Filtering partially observable diffusions up to the exit time from a domain

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N.V. Krylov

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Abstract

We consider a two-component diffusion process with the second component treated as the observations of the first one. The observations are available only until the first exit time of the first component from a fixed domain. We derive filtering equations for an unnormalized conditional distribution of the first component before it hits the boundary and give a formula for the conditional distribution of the first component at the first time it hits the boundary. We also derive a formula for the conditional distribution of the exit time if the observation is always available.

Contents

Chapter 1

Introduction

In this paper we study a filtering problem in a bounded domain. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete, with respect to $(\mathcal{F}, P)$, σ-fields $\mathcal{F}_t \subset \mathcal{F}$. Denote by $\mathcal{P}$ the predictable σ-field in $\Omega \times (0, \infty)$ associated with $\{\mathcal{F}_t\}$. Let $d \geq 1$ and $d_1 > d$, be integers and w_t be a d_1-dimensional Wiener process with respect to $\{\mathcal{F}_t\}$. Let $K, \delta > 0$ be fixed (finite) constants.

Consider a d_1-dimensional two-component process $z_t = (x_t, y_t)$ with x_t being d-dimensional and y_t $d_2 := (d_1 - d)$-dimensional. Let G be a bounded C^2 domain in $\mathbb{R}^d$. We assume that z_t is a diffusion process defined as a solution of the system

$$\begin{aligned} \mathrm{d}x_t &= b(t, z_t)\,\mathrm{d}t + \theta(t, z_t)\,\mathrm{d}w_t, \\ \mathrm{d}y_t &= B(t, z_t)\,\mathrm{d}t + \Theta(t, y_t)\,\mathrm{d}w_t \end{aligned} \tag{1.1}$$

with some initial data independent of the process w_t.

The coefficients of (1.1) are assumed to be vector- or matrix-valued functions of appropriate dimensions defined on $[0, \infty) \times \mathbb{R}^{d_1}$. Actually $\Theta(t, y)$ is assumed to be independent of x, so that it is a function on $[0, \infty) \times \mathbb{R}^{d_2}$ rather than $[0, \infty) \times \mathbb{R}^{d_1}$ but as always we may think of $\Theta(t, y)$ as a function of (t, z) as well. Finally, let $\tau := \tau_G$ be the first exit time of x_t from G, that is

$$\tau_G = \inf\{t \geq 0 : z_t \notin \mathbb{G}\}, \quad \mathbb{G} := G \times \mathbb{R}^{d_2}.$$

The component x_t is treated as unobservable and $y_{s \wedge \tau}$, $s \leq t$, is treated as the only available observations at time t. The problem is to find a way to compute the density

$\pi_t(x)$ (if it exists) of the conditional distribution of $(\tau \wedge \tau, x_{t\wedge\tau})$ in $[0,\infty) \times \bar{G}$ given $y_{s\wedge\tau}, s \le t$.

Notice that we will assume that y_t is a uniformly nondegenerate process (see Assumption 2.3). Hence, knowing $y_{s\wedge\tau}, s \le t$ we know exactly if $\tau \ge t$ or not because after τ the process $y_{t\wedge\tau}$ will not move, which is quite a noticeable difference with its behavior before τ. Therefore, our problem splits into two parts:

(i) Find the density of conditional distribution of x_t in G given $y_s, s \le t$, and given that $t < \tau$;

(ii) Find the density of conditional distribution $\alpha_t(dx)$ of x_τ on ∂G given $y_s, s \le \tau$ and given that $\tau = t$.

One can give the following interpretation of the problem. We have a moving target which moves as a diffusion process in $\mathbb{R}^d$ with observations y_t corrupted by a noise. In our setting the noise in the observations also enters the diffusion part of x_t. The setting is such that we can obtain the observations only until x_t lives in G. Once it hits ∂G we hear a loud bang, no observations are available after that, and we have to find the conditional distribution of x_τ given observations on $[0,\tau]$. We also want to know the conditional distribution of x_T given the observations on $[0,T]$ and given that there was no bang before T.

To the best of our knowledge E. Pardoux ([1], [2]) is the first and the only author who considered in the past filtering problem for diffusion processes with the signal restricted to move in a bounded region. However, he considered a process which either stops once it hits the boundary (the observations are still available after that moment) or reflects from the boundary. Our setting differs dramatically, in our scheme no observations after the hitting time are available.

The structure of the paper is as follows. In Chapter 2 we state our main results and a few auxiliary results needed to state them. It turns out that the conditional distribution of x_τ can be expressed through the "normal derivative" of a solution of an appropriate SPDE. We investigate this derivative in Chapter 3. In a rather short Chapter 4 we prove a simple but important for the future result that the conditional probability of $\{\tau > T\}$ given $y_t, t \le T$ is > 0 (a.s.) for any $T \in [0,\infty)$. The Chapter 5 is devoted to proving our main results.

Chapter 2

Main results

First we state and discuss our assumptions.

Assumption 2.1. The functions b, θ, B, and Θ are Borel measurable and bounded functions of their arguments. Each of them satisfies the Lipschitz condition in z with constant $K \in (0, \infty)$.

Introduce

$$\tilde{\theta}(t,z) = \begin{pmatrix} \theta(t,z) \\ \Theta(t,y) \end{pmatrix}, \quad \tilde{a}(t,z) = \frac{1}{2}\tilde{\theta}\tilde{\theta}^*(t,z), \quad \tilde{b}(t,z) = \begin{pmatrix} b(t,z) \\ B(t,z) \end{pmatrix},$$

$$\tilde{L}(t,z) = \tilde{a}^{ij}(t,z)\frac{\partial^2}{\partial z^i \partial z^j} + \tilde{b}^i(t,z)\frac{\partial}{\partial z^i}, \tag{2.1}$$

where $\tilde{\theta}^*$ is the transpose of $\tilde{\theta}$ and the summation convention is imposed.

Remark 2.2. System of equations (1.1) can be now written as

$$dz_t = \tilde{b}(t, z_t)dt + \tilde{\theta}(t, z_t)dw_t.$$

Assumption 2.3. The process z_t is uniformly nondegenerate: for any $\lambda, z \in \mathbb{R}^{d_1}$ and $t \in [0,T]$ we have

$$\tilde{a}^{ij}(t,z)\lambda^i\lambda^j \geq \delta|\lambda|^2.$$

Remark 2.4. Owing to Assumption 2.3 the symmetric matrix $\Theta\Theta^*$ is invertible and

$$\Psi := (\Theta\Theta^*)^{-\frac{1}{2}}$$

is a bounded function of (t, y).

Before stating the next assumption we introduce the space $\overset{0}{W}{}^1_2 = \overset{0}{W}{}^1_2(G)$ as the closure of $C_0^\infty = C_0^\infty(G)$ in the norm

$$\|u\|_{W_2^1} = \|u\|_{L_2} + \|Du\|_{L_2},$$

where Du is the gradient of u and $L_2 = L_2(G)$.

Assumption 2.5. The random vectors x_0 and y_0 are independent of the process w_t. The conditional distribution of x_0 in G given y_0 has a density, which we denote by $\pi_0(x) = \pi_0(\omega, x)$. More precisely, for any Borel $\Gamma \subset G$ we have (a.s.)

$$P\{x_0 \in \Gamma \mid y_0\} = \int_\Gamma \pi_0(x)\,\mathrm{d}x.$$

Finally, $\pi_0 \in L_2(\Omega, L_2)$ and $P(x_0 \in G) = 1$.

Next we introduce a few more notation. Let

$$\Psi_t = \Psi(t, y_t), \quad \Theta_t = \Theta(t, y_t), \quad a_t(x) = \frac{1}{2}\theta\theta^*(t, x, y_t), \quad b_t(x) = b(t, x, y_t),$$

$$\sigma_t(x) = \theta(t, x, y_t)\Theta_t^*\Psi_t, \quad \beta_t(x) = \Psi_t B(t, x, y_t).$$

Below we use the notation

$$D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j$$

only for $i, j = 1, ..., d$ and set

$$L_t(x) = a_t^{ij}(x)D_{ij} + b_t^i(x)D_i\,, \tag{2.2}$$

$$L_t^*(x)u_t(x) = D_{ij}(a_t^{ij}(x)u_t(x)) - D_i(b_t^i(x)u_t(x))$$

$$= D_j\big(a_t^{ij}(x)D_i u_t(x) - b_t^j(x)u_t(x) + u_t(x)D_i a_t^{ij}(x)\big), \tag{2.3}$$

$$\Lambda_t^k(x)u_t(x) = \beta_t^k(x)u_t(x) + \sigma_t^{ik}(x)D_i u_t(x), \tag{2.4}$$

$$\Lambda_t^{k*}(x)u_t(x) = \beta_t^k(x)u_t(x) - D_i(\sigma_t^{ik}(x)u_t(x))$$

$$= -\sigma_t^{ik}(x)D_i u_t(x) + (\beta_t^k(x) - D_i\sigma_t^{ik}(x))u_t(x), \tag{2.5}$$

where $t \in [0,\infty)$, $x \in \mathbb{R}^d$, $k = 1, ..., d_2$, and as above we use the summation convention over all "reasonable" values of repeated indices, so that the summation in (2.2), (2.3), (2.4), and (2.5) is done for $i, j = 1, ..., d$ (whereas in (2.1) for $i, j = 1, ..., d_1$). Observe that Lipschitz continuous functions have bounded generalized derivatives and by

$$D_i a_t^{ij}, \quad D_i \sigma_t^{ik}$$

we mean these derivatives. By assumption the operator L defined by (2.2) is uniformly elliptic with constant of ellipticity δ.

Finally, by $\mathcal{F}_t^y$ we denote the completion of $\sigma\{y_s : s \le t\}$ with respect to $P, \mathcal{F}$. By $\mathcal{P}^y$ we mean the predictable σ-field generated by $\{\mathcal{F}_t^y, t \ge 0\}$.

Let us consider the following initial value problem

$$\mathrm{d}\bar{\pi}_t(x) = L_t^*(x)\bar{\pi}_t(x)\,\mathrm{d}t + \Lambda_t^{k*}(x)\bar{\pi}_t(x)\,\mathrm{d}\tilde{y}_t^k, \tag{2.6}$$

$$\bar{\pi}_0(x) = \pi_0(x),$$

where $t \ge 0$, $x \in G$, $\bar{\pi}_t(x) = \bar{\pi}_t(\omega, x)$, and

$$\tilde{y}_t^k = \int_0^t \Psi_s^{kr}\,\mathrm{d}y_s^r.$$

In the theory of nonlinear filtering (2.6) is known as Zakai's equation for unnormalized conditional density.

To explain in which sense we understand this equation and the initial condition, for $T \in (0,\infty)$ introduce

$$G_T = (0,T) \times G, \quad \overset{0}{\mathbb{W}}{}_2^1(G_T) = L_2(\Omega \times (0,T), \mathcal{P}^y, \overset{0}{W}{}_2^1),$$

and introduce $\overset{0}{\mathcal{W}}{}_2^1(G_T)$ as the set of functions $u_t(x) = u_t(\omega, x)$ such that

(i) For each $(\omega, t) \in \Omega \times [0,T]$, u_t is a generalized function on G;

(ii) We have $u \in \overset{0}{\mathbb{W}}{}_2^1(G_T)$;

(iii) $u_0 \in L_2(\Omega, \mathcal{F}_0^y, L_2)$ and there exist $f^j, g^k \in L_2(\Omega \times (0,T), \mathcal{P}^y, L_2)$, $i = 0, ..., d$, $j = 1, \ldots, d_2$, such that for any $\zeta \in C_0^\infty$ $(= C_0^\infty(G))$ with probability one

$$(u_t, \zeta) = (u_0, \zeta) + \int_0^t \left[(f_s^0, \zeta) - (f_s^i, D_i\zeta)\right] \mathrm{d}s + \int_0^t g_s^k\,\mathrm{d}\tilde{y}_s^k$$

for all $t \in [0,T]$, where by (f,ζ) we mean the action of a generalized function f on ζ, in particular, if f is locally summable,

$$(f,\zeta) = \int_G f(x)\zeta(x)\,\mathrm{d}x.$$

In case (iii) holds we write

$$\mathrm{d}u_t = (D_i f_t^i + f_t^0)\,\mathrm{d}t + g_t^k\,\mathrm{d}\tilde{y}_t^k$$

for $t \in [0,T]$.

Accordingly, we are looking for a function $\bar{\pi} \in \cap_T \overset{0}{\mathcal{W}}{}_2^1(G_T)$ such that (2.6) holds for all T and $t \in [0,T]$, that is, for all $t \geq 0$. In particular, we require that for each $\zeta \in C_0^\infty$ with probability one for all $t \in [0,\infty)$ it hold that

$$(\bar{\pi}_t,\zeta) = (\pi_0,\zeta) - \int_0^t (a_s^{ij} D_i\bar{\pi}_s - b_s^j\bar{\pi}_s + \bar{\pi}_s D_i a_s^{ij}, D_j\zeta)\,\mathrm{d}s$$

$$- \int_0^t (\sigma_s^{ik} D_i\bar{\pi}_s + (D_i\sigma_s^{ik} - \beta_s^k)\bar{\pi}_s,\zeta)\,\mathrm{d}\tilde{y}_s^k \tag{2.7}$$

for all $t \geq 0$.

Observe that all expressions in (2.7) are well defined due to the fact that the coefficients of $\bar{\pi}$ and of $D_i\bar{\pi}$ are bounded and appropriately measurable and

$$\bar{\pi}, D_i\bar{\pi} \in \mathbb{L}_2(T) := L_2(\Omega \times (0,T), \mathcal{P}^y, L_2)$$

for any $T \in (0,\infty)$.

In all what follows we suppose that Assumptions 2.1, 2.3, and 2.5 are satisfied. Two of auxiliary results consist of the following.

Lemma 2.6. *There exists a unique solution $\bar{\pi}$ of (2.6) with initial condition π_0 in the sense explained above. In addition, $\bar{\pi}_t \geq 0$ for all $t \in [0,\infty)$ (a.s.). With probability one $\bar{\pi}_t$ is continuous in $L_1 = L_1(G)$ and in L_2.*

The existence, uniqueness, and the (a.s.) continuity in L_2 of $\bar{\pi}$ is a classical result proved in many places in a variety of settings (see, for instance, [3], [4], [5], and the references therein). That $\bar{\pi}_t$ is (a.s.) continuous as an L_1-function follows from its L_2-continuity and the boundedness of G. The fact that $\bar{\pi} \geq 0$ follows from the maximum principle (see, for instance, Theorem 1.1 of [6]) and the fact that, if $u \in \overset{0}{W}{}^1_2$, then $u^+ \in \overset{0}{W}{}^1_2$. However, it is still worth noting that all the above is formally true if we allow $\bar\pi$ to be predictable with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ and not $\{\mathcal{F}^y_t, t \geq 0\}$. The fact that, actually, $\bar\pi_t$ is $\mathcal{P}^y$-measurable is proved in a standard way by using Girsanov's theorem as, for instance, in [7].

Introduce

$$\tilde w_t = \int_0^t \Psi_s \Theta_s \, \mathrm{d}w_s, \quad \tilde\beta_s = \beta_s(x_s),$$

$$\rho_t = \exp\big(- \int_0^t \tilde\beta_s \, \mathrm{d}\tilde w_s - \frac{1}{2}\int_0^t |\tilde\beta_s|^2 \, \mathrm{d}s\big).$$

As is easily derived from Lévy's theorem, $\tilde w_t$ is an $\mathcal{F}_t$-Wiener process. In our view the following lemma is of independent interest.

Lemma 2.7. *For any Borel bounded or nonnegative function ϕ on G and $T \in [0,\infty)$ we have (a.s.)*

$$E\big\{ I_{\tau>T}\phi(x_T) \mid \mathcal{F}^y_T\big\} = (\bar\pi_T, \phi) m_T, \tag{2.8}$$

where

$$m_T := E\big\{\rho_T \mid \mathcal{F}^y_T\big\}.$$

In particular, for each $T \in [0,\infty)$ (a.s.)

$$P\{\tau > T \mid \mathcal{F}^y_T\} = (\bar\pi_T, 1) m_T. \tag{2.9}$$

Finally, (a.s.) for all $T \in [0,\infty)$ we have $(\bar\pi_T, 1) > 0$.

We prove Lemma 2.7 in Chapter 5. Here are two main results of the article. Introduce $\mathcal{G}^y_t$ as the completion of

$$\sigma\{y_{s\wedge\tau} : s \leq t\}$$

with respect to $P, \mathcal{F}$.

