The Tropical Geometry of Albouy-Chenciner Systems

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Abstract

The frequently studied n-body and n-vortex problems share a common structure, as they can both be taken as specific cases of the Albouy-Chenciner system. By studying this polynomial system, we can gain understanding of the whole class of systems. By exploiting its underlying tropical geometry, we can show the finiteness of the equilibrium solutions to these class of systems on 3 masses. We can also explore many basic properties of the solutions to these problems. Lastly it allows for further analysis of both the n-body and n-vortex problems.

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1 Background

1.1 Groebner Bases

One of the most commonly used tools used in the field of commutative algebraic systems is the Groebner basis. The Groebner basis of the ideal of a polynomial ring is a basis for that ideal that has certain "nice" properties, specifically in terms of the multivariate division algorithm. For the purposes we will interpret $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ so both $\mathbf{x}$ and α are vectors.

Definition: Given a monomial ordering, then a finite subset $G = \{g_1, g_2, \dots, g_t\}$ of an ideal I is said to be a **Groebner basis** if $\langle LT(g_1), LT(g_2), \dots, LT(g_t)\rangle = \langle LT(I)\rangle$, (where *LT(g)* is the leading term of g with regard to the given monomial ordering and $LT(I) = \{c\mathbf{x}^\alpha | \, there\ exists\ f \in I\ with\ LT(f) = c\mathbf{x}^\alpha\}$) [1].

Or equivalently,

Corollary: Given a monomial ordering, then a finite subset $G = \{g_1, g_2, \dots, g_t\}$ of an ideal *I*, then G is a **Groebner basis** if for every nonzero $f \in I$, $LT(I)$ is divisible by $LT(g_i)$ [2].

There are numerous orderings that can be applied to an ideal, and a few of the more important orderings are the lexicographic ordering and the degree reverse lexicographic ordering. Lexicographic order can be viewed as an a cousin to the alphabetic order you would find in a dictionary. In this ordering, $\mathbf{x}^\alpha >_{lex} \mathbf{x}^\beta$ if and only if the leftmost term non-trivial term in $(\alpha_1 - \beta_1, \alpha_2 - \beta_2, \dots, \alpha_n - \beta_n)$is positive [1]. It should be obvious that there is not a unique choice for a lexicographic order for a polynomial ring, since the order of the terms x_i in $(x_1, x_2, \dots, x_n) = \mathbf{x}$ is arbitrary. In fact there are $n!$ different lexicographic orderings possible for a polynomial ring in n variables.

Under the degree reverse lexicographic order, $\mathbf{x}^\alpha >_{degrevlex} \mathbf{x}^\beta$ if and only if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the rightmost nonzero entry of $(\alpha_1 - \beta_1, \alpha_2 - \beta_2, \dots, \alpha_n - \beta_n)$ is negative[1]. That is, terms are ordered in a descending fashion based mainly on the total degree, the sum of the degrees of all the variables in a monomial.

The lexicographic ordering is certainly the most intuitive orderings for anyone who attended elementary school. A lexicographically ordered Groebner basis eliminate variables from the basis. That is useful in studying the zeros of a system of polynomials. However, lexicographic basis can be quite computationally intense to compute and can be extremely long.

While unintuitive, the degree reverse lexicographic ordering is much friendly computationally and is generally more compact. A common method for studying polynomial systems is to compute a degree reverse lexicographic basis for the system, and then compute a lexicographic basis from that.

One final observation is that the for a given term ordering, the minimum Groebner basis of any ideal is unique[12].

1.2 Initial Form

Definition: The **initial form of a polynomial** $P = \sum c_\alpha \mathbf{x}^\alpha$, with respect to a *weight vector* ω (a vector of exponents of $\mathbf{x}$), denoted as $\text{in}_\omega(P)$, is the sum of all terms $c_\alpha \mathbf{x}^\alpha$ where $\alpha \cdot \omega$ is maximized [12]. An initial form is often called the reduced form. Frequently the initial form will consist of a single term, but that is not necessarily the case. For example, if $P = x + y = x^1y^0 + x^0y^1$, then $\text{in}_{(1,1)}(P) = P$.

Definition: The the **initial form of an ideal** I with respect to ω, is $\text{in}_\omega(I) = \langle \text{in}_\omega(P) \mid P \in I \rangle$. That is, it's the ideal generated by the initial forms with respect to ω of all the elements in the ideal I [12].

For this thesis we assume that these ω vectors are outward pointing (some authors assume inward pointing vectors). The initial form with respect to an inward pointing vector ω is the minimal terms with respect to $\mathbf{x}^\omega$.

1.3 Puiseux Series

Definition: A **Puiseux Series** is a formal power series of the form $\sum_{n=m}^{\infty} c_n t^{n/k}$ for $m, k \in \mathbb{Z}$ and $k \geq 1$ [8].

For a given polynomial system, any algebraic curve solution can be expanded as a Puiseux series, where the lead term is the solution to the initial form of the system with respect to some vector. This method was first described by Isaac Newton, and was completed by V. Puiseux [9]. Therefore, in order for an algebraic curve of solutions to a system of polynomials with respect to a weight vector ω, the initial form of the system must have a nontrivial solution for every polynomial.

1.4 Newton Polytope

The convex hull of a collection of points in $\mathbb{R}^n$ is the minimum convex set containing those points. The convex hull of a finite set of points is called a *polytope*[2].

Definition: The **Newton polytope** of an n-variable polynomial $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha \mathbf{x}^\alpha$, is the polytope in $\mathbb{R}^n$ formed by $\{\alpha \ : \ c_\alpha \neq 0\}$. Thus the Newton polytope describes the sparsity, or shape of the polynomial [2]. Note that two polynomials that differ only in the value of the coefficients will share the same Newton polytope.

Three dimensional polytopes consist of vertices, edges and faces. Edges are line segments connecting vertices on the boundary of the polytope. The normal cone of one of these faces is the normal vector for the supporting hyperplane of that face [2].

1.5 Tropical Varieties

Definition: The **tropical variety** of a **polynomial** f is the collection of all the normal cones of its Newton polytope [2]. Alternatively, it can also be defined as a collection of weight vectors $T(f) = \{\omega \ : \ \mathrm{in}_\omega(f) \textit{ is not a monomial}\}$ [23].

The **tropical variety** of a **polynomial ideal** $\mathcal{P}$ is the set of weight vectors

$T(\mathcal{P}) = \{\omega \mid in_\omega(\mathcal{P})\ is\ monomial - free\}$. That is, none of the initial forms of the members if $\mathcal{P}$, with regard to these vectors does not contain a monomial [23]. Equivaliently we can view it as the the intersection of the the intersection of the tropical varieties of each polynomial in $\mathcal{P}$. In either case, this is not computationaly feasible to work with since we are either taking the initial form of an infinite set of polynomials or taking an inifinte intersection. Computers have difficulty performing either tasks.

The tropical prevariety for a polynomial ideal $\mathcal{P}$ can be viewed as either the intersection of the Tropical Varieties of the generating set $\{f_1, f_2, \dots\}$ or as a collection of weight vectors $T(\mathcal{P}) = \{\omega\ :\ \text{in}_\omega(f_i)\ is\ monomial - free\ for\ all\ f_i\}$ Where again f_i is in the generating set of $\mathcal{P}$[2, 23]. Since we are operating on the generating sets and not the whole ideal, we often times only need to work with a finite collection of polynomials.

Since any solution curve must have a Puiseux expansion and any non-trivial Puiseux expansion must arise from a non-monomial initial form, the only possible curves of solutions must arise from one of the vectors in the Tropical Variety [5]. Furthermore, from the *balancing condition* we only need to consider vectors ω in a half-space in $\mathbb{R}^n$ [4].

1.6 BKK Theory

Thus far we have established the necessary conditions for the existence of a curve of solutions to a set of polynomials. Now to discuss the isolated solutions to a system.

Let $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ be a collection of polytopes. The n-dimensional *Mixed Volume* $MV_n(\mathcal{P})$ is the coefficient of the monomial $\lambda_1\lambda_2\dots\lambda_n$ in $Vol_n(\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_n P_n)$. Equivalently $MV_n(\mathcal{P}) = \sum_{k=1}^{n}(-1)^{n-k} \sum_{\substack{I\subset\{I,\dots,n\} \\ |I|=k}} Vol_n\left(\sum_{i\in I} P_i\right)$, where $\sum_{i\in I} P_i$ is the Minkowski sum of polytopes. [2].

So for a collection of 3 polytopes, the mixed volume would be $MV(\mathcal{P}) = Vol(P_{123}) -$

$Vol(P_{12}) - Vol(P_{13}) - Vol(P_{23}) + Vol(P_1) + Vol(P_2) + Vol(P_3)$ where$P_{ij} = P_i + P_j$.

There are 3 widely used bounds on the number of solutions grouped together as the BKK Theory (for Bernstein, Kushnirenko, and Khovanskii). The Bernstein Bound states that the total number of isolated, nonzero solutions is bounded by the Mixed Volume of the system (nonzero in that none of the variables can be 0). Moreover for sufficiently generic choices for the coefficients the number of solutions is exactly the Mixed Volume [2]. The Bernstein bound is often referred to as the BKK bound.

The Khovanskii Bound states that for a system of d polynomials in d variables involving n distinct monomials in total, the number of isolated roots in the Positive Orthant ($\mathbb{R}^d_+$) is at most $2^{\binom{n}{2}} \cdot (d+1)^n$ [3].

1.7 Syzygy

Definition: for a polynomial system $\mathcal{P} = (P_1, P_2, \ldots, P_s)$. A **syzygy** on the leading terms $LT(P_1)$, $LT(P_2)$, ... , $LT(P_s)$ of $\mathcal{P}$ *is an s-tuple of polynomials* $S = (h_1, h_2, \ldots, h_s)$ such that $\sum_{i=1}^{s} h_i \cdot LT(P_i) = 0$ [1].

Thus the syzygy are the coefficients for an homogenous combination of the Leading terms for the polynomial system. If we apply the same coefficients in a combination of the entire polynomial, that is $\sum_{i=1}^{s} h_i \cdot LT(P_i)$, it is said we **lift** the polynomial system by the syzygy [1]. Lifting a system in this manner eliminates the original leading terms from all the polynomials. We can use this lifted system to find a generating set with an smaller, easier to compute tropical prevariety.

1.8 Generalized n-Body/n-Vortex problem

The Newtonian n-body problem describes the dynamics of n point particles with masses $m_i > 0$ and positions $x_i \in \mathbb{R}^m$ moving according to Newton's law of gravitation and laws of motion, so the force acting on the ith particle is: [4]

$$m_i\ddot{\mathbf{x}}_i = \sum_{j\neq i}\frac{m_i m_j(\mathbf{x_j}-\mathbf{x_i})}{|\mathbf{x_j}-\mathbf{x_i}|^3} \quad 1\leq i\leq n$$

This problem, which was first formulated in Newton's *Principia*, arises from celestial mechanics, the motion of planetary bodies through space. An important class of solutions are the *central configurations*, where the acceleration of each body is directly proportional to it's displacement from the center of mass c. This condition gives rise to solutions that remain stationary when viewed in an appropriately rotating reference frame. The equation of state for this system is[4]

$$\ddot{\mathbf{x}}_i = \lambda(\mathbf{x}_i-\mathbf{c}) = \sum_{j\neq i}\frac{m_j(\mathbf{x_j}-\mathbf{x_i})}{|\mathbf{x_j}-\mathbf{x_i}|^3} \quad 1\leq i\leq n$$

A closely related problem is the n-vortex problem, which relates to the dynamics of n planar point vortices in a zero viscosity, incompressible fluid. The system has nonzero vortex strengths Γ_i and positions $z_i \in \mathbb{C}$ which move according to the equation: [6]

$$\mathbf{i}\dot{z}_i = \sum_{j\neq i}\frac{\Gamma_j(z_j-z_i)}{|z_j-z_i|^2} \quad 1\leq i\leq n$$

The dynamics of this system are much different from the n-body problem, but there is a clear commonality in their structure. This is clearer when we consider the central configurations of the n-vortex system. That is the velocity is proportional to the displacement from the "center of vorticity" c. This leads to the equation

$$\lambda(z_i-c) = \sum_{j\neq i}\frac{\Gamma_j(z_j-z_i)}{|z_j-z_i|^2} \quad 1\leq i\leq n$$

It should be clear that these two systems are virtually identical. If we extend the variables in both cases to be of the form $\mathbf{x_i}\in\mathbb{C}^{\mathbf{d}}$, the only remaining meaningful difference is the exponent on the denominator. This leads to a natural Generalization for these problems, for *central configuration*:

$$\lambda(x_i - c) = \sum_{j \neq i} \frac{m_i(\mathbf{x}_j - \mathbf{x_i})}{|\mathbf{x_j} - \mathbf{x_i}|^d} \quad 1 \leq i \leq n, \ d \in \mathbb{N}, \ \mathbf{x}_i \in \mathbb{C}^D$$

This system is the parent of what we will be addressing in this thesis. However, we first need to perform some algebraic manipulation of the system in order to cast this the system as a polynomial in the distances between the various masses. For this paper, we will take the convention of calling the coefficient parameters for these systems as masses except when dealing specifically with the the classic n-vortex case (where $d = 2$), where these coefficients are vorticity. However, aside from the physical meanings in the classic systems the concept of mass and vorticity are equivalent.

1.9 Albouy-Chenciner Polynomials

The Albouy-Chenciner polynomial casts this system for central configurations as a polynomial system, in terms of the mutual distances between the points r_{ij} and the variable $S_{ji} = \frac{1}{r_{ij}^d} - 1$. Then the resulting equations are [4].

$$\sum_{k=1}^{n} m_k \left[S_{ik}(r_{jk}^2 - r_{ik}^2 - r_{ij}^2) + S_{jk}(r_{ik}^2 - r_{jk}^2 - r_{ij}^2) \right] = 0 \quad for \ 1 \leq i \leq j \leq n$$

In order to clear the denominator, we will multiply each expression by the product $\prod_{i=1}^{n} \prod_{j=1}^{n} r_{ij}^d$.

Since the Albouy-Chenciner polynomials arise from the distances between the masses in the generalized n-mass problem, the only solutions that are physically meaningful for the n-mass problem are where all the r_{ij} values are positive real numbers.

1.10 Polynomial Homotopy Continuation (PHC)

The Polynomial Homotopy Continuation method is a numerical polynomial equation system solver that exploits the polynomial structure of the system to efficiently find all isolated solutions for a polynomial system. We will be using the method built in PHCPack, implemented in Sage. It starts by computing the BKK bound for the system, and then constructing a polynomial system with exactly that many known, isolated, nonzero solutions. This known system is then deformed by a *homotopy* into the target system. The original solutions are tracked as they converge, hopefully, into the solutions of the target system [20].

2 Finiteness of the Central Configurations of the Generalized Albouy-Chenciner problem

It will be shown that there are only finitely many central configurations in the generalized 3-mass Albouy-Chenciner system. This problem is described by the generalized Albouy-Chenciner polynomials given below [4].

$$0 = -2(r_{12}r_{13}r_{23})^d(m_1+m_2+m_3) + 2(r_{12}r_{13})^d(m_2+m_3)$$
$$+m_1(r_{23}^{d-2}r_{12}^dr_{13}^2 + r_{23}^{d-2}r_{12}^2r_{13}^d + r_{12}^dr_{23}^d + r_{13}^dr_{23}^d - r_{23}^{d-2}r_{12}^{d+2} - r_{23}^{d-2}r_{13}^{d+2})$$

$$0 = -2(r_{12}r_{13}r_{23})^d(m_1+m_2+m_3) + 2(r_{12}r_{23})^d(m_1+m_3)$$
$$+m_2(r_{13}^dr_{12}^d + r_{13}^{d-2}r_{12}^dr_{23}^2 + r_{13}^{d-2}r_{12}^2r_{23}^d + r_{13}^dr_{23}^d - r_{13}^{d-2}r_{12}^{d+2} - r_{23}^{d+2}r_{13}^{d-2})$$

$$0 = -2(r_{12}r_{13}r_{23})^d(m_1+m_2+m_3) + 2(r_{13}r_{23})^d(m_1+m_2)$$
$$+m_3(r_{12}^dr_{13}^d + r_{12}^{d-2}r_{13}^dr_{23}^2 + r_{12}^{d-2}r_{13}^2r_{23}^d + r_{12}^dr_{23}^d - r_{12}^{d-2}r_{13}^{d+2} - r_{12}^{d-2}r_{23}^{d+2})$$

Here is a rough sketch of the proof. By application of Bernstein's theorem, the number of isolated roots of this system must be finite [2]. Thus if the solution set for this system were infinite, there must exist an algebraic curve of solutions. This curve can be expanded in a Puiseux series, that is a series where each variable can be given by a series $\sum_{i=i_0}^{\infty} a_i t^{\frac{i}{q}}$ [2]. The leading coefficient of these solutions must be the solution to the reduced form of the Albouy-Chenciner polynomials with respect to one of the cones for the tropical prevariety for the system [4]. To show that no Puiseux series exists, it is sufficient to show that these reduced forms will be monomials. Since there are no Puiseux series solutions, there is no algebraic curve of solutions [4].

