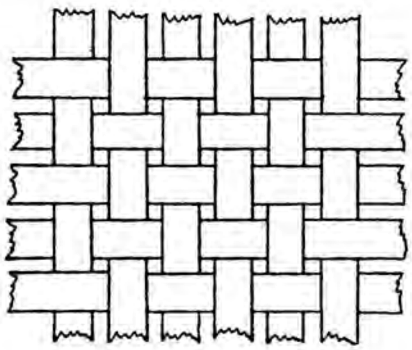


Fig. 3.17. Rectangular plaiting/weaving

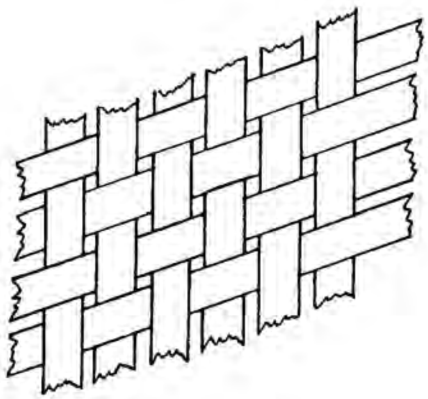


Fig. 3.18. Nonrectangular plaiting

2. Where did the idea of a regular hexagon arise?

Did the idea of a regular hexagon arise from direct observation—for example, of the honeycombs of bees—or was it the product of pure thought?

Old cultural elements with a hexagonal form are found in geographical regions of the world situated far from each other. For example, the Huarani (Ecuador), the Yekuana (Guyana), and the Ticuna and Omagua Indians in northwestern Brazil make big carrying baskets with hexagonal holes.[6] The Pukóbye Indians in the northeast of Brazil interlace their headbands hexagonally, just as the Micmac-Algonkin Indians of eastern Canada do with their snowshoes.[7] In the northern coastal zones of Mozambique, one weaves hexagonally the fish trap called *lema* and the carrying basket *litenga*. Cooking plates with hexagonal holes are plaited in Kenya, as are ladles used in boiling fruits among the Desana Indians of the northwest Amazon (see Somjee 1993, 96; Reichel-Dolmatoff 1985, 77). In Madagascar, fish traps and transport baskets are woven hexagonally, just as the Mbuti (Congo) plait their carrying baskets (see Faublée 1946, 28, 38; Meurant and Thompson 1995, 162). Hexagonally plaited baskets are also found among the Kha-ko in Laos (see photo in Grottanelli 1965, 8), as well as in China, India, Japan, Malaysia, and the Philippines.[8] On the island of Borneo (Indonesia), one meets hexagonally woven railings; and among the Munda, in India, a bird trap is interlaced in the same way.[9] Can we, perhaps, discover in the making of these woven objects *one* possible germ of the idea of a regular hexagon?

A practical problem that arises in the making of many kinds of baskets is how one can produce a border that is simultaneously strong, relatively smooth, and stable. Frequently, a nonsmooth border is bent (fig. 3.19a), or a separate smooth and firm border is fastened to the basket, in order to solve the problem (fig. 3.19b). Let us now see how hexagonal weaving solves the same problem.

Imagine the situation where both the border and walls of the basket are made out of the same material. To fasten the border well, one may try to wrap the other strands of plant around the border strand, as displayed in figs. 3.20, 3.21, and 3.22 for the case of one strand.

It may be noted that this folding forces the artisans to *symmetrical* forms, whether or not they wish to do so. Initially, they are probably not conscious of the idea of symmetry, but the beginning of the concept of symmetry has begun to emerge. One or two folds only are little use. In the first case (fig. 3.23), the border strand is free to slide down. In the second case, the border

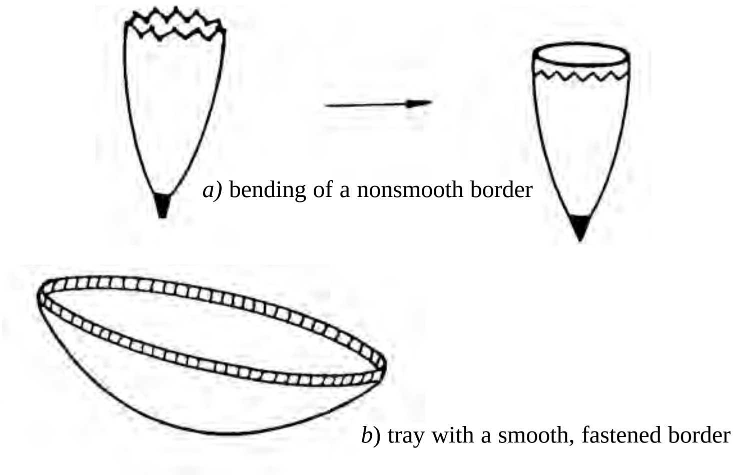


Fig. 3.19. Basket borders

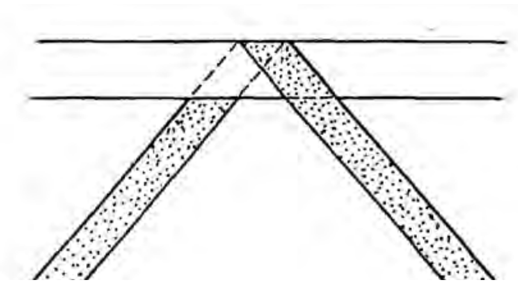


Fig. 3.20. One fold

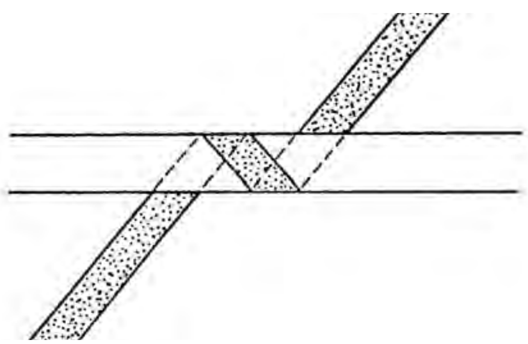


Fig. 3.21. Two folds

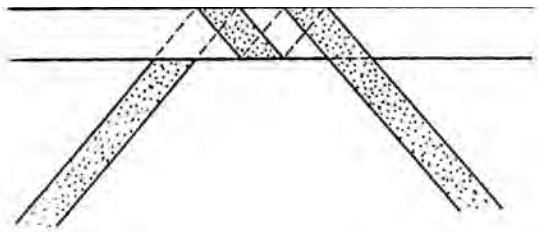


Fig. 3.22. Three folds

loses its limiting function. At least three folds are necessary. What can our artisans then still freely choose (see figs. 3.26a and 3.26b)? The angle of incidence is still variable. With a relatively small angle of incidence, the border can come quickly undone. Therefore, one needs the maximum possible angle of incidence, realized *materially* when, at the moment the second fold is made, one side of the strand touches the other. Figure 3.27 shows that

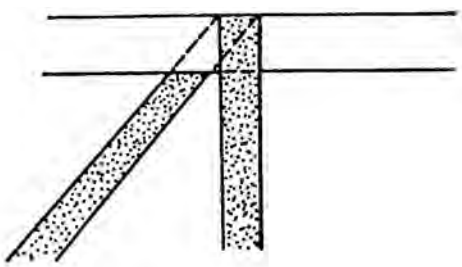


Fig. 3.23. A materially impossible fold

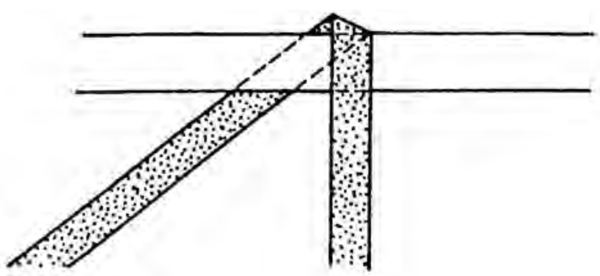


Fig. 3.24. A possible fold that, however, does not lie parallel to the border

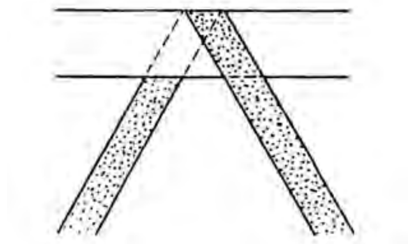


Fig. 3.25. A possible and necessarily symmetrical fold

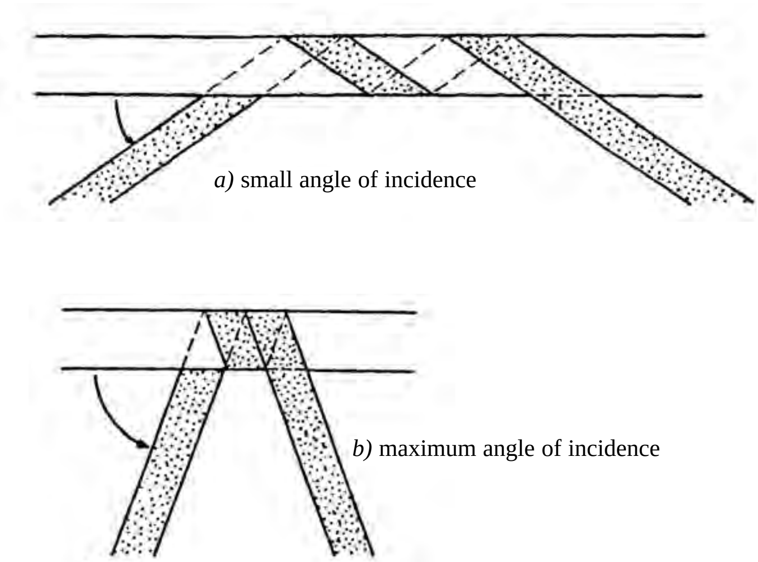


Fig. 3.26. Angles of incidence

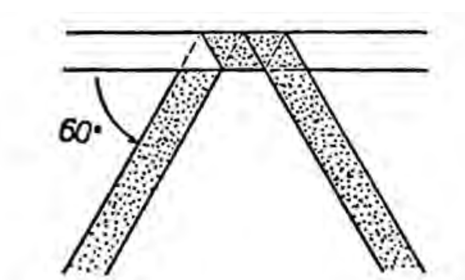


Fig. 3.27. Maximum angle of incidence of 60º

this maximum angle of incidence measures 60° if the border and wall strands have the same width. If, afterward, other wall strands are fastened to the border, and one links them together, then one sees an image like the one in fig. 3.28.

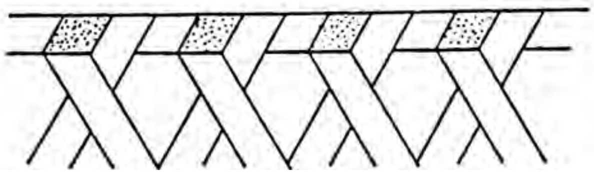


Fig. 3.28. Several strands fastened to the border

Interlacing further the horizontal strands, one obtains *automatically* a regular hexagonal pattern (fig. 3.29), or, if one skips over one horizontal strand each time, a regular pattern appears like that found among the Caraib Indians (fig. 3.30; see also Kästner 1978, 101). Both weaving patterns are very stable; the resulting holes are almost impossible to enlarge or reduce.

After this pattern is found in the context of fastening a border, it proves possible to produce similar interlacing without a border

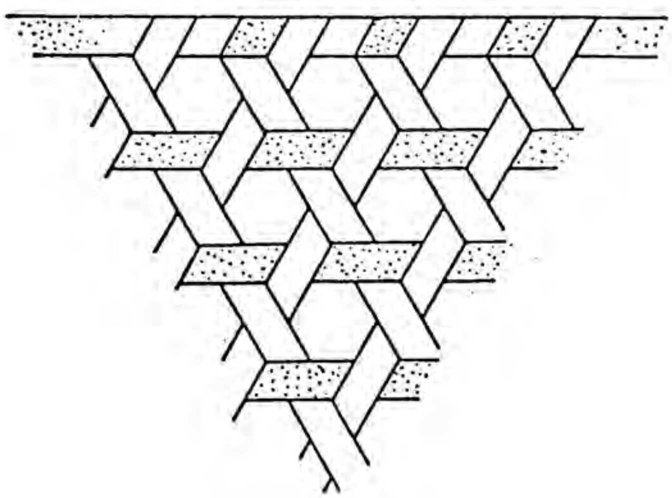


Fig. 3.29. A regular hexagonal pattern

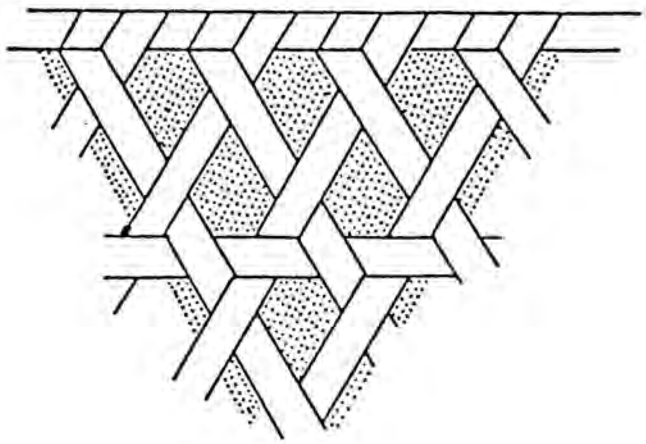


Fig. 3.30. A semiregular pentagonal pattern

(fig. 3.31). This plane pattern can be used for the vertical wall of a basket—for example, as among the Kha-ko for a cylindrical wall. But if the hexagonal pattern is applied to the bottom of a basket, what form must this base display? An equilateral triangle, an isosceles trapezium, and a rhombus belong to the materially possible forms, as our artisans discover. Nevertheless, as they know on the basis of their experiences, a *convex* and *symmetrical*, *rounder* form is more appropriate for making a well-balanced, handy basket. The hexagonal weaving pattern forces them to choose the hexagonal form for the whole bottom of the basket.[10] The similarity between the small hexagonal holes and the

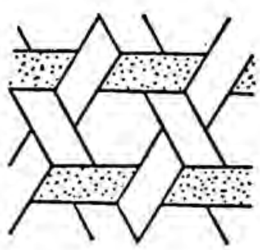


Fig. 3.31. Starting with a hexagonal hole

hexagonal base reinforces the growing idea of a regular hexagon without the basket weaver, as we may assume, being aware, at the first instance, of the *six* angles or of the *six* edges of the holes of his or her basket.

In a dialectical interplay between the choice of objective experimentation, and the nature and form of the material used, a first hexagon concept could have been e*labor*ated (labor!) in the way I described here. The feeling for *order* increases. Necessary for the production of a firm basket with holes is a repeatable, regular pattern. Through the repeated fabrication of each "cell" of the basket, the capacity to compare is developed further. The artisan may observe, in particular, the congruence of the small hexagonal holes and the similarity with the hexagonal bottom. This enables the artisan to see the similarity with naturally occurring hexagons and so to *learn* to observe hexagonality in nature—for example, of the honeycombs of bees. In other words, I should like to stress that the capacity to observe and recognize order and regular spatial forms in nature has been shaped *through* labor activity. But not only the capacity to observe. Simultaneously emerges the *appreciation* of the hexagonal pattern for the production of firm baskets and sweet honey.

The practical, valuable properties of the hexagonal pattern and the discovery of similar forms in nature stimulate further interest in this form as such, and in its characteristic elements like, for example, the angle of 60°. It cannot, after all, be accidental (a present from God or a product of pure thinking) that the Ticuna Indians, for whom honey is a welcome extra (Neumann and Kästner 1983, 42)—we saw already that they make hexagonally woven carrying baskets—link hexagonally the two skins of their drums *without any material necessity* forcing them to choose that form (fig. 3.32). The thinking that developed, enforced by active labor in order to produce something valuable, has here liberated itself from the "reign of necessity," since in this case there is no necessity to opt for an angle of incidence of 60°. This is an example, early in cultural history, of the emergence of a *relatively* independent "mathematical" thinking. The *diagonals* and the *center* of a regular hexagon have been discovered, along

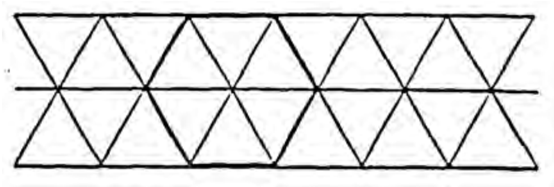


Fig. 3.32. Threads that link the top and bottom skins of a Ticuna drum

with the relationship between hexagons and equilateral triangles (see once again fig. 3.32).

How can one weave baskets with a pattern of holes or open spaces?

The hexagonal pattern can also have been discovered in another way, where the intention had not been so much the construction of a stable border, but the fabrication of a basket with a pattern of holes. To satisfy several daily necessities, baskets with small holes may be preferred: the water can stream outward through the fish trap, small animals such as birds need sufficient ventilation when they are transported in baskets, baskets with open spaces are lighter and need less material for their fabrication, etc.

The first possibility of making baskets lies in the well-known process of weaving at right angles over-one-under-one, in particular, when the strips are relatively thick in respect to their width (fig. 3.33, so-called wickerwork) as, for example, in some types of Mozambican sieves. A variant is obtained when one uses strands of different widths in the two directions, where multiple strands are twisted together as they pass over the strips with which they interlock (fig. 3.34, so-called twining). If the strands at the disposal of the basket maker are relatively thin and wide, then it would not be easy to weave a strong basket, because the strands could easy be pushed aside under those conditions.

If this solution of interlacing in two directions at right angles is not satisfactory because, for example, the holes are too small or the raw materials are too difficult to find in sufficient quantity,

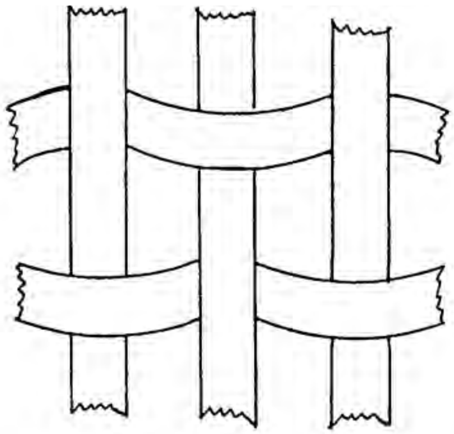


Fig. 3.33. Rectangular plaiting

the artisan may try to interlace in three directions. How then should one start? Three mutual arbitrary directions (as, for example, in fig. 3.35a), are possible as starting points; but how should one continue to interlace to obtain an object that does not fall apart? To avoid the strands moving apart, it is important that the second strand 4 go in the same direction as strand 1, passing next to the crossing point of *A* on the same side as strand 1. When one interlaces strand 5 in the same manner, one observes that this last

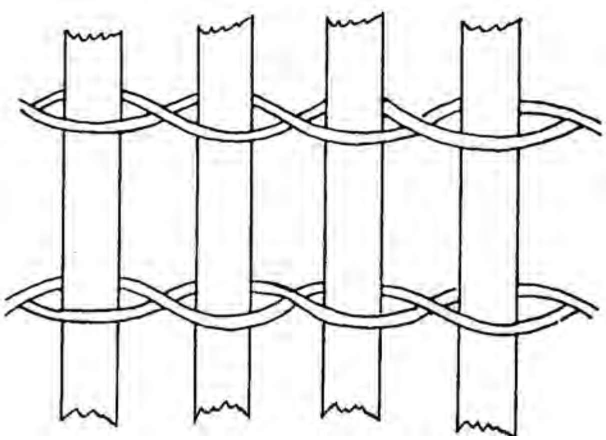


Fig. 3.34. Twining with strands of different widths

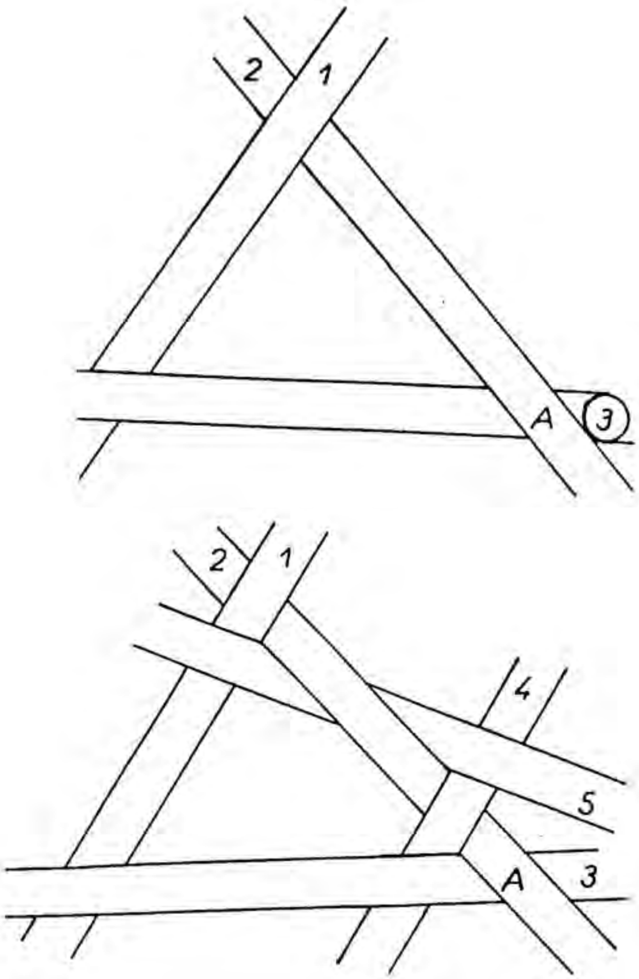


Fig. 3.35 (*a*–top, *b*–bottom). Experimenting with three directions

strand does not, in general, lie parallel to the third strand (fig. 3.35b), etc. Experimenting, one discovers a regular (fig. 3.29) or semiregular pattern (fig. 3.36). The Nambikwara Indians of Brazil start with strands in two directions at right angles, and interlace strands diagonally in the third direction as in fig. 3.36 (Levi-Strauss 1976, photos 19, 20).

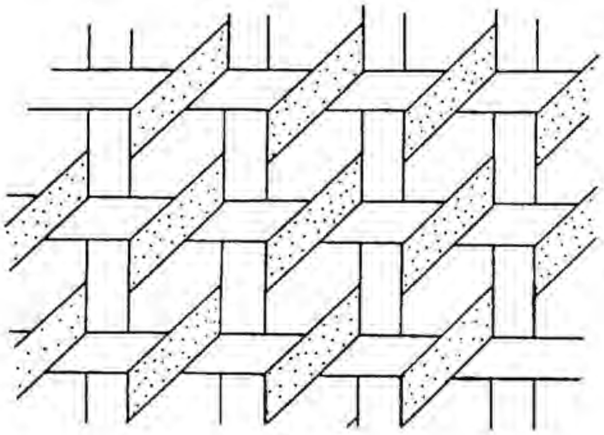


Fig. 3.36. Nambikwara plaiting in three directions

The practical optimality of the solution can contribute to the placing of an aesthetic value on the hexagonal pattern and to the isolation and the elaboration of its generating elements. For example, the Igbo and Efik in the south of Nigeria use fig. 3.37 as symbol of profound love.

The choice of three "equal" directions, leading to a regular hexagonal pattern, may have been suggested also by other experiences of production. Indians in South America discovered that the *equilateral triangle* is the optimal solution for the support of an instrument to press out manioc on a fine sieve (fig. 3.38).

Another possibility is found, for example, in Thailand. Here the solution uses four directions: two principal directions and two auxiliary directions (fig. 3.39). Once again a regular pattern is discovered. The form of the bigger holes is that of a semiregular *octagon*.

Fig. 3.37. Igbo and Efik symbol of profound love

Fig. 3.38. Manioc sieve

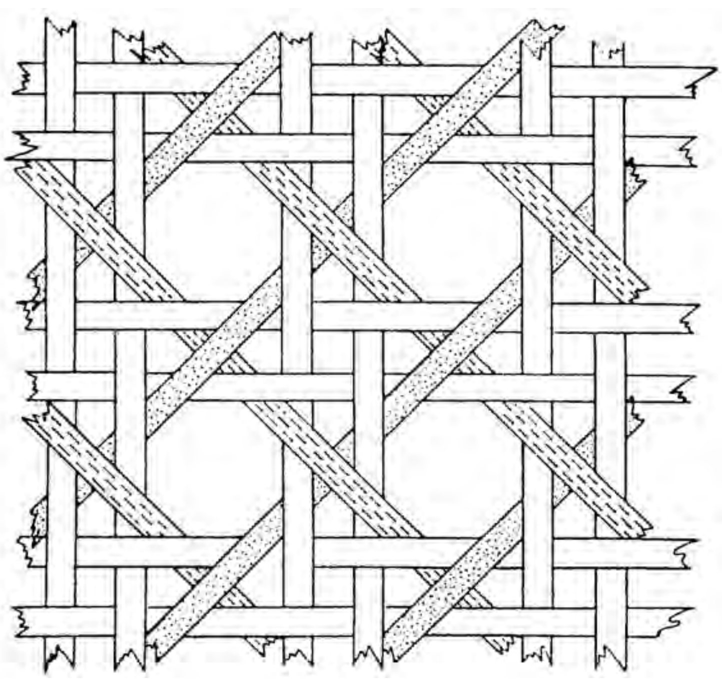


Fig. 3.39. Regular weaving in four directions

3. How can one braid strands together?

In many situations where one needs cords, for example, to bind a bundle of sticks, the individual plant strands are not strong enough. So there then arises the question *if* and *how* with more strands a better firmness might be achieved. To be able to raise this question already presupposes the experience that quantity and thickness are related to firmness—for example, the experience that over a bridge of three trunks one can drag more load than over a bridge of only one trunk; or the experience that a windscreen with thicker branches is firmer than one with thinner branches.

A first possibility of solving the problem consists of using various strands above and next to one another, as, for example, can be frequently observed in the way borders are fastened to a basket. On the other hand, it is also possible to interlace some strands into a braid and thereby establish the concept of *braiding.* Let us observe more closely this second possibility.

When one starts with only two strands, one may wrap one around the other as in fig. 3.40; or tighten it more to avoid it

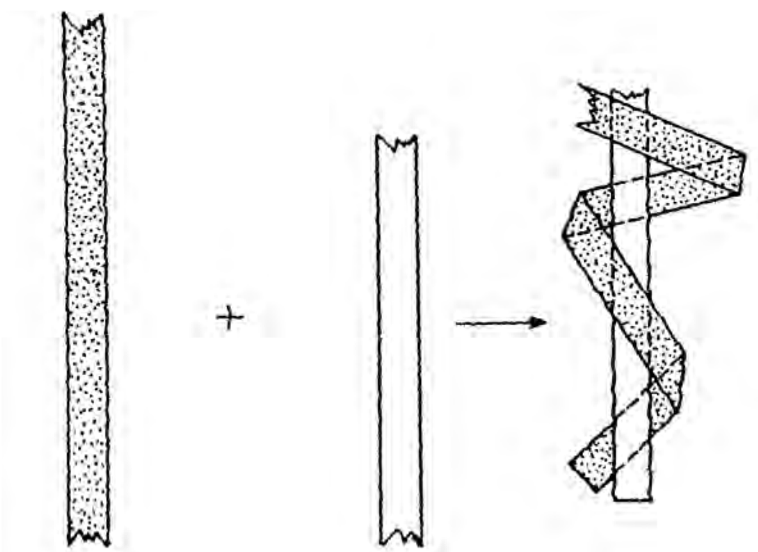


Fig.3.40. Wrapping a strand around another

slipping off, as in fig. 3.41. Once more we see that the material itself requires regularity and symmetry. In many cases, however, this interlacing does not correspond to the practical demands: both strands shrink and dilate differently, for example, as a result of warmth or humidity. For this reason, both strands should be interlaced *in the same manner.* How can one do this practically? In what sense *in the same manner*?—not to wrap one around the other, but to interlace simultaneously both strands around each

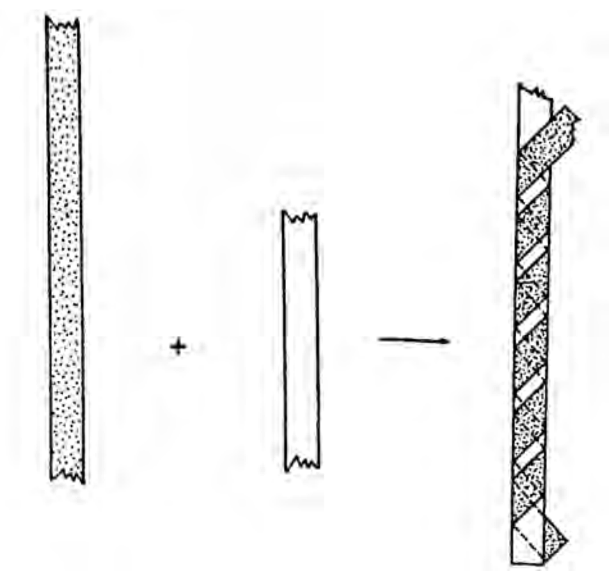


Fig. 3.41. One strand wrapped tightly around another strand

Fig. 3.42. A side view of a knotted strand

other. One can start knotting the two strands at right angles, perpendicularly, because this relationship already proved itself to be advantageous in other contexts, or because it enables a simple, fast knot (figs. 3.42 and 3.43).

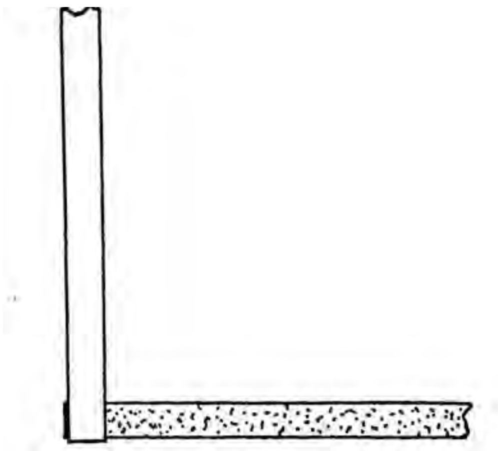


Fig. 3.43. Two strands knotted together perpendicularly as seen from above

Now both strands are linked to each other, and they simultaneously lie in perpendicular directions. If one next folds arbitrarily one strand over the other, it will be difficult to decide how to continue the interlacing “in the same manner” (fig. 3.44).

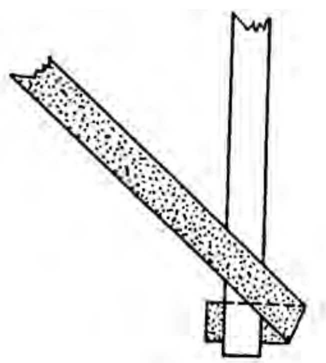


Fig. 3.44. How can one strand be folded around another?

Nevertheless, another possibility exists. One may fold the second strand in such a way that it remains parallel to the first (fig. 3.45). The perpendicular position can be recovered, folding the first strand, in the same way, over the second, as in fig. 3.46. Continuing in this manner, one obtains a braid (fig. 3.47).

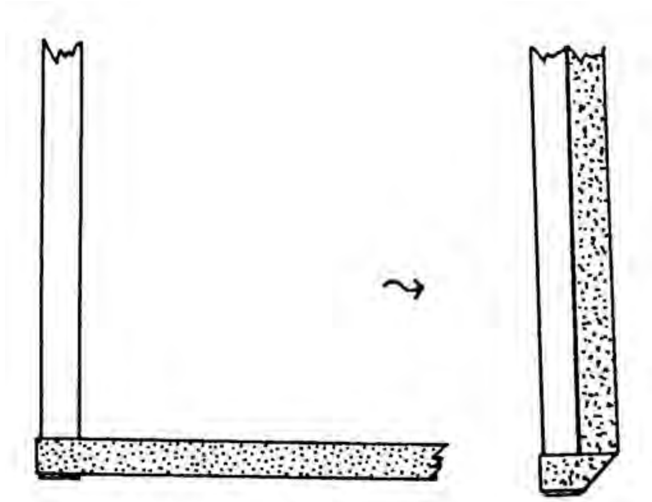


Fig. 3.45. After being folded, both strands lie parallel

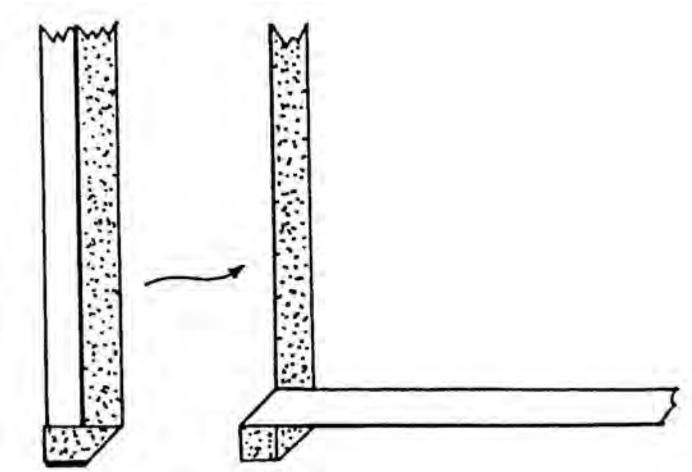


Fig. 3.46. Folding the first strand similarly around the second

The first is in fact a pseudobraid: one strand always lies on the other. The second braid is already much firmer, but comes undone when rotated around its axis. The question therefore arises whether, in view of the lack of success with two strands, success can be achieved with three strands. Figure 3.48 illustrates

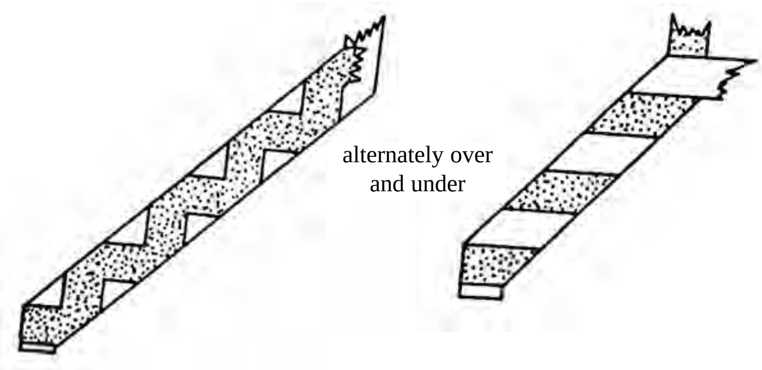


Fig. 3.47. A two-strand braid

what happens if one interlaces similarly three instead of two strands. In this active “braiding” labor, the artisans discover the advantages of an over-and-under alternation to increase the firmness of the braid.

