Local Cohomology Modules Over Polynomial Rings Of Prime Characteristic

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

YI ZHANG

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
Doctor of Philosophy

GENNADY LYUBEZNIK, Adviser

July, 2012

Acknowledgements

I would like to express my deepest gratitude to my adviser Professor Gennady Lyubeznik, who shared with me his wonderful insights and perspectives in mathematics and provided so much encouragement and every kind of support during my graduate study and towards graduation. I am grateful to Professor Craig Huneke for hosting my visit to University of Kansas and introducing me to several other topics which are not discussed here. I thank my wife, Huiqiong Deng, for her continued support and love.

Dedication

To my parents, Zhiguo Zhang and Yuanzhen Li, who would be very happy for me.

Abstract

This thesis studies the local cohomology modules over polynomial rings of prime characteristic.

Many great mathematicians have contributed to the theory of local cohomology. However, the structure of local cohomology modules is still full of mystery. In most cases $H^i_I(M)$ is not finitely generated. So it is very difficult to tell whether the local cohomology modules are 0 or not, and it is also the main reason it had been a long time before an algorithm [21] was found in characteristic 0 and implemented into computer programs.

We study the local cohomology modules of polynomial rings in prime characteristic. Let $R = k[x_1, \cdots, x_n]$ be a polynomial ring over a field k of characteristic $p > 0$. If I is an ideal of R, we denote $H^i_I(R)$ the i-th local cohomology module of R with support in I. After some introductory material on local cohomology, we will give a lower bound of the dimension of associated primes P of $H^i_I(R)$ in terms of the degrees of the generators of I. Let $\mathfrak{m} = (x_1, \cdots, x_n)$ be the maximal ideal generated by the variables and let $I_1, \cdots, I_s$ be homogeneous ideals of R. We will describe the grading of $H^i_{\mathfrak{m}}(H^{j_1}_{I_1}(\cdots(H^{j_s}_{I_s}(R))))$ and also give two algorithms to calculate it.

Contents

Chapter 1

Introduction

Local cohomology theory, developed by A. Grothendieck, played a key role in his remarkable solution to Samuel's Conjecture on factoriality [5]. Since then, local cohomology theory has become an important and interesting research area of its own, sitting in the crossroad of algebra and topology.

The thesis studies local cohomology modules over polynomial rings of prime characteristic. Chapter 2 reviews the fundamentals of local cohomology. Chapter 3 presents the local duality and then our adjointness theorem [15]. Chapter 4 summarizes some general vanishing theorems and then include our theorem in prime characteristic [23]. Chapter 5 introduces the graded version of F-module theory [24]. Chapter 6 describes two algorithms to compute the local cohomology modules with support in the irrelevant ideal in prime characteristic.

All rings in this thesis are commutative with identity element 1, all homomorphisms of rings take 1 to 1. Let R be a ring and $X = \mathrm{Spec}R$, then R-mod denotes the category of R-modules, $\mathfrak{M}(X)$ denotes the category of sheaves of $\mathcal{O}_X$-modules on the ringed space $(X, \mathcal{O}_X)$. We assume the facts that both of R-mod and $\mathcal{O}_X$-modules have enough injective objects, and for any R-module M, there exists a unique minimal injective module $E(M)$ containing it. Otherwise, the notations follow Atiyah-Macdonald [1] and Hartshorne [7]. Note that when results are cited, the first symbol indicates the chapter in the reference.

Chapter 2

Local Cohomology

2.1 Definitions

Algebraically, let R be a noetherian ring, I an ideal of R, M an R-module, define Γ_I the R-torsion functor from R-mod to itself as

$$\Gamma_I(M) = \bigcup_{n\in\mathbb{N}} (0 :_M I^n) = \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_R(R/I^n, M).$$

Geometrically, let $X = \mathrm{Spec}R$, let $Y = V(I)$ be the set of all prime ideals of R which contain I, define $\Gamma_Y(X, \cdot)$ the group of sections with support in Y from $\mathfrak{M}(X)$ to R-mod as

$$\Gamma_Y(X, \mathscr{F}) = \{s \in \mathscr{F} | \mathrm{Supp}(s) \subseteq Y\} = \mathrm{Hom}_X(\mathbb{Z}_{Y,X}, \mathscr{F}),$$

where $\mathbb{Z}_{Y,X}$ is the constant sheaf $\mathbb{Z}_Y$ of integers on Y extended by zero outside of Y. Then

$$\Gamma_I(M) = \Gamma_Y(X, \tilde{M}).$$

This is based on the following elementary observation. For any $m \in \Gamma_I(M) = \bigcup(0 : I^n)$, there is n such that $I^n m = 0$. Then for any $p \in X - Y$, there is $f \in I \setminus p$, then $m = f^n m/f^n = 0 \in M_p$, that is $\mathrm{Supp}(m) \subseteq Y$. Conversely, for any $m \in \Gamma_Y(X, \tilde{M})$, then $m = 0$ on $D(f_i)$ given $I = (f_1, \cdots, f_s)$, that is $X - Y = \bigcup D(f_i)$. Hence for each i, $f_i^{k_i} m = 0$ for some k_i and so $I^{\sum k_i} m = 0$, that is $m \in \Gamma_I(M)$.

Since $\mathrm{Hom}_R(R/I^n, \cdot)$ and $\mathrm{Hom}_X(\mathbb{Z}_{Y,X}, \cdot)$ are left exact and the direct limit functor is exact, the functors $\Gamma_I(\cdot)$ and $\Gamma_Y(X, \cdot)$ are left exact. Denote their right derived functors respectively on the proper categories by

$$H_I^i(\cdot) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}_R^i(R/I^n, \cdot) \text{ and } H_Y^i(X, \cdot) = \mathrm{Ext}_X^i(\mathbb{Z}_{Y,X}, \cdot).$$

For any injective module E, the associated quasi-coherent sheaf $\tilde{E}$ on X is flasque ([7, III.3.4]), and hence acyclic ([7, III.2.5]). Therefore, we can use the injective R-module resolutions to calculate both $H_I^i(\cdot)$ and $H_Y^i(X, \tilde{\cdot})$ ([22, Ex 2.4.3]), and so for any $M \in R$-mod,

$$H_I^i(M) = H_Y^i(X, \tilde{M}),$$

which is called the local cohomology module of M with support in I.

There are other different approaches to define local cohomology modules. Please refer to [2, 4, 8] for details.

If M is an R-module, $I = (f_1, \cdots, f_s)$ is an ideal of R, then define the cocomplex $C^\cdot(M)$ as follows. For each $k \geqslant 0$, let $C^0(M) = M$ and

$$C^k(M) = \prod_{i_1<\cdots<i_k} M_{f_{i_1}\cdots f_{i_k}}.$$

Thus an element $m \in C^k(M)$ is determined by giving an element

$$m_{i_1,\cdots,i_k} \in M_{f_{i_1}\cdots f_{i_k}},$$

for each k-tuple $i_1 < \cdots < i_k$. Define the coboundary map $d : C^k \to C^{k+1}$ by setting

$$(dm)_{i_1,\cdots,i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j-1} l(m_{i_1,\cdots,\hat{i_j},\cdots,i_{k+1}}),$$

where l are natural localizations, with the cohomology modules $H^i(C^\cdot(M))$.

Theorem 2.1.1. *(Čech cohomology)*

$$H^i(C^\cdot(M)) = H_I^i(M).$$

Proof. It is obvious that $(H^i(C^\cdot(\cdot)))_{i\geq 0}$ is a covariant δ-functor from R-mod to itself and $H^0(C^\cdot(M)) = H_I^0(M)$. For any injective I, each f_j acts on the indecomposable summand of I either nilpotently or as a unit. Then $C^\cdot(I)$ is acyclic, that is $(H^i(C^\cdot(\cdot)))_{i\geq 0}$ is universal by [7, III.1.3A]. Therefore, $(H^i(C^\cdot(\cdot)))_{i\geq 0}$ equals the right derived functor $(H_I^i(\cdot))_{i\geq 0}$. $\square$

This definition implies that for $i > 0$, the local cohomology $H_I^{i+1}(M)$ coincides with the Čech cohomology in sheaf theory $H^i(X - Y, \tilde{M})$, where $Y = V(I) \subset X$. See Proposition 2.2.4 for a generalization.

If M is an R-module, $I = (f_1, \cdots, f_s)$ is an ideal of R. Let $K^{\cdot}(f)$ be the cocomplex of free R-modules $0 \to Re_{\emptyset} \xrightarrow{f} Re_f \to 0 \to \cdots$, then define the cocomplex $K^{\cdot}(f_1^t, \cdots, f_s^t, M) = K^{\cdot}(f_1^t) \otimes \cdots \otimes K^{\cdot}(f_s^t) \otimes M$ with the cohomology modules $H^i(K^{\cdot}(f_1^t, \cdots, f_s^t, M))$. For any pair of positive integers $t \leqslant t'$, there is a natural map of the Koszul cocomplexes $K^{\cdot}(f_1^t, \cdots, f_s^t, M) \to K^{\cdot}(f_1^{t'}, \cdots, f_s^{t'}, M)$. This implies the maps of the cohomology modules for $t \leqslant t'$,

$$H^i(K^{\cdot}(f_1^t, \cdots, f_s^t, M)) \to H^i(K^{\cdot}(f_1^{t'}, \cdots, f_s^{t'}, M)).$$

Theorem 2.1.2. *(Koszul cohomology)*

$$\varinjlim_t H^i(K^{\cdot}(f_1^t, \cdots, f_s^t, M)) = H^i(C^{\cdot}(M)).$$

Proof. When $s = 1$, since $M_f = \varinjlim \frac{M}{f^t}$, $\varinjlim K^{\cdot}(f^t, M) = C^{\cdot}(M)$ with respect to the following diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f^t"'] & C^0(M)(= M) \arrow[d] \\ M \arrow[r, "1/f^t"] & C^1(M)(= M_f) \end{tikzcd}$$

In general,

$$\varinjlim_t K^{\cdot}(f_1^t, \cdots, f_s^t, M) = \varinjlim_t K^{\cdot}(f_1^t) \otimes \cdots \otimes \varinjlim_t K^{\cdot}(f_s^t) \otimes M = C^{\cdot}(M)$$

by [1, Ex 2.20]. Therefore $\varinjlim_t H^i(K^{\cdot}(f_1^t, \cdots, f_s^t, M)) = H^i(C^{\cdot}(M))$. □

Since these four approaches to local cohomology are equivalent, we will switch freely in between them, and especially in between the algebraic and geometric languages.