Theorem 2.8. *For any Borel bounded or nonnegative function ϕ on G and $T \in (0,\infty)$ we have (a.s.) on $\{\omega : \tau > T\}$ that*

$$E\big\{\phi(x_T) \mid \mathcal{G}^y_T\big\} = \frac{(\bar\pi_T, \phi)}{(\bar\pi_T, 1)}.$$

This theorem says that (by definition) the conditional density of the distribution of x_T given $y_{t\wedge\tau}, t \le T$, on the set $\{\tau > T\}$ is

$$\pi_T = \frac{\bar{\pi}_T}{(\bar{\pi}_T, 1)}.$$

We prove Theorem 2.8 in Chapter 5.

If we have a Borel subset C of a Euclidean space, by $\mathcal{B}(C)$ we denote the σ-field of Borel subsets of C.

Theorem 2.9. *Let ν be the interior normal to ∂G. Then*

$$\alpha_t(B) = (a_t^{ij}\nu^i\nu^j \frac{\partial \bar{\pi}_t}{\partial \nu})(B)\big[(a_t^{ij}\nu^i\nu^j \frac{\partial \bar{\pi}_t}{\partial \nu})(\partial G)\big]^{-1} \tag{2.10}$$

is well defined as a function on $\Omega \times [0,\infty) \times \mathcal{B}(\partial G)$ such that it is $\mathcal{P}^y$-measurable for each $B \in \mathcal{B}(\partial G)$, it is a probability measure with respect to B for any (ω, t), and for any Borel bounded or nonnegative ϕ given on ∂G and $T \in (0,\infty)$ with probability one on the set $\{\tau \le T\}$ we have

$$E\{\phi(x_\tau) \mid \mathcal{G}_T^y\} = \int_{\partial G} \phi(x)\, \alpha_\tau(dx). \tag{2.11}$$

Theorem 2.9 is proved in Chapter 5.

It is to be said that the notation (2.10) is understood in a certain generalized sense (see Remark 3.7). The point is that even the continuity properties of $\bar{\pi}_t$ near the boundary can be rather poor if $\sigma \not\equiv 0$ (see, for instance, [8] so that there no hope to define its normal derivative in a usual way. The situation here is similar to the local time of the one-dimensional Wiener process which is often written as

$$\int_0^t \delta_0(w_s)\, \mathrm{d}s,$$

where δ_0 is the delta function concentrated at zero, although $\delta_0(w_s)$, understood literally, should be zero with probability one since $P(w_t = 0) = 0$.

Not surprisingly, the right-hand side of (2.11) does not depend on T. Indeed, its left-hand side stops changing after τ since no new observations are coming in.

Theorem 2.9 says that for any $dx \in \mathcal{B}(\partial G)$ and $T \in [0,\infty)$ with probability one on the set $\{\tau \le T\}$ we have (a.s.)

$$P(x_\tau \in dx \mid y_{s\wedge\tau}, s \le T\} = \alpha_\tau(dx).$$

We derive the above results quite formally without using filtering theory, which at this stage seems not to be applicable in our situation. Anyhow, it would be very interesting to find any heuristic explanation of Theorem 2.9.

Chapter 3

Defining the normal derivative of $\bar{\pi}$ on ∂G

Introduce

$$\psi(x) := \operatorname{dist}(x, \partial D)$$

and for $\varepsilon > 0$ denote

$$G_\varepsilon = \{x \in G : \operatorname{dist}(x, \partial G) > \varepsilon\}, \quad \delta_\varepsilon G = G \setminus G_\varepsilon.$$

It is well known that for sufficiently small $\varepsilon_0 > 0$ the function ψ is twice continuously differentiable in the closure of $\delta_{\varepsilon_0} G$. In this set we introduce

$$\nu(x) := \operatorname{grad} \psi(x)$$

which is a natural extension of the interior normal to ∂G into $\delta_{\varepsilon_0} G$. By the above $\nu(x)$ is continuously differentiable in the closure of $\delta_{\varepsilon_0} G$.

We start with two remarks and a technical results.

Remark 3.1. Equation (2.7) also holds for $\zeta \in \overset{0}{W}{}^1_2$. Indeed, if we approximate $\zeta \in \overset{0}{W}{}^1_2$ by $\zeta^n \in C_0^\infty$ in W_2^1-norm, then, as is easy to see, the terms in (2.7) with ζ^n in place of ζ will converge to the corresponding term of (2.7) uniformly on finite time intervals in probability.

Remark 3.2. For any $T \in [0, \infty)$ we have

$$E \int_0^T \|\bar{\pi}_t / \psi\|_{L_2}^2 \, \mathrm{d}t < \infty.$$

This fact follows from Hardy's inequality: for any $u \in \overset{0}{W}{}^1_2$

$$\|u/\psi\|_{L_2} \leq N\|u\|_{W^1_2},$$

where the constant N is independent of u. In turn this inequality is obtained by using flattening the boundary and partitions of unity from the one-dimensional Hardy inequality

$$\int_0^\infty |u(x)/x|^2 \, \mathrm{d}x = \int_0^\infty \Big| \int_0^1 u'(xt) \, \mathrm{d}t \Big|^2 \mathrm{d}x$$

$$\leq \Big(\int_0^1 \Big(\int_0^\infty |u'(xt)|^2 \, \mathrm{d}x \Big)^{1/2} \mathrm{d}s \Big)^2 = \int_0^\infty |u'(x)|^2 \, \mathrm{d}x \Big(\int_0^1 t^{-1/2} \, \mathrm{d}t \Big)^2,$$

which is valid for smooth functions on $[0,\infty)$ vanishing at 0, where the inequality is just Minkowski's inequality.

Lemma 3.3. *With probability one for all $t \geq 0$*

$$m^2 \int_0^t (\bar{\pi}_s, (1-m\psi)_+)^2 \, \mathrm{d}s \to 0. \tag{3.1}$$

In particular,

$$m \int_0^t (\bar{\pi}_s, (1-m\psi)_+) \, \mathrm{d}s \to 0.$$

To prove the lemma it suffices to observe that the left-hand side of (3.1) is majorated by

$$\int_0^t (\psi^{-1}\bar{\pi}_s, (1-m\psi)_+)^2 \, \mathrm{d}s,$$

$(1-m\psi)_+ \to 0$ and $\bar{\pi}_s/\psi$ is in $\mathbb{L}_2(T)$ for any $T \in (0,\infty)$ in light of Remark 3.2.

Recall that

$$\tilde{y}^k_t = \int_0^t \Psi^{kr}_s \, \mathrm{d}y^r_s.$$

Next we introduce a process $\mathcal{A}_t$ which characterizes the charge $\bar{\pi}$ puts near the boundary up to time t.

Lemma 3.4. *The process*

$$\mathcal{A}_t := 1 - (\bar{\pi}_t, 1) + \int_0^t (\bar{\pi}_s, \beta^k_s) \, d\tilde{y}^k_s \tag{3.2}$$

is an increasing continuous $\mathcal{F}_t^y$-adapted process. Furthermore, uniformly on finite time intervals in probability

$$\mathcal{A}_{mt} := 2m^2 \int_0^t \int_{\delta_{1/m}G} a_s^{ij}\nu^i\nu^j\bar{\pi}_s \,\mathrm{d}x\,\mathrm{d}s \to \mathcal{A}_t \tag{3.3}$$

as $m \to \infty$.

Proof. It suffices to prove only the second assertion. Indeed, it implies that (a.s.) $\mathcal{A}_t \geq \mathcal{A}_s$ whenever $t \geq s$. This "almost surely" can be removed by modifying the stochastic integral in (3.2) on an event of zero probability.

Introduce $f(r) = (r-1)^2$ for $r \in [0,1]$, $f(r) = 0$ for $r \geq 1$. Also, for $m = 1, 2, \ldots$, set $F_m = f(m\psi)$. Then for $m > 1/\varepsilon_0$, in $\delta_{1/m}G$ ($\subset \delta_{\varepsilon_0}G$) we have

$$2m^2 a_s^{ij}\nu^i\nu^j = a_s^{ij}D_{ij}F_m + 2m(1-m\psi)_+ a_s^{ij}D_{ij}\psi. \tag{3.4}$$

Also, obviously, the right-hand side is zero in $G_{1/m}$.

Next, if $\zeta \in C_0^\infty$, then, owing to the fact that DF_m has bounded derivatives, integrating by parts yields

$$(D_j\zeta, a_s^{ij}D_iF_m) = -(\zeta, (D_ja_s^{ij})D_iF_m + a_s^{ij}D_{ij}F_m).$$

By approximation, this formula extends to any $\zeta \in \overset{0}{W}{}_2^1(G)$. Hence, almost surely for all $m > 1/\varepsilon_0$ and $t \geq 0$

$$\begin{aligned}\mathcal{A}_{mt} = &-\int_0^t [(\bar{\pi}_sD_ja_s^{ij}, D_iF_m) + (D_j\bar{\pi}_s, a_s^{ij}D_iF_m)]\,\mathrm{d}s \\ &+2m\int_0^t (\bar{\pi}_s, (1-m\psi)_+ a_s^{ij}D_{ij}\psi)\,\mathrm{d}s \\ = &\, I_{mt}^1 - \int_0^t (D_jF_m, a_s^{ij}D_i\bar{\pi}_s + \bar{\pi}_sD_ia_s^{ij} - b_s^j\bar{\pi}_s)\,\mathrm{d}s,\end{aligned}$$

where

$$I_{mt}^1 := 2m\int_0^t (\bar{\pi}_s, (1-m\psi)_+L_s\psi)\,\mathrm{d}s$$

and $I_{mt}^1 \to 0$ uniformly on finite time intervals (a.s.) by Lemma 3.3.

Due to Remark 3.1, for $H_m = 1 - F_m$, we have

$$\mathcal{A}_{mt} = I_{mt}^1 + (\pi_0, H_m) - (\bar{\pi}_t, H_m) - \int_0^t (\sigma_s^{ik}D_i\bar{\pi}_s + (D_i\sigma_s^{ik} - \beta_s^k)\bar{\pi}_s, H_m)\,\mathrm{d}\tilde{y}_s^k$$

$$= \mathcal{A}_t + I^1_{mt} + J^2_{mt} + I^3_{mt} + I^4_{mt} + I^5_{mt}, \tag{3.5}$$

where,

$$I^2_{mt} = -(\pi_0, F_m),$$

$$I^3_{mt} = (\bar{\pi}_t, F_m),$$

$$I^4_{mt} = -\int_0^t (\beta^k_s \bar{\pi}_s, F_m)\, \mathrm{d}\tilde{y}^k_s$$

$$I^5_{mt} = 2m \int_0^t (\bar{\pi}_s, (1 - m\psi)_+ \sigma^{ik}_s D_i \psi)\, \mathrm{d}\tilde{y}^k_s.$$

By Hölder's inequality $|I^2_{mt}| \le \|\pi_0\|_{L_2} \|F_m\|_{L_2} \to 0$ as $m \to \infty$ since $F_m \downarrow 0$ as $m \to \infty$. Similar estimate and the fact that $\|\bar{\pi}_t\|_{L_2}$ is continuous in t (a.s.) shows that $I^3_{mt} \to 0$ uniformly on finite time intervals (a.s.).

The remaining terms I^4_{mt} and I^5_{mt} are stochastic integrals and to show that they tend to zero uniformly on finite time intervals in probability it suffices to prove that the same holds for their quadratic variations. In what concerns I^4_{mt} it suffices to observe that

$$\int_0^t |(\beta^k_s \bar{\pi}_s, F_m)|^2 \, \mathrm{d}s \le \|F_m\|^2_{L_2} \int_0^t \|\beta^k_s \bar{\pi}_s\|^2_{L_2} \, \mathrm{d}s. \tag{3.6}$$

In the case of I^5_{mt} we get the result from Lemma 3.3.

We recall that $(\pi_0, 1) = 1$ and after that coming back to (3.5) we see that, indeed, the convergence in (3.3) is uniform on finite time intervals in probability and the lemma is proved. □

Remark 3.5. From (3.2) it follows that

$$E\mathcal{A}^2_t \le 1 + E\big(\int_0^t (\bar{\pi}_s, \beta^k_s)\, d\tilde{y}^k_s \big)^2$$

$$\le 1 + 2E\big(\int_0^t (\bar{\pi}_s, \beta^k_s)\, \mathrm{d}\tilde{w}^k_s \big)^2 + 2E\big(\int_0^t (\bar{\pi}_s, \beta^k_s) \tilde{\beta}^k_s \, \mathrm{d}s \big)^2,$$

where we recall that

$$\tilde{w}_t = \int_0^t \Psi_s \Theta_s \, \mathrm{d}w_s$$

is a Wiener process and $\tilde{\beta}_t = \beta_t(x_t)$. It follows by the isometry of stochastic integration and by Hölder's inequality that

$$E\mathcal{A}_t^2 \le 1 + N(1+t)E\int_0^t \|\bar{\pi}_s\|_{L_2}^2 \,\mathrm{d}s < \infty,$$

where the constant N is independent of t.

The main result of this chapter is the following theorem.

Theorem 3.6. *There exists a function $\alpha_t(B) = \alpha_t(\omega, B)$ defined on $\Omega \times [0,\infty) \times \mathcal{B}(\partial G)$ which is $\mathcal{P}^y$-measurable for each B, it is a probability measure with respect to B for any (ω, t), and is such that uniformly on finite time intervals in probability*

$$2m^2 \int_0^t \int_{\delta_{1/m}G} f_s a_s^{ij} \nu^i \nu^j \bar{\pi}_s \,\mathrm{d}x \,\mathrm{d}s \to \int_0^t \int_{\partial G} f_s \,\alpha_s(dx)\,\mathrm{d}\mathcal{A}_s \tag{3.7}$$

as $m \to \infty$ for any function $f_t = f_t(x) = f_t(\omega, x)$ possessing these properties:

(i) for any ω the function $f_t(x)$ is continuous with respect to $(t,x) \in [0,\infty) \times \bar{G}$;

(ii) for any $(t,x) \in [0,\infty) \times \bar{G}$, $f_t(x)$ is a random variable.

Remark 3.7. By definition for $dx \in \mathcal{B}(\partial G)$ we set

$$\alpha_t(dx) =: (a_t^{ij}\nu^i\nu^j \frac{\partial \bar{\pi}_t}{\partial \nu})(dx)\big[(a_t^{ij}\nu^i\nu^j \frac{\partial \bar{\pi}_t}{\partial \nu})(G)\big]^{-1}. \tag{3.8}$$

This definition is natural in the following sense. If we assume that the derivative $\partial\bar{\pi}_s/\partial\nu$ exists at the boundary, then for f independent of (ω, t)

$$2m^2 \int_0^t \int_{\delta_{1/m}G} f a_s^{ij}\nu^i\nu^j \bar{\pi}_s \,\mathrm{d}x\,\mathrm{d}s$$

is close to

$$I_{mt}(f) = 2m^2 \int_0^t \int_{\delta_{1/m}G} f a_s^{ij}\nu^i\nu^j \frac{\partial \bar{\pi}_s}{\partial \nu}\psi \,\mathrm{d}x\,\mathrm{d}s.$$

As is easy to see

$$2m^2 \int_{\delta_{1/m}G} \psi \,\mathrm{d}x$$

tends to the surface area of ∂G. Then naturally

$$I_{mt}(f) \to \int_0^t \int_{\partial G} f a_s^{ij}\nu^i\nu^j \frac{\partial \bar{\pi}_s}{\partial \nu}\,\Sigma(dx)ds =: J_t(f),$$

where Σ is the surface measure. Hence,

$$\frac{J_t(f)-J_s(f)}{J_t(1)-J_s(1)}=\frac{1}{A_t-A_s}\int_s^t\int_{\partial G} f\,\alpha_r(dx)\,\mathrm{d}\mathcal{A}_r.$$

By letting $s\uparrow t$ and using the arbitrariness of f, we naturally come to (3.8).

Remark 3.8. From the proof of Theorem 3.6 it will be seen that we do not use any relation of $\bar\pi$ to the filtering problem at hand. Similar results can be obtained for any SPDE in divergence form.

To prove Theorem 3.6 we need a few auxiliary results. Here is a generalization of Lemma 3.4.