The braid obtained in this manner (fig. 3.49) is indeed very firm and, consequently, proves to be very useful for applications such as the construction of houses or in the weaving of round mats made out of braided sisal ropes in North Mozambique.

Practical need forced the working human being to the discovery process leading to the three-strand braid. The regularity of the over-and-under alternation of the fabricated braid is the result of creative human labor, and not its presupposition. The real, existent, practical advantages of the discovered regular form lead to a growing awareness of this order and regularity and stimulate comparison with other results of labor. The regularity of the braid simplifies its reproduction and thus reinforces the consciousness of its form and the interest in it. With the growing consciousness and interest, an appreciation of the discovered form emerges. Simultaneously, the form is also applied where it is not necessary as, for example, in the braiding of the beards of the pharaohs in ancient Egypt or in the decoration of bronze objects in Benin and of wooden cups in Congo (Crowe 1975, 26; 1971, 175)—the form conveys a sense of *beauty*.

The successful interlacing of three strands prepares the way for experiences with more strands. For example, with four

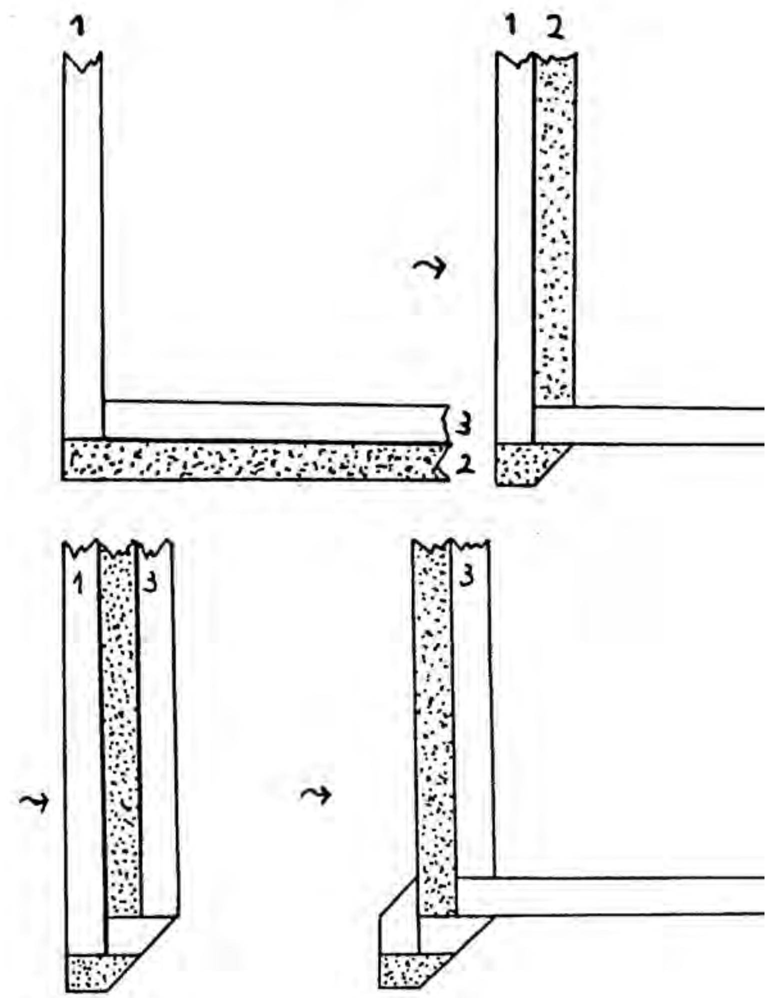


Fig. 3.48. Three-strand variant

strands it can become clear that a perpendicular crossing of the strands is not necessary (fig. 3.50). In practice, however, one sees that even with more strands, the artisans always choose the same angle of folding (fig. 3.51); it is the unique angle of incidence that guarantees that both parts of the strands, before and after the folding, stay perpendicular to each other. Putting two braids next to one another, one discovers that such a special angle of incidence

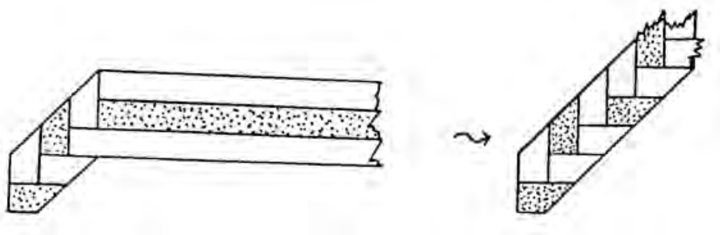


Fig. 3.49. Three-strand braid

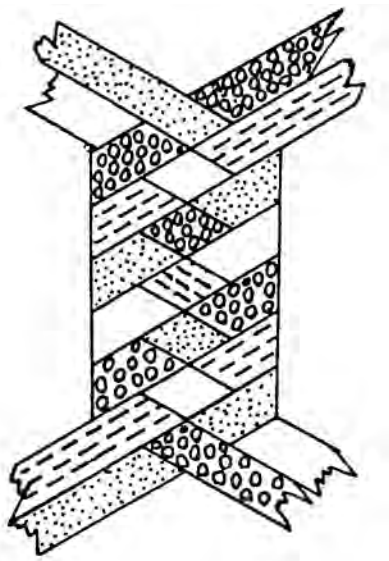


Fig. 3.50. Nonperpendicular crossing of the strands

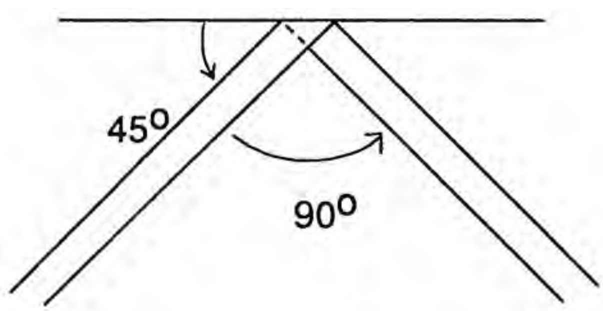


Fig. 3.51. Special angle of folding

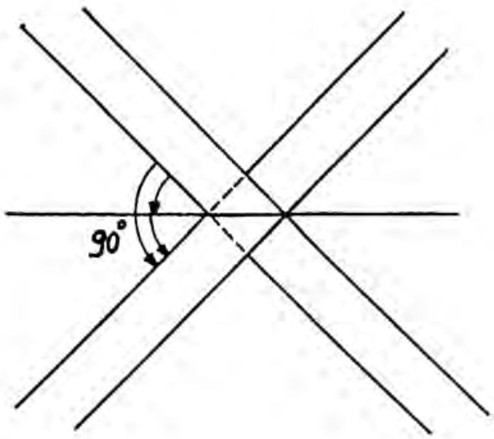


Fig. 3.52. Particular property of the special angle of folding

has still another relation to the right angle: this angle is the half of a right angle (fig. 3.52).

By comparing, one learns to recognize the same angle (45°) in other situations, for example, when binding together perpendicularly two sticks of equal thickness (fig. 3.53)[11] or in the simplest variation (over-two-under-two), where one interlaces at right angles strands of two distinct colors (fig. 3.54, so-called twill plaiting or twill work). Here the consciousness of this particular angle and the interest in it develop further. Thus it cannot

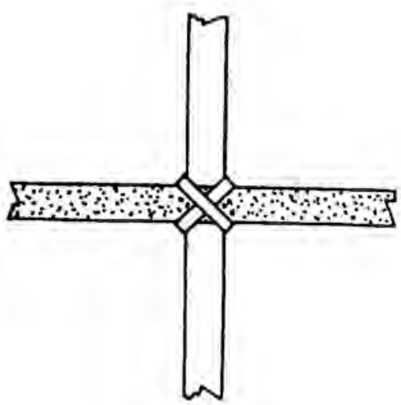


Fig. 3.53. Binding together two sticks of equal thickness

be accidental that the right angle and its half appear so often in many artistic forms (see some examples from Africa in fig. 3.55).

Using more than three or four strands, one obtains a rectangular mat with a relatively stable border (fig. 3.56). This type of mat is widespread.

It should be noted that other possibilities exist *theoretically*, as, for example, rectangular mats with a nonperpendicular crossing, where the angles of incidence are different for the

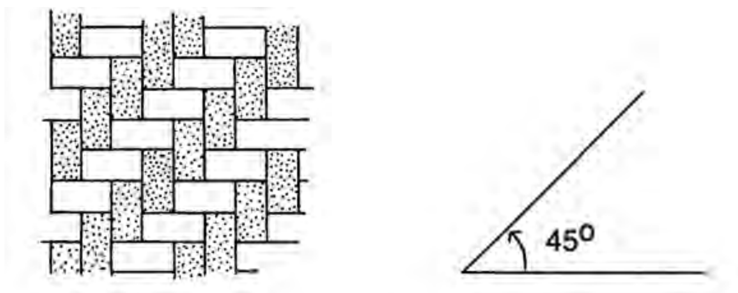


Fig. 3.54. Angle of 45° appearing in over-two-under-two weaving

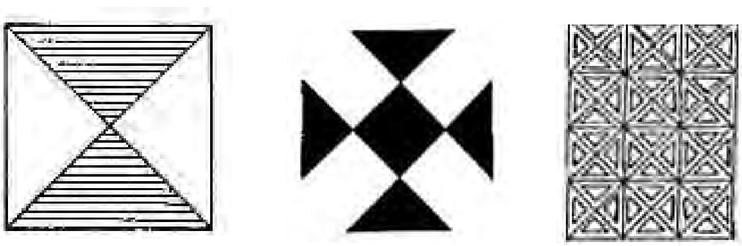


Fig. 3.55. Angle of 45° in African ornaments

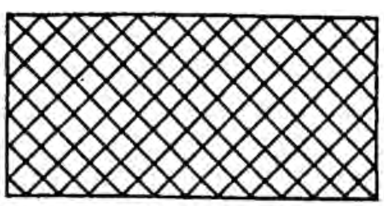


Fig. 3.56. Pattern of a rectangular mat

contiguous sides of the rectangle (fig. 3.57). This variant, however, is not found in practice. Preference is given to interlacing at right angles.

Upon discovery, the regular pattern (fig. 3.56) provokes, in turn, further reflection and application. For example, the Kuba in today's Congo used this pattern, called *mbolo,* in the colorful decoration of their textiles and in woodcarving. Their children were

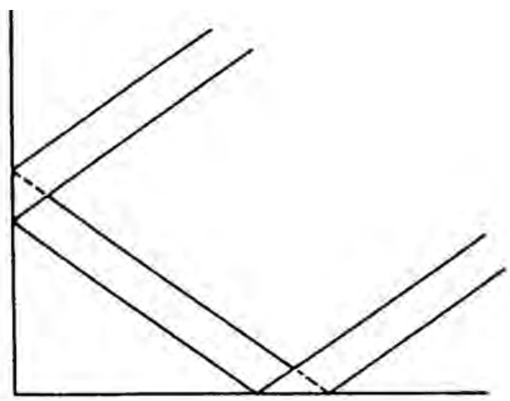


Fig. 3.57. Different angles of incidence for adjacent sides of the rectangle

observed playing a game with these patterns in the sand, the task being to draw these figures (fig. 3.58) in the sand without lifting a finger (Torday and Joyce 1911, 198).[12]

With these reflections in art and in games, early mathematical thought started to liberate itself from material necessity; form became more independent of matter, and thus the concept of *form* emerges—the way is made *free* for an *intramathematical* phase of development.

The rectangular braiding process we have described is not the only possible one. Another method that is very suitable for the fabrication, for instance, of hats (fig. 3.59) and bags is found in Africa (e.g., Mozambique and Nigeria), in Asia, in Hawaii, and among Brazilian Indians. The basic braiding angle measures 60°. Why 60°? An accidental discovery? The result of experimentation? Or an angle that proved itself already advantageous in other

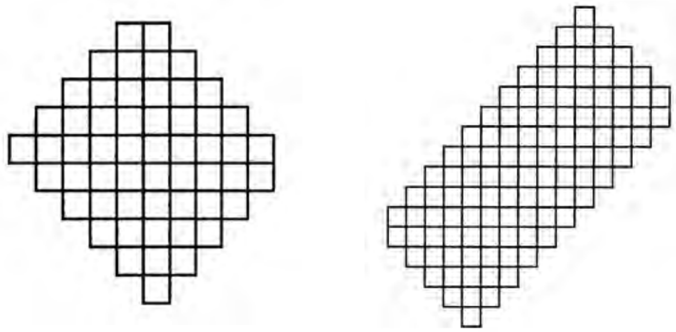


Fig. 3.58. *Mbolo* patterns drawn in the sand

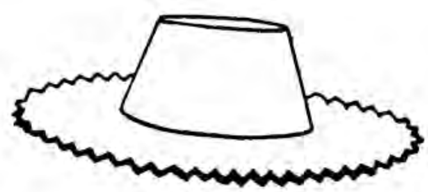


Fig. 3.59. A hexagonally woven hat

situations, like, for example, in the production of fish traps with hexagonal holes? Let us see how one may produce a braid *hexagonally*.

The artisan may start with two strands of equal width and fold one of them once around the other (fig. 3.60) in such a way that the angle of incidence measures 60°, as we saw in section 2. When one now folds *in the same way* the second strand around the first one, a figure is obtained that enables continuation with normal over-and-under plaiting (fig. 3.61).

Proceeding on this basis, one can weave several types of braid. Figure 3.62 presents two examples. A braid like the one in fig. 3.62a might constitute a stimulus to weave closed mats in three directions (fig. 3.63), as is common in the Far East. The zigzag braid in fig. 3.62b is used in Mozambique for the production of hats, starting first of all in a *spiral* from the future center at the top of the hat (see the start in fig. 3.64). The same zigzag braid is encountered in Kenya, Vietnam, China, and in the

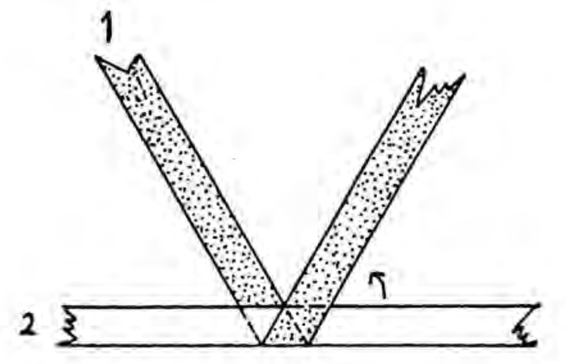


Fig. 3.60. Start position

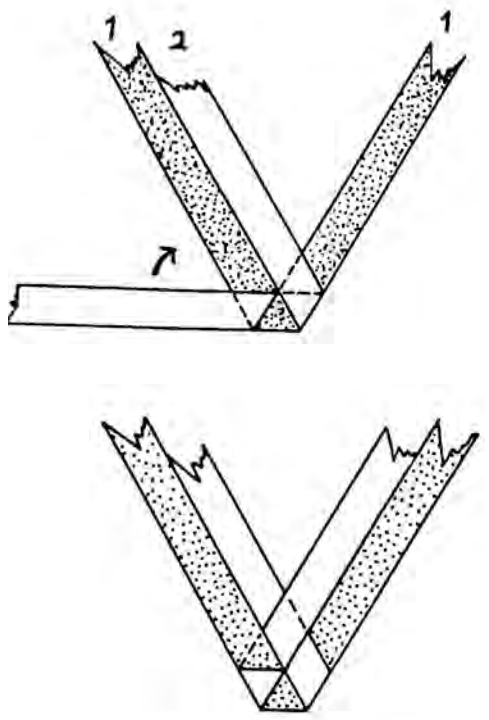


Fig. 3.61. The next steps

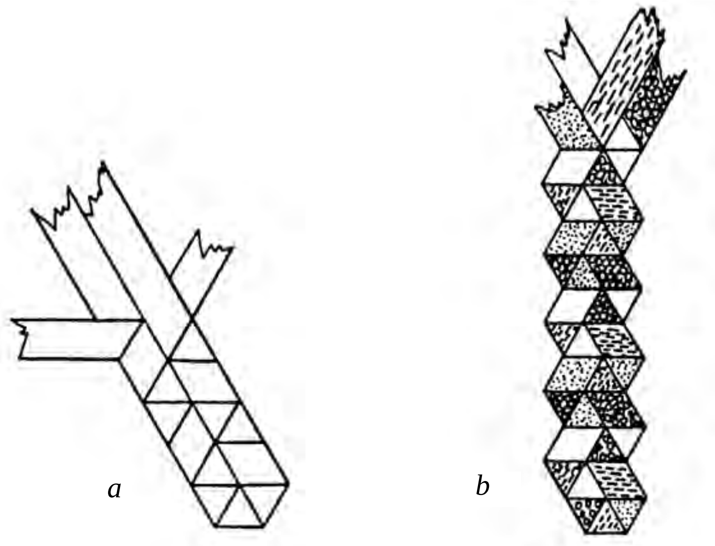


Fig. 3.62. Three-strand braid

Philippines as border ornamentation, e.g., for fans. On Hawaii it is used to weave a headband called *nihoniho* (Bird et al., 1952, 39–58). On the Palau Islands (Oceania) the strips of a woman's apron are braided in the same way. The same technique was also used among Indian peoples in South America, e.g., among the Esmeraldo in Ecuador (Christopher 1952, plate 13) and in Guyana (Roth 1970, 498–99).

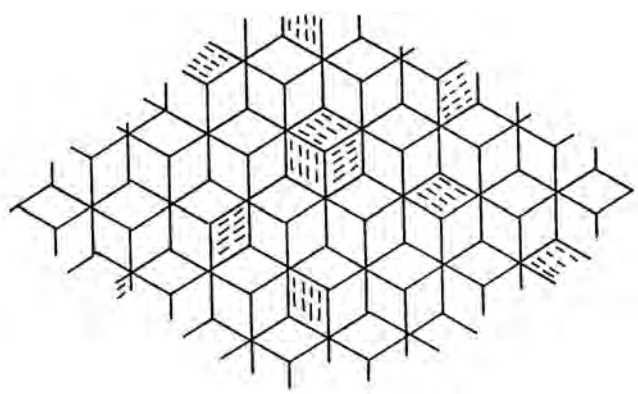


Fig. 3.63. Weaving in three directions

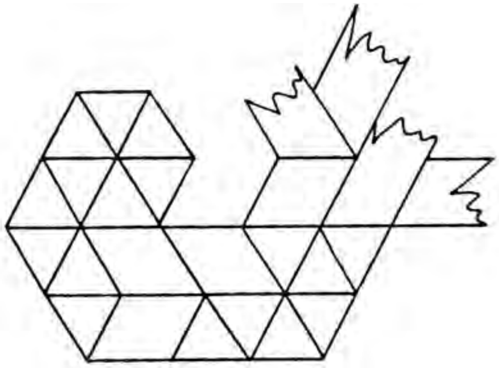


Fig. 3.64. The production of a hat starts at the center

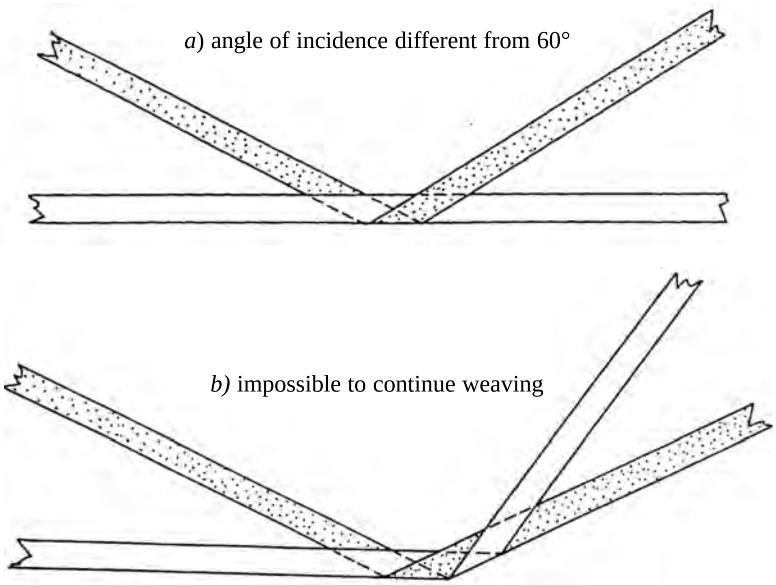


Fig. 3.65. Angle of incidence incompatible for weaving

If the angle of incidence was different from 60°, as in fig. 3.65a, then one would arrive at a next position (fig. 3.65b) that does not permit any further interlacing. In other words, the choice of the angle was not at all accidental; it was imposed by materials used for weaving. The angle of incidence here necessarily also has to be 60°. From the awareness of this necessity acquired through the achievement of certain aims arose the human *freedom* to produce hexagonally braided hats, fans, and bags, both useful and appraised as beautiful.

4. How can one weave a button?

A problem frequently met in basket-making is closing a basket with a lid. A possible solution is obtained in the following way. At least two laces are fastened to the lid as in fig. 3.66. An equal number of buttons are fixed to the wall of the basket. To close the basket, the laces are pulled around the respective buttons. To guarantee that the laces do not come off easily, what will be the most advantageous shape for the buttons? Practical human activity leads to the conclusion that a polygonal form is preferable to a round shape. For this purpose, a square shape is used among the Tsonga in the south of Mozambique (fig. 3.67).

Once the advantage of a square button is discovered and accepted for its fabrication, the artisan-inventor has the task of elaborating the successive steps for making a square button. Here he or she may start with a little square mat. At first sight, one understands that at least four strands are needed in order to plait a little mat (fig. 3.68). How may one advance from here and, without any further material, produce a strong, stable button that does not easily come undone?

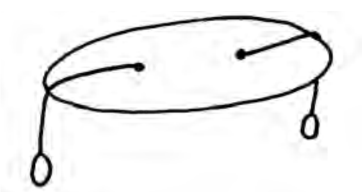


Fig. 3.66. Basket lid with two laces

Fig. 3.67. Lace pulled around a square knot

One verifies that it is useful to fold in two the part of a strand that goes under the part of another strand perpendicular to it, moving it first upward and then backward in the opposite direction. If this procedure is followed in the sequence indicated in fig. 3.69, with the folded-down part of the last strand passing under the doubled part of the first strand, then one obtains a stable "four-square" knot. If, at the end, one cuts off the protruding parts of the strands, then the stability would be lost and the knot would easily unravel. Therefore, the protruding parts are not cut off, but the previous phase of the production process is repeated, doubling and interlacing the protruding strands. One continues until one of the strands is finished. Next, one cuts off the protruding parts of the remaining strands. In this way a button is obtained in the shape of a block with a square base.

To avoid the necessity of cutting off some pieces of strand, it becomes preferable to choose immediately, at the start, four strands of equal length; the basket weaver puts them in a position with *rotational symmetry* (see the examples in fig. 3.70). To avoid the somewhat difficult interlacing with pairs of strand parts, an initial position like the one in fig. 3.70c may be preferred, which makes it possible, after doubling the little parts of strand, to continue weaving with only the longer parts. Nevertheless, now there appears another difficulty. The start, that is the first "layer" of the button, unravels relatively easily. The Mozambican basket weavers overcome this difficulty by starting

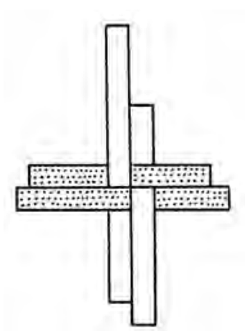


Fig. 3.68. Starting position

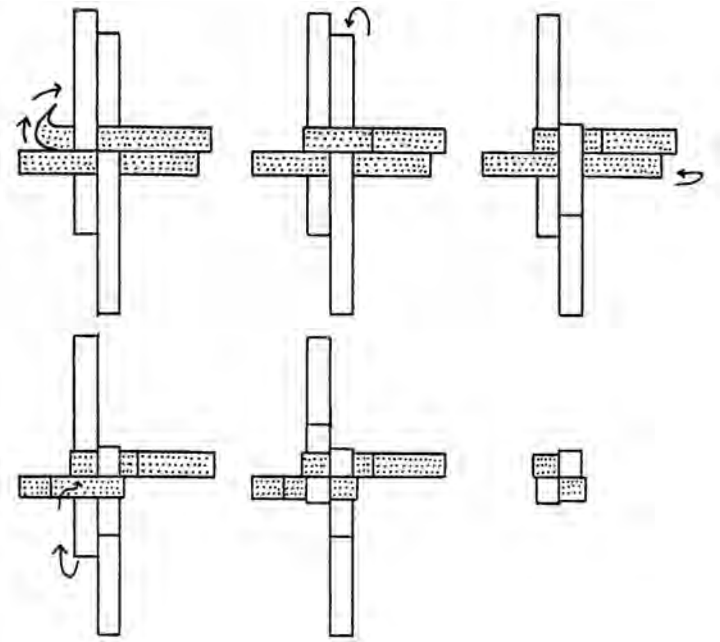


Fig. 3.69. Using four strands to produce a knot

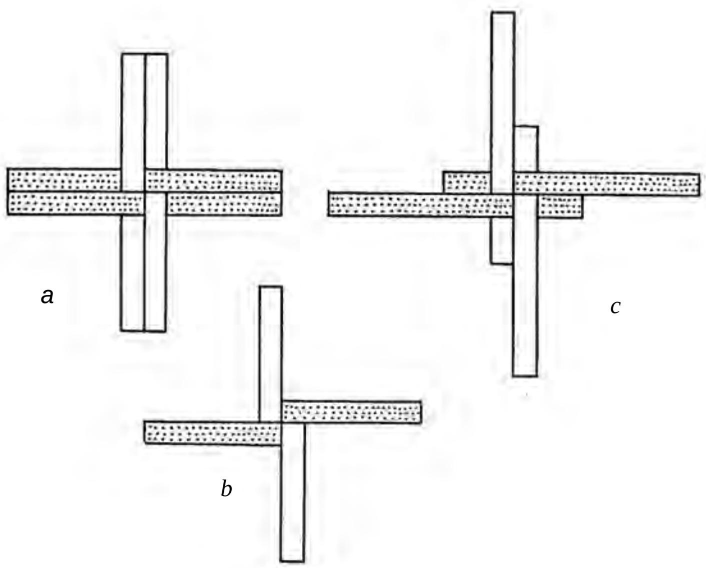


Fig. 3.70. Starting position with rotational symmetry

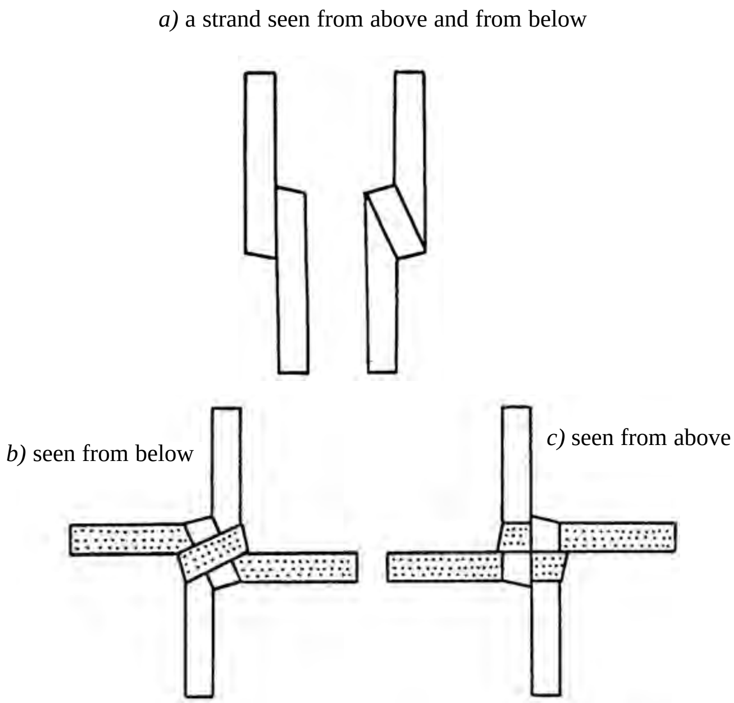


Fig. 3.71. Two interlaced strands

with two plant strands instead of four. Now they fold each strand twice, more or less in their middle (fig. 3.71a), in such a way that both protruding parts of one strand play the same role as two parallel strands in the case before. Next, one interlaces on the same side the protruding parts of strand, thereby building up the successive layers of the button (figs. 3.71b and c). The button obtained in this manner is very firm and stable.[13]

In the discovery process reconstructed here, several geometrical concepts like congruence, rotational symmetry, and square were developing and became more deeply understood by the active human being, the inventor. The proportional relationship between the height of the button, that is, the number of "layers," and the length the strands can be found.

The "double-S" knot, that is, the first "layer" of the button (figs. 3.71b and c), could also have been discovered or applied in a completely different context. In Kiribati in Micronesia (Koch 1965, 181ff.) and in Tuvalu in west Polynesia (Koch 1961, 148), it is put on the end of a stem, and with these little “windmills” as toys in their hands, children run against the wind (fig. 3.72).

Another variant of the same basic idea is met on Nonouti Island in Kiribati (Koch 1965, 185). When one joins a third strand to leave the plane of the “double-S” knot, one can fabricate a *cube*. In this way the exterior of the cubic ball is woven for the *bwebwe* game (fig. 3.73).

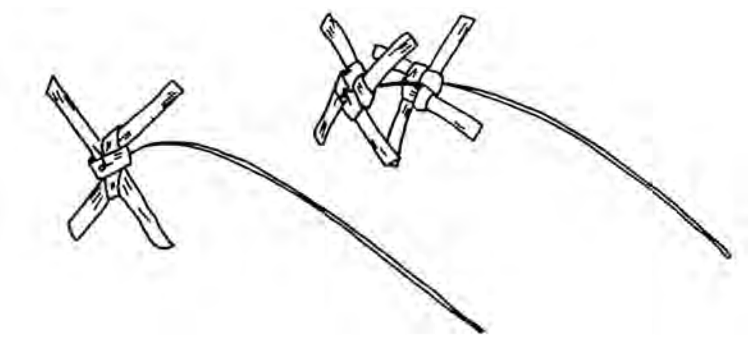


Fig. 3.72. Windmills from Tuvalu (formerly Ellice Islands)

Fig. 3.73. Ball for the *bwebw*e game at Kiribati (formerly Gilbert Islands)

5. The concept of a circle

In the practical activity of fabricating objects, people develop forms and continually improve them with increasing adequacy for their daily needs, Humans learn to recognize forms as such and distinguish not only between form and material, but also between changes of shape arising from the transformations due to their labor and the changes in form occurring in nature, such as the regular waxing and decreasing of the moon, the building up of a bird's nest, the centipede rolling itself into a spiral when it feels itself threatened, the web woven by the spider, etc. The clearly observed changes of form in nature can, in turn, lead people to new ideas and experimentation. The construction of a nest may stimulate the idea of a basket. The coiling up of a centipede (fig. 3.74), some types of flies depositing their eggs in a spiral (Sauer 1972, 32–33), or the rolling up of a rectangular mat into a cylinder may have contributed to the idea of the possible use of a spiral for the fabrication of circular mats from rope (fig. 3.81).

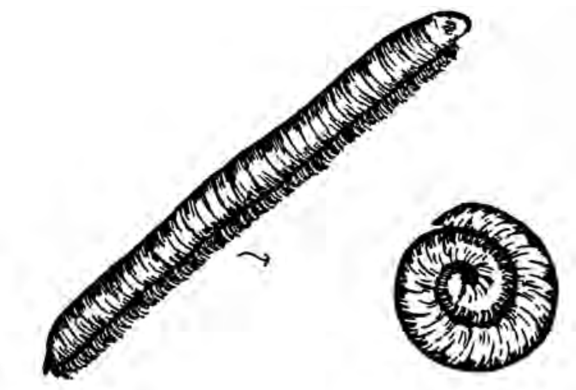


Fig. 3.74. Coiling up of a centipede

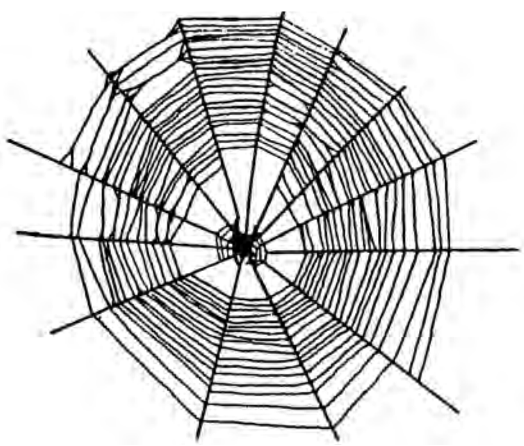


Fig. 3.75. Spider's web

The spinning of a web by a spider (fig. 3.75) could have had a stimulating effect on weaving activity. Various sticks or strands can be bound together and appear like the skeleton of a cobweb (fig. 3.76). When one interlaces fibers or strands up and down, "circling around" in the form of a spiral, starting from the center where the sticks are bound together, one then obtains a primary "web" (fig. 3.77). To increase the stability of the "web," it is necessary to draw the fibers tighter. The homogeneity of the fibers leads to the need to adjust and equalize the angles between the sticks. At the same time, one discovers that to produce a relatively flat horizontal "web," two fibers have to be interlaced simultaneously, circling around the center in the form of a spiral, but in opposite directions—that is, when one fiber passes over a

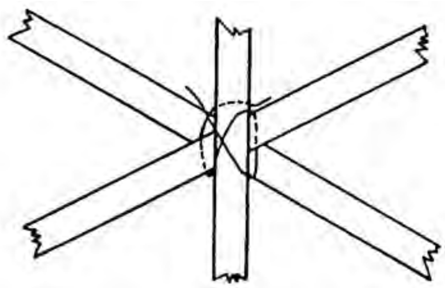


Fig. 3.76. Binding together some sticks

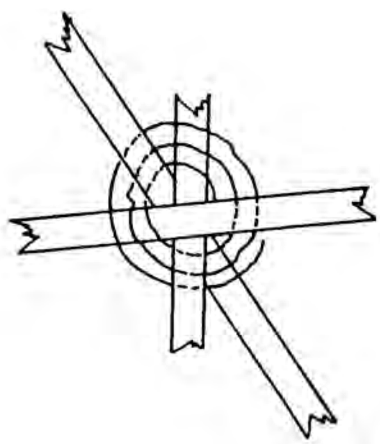


Fig. 3.77. Circling the fibers around the sticks

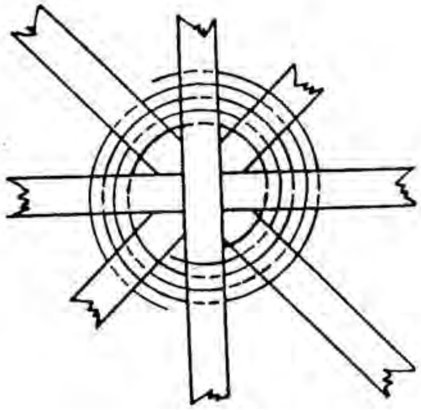


Fig. 3.78. Improvement

stick, the other passes under it (fig. 3.78). The "web" becomes circular. When it is big enough, one may break off the sticks at the border (fig. 3.79). With increasing experience, the necessity of choosing sticks of the same length and joining them together at their midpoints to avoid the necessity of breaking them off later becomes clear. Emerging in this process is the concept of the *radius* of a circle: in all directions the radii are *equal*; the circle acquires a *center* (fig. 3.80). This concept formation may also be stimulated when making round mats by coiling up a rope of sisal, where a "center" serves as starting point, and one can observe the distance between the perimeter and the center (fig. 3.81).