2.2 Basic Properties

We list here the basic properties of the local cohomology modules. We discuss in the category of R-mod, here R is a commutative ring with an identity element, and denote $Y = V(I)$ and $Z = V(b)$ for ideals R and b of R.

Proposition 2.2.1. *Let M be an R-module.*

1. $\Gamma_I(M) = \Gamma_{\sqrt{I}}(M)$, $\Gamma_J(\Gamma_I(M)) = \Gamma_{I+J}(M)$, *and* $\Gamma_{I\cap J}(M) = \Gamma_{IJ}(M)$.

2. *If* $(R, \mathfrak{m})$ *is local and* $\dim M > 0$, *then* $\dim M/\Gamma_{\mathfrak{m}}(M) = \dim M$.

3. *For any* $i > 0$, $H^i_I(\Gamma_I(M)) = 0$ *and* $H^i_I(M/\Gamma_I(M)) = H^i_I(M)$.

Proof. 1. These follow from the facts that $V(I) = V(\sqrt{a})$, $\Gamma_Z(X, \Gamma_Y(X, \tilde{M})) = \Gamma_{Y\cap Z}(X, \tilde{M})$ and $V(I \cap J) = V(IJ) = Y \cup Z$.

2. Since $(M/\Gamma_{\mathfrak{m}}(M))_p = M_p$ when $p \neq \mathfrak{m}$ and $(M/\Gamma_{\mathfrak{m}}(M))_{\mathfrak{m}} = M/\Gamma_{\mathfrak{m}}(M) \neq 0$ by $\dim M > 0$, we have that $\mathrm{Supp}(M)/\Gamma_{\mathfrak{m}}(M) = \mathrm{Supp}(M)$.

3. If E is an injective R-module, then $\Gamma_I(E)$ is also injective by [7, III.3.1]. Therefore, we can inductively construct an injective resolution $E^{\cdot}$ of $\Gamma_I(M)$ such that $\Gamma_I(E^i) = E^i$ for each i by the left exactness of $\Gamma_I(\cdot)$. This proves that $H^i_I(\Gamma_I(M)) = 0$ for all $i > 0$. Applying this fact to the short exact sequence

$$0 \to \Gamma_I(M) \to M \to M/\Gamma_I(M) \to 0$$

shows $H^i_I(M/\Gamma_I(M)) = H^i_I(M)$ for all $i > 0$. □

Proposition 2.2.2. *The local cohomology commutes with direct limit.*

Proof. The direct limit functor is exact in R-mod. □

Proposition 2.2.3. *Let $R \to R'$ be a homomorphism of rings and I be an ideal of R, then for all i, there are isomorphisms of local cohomology functors*

$$H^i_{IR'}(\,\cdot\,)|_R \cong H^i_I(\,\cdot\,|_R).$$

If in addition R' is flat over R, then there are isomorphisms

$$H^i_I(\,\cdot\,) \otimes_R R' \cong H^i_{IR'}(\,\cdot\, \otimes R').$$

Proof. Both the restriction of scalars and the flat base change functors are exact in R-mod. □

Proposition 2.2.4. *For any R-module M, there are an exact sequence*

$$0 \to H_I^0(M) \to M \to H^0(X - Y, \tilde{M}) \to H_I^1(M) \to 0$$

and an isomorphism for each $i > 0$

$$H^i(X - Y, \tilde{M}) = H_I^{i+1}(M).$$

Proof. Suppose $\mathscr{F}$ is flasque. The inclusion $\Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F})$ is obviously injective. If $s \in \Gamma_Y(X, \mathscr{F})$, then $s_{X-Y} = 0$; conversely, if $s_{X-Y} = 0$, then $\mathrm{Supp}(s) \subseteq Y$. If $\mathscr{F}$ is flasque, then $\Gamma(X, \mathscr{F}) \to (X - Y, \mathscr{F})$ is surjective by definition. Hence the sequence

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F}) \to 0$$

is exact.

Let $0 \to I^0 \to I^1 \to \cdots$ be an injective resolution of M in R-mod. By [7, III.3.4], $\tilde{I}^i$ is flasque for each i, then

$$0 \to \Gamma_Y(X, \tilde{I}^i) \to \Gamma(X, \tilde{I}^i) \to \Gamma(X - Y, \tilde{I}^i) \to 0$$

is exact for each i. Hence there is an associated long exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H_Y^0(X, \tilde{M}) \to H^0(X, \tilde{M}) \to H^0(X - Y, \tilde{M}) \to \\ &\to H_Y^1(X, \tilde{M}) \to H^1(X, \tilde{M}) \to H^1(X - Y, \tilde{M}) \to \\ &\to H_Y^2(X, \tilde{M}) \to \cdots . \end{aligned}$$

Now the conclusion follows from Serre's affineness criterion. $\square$

Proposition 2.2.5. *(Excision) Let U be an open subset of X containing Y, then for all i and $\mathscr{F}$,*

$$H_Y^i(X, \mathscr{F}) = H_Y^i(U, \mathscr{F}|_U).$$

Proof. This follows from the equivalence of the functors $\Gamma_Y(U, \cdot|_U)$ and $\Gamma_Y(X, \cdot)$ from $\mathfrak{M}(X)$ to R-mod. $\square$

Proposition 2.2.6. *(Mayer-Vietoris) For any R-module M, there is a long exact sequence*

$$\begin{aligned}0 \to H^0_{I+J}(M) &\to H^0_I(M) \oplus H^0_J(M) \to H^0_{I\cap J}(M) \to \\ \to H^1_{I+J}(M) &\to H^1_I(M) \oplus H^1_J(M) \to H^1_{I\cap J}(M) \to \\ \to H^2_{I+J}(M) &\to \cdots .\end{aligned}$$

Proof. Applying $\operatorname{Hom}_X(\cdot, \mathscr{I}^{\cdot})$, where $\mathscr{I}^{\cdot}$ is an injective resolution of $\tilde{M}$ in $\mathfrak{M}(X)$, to the short exact sequence

$$0 \to \mathbb{Z}_{Y\cup Z,X} \to \mathbb{Z}_{Y,X} \oplus \mathbb{Z}_{Z,X} \to \mathbb{Z}_{Y\cap Z,X} \to 0$$

induces the long exact sequence

$$\begin{aligned}0 \to H^0_{Y\cap Z}(X,\tilde{M}) &\to H^0_Y(X,\tilde{M}) \oplus H^0_Z(X,\tilde{M}) \to H^0_{Y\cup Z}(X,\tilde{M}) \to \\ \to H^1_{Y\cap Z}(X,\tilde{M}) &\to H^1_Y(X,\tilde{M}) \oplus H^1_Z(X,\tilde{M}) \to H^1_{Y\cup Z}(X,\tilde{M}) \to \\ \to H^2_{Y\cap Z}(X,\tilde{M}) &\to \cdots .\end{aligned}$$

□

Definition 2.2.7. *The arithmetic rank $ara(I)$ of $I \subset R$ is defined either geometrically as the least number of hypersurfaces whose intersections is $Y = V(I)$ or algebraically as the least number of generators of any ideal J such that $\sqrt{J} = \sqrt{I}$.*

Proposition 2.2.8. *The ith local cohomology module vanishes whenever i exceeds $ara(I)$.*

Proof. This follows from Proposition 2.2.1 and the definition of the Čech cohomology. □

Proposition 2.2.9. *If $(R, \mathfrak{m})$ is local noetherian and M is non-zero finitely generated, then $H^i_{\mathfrak{m}}(M)$ is artinian for all i.*

Proof. For any prime ideal $\mathfrak{p}$ of R, let $E(R/\mathfrak{p})$ denote the injective hull of $R/\mathfrak{p}$. If $I^{\cdot}$ is the minimal injective resolution of M, then

$$I^i \cong \bigoplus_{\mathfrak{p}\in\operatorname{Spec} R} E(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)},$$

where $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$ is the ith Bass number of M with respect to $\mathfrak{p}$. Then $\Gamma_{\mathfrak{m}}(I^{\cdot})$ is the cocomplex

$$0 \to E(k)^{\mu_0(\mathfrak{m},M)} \to E(k)^{\mu_1(\mathfrak{m},M)} \to \cdots \to E(k)^{\mu_i(\mathfrak{m},M)} \to \cdots .$$

Since $E(k)$ is artinian, the local cohomology modules $H^i_{\mathfrak{m}}(M)$, as subquotients of $E(k)^{\mu_i(\mathfrak{m},M)}$, are artinian. $\square$

Chapter 3

Duality

3.1 Local Duality

Let $(R, \mathfrak{m}, k)$ be a local ring, then there is a remarkable functor from R-mod to itself, namely the Matlis dual functor $D(\cdot) = \mathrm{Hom}(\cdot, E(k))$, where $E(k)$ denotes the injective hull of $R/\mathfrak{m}$. Since $E = E(k)$ is injective, this functor is exact. The basic properties needed for the local duality are listed below:

1. If M is an R-module, then the canonical map $M \xrightarrow{\theta} DD(M)$ defined by $\theta(m)(\varphi) = \varphi(m)$ for $m \in M$ and $\varphi \in D(M)$ is injective;

2. If M is of finite length, then $D(M)$ is of the same finite length, and so $DD(M) = M$;

3. Let $\hat{R}$ be the completion of R, then E is also an $\hat{R}$-module, and is an injective hull of k as $\hat{R}$-module;

4. $D_R(E) = D_{\hat{R}}(E) = \hat{R}$;

5. E is artinian as both an R-module and an $\hat{R}$-module;

6. Assume R is complete, then for any noetherian module M, $D(M)$ is artinian; for any artinian module M, $D(M)$ is noetherian. In either case, $DD(M) = M$.