We will be generally assuming that none of the masses or the distances equals zero, and that the total mass of the system is non-zero [4]. Additionally, since if any pair of masses sum to zero, one of the terms in one of the above polynomials will be eliminated changing the underlining geometry of the system.

Lemma 1: *There are finitely many isolated equilibrium solutions to the Albouy - Chenciner system (Bernstein's theorem).*

This is a basic observation from Bernstein's Theorem [2]. Thus if there were an infinite number of solutions, there must be an algebraic curve of solutions. Any algebraic curve can be decomposed as Puiseux series for each of the variables [2]. If such an algebraic curve solution exists, we can use the following algorithm to generate it's Puiseux series expansion [9].

For the general case (that is where no pair of masses sum to 0), the outward pointing rays of the tropical prevariety are $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ (as well as the ray $(-1,-1,-1)$, but from the balancing argument we can ignore this ray) . The initial forms of the polynomials, with respect for each of the following are [18]

$$2r_{13}^d r_{23}^d \left((m_1+m_2)-(m_1+m_2+m_3)r_{12}^d\right)$$
$$r_{13}^{d-2} r_{23}^d \left(m_2(r_{13}^2-r_{23}^2)-2r_{12}^d r_{13}^2(m_1+m_2+m_3)\right)$$
$$r_{13}^d r_{23}^{d-2} \left(m_1(r_{23}^2-r_{13}^2)-2r_{12}^d r_{23}^2(m_1+m_2+m_3)\right)$$

$$r_{12}^{d-2} r_{23}^d \left(m_3(r_{12}^2-r_{23}^2)-2r_{12}^2 r_{13}^d(m_1+m_2+m_3)\right)$$
$$2r_{12}^d r_{23}^d \left((m_1+m_3)-(m_1+m_2+m_3)r_{13}^d\right)$$
$$r_{12}^d r_{23}^{d-2} \left(m_1(r_{23}^2-r_{12}^2)-2r_{13}^d r_{23}^2(m_1+m_2+m_3)\right)$$

$$r_{12}^{d-2}r_{13}^{d}\left(m_3(r_{12}^2-r_{13}^2)-2r_{12}^2r_{23}^d(m_1+m_2+m_3)\right)$$
$$r_{13}^{d-2}r_{12}^{d}\left(m_2(r_{13}^2-r_{12}^2)-2r_{13}^2r_{23}^d(m_1+m_2+m_3)\right)$$
$$2r_{13}^{d}r_{12}^{d}\left((m_2+m_3)-r_{23}^d(m_1+m_2+m_3)\right)$$

The three initial forms are invariant under permutation of indices, so we need only show that one of these sets of polynomials has a finite number of roots. This is due to the fact that the leading term of any Puisuex series solution must come from one of these initial forms [9].

Without lack of generality, we will explore the first set of polynomials and factor out any monomial factors that appear, leaving.

$$(m_1+m_2)-(m_1+m_2+m_3)r_{12}^d=0$$
$$m_2(r_{13}^2-r_{23}^2)-2r_{12}^dr_{13}^2(m_1+m_2+m_3)=0$$
$$m_1(r_{23}^2-r_{13}^2)-2r_{12}^dr_{23}^2(m_1+m_2+m_3)=0$$

From the first equation, $(m_1+m_2)=(m_1+m_2+m_3)r_{12}^d$. Substituting this into the second and third equations, returns the new equations.

$$m_2(r_{13}^2-r_{23}^2)-2r_{13}^2(m_1+m_2)=-m_1r_{13}^2-m_2r_{23}^2-(m_1+m_2)r_{13}^2=0$$
$$m_1(r_{23}^2-r_{13}^2)-2r_{23}^2(m_1+m_2)=-m_1r_{13}^2-m_2r_{23}^2-(m_1+m_2)r_{23}^2=0$$

So it is clear that $-m_1r_{13}^2-m_2r_{23}^2=(m_1+m_2)r_{13}^2=(m_1+m_2)r_{23}^2$, or $r_{13}^2=r_{23}^2$.

Substituting this identity into the second equation we get $-2(m_1+m_2)r_{13}^2=0$. Thus implying that $m_1+m_2=0$, which violates the stated conditions for the general case addressed here. Therefore the only possible nontrivial solutions to the initial forms induced by the tropical prevariety for the general system actually exist in one of the 2 special cases addressed below.

First Special Case: We will now consider the system where one and only one pair of masses cancel out. Without loss of generality we will take $m_2 = 1$, $m_3 = -1$, and $m_1 \neq \pm 1$. The outward pointing rays of this tropical prevariety are $(0, 1, 1)$, $(1, 0, 1)$, and $(2, 2, -d + 2)$, with the initial forms with respect to these rays are [18].:

$$2r_{13}^{d}r_{23}^{d}\left(1 + m_1(1 - r_{12}^{d})\right)$$
$$r_{13}^{d-2}r_{23}^{d}\left(r_{13}^{2} - r_{23}^{2} - 2r_{12}^{d}r_{13}^{2}m_1\right)$$
$$m_1r_{13}^{d}r_{23}^{d-2}\left(r_{23}^{2} - 2r_{12}^{d}r_{23}^{2} - r_{13}^{2}\right)$$

$$r_{12}^{d-2}r_{23}^{d}\left(r_{23}^{2} - 2r_{12}^{2}r_{13}^{d}m_1 - r_{12}^{2}\right)$$
$$2r_{23}^{d}r_{12}^{d}\left(m_1(1 - r_{13}^{d}) - 1\right)$$
$$m_1r_{23}^{d-2}r_{13}^{d}\left(r_{23}^{2} - 2r_{12}^{d}r_{23}^{2} - r_{12}^{2}\right)$$

$$\left(r_{13} - r_{12}\right)\left(r_{12} + r_{13}\right)r_{12}^{d-2}r_{13}^{d}$$
$$\left(r_{13} - r_{12}\right)\left(r_{12} + r_{13}\right)r_{13}^{d-2}r_{12}^{d}$$
$$m_1r_{23}^{d-2}\left(r_{12}^{d}r_{13}^{2} + r_{12}^{2}r_{13}^{d} - 2r_{12}^{d}r_{13}^{d}r_{23}^{2} - r_{12}^{d+2} - r_{13}^{d+2}\right)$$

Clearly there are two distinct polynomial systems in this Groebner fan, specifically the first two sets of polynomials and the last set. Removing the monomial factors from the first set of equations, we have:

$$1 + m_1(1 - r_{12}^d) = 0$$
$$r_{13}^2 - r_{23}^2 - 2r_{12}^d r_{13}^2 m_1 = 0$$
$$r_{23}^2 - 2r_{12}^d r_{23}^2 - r_{13}^2 = 0$$

Adding the last 2 polynomials together, we get $-2r_{12}^d r_{13}^2(m_1 + 1) = 0$. So the only non trivial solution is for $m_1 = \pm 1$, which is outside the scope of this case.

The last set of polynomials, after removing the monomial factors is:

$$(r_{13} - r_{12})(r_{12} + r_{13}) = 0$$
$$(r_{13} - r_{12})(r_{12} + r_{13}) = 0$$
$$r_{12}^d r_{13}^2 + r_{12}^2 r_{13}^d - 2r_{12}^d r_{13}^d r_{23}^2 - r_{12}^{d+2} - r_{13}^{d+2} = 0$$

From the first 2 equations, it's clear that $r_{13} = \pm r_{12}$. We can make the substitution $r_{13}^2 = r_{12}^2$ into the last equation returns $r_{12}^{d+2} + r_{13}^{d+2} - 2r_{12}^d r_{13}^d r_{23}^2 - r_{12}^{d+2} - r_{13}^{d+2} = -2r_{12}^d r_{13}^d r_{23}^2 = 0$. Which forces either r_{12}, r_{13} or r_{23} to be 0. So there are no 1-dimensional ray solutions for this case.

There is also a 2-dimensional cone that must also be considered, an edge connecting the vertices $(-1,-1,-1)$ and $(2,2,-d+2)$ [18]. We will deal with this edge case by taking a Syzygy between the Albouy-Chenciner polynomials, and take the leading term of the resulting polynomial with respect to the ray connecting those vertices (0,0,-d) . This initial form must be the initial coefficient to a Puiseux series expansion of any curve of solutions in the direction of this edge. The resulting initial form is a monomial, so since the lead coefficient must be a solution there are no nontrivial series of solutions for this edge [10].

The Albouy-Chenciner equations, for this special case:

$-2(r_{12}r_{13}r_{23})^d m_1+m_1(r_{23}^{d-2}r_{12}^d r_{13}^2+r_{23}^{d-2}r_{12}^2 r_{13}^d+r_{12}^d r_{23}^d+r_{13}^d r_{23}^d-r_{23}^{d-2}r_{12}^{d+2}-r_{23}^{d-2}r_{13}^{d+2})$

$-2(r_{12}r_{13}r_{23})^d m_1+2(r_{12}r_{23})^d(m_1-1)-(r_{13}^d r_{12}^d+r_{13}^{d-2}r_{12}^d r_{23}^2+r_{13}^{d-2}r_{12}^2 r_{23}^d+r_{13}^d r_{23}^d-r_{13}^{d-2}r_{12}^{d+2}-r_{23}^{d+2}r_{13}^{d-2})$

$-2(r_{12}r_{13}r_{23})^d m_1+2(r_{13}r_{23})^d(m_1+1)+(r_{12}^d r_{13}^d+r_{12}^{d-2}r_{13}^d r_{23}^2+r_{12}^{d-2}r_{13}^2 r_{23}^d+r_{12}^d r_{23}^d-r_{12}^{d-2}r_{13}^{d+2}-r_{12}^{d-2}r_{23}^{d+2})$

We will take a Syzygy between the first and second polynomial, of the form $(1)\cdot r_{12}^2-(2)\cdot r_{13}^2$.

The result of this is [19]:

$-2r_{12}^{d+2}r_{13}^d r_{23}^d m_1+2r_{13}^d r_{23}^d r_{12}^2(m_1+1)-(r_{12}^{d+2}r_{13}^d+r_{12}^d r_{13}^d r_{23}^2+r_{12}^d r_{13}^2 r_{23}^d+r_{12}^{d+2}r_{23}^d-r_{12}^d r_{13}^{d+2}-r_{12}^d r_{23}^{d+2})$

$+2r_{12}^d r_{13}^{d+2}r_{23}^d m_1-2r_{12}r_{23}^d r_{13}^2(m_1-1)-(r_{13}^{d+2}r_{12}^d+r_{13}^d r_{12}^d r_{23}^2+r_{13}^d r_{12}^2 r_{23}^d+r_{13}^{d+2}r_{23}^d-r_{13}^d r_{12}^{d+2}-r_{23}^{d+2}r_{13}^d)$

We now take the initial form of this new polynomial with respect to the ray (0,0,1) [18]. Thus the initial form is $-2r_{13}^d r_{12}^d r_{23}^2$. This initial form will have only trivial solutions, so there is no nontrivial Puiseux series solution for this edge [10].

Second Special case: The final case we need to consider is when one of the masses cancels with both the other masses. Without loss of generality we will take the masses to be $m_1=m_2=1$ and $m_3=-1$. The outward pointing rays for this tropical prevariety are $(0,1,1)$, $(2,-d+2,2)$, and $(2,2,-d+2)$ [18]. There are also 2 separate 2-dimensional cones, from $(-1,-1,-1)$ to $(2,-d+2,2)$ and $(2,2,-d+2)$ respectively [18]. We will address the rays first, and then the 2-dimensional cones. The initial form for these rays are [18].

$$2r_{13}^d r_{23}^d \left(2 - r_{12}^d\right))$$

$$r_{13}^{d-2} r_{23}^d \left(r_{13}^2 - r_{23}^2 - 2r_{12}^d r_{13}^2\right)$$

$$r_{13}^d r_{23}^{d-2} \left(r_{23}^2 - 2r_{12}^d r_{23}^2 - r_{13}^2\right)$$

$$\left(r_{23} - r_{12}\right)\left(r_{12} + r_{23}\right) r_{12}^{d-2} r_{23}^d$$

$$r_{13}^{d-2} \left(r_{12}^d r_{23}^2 + r_{12}^2 r_{23}^d - 2r_{12}^d r_{12}^2 r_{23}^d - r_{12}^{d+2} - r_{23}^{d+2}\right)$$

$$\left(r_{23} - r_{12}\right)\left(r_{12} + r_{23}\right) r_{23}^{d-2} r_{12}^d$$

$$\left(r_{13} - r_{12}\right)\left(r_{12} + r_{13}\right) r_{12}^{d-2} r_{13}^d$$

$$\left(r_{13} - r_{12}\right)\left(r_{12} + r_{13}\right) r_{13}^{d-2} r_{12}^d$$

$$r_{23}^{d-2} \left(r_{12}^d r_{13}^2 + r_{12}^2 r_{13}^d - 2r_{12}^d r_{13}^d r_{23}^2 - r_{12}^{d+2} - r_{13}^{d+2}\right)$$

Clearly the last two equations are symmetrically equivalent under permutation of indices. What's more they are also equivalent to the third initial form of the first special case, addressed above. Thus it is known that there are no nontrivial solutions for those rays, leaving just the first ray.

From the first polynomial in that set, it is clear that $2 = r_{12}^d$. Making this substitution into the second and third equations (and removing monomial factors), we get

$$r_{13}^2 - r_{23}^2 - 4r_{13}^2 = 0$$

$$r_{23}^2 - 4r_{23}^2 - r_{13}^2 = 0$$

We can then sum these together, returning $-4(r_{13}^2 + r_{23}^2) = 0$. So $r_{13}^2 + r_{23}^2 = 0$.

We can now substitute the identities $r_{13}^2 = -r_{23}^2$ and $2 = r_{12}^d$ into the second and third polynomials, returning $r_{13}^2 + r_{13}^2 - 4r_{13}^2 = -2r_{13}^2 = 0$ and $r_{23}^2 - 4r_{23}^2 + r_{23}^2 = -2r_{23}^2 = 0$. From this, we can clearly see that r_{13} and r_{23} is 0. So there are no nontrivial ray solutions for this case.

Now to turn our attention to the 2-dimensional edge cones. One edge connects the vertices $(-1,-1,-1)$ and $(2,2,-d+2)$, while the other connects $(-1,-1,-1)$ and $(2,-d+2,2)$ [18]. We will address these edges similarly to the edge in the first special case. We will take a Syzygy between the Albouy-Chenciner polynomials, and show that the leading term with respect to an appropriate choice of ray will be monomial. Thus, there can be no nontrivial solutions to the system for that edge [10].

The Albouy-Chenciner equations, for this special case:

$-2(r_{12}r_{13}r_{23})^d + (r_{23}^{d-2}r_{12}^d r_{13}^2 + r_{23}^{d-2}r_{12}^2 r_{13}^d + r_{12}^d r_{23}^d + r_{13}^d r_{23}^d - r_{23}^{d-2}r_{12}^{d+2} - r_{23}^{d-2}r_{13}^{d+2})$

$-2(r_{12}r_{13}r_{23})^d - (r_{13}^d r_{12}^d + r_{13}^{d-2}r_{12}^d r_{23}^2 + r_{13}^{d-2}r_{12}^2 r_{23}^d + r_{13}^d r_{23}^d - r_{13}^{d-2}r_{12}^{d+2} - r_{23}^{d+2}r_{13}^{d-2})$

$-2(r_{12}r_{13}r_{23})^d + 4(r_{13}r_{23})^d + (r_{12}^d r_{13}^d + r_{12}^{d-2}r_{13}^d r_{23}^2 + r_{12}^{d-2}r_{13}^2 r_{23}^d + r_{12}^d r_{23}^d - r_{12}^{d-2}r_{13}^{d+2} - r_{12}^{d-2}r_{23}^{d+2})$

We will take a Syzygy between the first and second polynomial, of the form $(1) \cdot r_{12}^2 - (2) \cdot r_{13}^2$ [19].

The result of this is [19]:

$-2r_{12}^{d+2}r_{13}^d r_{23}^d + 4r_{13}^d r_{23}^d r_{12}^2 - (r_{12}^{d+2}r_{13}^d + r_{12}^d r_{13}^d r_{23}^2 + r_{12}^d r_{13}^2 r_{23}^d + r_{12}^{d+2}r_{23}^d - r_{12}^d r_{13}^{d+2} - r_{12}^d r_{23}^{d+2})$
$+2r_{12}^d r_{13}^{d+2} r_{23}^d - (r_{13}^{d+2}r_{12}^d + r_{13}^d r_{12}^d r_{23}^2 + r_{13}^d r_{12}^2 r_{23}^d + r_{13}^{d+2}r_{23}^d - r_{13}^d r_{12}^{d+2} - r_{23}^{d+2}r_{13}^d)$.

We now take the initial form of this polynomial with respect to an appropriate choice of ray for each edge, $(0,0,1)$ for the first and $(0,1,0)$ for the second [18]. That gives us the initial forms of $-2r_{12}^d r_{13}^d r_{23}^2$ and $-2r_{12}^d r_{13}^2 r_{23}^d$[18]. Since these are both monomials, they have no nontrivial solutions. Hence, there are no nontrivial solutions along either 2-dimensional cone.

Thus there can be no algebraic curve of solutions to the Albouy-Chenciner system, and therefore there can only be finitely many solutions.