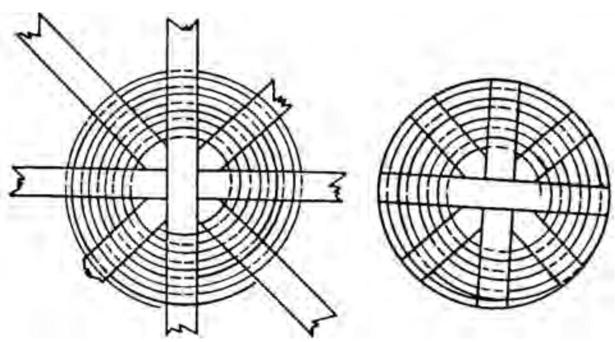


Fig. 3.79. Breaking off the protruding stick parts

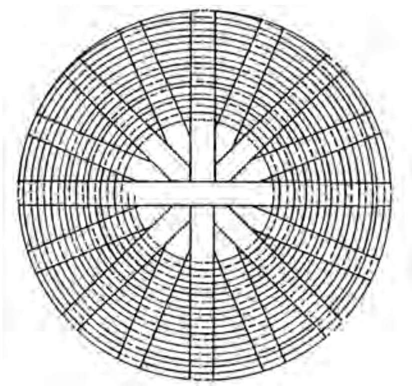


Fig. 3.80. "Spiral-radius-circle"

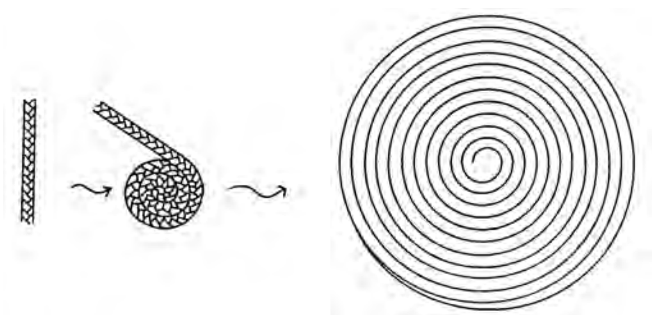


Fig. 3.81. A circular mat made by coiling a braid of sisal (Northern Mozambique)

When one folds the sticks upward—as done by Makhuwa basket weavers in northeastern Mozambique—instead of breaking them off at the edge, and continues the spiral interlacing of the fibers, the spatial figure that emerges, usually a basket, has the form of a cylinder or a truncated cone (fig. 3.82). Also, as it happens here, the homogeneous material forces a regular form in the shape of a cylinder or a cone (constant surface tension).

Once the concept of radius is e*labor*ated, the possibility arises for a new circular construction, apart from the spiral and constant-surface-tension methods (figs. 3.81 and 3.83). The new construction can be applied immediately with a pair of compasses wherever equal distance to a center plays an important role.

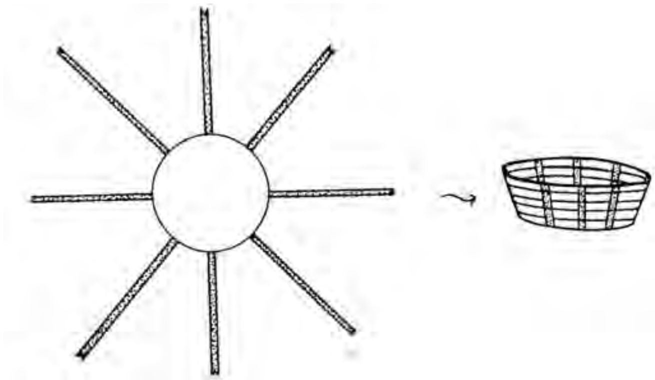


Fig. 3.82. A conical basket

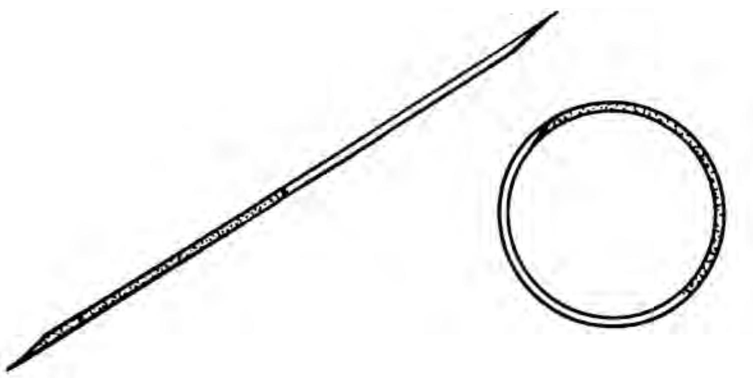


Fig. 3.83. Circular bending of homogeneous material

One method of drying fish in the northern coastal zones of Mozambique is to attach the fish to little sticks in the sand at an equal distance from a fire. That the fish should be "at the same distance" from the fire when there is no wind is already a result of experimentation and reflection. How can one ensure that the fish are "equidistant" from the fire? The spiral method can be applied, but it is a relatively roundabout way to do so in this context. A basket with the shape of a truncated cone can be closed in various ways with a lid in the form "spiral circle" (figs. 3.80 and 3.84), its "rotational" symmetry already presenting a suggestion for the construction of a circle with the help of a "pair of compasses." Figures 3.85 and 3.86 illustrate how fishermen on Mozambique Island apply this construction to dry their fish.

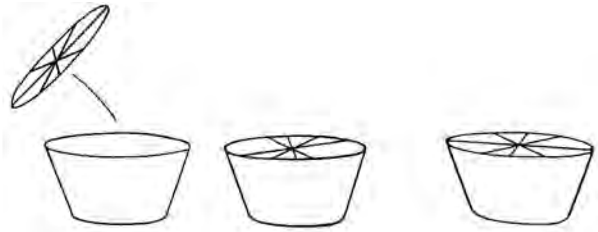
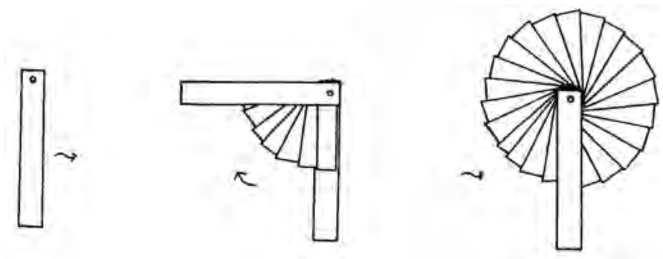


Fig. 3.84. Various lid positions

Fig. 3.85. Drying fish around a fire at Mozambique Island

Fig. 3.86. Circle construction for the fish drying

The making of a circular fan in the same coastal zone constitutes an interesting variation of the already cited construction with the help of a "pair of compasses," as displayed in fig. 3.87.



a) closed fan *b)* opening of the fan *c)* opened fan

Fig. 3.87. Circular fan

The discovery of construction with the help of a "pair of compasses" can be applied in situations where there is no immediate necessity for a circular form as, for example, in construction of houses. Why should a house have a circular base?

In his book on the construction of houses in Southern Africa, Frescura (1981) sketches the historical development of housing, from cave shelters and tents to the beehive and cylindrical-conical house. What he does not elucidate, however, is why, at a certain moment, a circular base was chosen.

The construction of a type of grass house among the Ngwane in South Africa may serve as an example of the concrete dialectical interplay between active life and abstract thinking. Consider a widely used way of producing round basket bowls that schematically can be summarized as follows: a woven square mat is fastened to a circular border (see fig. 3.88 and section 8 of this chapter). The border is circular as a consequence of the homogeneity of the material that was bent; the bottom is interlaced at right angles, not because it is materially necessary (fig. 3.89 displays an alternative construction in the form of a spiral), but because it is an already known way of making mats (transplantation of the idea). The basket bowl fabricated in this way can also be turned upside down to cover something (fig. 3.90).

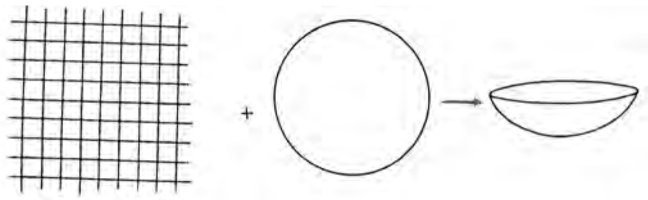


Fig. 3.88. Schematic display of the production of a type of circular tray

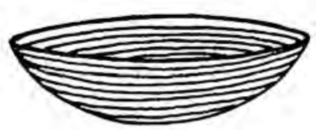


Fig. 3.89. A basket coiled in a spiral

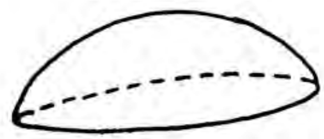


Fig. 3.90. Basket turned upside down

The experiences with basket bowls that are turned upside down and with the older construction of tents could, upon being united in human reflection, lead to the idea of an “inverted bowl-house,” as illustrated by the construction of the house of the Ngwane (fig. 3.91; see Knuffel 1973) or by the construction of spiral igloos by the Inuit.

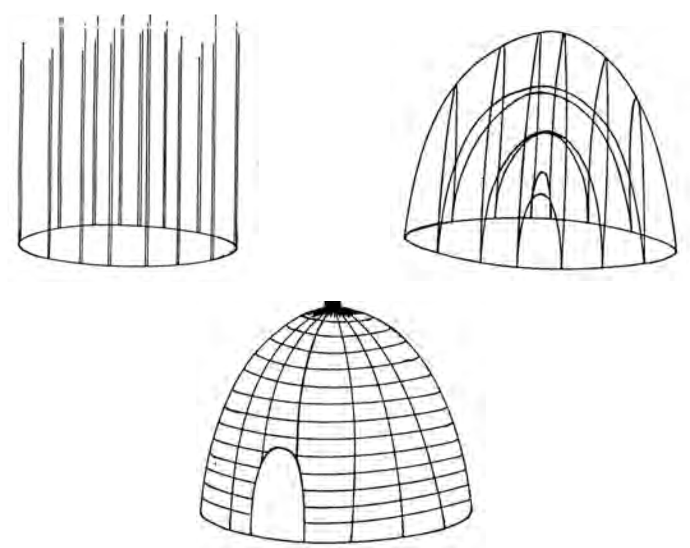


Fig. 3.91. Phases in the construction of a Ngwane house

In the construction of houses, no immediate practical necessity exists either to plait the skeleton rectangularly or to choose a circular base. These ideas had already been developed in another context like the one described above, and were applied here *freed from material necessity*. The manner in which the circular border of a basket bowl is made is not suitable for the construction of the circular base of a house. On the other hand, construction with the help of a pair of compasses is much easier, and, in fact, occurs.

In an analogous way, one may try to sketch the possible development of ideas that led from the conical basket to the conical hat and to the cylinder-conical house (fig. 3.92).

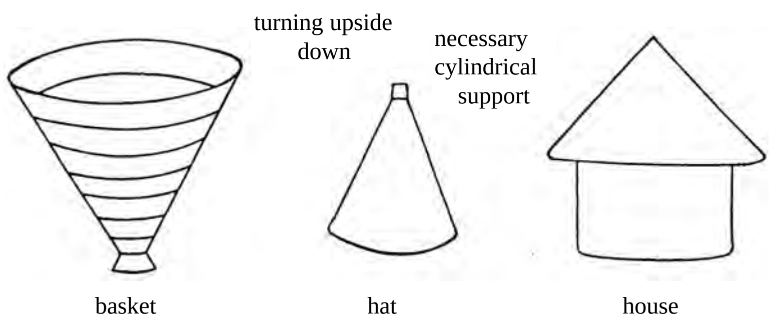


Fig. 3.92. The idea of the cylinder-conical house

6. The idea of a regular pentagon

Could the idea of a pentagram have been the result of direct observation—for example, of starfish (fig. 3.93)? Or could it have been derived from the diagonals drawn in a regular pentagon (fig. 3.94)?

Could the idea of a regular pentagon correspond to an intra-mathematical development of the concepts of the equilateral triangle, square, and regular hexagon? Or did it always exist, independent of the real world of objects, in a Platonic world of ideas?

Fig. 3.93. Starfish

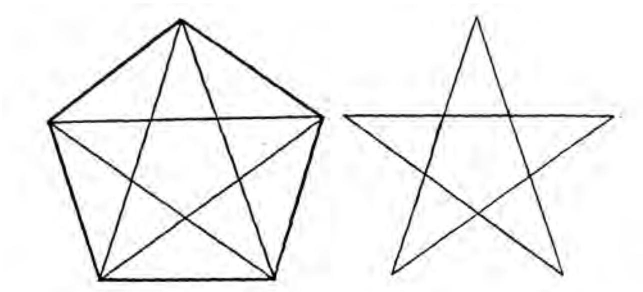


Fig. 3.94. Regular pentagon and pentagram

The pentagram was the emblem of the Pythagoreans. In medieval Europe, the pentagonal star was viewed as a protection for human beings against druids and angry spirits. Why did people think pentagons provided such protection? Let us examine the following practical problem that arises when harvesting cereals to discover a possible real relationship—and thus not a magical one—between protection and the form of the regular pentagon.

The kernels of grain are often torn off the stalk with the hands. How can one protect the hands against cuts during the harvest without gloves?

What happens if one tries to weave a "thimble" with a flat strand from a plant?

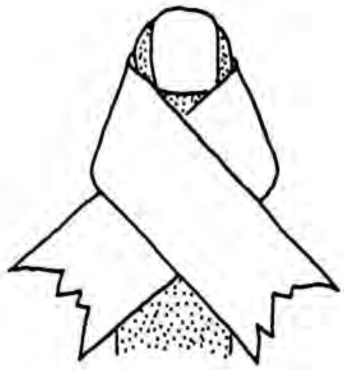


Fig. 3.95. A strand folded around the fingertip

It is possible to fold a strand around the finger, but how can one weave it into a thimble (fig. 3.95)? How can the tip of the finger be reliably protected?

Certainly it is necessary to join in one way or another both parts of the strand. Let us remove the strand from the finger and try to make the simplest knot (fig. 3.96). If one now carefully tightens the knot, one obtains a figure that simplifies further weaving (fig. 3.97a). Once proceeding in this way, it turns out that it is not difficult to continue to interlace the strand, obtaining a stable and solid thimble closed on four sides; the fifth lets the finger enter (figs. 3.97 and 3.98). The finger protector made in

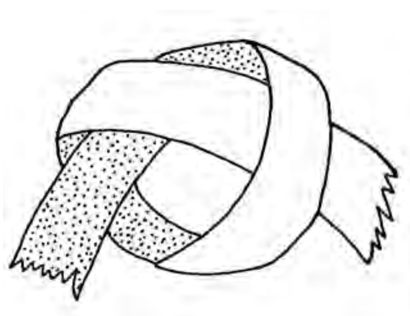


Fig. 3.96. Simple knot

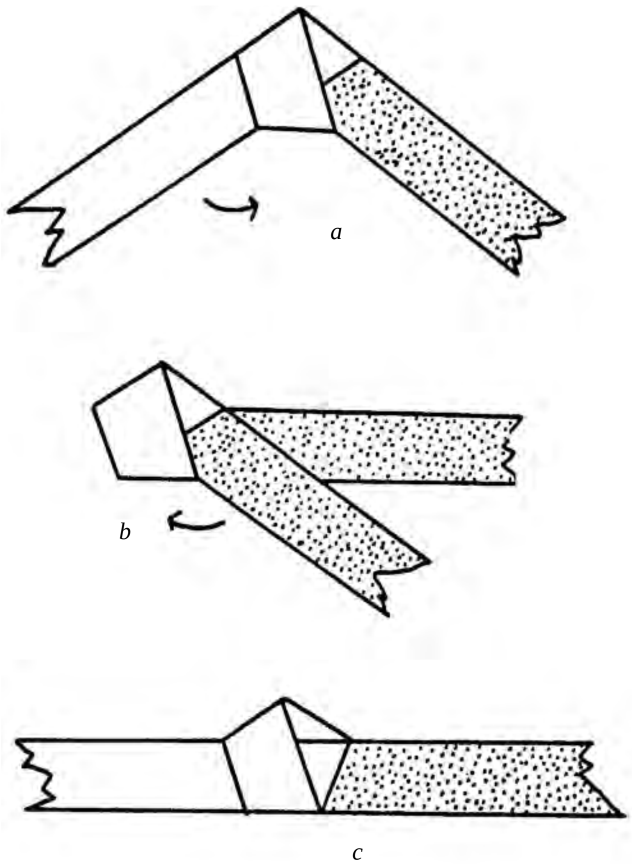


Fig. 3.97. Interlacing a finger protector

Fig. 3.98. The thimble put on the finger

this manner is pentagonal and regular, not because a God or anybody else desired it a priori, not because a Pythagorean imagined it, but as a result of human activity for dealing with the problem of avoiding cuts on the fingers. The solution of the problem of producing a finger protector presented itself, and its solution was found, from the material possibilities.

On the Indonesian island of Roti, regular-pentagonal thimbles made out of strands of Lontar leaves are placed on the index finger and thumb when grains are torn off the stalk (Hirschenberg and Janata 1986, 263).

The type of problem solved here already appeared very early in human history when an economy of harvesting emerged during the end of the Paleolithic and Mesolithic periods in some limited geographical regions as a result of the appearance of a massive presence of wild plants. It is thus possible that the solution by means of a regular pentagonal woven thimble dates from this period. If this is the case, then the appearance of the regular pentagon in cuneiform tablets from ancient Mesopotamia[14] would be less surprising.

Other weaving and knotting work could also have contributed to the emergence of the concept of a (regular) pentagon.

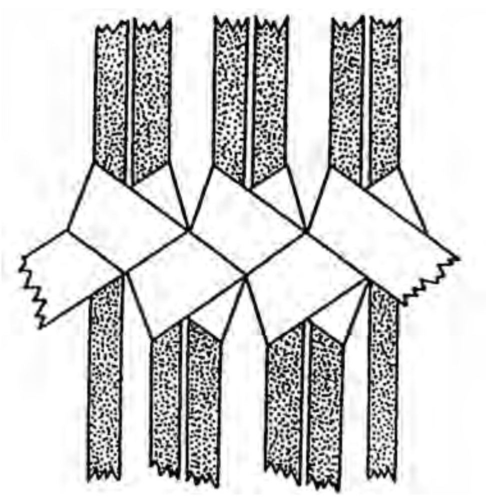
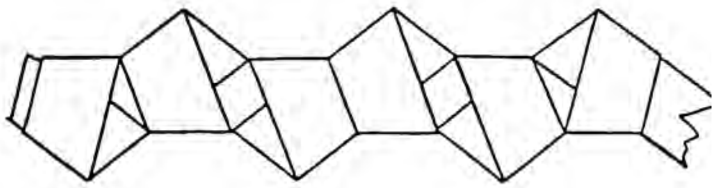
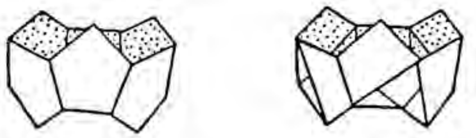


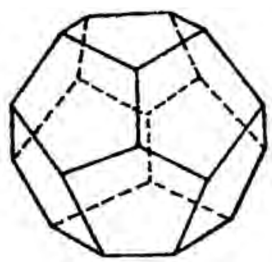
Fig. 3.99. The strands of a broom held together by pentagonally woven knots



a) interlacing various knots on a row



b) ring of six pentagonal knots



c) regular dodecahedron

Fig. 3.100. Symbolic uses of pentagons

The strands of Mozambican brooms are held together by pentagonally woven knots (fig. 3.99). Does another reason for the possible connection between the ideas of "pentagon" and "protection" reside here?[15]

The invention of the regular pentagonal, woven thimble may stimulate playful activities like interlacing various knots on a row (fig. 3.100a) or symbolic use in traditional Japanese family crests (fig. 3.101). When one produces six pentagons and joins the first with the sixth, one obtains a ring, as in fig. 3.100b. In this continuing experimentation, one has already left the "reign of

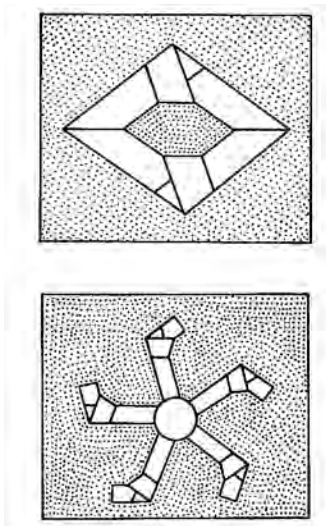


Fig. 3.101. Two traditional Japanese family crests

necessity," the reign of immediate material satisfaction of human necessities. One may now try to weave a "round" closed figure. With further experimenting and attempts to create a closed spatial object, one discovers, without anticipating it, that there is only one possibility, the dodecahedron—that is, a solid with twelve faces of regular pentagons (fig. 3.100c). The liberty to experiment and find many forms is thus lost; only one solution is materially possible.

"Knowledge of the dodecahedron by the Pythagoreans in antiquity is surprising," writes Wussing, "perhaps related to the fact that pyrite found in Italy crystallizes in dodecahedrons" (1979, 47). An Etruscan dodecahedron made out of soapstone goes back to the sixth century B.C. (van der Waerden 1954, 100). Dodecahedrons of bronze and iron served as burial goods in Celtic and Etruscan tombs. Were these dodecahedrons the product of pure fantasy? Or did the pyrite crystal serve as model? Perhaps, however, earlier situations of social activity leading to the discovery of the dodecahedron, such as the one I described with the pentagon, might have occurred. If this were the case,

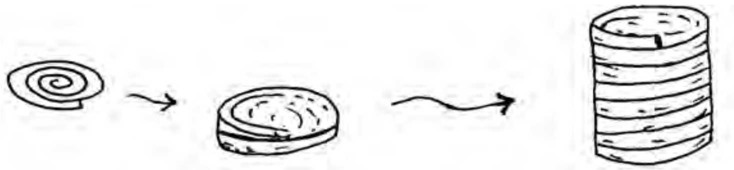


Fig. 3.102. Production of a cylindrical basket using the coiling technique

then the Etruscan dodecahedron and the Pythagorean knowledge would also appear less surprising.[16]

7. How can one weave baskets with a flat bottom?

Knowledge of the use of the coiling technique to make circular mats or flat dishes leads almost automatically to the ability to produce baskets with a *round* bottom of the desired diameter. One then sews a spiral upward onto the last spiral of the bottom (fig. 3.102). As one proceeds upward, the form can be chosen at will. This choice proves to be free only in a certain measure, as the coiling method always forces a *rotational symmetry* on the vessels when one starts with a circular base (see section 5 in this chapter).

The coiling technique is reflected in the older pottery techniques, whereby the clay is first rolled into cylindrical bars, which are then joined one behind another into a spiral. The pot is then shaped (fig. 3.103), retaining rotational symmetry. The origin of the symmetry of the pots made in this way lies in the imitation of the coiling technique from basketry. Although the clay, as a new material, gave people the liberty to choose other shapes, their thinking initially remained too influenced by the basket-weaving tradition to allow the imagination of other shapes for pot production. Once the restrictedness of this thinking was

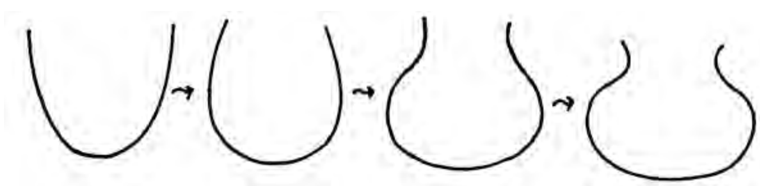


Fig. 3.103. Shaping a pot

surmounted, new forms could be tested. Completely unforeseen circumstances, however, forced a return to the traditions of rotational symmetry: sharp-edged vessels break more easily than round-shaped vessels during firing, unbalanced pots are more difficult to carry on the head, etc.[17] Would not a principal reason for the relative *uniformity* of clay pots all over the world lie here? Should one not look here for an important cause of what some sociologists call the conservatism of potters (Foster 1961)? Would not idealistic exaggeration, loosened from societal activity, become understandable, if one considers that the potters in southern Mozambique produce circular shapes "instinctively" (see, for example, Junod 1974, 2:106)?. Furthermore, would the earlier making of clay vessels with rotational symmetry not have been a precondition for the invention of the potter's wheel, rather than the other way around? Once discovered, the potter's wheel for the reproduction of vessels with rotational symmetry became simplified with increased precision, further stimulating, in turn, the formation of this symmetry concept.

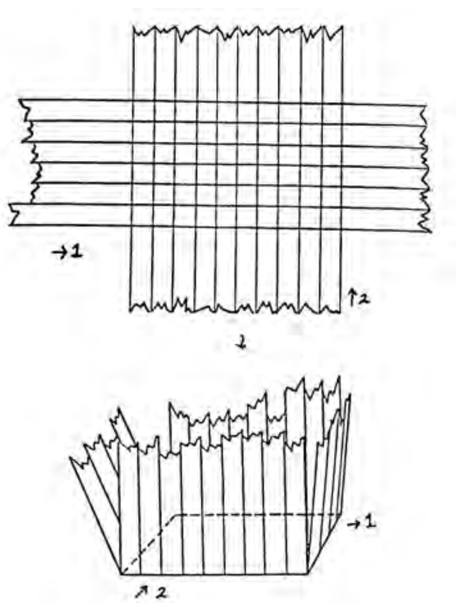


Fig. 3.104. Bending upward the protruding parts of strand

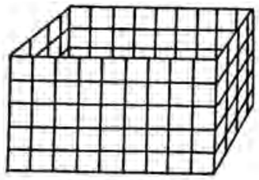


Fig. 3.105. Block-shaped basket

Let us return to the question of how one can weave baskets with a flat base.

We already met the widespread mat-making technique that consists of interlacing at right angles two groups of strands of equal width. How can one continue to weave upward to get a basket with a flat bottom? When one bends upward the protruding parts of strand (fig. 3.104), a "skeleton" of a basket appears. How can one now link together the parts of strand protruding upward? Without auxiliary strands or other material, this is not possible. But when one interlaces horizontally or spirally new strands with

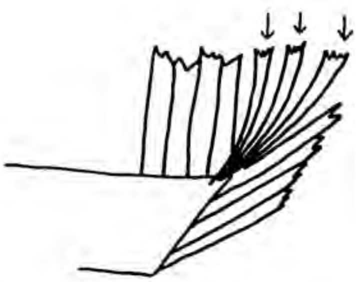


Fig. 3.106. Adding strands at a corner

Fig. 3.107. Shape of a carrying basket

the vertically protruding pieces of strand, a basket in the shape of a block is automatically obtained (fig. 3.105). The walls are necessarily perpendicular to the bottom. For example, the *cabaz* basket from Portugal is made in this manner (Silva 1961, 53, 76). If one joins some extra vertical strands at the four corners (fig. 3.106), then one gets a more spacious basket, like the common carrying basket in southern Mozambique seen in fig. 3.107.

The symmetries of the base reflect themselves in the particular shape of the labor product almost independently of the will

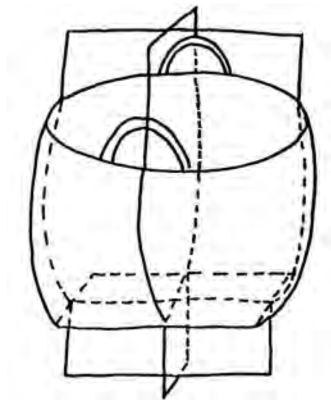


Fig. 3.108. Two perpendicular planes of symmetry

of the artisan—almost independently, as the artisan still has the liberty to choose the number of auxiliary strands at the four corners. Once given the preference for an equal number of strands at all four corners, then the two line symmetries of the rectangular base generate two *bilateral symmetries in space* (fig. 3.108).

In this basketry labor, several symmetry concepts are further developed: a certain connection between *equality of quantities* of strands at the corners and "form equality" or symmetry properties of the produced basket.

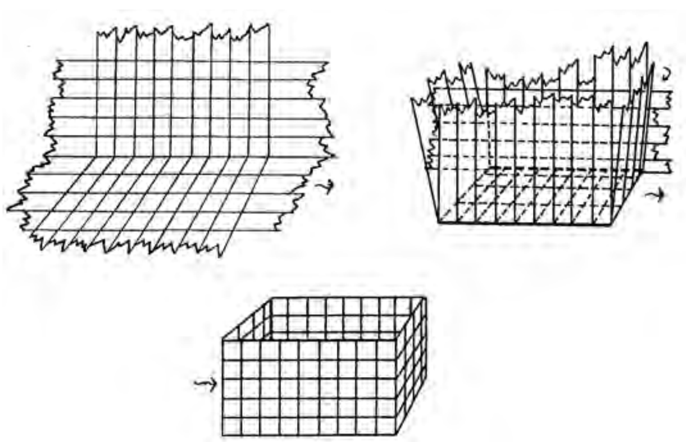


Fig. 3.109. Folding a rectangular part

On the other hand, the question arises whether it is possible, with this same technique of interlacing at right angles, to weave baskets with a flat bottom at once without auxiliary strands? Obviously, such a possibility arises not just with the protruding vertical pieces of strand (fig. 3.104), but when more parts of the already plaited base are bent in the same direction. But which parts?

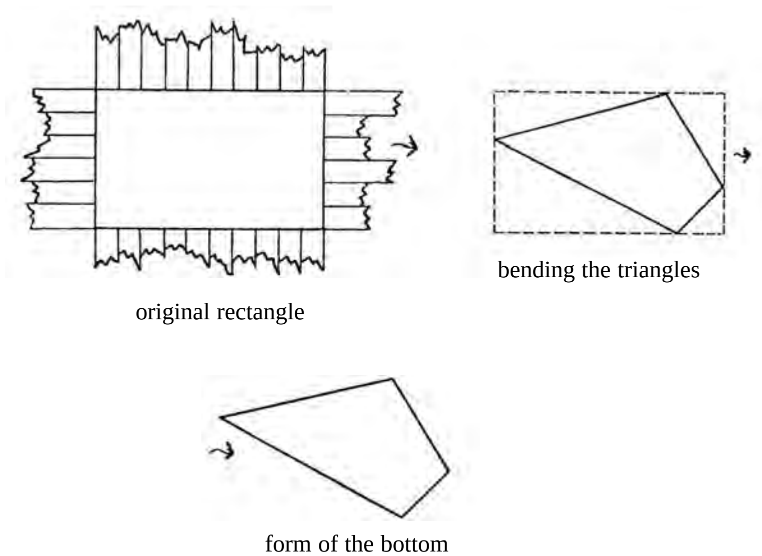


Fig. 3.110. Bending the triangles to form the bottom

When one folds a rectangular part, then once again it becomes possible to make a block-shaped basket (fig. 3.109). One can, however, fold a triangle upward, rather than a rectangle. This has to be done simultaneously with four right triangles at the corners, with adjacent hypotenuses (fig. 3.110); otherwise one would obtain a basket with rather big holes.

In principle, it proves to be possible to make baskets with such an irregular base as in fig. 3.110, even if one demands that the wall should be interlaced at right angles. Initially, when the walls are woven, the basket becomes still more irregular than the bottom. Where the interior angles are acute, the basket leans outward; where they are obtuse, the basket leans inward. Moreover, the more obtuse the interior angle, the more difficult it is to weave the corresponding part of the wall. This experience may lead our hypothetical artisan to search for equal interior angles, as acute as possible. The artisan then finds that the angles should be right angles, that is, the inscribed quadrilateral must be a *rectangle*.

As was already noted, the basket would initially tend to become even more irregular than its bottom. But with further weaving upward, its shape becomes ever more regular; it becomes *cylindrical* (fig. 3.111). To guarantee an upright basket, one may now try to choose a more balanced, more regular form for the base—at least a rectangle. In this way, once more, the artisan arrives at the conclusion that the inscribed quadrilateral has to be a rectangle.

When one starts with an arbitrary rectangle, it is generally not easy to inscribe in it another rectangle. This stimulates the

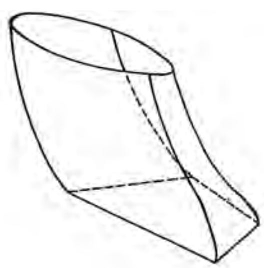


Fig. 3.111. Basket with a nonrectangular bottom

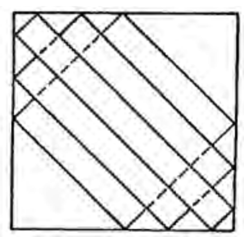


Fig. 3.112. Rectangles inscribed into a square

artisan to choose the initial rectangle as balanced as possible—a *square*. Now many inscribed rectangles turn out to be possible, with sides making angles of 45° with the initial square (fig. 3.112).[18]

Our hypothetical artisan could have arrived at the same conclusion also in another way. Perhaps the artisan was already acquainted with angles of 45° in another context—for example, the three-strand braid. Or the artisan, skipping almost all the phases of experimentation that I sketched, chooses a square as

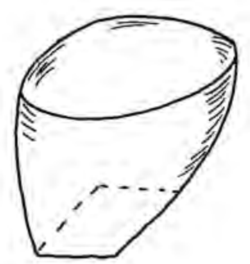


Fig. 3.113. Cylindrical basket with rectangular bottom

Fig. 3.114. Cylindrical basket with hexagonal base

the initial rectangle, on the basis of personal or societally transmitted experience and related formation of aesthetic-utilitarian values, and doubles up such triangles so that the bottom of the basket to be produced becomes a rectangle or even a square (fig. 3.113).