Proposition 3.1.1. *Let $(R, \mathfrak{m})$ be a local noetherian ring, then for any finitely generated R-module M with annihilator ideal I, $D(M)$ has the same annihilator ideal I.*

Proof. Suppose $M = \sum Rx_i$, then $0 \to R/\mathrm{ann}M \to \oplus R/\mathrm{ann}x_i \to \sum Rx_i \to 0$ implies that $0 \to R\hat{}/(\mathrm{ann}M)\hat{} \to \oplus R\hat{}/(\mathrm{ann}x_i)\hat{} \to \sum R\hat{}\,x_i \to 0$ and hence $(\mathrm{ann}M)\hat{} = \mathrm{ann}\hat{M}$. Since $\hat{R}$ is faithfully flat over R, $\mathrm{ann}_R M = \mathrm{ann}_{\hat{R}}\hat{M} \cap R$. Therefore, $\hat{M} = D_{\hat{R}}D_{\hat{R}}(\hat{M})$, hence $\mathrm{ann}M \subseteq \mathrm{ann}D_R(M)$ implies that $\mathrm{ann}D_R(M) = I$. □

Definition 3.1.2. *If $(R, \mathfrak{m})$ is a local Cohen-Macaulay ring with the residue field k, then R is called Gorenstein if $\mathrm{Ext}^i(k, R) \neq 0$ only when $i = \dim R$. Equivalently, R is Gorenstein if and only if the minimal injective resolution $I^{\cdot}$ of R satisfies $I^i = \oplus_{\mathrm{ht}p=i} E(R/\mathfrak{p})$.*

Example 3.1.3. *Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring, then $H^n_{\mathfrak{m}}(R) = E$.*

Theorem 3.1.4. *([4, 6.3]) Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring of dimension n, then for any finitely generated R-module M, the pairing*

$$H^i_{\mathfrak{m}}(M) \times \mathrm{Ext}^{n-i}_R(M, R) \to E$$

gives rise to isomorphisms

$$\phi_i : H^i_{\mathfrak{m}}(M) \xrightarrow{\sim} D(\mathrm{Ext}^{n-i}_R(M, R))$$

and

$$\psi_i : \mathrm{Ext}^{n-i}_R(M, R)\hat{} \xrightarrow{\sim} D(H^i_{\mathfrak{m}}(M)).$$

3.2 Adjointness of Frobenius

In this section, $R = k[x_1, \cdots, x_n]$ is the ring of polynomials in n variables over a field k of characteristic $p > 0$. Denote the multi-index $(i_1, \cdots, i_n)$ by $\bar{i}$, especially $\overline{p^l - 1} = (p^l - 1, \cdots, p^l - 1)$ where l is a positive integer. Since all the results in this dissertation concern the vanishing of local cohomology modules and since extending the field is faithfully flat, in the sequel, we can always enlarge the field to make it perfect and infinite. Then R is a free R^{p^l}-module on the p^{ln} monomials $e_{\bar{i}} = x_1^{i_1} \cdots x_n^{i_n}$ where $0 \leqslant i_j < p^l$ for every j. Let $F : R \xrightarrow{r \mapsto r^{p^l}} R$ be the Frobenius homomorphism and denote the source and target of F by R_s and R_t respectively, that is $F : R_s \to R_t$. There are two associated functors

$$F^* : R_s\text{-mod} \to R_t\text{-mod}$$

such that $F^*(-) = R_t \otimes_{R_s} -$, and

$$F_* : R_t\text{-mod} \to R_s\text{-mod}$$

which is the restriction of scalars. For each R_s-module N, we have

$$F^*(N) = R_t \otimes_{R_s} N = \bigoplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} N).$$

For each $f \in \mathrm{Hom}_{R_t}(M, F^*(N))$, define $f_{\bar{i}} = p_{\bar{i}} \circ f : F_*(M) \to N$, where

$$F^*(N) \xrightarrow{y \mapsto e_{\bar{i}} \otimes p_{\bar{i}}(y)} e_{\bar{i}} \otimes_{R_s} N$$

is the natural projection to the $\bar{i}$-component. There is a duality theorem in [15]:

Theorem 3.2.1. *(Theorem 3.3 in [15]) For every R_t-module M and every R_s-module N there is an R_t-linear isomorphism*

$$\begin{aligned} Hom_{R_s}(F_*(M), N) &\cong Hom_{R_t}(M, F^*(N)) \\ g_{\overline{p^l-1}}(-) &\leftarrow (g = \oplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} g_{\bar{i}}(-))) \\ g &\mapsto \oplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} g(e_{\overline{p^l-1-\bar{i}}}(-))). \end{aligned}$$

Proof. Denote by $\psi : R_t \to R_s$ the natural R_s-linear surjection to the coefficient of $e_{\overline{p^l-1}}$ and by $\mu : \mathrm{Hom}_{R_s}(R_t \otimes_{R_t} M, N) \cong \mathrm{Hom}_{R_t}(M, \mathrm{Hom}_{R_s}(R_t, N))$ the natural adjointness isomorphism. Let $\gamma : \mathrm{Hom}_{R_s}(R_t, R_s) \otimes_{R_s} N \to \mathrm{Hom}_{R_s}(R_t, N)$ be the isomorphism defined by $f \otimes n \xrightarrow{\gamma} n \cdot f$ and $\sum g_{\bar{i}} \otimes g(e_{\bar{i}}) \xleftarrow{\gamma^{-1}} g$.

Let $e'_{\bar{i}} : \mathrm{Hom}_{R_s}(R_t, R_s)$ be the dual of $e_{\bar{i}}$ such that $e'_{\bar{i}}(e_{\bar{i}'}) = \delta^i_{i'}$. Fix the R_t-module isomorphism $\phi : R_t \to \mathrm{Hom}_{R_s}(R_t, R_s)$ which sends $e_{\bar{i}}$ to $e'_{\overline{p^l-1-\bar{i}}}$. For any R_s-module N, let ϕ_N denote $\phi \otimes_{R_s} N : R_t \otimes_{R_s} N \to \mathrm{Hom}_{R_s}(R_t, R_s) \otimes_{R_s} N$.

There are natural morphisms between

$$\mathrm{Hom}_{R_s}(F_*M, N) \quad \text{and} \quad \mathrm{Hom}_{R_t}(M, F^*N)$$

such that

$$g \to \phi_N^{-1} \circ \gamma^{-1} \circ \mu(g)$$

and

$$\mu^{-1} \circ \gamma \circ \phi_N(g) \leftarrow g.$$

If $g \in \mathrm{Hom}_{R_s}(F_*M, N)$, for any $m \in F_*M$,

$$\begin{aligned}
\phi_N^{-1} \circ \gamma^{-1} \circ \mu(g)(m) &= \phi_N^{-1} \circ \gamma^{-1}(g(- \cdot m)) \\
&= \phi_N^{-1}(\sum e'_{\bar{i}} \otimes g(e_{\bar{i}} m)) \\
&= \sum e_{\overline{p^n-1}-\bar{i}} \otimes g(e_{\bar{i}} m).
\end{aligned}$$

Conversely, if $g \in \mathrm{Hom}_{R_t}(M, F^*N)$, then

$$\begin{aligned}
\mu^{-1} \circ \gamma \circ \phi_N(g) &= \mu^{-1} \circ \gamma \circ \phi_N(\oplus(e_{\bar{i}} \otimes g_{\bar{i}})) \\
&= \mu^{-1} \circ \gamma(\oplus(e'_{\overline{p^n-1}-\bar{i}} \otimes g_{\bar{i}})) \\
&= \mu^{-1}(\oplus(e'_{\overline{p^n-1}-\bar{i}} \cdot g_{\bar{i}})) \\
&= g_{\overline{p^n-1}}.
\end{aligned}$$

Moreover, these two morphisms are inverses to each other. $\square$

Chapter 4

Vanishing Theorem

4.1 General Case

Theorem 4.1.1 (Grothendieck)**.** *Let R be a noetherian ring and M is an R-module of dimension d, then $H_I^i(M) = 0$ for all $i > d$. Moreover, if $(R, \mathfrak{m})$ is a Noetherian local ring and M is a finite R-module of depth t and dimension d, then*

1. *$H_\mathfrak{m}^i(M) = 0$ for $i < t$ and $i > d$,*
2. *$H_\mathfrak{m}^t(M) \neq 0$ and $H_\mathfrak{m}^d(M) \neq 0$.*

Proposition 4.1.2. *If R is noetherian, R is an ideal of R and M is a finitely generated R-module such that $IM \neq M$, then*

$$\mathrm{depth}_I M = \min\{\, i \mid H_I^i(M) \neq 0 \,\}.$$

In particular, $\mathrm{depth}_I M/\Gamma_I(M) > 0$.