Lemma 3.9. *Let f be a twice continuously differentiable function in $\bar G$. Introduce*

$$\mathcal{A}_t(f)=(\pi_0,f)-(\bar\pi_t,f)+\int_0^t(\bar\pi_s,L_sf)\,\mathrm{d}s$$

$$-\int_0^t(\sigma_s^{ik}D_i\bar\pi_s+(D_i\sigma_s^{ik}-\beta_s^k)\bar\pi_s,f)\,\mathrm{d}\tilde y_s^k.$$

Then uniformly on finite time intervals in probability

$$2m^2\int_0^t\int_{\delta_{1/m}G} f a_s^{ij}\nu^i\nu^j\bar\pi_s\,\mathrm{d}x\,\mathrm{d}s\to\mathcal{A}_t(f).$$

Proof. Take the functions ψ and F_m from the proof of Lemma 3.4 and observe that for $m>1/\varepsilon_0$, in $\delta_{1/m}G$ we have

$$2m^2fa_s^{ij}\nu^i\nu^j=a_s^{ij}fD_{ij}F_m+2m(1-m\psi)_+fa_s^{ij}D_{ij}\psi$$

$$=a_s^{ij}D_{ij}(fF_m)-F_ma_s^{ij}D_{ij}f+2m(1-m\psi)_+[fD_{ij}\psi+2(D_if)D_j\psi]$$

and the last expression vanishes in $G_{1/m}$. Therefore, as in the proof of Lemma 3.4 we see that uniformly on finite time intervals in probability

$$2m^2\int_0^t\int_{\delta_{1/m}G} fa_s^{ij}\nu^i\nu^j\bar\pi_s\,\mathrm{d}x\,\mathrm{d}s-Q_{mt}\to 0$$

as $m\to\infty$, where

$$Q_{mt}=\int_0^t(\bar\pi_s,a_s^{ij}D_{ij}(fF_m))\,\mathrm{d}s=-Q_{mt}^1+Q_t^2,$$

$$Q^1_{mt} = \int_0^t (\bar{\pi}_s, a^{ij}_s D_{ij}(fH_m))\,\mathrm{d}s, \quad Q^2_t = \int_0^t (\bar{\pi}_s, a^{ij}_s D_{ij} f)\,\mathrm{d}s.$$

Next, from (2.7) we infer that

$$Q^1_{mt} = -\int_0^t (a^{ij}_s D_i \bar{\pi}_s - b^j \bar{\pi}_s + \bar{\pi}_s D_i a^{ij}_s, D_j(fH_m))\,\mathrm{d}s$$

$$-\int_0^t (b^j \bar{\pi}_s, D_j(fH_m))\,\mathrm{d}s = (\bar{\pi}_t, fH_m) - (\pi_0, fH_m)$$

$$+\int_0^t (\sigma^{ik}_s D_i \bar{\pi}_s + (D_i \sigma^{ik}_s - \beta^k_s)\bar{\pi}_s, fH_m)\,\mathrm{d}\tilde{y}^k_s - \int_0^t (b^j \bar{\pi}_s, D_j(fH_m))\,\mathrm{d}s.$$

Hence, by simple manipulations we get that

$$Q_{mt} = \mathcal{A}_t(f) + Q^3_{mt} + Q^4_{mt} + Q^5_{mt},$$

where

$$Q^3_{mt} = (\bar{\pi}_t, fF_m) - (\pi_0, fF_m)$$

$$Q^4_{mt} = \int_0^t (\sigma^{ik}_s D_i \bar{\pi}_s + (D_i \sigma^{ik}_s - \beta^k_s)\bar{\pi}_s, fF_m)\,\mathrm{d}\tilde{y}^k_s$$

$$Q^5_{mt} = -\int_0^t (b^j_s \bar{\pi}_s, F_m D_j f)\,\mathrm{d}s, \quad Q^6_{mt} = 2m \int_0^t (b^j_s \bar{\pi}_s, f(1 - m\psi)_+)\,\mathrm{d}s.$$

As in the proof of Lemma 3.4 we show that $Q^3_{mt} + Q^4_{mt} + Q^5_{mt} \to 0$ uniformly on finite time intervals in probability and this proves the lemma. □

Now we further generalize these results for continuous f.

Lemma 3.10. *There exists a function $\alpha_t(B) = \alpha_t(\omega, B)$ possessing the properties listed in Theorem 3.6 and is such that uniformly on finite time intervals in probability*

$$2m^2 \int_0^t \int_{\delta_{1/m}G} f a^{ij}_s \nu^i \nu^j \bar{\pi}_s \,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int_{\partial G} f\,\alpha_s(dx)\,\mathrm{d}\mathcal{A}_s$$

as $m \to \infty$ for any bounded and continuous function f given on $\bar{G}$.

Proof. We will be arguing in the same way as in [9] in a similar situation. Let $C(\bar{G})$ denote the Banach space of continuous functions on $\bar{G}$, let $C_+(\bar{G})$ be its subset consisting of nonnegative functions, and let $C_0(\bar{G})$ be the subspace of $C(\bar{G})$ consisting of functions vanishing on ∂G. Let $\mathbb{F}_+$ be a countable dense subset of $C_+(\bar{G})$ such that $\mathbb{F}_+ \cap C_0(\bar{G})$ is dense in $C_+(\bar{G}) \cap C_0(\bar{G})$ and $1 \in \mathbb{F}_+$.

Take an $f \in \mathbb{F}_+$ and take the processes $\mathcal{A}_t(f)$ introduced in Lemma 3.9 and the process $\mathcal{A}_t = \mathcal{A}_t(1)$ from Lemma 3.4. It follows from Lemmas 3.4 and 3.9 that $\mathcal{A}_t(f)$ and $\mathcal{A}_t$ are continuous and satisfy (a.s.)

$$0 \le \mathcal{A}_t(f) - \mathcal{A}_s(f) \le \sup_{\bar{G}} |f| (\mathcal{A}_t - \mathcal{A}_s)$$

for all s, t such that $0 \le s \le t$.

Due to an appropriate measurability of $\mathcal{A}_t(f)$ with respect to ω, the quantities

$$\nu_f(B) = E \int_0^\infty I_B(\omega, t) \, \mathrm{d}\mathcal{A}_t(f), \quad \mu(B) = E \int_0^\infty I_B(\omega, t) \, \mathrm{d}\mathcal{A}_t$$

are measures on $(\Omega \times (0, \infty), \mathcal{F} \otimes \mathcal{B}(0, \infty))$ and the first one is absolutely continuous with respect to the second one. However, we will be only interested in their values on the σ-field $\mathcal{P}^y$. Observe that on this σ-field μ is σ-finite, since for

$$\gamma_n = \inf\{t \ge 0 : \mathcal{A}_t \ge n\}$$

we have that $(\!(0, \gamma_n]\!] \in \mathcal{P}^y$, $\Omega \times (0, \infty) = \cup_n (\!(0, \gamma_n]\!]$, and $\mu((\!(0, \gamma_n]\!]) \le n$. In addition, $\nu_f \le \sup_{\bar{G}} |f| \mu$, which by the Radon-Nikodým theorem allows us to conclude that there exists a $\mathcal{P}^y$-measurable process $\alpha_t(f)$ such that

$$0 \le \alpha_t(f) \le \sup_{\bar{G}} |f|,$$

$$E \int_0^\infty I_B(\omega, t) \, \mathrm{d}\mathcal{A}_t(f) = E \int_0^\infty I_B(\omega, t) \alpha_t(f) \, \mathrm{d}\mathcal{A}_t$$

for any $B \in \mathcal{P}^y$. In particular, for any $n \ge 1$ and $\mathcal{F}_t^y$-stopping time γ we have

$$E\big(I_{\gamma \wedge \gamma_n}(f) - \int_0^{\gamma \wedge \gamma_n} \alpha_t(f) \, \mathrm{d}\mathcal{A}_t\big) = 0,$$

which implies that

$$\mathcal{A}_{t \wedge \gamma_n}(f) - \int_0^{t \wedge \gamma_n} \alpha_s(f) \, \mathrm{d}\mathcal{A}_s$$

is an $\mathcal{F}_t^y$-martingale. Since it is continuous and has locally bounded variation, it is zero for all $t \ge 0$ (a.s.). By letting $n \to \infty$ we obtain that (a.s.)

$$\mathcal{A}_t(f) = \int_0^t \alpha_s(f) \, \mathrm{d}\mathcal{A}_s, \quad \forall t \ge 0. \tag{3.9}$$

The above construction of $\alpha_s(f)$ is obviously valid with trivial modifications for any $f \in C(\bar{G})$. In particular, for any numbers $n \geq 1$, $r_i, i = 1, ..., n$, and $f_i \in \mathbb{F}_+$, $i = 1, ..., n$, we have a well-defined process

$$\alpha_s(g), \quad g = \sum_{i=1}^{n} r_i f_i.$$

Furthermore, by uniqueness of Radon-Nikodým derivatives we have that

$$\alpha_s\big(\sum_{i=1}^{n} r_i f_i\big) = \sum_{i=1}^{n} r_i \alpha_s(f_i) \tag{3.10}$$

almost everywhere with respect to μ. Also

$$\big|\alpha_s\big(\sum_{i=1}^{n} r_i f_i\big)\big| \leq \sup_{\bar{G}} \big|\sum_{i=1}^{n} r_i f_i\big| \tag{3.11}$$

almost everywhere with respect to μ. Now we define Γ as the set of (ω, s) such that equations (3.10) and (3.11) hold for all $n \geq 1$, rational r_i, $i = 1, ..., n$, and $f_i \in \mathbb{F}_+$. Since $\alpha_s(f)$ are $\mathcal{F}_t^y$-predictable the same holds for Γ and

$$0 = \mu((\Omega \times (0, \infty)) \setminus \Gamma) = E \int_0^\infty I_{\Gamma^c}(\omega, s) \, \mathrm{d}\mathcal{A}_s. \tag{3.12}$$

Furthermore, for any $(\omega, s) \in \Gamma$ we have a continuous functional $\alpha_s(g)$ defined on a dense subset of $C(\bar{G})$ and linear over the set of rational numbers. It extends uniquely by continuity to become a linear bounded functional $\alpha_s(f)$ on the whole of $C(\bar{G})$ and by the Riesz representation theorem there exists a (signed) measure α_s on $\bar{G}$ such that $|\alpha_s|(\bar{G}) \leq 1$ and for any $f \in C(\bar{G})$ and $(\omega, s) \in \Gamma$

$$\alpha_s(f) = \int_{\bar{G}} f(x) \, \alpha_s(dx).$$

Extension by continuity preserves measurability properties, so that $\alpha_s(f)$ is a $\mathcal{P}^y$-measurable function on Γ for any bounded continuous f implying the same property for $\alpha_s(B)$ for any $B \in \mathcal{B}(\bar{G})$. By substituting $f = 1$ into (3.9) we see that $\alpha_s(\bar{G}) = 1$ almost everywhere with respect to μ.

Next, observe that the set $\hat{\Gamma} = \{(\omega, s) \in \Gamma : \alpha_s(\bar{G}) = 1\}$ is $\mathcal{P}^y$-measurable. On this set $\alpha_s(\bar{G}) = 1$ and $|\alpha_s|(\bar{G}) \leq 1$, which implies that α_s is a probability measure on $\bar{G}$ for $(\omega, s) \in \hat{\Gamma}$. Now we define $\hat{\alpha}_s(B) = \alpha_s(B)$ if $(\omega, s) \in \hat{\Gamma}$ and for $(\omega, s) \notin \hat{\Gamma}$ we set $\hat{\alpha}_s(B)$

to be any fixed nonrandom probability distribution on ∂G. We will show that $\hat{\alpha}_s(B)$ is almost the function we need.

Its $\mathcal{P}^y$-measurability follows from the above. Then for $f \in C(\bar{B})$ set

$$\hat{\mathcal{A}}_t(f) = \int_0^t \int_G f(x)\,\bar{\alpha}_s(dx)\,\mathrm{d}\mathcal{A}_s.$$

Equation (3.12) holds also if we replace Γ with $\hat{\Gamma}$, which shows that with probability one

$$\hat{\mathcal{A}}_t(f) = \int_0^t \int_G f(x)\,\alpha_s(dx) I_{\hat{\Gamma}}\,\mathrm{d}\mathcal{A}_s = \int_0^t I_{\hat{\Gamma}}\alpha_s(f)\,\mathrm{d}\mathcal{A}_s \tag{3.13}$$

for all $t \geq 0$. If $f \in \mathbb{F}_+$, the process $\alpha_s(f)$ was defined on a larger set than Γ and it satisfied (3.9) for all $t \geq 0$ (a.s.). For such f one can harmlessly drop the indicator in the last term of (3.13) and then one sees that (a.s.) $\hat{\mathcal{A}}_t(f) = \mathcal{A}_t(f)$ for all $t \geq 0$, which implies that uniformly on finite time intervals in probability

$$2m^2 \int_0^t \int_{\delta_{1/m}G} f a_s^{ij} \nu^i \nu^j \bar{\pi}_s\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int_{\bar{G}} f\,\hat{\alpha}_s(dx)\,\mathrm{d}\mathcal{A}_s \tag{3.14}$$

as $m \to \infty$ for any function $f \in \mathbb{F}_+$. Owing to the denseness of $\mathbb{F}_+$ in $C_+(\bar{G})$, (3.14) holds for any nonnegative $f \in C(\bar{G})$ and, by linearity, for any $f \in C(\bar{G})$.

For $f \in \mathbb{F}_+ \cap C_0(\bar{G})$ the left-hand side of (3.14) is obviously going to zero. Therefore, the right-hand side is zero (a.s.) for such f. It follows that

$$\int_0^t \hat{\alpha}_s(G)\,\mathrm{d}\mathcal{A}_s = 0$$

(a.s.), $\hat{\alpha}_s(G) = 0$ almost everywhere with respect to μ and then the function $\check{\alpha}_s(B)$ defined as $\hat{\alpha}_s(B)$ for those (ω, s) for which $\hat{\alpha}_s(G) = 0$ and defined elsewhere to be any fixed nonrandom probability measure on ∂G will possess all the properties claimed in the lemma, which is thus proved. □

Proof of Theorem 3.6. We take $\alpha_t(B)$ from Lemma 3.10 and first let f_s be independent of ω and be piecewise constant in s that is $f_s(x) = f^k(x)$ for $t \in (t_k, t_{k+1}]$, where $0 = t_0 < t_1 < t_2 < ...$ and $t_k \to \infty$ as $k \to \infty$. Observe that then the left hand-side of (3.7) equals

$$\sum_{k=0}^{\infty} 2m^2 \int_{t\wedge t_k}^{t\wedge t_{i+1}} \int_{\delta_{1/m}G} f^k a_s^{ij} \nu^i \nu^j \bar{\pi}_s\,\mathrm{d}x\,\mathrm{d}s.$$

In this situation we get (3.7) uniformly on finite time intervals in probability by Lemma 3.10 because on each finite time interval the series can contain only finitely many nonzero terms. By using uniform approximations of continuous functions by piecewise constant ones we convince ourselves that that the assertion of the theorem is true if f is independent of ω.

In the general case take a $T \in [0, \infty)$ and let $C([0,T] \times \bar{G})$ be the Banach space of (nonrandom) continuous functions on $[0,T] \times \bar{G}$. Since $C([0,T] \times \bar{G})$ is a Polish space, for any random element $f_s(x) = f_s(\omega, x)$ with values in this space and $\varepsilon > 0$ one can find finitely many $f^k \in C([0,T] \times \bar{G})$, $k = 1, .., n(\varepsilon)$, and an event Ω_ε such that $P(\Omega_\varepsilon) \geq 1 - \varepsilon$ and for each $\omega \in \Omega_\varepsilon$ one can find a $k = k(\omega)$ such that $|f - f^k| \leq \varepsilon$ on $[0,T] \times \bar{G}$. In other words, there are events $\Omega_\varepsilon^k \subset \Omega_\varepsilon$ such that $|f_s(x) - f_s^\varepsilon(x)| \leq \varepsilon$ for $\omega \in \Omega_\varepsilon$ and $(s, x) \in [0,T] \times \bar{G}$, where

$$f_s^\varepsilon(x) = \sum_k f_s^k(x) I_{\Omega_\varepsilon^k}.$$

Now denote by $I_{mt}(f)$ and $J_t(f)$ the left-hand side and the right-hand side, respectively, of (3.7) and observe that, by the above $I_{mt}(f^\varepsilon) - J_t(f^\varepsilon) \to 0$ uniformly on finite time intervals in probability. Also for $t \leq T$

$$|I_{mt}(f - f^\varepsilon)| \leq I_{\Omega_\varepsilon^c} \sup_{[0,T] \times \bar{G}} |f| I_{mT}(1) + \varepsilon I_{mT}(1),$$

$$|J_t(f - f^\varepsilon)| \leq I_{\Omega_\varepsilon^c} \sup_{[0,T] \times \bar{G}} |f| J_t(1) + \varepsilon J_T(1).$$

This after being combined with

$$I_{mt}(f) - J_t(f) = I_{mt}(f^\varepsilon) - J_t(f^\varepsilon) + I_{mt}(f - f^\varepsilon) + J_t(f - f^\varepsilon)$$

and the fact that $I_{mT}(1) \to J_T(1)$ in probability shows that for any $\kappa > 0$

$$\overline{\lim_{m \to \infty}} P(\sup_{t \leq T} |I_{mt}(f) - J_t(f)| > \kappa) \leq P(2 I_{\Omega_\varepsilon^c} \sup_{[0,T] \times \bar{G}} |f| J_t(1) + 2\varepsilon J_T(1) > \kappa)$$

$$\leq \varepsilon + P(2\varepsilon J_T(1) > \kappa/2).$$

Here the last expression tend to zero as $\varepsilon \downarrow 0$, which shows that the first expression (independent of ε) is zero and brings the proof of the theorem to an end. □

Here is a convenient reformulation of Theorem 3.6.