Consider cylindrical baskets with axes perpendicular to the bottom. When the basket is woven upward, the rectangular or square form of the base is, independently of the will of the artisan, naturally suppressed; the most homogeneous closed curve, the equipotential curve—that is, the *circle*—carries itself through cylindrically. In regions of the world situated far from each other, such cylindrical baskets with square bases may be encountered. They are found, for example, among the Carajá, Timbira, and Guajajara Indians of eastern Brazil;[19] among the North American Cherokee and Chitimacha Indians (Mervin 1919); on Borneo and Lombok (Indonesia); in the Philippines and in Laos (see photos in LaPlantz 1993, 24, 47, 65); and among the Makonde of northern Mozambique (see also section 8 in this chapter).

A similar process can be observed in the case of interlacing in three directions at angles of 60°. To make the upward weaving easier, the artisan has to choose "automatically" a regular hexagon as the base. This time, the circle supersedes the hexagonal bottom. One obtains a cylindrical basket with hexagonal base (fig. 3.114), as is produced, for instance, in the Philippines. If one

Fig. 3.115. Composition of the cranium and bones of a cave bear

does not add horizontal strands for weaving the wall, then the basket will display rhombic holes on its wall.

Vessels often acquire the form of a cylinder, cone, or other symmetrical shape. At first sight, this may appear the result of instinctive impulses or of an innate feeling for these forms or—in another idealistic variant—as generated by a collective "cultural spirit" or as an imitation of natural phenomena. People, however, create these forms in their practical activity, to satisfy their daily needs. They e*labor*ate them; they *work* them out. The understanding of these forms develops further through the discovery, reproduction, and social transmission of the methods for the fabrication of baskets and other objects. This understanding grows in the struggle with the material being used to produce something useful: bows, boats, hand axes, baskets, pots. This growing comprehension already won a certain independence very early in history. In fig. 3.115 one sees an artistic composition of a cave bear. This Neanderthal construction, found in the Drachenloch cave in Switzerland, demanded "from its creator, independently of its simplicity, rather precise images of order and symmetry," as Panow underlines (1985, 69). Simultaneously, this composition shows that its creator had already learned to

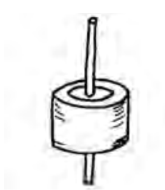


Fig. 3.116. *Ndona* lip ornament

recognize symmetries in nature. The continual recourse to the old and new practical advantages of symmetrical forms contributed to their aesthetic appreciation and to their application even where they were not necessary or immediately useful, as, for example, in the case of the cylindrical *ndona* lip ornament among the Makonde (fig. 3.116).

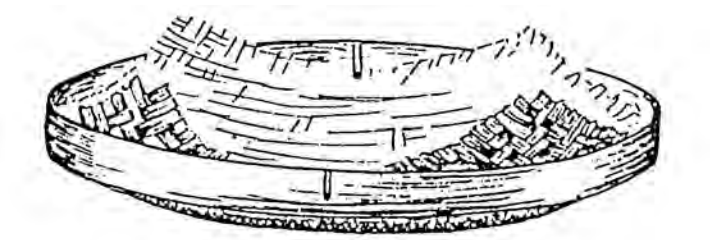


Fig. 3.117. Fastening the edges of the mat at their midpoints

8. The origin of some plaiting patterns and a unit for the measurement of volume

Already in section 5 we met a sort of basket bowl called *ungo* among the Swahili or *chelo* among the Makonde that is widespread, at least in eastern and southern Africa (Stuhlman 1910, 4). It is used as a sieve or as a dish for food. Corn is shaken onto it to scatter the chaff before the wind. How do the artisans make such a useful basket bowl?

The weaver starts by plaiting a square mat. To produce the border of the basket, a wooden board is bent and its two ends are bound one to another. Now the artisan fastens the sides of the mat at their respective midpoints to the border (fig. 3.117). Then the weaver wets the mat to make it more flexible, and with a foot presses the mat uniformly inward (fig. 3.118). To finish, the weaver cuts off the protruding parts of the mat and fastens the rest of the bottom to the border as, for example, in fig. 3.119.

The border of the *chelo* basket is necessarily circular as a consequence of the homogeneity of the bent material. The perpendicularly woven mat has to be a square. Experience shows

Fig. 3.118. Pressing the mat uniformly inward

that if the mat were not square, it would be more difficult to fasten it to the border, and the basket would fall over rather easily. To be able to round the mat, it is necessary that the basket weaver fasten it at four sides and not at two or three. Here, experience plays an important role: the sides should not be fastened to the border at arbitrary places, but exactly at their midpoints. Were this not to be done, then practice shows that it becomes more difficult to round the mat on the shortest end; the distortion of the initial right angles between the plant strands becomes greater.

How can the weaver guarantee that the mat really acquires a square form? How does the weaver determine the midpoints of the sides?

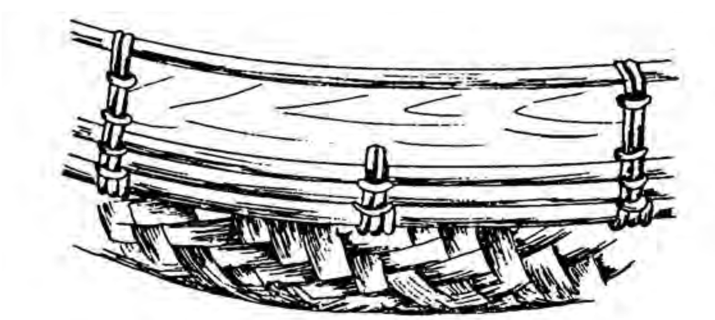


Fig. 3.119. Part of the bottom fastened to the border

The artisan may use the palms of his or her hands to compare the lengths of the sides, or count the number of strands in each direction and compare the numbers. Another type of solution seems to be more common: the midpoints of the sides are not determined after weaving the mat, but before. The artisan from the outset makes visible in one or another manner the line segments that will link the future midpoints of the opposite sides of the square mat (fig. 3.120). Two basic methods are used, sometimes simultaneously, as in fig. 3.121.

Fig. 3.120. Line segments that link the midpoints of the opposite sides of the square mat

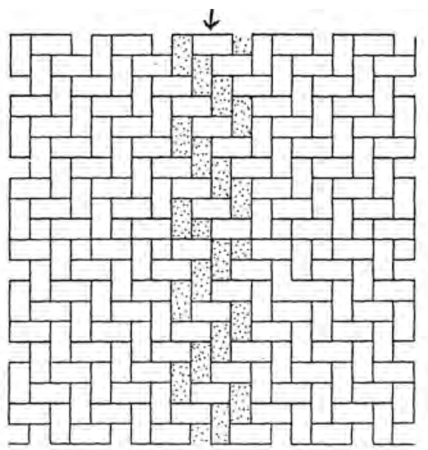


Fig. 3.121. Center of a square mat

In one method, a middle line is characterized by means of change in the coloring, whereas in the other method, the manner of plaiting is modified. Plaiting out from the center in this way, the artisan can observe more easily whether or not the mat is really becoming square—the middle line helps the eye.

Another possibility consists in a systematic step-by-step "plaiting around the center," as fig. 3.122 illustrates. If, on the contrary, the artisan interlaces two different colors permanently in orthogonal directions, then the colored plaiting pattern can give so much support that rigorous counting or measuring becomes completely superfluous (see the examples in fig. 3.123).

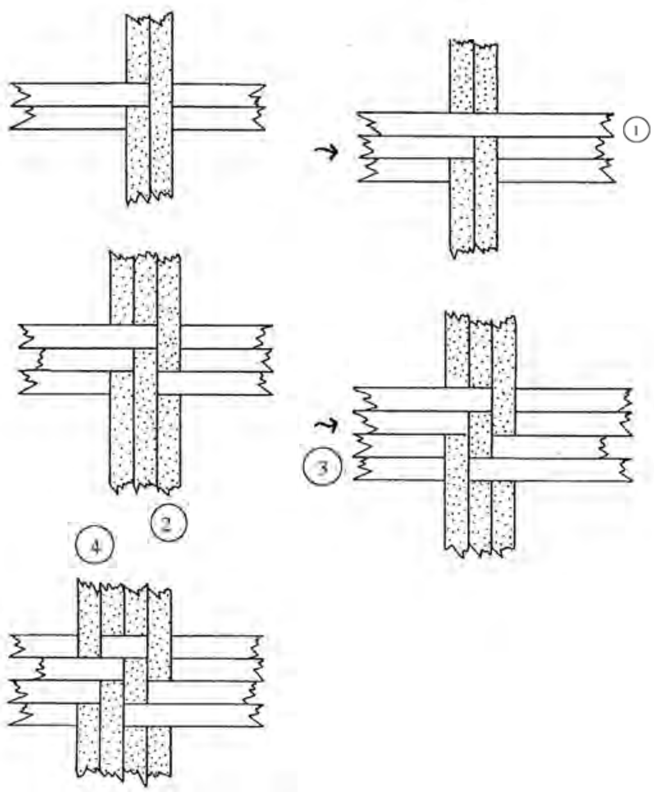


Fig. 3.122. Systematic plaiting around the center

The pattern does not allow any doubt about the squareness of the mat. Moreover, the midpoints of its sides are immediately recognizable.

The efficiency of these colored plaiting patterns contributes to their being viewed as beautiful in the society in which they arose. They become an aim in themselves and lead to new experiments with colored plaiting patterns. The weaver discovers new variations of form. Figure 3.124 presents some centers of square mats, and fig. 3.125 shows some examples of basket bowls. Once the designs are discovered, the artisan can also apply them to other weaving techniques, where the production process does not require a search for new forms. The color

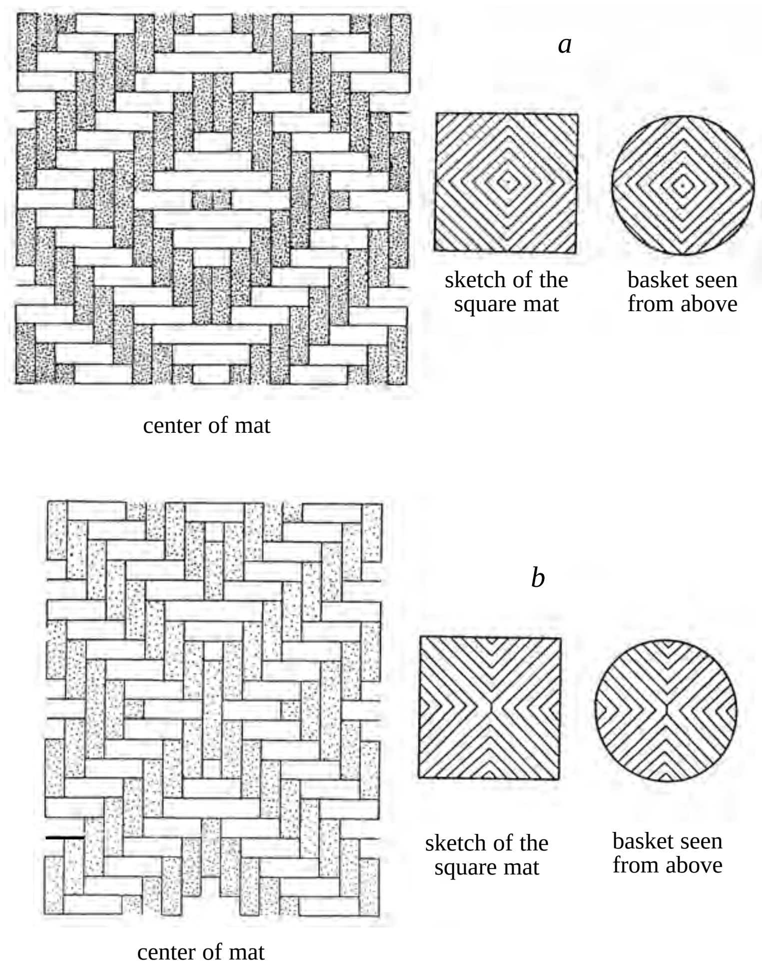


Fig. 3.123. Two examples of colored plaiting patterns

variations are emancipating. They become ornamental. Initially transferred from one type of basket to another, they may liberate themselves further from the original concrete context and finally find a new expression in other materials—for example, in the decoration of clay pots or wooden objects (see the examples in fig. 3.126).

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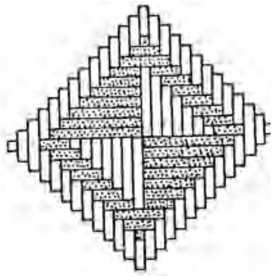
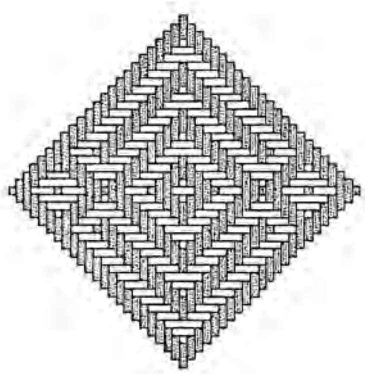
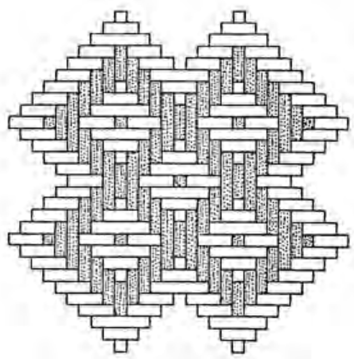
b) China



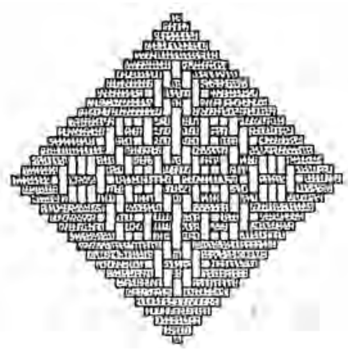
c) Angola



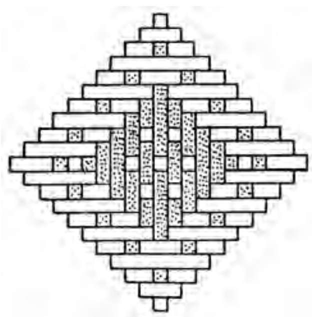
d) Kenya



e) Kenya



f) Halmahera (Indonesia)



g) Chitimacha (Louisiana, USA)

Fig. 3.124. Examples of centers of square mats

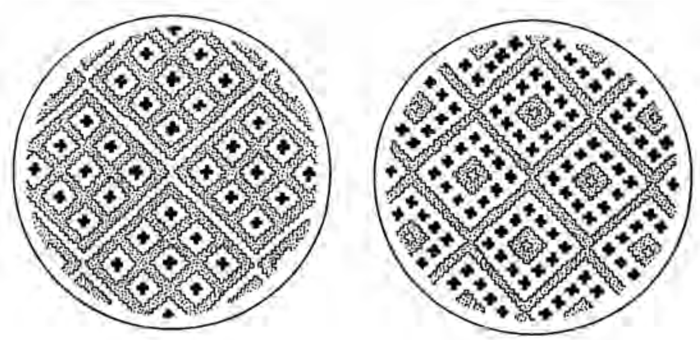
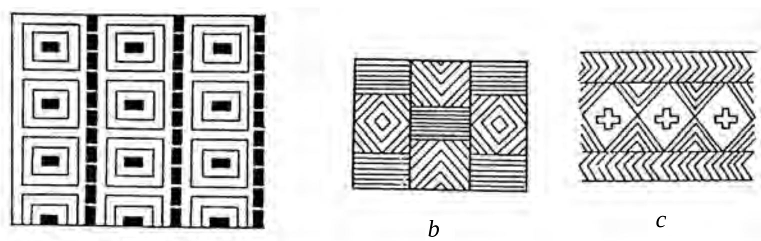


Fig. 3.125. Two circular trays from Guyana (seen from above)



a b c

Fig. 3.126. New expression in other materials. a) woodcarving (Angola), b) ceramic decoration (Cameroon), c) woodcarving (Cameroon)

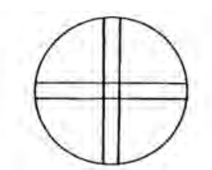


Fig. 3.127. Basic structure of a circular tray

When one observes a *chelo* basket from above, two perpendicular diameters are visible. The resulting figure (fig. 3.127) gives rise to many artistic activities (see the examples in fig. 3.128).

In section 7, we saw how after folding four congruent isosceles triangles upward, artisans weave a cylindrical basket with a square base. The Makonde of northern Mozambique produce

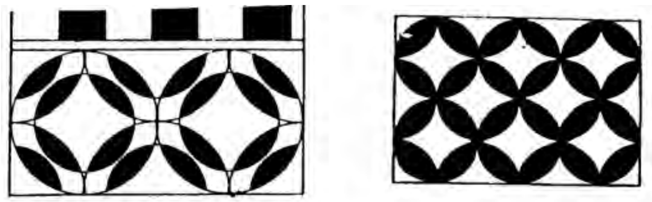


Fig. 3.128. Woodcarvings from Madagascar

their *likalala* basket in this way. How do the *likalala* weavers know that their mats are square and locate the midpoints of the sides of the square mats? In other words, they have to solve essentially the same problem that we encountered with the *chelo* basket bowl. The artisan first plaits both line segments that have to link the midpoints of the opposite sides of the square mat, that

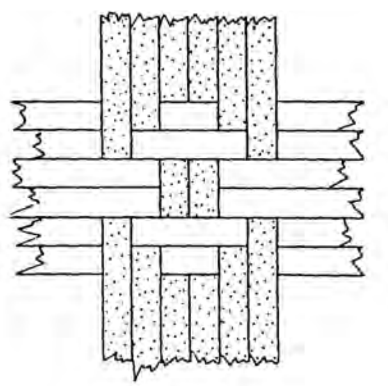


Fig. 3.129. Center of the bottom of a *likalala* basket

is, the diagonals of the future square base of the basket. The following procedure is used (Dias and Dias 1964, 139). With two groups of six strands of the same width, the weaver fabricates the center of the mat (fig. 3.129). The center displays a double bilateral symmetry. The center is called *yuyumunu,* that is, the mother who generates the mat with indication of the desired elements. After the center, the middle lines are woven by partial repetition of the "center-mother": each time four strands are interlaced (fig. 3.130, where four groups have been interlaced around

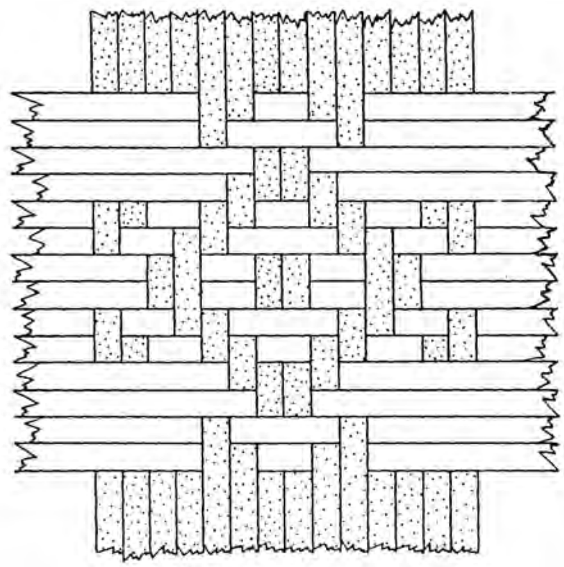


Fig. 3.130. Second phase

the center—one on the left, one on the right, and one above and one under the center). If the number of these repetitions in each of the four directions is equal, the mat, after the weaving is finished, will be square.

Once the middle lines are woven, the mat is completed quadrant by quadrant (figs. 3.131 and 3.132).

The direct counting of the strands has been avoided here. The artisan only counts the number of repetitions: the strands are thus grouped in sets of four. Perhaps these or similar experiences have contributed in earlier phases of cultural development to counting with 3, 4, 5, or 6 as base.

This way of producing mats starting with the “mother” enables standardization of the *likalala* baskets. As soon as the corresponding societal need arises, this possibility can be realized, as indeed happened among the Makonde. For example, it became necessary to know the yield of corn, sorghum, beans, etc., in order to decide what part could be used for exchange. The yield is measured in *likalala*-units by pouring the corn or

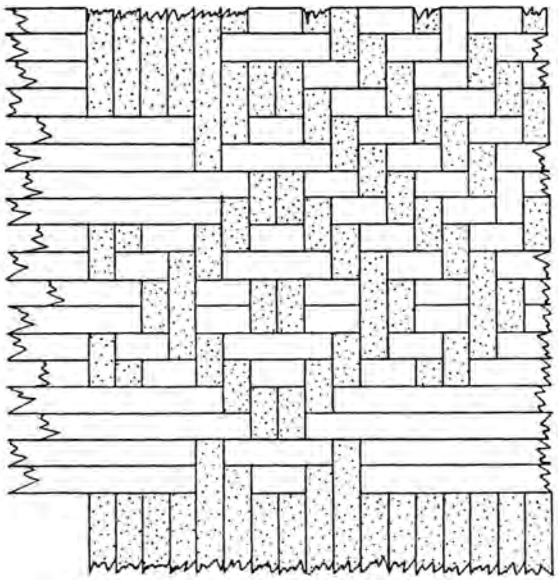


Fig. 3.131. Completion of the first quadrant

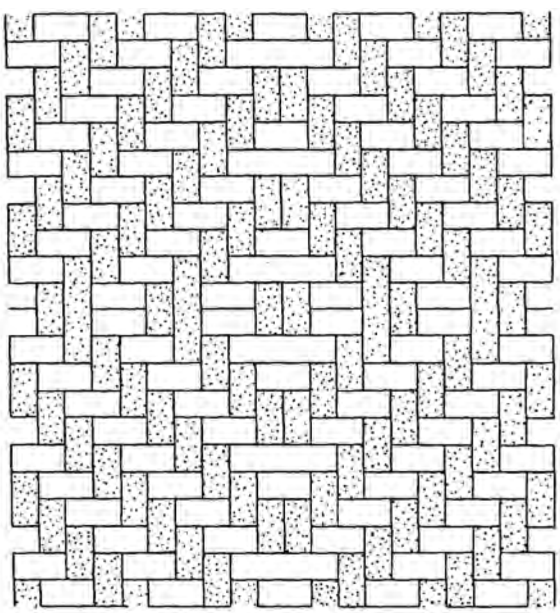


Fig. 3.132. Completion of the four quadrants

sorghum into a grain silo. In the case of the *likalala* basket, the basic pattern is repeated five times; the strands have a width of approximately 16 mm. Once the base and the height of the basket are standardized, one obtains the *likalala*-unit for the measurement of volumes (approximately 50 liters). A smaller unit for the measurement of volumes is the *ipichi* basket, which has the same shape (approximately 13 liters), where the basic pattern is repeated only four times; the strands are also smaller, approximately 9 mm (Guerreiro 1966, 17).

The plaiting patterns that appear in the making of the *chelo* and *likalala* baskets can be used and elaborated not only in art, but also in geometry, as suggested, for example, by some drawings in cuneiform texts from ancient Mesopotamia (fig. 3.133; see Neugebauer 1935, 139, 140).

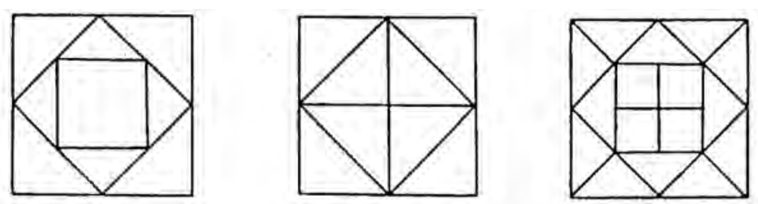


Fig. 3.133. Some drawings in cuneiform text from ancient Mesopotamia

9. How can one determine the rectangular base of a building?

The hypothesis presented by Moritz Cantor, according to which land surveyors and "cord-stretchers" in ancient Egypt laid out the right angles of the bases of their temples and pyramids with the help of a rope with 3 + 4 + 5 = 12 knots (1922, 105–6) is copied in ninety percent of the books as if it were a "universally known fact" (van der Waerden 1954, 6). No evidence exists, however, that mathematicians in ancient Egypt were aware of this possibility, which is based upon the relationship that we designate today as the *Theorem of Pythagoras* (Gillings 1982, 238, 242). The determination of the base of a temple in a solemn ceremony is described by Kurt Vogel: "The orientation was realized by means of determining the direction of the north with the help of the Great Bear. . . . *Nothing was said about* how the second basic direction perpendicular to the meridian was found" (1959, 59, 60). Why was this not explained? Perhaps it was already such a well-known construction that it did not seem worthwhile describing. What type of construction could have been so well known at that time?

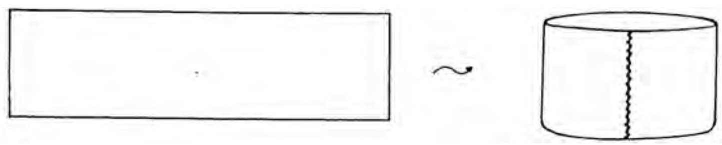


Fig. 3.134. In several zones of Mozambique, a rectangle of bark is cut off a tree to make a cylindrical vessel to store food.

Examples of early knowledge about rectangles

With the emergence of agriculture began a new phase in the construction of houses. The sedentary mode of life made it possible to construct bigger and more durable houses. In Hacilar in western Anatolia and Çatal Hüyük in central Anatolia (now Turkey), houses with assorted rectangular rooms were already being built in the eighth millennium B.C. (Herrmann 1984, 1:173, 335). The rectangular form itself—for example, rectangular mats—and its use (e.g., fig. 3.134) probably reach back to much earlier times (see section 1). What knowledge about rectangles could have been acquired from this experience?

From the production of rectangular mats, it almost immediately follows that their opposite sides have equal length; on the one hand, all the reeds are of equal length and, on the other hand, their number does not change when one goes from one side to the other (fig. 3.135). When one folds a mat at its middle, the resulting mat has half the length; when one turns the mat over by rotating it about one of its sides—length or width—it still occupies the same place. In other words, the rectangle has two axes of symmetry (fig. 3.136). Whether other conclusions are drawn from the symmetry of the rectangle—for example, that its

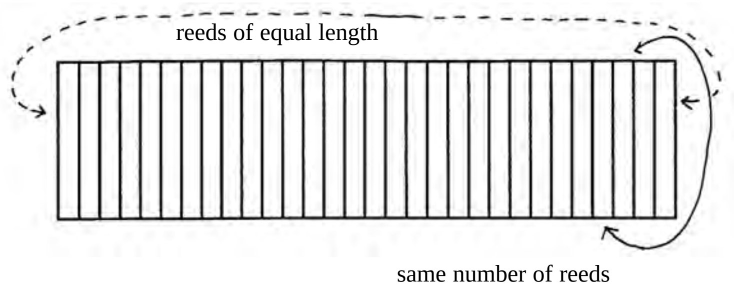


Fig. 3.135. Reeds of equal length.

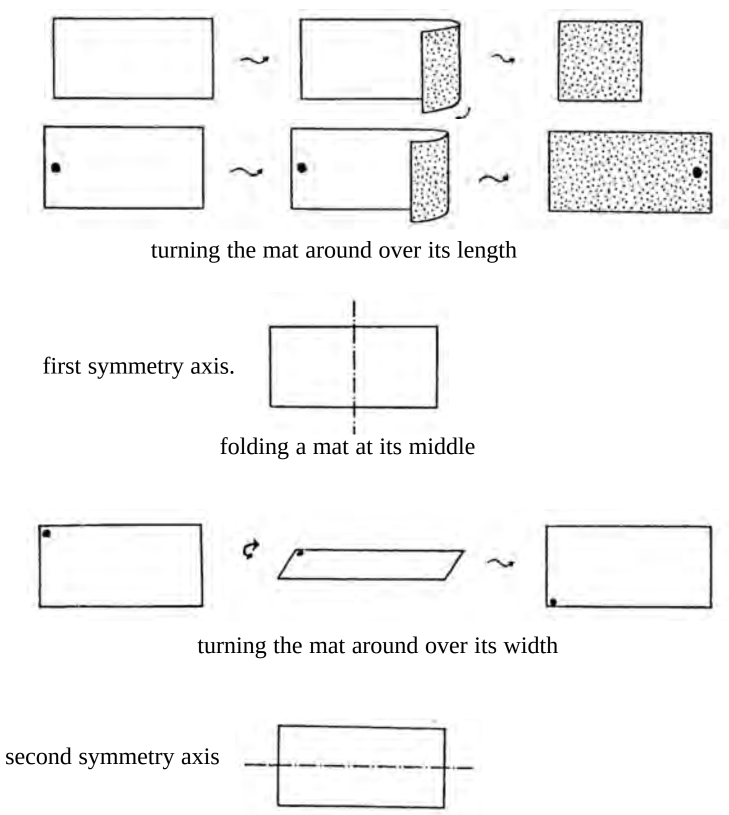


Fig. 3.136. The symmetry axes of a rectangular mat

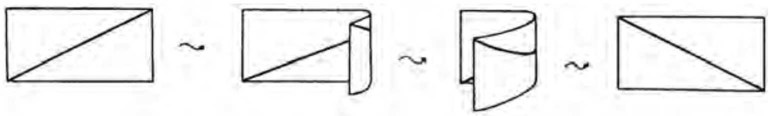


Fig. 3.137. Discovering that the diagonals have equal length

diagonals have the same length (fig. 3.137)—will probably depend on the experiences with symmetry in the particular cultures.

When mats of the same size are put next to one another—for example, when eating, working, or sleeping, as in fig. 3.138—bigger rectangles are obtained. This already gives the possibility of determining the rectangular base of a building.

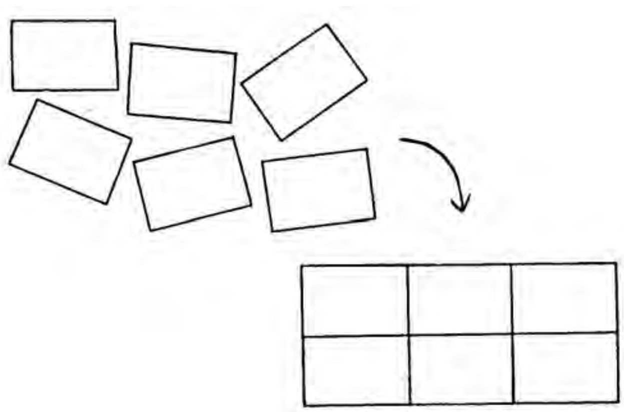


Fig. 3.138. Joining mats of the same size

One often stores ropes or threads by wrapping them up around some crossed sticks (fig. 3.139). In this way knots are avoided. This experience clears the way for an interesting toying with threads. When one coils the thread around two crossed sticks and, upon arriving at a stick, straightens it between the two sticks, the thread—unanticipatedly and independently of human will—forms a *rectangular* "thread cross" (fig. 3.140). Looking at the result, the person doing this concludes that the four stick radii, after having been broken off at the corners of the rectangle, have

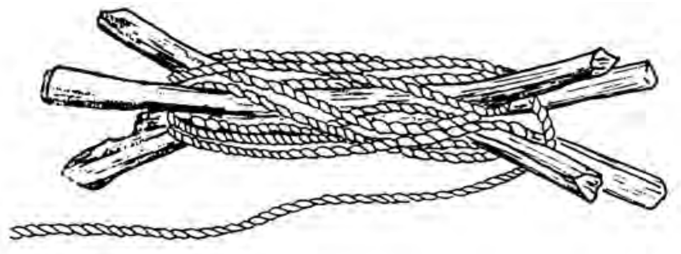


Fig. 3.139. Coiled rope

the same length and that the threads run around them the same number of times.

It can be discovered not only that the diagonals of a rectangle are equal, but also that they intersect at their midpoints and that the rectangle has a center of rotation (fig. 3.141). And if, besides this, one has already appropriated the concept of a circle

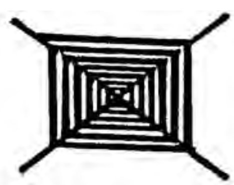


Fig. 3.140. Rectangular thread cross

as a set of points (in the same plane) equidistant from a center, then it can be discovered that the vertices of a rectangle are situated on a circle whose center coincides with the center of the rectangle, leading to the fundamental figure of Thales of Miletus (fig. 3.142).

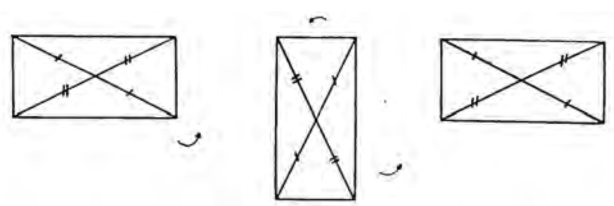


Fig. 3.141. Center of rotation of a rectangle

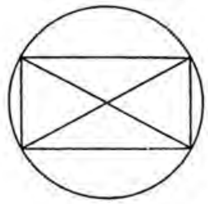


Fig. 3.142. Thales' fundamental figure

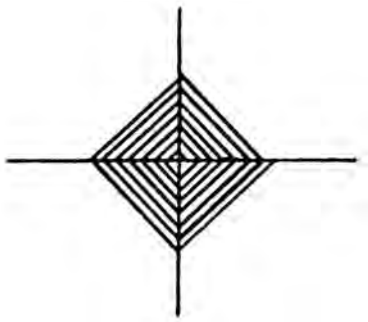


Fig. 3.143. Square thread cross

If the two sticks were initially crossed perpendicularly, a square will result automatically and independently of the will (fig. 3.143), like, for example, the thread crosses among the Guajajara Indians in northeastern Brazil (Neumann and Kästner 1983, 36),[21] among the Marind-anim in Papua New Guinea (Wirz 1978, 82), and among the Garadjeri in northwestern Australia.[22] The Guajajara thread square served as a toy and represented the wings of a bat. These wings could never have been the model for the thread square; on the contrary, only the created thread cross could have enabled human beings to compare and interpret it as the wings of a bat. Not only were interpretations enabled, but also new knowledge could be acquired:

a) If the diagonals of a rectangle are perpendicular to each other, then the rectangle will be a square, or, inversely, the diagonals of a square intersect each other perpendicularly;
b) the diagonals of a square are of equal length and intersect in their midpoints;

c) the vertices of a square lie on a circle whose center coincides with the point of intersection of the diagonals.