Given the origins of this system, it seems appropriate to consider the subset of solutions that are all positive and real. These are the only solutions that physically realizable Khovanskii's Theorem to state that there are at most $2^{\binom{22}{2}} \cdot (4)^{22} = 2^{275}$ positive real solutions to the general case. For the first special case the bound is $2^{\binom{21}{2}} \cdot (4)^{21} = 2^{252}$ and for the second special case the bound is $2^{\binom{20}{2}} \cdot (4)^{20} = 2^{230}$ [3]. While this bound is ridiculously large (roughly equal to the number of hydrogen atoms in the Milky Way), it is independent of d. This implies that the number of positive real solutions is not increasing with d.

3 Mixed Volume of the 3-mass Albouy-Chenciner polynomials

Theorem: *The mixed volume for the Newton Polyhedron for the 3-mass Albouy-Chenciner polynomials is* $d^3 + 12d^2 - 12d$.

Proof:

The volume of a 3-simplex on vertices **r, s, t,** and **u** is [25]:

$$V = \frac{1}{3!}\text{abs}\left(\begin{vmatrix} r_1 & r_2 & r_3 & 1 \\ s_1 & s_2 & s_3 & 1 \\ t_1 & t_2 & t_3 & 1 \\ u_1 & u_2 & u_3 & 1 \end{vmatrix}\right)$$

Let all the coordinates for all the vertices be some polynomial in d with degree no more than n. Then by computing the determinant by expanding down the right-most column, we can see the volume of the 3-simplex is some polynomial with degree at most $3n$.

Now consider a 3-dimensional Polytope on the above class of vertices that can be triangulated without added vertices. The volume of the polytope will be the sum of the volume of the 3-simplicies. Since the volume of each of those simplices will be a polynomial with degree at most *3n*, the volume of the polytope will also be a polynomial with degree at most *3n*.

Now consider the Mixed Volume of m polytopes in 3 dimensions, like those described above. Furthermore, lets assume that the Minkowski Sum of any combination of those polytopes is also a polytope of that form. Then by the above argument we can write the volumes of all the polytopes and the Minkowski Sum of any combination of them as a polynomial with degree at most $3n$. Finally, recall that we can write the

Mixed Volume of these polytopes as the sums and differences of the volumes of the original polytopes and Minkowski Sums of combinations of the polytopes as [2]:

$$MV_n(\mathcal{P}) = \sum_{k=1}^{n}(-1)^{n-k} \sum_{\substack{I\subset\{I,\ldots,n\} \\ |I|=k}} Vol_n \left(\sum_{i\in I} P_i\right)$$

Since each of those volumes $Vol_n \left(\sum_{i\in I} P_i\right)$ is a polynomial of degree at most $3n$ and all the Minkowski Sums are the above class of polytopes, the Mixed Volume must also be a polynomial of degree at most $3n$ [18].

Since the coordinates for all the polytopes in the Albouy-Chenciner 3-mass equations are linear polynomials in d, the mixed volume must be a polynomial in d with degree at most 3.

By applying a least squares polynomial fit to the values for the Mixed Volume for the system for values of d, we were able to compute this polynomial. Since the above equation has been verified for more then 4 points, the Mixed Volume must be $d^3 + 12d^2 - 12d$ [20, 22].

Since we have previously established the finiteness of the solution set, we can use Bernstein's theorem to state.

Theorem: *There are at most $d^3 + 12d^2 - 12d$ solutions to the Albouy-Chenciner polynomial system.*

4 Numerical Solutions to Generalized Albouy-Chenciner System 3-Mass system.

4.1 Solution types

We will start our numerical explorations by using PHCPack in Sage to compute the types of solutions for various choices of masses. It seems logical to start with the cases described in the previous section. We will classify solutions as real, complex or failures and state the mixed volume for the system, which if you recall is the total number of isolated solutions for a sufficiently generic choice of coefficients (note that our system is far from generic) .

d	2	3	4	5	6	7	8	9	10	11	12
Mixed Volume	32	99	208	365	576	847	1184	1593	2080	2651	3312
Solutions in $\mathbb{R}^3$	32	7	32	7	32	7	32	7	32	7	32
Solutions in $\mathbb{R}^3_+$	4	4	4	4	4	4	4	4	4	4	4
Solutions in $\mathbb{C}^3 \sim \mathbb{R}^3$	0	92	176	358	544	840	1152	1586	2048	2644	3280
Failures	0	0	0	0	0	0	0	0	0	0	0

Table 1: Solution types for $\mathbf{m} = (1, 2, 4)$[20].

The computations was repeated for $\mathbf{m} = (1, 1, 1)$ and $\mathbf{m} = (1, 1, 4)$ with similar results.

Now lets consider examples from the first special case addressed above, that is where exactly two of the masses cancel out. For each d, PHC was run multiple times, the number of failures is shown, the mean and standard deviation of these are presented, the average number of Complex solutions is shown, and a "best guess" value for the number of complex solutions is computed. The "best guess" is the value closest to the average number of solutions, where the total number of solutions real and complex is divisible by d. The rational for the divisibility condition is discussed at the end of this section. The validity of these best guesses seems dubious for several values of d, as evidenced by the distribution of values for the number of failures and their standard deviation. However it is included here for completeness.

d	2	3	4	5	6	7
Mixed Volume	32	99	208	365	576	847
# of Solutions in $\mathbb{R}^3$	16	9	16	9	16	9
# of Solutions in $\mathbb{R}^3_+$	2	2	2	2	2	2
# of PHC Failures	7	9	16	11	6	5
	0	9	6	15	16	10
	1	8	8	16	2	6
	1	8	15	15	10	1
	0	9	13	11	6	5
	6	9	1	8	10	8
	0	8	16	12	6	6
	5	10	11	14	11	0
	0	8	7	7	1	4
	1	10	0	10	4	1
	0	7	1	8	2	5
	1	10	2	7	0	1
	3	6	2	7	0	1
	0	12	14	3	0	13
	0	12	14	9	1	7
	1	10	4	7	4	3
	3	9	16	13	2	7
	3	4	16	6	5	1
	0	6	10	9	9	7
	0	11	0	14	5	0
Average # of Failures	1.6	8.8	8.6	10.1	5.0	4.6
Standard Deviation in the # of Failures	2.2	2.0	6.2	3.6	4.4	3.6
Average # of Solutions in $\mathbb{C}^3 \sim \mathbb{R}^3$	14.4	81.2	183.4	345.9	555	833.4
Best Guess for # of Solutions in $\mathbb{C}^3 \sim \mathbb{R}^3$	14	81	182	346	556	831

Table 2: $\mathbf{m} = (1, -1, 2)$, d from 2 to 7 [20].

d	8	9	10	11	12
Mixed Volume	1184	1593	2080	2651	3312
# of Solutions in $\mathbb{R}^3$	16	9	16	9	16
# of Solutions in $\mathbb{R}^3_+$	2	2	2	2	2
# of PHC Failures	8	4	11	3	6
	3	2	1	3	4
	0	3	0	5	0
	4	1	0	2	9
	0	1	8	6	0
	0	0	3	0	22
	0	3	3	1	0
	0	1	0	2	0
	0	6	3	2	6
	10	0	11	2	0
	0	0	0	2	0
	0	10	0	0	0
	5	1	0	4	0
	8	3	0	3	0
	0	6	0	1	0
	0	0	8	0	0
	3	4	1	1	0
	14	3	0	2	1
	0	3	4	0	0
	9	4	0	3	15
Average # of Failures	3.2	2.8	2.6	2.1	3.2
Standard Deviation in the # of Failures	4.3	2.5	3.8	1.6	6.0
Average # of Solutions in $\mathbb{C}^3 \sim \mathbb{R}^3$	1164.8	1581.2	2061.4	2640.0	3292.8
Best Guess for # of Solutions in $\mathbb{C}^3 \sim \mathbb{R}^3$	1168	1584	2064	2642	3296

Table 3: $\mathbf{m} = (1, -1, 2)$ d from 8 to 12 [20].

Similar results were computed for $\mathbf{m} = (1, -1, 4)$.

Lastly, we compute the results for the last case above. Up to permutation of indices and scaling of the masses of the system, there is only one set of masses for this case:

d	2	3	4	5	6	7
Mixed Volume	32	99	208	365	576	847
# of Solutions in $\mathbb{R}^3$	8	9	8	9	8	9
# of Solutions in $\mathbb{R}^3_+$	1	1	1	1	1	1
# of PHC Failures	0	13	22	21	9	10
	1	21	12	24	14	10
	8	18	32	12	10	7
	6	16	20	19	2	14
	4	20	19	15	23	14
	6	19	16	18	8	8
	1	18	20	20	16	5
	2	15	9	16	10	11
	4	20	24	16	28	9
	1	19	17	18	7	4
	2	18	27	23	2	14
	1	17	15	17	19	8
	0	17	0	17	23	9
	5	17	22	24	8	6
	2	17	16	22	2	10
	2	19	21	17	13	9
	3	17	11	12	21	6
	3	18	14	21	24	10
	1	13	30	22	9	6
	0	20	7	19	1	5
Average # of Failures	2.6	17.9	18.2	18.7	12.4	8.8
Standard Deviation in the # of Failures	2.3	1.9	7.8	3.5	8.1	3.0
Average # of Solutions in $\mathbb{C}^3 \sim \mathbb{R}^3$	21.4	72.1	181.8	337.4	555.6	829.2
Best Guess for # of Solutions in $\mathbb{C}^3 \sim \mathbb{R}^3$	22	72	180	336	556	831

Table 4: $\mathbf{m} = (1, -1, -1)$, d from 2 to 7 [20]

d	8	9	10	11	12
Mixed Volume	1184	1593	2080	2651	3312
# of Solutions in $\mathbb{R}^3$	8	9	8	9	8
# of Solutions in $\mathbb{R}^3_+$	1	1	1	1	1
# of PHC Failures	2	9	11	4	0
	0	8	14	5	19
	3	2	3	3	1
	21	5	4	4	1
	9	3	10	6	5
	5	7	0	4	4
	18	6	2	5	0
	1	12	0	2	0
	8	3	8	3	1
	3	7	0	3	0
	7	2	0	8	4
	5	8	4	1	2
	29	3	8	4	0
	0	5	29	2	0
	3	6	10	3	0
	1	3	5	2	9
	33	5	7	4	1
	21	4	0	3	0
	4	3	3	6	0
	16	12	4	2	16
Average # of Failures	9.4	5.6	6.1	3.7	3.2
Standard Deviation in the # of Failures	10.0	3.0	6.8	1.7	5.4
Average # of Solutions in $\mathbb{C}^3 \sim \mathbb{R}^3$	1166.6	1578.4	2065.9	2638.3	3300.8
Best Guess for # of Solutions in $\mathbb{C}^3 \sim \mathbb{R}^3$	1168	1575	2062	2642	3304

Table 5: $\mathbf{m} = (1, -1, -1)$ d from 8 to 12 [20].

The typical number of failures for this system was far greater than the previous cases. Case in point, the calculation for $d = 3$ was repeated over 100 times, and the least number of failures was 12. It is possible that the for this case, and others with failures, the system might be manipulated to find an equivalent system with a lower mixed volume (and thus without these failures).

We can make a few observations for these calculations. The immediate first observation is that, for a given set of masses, the number of real solutions alternates between two different values (depending on whether d is even or odd). These values are $(32, 7)$, $(16, 9)$, and $(8, 9)$ for each of the 3 cases in order. The number of positive real solutions for the 3 cases are fixed, at 4, 2, and 1 respectively. This should be no surprised, since the Khovanskii bound for the system was fixed with respect to d (although it is most decided not a sharp bound).

4.2 Solution Values

After the exploration of the types of solutions for this system, we will now explore the values of the solutions. Below are plots of the solutions for each of the mass sets used in the previous section, for every value of d from 2 to 12. They are plotted in the complex plane with the unit circle for reference. Each solution is actually a set of 3 ordered points (r_1, r_2, r_3), and for small values of d the three points are connected by lines to form a triangle.

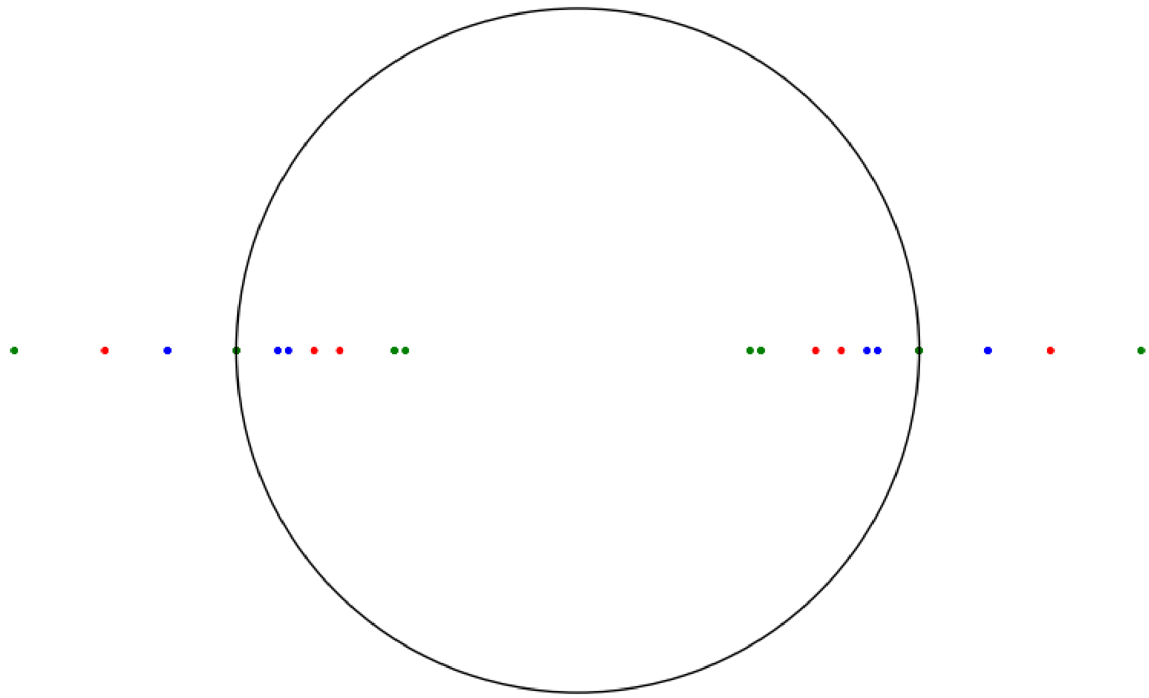

Figure 1: Solutions for $d = 2$ and $\mathbf{m} = (1, 2, 4)$, Case 1[20, 21]

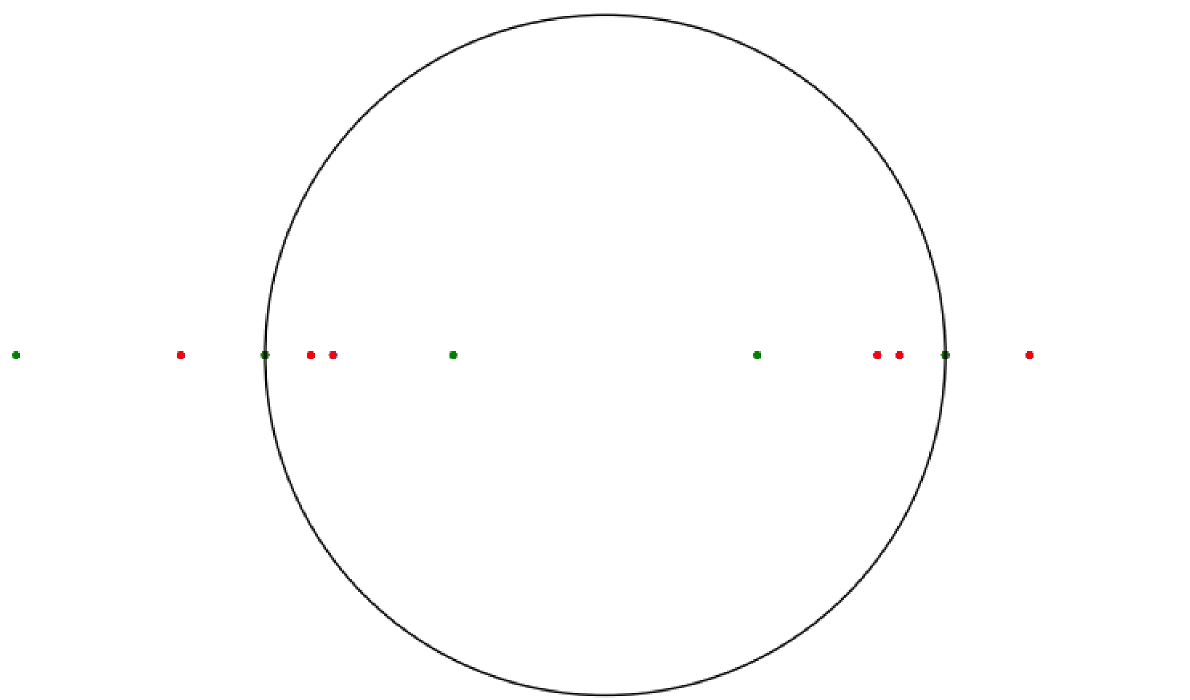

Figure 2: Solutions for $d = 2$ and $\mathbf{m} = (1, 1, 4)$, Case 1[20, 21]