Remark 3.11. Owing to (3.4) and Lemma 3.3 one can replace the left-hand side of (3.7) with

$$\int_0^t \int_G f_s \bar{\pi}_s L_s F_m \, \mathrm{d}x \, \mathrm{d}s \quad \text{or} \quad \int_0^t \int_G f_s \bar{\pi}_s a_s^{ij} D_{ij} F_m \, \mathrm{d}x \, \mathrm{d}s.$$

Chapter 4

On the conditional probability of $\{\tau > T\}$ given $\mathcal{F}_T^y$

The goal of this chapter is to prove the following intuitively clear result, which will be used in the proof of the last assertion of Lemma 2.7.

Theorem 4.1. *For any $T \in [0, \infty)$, we have $P\{\tau > T \mid \mathcal{F}_T^y\} > 0$ (a.s.).*

We start with an auxiliary result.

Lemma 4.2. *Consider system* (1.1) *in which we replace the first equation with*

$$dx_t = [b(t, z_t) + \gamma(x_t)]\,\mathrm{d}t + \theta(t, z_t)\,\mathrm{d}w_t,$$

where $\gamma = \gamma(x)$ is an $\mathbb{R}^d$-valued bounded function on $\mathbb{R}^d$ which is Lipschitz continuous. Then for any $p > 0$ and $T \in [0, \infty)$ there exists a function γ with the properties described above such that

$$P(\tau \leq T) \leq p. \tag{4.1}$$

Proof. Of course we will take γ pointing inside G near ∂G with sufficiently large $|\gamma|$. Take ψ and ε_0 from the beginning of Chapter 3, set $\varepsilon_1 = \varepsilon_0/2$, and for a constant $c > 0$, to be specified later, define

$$\gamma(x) = -c(\psi(x) - \varepsilon_1)\mathrm{grad}\,\psi(x)$$

for $x \in \delta_{\varepsilon_0} G$ and continue $\gamma(x)$ outside δ_{ε_0} arbitrarily preserving its boundedness and Lipschitz continuity. Then fix a constant $\kappa > 0$ and introduce

$$U(x) = \exp\big(\kappa(\psi(x) - \varepsilon_1)^4\big).$$

Observe that in $[0, \infty) \times \delta_{\varepsilon_0} G$ we have

$$\partial_t U + \tilde{L}_t U - U = 4U\kappa(\psi - \varepsilon_1)^2[4\kappa(\psi - \varepsilon_1)^4 a^{ij}(D_i\psi)D_j\psi$$

$$+3a^{ij}(D_i\psi)D_j\psi + (\psi - \varepsilon_1)a^{ij}D_{ij}\psi] - 4U\kappa c(\psi - \varepsilon_1)^4|\mathrm{grad}\,\psi|^2 - U.$$

$$+4b^i U\kappa(\psi(x) - \epsilon_1)^3 D_i\psi,$$

in which $\tilde{L}_t$ is understood as

$$\tilde{L}_t(z) = a_t^{ij}(z)D_{ij} + (\tilde{b}_t^i(x) + \gamma_t^i(x))D_i,$$

$$\gamma_t^i(x) = 0, \text{ for } i > d.$$

Since $|\mathrm{grad}\,\psi| = 1$ at ∂G, by reducing ε_0 if necessary we may assume that $|\mathrm{grad}\,\psi| \geq 1/2$ in $\delta_{\varepsilon_0} G$. Then in $[0, \infty) \times \delta_{\varepsilon_0} G$ it holds that

$$U^{-1}(\partial_t U + \tilde{L}_t U - U) \leq N(\kappa^2 + 1)(\psi - \varepsilon_1)^2 - \kappa c(\psi - \varepsilon_1)^4 - 1,$$

where the constant N is independent of κ (and (t, z), which are dropped for convenience of notation). It follows that for any $\kappa > 0$ there is a sufficiently large $c > 0$ such that $\partial_t U + \tilde{L}_t - U \leq 0$ in $[0, \infty) \times \delta_{\varepsilon_0} G$.

Now let

$$\tau_1 = \inf\{t \geq 0 : x_t \notin G_{\varepsilon_1}\},$$

$$\tau_2 = \inf\{t \geq \tau_1 : x_t \notin \delta_{\varepsilon_0} G\}.$$

By the way, observe that $\tau_2 \leq \tau < \infty$ (a.s.) because x_t is uniformly nondegenerate and G is bounded. Now by Itô's formula

$$Ee^{-\tau_2}\exp(\kappa\varepsilon_1^4) = Ee^{-\tau_2}U(x_{\tau_2})$$

$$= EI_{x_0 \in \delta_{\varepsilon_1} G} e^{-\tau_2} U(x_{\tau_2}) + EI_{x_0 \in G_{\varepsilon_1}} e^{-\tau_2} U(x_{\tau_2})$$

$$= EI_{x_0 \in \delta_{\varepsilon_1} G} e^{-\tau_2} U(x_{\tau_2}) + EI_{x_0 \in \delta_{\varepsilon_1} G} e^{-\tau_1} U(x_{\tau_1})$$

$$+EI_{x_0\in\delta_{\varepsilon_1}G}\int_{\tau_1}^{\tau_2} e^{-t}(\partial_t U+\tilde{L}_t U-U)(t,z_t)\,\mathrm{d}t$$
$$\le EI_{x_0\in\delta_{\varepsilon_1}G}e^{-\tau_2}U(x_{\tau_2})+EI_{x_0\in\delta_{\varepsilon_1}G}e^{-\tau_1}U(x_{\tau_1}).$$

Since $U(x_{\tau_1})=1$ provided that $x_0\in\delta_{\varepsilon_1}G$, the second term in the inequality is less than 1 and we conclude

$$P(\tau\le T)e^{-T}\exp(\kappa\varepsilon_1^4)\le Ee^{-\tau_2}\exp(\kappa\varepsilon_1^4)\le\exp(\kappa\varepsilon_1^4)P(x_0\in\delta_{\varepsilon_1}G)+1,$$
$$P(\tau\le T)\le e^T P(x_0\in\delta_{\varepsilon_1}G)+\exp(T-\kappa\varepsilon_1^4).$$

Finally, by reducing further ε_0, if necessary, we may assume that $P(x_0\in\delta_{\varepsilon_1}G)\le e^{-T}p/2$ and then, by choosing κ large enough (and choosing c accordingly) we can have $\exp(T-\kappa\varepsilon_1^4)\le p/2$. Then we come to (4.1) and the lemma is proved. □

Proof of Theorem 4.1. Take $T\in[0,\infty)$ and assume that our assertion is false. Then there is an event $H\in\mathcal{F}_T^y$ and a number $p>0$ such that

$$P(\tau>T,H)=0,\quad P(H)=p.$$

We are going to use Girsanov's theorem and, therefore, we need a more detailed notation. We write $\tau(x.)$ to specify that τ is the first exit time of x_t from G. Also each event $H\in\mathcal{F}_T^y$ can be written as $\{\omega : y.\in\Gamma\}$ where Γ is a Borel subset of the space of $\mathbb{R}^{d_1}$-valued continuous functions on $[0,T]$. Now our assumption becomes

$$P(\tau(x.)>T,y.\in\Gamma)=0,\quad P(y.\in\Gamma)=p. \tag{4.2}$$

Consider the following system

$$d\hat{x}_t=b(t,\hat{z}_t)\,\mathrm{d}t+\theta(t,\hat{z}_t)\,(dw_t-\Theta^*(t,\hat{y}_t)\Psi^2(t,\hat{y}_t)B(t,\hat{z}_t)\,\mathrm{d}t),$$
$$d\hat{y}_t=\Theta(s,\hat{y}_s)\,\mathrm{d}w_s,$$

where $\hat{z}_t=(\hat{x}_t,\hat{y}_t)$ with initial condition $\hat{z}_0=z_0$. If we introduce

$$\hat{w}_t=w_t-\int_0^t\Theta^*(s,\hat{y}_s)\Psi^2(s,\hat{y}_s)B(s,\hat{z}_s)\,\mathrm{d}s,$$

then, as is easy to check, $\hat{z}_t$ satisfies (1.1) with $\hat{w}_t$ in place of w_t. It follows by Girsanov's theorem that

$$P(\tau(\hat{x}.)>T,\hat{y}.\in\Gamma)=0,\quad P(\hat{y}.\in\Gamma)>0.$$

We see that, while proving that (4.2) is impossible, we may assume that $B \equiv 0$. We proceed further under this assumption.

Our next step is to change the underlying probability measure in such a way that the second relation in (4.2) would remain unchanged and τ would be almost infinite. Then the two relations in (4.2) will become incompatible.

Take the function $\gamma = \gamma(x)$ from Lemma 4.2, where we take $p/2$ in place of p, and consider the system

$$d\check{x}_t = [b(t, \check{z}_t) + \gamma(\check{x}_t)]\,\mathrm{d}t + \theta(t, \check{z}_t)\,\mathrm{d}w_t, \quad d\check{y}_t = \Theta(s, \check{y}_s)\,\mathrm{d}w_s,$$

where $\check{z}_t = (\check{x}_t, \check{y}_t)$ with initial condition $\check{z}_0 = z_0$. Obviously, $\check{y}_t = y_t$, so that $P(\check{y}. \in \Gamma) = p$. On the other hand, by Lemma 4.2 we have $P(\tau(\check{x}.) \le T) \le p/2$. It follows that

$$P(\tau(\check{x}.) > T, \check{y}. \in \Gamma) = P(\check{y}. \in \Gamma) - P(\tau(\check{x}.) \le T, \check{y}. \in \Gamma) \ge p/2.$$

However, if we knew that the distribution of $\{\check{z}_t, t \in [0, T]\}$ is absolutely continuous with respect to the distribution of $\{z_t, t \in [0, T]\}$, then the first relation in (4.2) would imply that $P(\tau(\check{x}.) > T, \check{y}. \in \Gamma) = 0$ and this would lead to the desired contradiction.

By Girsanov's theorem the above mentioned absolute continuity holds if there is a bounded $\mathbb{R}^{d_1}$-valued $\mathcal{P}$-measurable process $\check{\gamma}_t$ such that

$$\begin{pmatrix} \gamma(\check{x}_t) \\ 0 \end{pmatrix} = \tilde{\theta}(t, \check{z}_t)\check{\gamma}_t. \tag{4.3}$$

To find an appropriate $\check{\gamma}_t$ observe that as we know (see, for instance, [10]), owing to the fact that $\tilde{\theta}\tilde{\theta}^*$ is uniformly nondegenerate, we have that the $d \times d$-matrix valued function

$$U = U(t, z) := \theta[1 - \Theta^* \Psi^2 \Theta]\theta^*$$

is also uniformly nondegenerate. Now we claim that the bounded process

$$\check{\gamma}_t = [1 - \Theta_t^* \Psi_t^2 \Theta_t]\theta^*(t, \check{z}_t) U^{-1}(t, \check{z}_t)\gamma(\check{x}_t)$$

satisfies (4.3) (keep in mind that $\check{y}_t = y_t$). Indeed, by the definition of U we have

$$\theta(t, \check{z}_t)\check{\gamma}_t = \theta(t, \check{z}_t)[1 - \Theta_t^* \Psi_t^2 \Theta_t]\theta^*(t, \check{z}_t) U^{-1}(t, \check{z}_t)\gamma(\check{x}_t) = \gamma(\check{x}_t).$$

Furthermore, $\Theta\Theta^*\Psi^2 = 1$ so that $\Theta[1 - \Theta^*\Psi^2\Theta] = 0$ and

$$\Theta(t, \check{z}_t)\check{\gamma}_t = 0.$$

The theorem is proved. $\square$

Chapter 5

Proof of the main results

We are going to use the approach from [10] which allows us to derive all our results about filtering without using anything from filtering theory itself. According to this approach we first solve an appropriate deterministic problem.

Take a function $\phi(t,z)$ of class $C_0^\infty(\mathbb{R}\times\mathbb{R}^{d_1})$ and a Borel bounded function $c(t,z)$ on $\mathbb{R}\times\mathbb{R}^{d_1}$, take a $T\in[0,\infty)$, set $\mathbb{G}=G\times\mathbb{R}^{d_2}$, set

$$\partial_t = \frac{\partial}{\partial t}$$

and consider the following equation

$$\partial_t v(t,z) + \tilde{L}_t v(t,z) + c(t,z)v(t,z) = 0 \tag{5.1}$$

in $\mathbb{G}_T := (0,T)\times\mathbb{G} = (0,T)\times G\times\mathbb{R}^{d_2}$ with boundary and terminal condition equal to ϕ:

$$\begin{aligned} v = \phi \quad &\text{on} \quad [0,T]\times\partial\mathbb{G}, \\ v(T,z) = \phi(T,z) \quad &\text{on} \quad \mathbb{G}. \end{aligned} \tag{5.2}$$

For $p\geq 1$ introduce $W_p^{1,2}(\mathbb{G}_T)$ as the closure of $C^{1,2}(\bar{\mathbb{G}}_T)$ with respect to the norm

$$\|u\|_{W_p^{1,2}(\mathbb{G}_T)} = \|\partial_t u\|_{L_p(\mathbb{G}_T)} + \|D^z u\|_{L_p(\mathbb{G}_T)} + \|(D^z)^2 u\|_{L_p(\mathbb{G}_T)} + \|u\|_{L_p(\mathbb{G}_T)},$$

where $D^z u$ is the gradient and $(D^z)^2 u$ is the Hessian matrix with respect to z of the function $u(t,z)$.

Lemma 5.1. *There exists a unique function v on $\bar{\mathbb{G}}_T$ such that $v \in W_p^{1,2}(\mathbb{G}_T) \cap C(\bar{\mathbb{G}}_T)$ for all $p \in ((d+2)/2, \infty)$, v satisfies* (5.1) *(a.e.) in $\mathbb{G}_T$ and satisfies* (5.2)*. Moreover, $D^z v(t,z)$ is bounded and continuous in $\bar{\mathbb{G}}_T$.*

Proof. The first assertion, in what concerns the solvability in the space $W_p^{1,2}(\mathbb{G}_T)$, is classical if the coefficients of $\tilde{L}_t$ are uniformly continuous in (t,z) (see, for instance, Theorem IV.9.1, its corollary, and the comment about (IV.9.31) in [11]. However, we are only given that they are Lipschitz continuous in z and no regularity with respect to t is assumed. In any case the coefficients are measurable in t and VMO in z. In such a situation interior and boundary estimates in half spaces needed for the arguments in [11] to work are obtained in [12]. Actually, in [12] higher order elliptic and parabolic systems are considered, so that applying the results from there looks somewhat strange but yet we could not find in the literature an appropriate result for second order single parabolic equation. On the other hand, it is worth mentioning that in [12] the boundary values are assumed to be zero. As always, one reduces our situation to theirs by considering $v - \phi$ which will lead only to appearing of a free term in (5.1). One more comment is that neither in [11] nor in [12] it is mentioned that the solution is independent of p as in $W_p^{1,2}(\mathbb{G}_T)$, the space where it is looked for. This fact is absolutely trivial for equations in bounded domains and proved in a standard way for unbounded domains by using embedding theorems. In this argument one can take any $p \in (1,\infty)$.

Once we know that $v \in W_p^{1,2}(\mathbb{G}_T)$ its boundedness and continuity in $\bar{\mathbb{G}}_T$ follow from embedding theorems if we take $p > (d+2)/2$ (see, for instance, Lemma II.3.3 of [11]). The boundedness and continuity in $\bar{\mathbb{G}}_T$ of $D^z v$ also follow from embedding theorems if we take $p > d+2$. The lemma is proved. □

The reader might have noticed that if we have a function $\psi(t,z)$, then by ψ_t or $\psi_t(x)$ we denote the function $\psi(t,x,y_t)$. We are going to use this stipulation quite often.