Proof. If $\mathrm{depth}_I M = 0$, then $I \subseteq \cup_{\mathfrak{p} \in \mathrm{Ass} M} \mathfrak{p}$, hence $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass} M$ by the prime avoidance theorem. Therefore,

$$0 \neq \mathrm{Hom}(R/\mathfrak{p}, M) \hookrightarrow \mathrm{Hom}(R/I, M) \hookrightarrow H_I^0(M)$$

implies that $H_I^0(M)$ is not 0. Conversely, if $H_I^0(M) \neq 0$, then $I^n \subseteq \mathfrak{p}$ for some $n \in \mathbb{Z}$ and $p \in \mathrm{Ass} M$. Hence $I \subseteq \sqrt{I^n} \subseteq \mathfrak{p}$, that is $\mathrm{depth}_I M = 0$.

Assume that $d = \text{depth}_I M > 0$, let $f_1, \cdots, f_d$ be an M-regular sequence in I, then $f_1, \cdots, f_{d-1}$ is an M/f_dM-regular sequence in I. The short exact sequence $0 \to M \xrightarrow{f_d} M \to M/f_dM \to 0$ induces the long exact sequence

$$\cdots \to H_I^{i-1}(M/f_dM) \to H_I^i(M) \xrightarrow{f_d} H_I^i(M) \to H_I^i(M/f_dM) \to \cdots.$$

Since $H_I^i(M/f_dM) = 0$ for all $i < d-1$ by induction and $f_d \in a$ is a zero divisor of $H_I^i(M)$, we get $H_I^i(M) = 0$ for all $i < d$. Conversely, assume $d = \min\{i | H_I^i(M) \neq 0\} > 0$, and f is M-regular, then $0 \to M \xrightarrow{f} M \to M/fM \to 0$ induces the long exact sequence

$$\cdots \to H_I^{i-1}(M) \to H_I^{i-1}(M/fM) \to H_I^i(M) \xrightarrow{f} \cdots.$$

Therefore $H_I^i(M/fM) = 0$ for all $i < d-1$, then $\text{depth}_I M/fM \geqslant d-1$ by induction, that is $\text{depth}_I M \geqslant d$.

The last claim follows from the fact that $H_I^0(M/\Gamma_I(M)) = 0$. $\square$

4.2 Prime Characteristic

Throughout this section, $R = k[x_1, \cdots, x_n]$ is the ring of polynomials in n variables over a field k of characteristic $p > 0$.

Proposition 4.2.1. *[23, Proposition 3] Assume $I = (f_1, \cdots, f_s)$ is an ideal of R such that $\sum \deg f_i < n$. Then $H^0_{\mathfrak{m}}(H_I^i(R)) = 0$ for every maximal ideal $\mathfrak{m}$.*

Proof. Let $K^{\cdot}(\underline{f}^t, R)$ be the Koszul cocomplex of R on $f_1^t, \cdots, f_s^t$, that is

$$0 \to R \xrightarrow{d^0} \bigoplus_{1 \leqslant \alpha \leqslant s} R_\alpha \xrightarrow{d^1} \bigoplus_{1 \leqslant \alpha_1 < \alpha_2 \leqslant s} R_{\alpha_1, \alpha_2} \xrightarrow{d^2} \cdots \xrightarrow{d^{s-1}} R_{1, \cdots, s} \to 0$$

where each $R_{\alpha_1, \cdots, \alpha_j}$ is just a copy of R indexed by the tuple $(\alpha_1, \cdots, \alpha_j)$ and the differentials

$$d^j : \bigoplus_{1 \leqslant \alpha_1 < \cdots < \alpha_j \leqslant s} R_{\alpha_1, \cdots, \alpha_j} \to \bigoplus_{1 \leqslant \alpha_1 < \cdots < \alpha_{j+1} \leqslant s} R_{\alpha_1, \cdots, \alpha_{j+1}}$$

are given by

$$(d^j(r))_{\alpha_1, \cdots, \alpha_{j+1}} = \sum_{v=1}^{v=j+1} (-1)^v f_v^t r_{\alpha_1, \cdots, \hat{\alpha}_v, \cdots, \alpha_{j+1}}.$$

Let $l > 1$ be an integer and $F : R_s \xrightarrow{r \mapsto r^{p^l}} R_t$ be the Frobenius homomorphism. Let

$$\phi : H^i(K^\cdot(\underline{f}, R)) \to F^*(H^i(K^\cdot(\underline{f}, R))) = H^i(F^*(K^\cdot(\underline{f}, R)))$$

be the map induced by the chain map

$$\phi^\cdot : K^\cdot(\underline{f}, R) \to F^*(K^\cdot(\underline{f}, R)) = K^\cdot(\underline{f}^{p^l}, F^*(R))$$

that sends each $R_{\alpha_1,\cdots,\alpha_j}$ to $F^*(R_{\alpha_1,\cdots,\alpha_j}) \cong R_{\alpha_1,\cdots,\alpha_j}$ via multiplication by $f_{\alpha_1}^{p^l-1} \cdots f_{\alpha_j}^{p^l-1}$. By Proposition 1.11 in [12], $H_I^i(R)$ is the direct limit of

$$H^i(K^\cdot(\underline{f}, R)) \xrightarrow{\phi} F^*(H^i(K^\cdot(\underline{f}, R))) \xrightarrow{F^*(\phi)} (F^*)^2(H^i(K^\cdot(\underline{f}, R))) \xrightarrow{(F^*)^2(\phi)} \cdots.$$

Since $H^i(K^\cdot(\underline{f}, R))$ is a subquotient of $\bigoplus_{1 \leqslant \alpha_1 < \cdots < \alpha_i \leqslant s} R_{\alpha_1,\cdots,\alpha_i}$, assume that $H^i(K^\cdot(\underline{f}, R)) \subseteq \bigoplus_{1 \leqslant \alpha_1 < \cdots < \alpha_i \leqslant s} R_{\alpha_1,\cdots,\alpha_i}/Q$ for a submodule Q. Notice that ϕ is the multiplication by $f_{\alpha_1}^{p^l-1} \cdots f_{\alpha_i}^{p^l-1}$ in the $(\alpha_1, \cdots, \alpha_i)$ component. For each $((g_{\alpha_1,\cdots,\alpha_i}) + Q) \in \bigoplus_{1 \leqslant \alpha_1 < \cdots < \alpha_i \leqslant s} R_{\alpha_1,\cdots,\alpha_i}/Q$, suppose $l > \sum \deg g_{\alpha_1,\cdots,\alpha_i}$, then

$$\begin{aligned}
&\deg(g_{\alpha_1,\cdots,\alpha_i} \cdot f_{\alpha_1}^{p^l-1} \cdots f_{\alpha_i}^{p^l-1}) \\
\leqslant &\deg g_{\alpha_1,\cdots,\alpha_i} + (p^l - 1) \cdot \deg(f_1 \cdots f_s) \\
< &l + (p^l - 1) \cdot (n - 1) \leqslant n \cdot (p^l - 1) \\
= &\deg x_1^{p^l-1} \cdots x_n^{p^l-1},
\end{aligned}$$

where the second inequality holds since $\deg g_{\alpha_1,\cdots,\alpha_i} < l$ and $\deg(f_1 \cdots f_s) < n$. In other words, all $g_{\alpha_1,\cdots,\alpha_i} \cdot f_{\alpha_1}^{p^l-1} \cdots f_{\alpha_i}^{p^l-1}$ have zero $e_{\overline{p^l-1}} = x_1^{p^l-1} \cdots x_n^{p^l-1}$ components in $F^*(R_{\alpha_1,\cdots,\alpha_i})$. Since

$$F^*(\bigoplus_{1 \leqslant \alpha_1 < \cdots < \alpha_i \leqslant s} R_{\alpha_1,\cdots,\alpha_i}/Q) \cong (\bigoplus_{1 \leqslant \alpha_1 < \cdots < \alpha_i \leqslant s} F^*(R_{\alpha_1,\cdots,\alpha_i}))/F^*(Q),$$

$\phi((g_{\alpha_1,\cdots,\alpha_i}) + Q) = ((g_{\alpha_1,\cdots,\alpha_i} \cdot f_{\alpha_1}^{p^l-1} \cdots f_{\alpha_i}^{p^l-1}) + F^*(Q))$ has zero $e_{\overline{p^l-1}}$ component in $F^*(H^i(K^\cdot(\underline{f}, R)))$. Therefore

$$\phi_{\overline{p^l-1}}((g_{\alpha_1,\cdots,\alpha_i}) + Q) = p_{\overline{p^l-1}}((g_{\alpha_1,\cdots,\alpha_i} \cdot f_{\alpha_1}^{p^l-1} \cdots f_{\alpha_i}^{p^l-1}) + F^*(Q)) = 0.$$

Since the corresponding morphism

$$\psi : F_*(H^i(K^\cdot(\underline{f}, R))) \to H^i(K^\cdot(\underline{f}, R))$$

under Theorem 3.2.1 is exactly $\phi_{\overline{p^l-1}}$, we see that $\psi((g_{\alpha_1,\cdots,\alpha_i})+Q)=0$.

Since $H^0_{\mathfrak{m}}(H^i(K^{\cdot}(\underline{f},R)))$ is of finite length, we can choose l big enough such that ψ sends a k-basis of $H^0_{\mathfrak{m}}(H^i(K^{\cdot}(\underline{f},R)))$ to 0. Hence $\phi=0$ on $H^0_{\mathfrak{m}}(H^i(K^{\cdot}(\underline{f},R)))$ by Theorem 3.2.1.