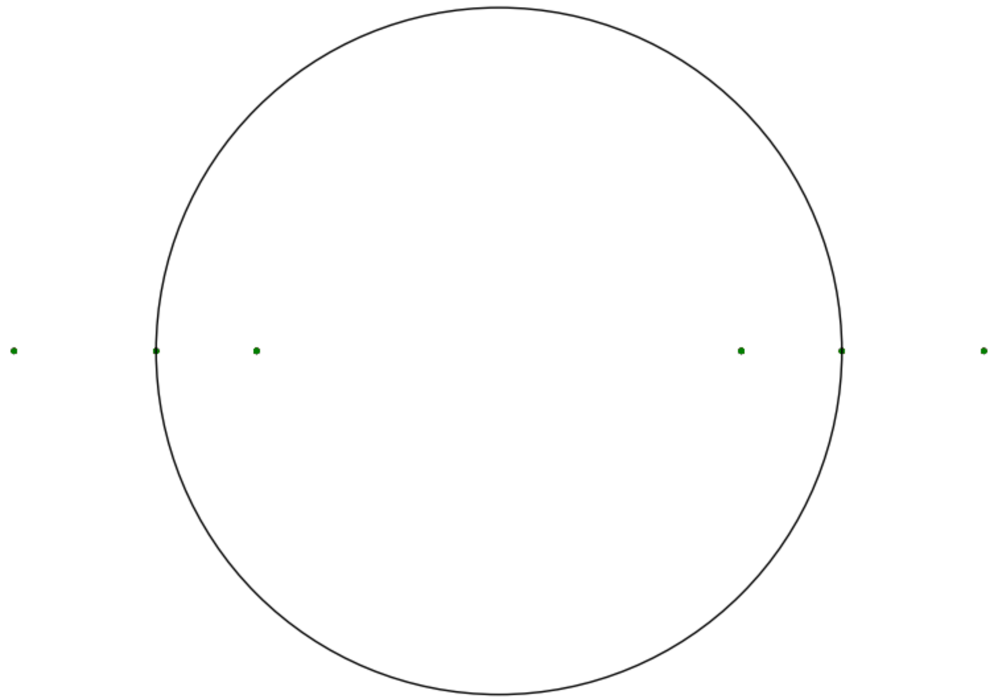

Figure 3: Solutions for $d = 2$ and $\mathbf{m} = (1, 1, 1)$, Case 1[20, 21]

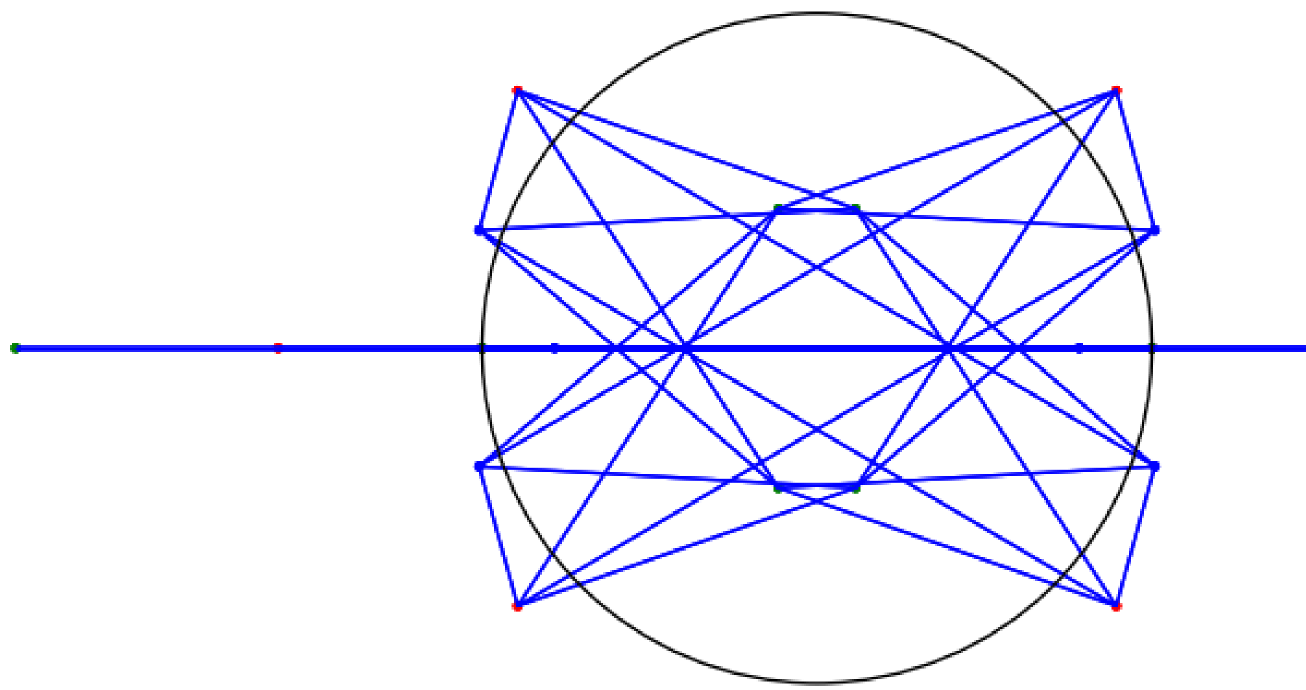

Figure 4: Solutions for $d = 2$ and $\mathbf{m} = (1, -2, 4)$, Case 1[20, 21]

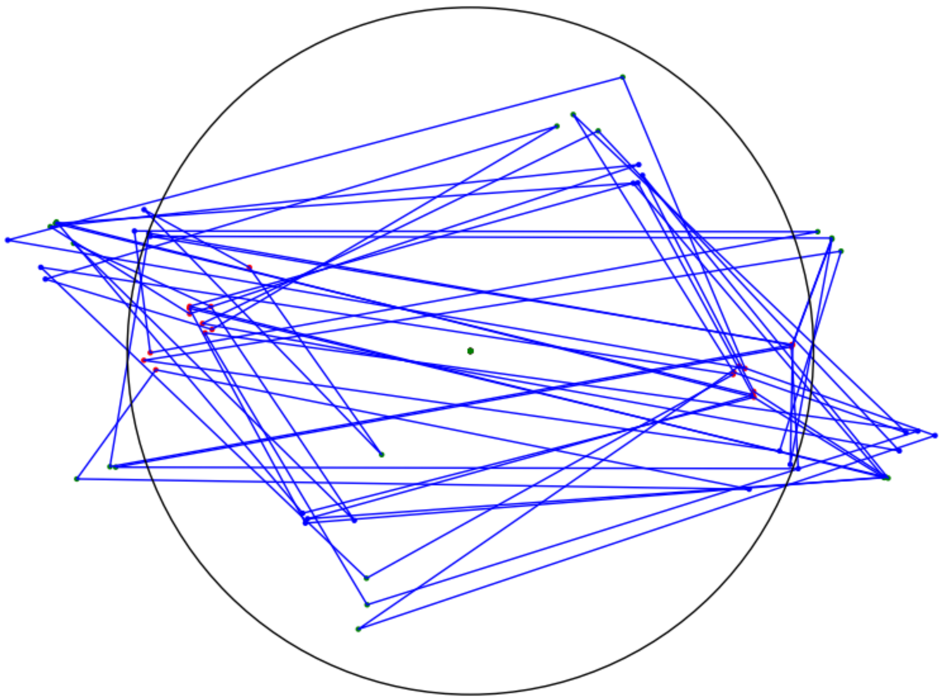

Figure 5: Solutions for $d = 2$ and $\mathbf{m} = (2, -1, 2)$, Case 1[20, 21]

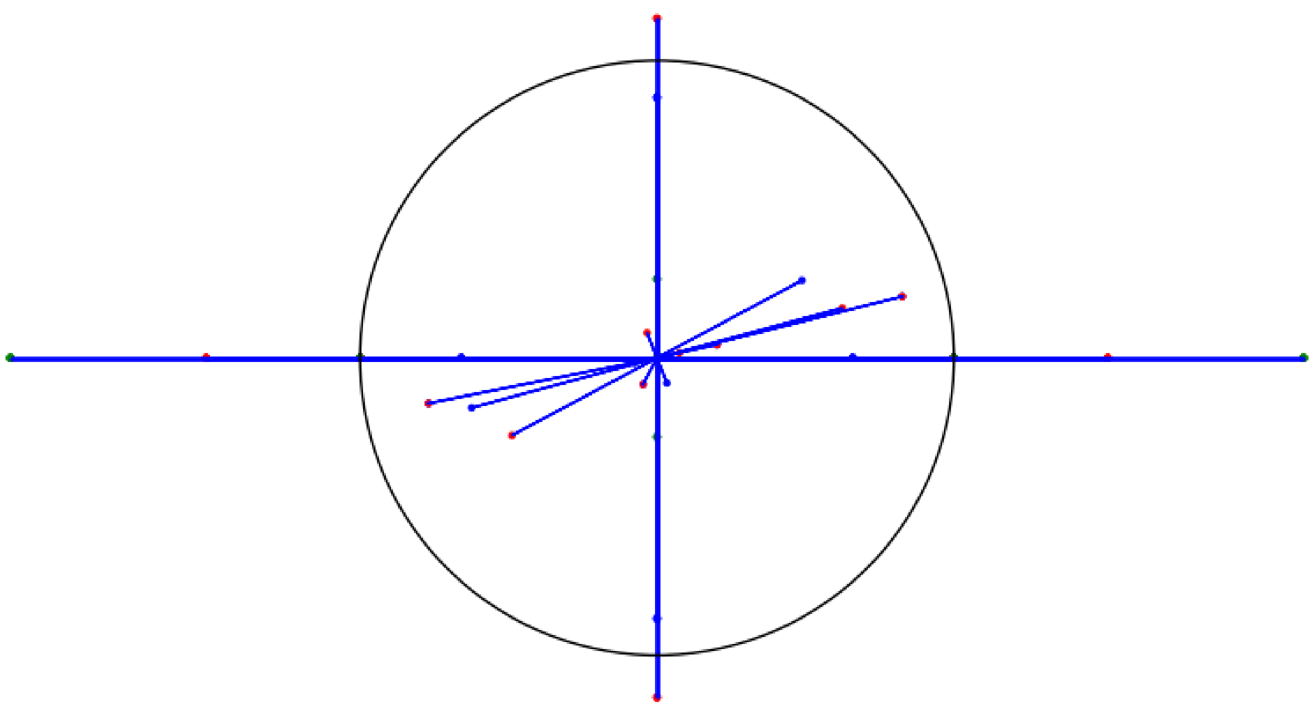

Figure 6: Solutions for $d = 2$ and $\mathbf{m} = (1, -1, 2)$, Case 2[20, 21]

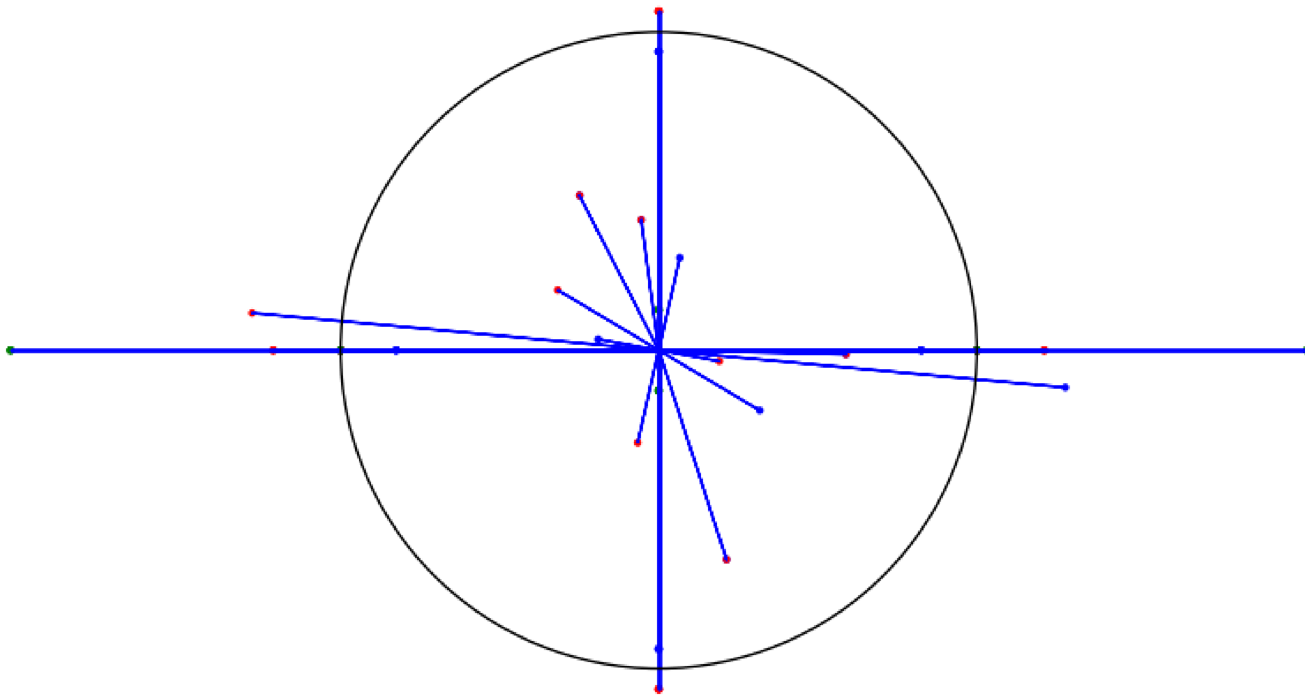

Figure 7: Solutions for $d = 2$ and $\mathbf{m} = (1, -1, 4)$, Case 2[20, 21]

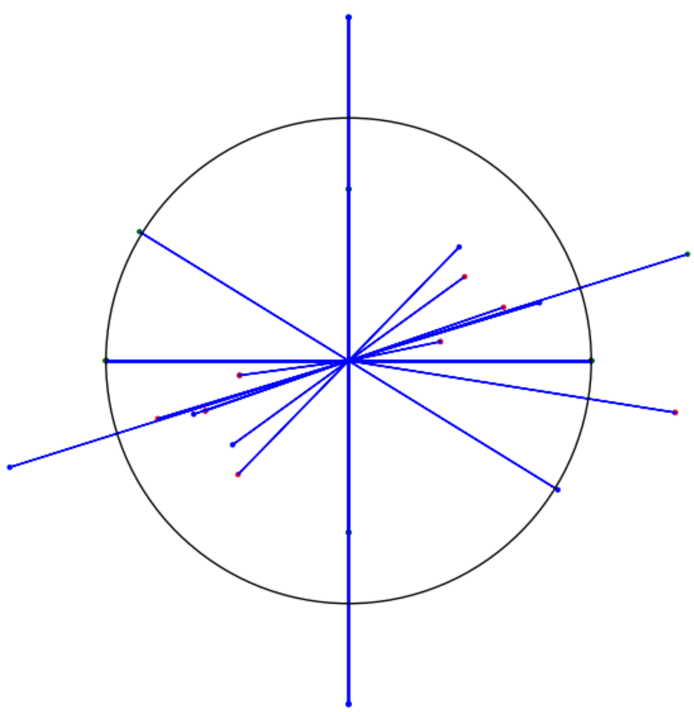

Figure 8: Solutions for $d = 2$ and $\mathbf{m} = (1, -1, -1)$, Case 3[20, 21]

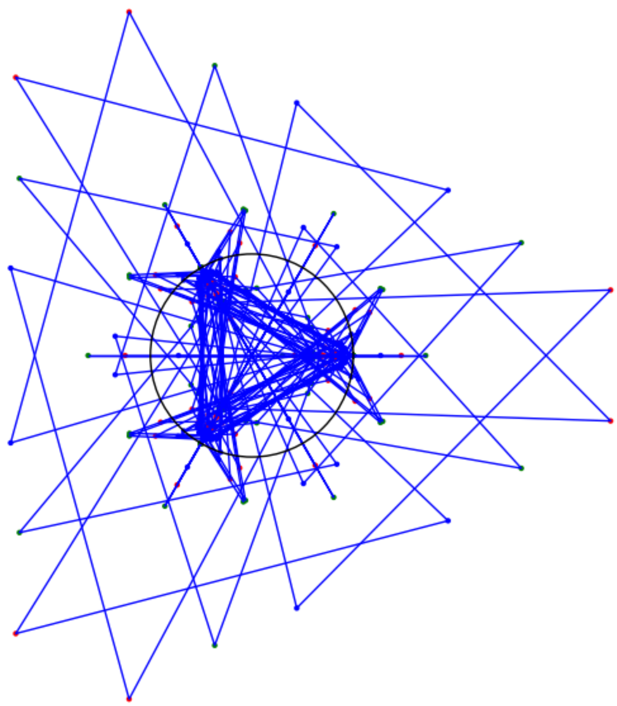

Figure 9: Solutions for $d = 3$ and $\mathbf{m} = (1, 2, 4)$, Case 1[20, 21]