We want to relate $\bar{\pi}_t$ to our filtering problem by considering $(\bar{\pi}_t, v_t)$. In order to do that we first cut off $v_t(x)$ near ∂G.

Lemma 5.2. *Take v from Lemma 5.1 and F_m and H_m from the proof of Lemma 3.4. Set $v_{mt}(x) = H_m(x)v(t,x,y_t)$. Then $v_m \in \overset{0}{\mathcal{W}}{}_2^1(T)$ and on $[0,T]$ we have*

$$\mathrm{d}(v_{mt}) = H_m D_i^y v_t \Theta_t^{ik}\, \mathrm{d}w_t^k$$

$$+H_m\big[-c_tv_t - L_tv_t - 2\tilde{a}_t^{i,d+j}D_iD_j^yv_t - B_t^iD_i^yv_t\big]\,dt$$

in the sense of generalized functions, where

$$D_i^y = \frac{\partial}{\partial y^i}$$

and we first apply the differentiations and then plug in the argument (t, x, y_t)*.*

Proof. First we take a $\xi \in C^{1,2}(\bar{\mathbb{G}}_T)$ and use Itô's formula to write that for any $x \in G$

$$d\xi(t,x,y_t) = \big[\partial_t\xi(t,x,y_t) + \tilde{a}^{d+i,d+j}(t,x,y_t)D_{ij}^y\xi(t,x,y_t)$$

$$+\tilde{b}^{d+i}(t,x,y_t)D_i^y\xi(t,x,y_t)\big]\,dt + D_i^y\xi(t,x,y_t)\Theta_t^{ik}\,dw_t^k, \tag{5.3}$$

where $D_{ij}^y = D_i^yD_j^y$.

Next, we take a test function $\zeta \in C_0^\infty$, multiply both part of (5.3) by ζH_m and integrate with respect to x over G, that is, use the stochastic Fubini theorem (see, for instance, Lemma 2.7 of [13]). Then we obtain that, with probability one, for all $t \geq 0$

$$(H_m\xi_t, \zeta) = (H_m\xi_0, \zeta) + \int_0^t (H_mD_i^y\xi_s, \zeta)\Theta_s^{ik}\,dw_s^k$$

$$+\int_0^t \big(H_m\big[\partial_s\xi_s + \tilde{a}_s^{d+i,d+j}D_{ij}^y\xi_s + \tilde{b}_s^{d+i}D_i^y\xi_s\big], \zeta\big)\,ds. \tag{5.4}$$

Now we take a sequence $\xi^n \in C^{1,2}(\bar{\mathbb{G}}_T)$, $n = 1, 2, ...$, which converges to v in $W_p^{1,2}(\mathbb{G}_T)$ as $n \to \infty$ for $p = d+1$ and $p = 2(d+1)$, so that $\xi^n \to v$ also in $C(\bar{\mathbb{G}}_T)$. We plug ξ^n in place of ξ into (5.4) and pass to the limit as $n \to \infty$. Observe that by Hölder's inequality

$$I_n := E\int_0^T |(H_mD_i^y(\xi_s^n - v_s), \zeta)|^2\,ds$$

$$\leq N\int_G E\int_0^T |D^y(\xi^n(s,x,y_s) - v(s,x,y_s))|^2\,ds.$$

By Theorem 2.3.3 of [14] the right-hand side is dominated by

$$N\int_G \big(\int_{[0,T]\times\mathbb{R}^d} |D^y(\xi^n(s,x,y) - v(s,x,y))|^{2(d+1)}\,dyds\big)^{1/(d+1)}dx,$$

which again by Hölder's inequality is dominated by

$$N\Big(\int_G\int_{[0,T]\times\mathbb{R}^d}|D^y(\xi^n(s,x,y)-v(s,x,y))|^{2(d+1)}\,dyds\Big)^{1/(d+1)}dx$$

$$\leq N\|\xi^n-v\|^2_{W^{1,2}_{2(d+1)}(\mathbb{G}_T)},$$

and the latter tends to zero as $n\to\infty$. Therefore,

$$\int_0^t(H_mD_i^y\xi_s^n,\zeta)\Theta_s^{ik}\,dw_s^k\to\int_0^t(H_mD_i^yv_s,\zeta)\Theta_s^{ik}\,dw_s^k$$

uniformly on $[0,T]$ in probability. Uniform convergence in probability of the usual integrals with respect to s in (5.4) is proved similarly and, since $\xi^n\to v$ uniformly in $\bar{\mathbb{G}}_T$, the remaining terms in (5.4) with ξ^n in place of ξ will also converge uniformly on $[0,T]$ (for any ω).

Hence, with probability one for all $t\in[0,T]$

$$(H_mv_t,\zeta)=(H_mv_0,\zeta)+\int_0^t(H_mD_i^yv_s,\zeta)\,dy_s^i$$

$$+\int_0^t\big(H_m\big[\partial_sv_s+\tilde{a}_s^{d+i,d+j}D_{ij}^yv_s\big],\zeta\big)\,ds.$$

Here the expression in the brackets can be transformed on account of (5.1). This yields

$$(H_mv_t,\zeta)=(H_mv_0,\zeta)+\int_0^t(H_mD_i^yv_s,\zeta)\,dy_s^i$$

$$+\int_0^t\big(H_m\big[c_sv_s-L_sv_s-2\tilde{a}_s^{i,d+j}D_iD_j^yv_s-B_s^iD_i^yv_s\big],\zeta\big)\,ds.$$

The lemma is proved. □

The following theorem concludes the first step in relating $\bar{\pi}_t$ to the conditional distribution of x_t.

Theorem 5.3. *Take v from Lemma 5.1 with c depending only on (t,y) and introduce*

$$\kappa_t=\kappa_t(c)=\exp\int_0^tc_s\,ds.$$

Then with probability one for all $t\in[0,T]$ we have

$$(\bar{\pi}_t,v_t)\kappa_t\rho_t=(\pi_0,v_0)-\int_0^t\kappa_s\rho_s\int_{\partial G}\phi_s\,\alpha_s(dx)\,d\mathcal{A}_s$$

$$+\int_0^t \kappa_s\rho_s(\bar\pi_s, D^y_j v_s\Phi^{jk}_s + \Lambda^k_s v_s - \tilde\beta^k_s v_s)\,d\tilde w^k_s. \tag{5.5}$$

Proof. Take H_m from the proof of Lemma 3.4 and set $v_{mt}(x) = H_m(x)v(t,x,y_t)$. We remind the reader that Itô's formula for the L_2-norm of $\overset{0}{\mathcal{W}}{}^1_2$-processes is a classical result proved in various settings in many places. Both $\bar\pi_t$ and v_{mt} are such processes.

By applying Itô's formula for $\|\bar\pi_t - \lambda v_{mt}\|^2_{L_2}$ and comparing the coefficients of λ we see that a natural result holds for $(\bar\pi_t, v_{mt})$. Namely,

$$(\bar\pi_t, v_{mt}) = (\pi_0, v_{m0}) + I_{mt} + J_{mt}, \tag{5.6}$$

where

$$I_{mt} := \int_0^t \big(\bar\pi_s, L_s v_{ms} + H_m\big[-c_s v_s - L_s v_s - 2\tilde a^{i,d+j}_s D_i D^y_j v_s - B^i_s D^y_i v_s\big]\big)\,ds$$

$$+\int_0^t (\Lambda^{k*}_s\bar\pi_s\Psi^{kr}_s\Theta^{rj}_s, H_m D^y_i v_s)\Theta^{ij}_s\,ds,$$

$$J_{mt} := \int_0^t \big[(\bar\pi_s, H_m D^y_i v_s) + (\Lambda^{k*}_s\bar\pi_s, v_{ms})\Psi^{ki}_s\big]\,dy^i_s.$$

Denote $\Phi_s = \Psi^{-1}_s$ and notice that

$$\Psi_s\Theta_s\theta^*_s = \Phi_s, \quad \Psi^{kr}_s\Theta^{rj}_s\theta^{ij}_s = \Phi^{ki}_s.$$

We also use the fact that by definition

$$2\tilde a^{i,d+j}_t\Psi^{jk} = \sigma^{ik}_t, \quad \beta^i_t = \Psi^{ik}_t B^k_t.$$

Consequently,

$$2\tilde a^{i,d+j}_t = \sigma^{ik}_s\Phi^{kj}_t, \quad B^i_t = \Phi^{ki}_s\beta^i_s$$

Then we find that

$$(\Lambda^{k*}_s\bar\pi_s\Psi^{kr}_s\Theta^{rj}_s, H_m D^y_i v_s)\Theta^{ij}_s = (\bar\pi_s, \Lambda^k_s(H_m D^y_i v_s))\Phi^{ki}_s$$

$$= (\bar\pi_s, \beta^k_s H_m D^y_i v_s)\Phi^{ki}_s + (\bar\pi_s, H_m\sigma^{jk}_s D_j D^y_i v_s)\Phi^{ki}_s + (\bar\pi_s, \sigma^{jk}_s(D_j F_m)D^y_i v_s)\Phi^{ki}_s$$

$$= \big(\bar\pi_s, H_m\big[B^i_s D^y_i v_s + 2\tilde a^{i,d+j}_s D_i D^y_j v_s\big]\big) + 2(\bar\pi_s, \tilde a^{i,d+j}_s(D_i F_m)D^y_j v_s).$$

This allows us to cancel certain terms in the definition of I_{mt}. Furthermore, observe that

$$L_s v_{ms} = H_m L_s v_s + v_s L_s H_m + 2a_s^{ij}(D_i F_m) D_j v_s.$$

Then we see that

$$I_{mt} = \int_0^t \big(\bar{\pi}_s, v_s L_s H_m - c_s v_{ms} + 2a_s^{ij}(D_i F_m) D_j v_s + 2\tilde{a}_s^{i,d+j}(D_i F_m) D_j^y v_s\big)\, ds.$$

By Remark 3.11 and Lemma 3.3 we obtain that uniformly on $[0,T]$ in probability

$$I_{mt} \to -\int_0^t \int_{\partial G} \phi_s\, \alpha_s(dx)\, d\mathcal{A}_s - \int_0^t c_s(\bar{\pi}_s, v_s)\, ds$$

(here we also used the fact that $c = c(t,y)$, so that $c_t = c(t, y_t)$ is independent of x, and the fact that $v_s(x) = \phi_s(x)$ if $x \in \partial G$).

Coming to J_{mt} we notice that

$$(\Lambda_s^{k*}\bar{\pi}_s, v_{ms}) = (\bar{\pi}_s, \Lambda_s^k v_{ms}) = (\bar{\pi}_s, H_m \Lambda_s^k v_s) + (\bar{\pi}_s, v_s \sigma_s^{ik} D_i F_m).$$

After that an already familiar argument convinces us that uniformly on $[0,T]$ in probability

$$J_{mt} \to \int_0^t (\bar{\pi}_s, D_i^y v_s + \Lambda_s^k v_s \Psi_s^{ki})\, dy_s^i.$$

It follows from (5.6) that with probability one for all $t \in [0,T]$

$$(\bar{\pi}_t, v_t) = (\pi_0, v_0) + \int_0^t (\bar{\pi}_s, D_j^y v_s \Phi_s^{jk} + \Lambda_s^k v_s)\Psi_s^{ki}\, dy_s^i$$

$$-\int_0^t \int_{\partial G} \phi_s\, \alpha_s(dx)\, d\mathcal{A}_s - \int_0^t c_s(\bar{\pi}_s, v_s)\, ds.$$

Now our assertion follows directly from Itô's formula and the theorem is proved. $\square$

Lemma 5.4. *Under the assumptions of Theorem 5.3 the stochastic integral in* (5.5) *and the process*

$$m_t := \int_0^t \kappa_s \rho_s (\bar{\pi}_s, \beta_s^k - \tilde{\beta}_s^k)\, d\tilde{w}_s^k$$

are martingales on $[0,T]$. *Furthermore, with probability one, for all* $t \geq 0$

$$(\bar{\pi}_t, 1)\rho_t = 1 - \int_0^t \rho_s\, d\mathcal{A}_s + m_t. \tag{5.7}$$

Proof. One knows that the stochastic integral in (5.5) is a martingale on $[0,T]$, if

$$E\Big(\int_0^T \kappa_s^2\rho_s^2(\bar\pi_s, D_j^y v_s\Phi_s^{jk} + \Lambda_s^k v_s - \tilde\beta_s^k v_s)^2\,ds\Big)^{1/2} < \infty. \tag{5.8}$$

Here

$$(\bar\pi_s, D_j^y v_s\Phi_s^{jk} + \Lambda_s^k v_s - \tilde\beta_s^k v_s)^2 \le N\|\bar\pi_s\|^2 \sup_{\mathbb{G}_T}(|D^z v|^2 + |v|^2),$$

where and below by N we denote constants independent of ω.

Hence the left-hand side of (5.8) is less than

$$NE\Big(\int_0^T \kappa_s^2\rho_s^2\|\bar\pi_s\|^2\,ds\Big)^{1/2} \le NE\sup_{[0,T]}\kappa_t\rho_t\Big(\int_0^T \|\bar\pi_s\|^2\,ds\Big)^{1/2}$$

$$\le N\big(E\sup_{[0,T]}\kappa_t^2\rho_t^2\big)^{1/2}\|\bar\pi\|_{\mathbb{L}_2(T)}.$$

The last expression is finite since $\bar\pi \in \overset{0}{\mathbb{W}}{}_2^1(G_T)$ and c and $\tilde\beta$ are bounded, so that κ is bounded and $\sup_{[0,T]}\rho_t$ has all moments finite. It is much easier to prove that m_t is a martingale.

Finally, one obtains (5.7) from (3.2) by applying Itô's formula and the lemma is proved.

In Theorem 5.3 the distribution of x_T is not involved. The following result makes it enter the picture and excludes v from further investigation.

Theorem 5.5. *Let c be a Borel bounded function of $(t,y) \in [0,T]\times\mathbb{R}^{d_2}$ and let ϕ be a Borel bounded or nonnegative function of $(t,z) \in [0,T]\times\mathbb{R}^{d_1}$. Then*

$$EI_{\tau>T}\phi(T,z_T)\kappa_T(c) = E\big(\bar\pi_T, \phi(T,\cdot,y_T)\big)\kappa_T(c)\rho_T, \tag{5.9}$$

$$EI_{\tau\le T}\phi(\tau,z_\tau)\kappa_\tau(c) = E\int_0^T \kappa_s(c)\rho_s\int_{\partial G}\phi(s,x,y_s)\,\alpha_s(dx)\,d\mathcal{A}_s. \tag{5.10}$$

Proof. Since the values of ϕ on $\{T\}\times G$ and $[0,T]\times\partial G$ are unrelated it suffices to prove that

$$E\phi(T\wedge\tau, z_{T\wedge\tau})\kappa_{T\wedge\tau} = E\big(\bar\pi_T, \phi(T,\cdot,y_T)\big)\kappa_T\rho_T$$

$$+E\int_0^T \kappa_s\rho_s\int_{\partial G}\phi(s,x,y_s)\,\alpha_s(dx)\,d\mathcal{A}_s. \tag{5.11}$$

Standard measure-theoretic arguments show that we may concentrate on smooth c and ϕ with compact support. In that case take v from Lemma 5.1 and notice that $v \in W^{1,2}_{d+1}(\mathbb{G}_T)$ so that by Itô's formula

$$E(\pi_0, v_0) = E\int_G v(0,x,y_0)\pi_0(x)\,dx$$
$$= Ev(0,z_0) = E\kappa_{T\wedge\tau}\phi(T\wedge\tau, z_{T\wedge\tau})$$

Now it only remains to use Theorem 5.3 and Lemma 5.4. The theorem is proved.

Remark 5.6. Observe that that by Doob's inequality for any $p > 1$

$$E\sup_{t\le T}\rho_t^p \le N(p)E\rho_T^p$$

and the latter is finite because $\tilde\beta$ is bounded.

Next, by taking $c = 0$ in (5.10) we get that

$$P(\tau \le T) = E\int_0^T \rho_s\,d\mathcal{A}_s. \tag{5.12}$$

Here the right-hand side is continuous owing to the dominated convergence theorem and the fact that

$$E\mathcal{A}_T\sup_{t\le T}\rho_t \le \big(E\mathcal{A}_T^2\big)^{1/2}\big(E\sup_{t\le T}\rho_t^2\big)^{1/2},$$

which is finite by Remark 3.5 and the fact mentioned above. It follows from (5.12) that the left-hand side is also continuous, that is, $P(\tau = T) = 0$ for any $T \ge 0$. This is, of course, a very well known result from the theory of uniformly nondegenerate processes.