Since local cohomology commutes with direct limit, $H^0_{\mathfrak{m}}(H^i_I(R))$ has the "generating morphism", see [12, Definition 1.9],

$$\phi: H^0_{\mathfrak{m}}(H^i(K^{\cdot}(\underline{f},R)))\to F^*(H^0_{\mathfrak{m}}(H^i(K^{\cdot}(\underline{f},R)))),$$

which is zero as has just been shown. Therefore, $H^0_{\mathfrak{m}}(H^i_I(R))=0$ by Proposition 2.3 in [12]. $\square$

This theorem has an equivalent statement in the dimension of associated primes.

Theorem 4.2.2. *[23, Theorem 1] Assume $I=(f_1,\cdots,f_s)$ is an ideal of R such that $\sum degf_i<n$. If P is an associated prime of $H^i_I(R)$, then $dimR/P\geqslant n-\sum degf_i$.*

Proof. Let $h=\text{height}P$. Suppose on the contrary we have $\dim R/P<n-\sum \deg f_i$. By Noether's normalization lemma ([1, Exercise 5.16]), there are elements $y_1,\cdots,y_{n-h}\in R$ which are algebraically independent over k and such that $k[y_1,\cdots,y_{n-h}]\cap P=0$ and R/P is integral over $k[y_1,\cdots,y_{n-h}]$. Moreover, since k is infinite, these $y_1,\cdots,y_{n-h}$ can be chosen to be linear combinations of $x_1,\cdots,x_n$. Therefore, we can assume without loss of generality that $x_1,\cdots,x_n$ are such that $k[x_1,\cdots,x_{n-h}]\cap P=0$ and P is a maximal ideal in $R'=K[x_{n-h+1},\cdots,x_n]$, where K is the fraction field of $k[x_1,\cdots,x_{n-h}]$. By Theorem 4.2.1, we get $H^0_P(H^i_I(R'))=0$, which contradicts the assumption that P is an associated prime of $H^i_I(R)$. $\square$

Chapter 5

F-modules

5.1 F-modules

In this section, R is a regular ring containing a field of characteristic $p > 0$. Let $F : R_s(= R) \xrightarrow{r \to r^p} R_t(= R)$ be the Frobenius homomorphism and let $F^*(-) = R_t \otimes_{R_s} -$ as defined in Section 3.2.

Proposition 5.1.1. *([12]) We list here several properties of the functor F^*. Some of them depend on Kunz's theorem [11], which establishes the equivalence of regularity of R and the flatness and reducedness of R_t over R_s.*

1. *$F^*(R/I) = R/I^{[p]}$ for any ideal I, where $I^{[p]}$ is the ideal of R generated by the p-th powers of all the elements of I;*

2. *F^* commutes with formation of direct sums, direct limit and cohomology of complexes;*

3. *For all R-modules M and N such that M is finitely generated,*

$$F^*(\mathrm{Ext}^i_R(M, N)) = \mathrm{Ext}^i_R(F^*(M), F^*(N));$$

4. *For any ideal I, Γ_I commutes with F for all R-modules;*

5. *F^* commutes with localization with respect to any multiplicative closed set $S \subset R$ for all R-modules;*

6. F^* maps injective modules to injective modules, see [10].

Definition 5.1.2. *An F-module is an R-module $\mathscr{M}$ equipped with an R-module isomorphism $\theta : \mathscr{M} \to F^*(\mathscr{M})$ which we call the structure morphism of $\mathscr{M}$. A homomorphism of F-modules is an R-module homomorphism $f : \mathscr{M} \to \mathscr{M}'$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "f"] \arrow[d, "\theta"'] & \mathscr{M}' \arrow[d, "\theta'"] \\
F^*(\mathscr{M}) \arrow[r, "F(f)"'] & F^*(\mathscr{M}'),
\end{tikzcd}$$

where θ and θ' are the structure morphisms of $\mathscr{M}$ and $\mathscr{M}'$.

Example 5.1.3. *The canonical F-module structure on R is defined by the R-module isomorphism $\theta : R_s \xrightarrow{r \mapsto r\otimes 1} R_t \otimes R_s = F^*(R_s)$. However, the F-module structure on R is not unique, any R-module isomorphism $\theta : R \to F^*(R)$ gives one.*

Definition 5.1.4. *A generating morphism of an F-module $\mathscr{M}$ is an R-module homomorphism $\beta : M \to F^*(M)$, where M is some R-module, such that $\mathscr{M}$ is the limit of the inductive system in the top row of the commutative diagram*

$$\begin{tikzcd}
M \arrow[r, "\beta"] \arrow[d, "\beta"'] & F^*(M) \arrow[r, "F^*(\beta)"] \arrow[d, "F^*(\beta)"'] & F^{*^2}(M) \arrow[r, "F^{*^2}(\beta)"] \arrow[d, "F^{*^2}(\beta)"'] & \cdots \\
F^*(M) \arrow[r, "F^*(\beta)"] & F^{*^2}(M) \arrow[r, "F^{*^2}(\beta)"] & F^{*^3}(M) \arrow[r, "F^{*^3}(\beta)"] & \cdots
\end{tikzcd}$$

and $\theta : \mathscr{M} \to F^*(\mathscr{M})$, the structure isomorphism of $\mathscr{M}$, is induced by the vertical arrows in this diagram.

Example 5.1.5. *If $I = (f_1, \cdots, f_s) \subset R$, and $\mathscr{M}$ is an F-module with structure morphism $\theta : \mathscr{M} \to F^*(\mathscr{M})$, then the i-th local cohomology module $H^i_I(\mathscr{M})$ of $\mathscr{M}$ inherits an F-module structure*

$$\theta_{H^i_I(\mathscr{M})} : H^i_I(\mathscr{M}) \to F^*(H^i_I(\mathscr{M})),$$

which has the following different generating morphisms:

1. *The R-module homomorphism*

$$\beta : Ext^i_R(R/I, \mathscr{M}) \to F^*(\mathrm{Ext}^i_R(R/I, \mathscr{M})) = \mathrm{Ext}^i_R(F^*(R/I), F^*(\mathscr{M}))$$

induced by the maps

$$\theta : \mathscr{M} \to F^*(\mathscr{M}) \quad \text{and} \quad F^*(R/I) = R_t \otimes_{R_s} (R_s/I) \xrightarrow{f' \otimes f \mapsto f^p f'} R_s/I$$

is a generating morphism of $H^i_I(\mathscr{M})$;

2. *Let* $K^{\cdot}(f_1, \cdots, f_s, \mathscr{M})$ *be the Koszul cocomplex of* $\mathscr{M}$ *on* $f_1, \cdots, f_s$, *that is*

$$0 \to \mathscr{M} \xrightarrow{d^0} \bigoplus_{1 \leqslant i \leqslant s} \mathscr{M}_i \xrightarrow{d^1} \bigoplus_{1 \leqslant i_1 < i_2 \leqslant s} \mathscr{M}_{i_1 i_2} \xrightarrow{d^2} \cdots \xrightarrow{d^{s-1}} \mathscr{M}_{1 \cdots s} \to 0,$$

where each $\mathscr{M}_{i_1, \cdots, i_j}$ *is just a copy of* $\mathscr{M}$ *indexed by the tuple* $(i_1, \cdots, i_j)$ *and the differentials*

$$d^j : \bigoplus_{1 \leqslant i_1 < \cdots < i_j \leqslant s} \mathscr{M}_{i_1 \cdots i_j} \to \bigoplus_{1 \leqslant i_1 < \cdots < i_{j+1} \leqslant s} \mathscr{M}_{i_1 \cdots i_{j+1}}$$

are given by

$$(d^j m)_{i_1 \cdots i_{j+1}} = \sum_{v=1}^{v=j+1} (-1)^v f_v m_{i_1 \cdots \hat{i}_v \cdots i_{j+1}}.$$

Let

$$\beta : H^i(K^{\cdot}(f_1, \cdots, f_s, \mathscr{M})) \to F^*(H^i(K^{\cdot}(f_1, \cdots, f_s, \mathscr{M}))) = H^i(F^*(K^{\cdot}(f_1, \cdots, f_s, \mathscr{M})))$$

be the map induced by the chain map

$$\beta^{\cdot} : K^{\cdot}(f_1, \cdots, f_s, \mathscr{M}) \to F^*(K^{\cdot}(f_1, \cdots, f_s, \mathscr{M})) = K^{\cdot}(f_1^p, \cdots, f_s^p, F^*(\mathscr{M}))$$

that sends each $\mathscr{M}_{i_1 \cdots i_j}$ *to* $F^*(\mathscr{M}_{i_1 \cdots i_j})$ *via* $\theta \circ (f_{i_1}^{p-1} \cdots f_{i_j}^{p-1})$. *Then* β *is a generating morphism of* $H^i_I(\mathscr{M})$.

Definition 5.1.6. *An* F*-module* $\mathscr{M}$ *is called* F*-finite if* $\mathscr{M}$ *has a generating morphism* $\beta : M \to F^*(M)$ *with* M *a finitely generated* R*-module. If in addition* β *is injective, both* M *and the image of* M *in* $\mathscr{M}$ *are called the roots of* $\mathscr{M}$, *and* β *is called the root morphism of* $\mathscr{M}$.