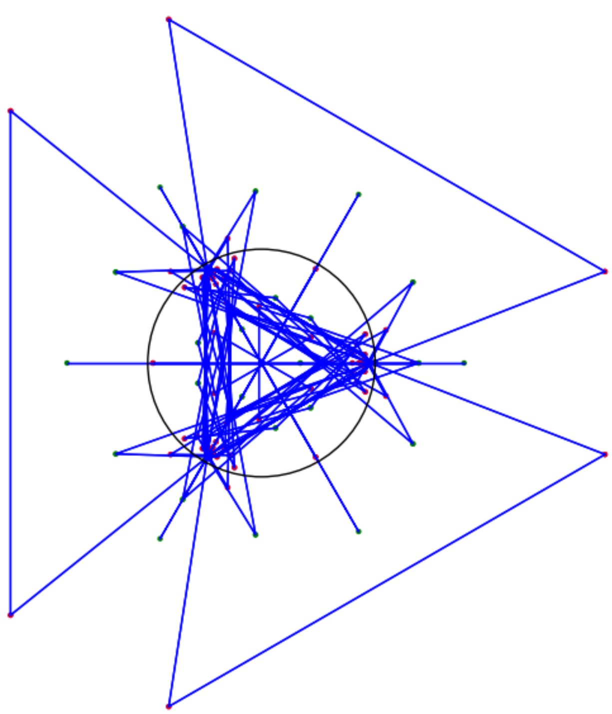

Figure 10: Solutions for $d = 3$ and $\mathbf{m} = (1, 1, 4)$, Case 1[20, 21]

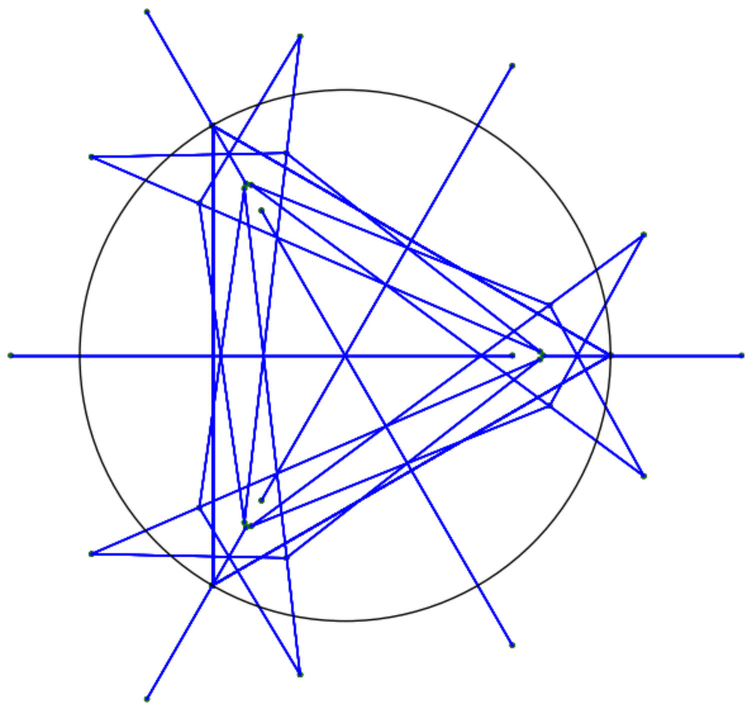

Figure 11: Solutions for $d = 3$ and $\mathbf{m} = (1, 1, 1)$, Case 1[20, 21]

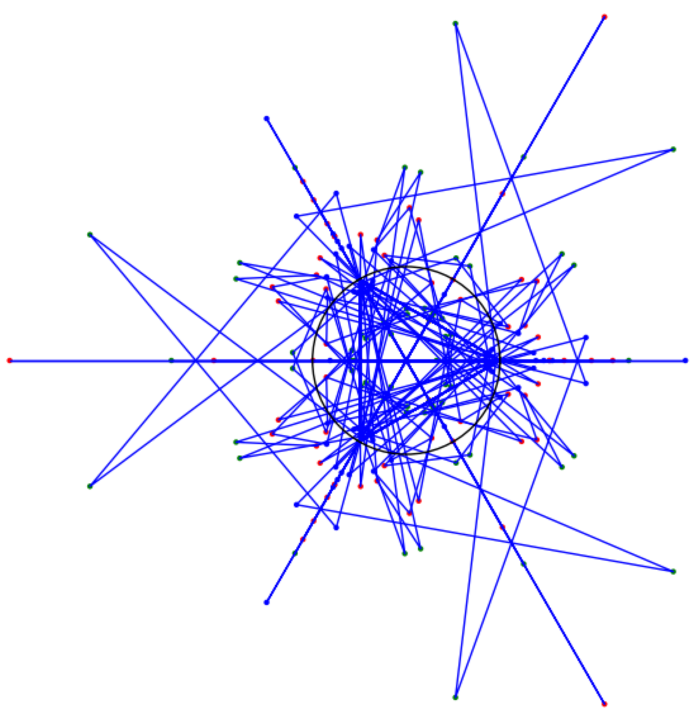

Figure 12: Solutions for $d = 3$ and $\mathbf{m} = (1, -2, 4)$, Case 1[20, 21]

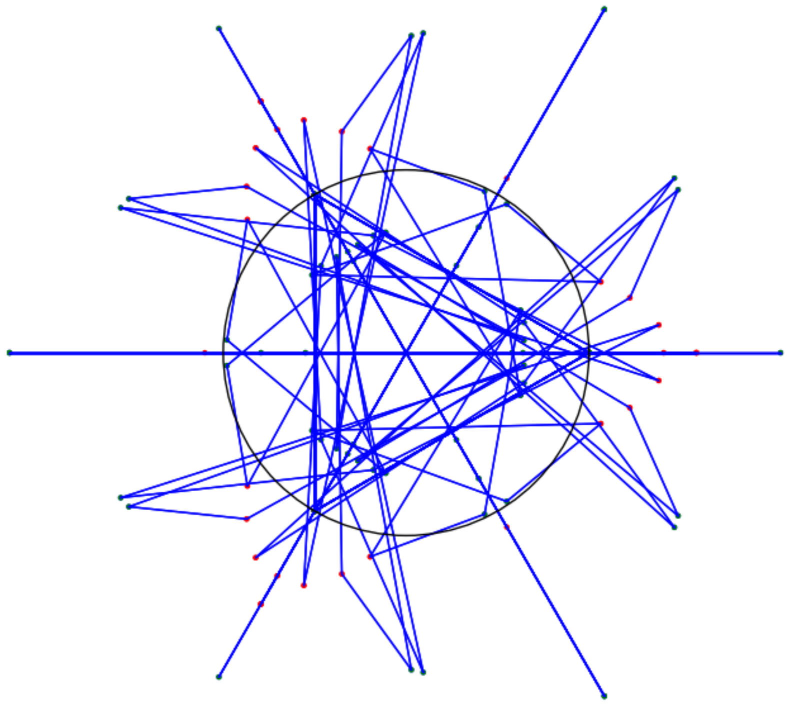

Figure 13: Solutions for $d = 3$ and $\mathbf{m} = (2, -1, 2)$, Case 1[20, 21]

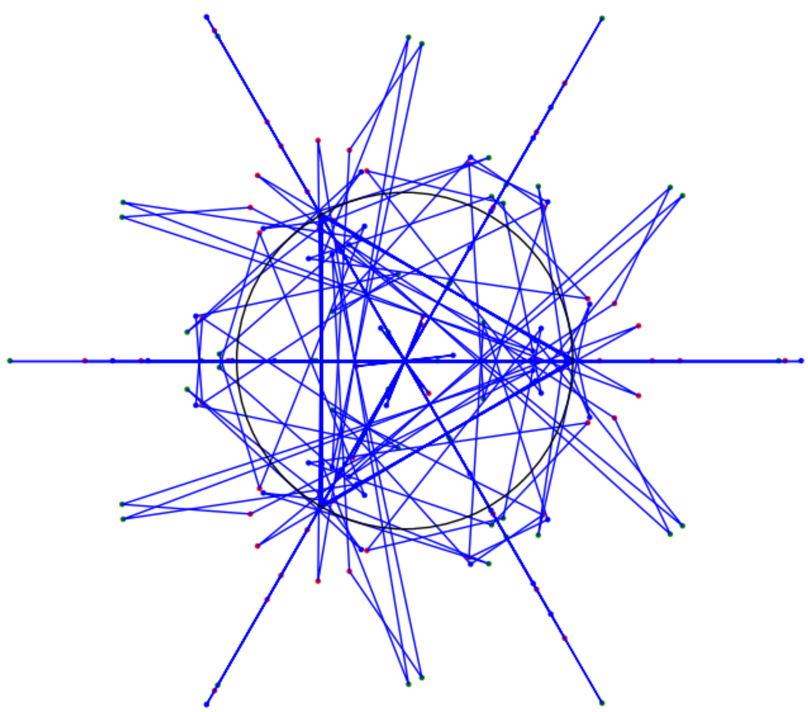

Figure 14: Solutions for $d = 3$ and $\mathbf{m} = (1, -1, 2)$, Case 2[20, 21]

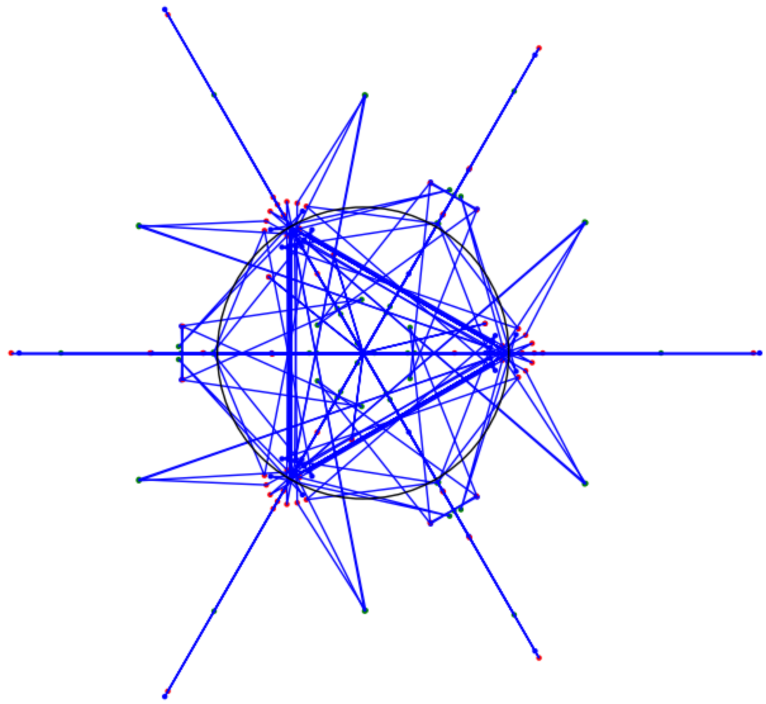

Figure 15: Solutions for $d = 3$ and $\mathbf{m} = (1, -1, 4)$, Case 2[20, 21]

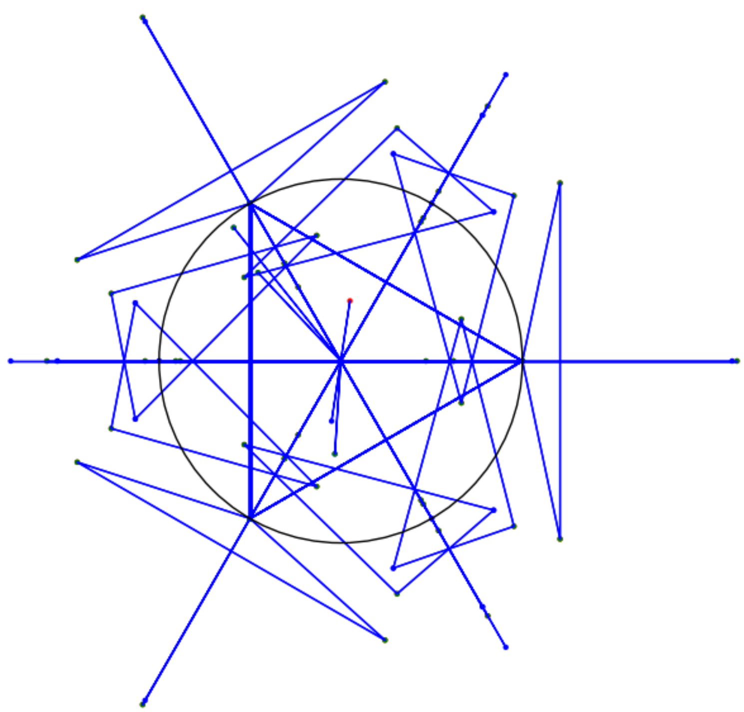

Figure 16: Solutions for $d = 3$ and $\mathbf{m} = (1, -1, -1)$, Case 3[20, 21]

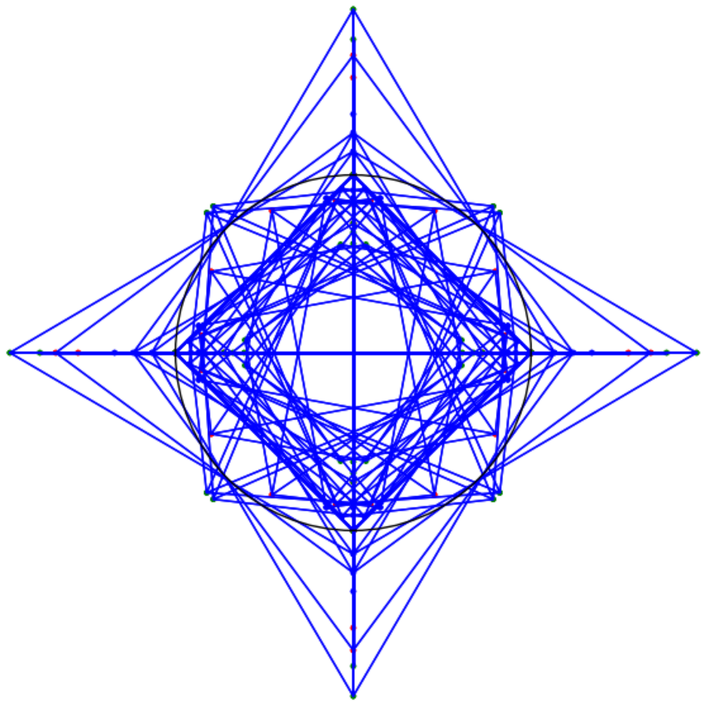

Figure 17: Solutions for $d = 4$ and $\mathbf{m} = (1, 2, 4)$, Case 1[20, 21]

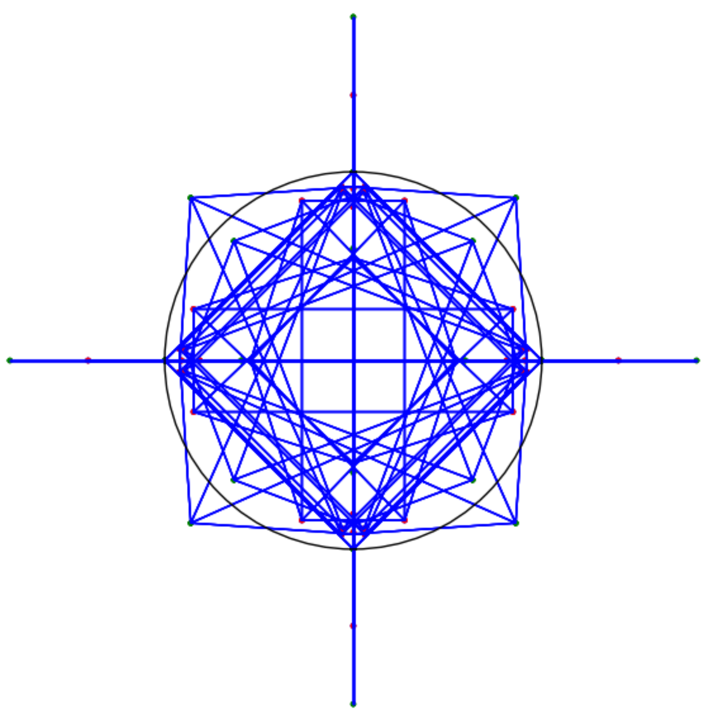

Figure 18: Solutions for $d = 4$ and $\mathbf{m} = (1, 1, 4)$, Case 1[20, 21]

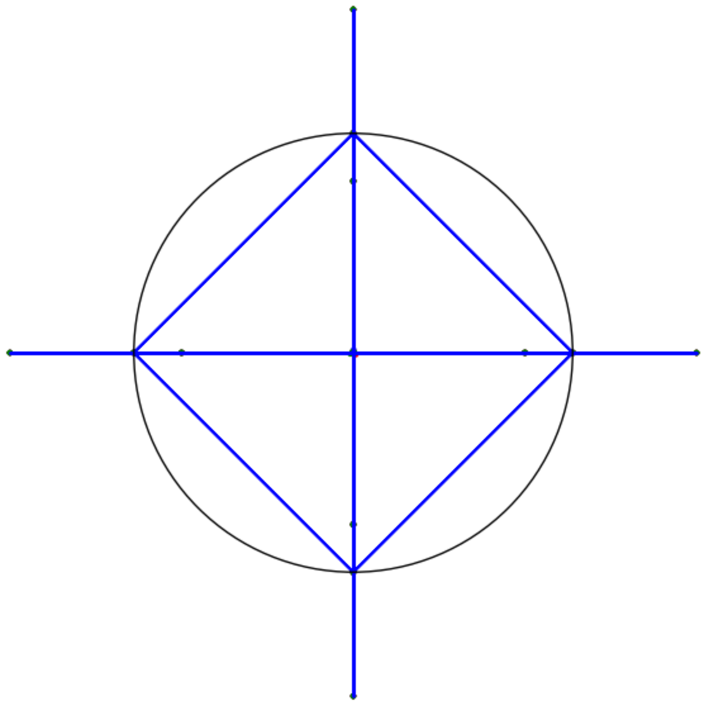

Figure 19: Solutions for $d = 4$ and $\mathbf{m} = (1, 1, 1)$, Case 1[20, 21]