In order to be able to use the arbitrariness of c in Theorem 5.5 and conclude from (5.9) that certain conditional expectations given $\mathcal{F}^y_T$ coincide, we need one more auxiliary result. The following lemma is probably well known. We give it with a proof for completeness.

Lemma 5.7. *Let ξ_t be an $\mathbb{R}^k$-valued random process on $[0,\infty)$ which is continuous at zero and left-continuous on $(0,\infty)$. Let η be a random variable with finite expectation. Fix $T \in [0,\infty)$, define $\mathcal{F}^\xi_T$ as the completion with respect to $\mathcal{F}, P$ of the σ-field generated by $\xi_s, s \le T$, and assume that*

$$E\eta\exp\int_0^T c(t,\xi_t)\,dt = 0$$

for any continuous bounded function c on $[0,T]\times\mathbb{R}^k$. Then (a.s.)

$$E\{\eta \mid \mathcal{F}^{\xi}_T\} = 0. \tag{5.13}$$

Proof. For a complex parameter λ introduce

$$\chi(\lambda) = E\eta \exp \int_0^T \lambda c(t,\xi_t)\,dt.$$

Then χ is a continuous function whose integral over any circle is zero. Therefore χ is an analytic function. Since it is zero on the real axis, it also vanishes on the imaginary axis. Hence

$$E\eta \exp \int_0^T ic(t,\xi_t)\,dt = 0$$

for any continuous bounded function c. Here the boundedness requirement can be dropped on account of the dominated convergence theorem. In particular, for any continuous $\mathbb{R}^k$-valued $c = c(t)$ we have

$$E\eta \exp \int_0^T i(c(t),\xi_t)\,dt = 0. \tag{5.14}$$

Simple approximations show that one can allow $c(t)$ in (5.14) to be piecewise constant. Then from the fact that, if $s > 0$, then

$$n\int_{s-1/n}^{s} \xi_t\,dt \to \xi_s$$

because ξ_t is left-continuous, it follows that, if $0 < t_1 < ... < t_m \le T$ and $c_1,...,c_m \in \mathbb{R}^k$, then

$$E\eta \exp \sum_{j=1}^{m} ic_j\xi_{t_j} = 0.$$

The continuity at zero of ξ_t allows us to relax the restriction on t_i to $0 \le t_1 < ... < t_m \le T$ Since one can approximate any bounded continuous function by trigonometric polynomials we get that

$$E\eta f(\xi_{t_1},...,\xi_{t_m}) = 0$$

for any continuous bounded and then, by a standard measure-theoretic argument, for any Borel bounded f. This means that (a.s.)

$$E\{\eta \mid \xi_{t_1},...,\xi_{t_m}\} = 0.$$

Now for $n = 1, 2, ...$ we see that

$$E\{\eta \mid \xi_{T/2^n}, \xi_{2T/2^n}, ..., \xi_{(2^n-1)T/2^n}, \xi_T\} = 0. \tag{5.15}$$

The σ-fields $\sigma(\xi_{T/2^n}, \xi_{2T/2^n}, ..., \xi_{(2^n-1)T/2^n}, \xi_T)$ are increasing with n and the smallest σ-field containing the completion of their union is $\mathcal{F}^{\xi}_T$, since ξ_t is left continuous. By Lévy's theorem (5.15) implies (5.13) and the lemma is proved.

One more lemma serves the purpose of facilitating dealing with $\mathcal{G}^y_t$, which is by definition the completion of $\sigma(y_{s\wedge\tau}; s \le t)$ with respect to $\mathcal{F}, P$.

Lemma 5.8. *(i) The random variable τ is a stopping time with respect to the filtration $\mathcal{G}^y_t$, $t \ge 0$.*

(ii) If $t \in [0, \infty)$ and η is an $\mathcal{F}^y_t$-measurable random variable, then $\eta I_{\tau>t}$ is $\mathcal{G}^y_t$-measurable.

(iii) If ξ_t is a $\mathcal{P}^y$-measurable process, then $\xi_\tau I_{\tau\le t}$ is $\mathcal{G}^y_t$-measurable for any $t \in [0, \infty)$.

Proof. (i) First observe that as is well known for any $0 \le s < t$ in the mean square sense

$$J_n(s,t) := \sum_{i=1}^{n-1} |y_{\tau\wedge t_{i+1,n}} - y_{\tau\wedge t_{i,n}}|^2 \to \int_{s\wedge\tau}^{t\wedge\tau} \|\Theta(r, y_r)\|^2 \, dr,$$

where $t_{i,n} = i/n$. Since Θ is nondegenerate (or just $\ne 0$), the last expression, which is $\mathcal{G}^y_t$-measurable by the above formula, is > 0 if and only if $\tau > s$. It follows

$$\{\omega : \tau > s\} \in \mathcal{G}^y_t,$$

whenever $t > s \ge 0$. By letting $s \uparrow t$ along rational s we see that $\{\omega : \tau \ge t\} \in \mathcal{G}^y_t$ for any $t > 0$. Now Remark 5.6 implies that $\{\omega : \tau > t\} \in \mathcal{G}^y_t$ for any $t > 0$. The same holds for $t = 0$ since $P(\tau > 0) = 1$.

(ii) Denote by Λ the set of events A such that $A \cap \{\tau > t\} \in \mathcal{G}^y_t$. Obviously Λ is a λ-system. Furthermore, for any integer n and $t_1, ..., t_n \in [0, t]$ and Borel $\Gamma_1, ..., \Gamma_n \subset \mathbb{R}^{d_2}$ for the event

$$\{y_{t_1} \in \Gamma_1, ..., y_{t_n} \in \Gamma_n\} \tag{5.16}$$

we have that

$$\{y_{t_1} \in \Gamma_1, ..., y_{t_n} \in \Gamma_n\} \cap \{\tau > t\}$$
$$= \{y_{t_1\wedge\tau} \in \Gamma_1, ..., y_{t_n\wedge\tau} \in \Gamma_n\} \cap \{\tau > t\} \in \mathcal{G}^y_t.$$

The collection of events of type (5.16) is a π-system which is contained in Λ. By the lemma about λ- and π-systems, Λ contains the σ-field generated by the π-system. Hence, for any $A \in \mathcal{F}^y_t$ we have that $A \cap \{\tau > t\} \in \mathcal{G}^y_t$. A standard measure-theoretic argument finishes proving (ii).

(iii) Recall that $\mathcal{P}^y$ is the σ-field of subsets of $\Omega \times (0,\infty)$ (not of $\Omega \times [0,\infty)$) generated by the sets $B \times (r_1, r_2]$, where $0 \le r_1 \le r_2$ and $B \in \mathcal{F}^y_{r_1}$ are arbitrary. As usual, it suffices to prove assertion (iii) for the indicator functions of generating sets. Thus take such a set, denote by ξ_t its indicator and use the fact that

$$\xi_\tau I_{\tau \le t} = (I_B I_{\tau > r_1, r_1 \le t}) I_{r_1 \wedge t < \tau \le r_2 \wedge t}.$$

Here the first factor on the right is $\mathcal{G}^y_{r_1}$- and $\mathcal{G}^y_t$-measurable by assertion (ii) and the second factor is $\mathcal{G}^y_t$-measurable by assertion (i). The lemma is proved.

Proof of Lemma 2.7 and Theorem 2.8. By (5.9) for any Borel bounded $c(t,y)$ and Borel bounded $\phi(x)$

$$EI_{\tau > T}\phi(x_T)\kappa_T(c) = E(\bar\pi_T, \phi)\rho_T\kappa_T(c).$$

Owing to Lemma 5.7 and the continuity of y_t, we have that (a.s.)

$$E\{I_{\tau > T}\phi(x_T) \mid \mathcal{F}^y_T\} = (\bar\pi_T, \phi)m_T$$

for each T, where $m_T = E\{\rho_T \mid \mathcal{F}^y_T\}$. This proves (2.8).

By taking $\phi \equiv 1$ we obtain (2.9), which along with Theorem 4.1 implies that, for any $T \in [0,\infty)$ we have $(\bar\pi_T, 1) > 0$ (a.s.). Recall that $(\bar\pi_t, 1)$ is a continuous $\mathcal{F}^y$-adapted process, set

$$\gamma = \inf\{t \ge 0 : (\bar\pi_t, 1) = 0\},$$

and define $\check\pi_t = \bar\pi_{t\wedge\gamma}$. Obviously, $\check\pi_t$ satisfies (2.6) and has the same initial value as $\bar\pi_t$, By uniqueness, with probability one, $\check\pi_t = \bar\pi_t$ for all $t \ge 0$, that is $\bar\pi_t = \bar\pi_{t\wedge\gamma}$. If we assume that $P(\gamma < \infty) > 0$, then for some $T \in [0,\infty)$ we would have that $\bar\pi_T = 0$ with nonzero probability. This however contradicts (2.9) owing to Theorem 4.1. This proves the last assertion of Lemma 2.7 and finishes its proof.

It follows from (2.8) that for any Borel bounded $f(y_1, ..., y_n)$ and $0 \le t_1 \le t_2 \le ... \le t_n \le T$

$$EI_{\tau > T}\phi(x_T)f(y_{t_1}, ..., y_{t_n}) = E(\bar\pi_T, \phi)m_T f(y_{t_1}, ..., y_{t_n})$$

and

$$EI_{\tau>T}\phi(x_T)f(y_{t_1\wedge\tau},...,y_{t_n\wedge\tau}) = E(\bar\pi_T,\phi)m_T f(y_{t_1},...,y_{t_n}). \tag{5.17}$$

One encounters here a paradoxical situation: the left-hand side involves $(y_{t_1\wedge\tau},...,y_{t_n\wedge\tau})$ and on the right-hand side we are dealing with $(y_{t_1},...,y_{t_n})$.

Help comes from (2.9) and the last statement of Lemma 2.7 by which

$$E(\bar\pi_T,\phi)m_T f(y_{t_1},...,y_{t_n}) = E\frac{(\bar\pi_T,\phi)}{(\bar\pi_T,1)}f(y_{t_1},...,y_{t_n})P\{\tau>T\mid\mathcal{F}^y_T\}$$

$$= EI_{\tau>T}\frac{(\bar\pi_T,\phi)}{(\bar\pi_T,1)}f(y_{t_1},...,y_{t_n})I_{\tau>T} = EI_{\tau>T}\frac{(\bar\pi_T,\phi)}{(\bar\pi_T,1)}f(y_{t_1\wedge\tau},...,y_{t_n\wedge\tau})$$

After that by coming back to (5.17) and using the arbitrariness of $f,n,t_1,...,t_n$ and Lemma 5.8, we conclude that on the set $\{\tau>T\}$ (a.s.) we have

$$E\{\phi(x_T)\mid\mathcal{G}^y_T\} = \frac{(\bar\pi_T,\phi)}{(\bar\pi_T,1)}.$$

This brings to an end the proof of Lemma 2.7 and Theorem 2.8.

The proof of Theorem 2.9 is based on (5.10) the right-hand side of which looks even more puzzling than (5.17), since it contains κ_s for all $s\le T$, whereas its left-hand side contains only κ_τ. In order to overcome this difficulty we prove the following lemma which, actually, says that the process

$$P\{\tau\le t\mid\mathcal{F}^y_t\} - \int_0^t m_s\,d\mathcal{A}_s$$

is an $\mathcal{F}^y_t$-martingale.

Lemma 5.9. *For any bounded or nonnegative $\mathcal{P}^y$-measurable process f_t we have*

$$E\int_0^T f_t\rho_t\,d\mathcal{A}_t = Ef_\tau I_{\tau\le T}. \tag{5.18}$$

Proof. As usual, it suffices to prove (5.18) for the indicator functions of sets $B\times(r_1,r_2]$, where $0\le r_1\le r_2$ and $B\in\mathcal{F}^y_{r_1}$ are arbitrary, which generate $\mathcal{P}^y$. Take such a set and use (5.7) to write

$$I := E\int_0^T \rho_t I_{B\times(r_1,r_2]}\,d\mathcal{A}_t = EI_B\int_{r_1\wedge T}^{r_2\wedge T}\rho_t\,d\mathcal{A}_t$$

$$= E\rho_{r_1\wedge T}I_B(\bar\pi_{r_1\wedge T},1) - E\rho_{r_2\wedge T}I_B(\bar\pi_{r_2\wedge T},1).$$

If $r_1 \geq T$, then $r_1 \wedge T = r_2 \wedge T = T$, and

$$I = 0 = EI_{B\times(r_1,r_2]}(\omega,\tau)I_{\tau\leq T}.$$

In case $r_1 < T$ we have $r_1 \wedge T = r_1$ and we can use (2.8). Then we find that

$$I = EI_B(\bar{\pi}_{r_1},1)E\{\rho_{r_1} \mid \mathcal{F}^y_{r_1}\} - EI_B(\bar{\pi}_{r_2\wedge T},1)E\{\rho_{r_2\wedge T} \mid \mathcal{F}^y_{r_2\wedge T}\}$$

$$= EI_B E\{I_{\tau>r_1} \mid \mathcal{F}^y_{r_1}\} - EI_B E\{I_{\tau>r_2\wedge T} \mid \mathcal{F}^y_{r_2\wedge T}\}$$

$$= EI_B I_{r_1<\tau\leq r_2\wedge T} = EI_{B\times(r_1,r_2]}(\omega,\tau)I_{\tau\leq T}.$$

Hence, the equality between the extreme terms holds in all the cases and the lemma is proved.

Proof of Theorem 2.9. First of all, for any Borel bounded or nonnegative function $\phi(t,z)$, Theorem 3.6 shows that

$$\int_{\partial G} \phi(s,x,y_s)\alpha_s(dx)d\mathcal{A}_s$$

is a $\mathcal{P}^y$-measurable process. We use Theorem 5.5 and Lemma 5.9 to see that for any Borel bounded or nonnegative function $\phi(t,z)$ and any Borel bounded function $c(t,y)$ we have

$$EI_{\tau\leq T}\phi(\tau,z_\tau)\kappa_\tau(c) = E\kappa_\tau(c)\int_{\partial G}\phi(\tau,x,y_\tau)\,\alpha_\tau(dx)I_{\tau\leq T},$$

that is

$$E\kappa_\tau(c)\eta = 0, \tag{5.19}$$

where

$$\eta := I_{\tau\leq T}\phi(\tau,z_\tau) - I_{\tau\leq T}\int_{\partial G}\phi(\tau,x,y_\tau)\,\alpha_\tau(dx).$$

We observe that on the event $\{\tau \leq T\}$ we have

$$\kappa_\tau(c) := \int_0^T c(s,y_s)I_{s\leq\tau}\,ds$$

and then exactly as in the proof of Lemma 5.7 we obtain that (5.14) holds for any Borel R^{d_2}-valued function $c(t)$ if we set $\xi_t := y_t I_{t\leq\tau}$. Then as in the proof of Lemma 5.7 this leads to (5.13).

We now claim that

$$\mathcal{F}^\xi_T = \mathcal{G}^y_T. \tag{5.20}$$

To prove the claim observe that $\xi_t = y_{t\wedge\tau} I_{t\le\tau}$, which is $\mathcal{G}^y_t$-measurable by the definition of $\mathcal{G}^y_t$ and Lemma 5.8 (i). Therefore, $\mathcal{F}^\xi_T \subset \mathcal{G}^y_T$. On the other hand, notice that $\xi_{t+} = \xi_t$ (a.s.) for any $t \ge 0$ since $P(\tau = t) = 0$ by Remark 5.6. It follows that $\mathcal{F}^\xi_T$ is also generated by $\xi_{t+}, t \le T$. Next,

$$\{\omega : \xi_{t+} = \xi_t\} = \{\omega : t < \tau\} \cup \{\omega : y_t = 0, t > 0\},$$

where, for $t > 0$, the event $\{\omega : y_t = 0\}$ has probability zero since the nondegenerate diffusion process z_t has a density of distribution at any $t > 0$. It follows that τ is an $\mathcal{F}^\xi_t$ stopping time. In particular, $\xi_s I_{t\le\tau}$ is $\mathcal{F}^\xi_t$-measurable whenever $s \le t$. Now from

$$y_{t\wedge\tau} = \xi_t + y_\tau I_{\tau<t}$$

and the fact that

$$y_\tau I_{\tau<t} = \lim_{n\to\infty} \sum_{k=0}^{n-1} y_{kt/n} I_{kt/n\le\tau<(k+1)t/n} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \xi_{kt/n}(1 - I_{(k+1)t/n\le\tau})$$

we infer that $y_{t\wedge\tau}$ is $\mathcal{F}^\xi_t$-measurable and hence $\mathcal{F}^\xi_T \supset \mathcal{G}^y_T$. This proves (5.20).