Theorem 5.1.7. *([12]) The F-finite modules form a full abelian subcategory of the category of F-modules which is closed under formation of submodules, quotient, extensions and localization at a single element of R. If in addition R is local, then every F-finite module has finite length in the category of F-module.*

Example 5.1.8. *If R is an ideal of R and $\mathscr{M}$ is an F-finite module, then $H^i_I(\mathscr{M})$ with its induced F-module structure is F-finite.*

Theorem 5.1.9. *([12]) If $\mathscr{M}$ is F-finite with a generating morphism $\beta : M \to F^*(M)$, let $\beta_i : M \to F^{*^i}(M)$ be the composition*

$$M \xrightarrow{\beta} F^*(M) \xrightarrow{F^*(\beta)} \cdots \xrightarrow{F^{*^{i-1}}(\beta)} F^{*^i}(M),$$

then we have:

1. *The ascending chain $\ker\beta_1 \subset \ker\beta_2 \subset \cdots$ of submodules of M eventually stabilizes. Let $C \subset M$ be the common value of $\ker\beta_i$ for sufficiently big i;*

2. *If i is the first integer such that $\ker\beta_i = \ker\beta_{i+1}$, then $\ker\beta_i = C$, that is $\ker\beta_i = \ker\beta_t$ for all $t \geqslant i$;*

3. *$\mathrm{im}\beta_i = M/C$ is a root of $\mathscr{M}$. Hence, every F-finite module has a root;*

4. *$\mathscr{M} = 0$ if $\mathscr{M}$ has a zero root and $\mathscr{M} \neq 0$ if $\mathscr{M}$ has a non-zero root.*

5.2 Graded F-modules

Observe that the ring $R = k[x_1, \cdots, x_n]$ has a natural grading $R = \bigoplus_{i\in\mathbb{N}} R_i$ (as a $\mathbb{Z}$-module) such that R_i consists of all homogeneous polynomials in $x_1, \cdots, x_n$ of degree i. Recall that a graded R-module is an R-module M together with a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ (as a $\mathbb{Z}$-module) such that $R_iM_j = M_{i+j}$ for all $i, j \in \mathbb{Z}$. Recall if M and N are both graded R-modules, then a homomorphism $\varphi : M \to N$ is degree preserving if $\varphi(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$.

If $\mathscr{M}$ is graded, we define the grading of $F^*(\mathscr{M})$ by $\deg r \otimes x = \deg r + p \cdot \deg x$ for all homogeneous $r \in R$ and $x \in \mathscr{M}$. Now we introduce a definition of graded F-modules as follows:

Definition 5.2.1. *An F-module $(\mathscr{M}, \theta)$ is graded if $\mathscr{M}$ is a graded R-module and the structure isomorphism $\theta : \mathscr{M} \to F(\mathscr{M})$ is degree preserving. A homomorphism of graded F-modules $f : \mathscr{M} \to \mathscr{M}'$ is a degree preserving F-module homomorphism.*

Example 5.2.2. *The canonical F-module structure on R defined by the R-module isomorphism $\theta : R \xrightarrow{r \to r \otimes 1} F^*(R)$ [12, Page 72] makes (R, θ) a graded F-module.*

The theory of F-modules developed in [12] can be developed in this graded version without difficulty. In particular, it is easily seen that the category of graded F-modules is abelian. The facts we need are the following with the use of the standard terminology in [3, Section 3.6]:

Theorem 5.2.3. *If $\mathscr{M}$ is a graded F-module, then there is an induced graded F-module structure on the local cohomology modules $H^i_I(\mathscr{M})$ for any homogeneous ideal I of R.*

Proof. Since the ordinary local cohomology can be computed using *injective resolutions [2, Corollary 12.3.3], the proof is basically the same as in [12, Example 1.2(b)] except that instead of injective resolutions one uses *injective ones. $\square$

Theorem 5.2.4. *If $\mathscr{M}$ is a graded F-module such that $\dim_R \mathit{Supp}(\mathscr{M}) = 0$, then $\mathscr{M}$ is a *injective R-module.*

Proof. A proof of this is, with minor and straightforward modifications, the same as the proof of the $\dim_R \mathrm{Supp}(\mathscr{M}) = 0$ case of [12, Theorem 1.4]. Modifications involve choosing the elemnents e_i and $e_{i,j}$ homogeneous and in the last step showing that M_i is isomorphic to $^*E(R/\mathfrak{m})(t)$ where $t = \deg e_i$ (rather than just $E(R/\mathfrak{m})$, as in [12, Theorem 1.4]). $\square$

From this section on, we will adopt the notation F^* to represent the Frobenius functor F. We use the duality Theorem 3.2.1 to prove the following striking result.

Theorem 5.2.5. *Let M be a graded R-module. Assume*

1. $\{ d \in \mathbb{Z} \mid M_d \neq 0 \}$ *is finite;*

2. $M_{-n} = 0$.

Then there is $s \in \mathbb{N}$ (that depends only on the set $\{ d \in \mathbb{Z} \mid M_d \neq 0 \}$) such that for any $l \geqslant s$ and for any graded R-module N, the only degree preserving R-module map $f : M \to F^{*^l}(N)$ is the zero map.

Proof. Let K be the perfect closure of k. Viewing $K \otimes_k R$, $K \otimes_k M$ and $K \otimes_k N$ as the new R, M and N, we may assume that k is perfect. Therefore Theorem 3.2.1 applies and it is sufficient to study the $\overline{p^l - 1}$ component of the image of M. Recall that the Frobenius functor F^{*^l} multiplies the grading by p^l, i.e.

$$\deg r \otimes x = \deg r + p^l \cdot \deg x, \quad (r \in R_t,\ x \in N \text{ and } r \otimes x \in F^{*^l}(N)).$$

Since $F^{*^l}(N) = \bigoplus_{\bar{i}}(e_{\bar{i}} \otimes_{R_s} N)$ and $\deg e_{\bar{i}} = \sum_j i_j$, for every $d \in \mathbb{Z}$ we have

$$F^{*^l}(N)_d = \bigoplus_{\bar{i}}(e_{\bar{i}} \otimes_{R_s} N)_d = \bigoplus_{\bar{i}} e_{\bar{i}} \otimes_{R_s} N_{(d - \deg e_{\bar{i}})/p^l},$$

where the direct sum is taken over those $\bar{i}$ for which $(d - \deg e_{\bar{i}})/p^l$ is an integer. Clearly, $\deg e_{\overline{p^l-1}} = n(p^l - 1)$. When $d \neq -n$ and l is sufficiently large, the fraction $(d - n(p^l - 1))/p^l$ is not an integer. Hence the coefficient of $e_{\overline{p^l-1}}$ in $F^{*^l}(N)_d$ is 0. Let d run through the finite set $\{ d \in \mathbb{Z} \mid M_d \neq 0 \}$ and enlarge l correspondingly, we see that the $\overline{p^l - 1}$ component of the image of M is 0. Moreover, it is obvious that the selection of l depends only on the set $\{ d \in \mathbb{Z} \mid M_d \neq 0 \}$. Now Theorem 3.2.1 induces the conclusion. $\square$

Theorem 5.2.6. *Let $\mathscr{M}$ be a graded F-module supported on $\mathfrak{m} = (x_1, \cdots, x_n)$. Then $\mathscr{M}$ as a graded R-module is a direct sum of a (possibly infinite) number of copies of ${}^*E(n)$.*

Proof. Since $\mathscr{M}$ is supported on $\mathfrak{m}$, it is *injective by Theorem 5.2.4. By [3, Theorem 3.6.3], every *injective module can be decomposed into a direct sum of modules ${}^*E(R/\mathfrak{p})(i)$ for graded prime ideals $\mathfrak{p} \in \mathrm{Spec}R$ and integers $i \in \mathbb{Z}$. Since $\mathscr{M}$ is supported on $\mathfrak{m}$, the only $\mathfrak{p}$ that appears in the decomposition is $\mathfrak{p} = \mathfrak{m}$, i.e. $\mathscr{M} = \oplus_i^* E(i)^{\alpha(i)}$ where ${}^*E = {}^*E(R/\mathfrak{m})$ and $\alpha(i)$ is the (possibly infinite) number of copies of ${}^*E(i)$. Let $\theta : \mathscr{M} \to F^*(\mathscr{M})$ be the structure isomorphism of $\mathscr{M}$. Fix $i \neq n$, assume $\alpha(i) \neq 0$, i.e. $\mathrm{soc}^* E(i)^{\alpha(i)} \neq 0$, and apply Theorem 5.2.5 to $M = \mathrm{soc}^* E(i)^{\alpha(i)}$ and $N = \mathscr{M}$. Since the degree of the socle of ${}^*E(i)$ is $-i \neq -n$, we see that the composition of isomorphisms $F^{*^l}(\theta) \circ F^{*^{l-1}}(\theta) \circ \cdots \circ \theta : \mathscr{M} \to F^{*^l}(\mathscr{M})$ vanishes on M, i.e. θ is not an isomorphism. That is a contradiction. Hence $\alpha(i) = 0$ when $i \neq n$. $\square$

Theorem 5.2.7. *If I is a homogeneous ideal of R, then as a graded R-module, $H^i_{\mathfrak{m}}(H^j_I(R))$ is isomorphic to ${}^*E(n)^c$ for some $c < \infty$.*

Proof. The graded F-module structure on R in Example 5.2.2 induces the graded F-module structures on $H^j_I(R)$ and $H^i_{\mathfrak{m}}(H^j_I(R))$ by Theorem 5.2.3. Now Theorem 5.2.6 gives the desired result. $\square$

More generally, assume $\mathscr{M}$ is a graded F-module that is F-finite (see [12, Definition 2.1] for a definition of F-finite modules). If $\mathscr{M}$ is supported on $\mathfrak{m} = (x_1, \ldots, x_n)$, then $\mathscr{M}$ as a graded R-module is isomorphic to a direct sum of a finite number of copies of ${}^*E(n)$. Indeed it follows from [12, Theorems 1.4 and 2.11] that simply as an R-module, i.e. disregarding the grading, $\mathscr{M}$ is isomorphic to a direct sum of a finite number of copies of E, and according to Theorem 5.2.6 $\mathscr{M}$, as a graded R-module, is isomorphic to a possibly infinite number of copies of ${}^*E(n)$. But the number of copies of E is the same as the number of copies of ${}^*E(n)$ because both are equal to the dimension of the socle of $\mathscr{M}$ (equivalently, the Bass number of $\mathscr{M}$ with respect to $\mathfrak{m}$).