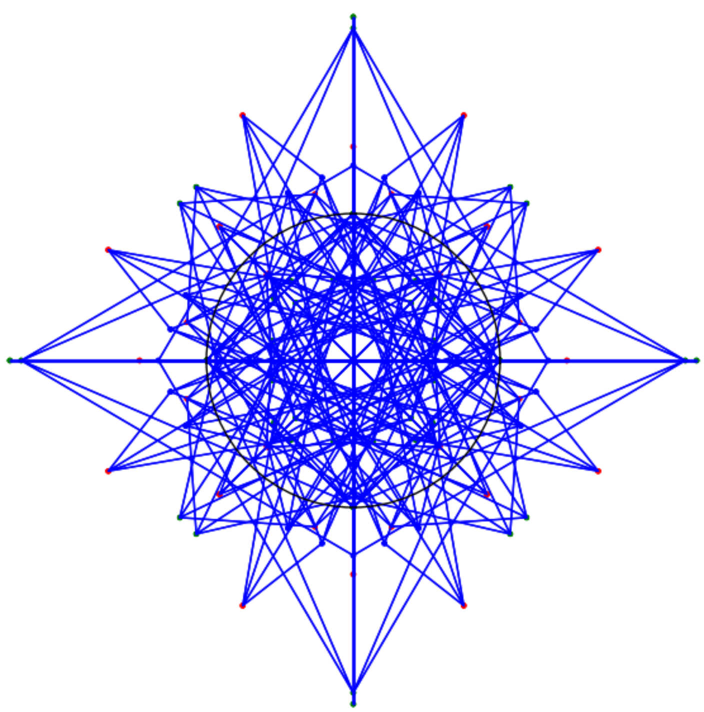

Figure 20: Solutions for $d = 4$ and $\mathbf{m} = (1, -2, 4)$, Case 1[20, 21]

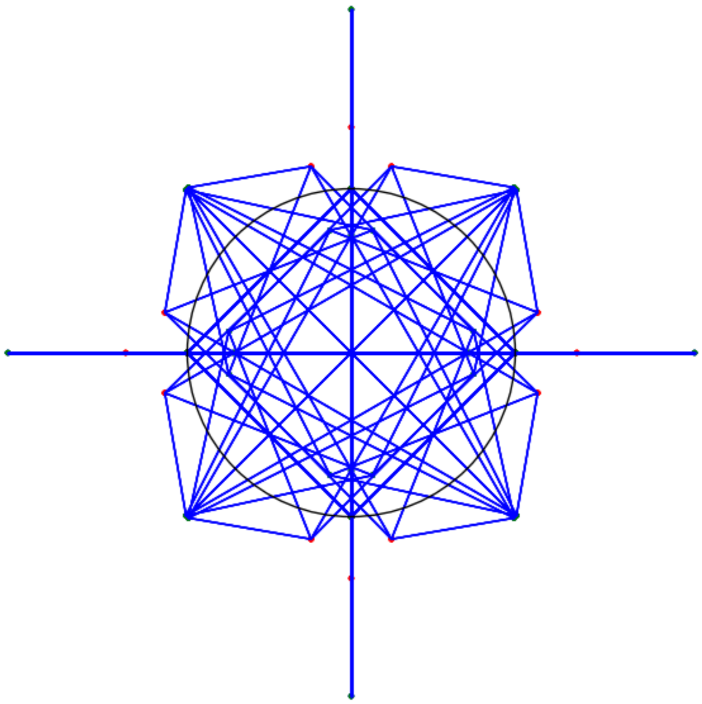

Figure 21: Solutions for $d = 4$ and $\mathbf{m} = (2, -1, 2)$, Case 1[20, 21]

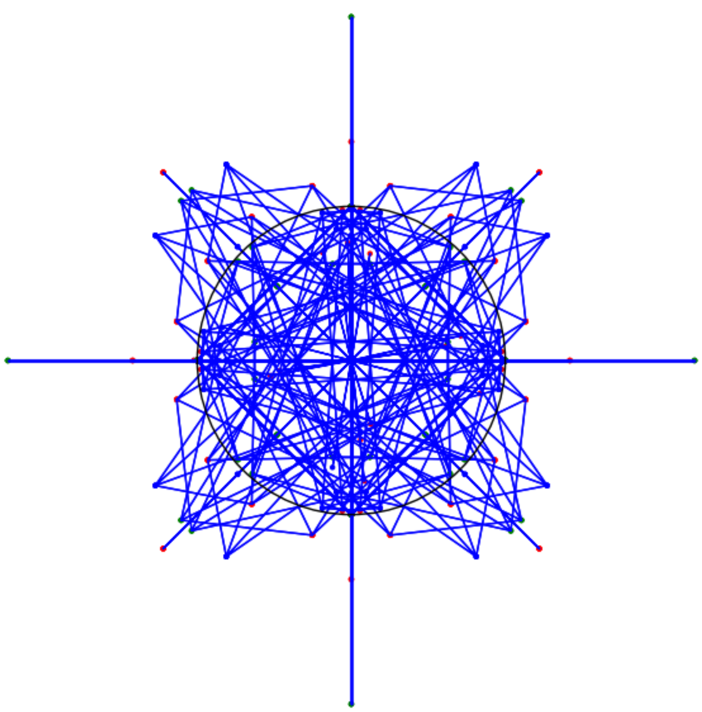

Figure 22: Solutions for $d = 4$ and $\mathbf{m} = (1, -1, 2)$, Case 2[20, 21]

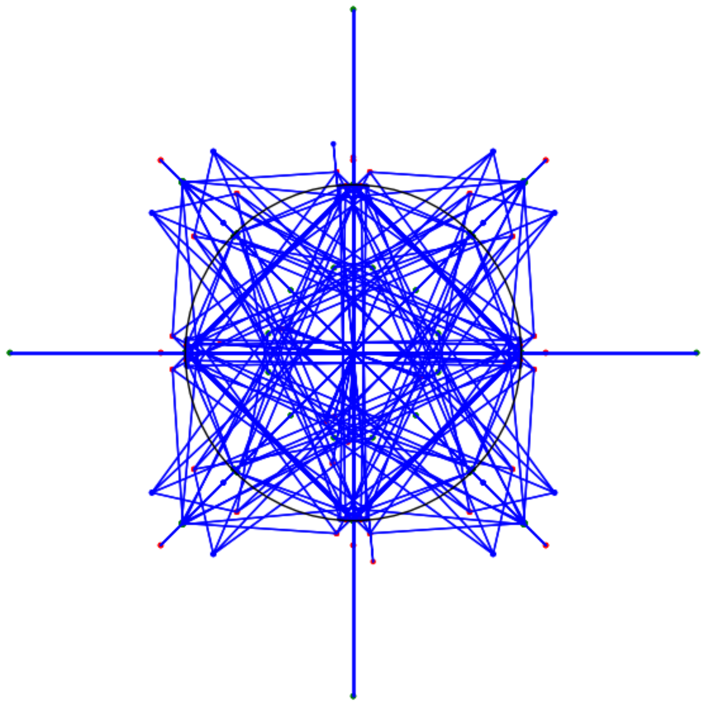

Figure 23: Solutions for $d = 4$ and $\mathbf{m} = (1, -1, 4)$, Case 2[20, 21]

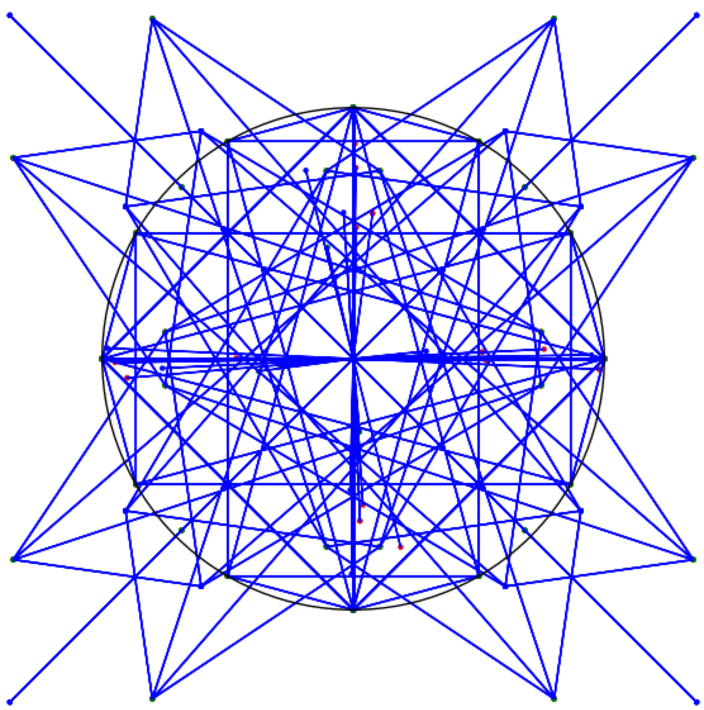

Figure 24: Solutions for $d = 4$ and $\mathbf{m} = (1, -1, -1)$, Case 3[20, 21]

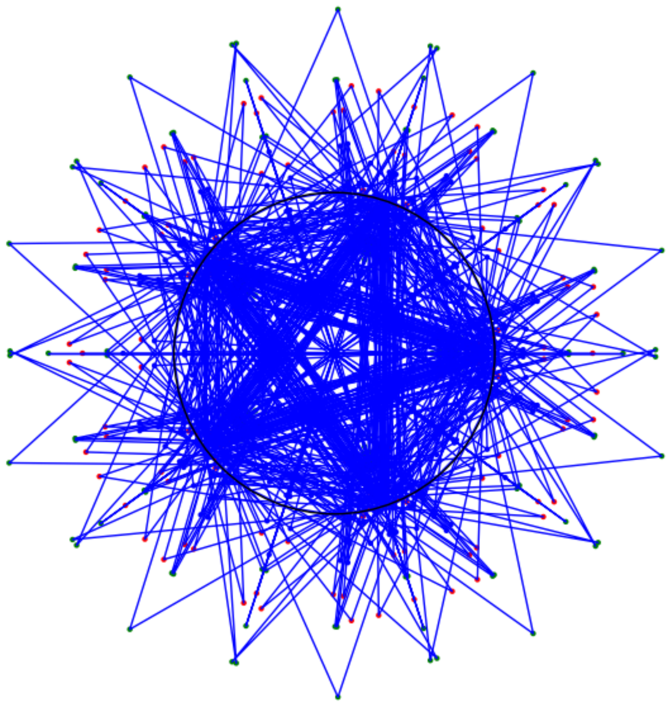

Figure 25: Solutions for $d = 5$ and $\mathbf{m} = (1, 2, 4)$, Case 1[20, 21]

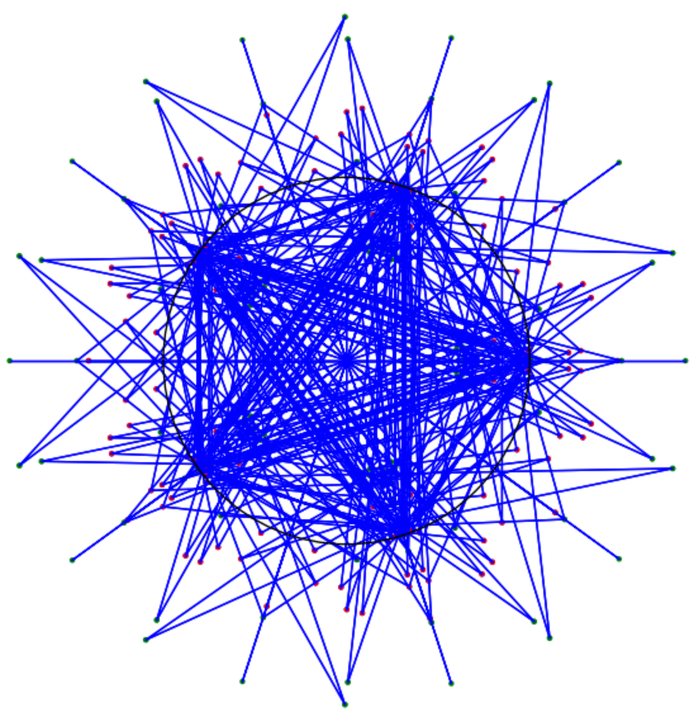

Figure 26: Solutions for $d = 5$ and $\mathbf{m} = (1, 1, 4)$, Case 1[20, 21]

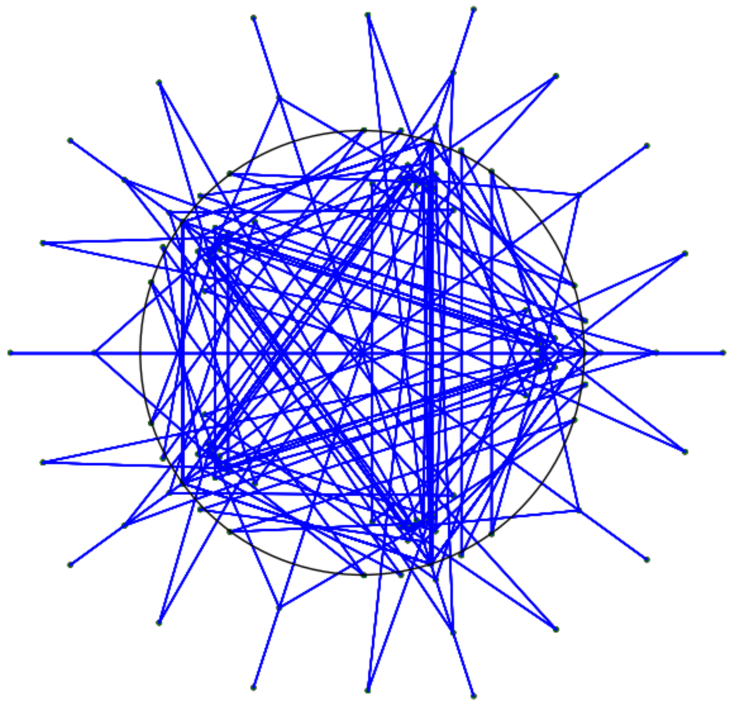

Figure 27: Solutions for $d = 5$ and $\mathbf{m} = (1, 1, 1)$, Case 1[20, 21]

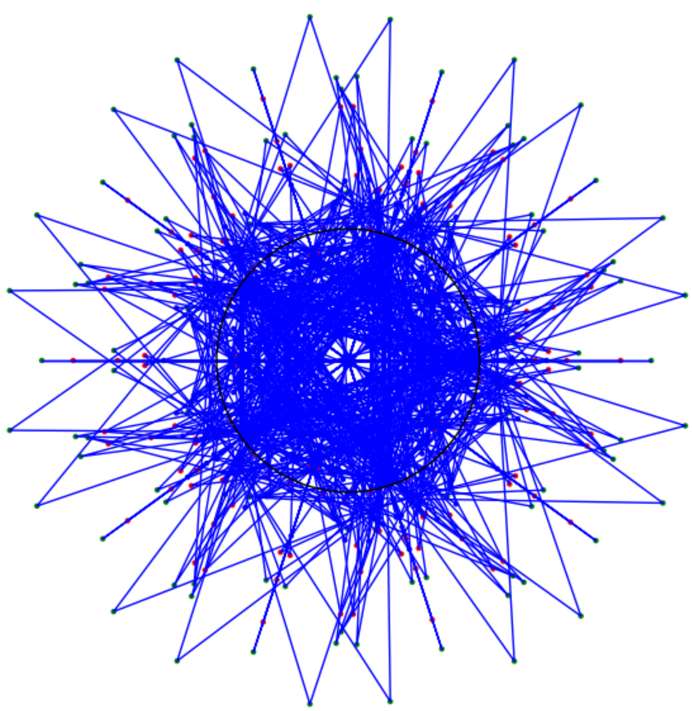

Figure 28: Solutions for $d = 5$ and $\mathbf{m} = (1, -2, 4)$, Case 1[20, 21]