Now (5.13) and (5.20) imply that

$$E\{I_{\tau\le T}\phi(\tau, x_\tau) \mid \mathcal{G}^y_T\} = E\{I_{\tau\le T} \int_{\partial G} \phi(\tau, x, y_\tau)\, \alpha_\tau(dx) \mid \mathcal{G}^y_T\}$$

and to obtain (2.11) it only remains to use Lemma 5.8. The theorem is proved.

Chapter 6

On the conditional distribution of τ_G given $\mathcal{F}^y_{t_0}$

Starting from this chapter, we fix $t_0 > 0$. Our goal is to deduce the conditional distribution of the exit time τ_G given that $y_s, s \le t_0$ is the only observation.

Before stating the main results in this chapter, we need a more detailed notaiton to denote the inner product in different spaces. Given any two functions, f and g, we write $(f,g)_G$ to specify that we integrate fg over G. Similarly, $(f,g)_{\mathbb{R}^d}$ is the integration of fg over $\mathbb{R}^d$.

Let us consider the following initial value problem

$$\mathrm{d}\bar{\pi}^w_t(x) = L^*_t(x)\bar{\pi}^w_t(x)\,\mathrm{d}t + \Lambda^{k*}_t(x)\bar{\pi}^w_t(x)\,\mathrm{d}\tilde{y}^k_t, \tag{6.1}$$

$$\bar{\pi}^w_0(x) = \pi_0(x) I_{x\in G},$$

where $t \ge 0$, $x \in \mathbb{R}^d$, $\bar{\pi}^w_t(x) = \bar{\pi}^w_t(\omega, x)$.

Remark 6.1. Actually, if we consider the initial value probelm (2.6) in $\mathbb{R}^d$, we get (6.1).

To explain in which sense we understand this equation and the initial condition, we introduce the space $W^1_2(\mathbb{R}^d)$ as the closure of the set of continuously differentiable functions with compact support in $\mathbb{R}^d$ in the norm

$$\|u\|_{W^1_2(\mathbb{R}^d)} = \|u\|_{L_2(\mathbb{R}^d)} + \|Du\|_{L_2(\mathbb{R}^d)}.$$

For $T \in (0,\infty)$ introduce

$$\mathbb{W}_2^1((0,T)\times\mathbb{R}^d) = L_2(\Omega\times(0,T),\mathcal{P}^y, W_2^1(\mathbb{R}^d))$$

and introduce $\mathcal{W}_2^1((0,T)\times\mathbb{R}^d)$ as the set of functions $u_t(x)=u_t(\omega,x)$ such that

(i) For each $(\omega,t)\in\Omega\times(0,T)$, u_t is a generalized function on $\mathbb{R}^d$;

(ii) We have $u\in\mathbb{W}_2^1((0,T)\times\mathbb{R}^d)$;

(iii) $u_0\in L_2(\Omega,\mathcal{F}_0^y,L_2(\mathbb{R}^d))$ and there exists f^j, $g^k\in L_2(\Omega\times(0,T),\mathcal{P}^y,L_2(\mathbb{R}^d))$, for $i=0,\dots,d$, $k=1,\dots,d_2$, such that for any $\zeta\in C_0^\infty(\mathbb{R}^d)$ with probability one

$$(u_t,\zeta)_{\mathbb{R}^d} = (u_0,\zeta)_{\mathbb{R}^d} + \int_0^t [(f_s^0,\zeta)_{\mathbb{R}^d} - (f_s^i, D_i\zeta)_{\mathbb{R}^d}]\,\mathrm{d}s + \int_0^t g_s^k\,\mathrm{d}\tilde{y}_s^k$$

for all $t\in[0,T]$.

In case (iii) holds, we write

$$\mathrm{d}u_t = (D_i f_t^i + f_t^0)\,\mathrm{d}t + g_t^k\,\mathrm{d}\tilde{y}_t^k$$

for $t\in[0,T]$.

Accordingly, we are looking for a fucntion $\bar{\pi}^w\in\cap_T\mathcal{W}_2^1((0,T)\times\mathbb{R}^d)$ such that (6.1) holds for all T and $t\in[0,T]$, that is, for all $t\geq 0$. To be more specific, we require that for each $\zeta\in C_0^\infty(\mathbb{R}^d)$ with probability one, it holds that

$$(\bar{\pi}_t^w,\zeta) = (\pi_0,\zeta) - \int_0^t (a_s^{ij}D_i\bar{\pi}_s^w - b_s^j\bar{\pi}_s^w + \bar{\pi}_s^w + \bar{\pi}_s^w D_i a_s^{ij}, D_j\zeta)\,\mathrm{d}s \tag{6.2}$$

$$-\int_0^t (\sigma_s^{ik}D_i\bar{\pi}_s^w + (D_i\sigma_s^{ik} - \beta_s^k)\bar{\pi}_s^w,\zeta)\,\mathrm{d}\tilde{y}_s^k.$$

Lemma 6.2. *Let Assumptions 2.1– 2.5 be satisfied. There exists a unique solution $\bar{\pi}^w$ of* (6.1) *with initial condition $\pi_0 I_{x\in G}$ in the sense explained above. In addition, $\bar{\pi}_t^w\geq 0$ for all $t\in[0,\infty)$ (a.s.). With probability one, $\bar{\pi}_t^w$ is continuous in $L_2(\mathbb{R}^d)$.*

The existence, uniqueness, and the (a.s.) continuity in $L_2(\mathbb{R}^d)$ of $\bar{\pi}^w$ is a classical result (see, for instance, Theorem 5.1, Theorem 7.1 of [15]). The fact that $\bar{\pi}_t^w\geq 0$ follows from the maximum principle (see, for instace, Theorem 5.12 of [15])

However, it is worth noting that, as the explanation following Lemma 2.6, the fact that, actually, $\bar{\pi}^w_t$ is $\mathcal{P}^y$ measurable can be proved by using Girsanov's theorem.

Recall that $\bar{\pi}$ is the solution of the initial value problem (2.6), for any $0 < t < t_0$, we consider (6.1)

$$\mathrm{d}\bar{\pi}^w_{t,s}(x) = L^*_{t+s}(x)\bar{\pi}^w_{t,s}(x)\,\mathrm{d}s + \Lambda^{k*}_{t+s}(x)\bar{\pi}^w_{t,s}(x)\,\mathrm{d}\tilde{y}^k_{t+s}, \tag{6.3}$$

with initial condition

$$\bar{\pi}^w_{t,0}(x) = \bar{\pi}_t(x)I_{x\in G}, \tag{6.4}$$

where $0 < s < t_0 - t$, $x \in \mathbb{R}^d$, $\bar{\pi}^w_{t,s}(x) = \bar{\pi}^w_{t,s}(\omega, x)$.

Notice that if we write $s' = s + t$ and $\hat{\pi}^w_{s'} = \bar{\pi}^w_{t,s}$, then $\hat{\pi}^w_{s'}$ satisfies (6.1) with initial condition (6.4). Since $\bar{\pi}_t$ is continuous in L_2, the initial condition (6.4) satisfies Assumption 2.5. As a result, there exists a unique solution $\hat{\pi}^w_{s'} = \bar{\pi}^w_{t,s}$ which is continuous in $L_2(\mathbb{R}^d)$. Moreover $\bar{\pi}^w_{t,s} \geq 0$ for all $s \in [0,\infty)$ (a.s.).

We are now ready to give the first result of this chapter:

Theorem 6.3. *For any Borel bounded or nonnegative function ϕ on $\mathbb{R}^d$ and $0 \leq t \leq t_0$ we have (a.s.) that*

$$E\{I_{\tau\geq t}\phi(x_{t_0}) \mid \mathcal{F}^y_{t_0}\} = \frac{(\bar{\pi}^w_{t,t_0-t}, \phi)_{\mathbb{R}^d}}{(\bar{\pi}^w_{t_0}, 1)_{\mathbb{R}^d}}. \tag{6.5}$$

Remark 6.4. If $\phi \equiv 1$, this theorem gives, by definition, the conditional probability:

$$P\{\tau \geq t \mid \mathcal{F}^y_{t_0}\} = \frac{(\bar{\pi}^w_{t,t_0-t}, 1)_{\mathbb{R}^d}}{(\bar{\pi}^w_{t_0}, 1)_{\mathbb{R}^d}}, \tag{6.6}$$

in the case $t \leq t_0$.

We will prove Theorem 6.3 in Chapter 7. It is also interesting to note that $P_{t_0+s} := E\{I_{\tau\geq t} \mid \mathcal{F}^y_{t_0+s}\}$ is a $\mathcal{F}^y_{t_0+s}$-martingale for $s \geq 0$, since for any $s > s_0 > 0$, we have

$$E\{P_{t_0+s} \mid \mathcal{F}^y_{t_0+s}\} = E\{E\{I_{\tau\geq t} \mid \mathcal{F}^y_{t+s}\} \mid \mathcal{F}^y_{t+s_0}\} = E\{I_{\tau\geq t} \mid \mathcal{F}^y_{t+s_0}\} = P_{t+s_0}.$$

Clearly $P_{t_0+s} \leq 1$ for any $(\omega, s) \in \Omega \times [0,\infty)$. Therefore

$$P\{\tau \geq t \mid \mathcal{F}^y_{t_0}\} = E\{I_{\tau\geq t} \mid \mathcal{F}^y_{t_0}\}$$

is convergent as $t_0 \to \infty$ by martingale convergent theorem. Therefore,

Theorem 6.5. *Let Assumptions 2.1– 2.5 be satisfied.*

$$P\{\tau \geq t \mid \mathcal{F}^y_\infty\} := \lim_{t_0 \to \infty} P\{\tau \geq t \mid \mathcal{F}^y_{t_0}\} = \lim_{t_0 \to \infty} \frac{(\bar{\pi}^w_{t,t_0-t}, 1)_{\mathbb{R}^d}}{(\bar{\pi}^w_{t_0}, 1)_{\mathbb{R}^d}}. \tag{6.7}$$

Remark 6.6. Generally, neither denominator or numerator of the left-hand side of (6.7) converges as $t_0 \to \infty$.

Chapter 7

Proof of Theorem 6.3

We are going to use the approach from [10] to derive Theorem 6.3. For the same t_0 as fixed in Chapter 6, take a function $\phi(z)$ of class $C_0^\infty(\mathbb{R}^{d_1})$ and a function $c(s,y) \in C^1([0,t_0] \times \mathbb{R}^{d_2})$.

We start to solve an deterministic problem:

$$\partial_s v(s,z) + \tilde{L}_s v(s,z) + c(s,y)v(s,z) = 0 \tag{7.1}$$

in $(0,t_0) \times \mathbb{R}^{d_1}$ with "terminal" condition equal to ϕ:

$$v(t_0,z) = \phi(z) \quad \text{on} \quad \mathbb{R}^{d_1}. \tag{7.2}$$

As we explain in Lemma (5.1), since we are only given that the coefficients of $\tilde{L}_s$ are Lipschitz continuous in z and no regularity with respect to t is assumed, we cannot use classical results to prove the existence and uniqueness of the solution for (7.1). We need the following results:

Lemma 7.1 (Lemma 4.11 in [10])**.** *Let $\phi \in C_0^\infty(\mathbb{R}^{d_1})$ and $c \in C^1([0,t_0] \times \mathbb{R}^{d_2})$, $c \geq 0$. Then there exists a unique function v on $[0,t_0] \times \mathbb{R}^{d_1}$ possessing the following properties:*

(i) For any $s \in [0,t_0]$, $\epsilon \in (0,1)$ the function v_t is continuous as a $C^{2+\epsilon}(\mathbb{R}^{d_1})$-valued function on $[0,t_0]$;

(ii) For any $s \in [0,t_0]$ and $z \in \mathbb{R}^{d_1}$, it holds that

$$v_s(z) = \phi(z) + \int_s^{t_0} [\tilde{L}_r(z)v_r(z) - c_r(y)v_r(z)]\,\mathrm{d}r.$$

(iii) There exist constants $N < \infty$, $\epsilon > 0$ such that

$$|v_s(z)| + |v_{sz}(z)| + |v_{szz}(z)| < Ne^{-\epsilon|z|}$$

for all $s \in [0, t_0]$ and $z \in \mathbb{R}^{d_1}$.

To proceed further, we introduce

$$\kappa_{t,s} := \kappa_{t,s}(c) = \kappa_{t+s}/\kappa_t,$$

and

$$\rho_{t,s} = \rho_{t+s}/\rho_t$$

for $s \geq 0$.

Lemma 7.2. *For any $0 \leq t \leq t_0$, $v_{t+s}(z_{t+s}) \cdot \kappa_{t,s}$ is a martingale with respect to filtration $\mathcal{F}_{t+s}$ for $0 \leq s \leq t_0 - t$.*

Proof. Because $v_{t+s}(z)$ is a bounded function, it is sufficient to prove that it is a local martingale. Here we apply Itô's formula,

$$\mathrm{d}v_{t+s}(z_{t+s}) = \partial_s v_{t+s}(z_{t+s})\,\mathrm{d}s + D_i^z(z_{t+s})\,\mathrm{d}z_{t+s}^i + \frac{1}{2} D_{ij}^z v_{t+s}(z_{t+s})\,\mathrm{d}z_{t+s}^i\,\mathrm{d}z_{t+s}^j.$$

Hence,

$$\begin{aligned}\mathrm{d}v_{t+s}(z_{t+s})\kappa_{t,s} &= \kappa_{t,s}\,\mathrm{d}v_{t+s}(z_{t+s}) + v_{t+s}(z_{t+s})\,\mathrm{d}\kappa_{t,s} \\ &= c_{t+s}(y_{t+s})v_{t+s}(z_{t+s})\,\mathrm{d}s + D_i^z v_{t+s}(z_{t+s})\tilde{\theta}^{ik}(t+s, z_{t+s})\,\mathrm{d}w_{t+s}^k \\ &\quad - c_{t+s}(y_{t+s})v_{t+s}(z_{t+s})\,\mathrm{d}s \\ &= D_i^z v_{t+s}(z_{t+s})\tilde{\theta}^{ik}(t+s, z_{t+s})\,\mathrm{d}w_{t+s}^k,\end{aligned}$$

which proves $v_{t+s}\kappa_{t,s}$ is a $\mathcal{F}_{t+s}$-martingale for $s \geq 0$. □

We will need analogues of Theorem 5.5 and Lemma 2.7 for $\bar{\pi}^w$:

Theorem 7.3. *Let c be a Borel bounded function of $(t, y) \in [0, T] \times \mathbb{R}^{d_2}$ and let $\phi \in C_0^\infty(\mathbb{R}^d)$. Then, for any $0 < t < t_0$,*

$$E\phi(z_{t_0})\kappa_{t,t_0-t} = E(\bar{\pi}_{t,t_0-t}^w, \phi(\cdot, y_{t_0}))_{\mathbb{R}^{d_1}}\kappa_{t,t_0-t}\rho_{t,t_0-t}.$$

Proof. Actually, this is a main result of [10](see section 3 and 5). The main ingredient is to prove that

$$\xi_{t,s} = (\bar{\pi}^w_{t,s}, v_{t+s}(\cdot, y_{t+s}))_{\mathbb{R}^{d_1}} \kappa_{t,s} \rho_{t,s}$$

is a local martingale by using Itô's formula. Due to Lemma 6.2 and 7.1, the required integration by parts is legitimate. Lemma 7.1 and Doob's inequality also guarantee the local martingale is actually a martingale. Therefore

$$E\phi(z_{t_0})\kappa_{t,t_0-t} = Ev_t(z_t)$$

and

$$Ev_t(z_t) = E(\bar{\pi}_t, v_t(\cdot, y_t))_{\mathbb{R}^{d_1}} = E(\bar{\pi}^w_{t,t_0-t}, \phi(\cdot, y_{t_0}))_{\mathbb{R}^{d_1}} \kappa_{t,t_0-t} \rho_{t,t_0-t}.$$

The theorem is proved. □

Remark 7.4. Due to the fact of Lemma 7.2, we can slightly strengthen Theorem 7.3 as follows: under the assumption of c and ϕ, take v from Lemma 7.1 accordingly, we have

$$Ev_{t+s}(z_{t+s})\kappa_{t,s} = E(\bar{\pi}^w_{t,t_0-t}, \phi(\cdot, y_{t_0}))_{\mathbb{R}^{d_1}} \kappa_{t,t_0-t} \rho_{t,t_0-t}, \tag{7.3}$$

for any $s \in [0, t_0 - t]$.