The last conclusion could have been drawn by the Guajajara Indians and other populations also on the basis of their cylindrical baskets with square bases.

We already met another relationship between rectangles and cylinders: a rectangle of bark was sewn into a cylindrical vessel. New relationships may be discovered in the making of baskets and handbags. The Tsonga-speaking population in the south of Mozambique fabricate their *huama* or *funeco* baskets as follows. Two strands of palm leaf are knotted together, *pentagonally,* as illustrated in fig. 3.144.[23]

Let us observe this pentagonal knot more closely. The strands protrude on three sides; on two sides the knot is closed.

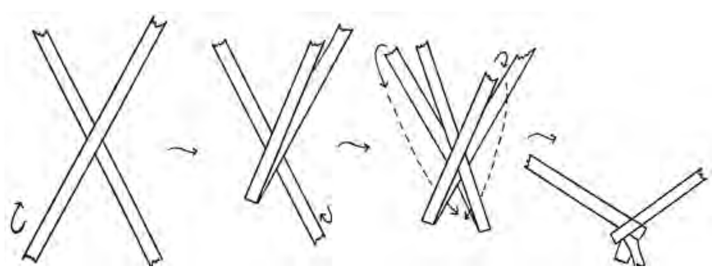


Fig. 3.144. Knotting together two strands

When flattened (for example, when made from paper), it takes on a semiregular shape with angles of 108°, 90°, 90°, 126°, and 126° (fig. 3.145a). In it is hidden, however, the regular pentagon and pentagram. If one interlaces such a knot with two relatively wide and thin strands and holds them up in front of the light, a regular pentagon and an almost complete pentagram appear at the same time (figs. 3.145b, 3.146) Does an alternative birthplace for the pentagram lie here?

To make *huama* and *funeco* baskets, one joins not two strands, but two pairs of strands (fig. 3.145b).[24] The result is shown in fig. 3.146: a pentagonal knot made of four strands.

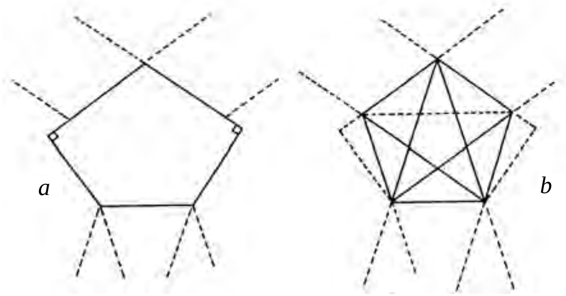


Fig. 3.145. Structure of the pentagonal knot

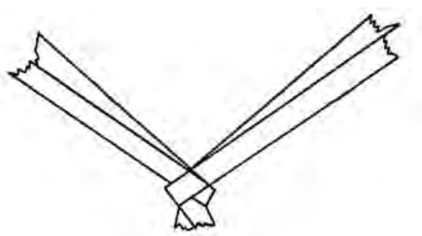


Fig. 3.146. Knotting together four strands

Now the artisan plaits these groups of four strands almost perpendicularly and in accordance with the "over-two-under-two" weave. A mat is obtained (fig. 3.147), but the weaver does not continue indefinitely. The border of the mat is bent into a circle; the first and the last knot become neighbors. Thereafter the basket maker continues to weave normally. In this way, one obtains

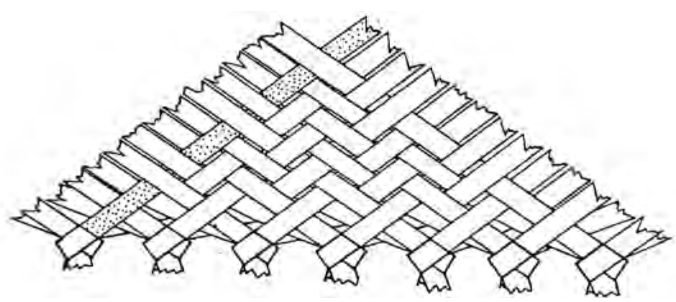


Fig. 3.147. Flat mat with border of pentagonal knots

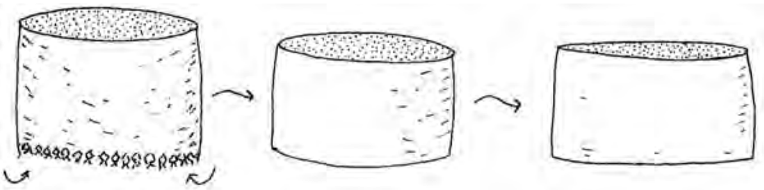


Fig. 3.148. Folding the knots inward, compressing and sewing the two sides at the bottom

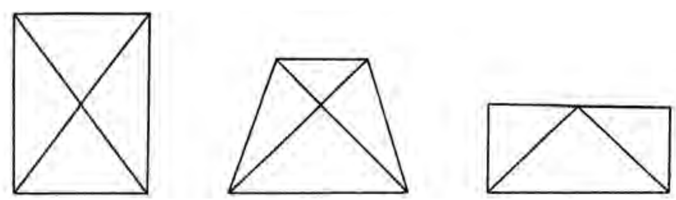
almost automatically a cylindrical wall. When the wall is sufficiently high, the series of knots at the underside is folded inward; one compresses the cylinder and sews together the two halves of the series of knots (fig. 3.148). The cylindrical form disappears and a flat rectangular handbag emerges.

Fig. 3.149. Conical wall and trapezium-shaped bag

Most probably, when this handbag form was discovered, there had been no intention beforehand to give it a rectangular form. The cylindrical shape that, in turn, replaced the plane mat was necessarily replaced by the rectangular shape of the final bag—it had to be closed at the underside. Use of strands that become gradually narrower, instead of strands with parallel edges, would have first resulted in a conical wall and, upon being closed at the underside, the bag would take on the shape of a trapezium (fig. 3.149). Possible ways of finishing this type of bag are shown in fig. 3.150, with two handles or with cover in the same form and a shoulder strap. As the figures already show, strands of different colorings are commonly used in such a way that diagonal bands become visible. Because of symmetry or because all strands have the same length, it can be shown that

Handbag opened shoulder bag closed shoulder bag

Fig. 3.150. Hand and shoulder bags



a *b* *c*

Fig. 3.151. Recognizing diagonals of equal length

both diagonals of each rectangle and of each isosceles trapezium have the same length (fig. 3.151).

Similarly, other knowledge can be acquired—for example, that the segments that link the vertices of two neighboring angles of a rectangle with the midpoint of the opposite side are congruent (fig. 3.151c). The interest in the rectangle with its diagonals is reflected in ornaments on ceramics not only in Mozambique, but also, for example, in the so-called "beaker-culture" at the end of the third millennium B.C. in west and southern Europe (van der Waerden 1983, 34).

Examples of the construction of rectangles

The interest in the rectangle and its diagonals is not only aesthetic, but, above all, also practical. If the opposite sides of a quadrilateral are equal and if its diagonals are also equal, then the quadrilateral is a rectangle. This "inverted" knowledge gave peasants of several regions of Mozambique and elsewhere, like the Kpelle of Liberia (Gay and Cole 1967, 61), an easy method for laying out the rectangular bases of their houses.

Vogel formulates the hypothesis that the ancient Egyptians might have used the carpenter's square (fig. 3.152) or might have realized the well-known construction based on the equilateral triangle to fix two perpendicular directions (1959, 60). The methods suggested by Vogel have the disadvantage that they have to

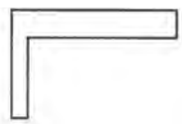


Fig. 3.152. Carpenter's square

be applied three times to construct a rectangle. The method of comparing diagonals of a parallelogram has the advantage of leading simultaneously to the four internal right angles. The symmetric skeletons of a table in the tomb of Tutankhamun (ca. 1346–1336 B.C.) allows me to suppose that the artisans of ancient Egypt certainly knew that the diagonals of a rectangle are of equal length (fig. 3.153). Perhaps they knew the method of comparing diagonals or a procedure like that used by the Kwakiutl Indians of Vancouver Island. The Kwakiutl determined the square base of their houses in the following way. From point *A* that was to become the midpoint of the front side of the house,

Fig. 3.153. Table skeletons from the tomb of Tutankhamum

they stretched a rope to the midpoint *B* of the rear side (fig. 3.154). Then they divided the rope into two halves and stretched it from the first point *A*, the first half of the rope being stretched leftward (endpoint *C*) and the other half rightward (endpoint *D*). Then, with the help of a second rope they compared the distance between C and *B* with the distance between *D* and *B*, and, if necessary, they adjusted the positions of *C* and *D* until *BC* and *BD* were equal. Once arrived at this equality, they had found the vertices of the front side. In the same way they found the vertices of the rear side (Struik 1948, 48; Seidenberg 1962b, 521–22).

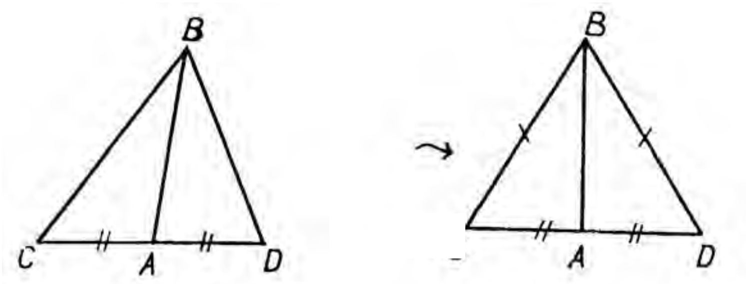


Fig. 3.154. Kwakiutl procedure

With a slight modification, the Kwakiutl method can also be applied to the construction of rectangles that are not square.[23] The Kwakiutl method and the method of comparing diagonals can achieve a high precision, in particular when the dimensions of the sides of the base are relatively big, as they were, for example, in the case of the pyramids of Gizeh in ancient Egypt. These methods present, however, a disadvantage that becomes increasingly problematic as the distances become larger: one has to walk quite a distance back and forth with the ropes to compare the distances. To avoid this inconvenience, one might work simultaneously with two ropes instead of only one. But in these circumstances, the awareness of the need to make comparison in another way arises: *measurement*! Once a measurement is necessary, there might then emerge another reason to compare the lengths of the sides and diagonals of rectangles. If this was the case, then it becomes more understandable why it is that

Pythagorean triplets appeared for the first time in an ancient Mesopotamian text, “Plimpton 322,” written during the dynasty of Hamurabi, about 1800 B.C. These triplets refer to the length, width, and diagonals of rectangles (van der Waerden 1983, 2).

It is also possible, nevertheless, to construct rectangles of large dimensions without making measurements. The experience of the Guajajara Indians with their thread square could have led to the following determination of a rectangle: when one stretches simultaneously two ropes of equal length that are attached to one another at their midpoints, the endpoints then constitute the vertices of a rectangle (fig. 3.155). In some zones of Mozambique this is the way the peasants lay out the rectangular base of their houses.

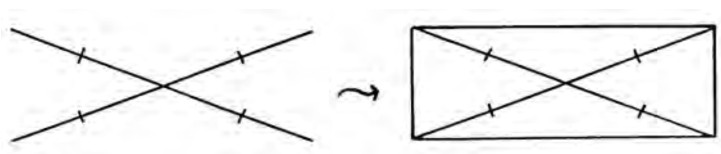


Fig. 3.155. Rectangle construction

NOTES

1. See Leroi-Gourhan 1983, 83, on the importance of the choice of the hew direction.

2. See photos in Grottanelli 1969, 3:227, 231, 236–37.

3. See photo 1 in UNESCO 1983, 15. See also, for example, Weule 1970, 196 (East Africa). Softwood on exhibit in the Egyptian Museum (Cairo) shows that also in ancient Egypt the fire drill was rotated at a right angle to the softwood.

4. Hauser writes: “The right angle is already therefore one of the oldest geometrical concepts, as it emerges out of the vertical position of the human being when standing.” Would humans not have become aware of the perpendicular character of this vertical position in relation to the ground after they have already elaborated through their activities an image of “perpendicular to one another” (1955, 11)?

5. For example, to weave a Hawaiian straight-edged headband, the strands are woven at an angle of 60° (see Bird et al. 1982, 59–69).

6. See, e.g., Guss 1989, 73; Roth 1970, 320-43; Neumann and Kästner 1983, 8, 43, 93; Mason 1904, 488, plate 240, for other examples from Brazilian Indians.

7. See photo in Grottanelli 1965, 45 (cf. also Mason 1904, 275). Baskets plaited in open hexagonal weave also appear among North American Indians—for example among the Delaware and among the Mashpee in the northeast (see photos in Turnbaugh and Turnbaugh 1986, 17, 19).

8. See, e.g., Ranjan et al. 1986, Dunsmore 1983, and Lane 1986.

9. See photos in Bodrogi 1978, 17, and Icke-Schwalbe 1983, 82. Other examples may be found, for example, in Roth 1970, 1:362; and Faublée 1946, 19, 28, 38.

10. See the description of the making of hexagonal baskets in Guyana in Roth 1970.

11. A frequent situation in the building of houses. Cf. the figures in Denyer 1982, 100.

12. For a reconstruction of the sand-drawing tradition in Africa south of the equator, see Gerdes 1994, 1995, 1997. Deacon (1934) described comparable drawings from Malekula (Oceania); cf. also Ascher (1988a; 1988b; 1991, chap. 2).

13. This “double-S” knot was also the distinctive start of basketry among the Pima and Papago Indians in Arizona (Whiteford 1988, 121).

14. For example, on the tablets from Susa (Bruins and Rutten 1961; Neugebauer 1970, 47).

15. Further examples of fivefold symmetry and (basket) weaving in various cultures are given in Gerdes 1992.

16. The woven *sepak raga* ball with its twelve pentagonal holes (Dunsmore 1983; Gerdes 1992) would be another possibility.

17. In his history of geometry, Mainzer remarks in this respect: “Regular pottery products with cylindrical or spherical shape prove themselves to be more stable and saving in the use of material” (1980, 19). More stable, yes. But how could one in practice have discovered that these shapes saved material?

18. Nonsquare rectangles are occasionally applied, for example, by the Chitimacha Indians in Louisiana (Mason 1904, plate 133) and by the Arawak Indians in Guyana (Mason 1904, plate 239; Roth 1970, plates 116, 117).

19. Neumann and Kästner 1983, 16, 24, 54.

20 Another example is provided by several zones of Mozambique, where a rectangle of bark is cut off a tree to make a cylindrical vessel for storing food (see fig. 3.134 on p. 88).

21. See Albaum 1972 on the making of thread crosses among North American Indians.

22. Exhibited, for example, at the Berlin Ethnographic Museum.

23 See also the making of *sipatsi* handbags in Mozambique’s Inhambane Province (Gerdes and Bulafo 1994).

Chapter Four
Societal Activity in the Formation of Ancient Geometry

1. Did geometry have a ritual origin?

According to Abraham Seidenberg, the dominant tendency in the historiography of mathematics prior to the 1970s viewed classical Greece as the source of the geometric or constructive tradition, and ancient Babylonia as the source of the algebraic or computational tradition in the history of mathematics (1978, 316). Seidenberg criticizes this view: "What are regarded as the two main sources of Western mathematics, namely Pythagorean mathematics and Old-Babylonian mathematics, both flow from a still older source" (1978, 329). Where may this older common source be found? This common source is to be sought, in his opinion, either in the oldest Indian—that is, Vedic—mathematics, or in a still older mathematics very much like it (329).

The *Sulvasutras*—that is, "cord rules"—give procedures of a geometric character for the construction of altars of various shapes, using cords and bamboo sticks. The shape of the altar depended on the ritual. There were square altars, circular altars, and also falcon-shaped altars. The basic falcon-shaped altar had an area of 7½ square purushas. The same shape was also required in the construction of larger altars, for the same ritual purpose, with areas of 8½, 9½, etc., square purushas. In order to solve these construction problems, one actually used explicit knowledge of the Theorem of Pythagoras for finding a square equal in area to two given squares. From this, Seidenberg concludes in his paper "The Ritual Origin of Geometry" that "in the successive augmentations of the falcon shaped altar, not only is the Theorem of Pythagoras used, but we see, in all likelihood, the *motive of its invention*" (1962b, 492; emphasis added). On the same page of his article, however, Seidenberg contradicts himself when he recognizes that "the *observation* that in a right triangle the square on the hypotenuse is the sum of the squares on the legs would have an *immediate theological application*" (emphasis added). What now? Motive or application? How could the Vedic ritualists, questions Seidenberg, seize the Theorem of Pythagoras from Greece or Babylonia and make it a vital part of their solemn rituals? The reverse process, that is, the *secularization of a ritual practice*, would be much easier to understand (501). In other words, Seidenberg decides finally for "motive."

In section 3, I shall return to the discovery of the Theorem of Pythagoras and try to show that it is not necessary to look for the cause of its discovery in theological speculations. I would now like to pose another question. Even if Seidenberg were right, and the Theorem of Pythagoras and the classical problems of "squaring the circle" and of the "duplication of the cube" were born as a kind of *theological geometry* from a ritual praxis (Seidenberg 1962b, 520), would this prove at the same time that the birth of all geometry was in ritual?

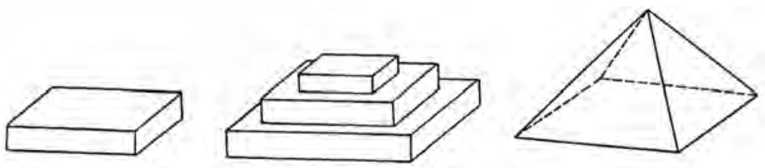
Maybe as an answer to this or to a similar question from one of the referees of the *Archive for the History of Exact Sciences,* Seidenberg closes his famous paper with a very short ethnographic section. In Madagascar, a ritual personage called *mpanandro* lays out the foundations of a house, square in shape; he finds the center as the intersection of the diagonals. The way of laying out the lines of a square house by the Kwakiutl of Vancouver Island (see chap. 3, sect. 9) is very reminiscent of the Sulvasutras. The Omaha Indians considered sacred the circular form of their houses. The Chavante Indians of Brazil have their houses in a perfect circle, and "as this shape could have *no useful purpose*,"—very doubtful!—"its origin in ritual is indicated" (1962b, 522; emphasis added). And then, without any further proof, Seidenberg concludes that "the circle and the square arise from ritual activities" (523). Here, however, Seidenberg became a victim of his own prejudice—my research has shown that many early geometrical forms are materially necessary. Or is matter, in Seidenberg's view, also of ritual origin?[1]

Seidenberg's attempts to demonstrate the ritual origin of geometry are not isolated. In an analogous way, he strives to prove the ritual origin of counting (1962a). He admits without any hesitation that his studies are "intended as a contribution to the general theory that civilization had a ritual origin" (1). This theory is closely related to the theory of diffusion of culture, according to which various widespread practices and beliefs may be explained on the basis of diffusion, in general from one unique center of birth. Thus, Seidenberg concludes in relation to counting that "counting was invented in a civilized center, in elaboration of the Creation ritual, as a means of calling participants in ritual onto the ritual scene, *once and only once*, and thence *diffused*" (37; emphasis added).

A footnote by the translator of the Russian edition of Struik's *Concise History of Mathematics* reads, "Readers may judge for themselves how improbable it is that counting among all peoples has a common origin, considering the isolation of prehistoric communities and the evident inequality of the development of counting among various peoples, and the fact that among one and the same people different words were used to designate the same quantity of objects of different sorts, etc." (Pogrybysskii 1978, 28). It therefore does not surprise us that from within ethnology, the prejudices and preconceptions of the theory of diffusion of culture have already been strongly criticized. For example, Marvin Harris, in his monumental work *The Rise of Anthropological Theory*, writes the following about diffusionism: "As soon as we admit, as the archaeology of the New World now compels, that independent invention has occurred on a massive scale, diffusion is by definition not only superfluous, but the very incarnation of antiscience" (1969, 378). Through the acceptance of a primacy of diffusion over independent invention, the representatives of diffusionism underestimate the creative capacities of humankind. Like other proponents of diffusion theory, Seidenberg overestimates the importance of diffusion for the development of geometry and exaggerates the possible connections between religious ideas and geometry. Having said this, we should not go to the other extreme. In fact, diffusion plays an important role in the development of mathematics.[2] The task is to analyze and understand historically the dialectics of diffusion and independent invention. Actually, magical or religious thinking may, in certain periods, reflect itself onto conceptions of number and of space. What matters is to try to explain the reasons for this phenomenon.

2. The possible formation of pyramid concepts

The greatest pyramid of Gizeh, 147 m in height and built in ancient Egypt under the reign of Cheops (ca. 2545–2520 B.C.), was considered, in antiquity, one of the seven wonders of the world. The pharaohs were honored at that time as gods. Was, therefore, the marvelous shape of their monumental tombs conceived in their godly fantasy and designed to remain incomprehensible to the people?



a) mastaba b) step pyramid c) pyramid

Fig. 4.1. Development from mastaba to pyramid

There were step pyramids (fig. 4.1b) on the Tonga and Tahiti islands in Polynesia, on the northwest coast of Peru, and also in the Mesoamerican cultures, where they constituted the bases for the temples. Just as in ancient Egypt, we are also dealing here with class societies. The Tahitian *marae* "represented a hierarchy just like the society" (Bellwood 1978, 82). Although the class character of such societies allows us to understand the differences in the dimensions of their pyramids, the source of the differences in shape is not immediately clear. Is the pyramidal shape of the temples of Central America perhaps an attempt to imitate the mighty volcanoes, as the cover of Wolf's book *Sons of the Shaking Earth* suggests (1969)?[3] But, if so, where does their regularity, such as the square base, come from? Could it have been an architectonic attempt to overcome chaos in nature, as Wertheim suggests, when he refers to the ceremonial center of Monte Albán in the Oaxaca Valley? "From its very beginning a departure from nature. The men who built it not only did not respect the lay of the land; they rejected it; they saw it as part of chaos on which man must impose order" (cited in Wolf 1969, 97).

As I have shown through various examples, thinking in terms of order does not fall from heaven, but reflects societal experience with production. Once this experience had been established to the extent that regularity took on an aesthetic value—and this was certainly the case in the "pyramid societies"—then new and, in a certain sense, ordered forms and shapes could have been created without the existence of an immediate, inescapable material obligation. In regard to the pyramids of ancient Egypt,

> their shape and basic idea must have come from the tomb made out of heaped up sand, of which the monumentalized form, the *mastaba* [fig. 4.1a], constituted the constructive starting point for configuration of the step pyramid of the third dynasty (Djoser complex, Saqqara, ca. 2600 B.C.). Under the sovereigns of the fourth dynasty (ca. 2570–2450 B.C.), the pyramids received their classical shape with flat faces and a square base. This was the result of a secular process in the history of building. (Herrmann 1984, 2:182)[4]

When I said, "without the existence of an immediate, inescapable material obligation," I meant by this that in this concrete case, on the one hand, other shapes were possible, such as the shape of a cone, and, on the other hand, that the choice of form was limited. It is easier to construct a step pyramid than a tower of the same height. Still, one may consider whether this involves a totally new figure, invented for the first time, or could an already known form have served as a model. Let us now consider two situations that might have already occurred at an earlier stage of cultural development than that of ancient Egypt

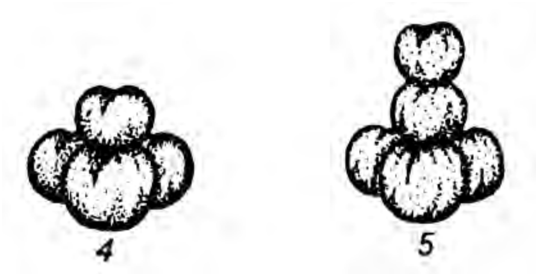


Fig. 4.2. Mandarin pyramids with three fruits at the base

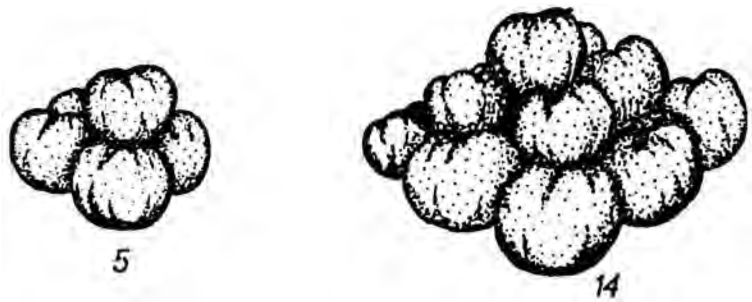


Fig. 4.3. Mandarin pyramids with four and nine fruits at the base

and that led to pyramid concepts.

Pyramids of fruits

Weight, volume, or number may be used as a measure when fruits are exchanged or sold. For example, in Mozambican markets, mandarins are sold frequently in groups of four or five.

To arrange the fruits so they can be examined, but not occupy too much space, they are piled up as in fig. 4.2. The same methods are also seen at other markets in Africa, Asia, and South America, sometimes with four fruits at the base (fig. 4.3). If one wishes, the fruits can be further piled up in groups of ten or twenty (fig. 4.4).[5] This experience may have contributed to the formation and development of a *tetrahedron* concept. It reflects

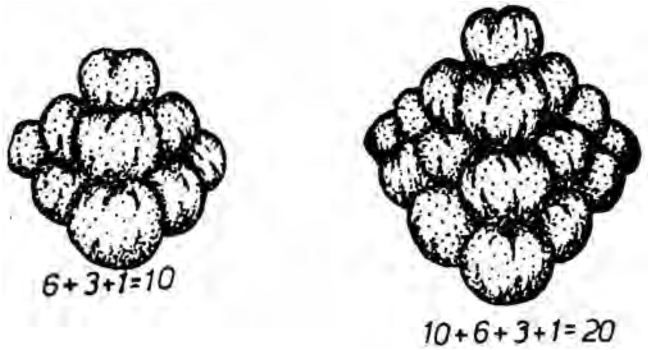


Fig. 4.4. Sets of 10 and 20 fruits

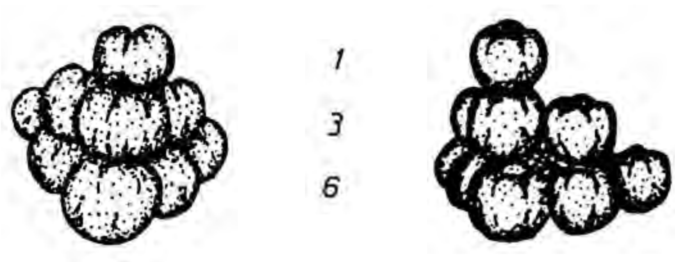


Fig. 4.5. Equal "heights," equal bases, and equal volumes

an optimal way of stacking round objects: each mandarin on a new layer rests only on the three mandarins of the layer beneath it, so that it will not roll down and a minimal base space is occupied. If one has sufficient space for the base, then it is possible to pile the objects up in a step pyramid with a square bottom, like, for example, flour sacks stacked up in West African harbors or oranges in pyramids of many layers in today's Egypt. We may ask whether Democritus (ca. 460–370 B.C.)—or other atomists before him—might have had such pyramids of fruits as an image model when he showed that two pyramids with equal base and equal height have the same volume. If, for example, one carefully piles up the ten mandarins of a "tetrahedron" in another way, with a "pyramid" of approximately the same height, both

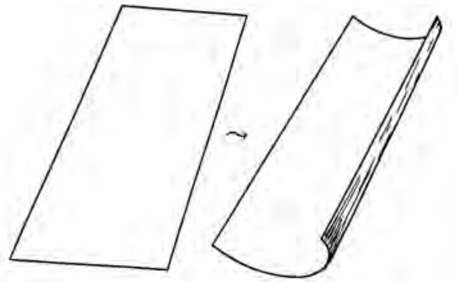


Fig. 4.6. Bending a mat in one direction

"pyramids" will have equal volumes, as an equal number of "atoms," that is, mandarins, lie in corresponding layers: 6, 3, and 1, etc. (fig. 4.5).

Woven pyramids

Let us suppose that one is able to use the widespread technique of plaiting mats with similar strands in two perpendicular directions. How can one, without any further material or auxil-

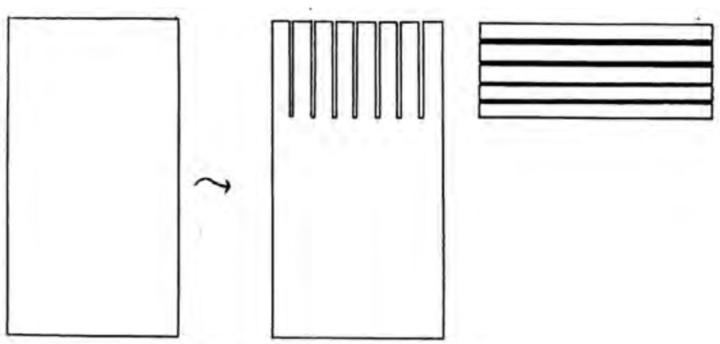


Fig. 4.7. Taking away some horizontal strands

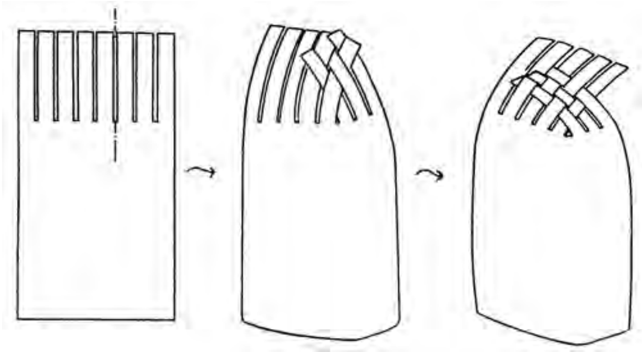


Fig. 4.8. Plaiting the protruding parts

iary means, make a bowl or a funnel out of such a mat? It is easy to bend or fold the mat in one direction (fig. 4.6). When one releases the bent mat, however, it returns to its initial plane shape. When one tries to bend the mat simultaneously in two directions, one feels how the material resists forcefully; there seems to be too much material to transform the plane mat into a bent object. The easiest way to reduce the quantity is to take away some parallel strands from one of its sides (fig. 4.7). And now something has to be done with the loose parts of the strands. But what? One group of loose parts can be plaited together with the other group of loose parts, as in fig. 4.8. To avoid gaps both groups necessarily, have to be woven perpendicularly (fig. 4.9). Actually, a

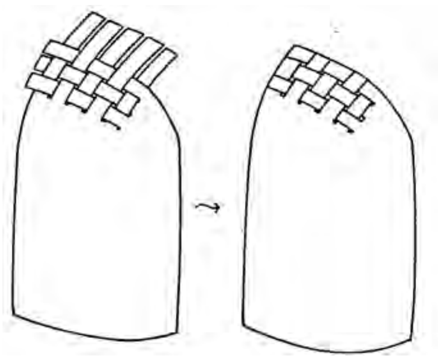


Fig. 4.9. To plait the protruding parts of strand perpendicularly

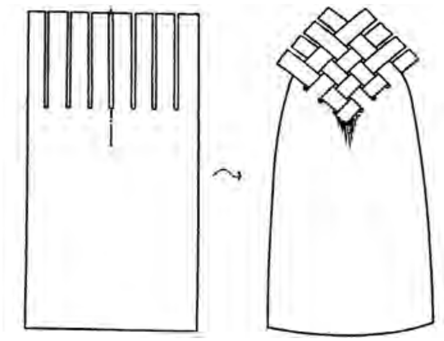


Fig. 4.10. The groups of strands on the left and the right have the same size

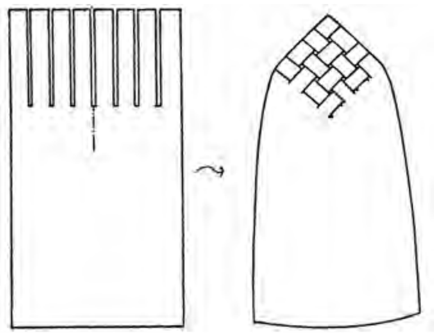


Fig. 4.11. Both groups of protruding strand parts are squares

"bowl" is obtained in this way. The production process can still be improved. To avoid an imbalance in the number of strands in each direction, after they have been woven together, one finds that both groups of strands should be of the same size (fig. 4.10). To avoid the necessity of cutting off the loose pieces of strand, the quantity of strands that were removed must correspond to half of the width of the mat (fig. 4.11), that is, both groups of initially loose strands are made square.

Thus far, the process of invention or discovery proceeded almost automatically, depending on the objective (weaving a "bowl" or "funnel" without any other auxiliary means), the material, and their reciprocal interaction. It turns out that the length of the mat can be freely chosen. But if one does not reflect

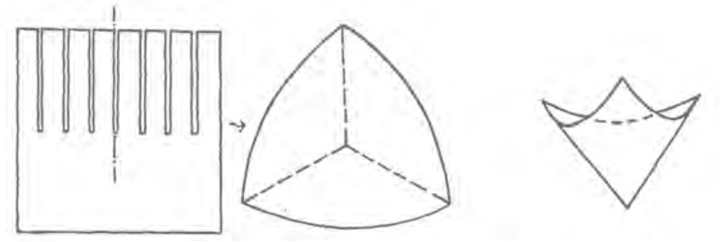


Fig. 4.12. Plaited basket “bowl” or “'funnel”'

consciously on this liberty, it is only relative; it is culturally embedded. Possibly, the artisan simply prefers the *square* as the form for the original rectangular mat, as this shape, in other contexts, had already shown itself to be advantageous, rational, or beautiful. An artisan preferring a “bowl” with rotational symme-

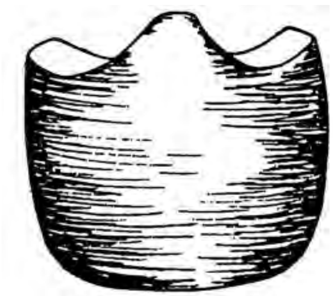


Fig. 4.13. New pot shape

try (fig. 4.12) discovers that the initial rectangular mat has to be a square. However this last step might have come about, the artisans from the Nyassa and Nampula provinces in northern Mozambique start with a square mat (see the scheme in fig. 4.12).