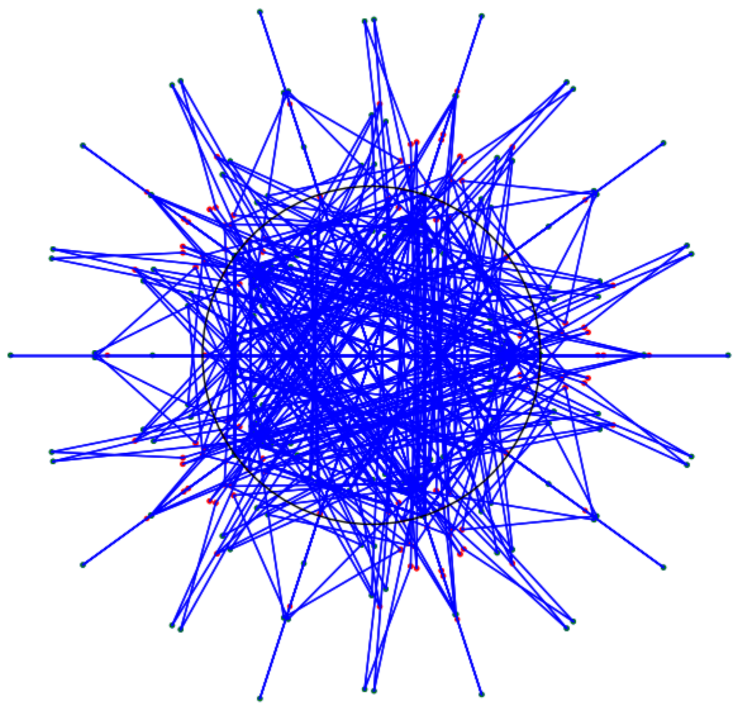

Figure 29: Solutions for $d = 5$ and $\mathbf{m} = (2, -1, 2)$, Case 1[20, 21]

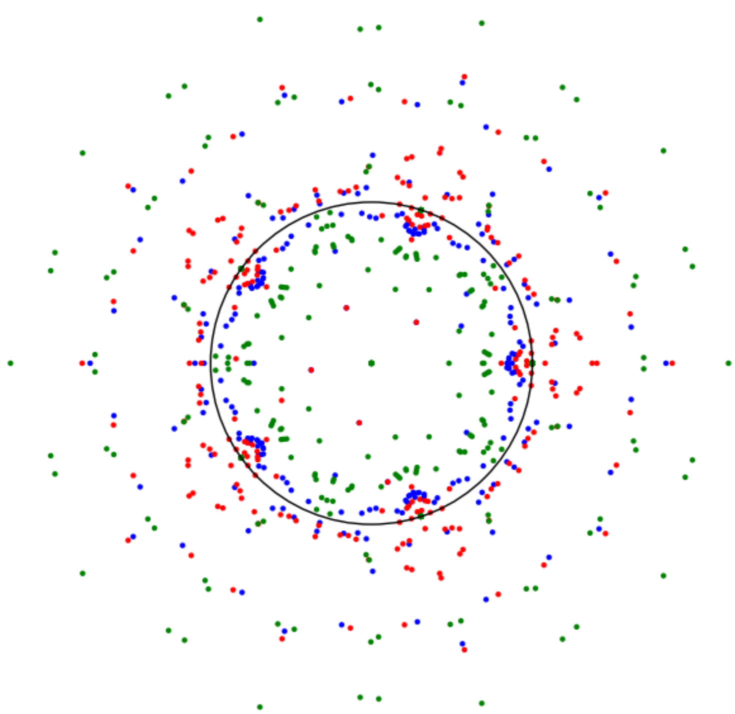

Figure 30: Solutions for $d = 5$ and $\mathbf{m} = (1, -1, 2)$, Case 2 [20, 21]

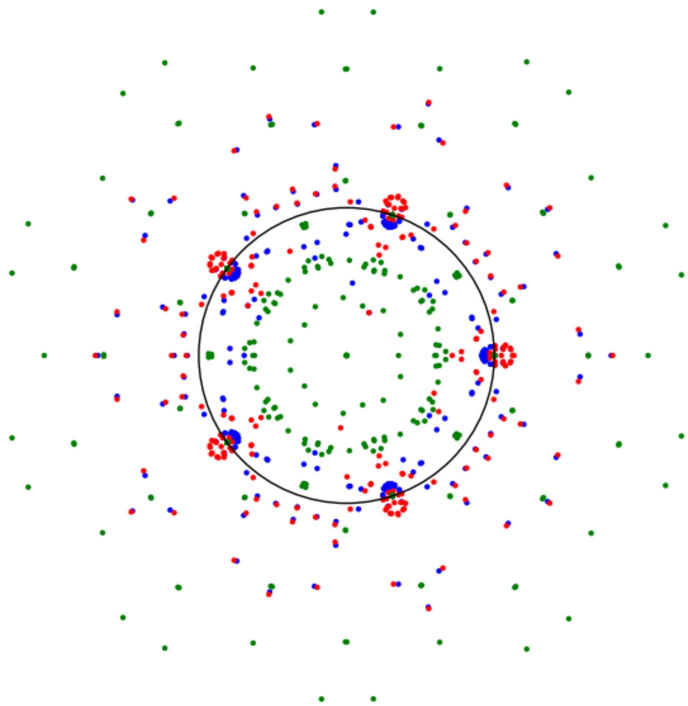

Figure 31: Solutions for $d = 5$ and $\mathbf{m} = (1, -1, 4)$, Case 2[20, 21]

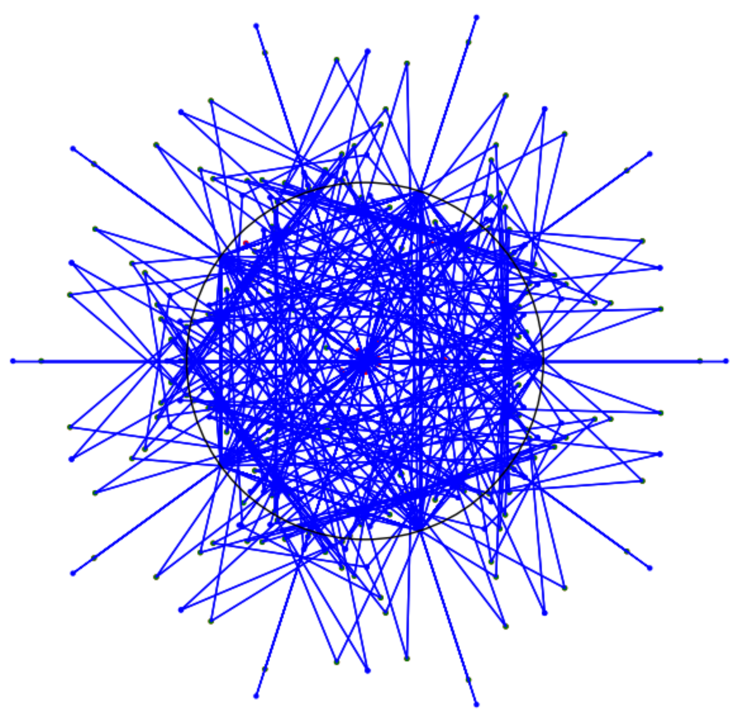

Figure 32: Solutions for $d = 5$ and $\mathbf{m} = (1, -1, -1)$, Case 3[20, 21]

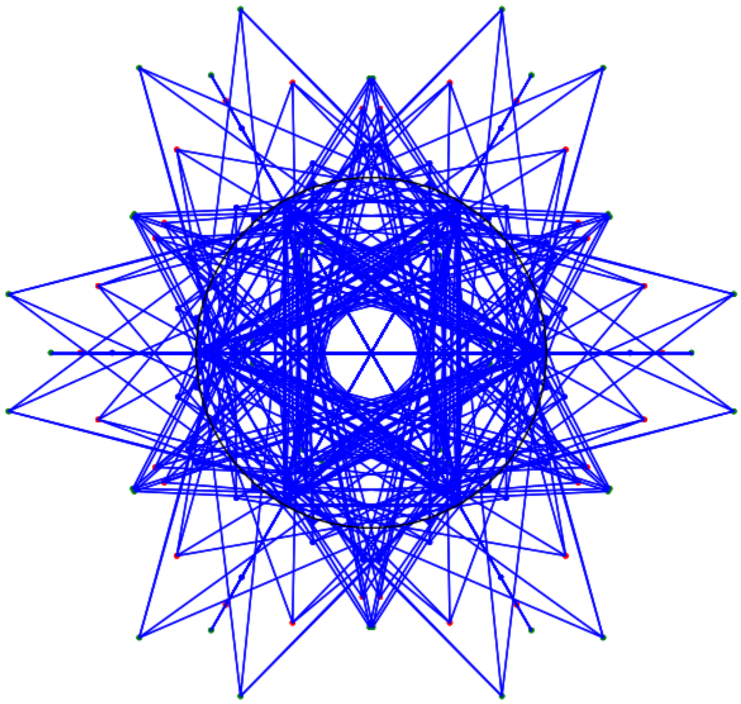

Figure 33: Solutions for $d = 6$ and $\mathbf{m} = (1, 2, 4)$, Case 1[20, 21]

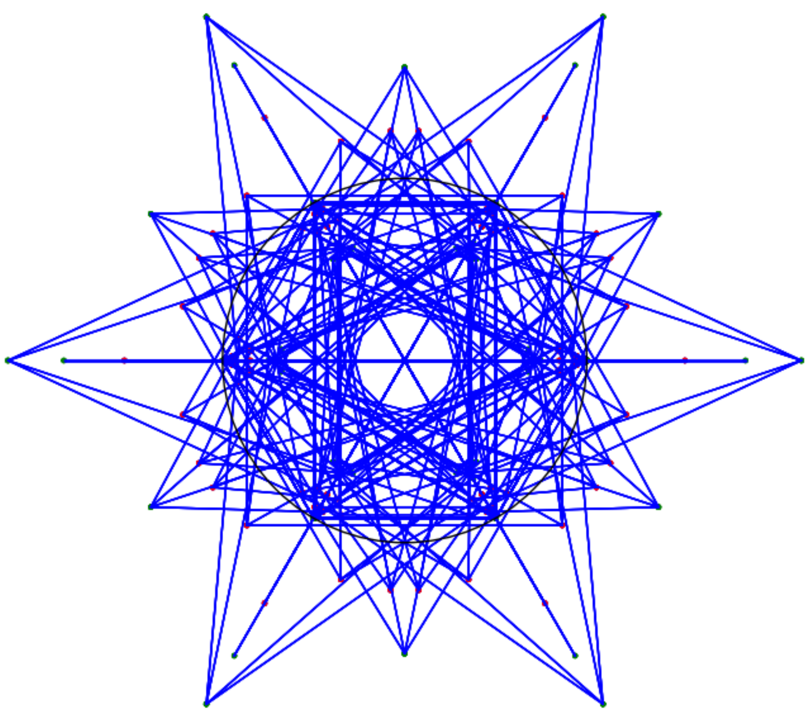

Figure 34: Solutions for $d = 6$ and $\mathbf{m} = (1, 1, 4)$, Case 1[20, 21]

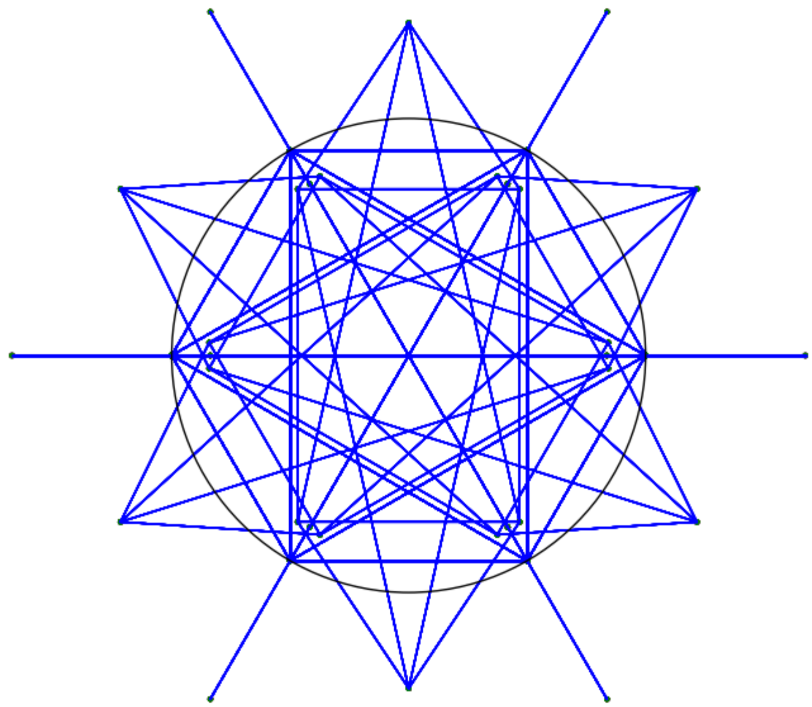

Figure 35: Solutions for $d = 6$ and $\mathbf{m} = (1, 1, 1)$, Case 1[20, 21]

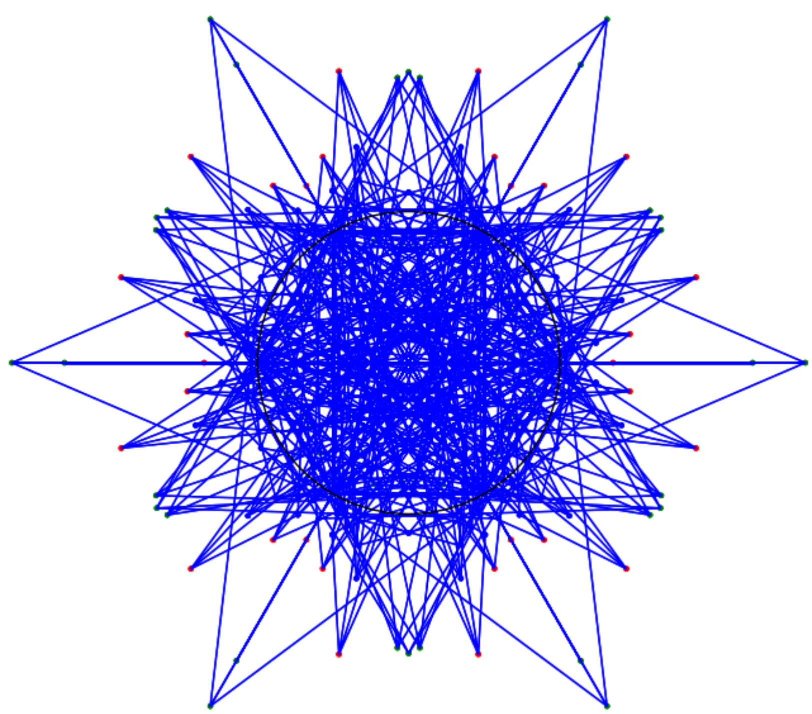

Figure 36: Solutions for $d = 6$ and $\mathbf{m} = (1, -2, 4)$, Case 1[20, 21]

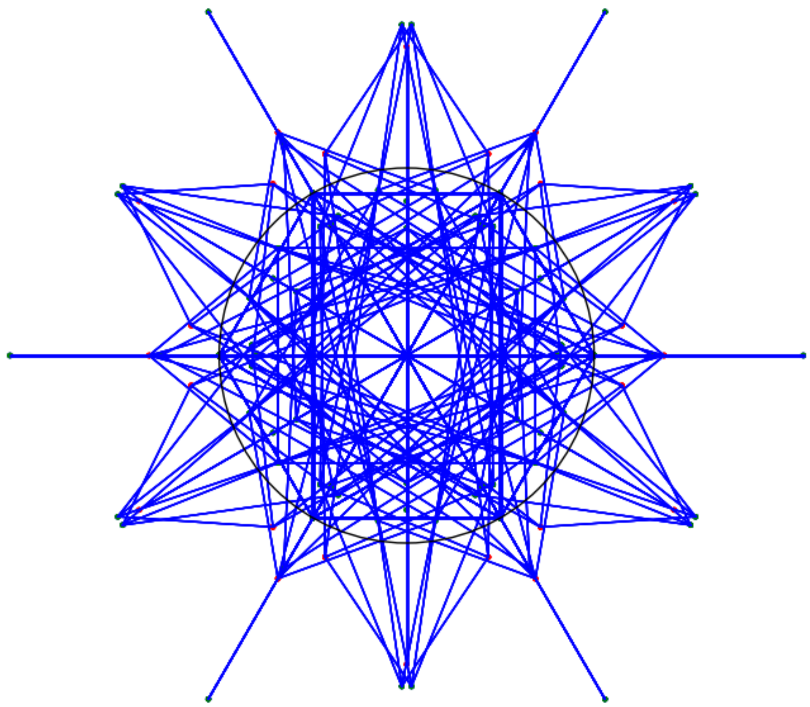

Figure 37: Solutions for $d = 6$ and $\mathbf{m} = (2, -1, 2)$, Case 1[20, 21]