This implies that the result of Theorem 5.2.7 holds for a considerably larger class of functors than just $T = H^j_I(-)$. For example, if $T_u = H^{j_u}_{I_u}(-)$ and $T = T_1 \circ \cdots \circ T_s$, where $I_1, \cdots, I_s$ are homogeneous ideals of R, then $H^i_{\mathfrak{m}}(T(R))$ is isomorphic to a direct sum of a finite number of copies of ${}^*E(n)$. This is because R is an F-finite module and if $\mathscr{M}$ is an F-finite module, then so is $H^j_I(\mathscr{M})$ by [12, Proposition 2.1].

Chapter 6

Algorithmic Computation

6.1 Generating Morphism

Fix the F-module structure on R,

$$\theta : R \xrightarrow{f \mapsto f\otimes 1} R_t \otimes_{R_s} R_s = F^*(R).$$

Example 5.1.5 provides two generating morphisms for $H^i_I(R)$ with its induced F-module structure, one of them is

$$\beta_2 : H^i(K^{\cdot}(f_1, \cdots, f_s, R)) \to F^{*^l}(H^i(K^{\cdot}(f_1, \cdots, f_s, R))) = H^i(K^{\cdot}(f_1^{p^l}, \cdots, f_s^{p^l}, R))$$

induced by the chain map $K^{\cdot}(f_1, \cdots, f_s, R) \to K^{\cdot}(f_1^{p^l}, \cdots, f_s^{p^l}, R)$ which sends each $R_{i_1\cdots i_j}$ to $R_{i_1\cdots i_j}$ via the multiplication by the product $f_{i_1}^{p^l-1} \cdots f_{i_j}^{p^l-1}$. Applying Proposition 5.1.9 to this morphism leads to an algorithm for deciding whether a given local cohomology module $H^i_I(R)$ is zero.

However, when p is comparatively large, say 31, the algorithms would take a tremendously long time and consume a huge amount of storage space, because so do the computations of $I^{[p]}$ and $R/I^{[p]}$ or the multiplication by $f_{i_1}^{p-1} \cdots f_{i_j}^{p-1}$.

We present an idea which will dramatically reduce the space needed for the computation of local cohomology of $R = k[x_1, \cdots, x_n]$, where k contains $\mathbb{Z}/p\mathbb{Z}$. Without loss of generality, we assume the field k perfect and infinite. And we fix the maximal ideal $\mathfrak{m} = (x_1, \cdots, x_n)$.

Let $K^{\cdot}(\underline{f}^t, R)$ be the Koszul cocomplex of R on $f_1^t, \cdots, f_s^t$, that is

$$0 \to R \xrightarrow{d^0} \bigoplus_{1\leqslant\alpha\leqslant s} R_\alpha \xrightarrow{d^1} \bigoplus_{1\leqslant\alpha_1<\alpha_2\leqslant s} R_{\alpha_1,\alpha_2} \xrightarrow{d^2} \cdots \xrightarrow{d^{s-1}} R_{1,\cdots,s} \to 0$$

where each $R_{\alpha_1,\cdots,\alpha_j}$ is just a copy of R indexed by the tuple $(\alpha_1,\cdots,\alpha_j)$ and the differentials

$$d^j : \bigoplus_{1\leqslant\alpha_1<\cdots<\alpha_j\leqslant s} R_{\alpha_1,\cdots,\alpha_j} \to \bigoplus_{1\leqslant\alpha_1<\cdots<\alpha_{j+1}\leqslant s} R_{\alpha_1,\cdots,\alpha_{j+1}}$$

are given by

$$(d^j(r))_{\alpha_1,\cdots,\alpha_{j+1}} = \sum_{v=1}^{v=j+1} (-1)^v f_v^t r_{\alpha_1,\cdots,\hat{\alpha}_v,\cdots,\alpha_{j+1}}.$$

Let

$$\phi : H^i(K^{\cdot}(\underline{f}, R)) \to F^*(H^i(K^{\cdot}(\underline{f}, R))) = H^i(F^*(K^{\cdot}(\underline{f}, R)))$$

be the map induced by the chain map

$$\phi^{\cdot} : K^{\cdot}(\underline{f}, R) \to F^*(K^{\cdot}(\underline{f}, R)) = K^{\cdot}(\underline{f}^p, F^*(R))$$

that sends each $R_{\alpha_1,\cdots,\alpha_j}$ to $F^*(R_{\alpha_1,\cdots,\alpha_j}) \cong R_{\alpha_1,\cdots,\alpha_j}$ via multiplication by $f_{\alpha_1}^{p-1}\cdots f_{\alpha_j}^{p-1}$. By [12, Proposition 1.11], $H^0_{\mathfrak{m}}(H^i_I(R))$ is the direct limit of

$$H^0_{\mathfrak{m}}(H^i(K^{\cdot}(\underline{f}, R))) \xrightarrow{\phi} F^*(H^0_{\mathfrak{m}}(H^i(K^{\cdot}(\underline{f}, R)))) \xrightarrow{F^*(\phi)} (F^*)^2(H^0_{\mathfrak{m}}(H^i(K^{\cdot}(\underline{f}, R)))) \xrightarrow{(F^*)^2(\phi)} \cdots.$$

Let $g_u = (g_u^{\alpha_1,\cdots,\alpha_i})$ be generators of $H^0_{\mathfrak{m}}(H^i(K^{\cdot}(\underline{f}, R)))$ over k, let $l > 1$ be an integer and let ψ_l be the composition of the maps $F^{*^{l-1}}(\phi) \circ \cdots \circ \phi$. Then

$$\psi_l(g_u) = (f_{\alpha_1}^{p^l-1}\cdots f_{\alpha_j}^{p^l-1} \cdot g_u^{\alpha_1,\cdots,\alpha_i}).$$

By [12, Proposition 2.3], to check whether the local cohomology module $H^0_{\mathfrak{m}}(H^i_I(R))$ vanishes we only need to check whether ψ_l is 0 on all g_u for large l. Since

$$\psi_l(g_u) = (f_{\alpha_1}^{p^l-1}\cdots f_{\alpha_j}^{p^l-1} \cdot g_u^{\alpha_1,\cdots,\alpha_i}),$$

it reduces to checking whether the $e_{\overline{p^l-1}}$ component of $f^{p^l-1}\cdot g$ for two homogeneous polynomials f and g vanishes by Theorem 3.2.1.

The first observation is that the nonzero $e_{\overline{p^{l+1}-1}}$ components of $f^{p^{l+1}-1}\cdot g = f^{p^{l+1}-p^l}\cdot(f^{p^l-1}\cdot g)$ are the $e_{\overline{p-1}}$ component of the nonzero $e_{\overline{p^l-1}}$ components of $f^{p^l-1}\cdot g$.

The second observation is not obvious, but is crucial. Suppose $\deg f = d$ and $\deg g = d'$ and suppose the $x_1^{p^l-1} \cdots x_n^{p^l-1}$-component of $f^{p^l-1} \cdot g$ is $x_1^{p^l i_1} \cdots x_n^{p^l i_n}$, then

$$p^l i_1 + p^l i_2 + \cdots + p^l i_n < d(p^l - 1) + d',$$

hence

$$i_1 + i_2 + \cdots + i_n < d + d'/p.$$

Therefore, there are less than $N = \binom{n+d+\lceil d'/p \rceil}{n}$ possibilities for the exponents $(i_1, \cdots, i_n)$ such that $x_1^{p^l i_1} \cdots x_n^{p^l i_n}$ is an $x_1^{p^l-1} \cdots x_n^{p^l-1}$-component of a monomial in $f^{p^l-1} \cdot g$. More importantly, the number of possibilities does not depend on l.

The idea follows from these two observations. Suppose the nonzero $e_{\overline{p^l-1}}$ components of $f^{p^l-1} \cdot g$ are $c_{i_1,\ldots,i_n} \cdot x_1^{p^l i_1} \cdots x_n^{p^l i_n}$, where there are less than $N = \binom{n+d+\lceil d'/p \rceil}{n}$ possibilities for the exponents $(i_1, \cdots, i_n)$. Then the nonzero $e_{\overline{p^{l+1}-1}}$ components of $f^{p^{l+1}-1} \cdot g = f^{p^{l+1}-p^l} \cdot (f^{p^l-1} \cdot g)$ are the nonzero $e_{\overline{p^{l+1}-1}}$ components of $(f^{p-1})^{p^l} \cdot (\sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} \cdot x_1^{p^l i_1} \cdots x_n^{p^l i_n}) \cdot e_{\overline{p^l-1}}$. Let $c'_{j_1,\ldots,j_n} \cdot x_1^{p^{l+1} j_1} \cdots x_n^{p^{l+1} j_n}$ denote the nonzero $e_{\overline{p^{l+1}-1}}$ components of $f^{p^{l+1}-1} \cdot g$, then each

$$c'_{j_1,\ldots,j_n} \cdot x_1^{p^{l+1} j_1} \cdots x_n^{p^{l+1} j_n} \cdot e_{\overline{p^{l+1}-1}}$$

must be a summand in $(f^{p-1})^{p^l} \cdot (c_{i_1,\ldots,i_n} \cdot x_1^{p^l i_1} \cdots x_n^{p^l i_n}) \cdot e_{\overline{p^l-1}}$. Therefore $c'_{j_1,\ldots,j_n} \cdot x_1^{p(j_1+1)} \cdots x_n^{p(j_n+1)}$ must be a summand in $f^{p-1} \cdot (\sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} \cdot x_1^{i_1+1} \cdots x_n^{i_n+1})$. Hence the determination of all $c'_{j_1,\ldots,j_n}$ comsumes the space linearly proportional to p. Notice this does not depend on l. If we repeat the above consideration by induction in l, the compuation consumes the space proportional to p.