The result has been considered beautiful and has inspired ceramists in the same region to find new shapes for their pots (fig. 4.13). This was not, however, a simple transplantation of form, not a mere imitating or copying. The model, the objective possibilities of clay as the new material, and the potter’s experi-

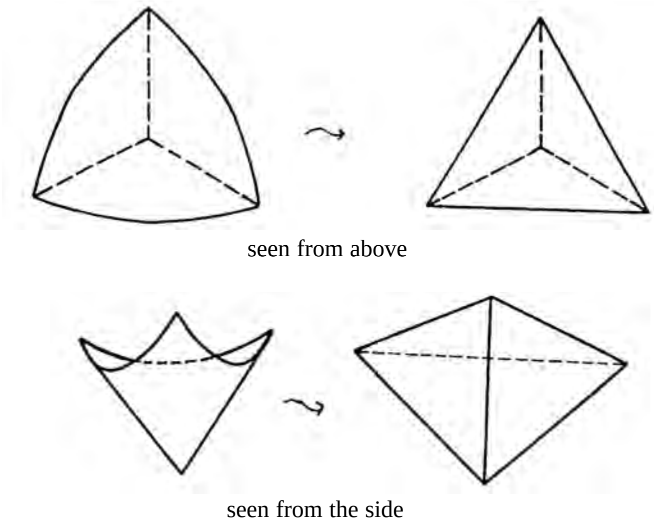


Fig. 4.14. Nyassa pyramid

Fig. 4.15. Shapes of a face and border

ence, united in the creative labor process, are reflected in the new pot shape.

The weaver did not only serve as a source of inspiration for potters. The weaver finds that as a consequence of intensive use, the strands on the sides easily become loose. To avoid this, it becomes necessary to fasten a border to the “bowl.” The plant strands are relatively flexible, so that when the artisan uses stiff sticks for the border, a very particular object is unexpectedly obtained (fig. 4.14). It has three plane faces; each face formed by a right isosceles triangle, while the border is formed by an equilateral triangle (fig. 4.15). The plaited square has been transformed into a pyramid; the original square border “generated” the equilateral, triangular border of the pyramid.[6] A pyramid concept

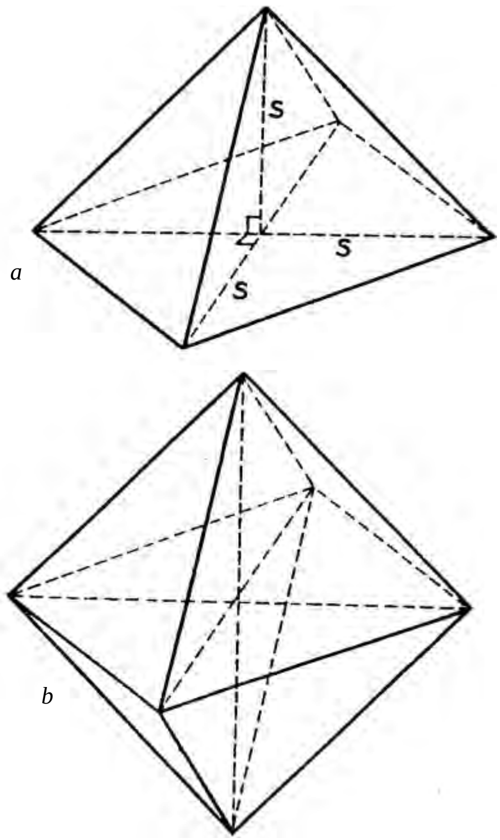


Fig. 4.16. Building up polyhedra with Nyassa pyramids: a) pyramid with square base built up with four Nyassa pyramids. b) regular octahedron built up with eight Nyassa pyramids

is thus born in this labor process. With this, the foundations are laid for further development. For example, when one joins four "Nyassa pyramids"—called *eheleo* in the Makhuwa language—of the same size, as in fig. 4.16a, then one obtains a pyramid with a square base.

And when one joins together two of these new pyramids, or eight "Nyassa pyramids," as in fig. 4.16b, then one obtains a *regular octahedron!*[7]

In the north of Mozambique, in the south of Tanzania (Weule

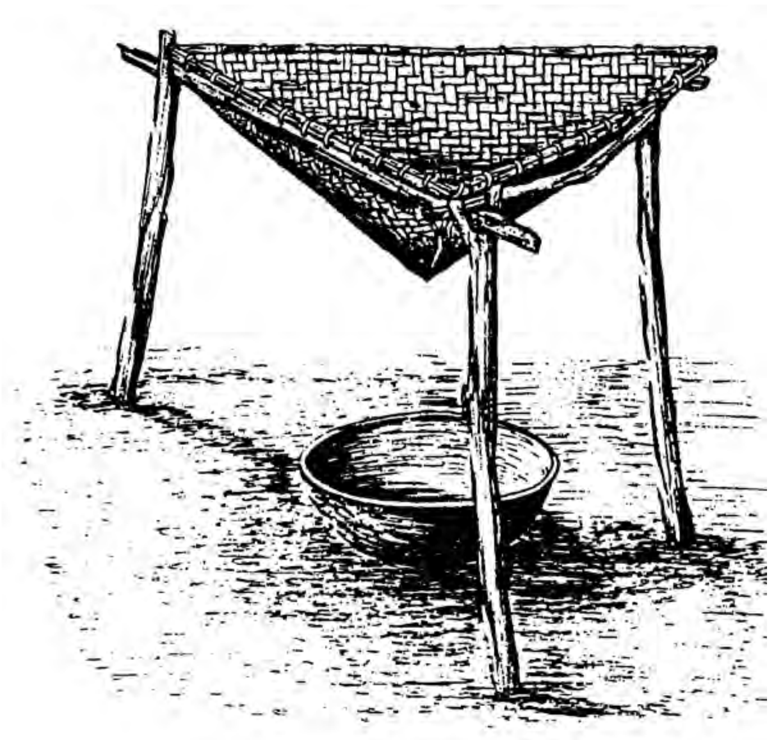


Fig. 4.17. *Eheleo* funnel

1908, 10 and table 19), and in Congo, a "Nyassa pyramid" is used as a funnel in the production of salt. As fig. 4.17 shows, the funnel is hung on a skeleton of sticks, and earth containing salt is put in it. Hot water is poured on the earth and the salt water is caught under the basket. After the water evaporates, the salt remains.

$$V = \frac{h}{3}(a^2 + ab + b^2)$$

3. The "pinnacle of achievement" of mathematics in ancient Egypt

The "pinnacle of achievement" of mathematics in ancient Egypt is the exact result for the volume of a truncated pyramid (with square base) corresponding to the application of the formula (Wussing 1979, 37). "An outstanding accomplishment," considers van der Waerden (1954, 34), or "the very acme of Egyptian mathematical achievement," according to Gillings (1982, 187). Unfortunately, the original text of the *Moscow Mathematical Papyrus,* in which the exercise about a truncated pyramid is included, does not permit us to conclude how the result had been found. Probably the formula was not discovered in a merely empirical way: "It must have been obtained on the basis of a the-

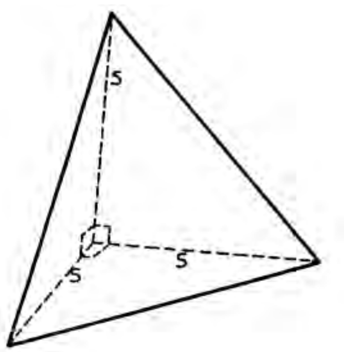


Fig. 4.18. Nyassa pyramid (*s* denotes the length of its edge)

oretical argument; how?" asks van der Waerden (1954, 34). To answer this question "how?" Coolidge (1963) and Gillings (1982) present a summary of earlier hypotheses. I will formulate a new hypothesis

Basket weaving had achieved a high level of development in ancient Egypt. Let us suppose that something like the Nyassa pyramid was already known at that stage.[8]

If a Nyassa pyramid filled with cereals is emptied several

(1) $$V_{np} = \frac{1}{6} s^3,$$

times in succession, say into a cubic basket of the same edge length, one finds that the basket is filled exactly to the brim after the sixth time. Or, inversely, a full cubic basket can completely fill six Nyassa pyramids with edges of the same length as that of

(2) $$V_p = 4 \times \frac{1}{6} s^3,$$

the cubic basket. Once the formula $V_c = s^3$ for the volume V_c of a cube of edge s is known, it follows immediately for the volume of a Nyassa pyramid V_{np} that

where s denotes the length of its edge (fig. 4.18).

Let us consider now a pyramid of square base that is composed of four equal Nyassa pyramids (as in fig. 4.16); its

(4) $$V_p = 4 \times \frac{1}{6} s^3 = \frac{1}{3} sa^2,$$

volume V_p is given by

where s denotes the length of the edges of the Nyassa pyramids.

(5) $$V_p = \frac{1}{3} ha^2,$$

If a denotes the length of the edge of the base of the new pyramid, then the following relation is obviously valid:

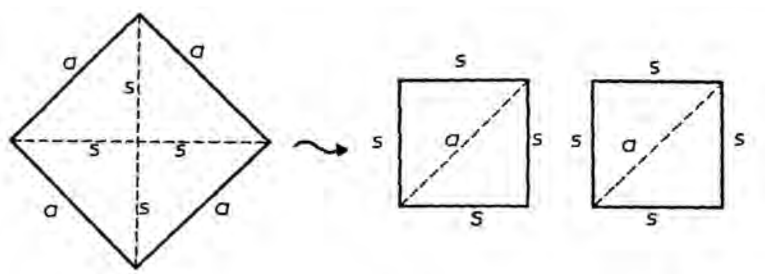


Fig. 4.19. Surface transformation

(3) $$a^2 = 2\,s^2,$$

as follows from the transformation in fig. 4.19. With this formula, (2) can be transformed to

or, if one now extrapolates to arbitrary pyramids with square

(6) $$V_{tp} = V_{p_1} - V_{p_2} = \frac{1}{3}(ma^2 - nb^2),$$

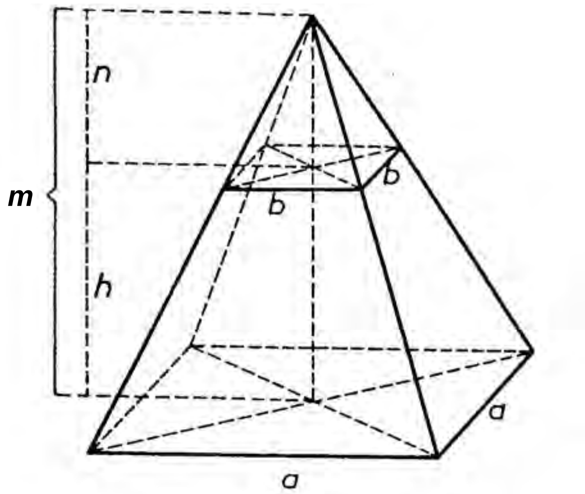


Fig. 4.20. Truncated pyramid as the "difference" of two pyramids obtained by "subtracting" a small pyramid from a larger one

base,
where h denotes the height of the pyramid. In other words, the volume of such a pyramid is found as one-third of the product of its height and the area of its base.

And let us now go to the "pinnacle of achievement." A truncated pyramid (tp) with square base may be considered as the "difference" of two pyramids (fig. 4.20). From formula (5), we have for the its volume V_{tp}:
where m and n denote their respective heights and a and b the lengths of the sides of the bases of the two pyramids. How can one apply this formula, however, if one does not know m and n, but only the height of the truncated pyramid, that is, $m - n$? It

(7) $$V_{tp_1} = V_{np_1} - V_{np_2} = \frac{1}{6}s^3 - \frac{1}{6}t^3 = \frac{1}{6}(s^3 - t^3),$$

seems very probable that one needs more knowledge—for example, of the similarity relation

$$h : n = (a - b) : b.$$

(8) $$V_{tp} = 4V_{tp_1} = 4 \times \frac{1}{6}(s^3 - t^3) = \frac{2}{3}(s^3 - t^3).$$

Thus we started from the general formula (5) and we ended in a "deadlock." Let us therefore first consider particular cases—for example, that of the truncated pyramid that may be conceived of

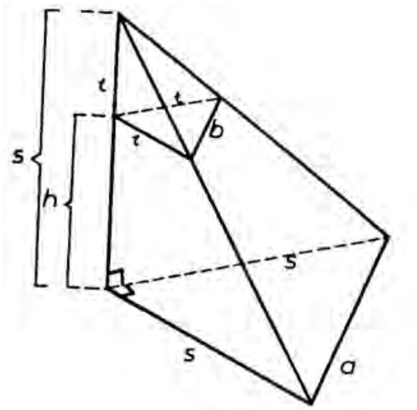


Fig. 4.21. "Difference" of two Nyassa pyramids

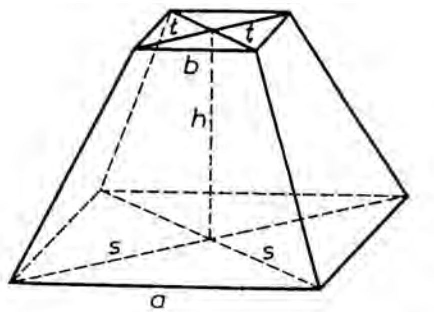


Fig. 4.22. Truncated pyramid with square base

as the "difference" of two Nyassa pyramids (fig. 4.21). Using formula (1), one obtains:
where s and t denote the respective lengths of the edges of the two Nyassa pyramids. When one now joins four of these truncated pyramids, one obtains a truncated pyramid with a square

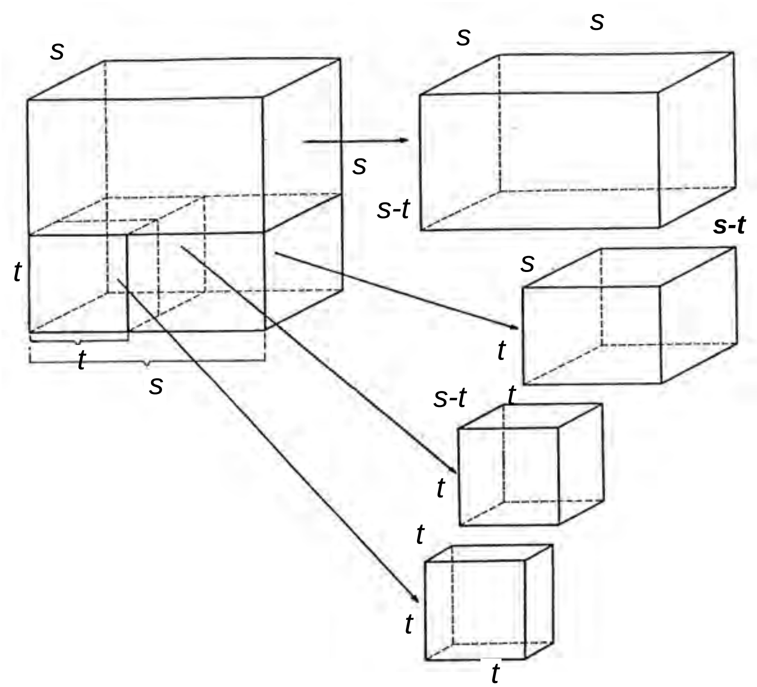


Fig. 4.23. Decomposition of a cube

base, the volume of which is fourfold that of the other (fig. 4.22):

How can one transform this formula into terms of the lengths *a* and *b* of the edges of the respective squares and of the height *h*?

Obviously one has $h = s - t$. What follows from here? The

10) $$V_{tp} = \frac{2}{3}(s^3 - t^3) = \frac{2}{3}h\,(s^2 + st + t^2)$$

$$= \frac{1}{3}\,h(2s^2 + 2st + 2t^2).$$

difference $s^3 - t^3$ corresponds to the difference of the volumes of two cubes. Is it possible to represent this difference in still another way? A simple decomposition of the bigger cube, as in fig. 4.23, shows:

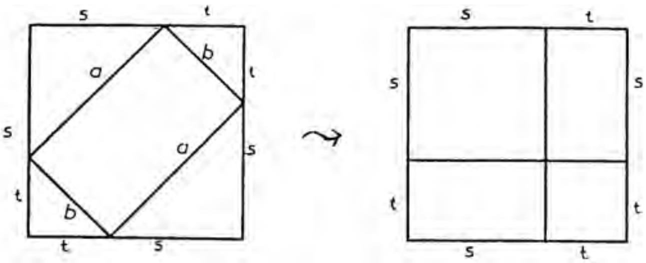


Fig. 4.24. Transformation of a square

$$s^3 = t^3 + (s - t)s^2 + (s - t)st + (s - t)t^2,$$

(11) $$V_{tp} = \frac{h}{3}(a^2 + ab + b^2)$$

that is,

(9) $$s^3 - t^3 = (s - t)(s^2 + st + t^2) = h(s^2 + st + t^2).$$

Therefore:

We saw already that $a^2 = 2s^2$ and, in the same way, we have $b^2 = 2t^2$. On the basis of fig. 4.24, one obtains

$$ab + s^2 + t^2 = 2st + s^2 + t^2,$$

(6) $$V_{tp} = V_{p_1} - V_{p_2} = \frac{1}{3}(ma^2 - nb^2),$$

that is, $ab = 2st$.

In this manner, one arrives at

for the volume of that truncated pyramid. Once formulated in this form, it can be easily generalized to the volume of an arbitrary truncated pyramid with square base.

Let us now compare my hypothesis with the well-known hypotheses of Gillings, Neugebauer, and van der Waerden.

Gillings considers a truncated pyramid with square base as

(12) $$V_{tp} = \frac{1}{3}ma^2 - \frac{1}{3}nb^2 = \frac{1}{3}[(h+n)a^2 - nb^2]$$

$$= \frac{1}{3}[ha^2 + na^2 - nb^2] = \frac{1}{3}h[a^2 + (a^2 - b^2)].$$

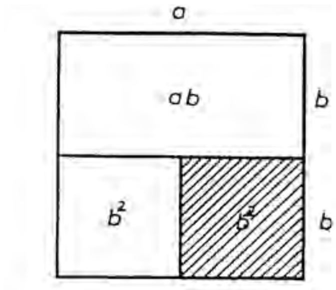


Fig. 4.25. Special case $a = 2b$

the difference of two pyramids (fig. 4.20). For its volume, we have, as we already saw,
where m and n denote the respective heights and a and b the lengths of the bases of the two pyramids. As I already showed, it is difficult in general to apply this formula if only a, b and the height h of the truncated pyramid are known. Therefore, as

(11) $$V_{tp} = \frac{h}{3}(a^2 + ab + b^2).$$

Gillings supposes, a scribe in ancient Egypt could have first analyzed a special case. If $a = 2b$, then $m = 2n$, that is, $h = n$. Under these circumstances, it is easy to transform formula (6) as follows:

From fig. 4.25, one obtains

(13) $$a^2 - b^2 = ab + b^2$$

if $a = 2b$. And in this manner one may realize "magically" the

(12) $$V_{tp} = \frac{1}{3}h[a^2 + (a^2 - b^2)],$$

transition from formula (12) into the "pinnacle of achievement"

$$V_{tp} = \frac{1}{3}h[2a^2 - b^2],$$

of Egyptian mathematics:

Afterward, the scribe could have verified the formula in other cases, like $a = 3b$, $a = 4b$, $a = 5b$, etc., and then he might have concluded that it is valid for all truncated pyramids with a square base (Gillings 1982, 191–93).

Very beautiful! Nevertheless, Gillings's reconstruction of the sequence of thought of an Egyptian scribe remains incomplete. The transition from (12) to (11) is not immediately necessary: for the time being, equation (12) will do:
that is,
as it is only formulated in terms of *a*, *b*, and *h*. It only needed to be transformed at the moment when one verifies that it is not true in other cases. And it is also not immediately clear at the outset what transformation is necessary to obtain a formula that is valid in general. The alternatives

$$V_{tp} = \frac{2}{3}(s^3 - t^3) \tag{8}$$

$$a^2 + (a^2 - b^2) = a^2 + 3b^2, \tag{14}$$

$$a^2 + (a^2 - b^2) = 3ab + b^2, \tag{15}$$

could have been chosen instead of

$$a^2 + (a^2 - b^2) = a^2 + ab + b^2. \tag{13}$$

In other words, Gillings's hypothesis presupposes on the part of the Egyptian scribes an experience with verification and, consequently, if necessary, the adaptation of formulas. Here resides an important distinction with my hypothesis. In my case, it is *immediately* verified that the formula
is not satisfactory, as it is not expressed in terms of the lengths *a* and *b* of the sides of the square bases and the height *h*, and for

$$V_{tp} = hb^2 + 2 \times \frac{h}{2} b(a-b) + \frac{h}{3}(a-b)^2. \tag{16}$$

that reason cannot be generalized. Therefore, its transformation is immediately necessary. This means that there is no presupposition of verification being needed to arrive at the intended conclusion.

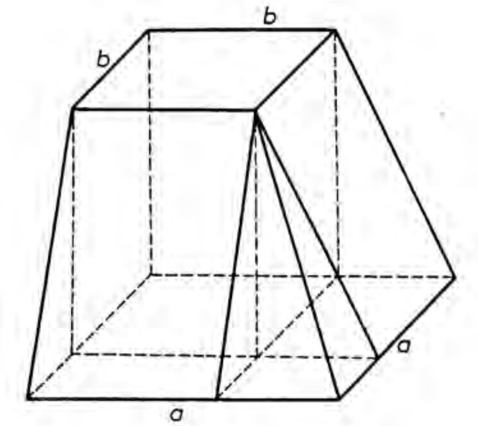


Fig. 4.26. Neugebauer's decomposition

In his book *Über vorgriechische Mathematik* [Pre-Greek mathematics] (1929), Otto Neugebauer considers the special case in which one of the edges is perpendicular to the base. He decomposes the truncated pyramid into four parts: one rectangular block, two prisms, and one pyramid, as shown in fig. 4.26. The volume of the truncated pyramid is equal to the sum of the volumes of its parts:

Neugebauer conjectures that equation (11) of ancient Egypt could have been deduced from here by means of an *algebraic*

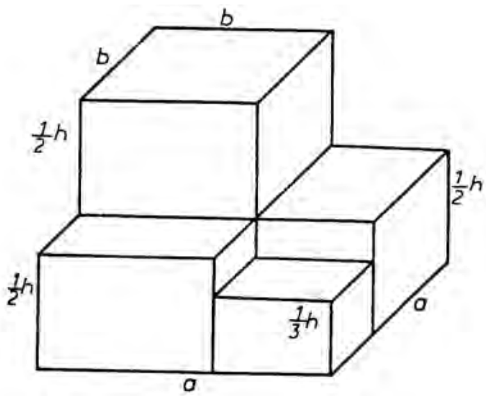


Fig. 4.27. First phase of van der Waerden's transformation

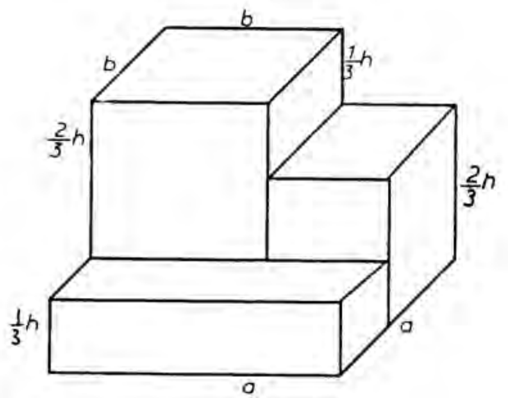


Fig. 4.28. Second phase of van der Waerden's transformation

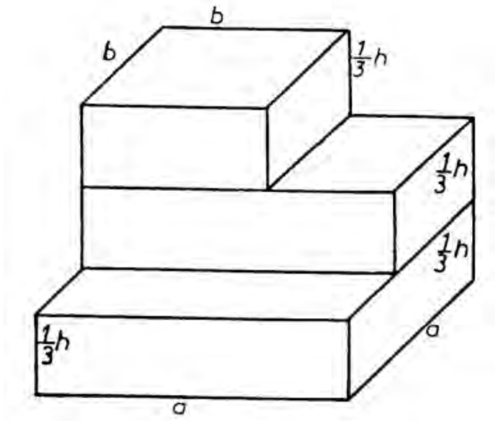


Fig. 4.29. Third phase of van der Waerden's transformation

transformation. But "can one justify the assumption that the Egyptians were able to make such an algebraic transformation?" questions van der Waerden. "They were able to calculate with concrete numbers, but not with general quantities" (1954, 34). Van der Waerden is led to the presupposition that this transfor-

(11) $$V_{tp} = \frac{h}{3}a^2 + \frac{h}{3}ab + \frac{h}{3}b^2 = \frac{h}{3}(a^2 + ab + b^2)\,.$$

mation would have been realized in a *geometrical* way.

The two prisms in fig. 4.26 are changed to rectangular blocks of half the height, and the pyramid is also transformed into such a block, but with one-third of the original height (fig. 4.27). Then the upper third of the first of these blocks is removed and placed on the top of the second one (fig. 4.28).

In this way there appears a solid that can be divided into three horizontal layers, each of which has a height $h/3$. The lowest of these layers has a base with area equal to a^2; the middle layer has a base with area equal to ab; and the top layer has a base with area equal to b^2. Therefore:

"This derivation of the formula does not transcend the level of Egyptian mathematics," observes van der Waerden in his book *Science Awakening* (1954, 35). However, one very important question is not answered by van der Waerden (and herein lies a distinction between his hypothesis and mine): why should the scribe have chosen exactly this transformation and not another? Was the choice of these geometrical transformations the result of experimenting, or alternatively, did the scribe already know the formula, in particular, the common factor $h/3$, and did he only look for a justification *post factum*? In his later book *Geometry and Algebra in Ancient Civilizations,* van der Waerden conjectures that the correct formula for the volume of a truncated pyramid, which one encounters both in ancient China and in ancient Egypt, had a "*pre-Babylonian* common source" (1983, 44).

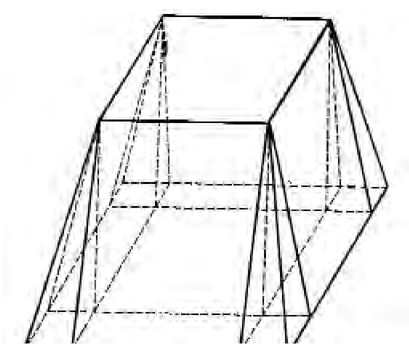


Fig. 4.30. Lui Hui's decomposition of a *fang-t'ing*

The mathematics of ancient China was summarized in the work *Mathematics in Nine Books.* This compilation survived in a version in the year 263 by Liu Hui. Liu Hui gives a derivation for the volume of a *fang-t'ing*, a truncated pyramid with a square base (fig. 4.30), in which he decomposes it into a block, four prisms, and four *yang-ma*'s (pyramids with a square base, where one edge is perpendicular to the base). To explain that the volume of a *yang-ma'* is equal to two thirds of the volume of a prism with the same base and same height, Liu Hui applies a limit process (Wagner 1979, 169, 173). "These explanations satisfy many of the criteria for what we would call a proof," asserts Wagner (164). A thought process in terms of limits could not arise at the beginning; the derivation of Liu Hui is, indeed, a justification of knowledge already acquired in another way. At least some parts of the *Mathematics in Nine Books*, because of the tradition of oral transmission, date to a much earlier era, perhaps even to a pre-Babylonian period.

My hypothesis may fill a gap here. My conjecture for the derivation of the formula for the volume of a truncated pyramid with square base has its starting point in material *products of human labor* (Nyassa pyramid and cubic baskets) and in their *empirically discovered relationships* [$V_{np} = (1/6)V_c$ or $V_c = 6\ V_{np}$], and each next step in the reasoning is *constructive* in the sense that it results, without any deviation, from the search for an answer to questions like "$s^3 - t^3 = ?$" or "$2st = ?$"

4. How could the Theorem of Pythagoras be discovered thousands of years before Pythagoras?

Discovered only once?

Both in Greek-Hellenistic antiquity and also in ancient Mesopotamia, India, and China, "Pythagorean triples" have not only been calculated, but the so-called Theorem of Pythagoras was also known. The research of Thom and Thom on the geometry of the megalithic gravesides on the British Isles suggests that its constructors used “Pythagorean triples” (1980; van der Waerden 1983, 16 ff). The relationship between these numerical triples (a,b,c) with $a^2 + b^2 = c^2$ and the right-triangle theorem, however, is not immediately evident. Moreover, van der Waerden assumes it is not so easy to discover the Theorem of Pythagoras (16).

Van der Waerden conjectures that “a common origin of the whole theory . . . [is] highly probable” (1983, 10, 45). The Theorem of Pythagoras was, according to van der Waerden, already known about 2000 B.C., when the ancestors of the Greeks and of the Aryan tribes that later invaded India still lived next to each other in the region of the Danube (14).

Simplest case

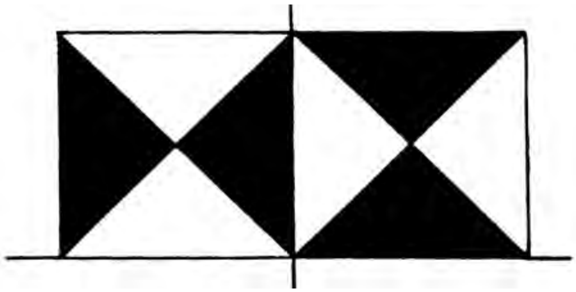
Is the Theorem of Pythagoras so difficult to find that it would be discovered only once in human history? Perhaps van der Waerden and Seidenberg are right. However, at least the simplest case of the theorem, where both sides of the right triangle have the same length, has been and continues to be discovered again and again. How this particular case of the theorem is formulated does not matter here (see the examples in figs. 4.31 and 4.32).

Surely one may ask how it could be possible to arrive at the general theorem from this special case.

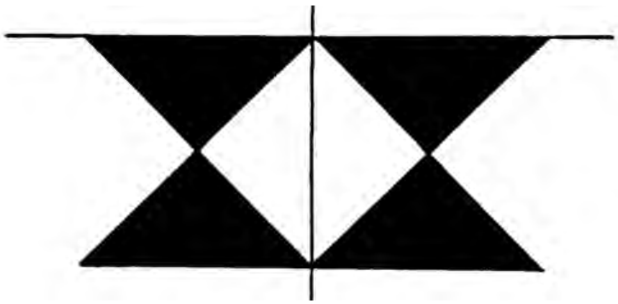
Twins

That only the Theorem of Pythagoras, but not Pythagorean triples, appeared later in the works of Euclid, Archimedes, and Apolonius does not mean, however, that a relationship between those numerical triples and the theorem with the same name had not been recognized before in human history, but only that, at some historical moment, knowledge of it may have disappeared. This does not mean that the mathematicians were not able to understand it. On the contrary, the Pythagorean triples became superfluous for the geometers. The Theorem of Pythagoras had been *freed*—generalized in such a way that any reference to these triples of whole numbers would again restrict it. The theorem and

Halaf ceramics (Mesopotamia, 6th millennium B.C.)



Malekula (Oceania)

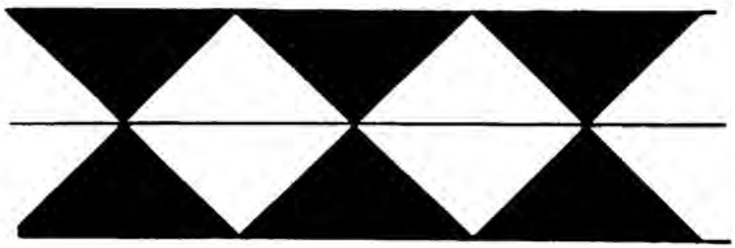


Apenayé Indians (Brazil)

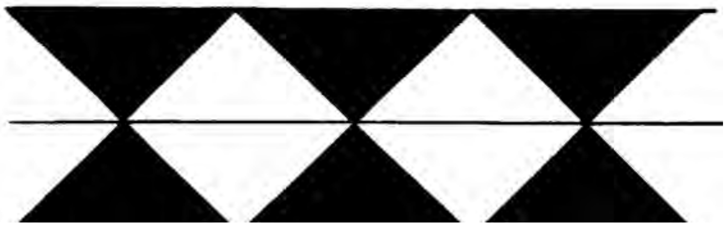
Fig. 4.31. Examples of ornaments composed of squares and triangles

the numerical triples might have born as twins—that is the conjecture that I should like to present here as a possibility—but later each went its own way: geometry and number theory.

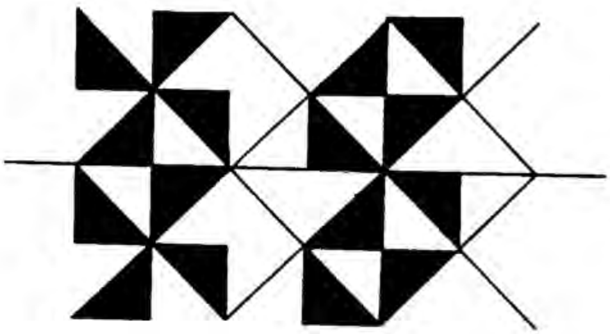
This reflection helps us to advance our tracing of the possi-



Dogon (Mali)



Igbo (Nigeria)



Madagascar

Fig. 4.32. Further examples of ornaments composed of squares and triangles

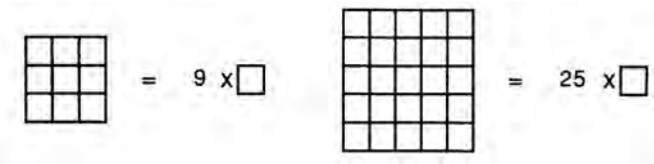


Fig. 4.33. Counting the unit squares

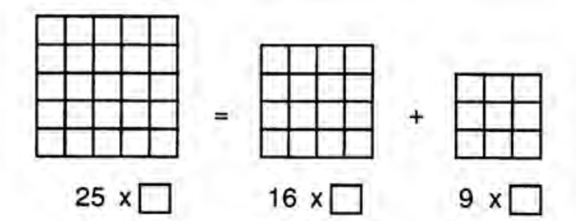


Fig. 4.34. Discovering an equality by counting unit squares

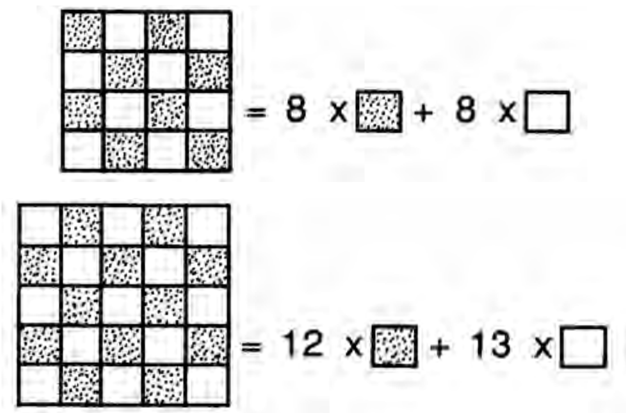


Fig. 4.35. Counting the unit squares of a chessboard pattern

ble birth(s) of the twins. One should look for situations where one encounters at the same time squares, sums of squares, right angles, and whole numbers. Most probably, the concept of a geometrical square is older than that of an arithmetical square. In which contexts do geometrical squares with countable areas appear?—in basket and textile weaving, in embroidery, and in laying tiles.