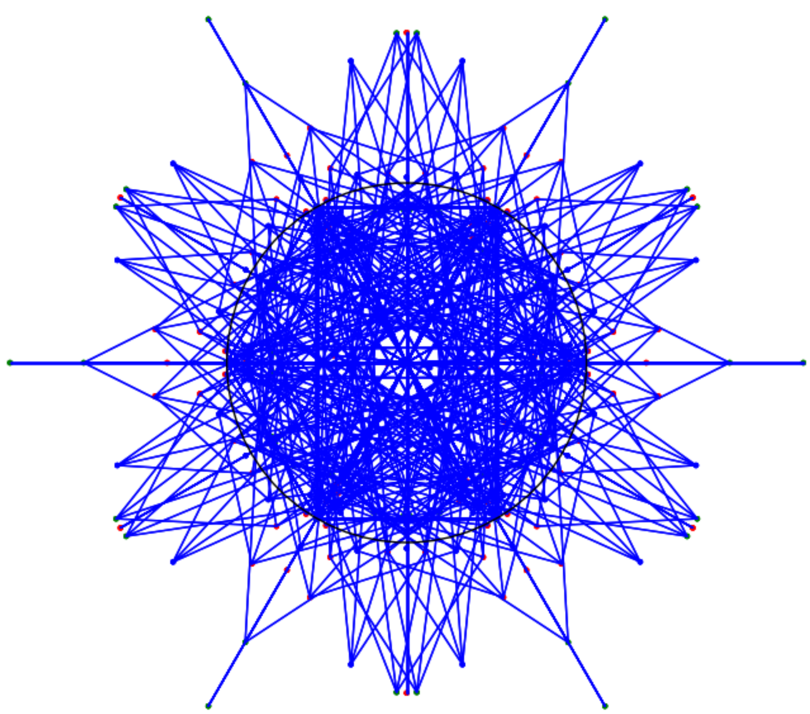

Figure 38: Solutions for $d = 6$ and $\mathbf{m} = (1, -1, 2)$, Case 2[20, 21]

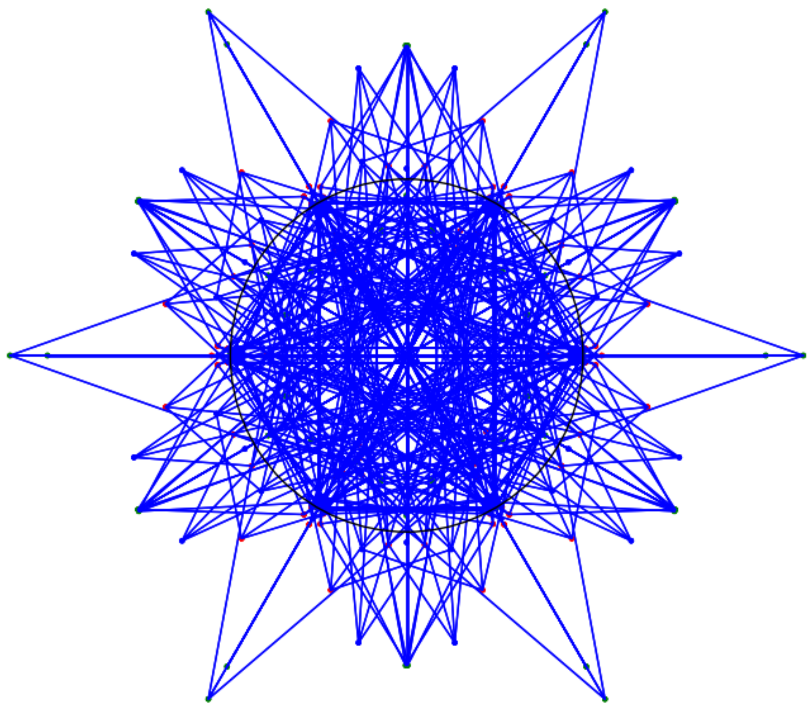

Figure 39: Solutions for $d = 6$ and $\mathbf{m} = (1, -1, 4)$, Case 2[20, 21]

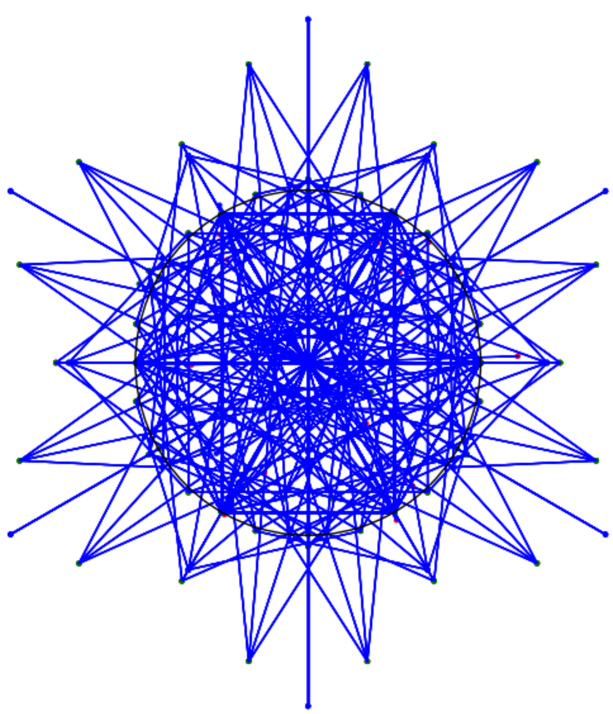

Figure 40: Solutions for $d = 6$ and $\mathbf{m} = (1, -1, -1)$, Case 3[20, 21]

4.3 Observations

Probably the most obvious feature of these sets of solutions is their rotational symmetry. Every solution for a given choice of d has d-fold rotational symmetry about the origin. That means that if $\mathbf{r}$ is a solution, then rotating $\mathbf{r}$ by $\frac{2\pi n}{d}$ radians will also be a solution. This can easily be proven. Let $e^{2\pi n/d} = z$ and $\mathbf{s} = z\mathbf{r} = (zr_1, zr_2, zr_3) = (s_1, s_2, s_3)$ (that is $\mathbf{r}$ times a dth root of unity).

$\mathbf{r}$ is solution to the system

$$0 = -2(r_{12}r_{13}r_{23})^d(m_1+m_2+m_3) + 2(r_{12}r_{13})^d(m_2+m_3)$$
$$+m_1(r_{23}^{d-2}r_{12}^{d}r_{13}^{2} + r_{23}^{d-2}r_{12}^{2}r_{13}^{d} + r_{12}^{d}r_{23}^{d} + r_{13}^{d}r_{23}^{d} - r_{23}^{d-2}r_{12}^{d+2} - r_{23}^{d-2}r_{13}^{d+2})$$

$$0 = -2(r_{12}r_{13}r_{23})^d(m_1+m_2+m_3) + 2(r_{12}r_{23})^d(m_1+m_3)$$
$$+m_2(r_{13}^{d}r_{12}^{d} + r_{13}^{d-2}r_{12}^{d}r_{23}^{2} + r_{13}^{d-2}r_{12}^{2}r_{23}^{d} + r_{13}^{d}r_{23}^{d} - r_{13}^{d-2}r_{12}^{d+2} - r_{23}^{d+2}r_{13}^{d-2})$$

$$0 = -2(r_{12}r_{13}r_{23})^d(m_1+m_2+m_3) + 2(r_{13}r_{23})^d(m_1+m_2)$$
$$+m_3(r_{12}^{d}r_{13}^{d} + r_{12}^{d-2}r_{13}^{d}r_{23}^{2} + r_{12}^{d-2}r_{13}^{2}r_{23}^{d} + r_{12}^{d}r_{23}^{d} - r_{12}^{d-2}r_{13}^{d+2} - r_{12}^{d-2}r_{23}^{d+2})$$

Now consider what happens when replacing r_{12}, r_{13}, and r_{23} with $s_{12} = zr_{12}$, $s_{13} = zr_{13}$, and $s_{23} = zr_{23}$ respectively.

Obviously, $r_{ij}^{nd} = s_{ij}^{nd}$ for any $n \in \mathbb{Z}$ and $s_{12}^a s_{13}^b s_{23}^c = z^a r_{12}^a \cdot z^b r_{13}^b \cdot z^c r_{23}^c = z^{a+b+c} r_{12}^a r_{13}^b r_{23}^c$. That is we convert from $\mathbf{r}$ to $\mathbf{s}$ by multiply each term by z^D where D is total degree of that term with respect to $\mathbf{r}$.

Since for any term in the system the total degree is either $3d$ or $2d$, we would end up with coefficients of z^{3d} and z^{2d}. But $z^{3d} = z^{2d} = 1$, so the equations for the rotated system is identical to the original one. Thus $\mathbf{s}$ is also a solution for the system.

There are other useful results that arise for this result. The foremost is that, since any solution $\mathbf{r}$ implies *d-1* rotated copies of $\mathbf{r}$ that are also solutions, the total number of solutions must be divisible by d. This is the rational behind the Best Guess for the number of complex solutions presented above.

This also explains why the number of real solutions alternates between 2 values depending on whether d is even or odd. If $\mathbf{r}$ is a solution in $\mathbb{R}$ (thus $\mathbf{r}$ is on the real axis) and d is even, if we rotate $\mathbf{r}$ by π radians (or equivalently take $\mathbf{s} = z^{\pi}\mathbf{r} = -\mathbf{r}$) we obtain another solution, that also falls on the real axis. If d is odd, then we don't get these rotated real solutions.

It should be noted that $\mathbf{r} = (1, 1, 1)$ is always a solution to the system, and for that matter (z_1, z_2, z_3), where z_i is a dth root of unity is also a solution. Recall that the Albouy-Chenciner equations can be written as [4]:

$$\sum_{k=1}^{n} m_k \left[S_{ik}(r_{jk}^2 - r_{ik}^2 - r_{ij}^2) + S_{jk}(r_{ik}^2 - r_{jk}^2 - r_{ij}^2)\right] \quad for\ 1 \leq i \leq j \leq n$$

$$S_{ij} = \frac{1}{r_{ij}^d} - 1$$

From this, it's clear that since $r_{ij}^d = 1$, $S_{ij} = 0$ for all *i,j* making every equation above equal to 0. Thus each of the d^3 number of points $\mathbf{r} = (z_1, z_2, z_3)$, are solutions to the Albouy-Chenciner equations.

4.3.1 Failures

As discussed in the introduction, PHC exploits the polynomial structure of a system to attempt to numerically find all the possible solutions to the system. The basis of this is the fact that the Mixed Volume of a polynomial is both the upper bound for the number of nonzero solutions and the actually number of solutions for a appropriately generic choice of coefficients [20]. However, our system is far from generic, so we should consider how solutions can arise.

PHC can fail in a couple ways. A solution to the initial system could fail to converge to a point, or could extend off towards infinity. This situation should be identified as a *failure* by PHC and presented as such. However, this is not not the only way PHC can fail to arrive at a valid solution. Recall that we are interested in the non-zero solutions to our system. It is possible for PHC solutions to converge to these solutions where at least one of the coordinates is 0.

The system seems very well behaved when the masses are all positive. As shown above, there were no failures [20]. However, this changes as the choice of masses change.

When we allow canceling masses, the behavior of PHC changes. We start getting large numbers of failures, as well as spurious solutions. For these cases, when one of the distances is 0 there is a curve of solutions that is attracting PHC solutions.

Let $m_1 + m_2 = 0$ and let $r_{12} = 0$. After making these substitutions into the generalized Albouy-Chenciner, we get [4]:

$$
\begin{aligned}
m_1(r_{13}^d r_{23}^d - r_{23}^{d-2} r_{13}^{d+2}) &= m_1 r_{13}^d r_{23}^{d-2}(r_{23}^2 - r_{13}^2) = m_1 r_{13}^d r_{23}^{d-2}(r_{23} + r_{13})(r_{23} - r_{13}) \\
m_2(r_{13}^d r_{23}^d - r_{23}^{d+2} r_{13}^{d-2}) &= m_2 r_{13}^{d-2} r_{23}^d(r_{13}^2 - r_{23}^2) = m_2 r_{13}^{d-2} r_{23}^d(r_{13} + r_{23})(r_{13} - r_{23}) \\
& \hphantom{= m_2 r_{13}^{d-2} r_{23}^d(r_{13}^2 - r_{23}^2) = m_2 r_{13}^{d-2} r_{23}^d(r_{13} + r_{23})(r_{13} - r_{23})} 0
\end{aligned}
$$

From this, it's clear that any choice of distances where $r_{23} = \pm r_{13}$ will be a solution to the system. On the included solution plots these will appear as a line through the center of the circle, with the center in the middle of the line, or with 2 points on top of each other next each other. An example of the first type can be seen below, particularly in the points arrayed around the center.

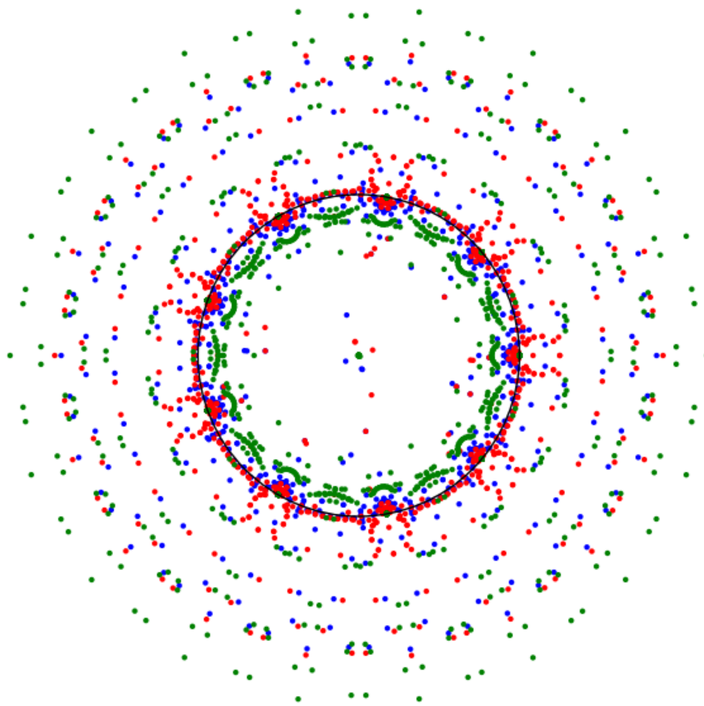

Figure 41: Solutions for $d = 9$ and $\mathbf{m} = (1, -1, 2)$ [20, 21]

In general, spurious solutions can be found by making use of the rotational symmetry of the true solutions. Thus if we find sets of solutions that obviously don't have this symmetry, they can not be true solutions to the system. Although, it should be noted that some of the solutions that complete the d-symmetric pattern may simply be missing.

5 Future and Ongoing work

There are two basic directions that future work in this area could take. The first is to continue the study of the General 3-Mass Albouy-Chenciner problem, while the second is to extend this work into 4 or more bodies.

For the 3-mass case, the remaining questions relate to how the solutions will change with the choice of masses. Only a cursory analysis has been performed in this paper, and there is certainly further work that could be done on the subject. One specific topic that can be addressed is the number of solutions in the cases where masses cancel out. It may be possible to find a simplification for the system that have a lower mixed volume, that will correspond to the number of genuine solutions for these cases. Even without this, a more complete study of the failed and spurious solutions might help to generate an accurate count of the true solutions.

Extending this study to higher number of bodies is a more complicated and difficult problem. Finiteness proofs have yet to be completed for the 4-vortex problems for all possible vorticity configurations. Current proofs omit certain vorticity conditions, where the vorticities cancel out in peculiar ways [6]. For higher numbers of bodies, the difficulty of these proofs becomes almost intractable. Amongst other concerns, the number of ways the coefficients can cancel amongst themselves increases rapidly. It is possible that some of the techniques used in this thesis may be able to patch some of the holes present in previous works, as well as leading to a better understand why these conditions cause difficulties.

By using these techniques, I was able to further complete the work of Hampton and Moeckel on the 4-vortex problem [6]. First, the tropical prevariety for the 4-vortex system was computed. Then for each of the cones in the tropical prevariety, the initial form of the system was computed. At this point, the Groebner basis for the initial form was generated and an elimination ideal for the Groebner basis was computed, eliminating any zero distances. This allowed many of the cones in the tropical prevariety to be eliminated immediately and to turn our attention to the more problematic rays [18].

The issues arise when either a pair or triplet of vorticities where their sum is 0, that is $\Gamma_i + \Gamma_j = 0$ or $\Gamma_i + \Gamma_j + \Gamma_k = 0$ [6, 18]. It seemed logical to first consider what occurs when both these conditions occur simultaneously, that is $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ and $\Gamma_1 = -\Gamma_4$. We can specify any two of the values $\Gamma_1, \Gamma_2, \Gamma_3$. The steps were then repeated for these vorticity conditions, and found two (up to permutation of indices) troublesome conditions for the remaining vorticities. The first of these is of the form $\Gamma_1 = \Gamma_2$, and thus $\Gamma_3 = -2\Gamma_1$ and $\Gamma_4 = -\Gamma_1$. We can choose the vorticity vector to be $(1, 1, -2, -1)$. The second special case is where $\Gamma_3^2 + \Gamma_3\Gamma_1 - \Gamma_1^2 = 0$. Choosing $\Gamma_1 = 1$, then $\Gamma_3 = \frac{-1\pm\sqrt{5}}{2} = \varphi - 1, -\varphi$ (where φ is the golden ratio, approx. $1.6180\ldots$). Thus the vorticity vector is $(1, -\varphi, \varphi - 1, -1)$ or $(1, \varphi - 1, -\varphi, -1)$ [18].

For the first vorticity condition, we were able to determine a lexicographic Groebner basis for the Albouy-Chenciner equations for the system [19]. This establishes an upper bound for the number of solutions for this vorticity condition. Thus the only way the 4-vortex system can have an infinite number of solutions is if the system fulfills a quadratic vorticity condition such as the one above.

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