We conclude that the idea discussed above will help to reduce the space complexity of the compuation of local cohomology in prime characteristic.

6.2 Frobenius Linear Algebra

Let us quote a well known lemma.

Lemma 6.2.1. *([18, Lemma III.4.13]) Let $\mathbb{M}$ be a square matrix over an algebraically closed field k of characteristic p, then there exists a matrix $\mathbb{P} = (\mathbb{P}_{ij})$ such that*

$$\mathbb{P}^{[p]}\mathbb{M}\mathbb{P}^{-1} = \begin{pmatrix} \mathbb{M}_{fs} & 0 \\ 0 & \mathbb{M}_{fn} \end{pmatrix},$$

$\mathbb{M}_{fs}$ is an identity matrix and $\mathbb{M}_{fn}$ is a nilpotent matrix. Here $\mathbb{P}^{[p]}$ is the position-wise exponent $(\mathbb{P}_{ij}^p)$.

Let $R = k[x_1, \cdots, x_n]$ be a polynomial ring over a field k of characteristic $p > 0$. Without loss of generality, we assume that k is algebraically closed. Let $\mathfrak{m} = (x_1, \cdots, x_n)$, let ${}^*E$ be the naturally graded injective hull of $R/\mathfrak{m}$ and let ${}^*E(n)$ be ${}^*E$ degree shifted downward by n. The following theorem describes a way to compute the c in Theorem 5.2.7.

Theorem 6.2.2. *If $\mathscr{M}$ is F-finite graded supported on m with a graded generating morphism $\beta : M \to F^*(M)$. Then M contains the socle of $\mathscr{M}$ and the number c of copies of ${}^*E(n)$ in $\mathscr{M}$ in Theorem 5.2.7 equals the rank of the matrix of the restriction $\beta : socM \to socF^*(M)$.*

Proof. Assume that $M \cong R^s/Q$ such that $Q = \sum Rg_u$, and $\mathrm{soc}M = \oplus_{v=1}^t R\bar{h}_v$. Since we can lift everything back in R^s, we can then talk about the p-th power as the component-wise p-th power. Then $F^*(M) \cong R \otimes_{R^p} ((R^p)^s/(\sum R^p g_u^p))$ and $F^*(\mathrm{soc}M) \cong R \otimes_{R^p} \sum R^p \bar{h}_v^p$. Since R is free over R^p such that $R = R^p \cdot 1 \oplus R^p \cdot x_1 \oplus \cdots \oplus R^p \cdot x_1^{p-1} \cdots x_n^{p-1}$,

$$\mathrm{soc}F^*(M) \supseteq \oplus R^p \cdot x_1^{p-1} \cdots x_n^{p-1} \bar{h}_v^p.$$

Since $\dim_K \mathrm{soc}M = \dim_K \mathrm{soc}F^*(M)$ by [14, Corollary 3.4],

$$\mathrm{soc}F^*(M) = \oplus R \cdot x_1^{p-1} \cdots x_n^{p-1} \bar{h}_v^p$$

and similarly

$$\mathrm{soc}F^{*^l}(M) = \oplus R \cdot x_1^{p^l-1} \cdots x_n^{p^l-1} \bar{h}_v^{p^l}$$

for all $l > 1$.

Suppose that $\mathbb{M}$ is the matrix of the restriction $\beta : \mathrm{soc}M \to \mathrm{soc}F^*(M)$, then

$$\beta \begin{pmatrix} \bar{h}_1 \\ \vdots \\ \bar{h}_t \end{pmatrix} = \mathbb{M} \begin{pmatrix} x_1^{p-1} \cdots x_n^{p-1} \cdot \bar{h}_1^p \\ \vdots \\ x_1^{p-1} \cdots x_n^{p-1} \cdot \bar{h}_t^p \end{pmatrix}.$$

Then

$$F^*(\beta)\begin{pmatrix} x_1^{p-1}\cdots x_n^{p-1}\cdot \bar{h}_1^p \\ \vdots \\ x_1^{p-1}\cdots x_n^{p-1}\cdot \bar{h}_t^p \end{pmatrix} = x_1^{p-1}\cdots x_n^{p-1}\begin{pmatrix} (\chi(\bar{h}_1))^p \\ \vdots \\ (\chi(\bar{h}_t))^p \end{pmatrix}$$
$$= x_1^{p-1}\cdots x_n^{p-1}\mathbb{M}^{[p]}\begin{pmatrix} x_1^{p^2-p}\cdots x_n^{p^2-p}\cdot \bar{h}_1^{p^2} \\ \vdots \\ x_1^{p^2-p}\cdots x_n^{p^2-p}\cdot \bar{h}_t^{p^2} \end{pmatrix}$$
$$= \mathbb{M}^{[p]}\begin{pmatrix} x_1^{p^2-1}\cdots x_n^{p^2-1}\cdot \bar{h}_1^{p^2} \\ \vdots \\ x_1^{p^2-1}\cdots x_n^{p^2-1}\cdot \bar{h}_t^{p^2} \end{pmatrix},$$

in another word, $F^*(\mathbb{M}) = \mathbb{M}^{[p]}$.

Let the matrix $\mathbb{P}$ be as in Lemma 6.2.1, and linearly transform the basis $(\bar{h}_1,\cdots,\bar{h}_u)$ of soc$N$ to $(\bar{h}_1,\cdots,\bar{h}_u)\cdot\mathbb{P}^T$. So we can assume that the basis $\bar{h}_1,\cdots,\bar{h}_t$ are such that $F^{*^l}(\beta)\circ\cdots\circ\beta(\bar{h}_v) = x_1^{p^l-1}\cdots x_n^{p^l-1}\bar{h}_v^{p^l}$ for $v=1,\cdots,c$ and all $l>0$, and $F^{*^l}(\beta)\circ\cdots\circ\cdots\beta(\bar{h}_v)=0$ for $v=c+1,\cdots,t$ and $l\gg 0$. It is now clear that the dimension of soc$\mathscr{M}$ equals c, which is the rank of the matrix $\mathbb{M}$ of the restriction $\beta:\mathrm{soc}M\to\mathrm{soc}F^*(M)$. In particular, this implies that M contains soc$\mathscr{M}$. $\square$

References

[1] M. F. Atiyah, I. G. Macdonald, Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.

[2] M. P. Brodmann, A. Y. Sharp, Local cohomology: an algebraic introduction with geometric applications. Cambridge Studies in Advanced Mathematics, 60. Cambridge University Press, Cambridge, 1998.

[3] W. Bruns, J. Herzog, Cohen-Macaulay rings. Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.

[4] A. Grothendieck, Local cohomology. Lecture Notes in Mathematics, No. 41 Springer-Verlag, Berlin-New York 1967.

[5] A. Grothendieck, Cohomologie locale des faisceaux cohérents et théorémes de Lefschetz locaux et globaux (SGA 2). Advanced Studies in Pure Mathematics, Vol. 2. North-Holland Publishing Co., Amsterdam; Masson & Cie, Éditeur, Paris, 1968.

[6] R. Hartshorne, Cohomological dimension of algebraic varieties. Ann. of Math. (2) 88 1968 403-450.

[7] R. Hartshorne, Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[8] C. L. Huneke, Lectures on local cohomology. Appendix 1 by Amelia Taylor. Contemp. Math., 436, Interactions between homotopy theory and algebra, 51–99, Amer. Math. Soc., Providence, RI, 2007.

[9] C. Huneke, G. Lyubeznik, On the vanishing of local cohomology modules. Invent. Math. 102 (1990), no. 1, 73-93.

[10] C. L. Huneke, R. Y. Sharp, Bass numbers of local cohomology modules. Trans. Amer. Math. Soc. 339 (1993), no. 2, 765-779.

[11] E. Kunz, Characterizations of regular local rings for characteristic p. Amer. J. Math. 91 1969 772-784.

[12] G. Lyubeznik, F-modules: applications to local cohomology and D-modules in characteristic $p > 0$. J. Reine Angew. Math. 491 (1997), 65-130.

[13] G. Lyubeznik, Finiteness properties of local cohomology modules: a characteristic-free approach. J. Pure Appl. Algebra 151 (2000), no. 1, 43-50.

[14] G. Lyubeznik, On the vanishing of local cohomology in characteristic $p > 0$. Compos. Math. 142 (2006), no. 1, 207-221.

[15] G. Lyubeznik, W. Zhang, Y. Zhang, A property of the Frobenius map of a polynomial ring. Contemporary Mathematics 555 (2011), 137-143.

[16] G. Lyubeznik, Notes in local cohomology (to appear).

[17] Mayr, Meyer, The complexity of the word problems for commutative semigroups and polynomial ideals. Adv. in Math. 46 (1982), no. 3, 305-329.

[18] J. Milne, Étale cohomology. Princeton Mathematical Series, 33. Princeton University Press, Princeton, N.J., 1980. xiii+323 pp.

[19] A. Ogus, Local cohomological dimension of algebraic varieties. Ann. of Math. (2) 98 (1973), 327-365.

[20] C. Peskine, L. Szpiro, Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck. (French) Inst. Hautes Études Sci. Publ. Math. No. 42 (1973), 47-119.

[21] U. Walther, Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties. J. Pure Appl. Algebra 139 (1999), no. 1-3, 303-321.

[22] C. Weibel, An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

[23] Y. Zhang, A property of local cohomology modules of polynomial rings. Proc. Amer. Math. Soc. 139 (2011), no. 1, 125-128.

[24] Y. Zhang, Graded F-modules and local cohomology. Bull. Lond. Math. Soc., accepted.