The right track?

When strands of equal width are plaited, squares that can be decomposed into small unit squares appear. Their areas are, therefore, relatively easily counted (fig. 4.33). In this way one, in fact, obtains squares, but not immediately those squares that constitute the *sum* of two squares. Only when one is accidentally lucky does one discover that $5^2 = 3^2 + 4^2$ (fig. 4.34) and be stimulated to search further for such particularities. It seems unlikely, however, that the Theorem of Pythagoras would have been discovered by this path.

If one plaits with strands of two colors, one may obtain a chessboard pattern that certainly enables one to consider squares

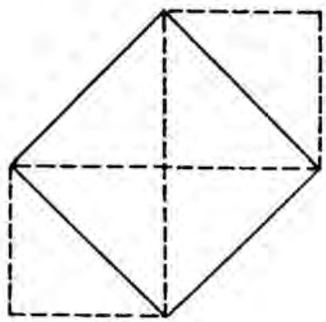


Fig. 4.36. The sides of the larger and the smaller squares are not parallel

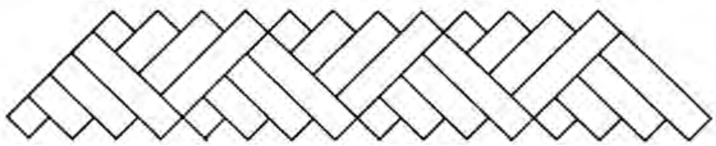


Fig. 4.37. A woven-strip pattern

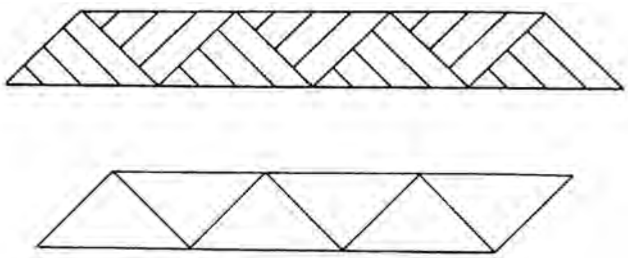


Fig. 4.38. Smooth ornaments

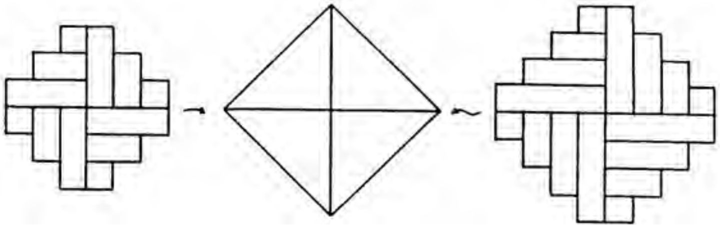


Fig. 4.39. Woven centers that correspond to the simplest case of the Theorem of Pythagoras

as sums (fig. 4.35), but not yet as sums of squares.

Did we come to a dead end? Or are we still on the right track?

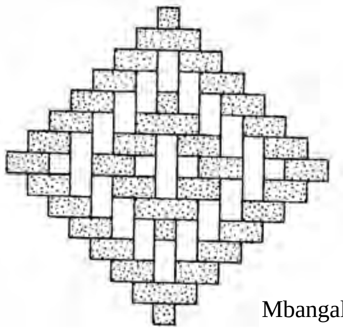
Second phase

Let us return to the simplest case of the Theorem of Pythagoras and note that the sides of the sum square and of the two minor squares are, in general, not parallel (fig. 4.36).

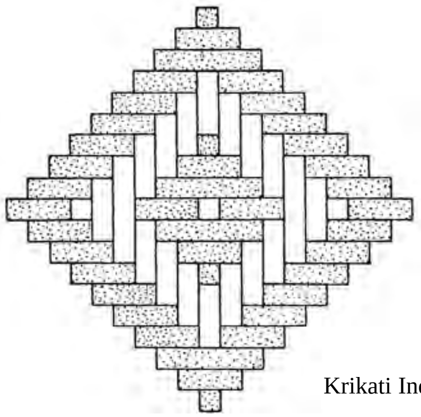
What weaving pattern could have served as a model for this nonparallel position of the three squares? The widespread “toothed” weaving pattern of fig. 4.37 reflects itself often like fig. 4.38 on smooth ornaments on pots and on wooden and metallic objects.

In the same way, the weaving patterns of fig. 4.39 correspond to the simplest case of the Theorem of Pythagoras.

Now we are probably on the right track: such toothed, woven squares of distinct colorings are well known, as the examples in fig. 4.40 underline.



Mbangala (Angola)



Krikati Indians (Brazil)

Fig. 4.40. Woven, toothed squares

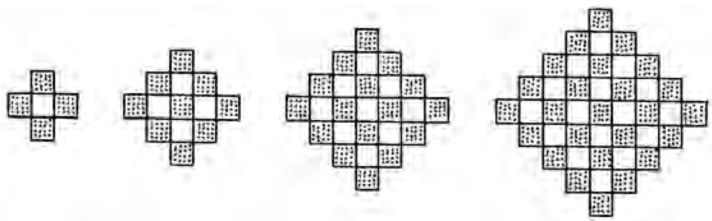


Fig. 4.41. Toothed squares with a chessboard coloring

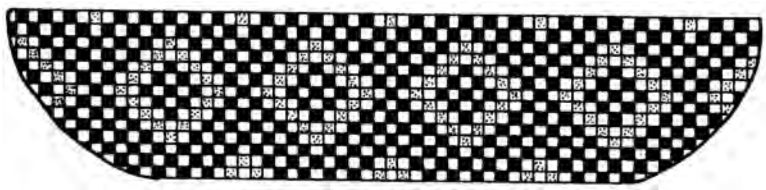


Fig. 4.42. Ancient Egyptian wall painting of a basket

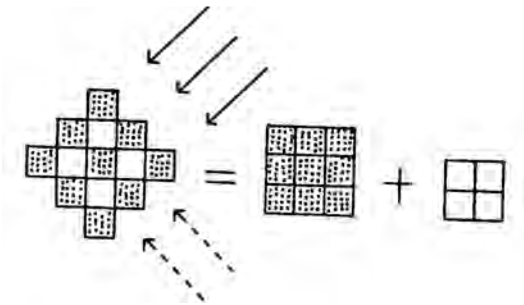


Fig. 4.43. Toothed square as the “sum” of two squares. First example

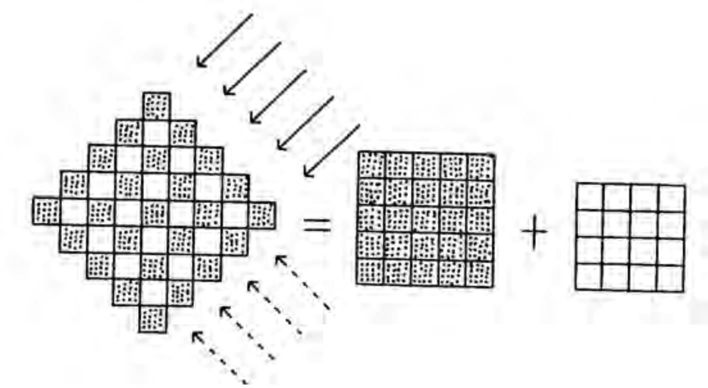


Fig. 4.44. Toothed square as the “sum”' of two squares. Second example

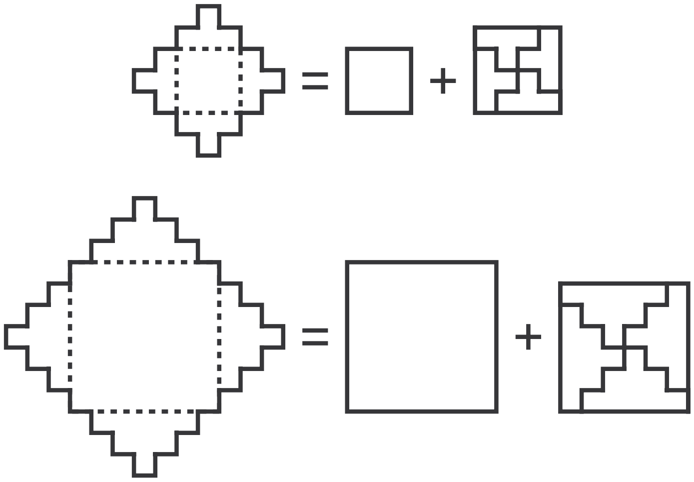


Fig. 4.45. Another geometric transformation of toothed squares

In other words, the toothed square is the sum of two squares, with the first composed of black unit squares and the second composed of white unit squares (fig. 4.43).

Would this observation be merely accidental? This result stimulates further searching. As the example in fig. 4.44 shows, it was not merely accidental.

Another geometric transformation (fig. 4.45) leads to the same conclusion that each toothed square is the sum of two smooth squares.

Third phase

A new phase emerges when our hypothetical "Pythagoras" discovers a particularity of the following case. On the one hand, it is observed that the toothed square in fig. 4.46 has the same area as two squares of 3 and 4 unit squares on each side, respectively, together. On the other hand, purely geometric (fig. 4.47a) or arithmetic-geometric reasoning (fig. 4.47b or 4.47c) leads to the conclusion (fig. 4.48) that two squares of 3 and 4 unit squares on each side, respectively, have together the same area as a square with 5 unit squares on each side, that is, $3^2 + 4^2 = 5^2$.

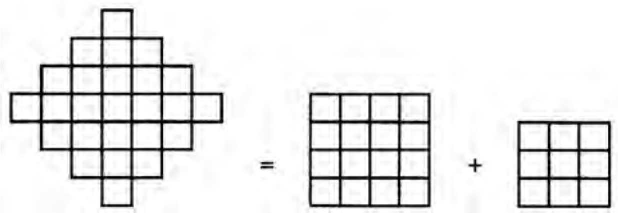


Fig. 4.46. A special case

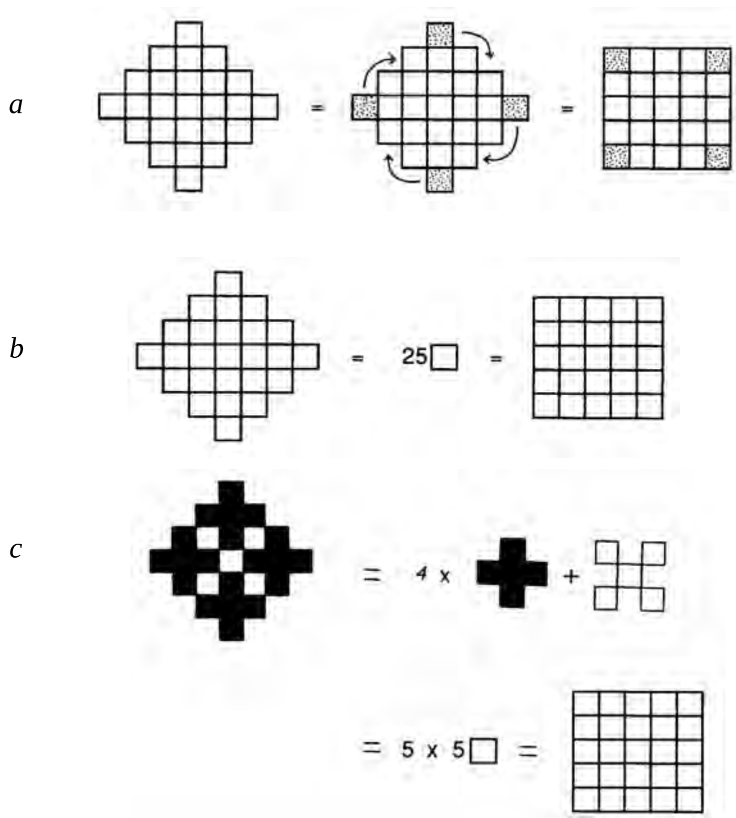


Fig. 4.47. Transformation of a particular toothed square

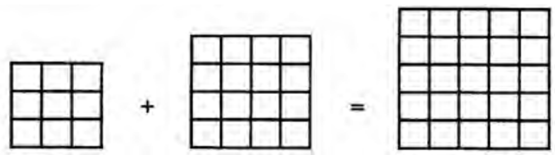
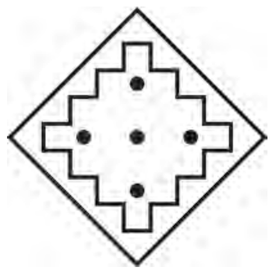


Fig. 4.48. Conclusion: $3^2 + 4^2 = 5^2$



4.49. Detail of an ancient Mesopotamian board game

The field of a board game found in one of the royal tombs of Ur (middle of the third millennium B.C.) lets me suppose that at that time this particularity of the toothed square with 7 unit squares on its diagonal (fig. 4.49) was already known.

Is this case ($3^2 + 4^2 = 5^2$) the only one, or do more exist?

Let us follow our "Pythagoras" in a further search for toothed squares that consist of a square number of unit squares or that can be transformed into a smooth square.

Possible further developments

From the point at which we have arrived, several further developments are possible. They can take place independently of one another or they may occur simultaneously, influencing one another.

Toward Pythagorean triplets (1)

One may compose a list of sums of squares of two consecutive (natural) numbers and, upon comparing them with a table of square numbers, discover that $20^2 + 21^2 = 29^2$, $119^2 + 120^2 = 169^2$, etc. This purely arithmetic way does not seem very probable.

Toward Pythagorean triplets (2)

One may try to transform in a manner that is analogous to fig. 4.47a other toothed squares into "real" smooth squares. A transformation done in an *arbitrary* way, then, will rarely be successful. A smooth square appears only for the second time when the diagonal of the toothed square is composed of 41 unit squares. However, in examining more closely the special case of fig. 4.47a, one may try to "cut off" from all the corners of the toothed square, equal "triangles" with a toothed cathetus[10] and smooth

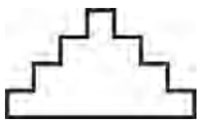


Fig. 4.50. "Triangle" with a toothed cathetus

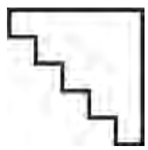


Fig. 4.51. "Triangle" with a toothed hypotenuse

hypotenuse (fig. 4.50) to try to transform them into "triangles" of equal area with a smooth cathetus and toothed hypotenuse (fig. 4.51), and afterward try to join these latter triangles to the original sides of the toothed square in order to obtain a smooth square (see the scheme in fig. 4.52).

This becomes possible for the first time when the side of the first "triangle" has 6 teeth (fig. 4.53). Therefore, a toothed square with 21 unit squares (6 + 8 + 1 + 6 = 21, fig. 4.54) on each side and with 41 unit squares on its diagonal can be transformed into a smooth square with 29 of these unit squares on each side (8 + 1 + 11 + 1 + 8 = 29). In other words: $20^2 + 21^2 = 29^2$

These and other results can also be obtained if one understands that the intended transformation is only possible when the

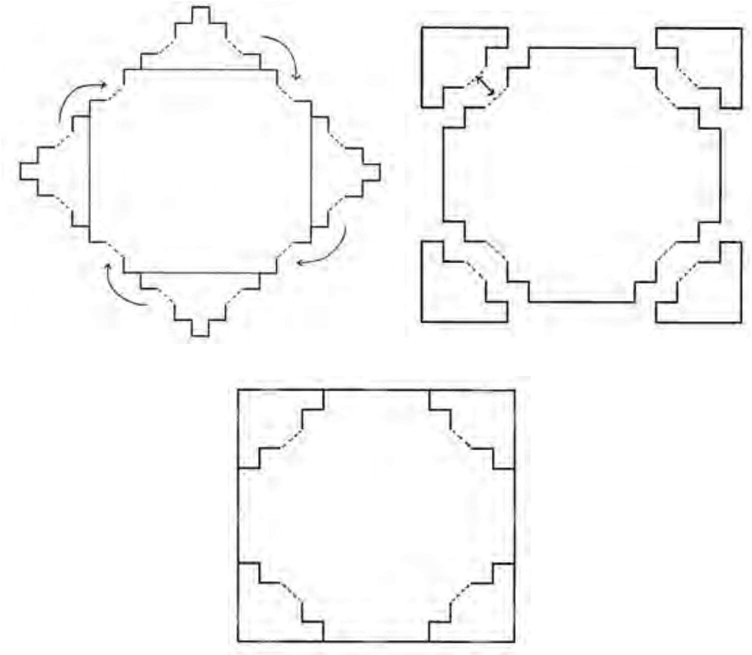


Fig. 4.52. Schematic representation of the transformation of a toothed square into a smooth square

number of unit squares of the first "triangle" (fig. 4.50) is equal to the number of unit squares of the second "triangle"—that is, to express it in another way, when a *square number* (fig. 4.55) is equal to a *triangular number*. It is only necessary to compare both lists of numbers (see table 1) to discover

$$(6 + 8 + 6)^2 + (6 + 8 + 6 + 1)^2 = [8 + 1 + (2 \times 6 - 1) + 1 + 8]^2,$$

that is,

$$20^2 + 21^2 = 29^2$$

and

$$(35 + 49 + 35)^2 + (35 + 49 + 35 + 1)^2$$
$$= [49 + 1 + (2 \times 35 - 1) + 1 + 49]^2,$$

that is,

$$119^2 + 120^2 = 169^2, \text{ etc.}$$

Table 1

Natural numbers	Square numbers	Triangular numbers
1	1	1
2	4	3
3	9	6
4	16	...
...	...	...
6	36	...
7	...	...
8	...	36
...	...	...
35	1225	...
...	...	...
49	...	1225
...	...	...

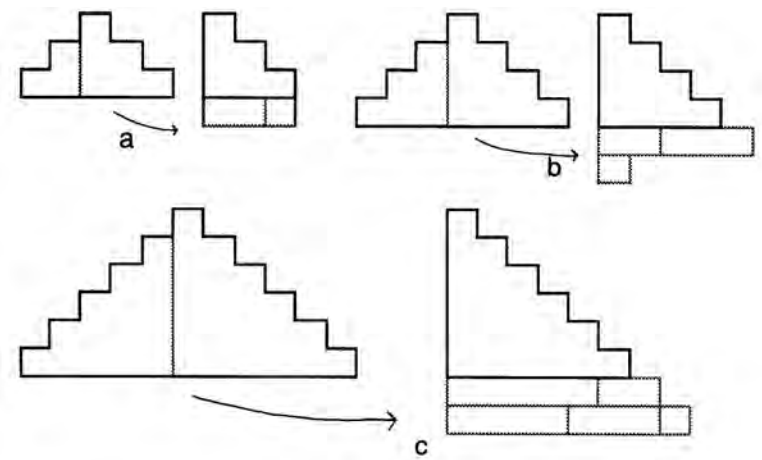


Fig. 4.53. Attempt to transform "triangles" with toothed catheti into "triangles" with toothed hypotenuses

I note that it may thus be discovered that the sum of the first n odd numbers is always equal to a square n^2 (fig. 4.55).

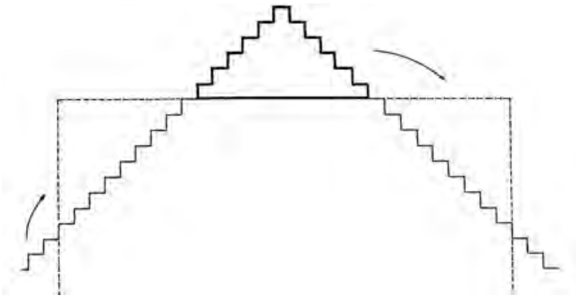


Fig. 4.54. Transformation of a toothed square with 21 unit squares on each side into a smooth square with 29 unit squares on each side

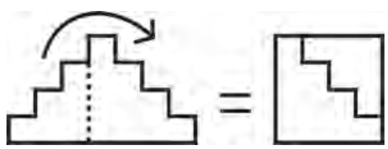


Fig. 4.55. Transformation of a “triangle” with toothed catheti into a square

Toward Pythagorean triplets (3)

Which observations may our “Pythagoras” arrive at by considering a toothed square together with its circumscribed smooth square (fig. 4.56)?

When counting the unit squares inside and outside the toothed square, it can be noted that in the interior there is always one more unit square than on the outside (fig. 4.57).

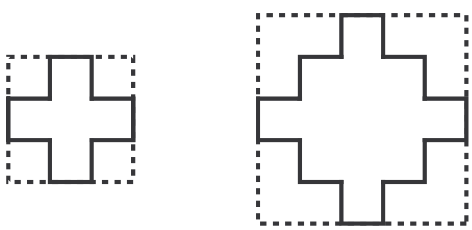


Fig. 4.56. Toothed squares together with their circumscribed smooth squares

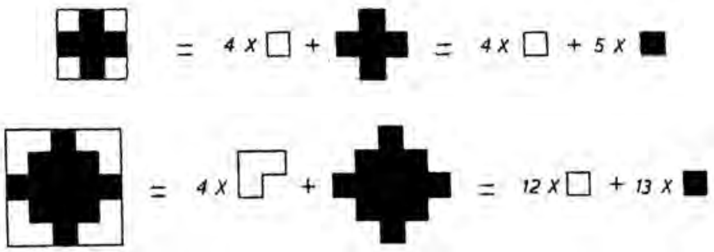


Fig. 4.57. Counting the unit squares inside and outside a toothed square

In a purely geometric way, it is also possible to arrive at the same conclusion when one decomposes a toothed square as in fig. 4.58. It follows that the number of unit squares of the circumscribed smooth square (= d^2) plus one is equal to two times the number N of unit squares inside the corresponding toothed square (fig. 4.59), that is,

$$d^2 + 1 = 2N.$$

In other words, N is a square number if half of $d^2 + 1$ is a square number. To encounter such N's, one elaborates a table (see table 2) and compares it with a table of squares (see table 3). In this way the desired solutions of the equation

$$d^2 + 1 = 2N$$

may be found:

1^2	+	1	=	2×1^2,
7^2	+	1	=	2×5^2,
41^2	+	1	=	2×29^2,
239^2	+	1	=	2×169^2, etc.

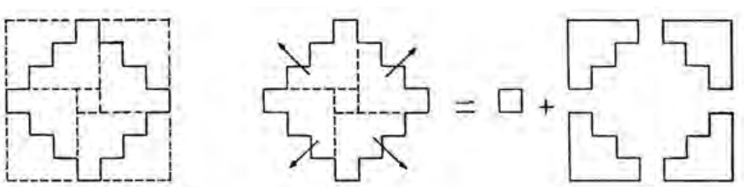


Fig. 4.58. Decomposition of a toothed square

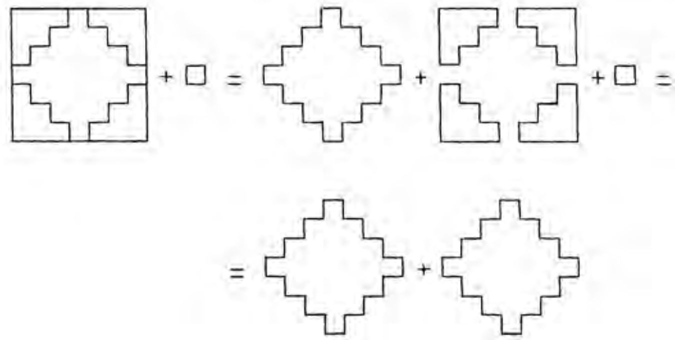


Fig. 4.59. Relationship between a smooth square and its corresponding toothed square

Table 2

d	d^2	$d^2 + 1$	$\frac{1}{2}(d^2 + 1)$
1	1	2	1
3	9	10	3
5	25	26	13
7	49	50	25
9	81	82	41
…	…	…	…
41	1681	1682	841
…	…	…	…
239	57121	57122	28561
…	…	…	…

Table 3

t	t^2
1	1
…	…
5	25
…	…
29	841
…	…
169	28561
…	…

Now, which are the toothed squares composed of 5^2, 29^2, 169^2, etc., unit squares? The length of the side of the circumscribed square is equal to the number of unit squares on the diagonals of the toothed square. And, in its turn, the number of unit squares on each diagonal is equal to the sum of the lengths of the sides of two smooth squares in which the toothed square can be decomposed (see figs. 4.43 and 4.44). Therefore, keeping in mind that 7 = 3 + 4, 41 = 20 + 21, 239 = 119 + 120, etc., we obtain the following results:

$$3^2 + 4^2 = 5^2,$$
$$20^2 + 21^2 = 29^2,$$
$$119^2 + 120^2 = 169^2, \text{ etc.}$$

Toward Pythagorean triplets (4)

It is possible that our "Pythagoras" arrives at the following observations. If one decomposes a toothed square into two smooth squares, then the difference between these new squares is equal to the number of unit squares on the diagonals. This observation can be the result of arithmetic or geometric reasoning:

$$4 - 1 = 3,$$
$$9 - 4 = 5,$$
$$16 - 9 = 7,$$
$$25 - 16 = 9, \text{ etc.},$$

or be the result of discovering first a new decomposition of the toothed square, whereby both parts have the same area as both smooth squares (fig. 4.60). It also may have been discovered in the same manner that the sum of the first n consecutive odd numbers is always equal to a square n^2.

On the other hand, the number of unit squares on the diagonals is equal to the sum of the lengths of the sides of the two smooth squares (figs. 4.43 and 4.44). Therefore, it is possible to arrive at the conclusion presented in fig. 4.61. If, in turn, this sum of the lengths of the sides of the two smooth squares, or better still, the sum of the unit squares of the sides of the two smooth squares, is equal to a square, then one finds once again that the sum of the areas of two squares is equal to the area of a third square. To find these cases, it is only necessary to look for odd square numbers: 9, 25, 49, 81, 121, 169, ... and then one obtains successively:

$$9 = 4 + 5 \rightarrow 9 + 4^2 = 5^2 \rightarrow 3^2 + 4^2 = 5^2,$$
$$25 = 12 + 13 \rightarrow 25 + 12^2 = 13^2 \rightarrow 5^2 + 12^2 = 13^2,$$
$$49 = 24 + 25 \rightarrow 49 + 24^2 = 25^2 \rightarrow 7^2 + 24^2 = 25^2, \text{ etc.}$$

The process of forming sums in fig. 4.61 could have been discovered outside the described context. Our context, however,

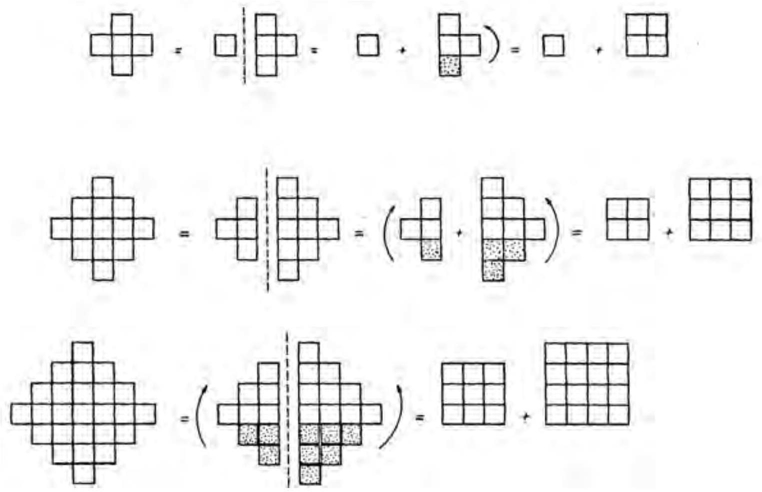


Fig. 4.60. Other decompositions of a toothed square

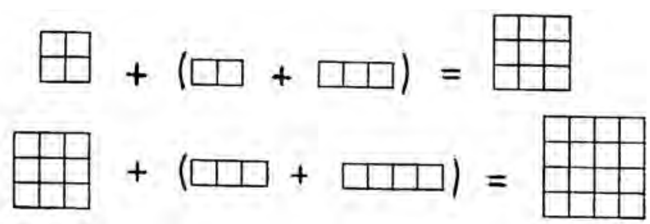


Fig. 4.61. Construction of bigger squares

explains *why* the idea of looking for squares that are the sum of other squares emerged intralogically. This search might have been reinforced by extramathematical ideas. Once discovered, this process can be generalized (fig. 4.62) and, analogously, new Pythagorean triplets can now be found:

$$8^2 + 15^2 = 17^2,$$
$$12^2 + 35^2 = 37^2, \text{ etc.}$$

In a more general way, it may be said that *all* Pythagorean triplets can be found, in principle, with the help of the geometric transformation in fig. 4.63, that is, with the help of

$$c^2 = b^2 + [c(c - b) + b(c - b)],$$
$$c^2 = b^2 + (c + b)(c - b).$$

To find Pythagorean triplets (a,b,c), one has to look for solutions of the equation

$$(c + b)(c - b) = a^2.$$

Toward the Theorem of Pythagoras

The particular experience with the toothed square, whose diagonal has length 7, and the square whose side has length 5—they have the same area (figs. 4.47 and 4.64)—can also stimulate further geometrical search: Every toothed square must be equal to a real (smooth) square.

Let us consider the smallest toothed square (fig. 4.65). How can it be transformed into a real square of the same area?

The dotted squares in figs. 4.66a and 4.66b are obviously too big. The dotted square in fig. 4.66c is too small; it takes away from the toothed square more than what it adds at the same time. Nevertheless, the way we began seems to be advantageous.

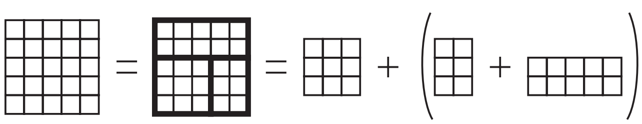


Fig. 4.62. Extension

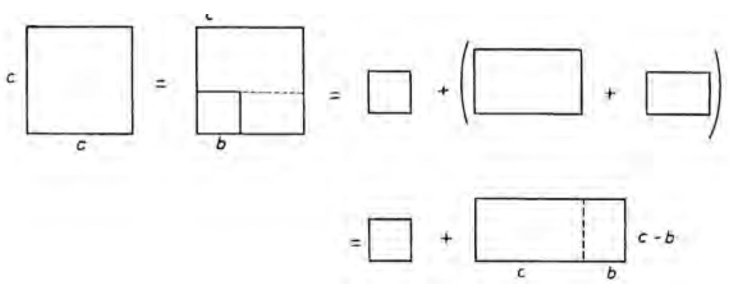


Fig. 4.63. Geometric transformation implying $c^2 = b^2 + (c + b)(c - b)$

When one continues to experiment, one finds a solution that takes away exactly as much as it adds (fig. 4.67). This very successful experience can be easily transferred to other cases, like the ones illustrated in figs. 4.68a and 4.68b.

The result obtained when the length of the diagonal of the toothed square is equal to 7 (unit squares) can be compared with the previous result (fig. 4.64). Both squares are really of the same

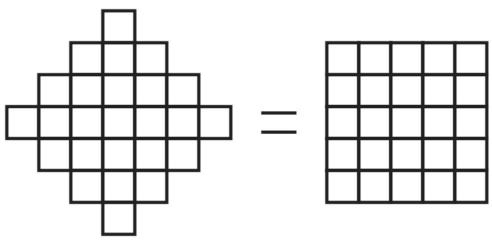


Fig. 4.64. Toothed square with a diagonal of length 7 (units) and a smooth square with a side of length 5 (units)

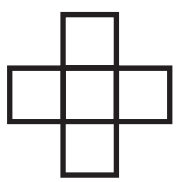


Fig. 4.65. The smallest toothed square

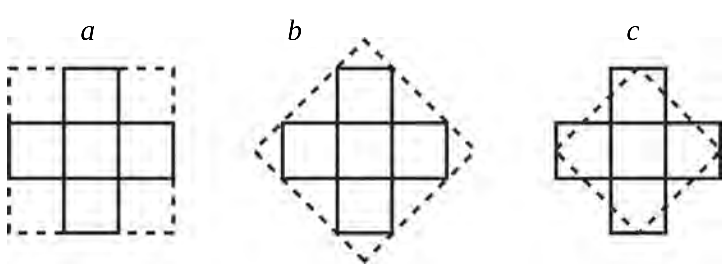


Fig. 4.66. Comparison of areas

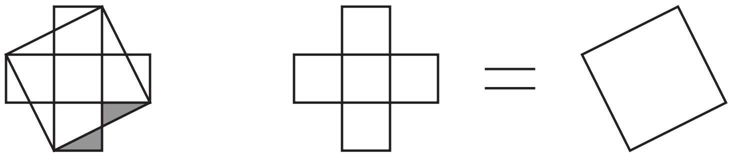


Fig. 4.67. Transformation of the smallest toothed square into a smooth square

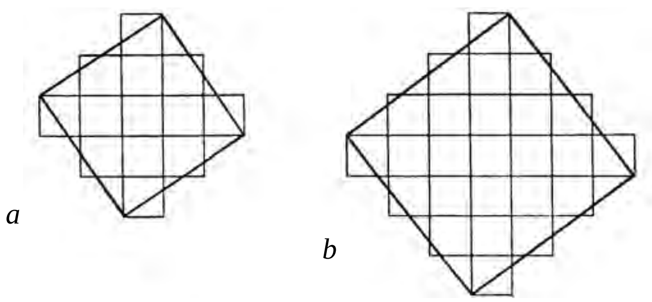


Fig. 4.68. Transformation of toothed squares into smooth squares of the same area

size; their sides are equal. A doubting "Pythagoras" would surely be convinced by now that a smooth square of the same area could always be obtained this way.

If our "Pythagoras" draws the new square in a rectangular array, together with the two squares of which the sum of the areas is equal to its area, then there exist only a few possibilities to do this in such a way that the three squares "touch" each other (fig. 4.69). A comparison with the simplest case (fig. 4.70) might have contributed to the choice of the position in fig. 4.69b.

Once drawn as in fig. 4.69b (compare with fig. 4.71), the Theorem of Pythagoras could have been conjectured in its general form. Simultaneously, the representation in a rectangular array offers a good starting point for the discovery of a proof of the Theorem of Pythagoras (fig. 4.72a), as in ancient China (see van der Waerden 1983, 27), leading to:

$$c^2 = (a + b)^2 - 2ab = a^2 + b^2,$$

or as by the Indian Bhâskara II (ca. 1169) (fig. 4.72b), leading to:

$$c^2 = (a - b)^2 + 2ab = a^2 + b^2,$$

a

b

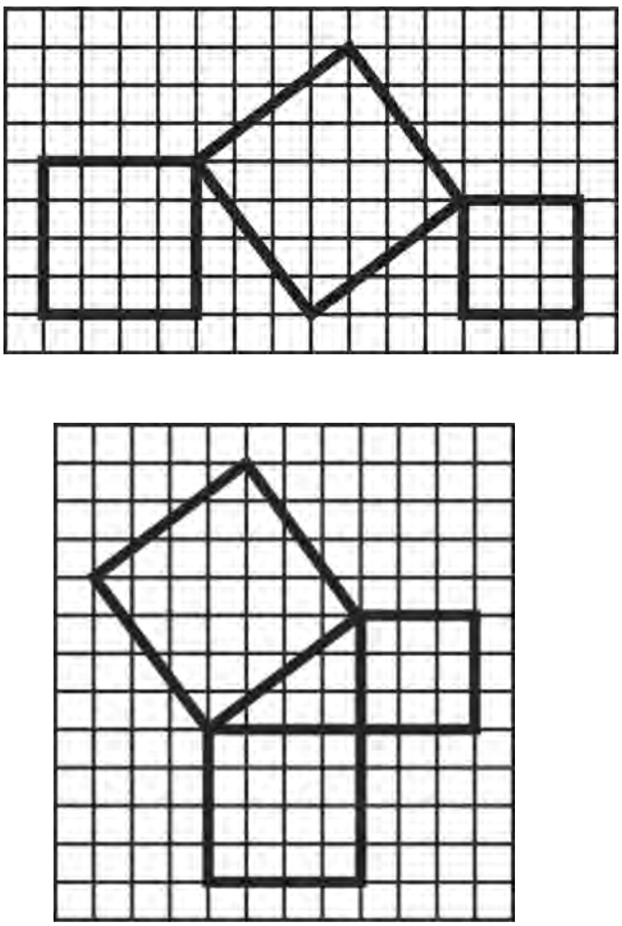


Fig. 4.69. Three squares drawn together in a rectangular grid

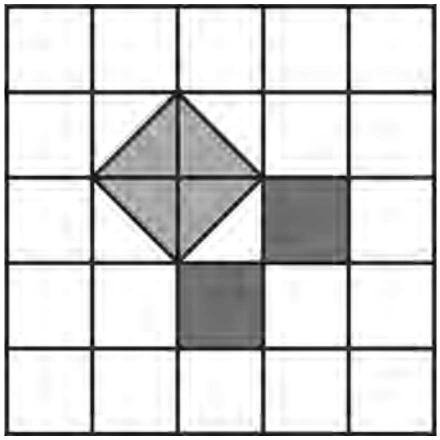


Fig. 4.70. Simplest case

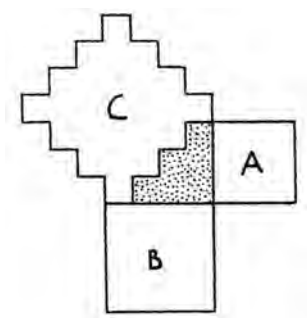
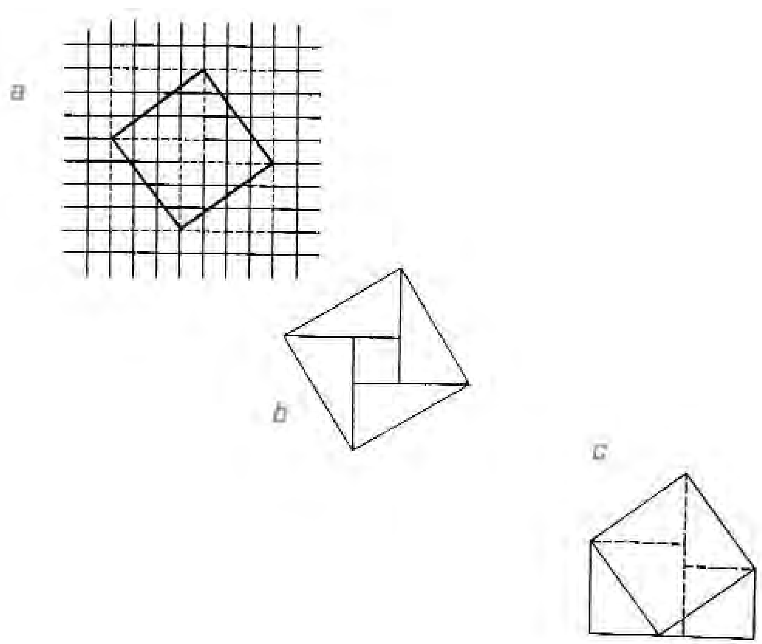


Fig. 4.71. A toothed square with an area equal to the sum of the areas of the two smooth squares: $C = A + B$

or still as the method of Tabit Ibn-Qurra (ca. 83–901), who worked in Baghdad (fig. 4.72c).

On approximations of √2

Not only Pythagorean triplets and the Theorem of Pythagoras could have been discovered in this way, but it could have led also to related considerations—for example, to good approximations of √2, as I should like to show now.



4.72. Designs that suggest ideas for a proof of the Theorem of Pythagoras

Let us return to fig. 4.68b. The diagonal of the smooth square of side 5 units is a little bigger than the diagonal (of length 7 unit squares) of the toothed square. Therefore, the ratio of the length of the diagonal and the length of the side of a smooth square is a little bit bigger than 7/5, that is

$$\sqrt{2} \approx 7/5 = 1.4.$$

Better approximations are found with bigger Pythagorean triplets (a,b,c), where $b = a + 1$. For example, with (20,21,29), one obtains

$$\sqrt{2} \approx (20+21)/29 = 1.41379....$$

The first triple that appears in “Plimpton 322” of ancient Mesopotamia, (119,120,169) (Neugebauer 1969, 37) leads to:

$$\sqrt{2} \approx (119+120)/169 = 1.414201....$$

This approximation value is already rather near the value that can be encountered in a cuneiform text of ancient Babylonia: 1.414213 (Neugebauer 1969, 35).

Retrospect

Taking widespread plaiting patterns as the starting point, I have shown how the factual relationship that today is called Theorem of Pythagoras could have been discovered step by step and *intramathematically* in direct connection with Pythagorean triplets. There were no jumps that could only be explained by extramathematical influences; neither the right angle nor the sum of squares that appears in the Theorem of Pythagoras fell from heaven. Completely superfluous is the supposition of Seidenberg that the motif for the discovery of this theorem has to be sought in the ritual identification of God with a square (1962b, 492, 498). According to Seidenberg, the religious idea of the unification of various gods into one God was the reason for seeking squares that are sums of squares.

Moreover, the possible precursors for the Theorem of Pythagoras described here are so simple (in particular, if one compares them with the level of mathematics of ancient Mesopotamia) that it may be conjectured that they were already known long before 2000 B.C. For the same reason, it seems to me too early to assume, as van der Waerden does, that the knowledge of ancient Mesopotamia, India, China, Greece, and Neolithic Great Britain would be of a common origin.

5. *How did ancient Mesopotamians and Egyptians determine the area of a circle?*

On the ancient Mesopotamian method

A series of clay tablets with mathematical content was found in 1933 in Susa (today Shush, Iran), the ancient capital of Elam, about 200 km east of Babylon. These cuneiform texts date from the end of the First Dynasty of Babylon, ca. 1894–1595 B.C. They have shown that the level of ancient Mesopotamian geometry was higher than the historians of mathematics had previously thought (Bruins and Rutten 1961, 18). Mesopotamian mathematics had really arrived at an understanding not only of the theoretical content of the Theorem of Pythagoras, but also of the connections between areas and perimeters of regular polygons and circles (Wussing, 1979, 41).

The first part of the surviving clay tablets consists of a table with geometric constants. The second row gives 5′ or 5/60 as the constant of a GAM circle[11]—that is, the area of a circle A_c is equal to 1/12 of the square of its perimeter p:

(1) $$A_c = \frac{1}{12}\, p^2$$

The third and fourth row give 20′ = 20/60 or 1/3 as the constant of the GAM-circle diameter d, and 10′ = 10/60 or 1/6 as that of the GAM-circle radius r, that is:

(2) $$d = \frac{1}{3} p,$$

and

(3) $$r = \frac{1}{6} p.$$

For a "real" circle, the relationships (1), (2), and (3) should be replaced by:

(4) $$A_c = \frac{1}{4\pi} p^2,$$

(5) $$d = \frac{1}{\pi} p,$$

and

(6) $$r = \frac{1}{2\pi} p.$$

In other words, in the first three rows, the approximate value $\pi \approx 3$ is used to calculate the area, diameter, and radius of a circle.

As was the case in ancient Egypt, calculations were often done with this approximate value. The scribes from Susa, however, were well aware that this value $\pi \approx 3$ constitutes only a first approximation, as row 30 of the same table points out.

Row 30 gives $57' \ 36'' = 57/60 + 36/60^2 = 24/25$ as a corrective value for the "perfect" SAR-circle,[12] that is, one corrects (1), (2), and (3) in the following way:

(7) $$A_c = \frac{24}{25} \times \frac{1}{12} p^2,$$

(8) $$d = \frac{24}{25} \times \frac{1}{3} p,$$

and

(9) $$r = \frac{24}{25} \times \frac{1}{6} p,$$

Comparison of (8) with (5), for example, immediately yields:

(10) $$\pi \approx 3\frac{1}{8}.$$

The question of how this better approximate value $\pi \approx 3\,{}^1\!/\!_8$ may have been discovered by the Mesopotamian mathematicians (or by their predecessors) is still open.

As possible answers to this question, I shall present two hypotheses. I assume, as a starting point, a mat-weaving technique that was probably known in ancient Mesopotamia (see Forbes 1956, 172 f, and its accompanying fig. 3.133).

First hypothesis

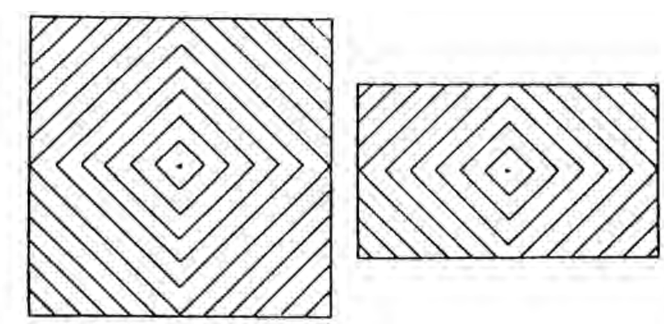
Circular mats may be made by coiling (fig. 3.81). Their diameter is easily countable if one uses the width of the spiral as the unit of measurement. The fifth spiral has a length of 25. When one starts this spiral, the diameter is 7 units; at the end of the fifth spiral, the diameter is 9 units (fig. 4.73). The average diameter is 8 units. In other words, a circle with a diameter of 8 units has a perimeter of approximately 25 units, that is:

(11) $$\pi = \frac{p}{d} \approx \frac{25}{8} = 3\frac{1}{8}.$$

Fig. 4.73. The fifth spiral has a length of 25 units

Second hypothesis

A very old, widespread way of making round baskets was described in chapter 3 (figs. 3.117–19). It may be summarized as follows: the four sides of a square mat are bound fast at their midpoints to a circular border; the weaver presses the mat uniformly inward, cuts off the protruding parts of the mat, and fastens the rest of the bottom to the border. In doing so, the basket weaver often selects the plaiting pattern such that it can be seen immediately whether or not the mat is square and where the midpoints of its sides fall (see the example in fig. 4.74). The depth or height of a basket woven in this way depends on the ratio of the



square mat nonsquare mat

Fig. 4.74. Support function of the weaving pattern in recognizing square mats

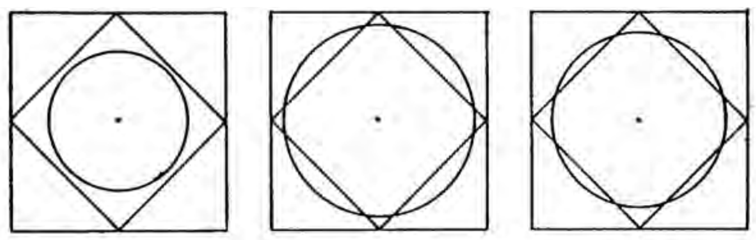


Fig. 4.75. Comparing the diameter of the circular border with the side length of the square mat

diameter of the circular border to the length of the side of the square mat. The basket maker may place a border on the mat before it is fastened (so that their centers coincide) in order to determine the right ratio, which guarantees the desired height of the basket (see the examples in fig. 4.75).

In this context, an artisan[13] could have noted that when the ratio of the radius of the circular mat to half the length of the side of the square mat is equal to 4:5 (under certain conditions of dimensions and plaiting pattern, this proportion is immediately visible, as the example in fig. 4.76a shows), then the areas of the circle A_c and of the smaller visible square A_{ss} that touches the bigger square at the midpoints of its sides are almost equal (fig. 4.76b).

In other words, when

(12) $$r : \frac{s}{2} = 4 : 5,$$

then

(13) $$A_c \approx A_{ss} \,,$$

where s denotes the length of the side of the square mat.

From (12) and the relation

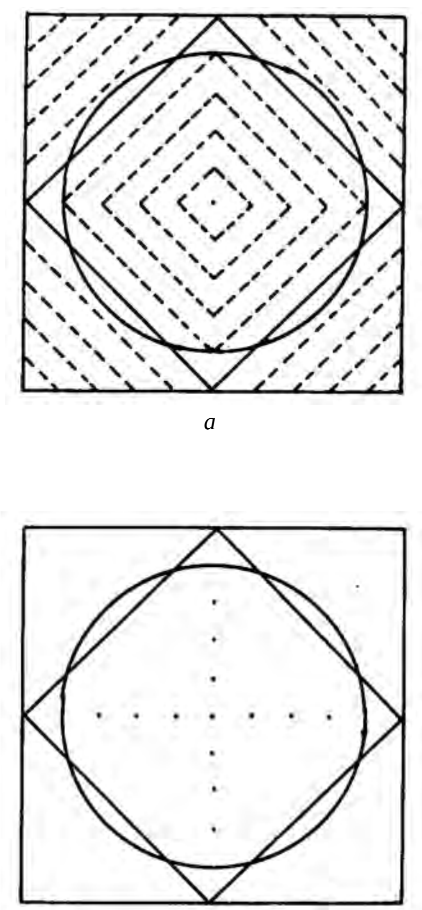
(14) $$A_{ss} = \frac{1}{2} A_{bs} = \frac{1}{2} s^2,$$

(where A_{bs} denotes the area of the bigger square—that is, of the square mat), it is possible to transform (13) into:

(15) $$A_c \approx \frac{1}{2}s^2 = \frac{1}{2}(\frac{5}{4}\times 2r)^2 = \frac{25}{8}r^2 = 3\frac{1}{8}r^2,$$

that is,

(16) $$\pi = \frac{A_c}{r^2} \approx 3\frac{1}{8}.$$



a

b

Fig. 4.76. The ratio of the radius of the circular border to half the length of a side of the square mat is 4:5

The basket weaver could, in this context, also arrive at the following equivalent conclusion. To fabricate the border of the basket, a wooden board is bent, and both its ends are fastened to one another. In doing so, the artisan could have observed that, if the wooden board measures 5/2 times the length of the side of the square mat, the areas of the circle and of the smaller visible square formed by the midpoints of the sides of the square mat are almost equal (see once more fig. 4.76). In other words, if

(17) $$p = \frac{5}{2}s,$$

then

(13) $$A_c \approx A_{ss}.$$

From (14) and (17), one then obtains for the area of a circle

(18) $$A_c \approx \frac{1}{2}s^2 = \frac{1}{2}(\frac{2}{5}p)^2 = \frac{2}{25}p^2,$$

or

(19) $$\pi = \frac{1}{4}\frac{p^2}{A_c} = \frac{1}{4} \times \frac{25}{2} = 3\frac{1}{8}.$$

On the ancient Egyptian method

Problems 48 and 50 of the papyrus having the name of its collector, Scott A. Rhind (1833–1863), give an indication of how areas of circles were calculated in ancient Egypt. The text of problem 50 reads: "*Example of the calculation of a circular field of [diameter] 9 [units]. What is its area? Take away 1/9 of its diameter. The remainder is 8. Multiply 8 by 8. It makes 64. Therefore it contains 64 setat of land*" (see Neugebauer 1931, 422–23; Gillings 1982, 140). In other words, the composer of this papyrus, the scribe Ahmose (ca. 1650 B.C.), determined the length of the side of the square that has approximately the same area as that of the circle to be 8/9 of the diameter of the circle, which implies for the value of π:

$$\pi \approx 4(\frac{8}{9})^2 = \frac{256}{81} = 3.1605.$$

How did the ancient Egyptians find this remarkably close approximation, which is indeed “a great accomplishment” (van der Waerden 1954, 32)?

To answer this question, interesting hypotheses have been formulated by Neugebauer (1931), Bruins (1946), Vogel (1959), Gillings (1982), Eganjan (1977), Hermann Engels (1977), and others.

The conjectures of Neugebauer, Bruins, Vogel, and Gillings are essentially rather similar. The circle is approximated in area by that of a semiregular octagon ABCDEFGH, which is produced on the basis of the circumscribed square by dividing each of its sides into three equal parts, and then by linking the vertices A, B, C, ..., as in fig. 4.77. The area of this octagon is obviously equal to 7/9, or 63/81 of the area of the square. Substituting for 63/81 the approximate value 64/81 or $(8/9)^2$, one is then led to the Egyptian formula. The aforementioned conjectures distinguish themselves in the way they explain the jump from 63 to 64 (from 63/81 to 64/81). For example, Neugebauer explains it as follows. The first correction (fig. 4.77),

$$d^2 - \frac{1}{9}d^2 - \frac{1}{9}d^2 = \frac{7}{9}d^2,$$

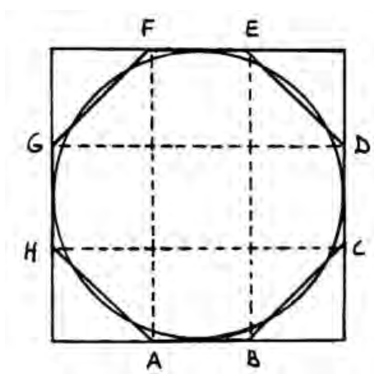


Fig. 4.77. Neugebauer's hypothesis

is too crude, as might have been verified, for example, in the case of the volume of a cylinder. But a mere formal iteration of the first step, by $(1/9)d^2$, leads immediately to the very good result:

$$(d^2 - \frac{1}{9}d^2) - \frac{1}{9}(d^2 - \frac{1}{9}d^2) = \frac{64}{81}d^2.$$

"Such a preference for formal repetitions is witnessed by the whole of Egyptian calculation procedures" (Neugebauer 1931, 429).

To Hermann Engels, the conjecture of the semiregular octagon seems too complicated (1977, 137). He suggests the following alternative. One knows that it was usual in Egyptian craft work to carry over drawings in a fixed ratio from one medium to another by means of a net of squares. What may be said about circles that arise in these nets of squares? The circle in fig. 4.78 has,

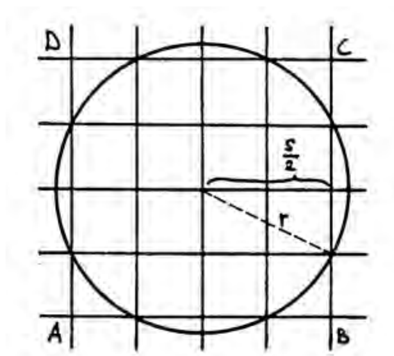


Fig. 4.78. Hermann Engels's hypothesis

intuitively, the same area as the square ABCD. By dividing every square into 16 equal subsquares, the Egyptian artisans could have observed that

$$r \approx \frac{9}{8} \times \frac{s}{2},$$

where r denotes the circle radius and s the length of the side of square ABCD. Then the following is valid for the area of the circle:

$$A_c \approx s^2 \approx (\frac{8}{9} \times 2r)^2 = (\frac{8}{9} \times d)^2.$$

The conjecture of Hermann Engels has the advantage that the derivation of the formula starts from a handwork technique known at that time. The explanation by Neugebauer and others finds support in the rough sketch accompanying problem 48 of the Rhind Papyrus that may be interpreted as a square with an inscribed octagon. I shall now propose another interpretation of this sketch that enables one to derive the ancient Egyptian formula without any jump from 63 to 64. In this proposal, Hermann Engels's reference to the square-net method will prove itself rather useful.

First hypothesis

The octagon, only roughly sketched by Ahmose, could correspond to a crenate or toothed figure like the one in fig. 4.79. Examples of this sort have already been met in this chapter and in chapter 3.

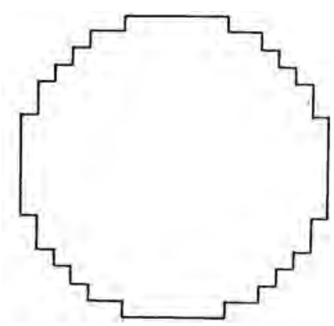


Fig. 4.79. A toothed "octagonal" figure

Now let us imagine a circle drawn into a square net as in fig. 4.80. The toothed figure (fig. 4.81) composed of all squares of the grid that lie completely or mostly inside the circle has approximately the same area as the circle. This last toothed design (fig. 4.81) coincides with the example in fig. 4.79. By means of a

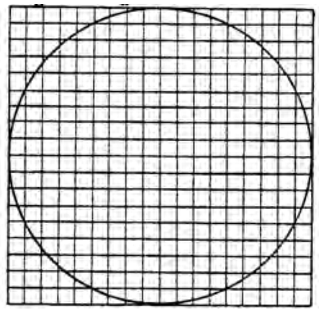


Fig. 4.80. A circle drawn in a grid; its diameter measures 18 units

simple geometric transformation (fig. 4.82), one sees that this toothed figure has the same area as a new square (fig. 4.83). The length of the side of this new square is 1/9 shorter than the diameter of the circle. Therefore it may be concluded:

$$\begin{aligned} \text{area of the circle} &\approx \text{area of the toothed figure} \\ &= \text{area of the new square} \\ &= (d - \frac{d}{9})^2. \end{aligned}$$

Second hypothesis[14]

The first hypothesis does not constitute the only possibility of relating the derivation of the ancient Egyptian formula for the area of a circle to production techniques. Just as in the case of the

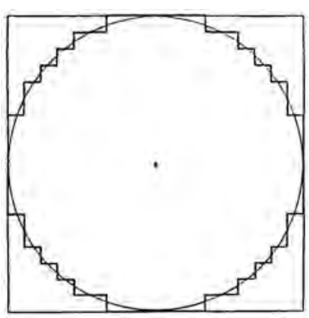


Fig. 4.81. The circle and the toothed figure drawn together

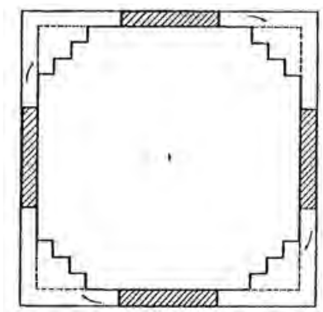


Fig. 4.82. Geometric transformation of the toothed “octagonal” figure

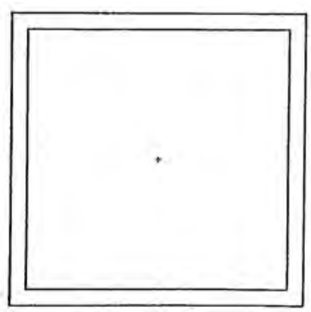


Fig. 4.83. Old and new squares

ancient Mesopotamian formula, further connections may be traced in the fabrication of circular mats.

The coiling technique was used in ancient Egypt for the making of circular mats (see Schmidl 1928). A string is rolled up around a fixed point and then sewn into successive spirals (fig. 4.84). Normally, a small piece of the end will be cut off in order to give the impression of an exact circle (fig. 4.85). The string can be considered a rectangle, whose width is taken as the unit of measurement. When the length L of the string is equal to a square n^2, the exposed surface area of the string is equal to that of a square with side n; it may be assumed that coiling the string does not essentially change its area. Now the diameter d of the spiral

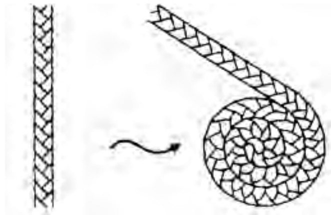


Fig. 4.84. Coiled rope

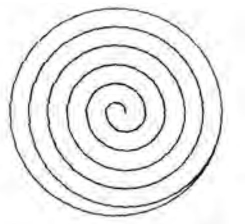


Fig. 4.85. "Spiral-circle"

circle can be easily counted. For $n = 8$, one gets *experimentally* $d = 9$. Therefore, the area of a circle of diameter $d = 9$ is approximately equal to a square of side $n = 8$. In other words, one finds:

$$\text{area of the circle} \approx L = n^2 = \frac{8}{9}\, d^2 = (d - \frac{1}{9} d)^2,$$

which is exactly the ancient Egyptian formula for the area of a circle.

NOTES

1. In his 1981 paper "The Ritual Origin of the Circle and Square," Seidenberg summarizes his view as follows: "In the Creation ritual the participants brought various objects onto the ritual scene and were identified with these objects. In elaboration of the ritual these objects, and in particular stars, were studied. The participants identified with stars moved in imitation of the stars, thereby giving the ritual scene a circular shape: this is the origin of the circle. The circle was bisected by the two sides of a dual organization taking up the two sides of a circular ritual scene. The two sides split, giving rise to quadrants. Then representatives of the four sections placed themselves about the center of the circle in positions corresponding to the positions of the four

sections, thereby giving rise to the square. The square was valued as a figure dual to the circle. Both the circle and the square are offspring of the ancient Creation ritual complex" (324). Cf. discussion here in chap. 3, sect. 8, on squares, circles, and quadrants in basket weaving.

2. See, for example, Wilder 1981, 48f.

3. Walter R. Fuchs formulates another hypothesis: "The construction of pyramids was closely related to the belief in afterlife and influenced by the cult of the sun. To arrive at heaven, the god-human Egyptian king had to go up enormous stairs or to use the sunrays. . . . The sunrays, when seen as they break through a thick layer of clouds, may be imitated by the geometric pure shape of the pyramid, as we see it in the Gizeh monument" (1976, 76–77). Does this explain the square base of the pyramid? Why not a circular base?

4. See also Watson 1987.

5. In this way, 4, 10, and 20 are *tetrahedron numbers*. Does herein lie another reason for the fact that so many number systems use 10 or 20 as a base?

6. If one fastens a circular border to the "dish," one obtains a conical object that may serve as a funnel or as a hat, as on the islands of Sumatra, Java, and Borneo (Indonesia), and in China.

7. It does not seem very likely that the regular octahedron would have first been discovered so late in human cultural history—-by Theaetetus of Athens (ca. 415–368 B.C.), as a scholium in Book XIII of Euclid's *Elements* wants us to believe (see van der Waerden 1954, 100, 173).

8. Cf. Schmidl 1928, in particular, photo 8 and table 2.

9. Cf. Bastin 1961, 116; Farrand 1900, 397; Barrett 1908, 199. In Japan, it appears as a traditional family crest (Adachi 1972, 12–13); in Nigeria and ancient Mexico, as a textile decoration (Picton and Mack 1979, 35, 75; Weitlanger-Johnson 1976, 63–64), etc.

10. A straight line falling perpendicularly on another straight line or surface—Ed.

11. See note 12.

12. GAM and SAR refer to different measurement systems found on the tables and used in the calculations.

13. Or a scribe, for example, when analyzing how this type of basket could be standardized.

14. I also formulate other hypotheses in Gerdes 1985.

Chapter Five
Conclusion: Awakening of Geometrical Thought

1. Methodology

The study of the awakening of geometrical thought is a relatively new field of research and demands the development of adequate methods. Almost no written sources exist, and oral tradition may give only partial answers to questions about early geometrical knowledge. Presented in this book are some elements for the elaboration of the methodology of research on the subject. According to a principal element of this methodology, the researcher studies first of all the usual production techniques (for example, weaving) of traditional labor products such as mats, baskets, traps, etc., and at each stage of the fabrication process asks which aspects of a geometrical nature play a role in arriving at the next stage. This methodological starting point proves to be fertile because readily hidden, "frozen" geometrical thought may be encountered.

The unfreezing of geometrical thought hidden or frozen in old and widespread techniques like those of basketry makes it possible to reflect on the early history of geometry. In this respect, the study presented here shows that the aspect of activity has been considered too little in the study of the origin of basic mathematical concepts.

2. Activity and the awakening of geometrical thought

The multiplicity of forms in nature is so large that it becomes necessary to explain how humans gradually acquired the ability to perceive certain forms in nature. There are no forms in nature that are a priori conducive to human observation. The capacity of human beings to recognize geometrical forms in nature and also in their own products was formed in activity.

The capacity to recognize order and regular spatial forms in nature has been developed through labor activity. *Regularity*—as has been shown by a series of examples in this book, such as the ordered up-and-down alternation of the three-strand braid—is the *result* of human creative labor and not its presupposition. It is the real, practical advantages of the invented regular form that lead to the growing awareness of order and regularity. The same advantages stimulate comparison with other products of labor and with natural phenomena. The regularity of the labor product simplifies its reproduction, and in that way the consciousness of its form and the interest in it become reinforced. With the growing awareness develops simultaneously a positive valuing of the discovered form: the form is then also used where it is *not necessary*; it is considered *beautiful*.

The cylinder, cone, or other symmetrical shapes of vessels, the hexagonal patterns of baskets, hats, and snowshoes, etc., may at first sight *appear* to be the result of instinctive impulses or of an innate feeling for these forms or—in another idealist variant—as generated by a collective "cultural spirit" or "archetype," or also, mechanically, as an imitation of natural phenomena—for example, of crystal structure or of honeycombs. In fact, however, humans *create* these forms in their practical activity to be able to satisfy their daily needs. They e*labor*ate them. The *understanding* of these materially necessary forms emerges and develops further through interaction with the given material in order to be able to produce something useful: bows, boats, hand axes, baskets, pots, etc. From the *recognition of these necessities* and of the thereby acquired possibilities of employing them to achieve certain aims emerged the human *freedom* to make things that are useful and considered beautiful.

Social activity plays an important role in the formation and development of early geometrical notions. We should not forget, however, that it assumes this role in diverse and rather different ways. No individual suddenly—that is without preconditions—

arrived at the conclusion that hand axes should be symmetrical. On the contrary. The symmetrical shape of the hand ax is the result of long historical development. The practical necessity of using a right angle, for example, to guarantee stability, was and continues to be "felt," that is, continues to be discovered daily by isolated individuals or by people working together. The discovery of the necessity of an angle of 60º to fasten the border of a basket firmly was the result of individual or collective experience and reflection. In this last case, the consciousness of the resulting form—a hexagonal pattern—remains probably more accentuated in the process of invention than in the other cases. More research is needed on how these different discovery processes entered (and enter) into the sociocultural knowledge of geometry.

Possibilities of intramathematical development

With the reflection in art and games of shapes formed in activity, early mathematical thinking started to liberate itself from material necessity: form becomes emancipated from matter, and thus emerges the *concept* and *understanding* of form; the way is made *free* for intramathematical development.

In the interplay of the needs important to a society, material possibilities, and experimental activity, certain shapes—for example, symmetrical forms—proved themselves to be optimal. Thinking in terms of order and symmetry does not need a mythical explanation. It reflects the societal experience of production. Once this experience has established itself to the degree that the regularity has acquired an aesthetic value, then also *new* and, in a certain sense, ordered shapes could be created without an immediate, inescapable material compulsion existing for them. In this process, the early geometrical thinking develops further—that is, the capacity to create *thinkable or imaginable* forms.

Possible sediment of magical thinking in the imagination of space

The emergence from human activity of certain basic forms—for example, the circle and the square—could have been "forgotten" in the course of history. It could be unknown not only to artisans—who imitate the materially necessary forms without being aware of why they are necessary—but also to those who remain outside the reproduction process. With the disappearance of the knowledge of the origin of these basic forms, there may arise magical or religious thinking—for example, the coupling of God and the square as a sediment in imagination and representation of space.

Relative uniformity of ideal structures

Often the relative uniformity of ideal structures is not really explained: one seeks refuge in God, in an objective world of ideas independent of human beings (Platonism) or in a general diffusion from a unique place of discovery. This study shows, however, that with respect to the awakening of geometrical thought, this relative uniformity of ideal structures reflects the unity of humankind, or more accurately, the unity of nature: equal situations led generally to equal problems with similar attempts to solve them, although possibly widely differing in detail. Corresponding societal activity, together with the general human constitution, enabled the elaboration of equal basic forms.

3. New hypotheses on the history of ancient geometry

On the basis of widespread early geometrical shapes and constructions, new hypotheses on the history of ancient geometry can be formulated. For instance, there could have been practical reasons for the significance of Thales' basic figures. The importance of the construction of rectangles in the daily life of many peoples—for example, for the building of houses—could have been reflected in Greek geometry. In this study I showed, for example, how one may derive in an easy way—more easily than in the hypotheses formulated so far—the correct ancient Egyptian formula for the volume of a truncated pyramid with a square base. In this derivation, one starts from certain early material products of labor and their empirically discovered mathematical relationships, and each next step in the thinking process is constructive in the sense, that it results, without any deviation, from the search for possible answers to still unsolved questions. Perhaps in this respect the most surprising result of this investigation lies in the explanation of how, starting from widespread weaving patterns, step by step, intramathematically, the relationship that today is called the Theorem of Pythagoras, with its direct connection with the Pythagorean numerical triplets, could have been discovered. The possibility of formulating this type of hypothesis reinforces the thesis about the unity of humankind with respect to the awakening of geometrical thought.

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