Lemma 7.5. *For each $t_0 \in [0, \infty)$ (a.s.)*

$$m_{t_0} := E\{\rho_{t_0} \mid \mathcal{F}^y_{t_0}\} = \frac{1}{(\bar{\pi}^w_{t_0}, 1)_{\mathbb{R}^d}}. \tag{7.4}$$

and (a.s.) for all $t_0 \in [0, \infty)$ we have $(\bar{\pi}^w_{t_0}, 1) > 0$.

Proof. The last asserion can be proved in the same way as in Lemma 2.7. Next, if we take $t = 0$ and $\phi \equiv 1$ in (7.3), then, by definition,

$$\bar{\pi}^w_{t,t_0-t} = \bar{\pi}^w_{t_0},$$

and

$$\rho_{t,t_0-t} = \rho_{t_0}, \quad \kappa_{t,t_0-t} = \kappa_{t_0}$$

Therefore, it follows that (a.s.)

$$1 = (\bar{\pi}^w_{t_0}, 1)_{\mathbb{R}^{d_1}} E\{\rho_{t_0} \mid \mathcal{F}^y_{t_0}\},$$

and the lemma is proved. □

Proof of Theorem 6.3. Due to the strong Markov property of the porcess z_t (see, for instance, Chapter 5, Section 5 of [16], we have

$$E\phi(z_{t_0})I_{\tau\geq t}\kappa_{t_0} = EI_{\tau\geq t}\kappa_t E\{\phi(z_{t_0})\kappa_{t,t_0-t} \mid \mathcal{F}_t\},$$

where κ_t was introduced in Theorem 5.3 and

$$\kappa_{t,s} = \kappa_{t+s}/\kappa_t$$

for any $t, s > 0$.

Here we can use Lemma 7.2 to get

$$E\phi(z_{t_0})I_{\tau\geq t}\kappa_{t_0} = EI_{\tau\geq t}v_t(z_t)\kappa_t.$$

In light of Lemma 2.7, this yields

$$E\phi(z_{t_0})I_{\tau\geq t}\kappa_{t_0} = E\rho_t(\bar{\pi}_t, v_t(\cdot, y_t))_G\kappa_t.$$

The inner product in the right-hand side can be transformed on account of (7.3), which gives

$$E\phi(z_{t_0})I_{\tau\geq t}\kappa_{t_0} = E\{E\{\rho_{t,t_0-t}\kappa_{t,t_0-t}(\phi, \bar{\pi}^w_{t,t_0-t})_{\mathbb{R}^d} \mid \mathcal{F}_t\}\rho_t\kappa_t\}.$$

By definition,

$$\rho_t \cdot \rho_{t,t_0-t} = \rho_{t_0},$$

$$\kappa_t \cdot \kappa_{t,t_0-t} = \kappa_{t_0},$$

therefore we obtain that

$$E\phi(z_{t_0})I_{\tau\geq t}\kappa_{t_0} = E\rho_{t_0}\kappa_{t_0}(\phi, \bar{\pi}^w_{t,t_0-t})_{\mathbb{R}^d}.$$

By Lemma 5.7,

$$E\{\phi(z_{t_0})I_{\tau\geq t} \mid \mathcal{F}^y_{t_0}\} = E\{\rho_{t_0}(\phi, \bar{\pi}^w_{t,t_0-t})_{\mathbb{R}^d} \mid \mathcal{F}^y_{t_0}\}.$$

Notice that $\bar{\pi}^w$, actually, is $\mathcal{P}^y$-predictable, we obtain that

$$E\{\phi(z_{t_0})I_{\tau\geq t} \mid \mathcal{F}^y_{t_0}\} = (\phi, \bar{\pi}^w_{t,t_0-t})_{\mathbb{R}^d}E\{\rho_{t_0} \mid \mathcal{F}^y_{t_0}\}. \tag{7.5}$$

To obtain (6.6) from (7.5), we only need to use (7.4). □

Chapter 8

Conditional distribution of the exit time: a worked example

Given a bounded domain $G = (-1, 1)$, for fixed $T > t_0 > 0$, let us consider a two-component process

$$\begin{aligned} \mathrm{d}x_t &= -x_t \,\mathrm{d}t + \mathrm{d}w_t, \\ \mathrm{d}y_t &= x_t \,\mathrm{d}t + \mathrm{d}B_t. \end{aligned} \tag{8.1}$$

As an example, we will apply Theorem (6.5) to deduce the conditional probability

$$P(I_{\tau \geq t_0} \mid \mathcal{F}_T^y)$$

where

$$\tau := \tau_G = \inf_{t \geq 0} \{x_t \notin G\}.$$

Let us start with the following Cauchy problems:

$$\mathrm{d}\bar{\pi}_t^w = [\frac{1}{2}\bar{\pi}_{txx}^w + (x\bar{\pi}_t^w)_x] \,\mathrm{d}t + (x\bar{\pi}_t^w) \,\mathrm{d}y_t, \tag{8.2}$$

for $x \in \mathbb{R}$ with initial condition $\bar{\pi}_0^w = \pi_0$,

$$\mathrm{d}\bar{\pi}_t = [\frac{1}{2}\bar{\pi}_{txx} + (x\bar{\pi}_t)_x] \,\mathrm{d}t + (x\bar{\pi}_t) \,\mathrm{d}y_t, \tag{8.3}$$

for $x \in (-1, 1)$ with initial condition $\bar{\pi}_0 = \pi_0$, and

$$\mathrm{d}\bar{\pi}_{t_0,t}^w = [\frac{1}{2}\bar{\pi}_{t_0,txx}^w + (x\bar{\pi}_{t_0,t}^w)_x] \,\mathrm{d}t + (x\bar{\pi}_{t_0,t}^w) \,\mathrm{d}y_{t_0+t}, \tag{8.4}$$

for $x \in \mathbb{R}$, with initial condition $\bar{\pi}^w_{t_0,0} = \bar{\pi}_{t_0}$.

It is worth while to pointing out that since x is not a bounded function, we need the ideas from [17], which allows us to use the theorems in this article. We will begin with finding a particular solution of the form e^{-Q_t} of (8.2), where $Q_t = \frac{1}{2}x^2 W_t + xV_t + U_t$ and $Q_0(x) = Q(x) = \frac{1}{2}x^2$.

On the one hand, we have that

$$\begin{aligned} \mathrm{d}e^{-Q_t} &= -\, e^{-Q_t}\,\mathrm{d}Q_t + \frac{1}{2}e^{-Q_t}\,\mathrm{d}Q_t\,\mathrm{d}Q_t \\ &= -\, e^{-Q_t}[\frac{1}{2}x^2\,\mathrm{d}W_t + x\,\mathrm{d}V_t + \,\mathrm{d}U_t] + \frac{1}{2}e^{-Q_t}\,\mathrm{d}Q_t\,\mathrm{d}Q_t \end{aligned} \tag{8.5}$$

On the other hand, since e^{-Q_t} is a solution of (8.2), we have that

$$\mathrm{d}e^{-Q_t} = [\frac{1}{2}(e^{-Q_t})_{xx} + (xe^{-Q_t})_x]\,\mathrm{d}t + (xe^{-Q_t})\,\mathrm{d}y_t \tag{8.6}$$

$$= [\frac{1}{2}e^{-Q_t}(xW_t+V_t)^2 - \frac{1}{2}e^{-Q_t}W_t + e^{-Q_t} - xe^{-Q_t}(xW_t+V_t)]\,\mathrm{d}t + (xe^{-Q_t})\,\mathrm{d}y_t$$

$$= e^{-Q_t}[(\frac{1}{2}W_t^2 - W_t)x^2 + (W_tV_t - V_t)x + (\frac{1}{2}V_t^2 - \frac{1}{2}W_t + 1)]\,\mathrm{d}t + (xe^{-Q_t})\,\mathrm{d}y_t.$$

Therefore, after comparing the coefficients, we can choose the following system to determine W_t, V_t and U_t

$$\begin{cases} \mathrm{d}W_t = (2W_t - W_t^2 + 1)\,\mathrm{d}t, \\ \mathrm{d}V_t = -\,\mathrm{d}y_t + (V_t - W_tV_t)\,\mathrm{d}t \\ \mathrm{d}U_t = (\frac{1}{2}W_t - \frac{1}{2}V_t^2 - 1)\,\mathrm{d}t, \end{cases} \tag{8.7}$$

where

$$W_0 = 1,\ V_0 = U_0 = 0.$$

We can solve (8.7) explicitly. Note

$$\frac{\mathrm{d}W_t}{2-(W_t-1)^2} = \mathrm{d}t,$$

therefore

$$\ln|\frac{\sqrt{2}+W_t-1}{\sqrt{2}-W_t+1}| = 2\sqrt{2}t + c,$$

and (c is a different constant from the one above)

$$\frac{\sqrt{2}+W_t-1}{\sqrt{2}-W_t+1} = c \cdot e^{2\sqrt{2}t}.$$

Since $W_0 = 1$, we have $c = 1$. Solving the above equation for W_t, we get

$$W_t = \frac{-\sqrt{2} + 1 + (\sqrt{2}+1)e^{2\sqrt{2}t}}{1 + e^{2\sqrt{2}t}}.$$

By the way, as $t \to \infty$, $W_t \uparrow \sqrt{2} + 1$.

To solve V_t, we can rewrite the equation as

$$\mathrm{d}V_t + (W_t - 1)V_t \,\mathrm{d}t = -\,\mathrm{d}y_t,$$

therefore

$$\mathrm{d}e^{\int_0^t (W_s - 1)\,\mathrm{d}s} V_t = -e^{\int_0^t (W_s-1)\,\mathrm{d}s}\,\mathrm{d}y_t.$$

Here we arrive at an explicit formula for V_t:

$$\begin{aligned} V_t &= -e^{-\int_0^t (W_s-1)\,\mathrm{d}s} \int_0^t e^{\int_0^s (W_r - 1)\,\mathrm{d}r}\,\mathrm{d}y_s \\ &= -\int_0^t e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}y_s \\ &:= V_{1t} + V_{2t}, \end{aligned}$$

where

$$\begin{aligned} V_{1t} &= -\int_0^t e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}B_s, \\ V_{2t} &= -\int_0^t e^{-\int_s^t (W_r-1)\,\mathrm{d}r} x_s \,\mathrm{d}s. \end{aligned}$$

For any fixed T_0, take any random variable $\xi \in L_2(\Omega, P, \mathcal{F}^{B_\cdot}_{T_0})$, then for any $t \geq T_0$

$$\begin{aligned} E[V_{1t}\xi] &= E[-\int_0^{T_0} e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}B_s\xi] + E[-\int_{T_0}^t e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}B_s\xi] \\ &= E[-\int_0^{T_0} e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}B_s\xi]. \end{aligned}$$

By Cauchy's inequality,

$$\{E[-\int_0^{T_0} e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}B_s \cdot \xi]\}^2 \leq \mathrm{Var}[-\int_0^{T_0} e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}B_s] E[\xi^2].$$

However, in the meanwhile, we have that

$$E[-\int_0^t e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}B_s] = 0$$

and

$$\mathrm{Var}[-\int_0^t e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}B_s] = \int_0^t e^{-2\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}s \to \frac{1}{2\sqrt{2}}.$$

Therefore as $t \to \infty$, $E[V_{1t}\xi]$ goes to 0. By arbitrariness of T_0 and ξ, we conclude V_{1t} converges weakly in $L(\Omega, P, \mathcal{F}^{B.}_\infty)$ to 0.

As to V_{2t}, we notice first

$$x_t = e^{-t}\cdot x_0 + \int_0^t e^{s-t}\,\mathrm{d}w_s,$$

so we have

$$\begin{aligned}V_{2t} &= -\int_0^t e^{-\int_s^t (W_r-1)\,\mathrm{d}r}(e^{-t}\cdot x_0 + \int_0^s e^{r-s}\,\mathrm{d}w_r)\,\mathrm{d}s\\ &= -\int_0^t e^r \int_r^t e^{-s-\int_s^t (W_u-1)\,\mathrm{d}u}\,\mathrm{d}s\,\mathrm{d}w_r - e^{-t}\cdot x_0\int_0^t e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}s.\end{aligned}$$

First of all, we can easily get

$$\lim_{t\to\infty} e^{-t}x_0 \int_0^t e^{-\int_s^t (W_r-1)\,\mathrm{d}r}\,\mathrm{d}s = 0$$

and for all $t \geq 0$

$$E[-\int_0^t e^r \int_r^t e^{-s-\int_s^t (W_u-1)\,\mathrm{d}u}\,\mathrm{d}s\,\mathrm{d}w_r] = 0.$$

Then, for any $T_0 \geq 0$, take arbitrary $\xi \in L_2(\Omega, P, \mathcal{F}^{w.}_{T_0})$, for any $t \geq T_0$,

$$\begin{aligned}E[\xi\cdot -\int_0^t e^r \int_r^t e^{-s-\int_s^t (W_u-1)\,\mathrm{d}u}\,\mathrm{d}s\,\mathrm{d}w_r] &= E[\xi\cdot -\int_0^{T_0} e^r \int_r^t e^{-s-\int_s^t (W_u-1)\,\mathrm{d}u}\,\mathrm{d}s\,\mathrm{d}w_r]\\ &\leq \{E[\xi^2]\cdot\int_0^{T_0} e^{2r}(\int_r^t e^{-s-\int_s^t (W_u-1)\,\mathrm{d}u}\,\mathrm{d}s)^2\,\mathrm{d}r\}^{\frac{1}{2}},\end{aligned}$$

while as $t \to \infty$, the right hand side goes to 0. By arbitrariness of T_0 and ξ, we have that V_{2t} converges weakly to 0 in $L_2(\Omega, P, \mathcal{F}^{w.}_\infty)$. Since B_t and w_t are independent, we have that V_t converges weakly to 0 in $L_2(\Omega, P, \sigma(\mathcal{F}^{B.}_\infty \cup \mathcal{F}^{w.}_\infty))$, also in $L_2(\Omega, P, \mathcal{F}^{y}_\infty)$.

By using the explicit formula for W_t and V_t, obtaining the explicit formula for U_t is straightforward

$$U_t = \int_0^t (\frac{1}{2}W_s - \frac{1}{2}V_s^2 - 1)\,\mathrm{d}s.$$

Let us denote

$$Q_t(x) = \frac{1}{2}W_t x^2 + V_t x + U_t,$$

and therefore

$$Q_{tx}(x) = xW_t + V_t.$$

Instead of solving (8.2) directly, let us consider $\hat{\pi}_t^w = e^{Q_t}\bar{\pi}_t^w$:

$$\mathrm{d}(e^{Q_t}\bar{\pi}_t^w) = e^{Q_t}\,\mathrm{d}\bar{\pi}_t^w + \bar{\pi}_t^w\,\mathrm{d}e^{Q_t} + \mathrm{d}(e^{Q_t})\,\mathrm{d}\bar{\pi}_t^w$$

$$= e^{Q_t}(\frac{1}{2}\bar{\pi}_{txx}^w + (x\bar{\pi}_t^w)_x)\,\mathrm{d}t + (x\hat{\pi}_t^w)\,\mathrm{d}y_t + \hat{\pi}_t^w\,\mathrm{d}Q_t + \frac{1}{2}\hat{\pi}_t^w(\,\mathrm{d}Q_t)^2 - x^2\hat{\pi}_t^w\,\mathrm{d}t.$$

Since

$$\hat{\pi}_{tx}^w = e^{Q_t}\bar{\pi}_{tx}^w + \hat{\pi}_t^w Q_{tx},$$

$$\hat{\pi}_{txx}^w = (e^{Q_t})_{xx}\bar{\pi}_t^w + 2(e^{Q_t})_x\bar{\pi}_{tx}^w + e^{Q_t}\bar{\pi}_{txx}^w$$

$$= Q_{txx}\hat{\pi}_t^w + 2Q_{tx}\hat{\pi}_{tx}^w - Q_{tx}^2\hat{\pi}_t^w + e^{Q_t}\bar{\pi}_{txx}^w,$$

we can simplify $\mathrm{d}\hat{\pi}_{tx}^w$:

$$\begin{aligned}\mathrm{d}\hat{\pi}_t^w &= [\frac{1}{2}\hat{\pi}_{txx}^w - \frac{1}{2}W_t\hat{\pi}_t^w - Q_{tx}\hat{\pi}_{tx}^w + \frac{1}{2}(xW_t + V_t)^2\hat{\pi}_t^w]\,\mathrm{d}t \\ &+[\hat{\pi}_t^w + (x\hat{\pi}_t^w - x^2W_t\hat{\pi}_t^w - xV_t\hat{\pi}_t^w]\,\mathrm{d}t + (x\hat{\pi}_t^w)\,\mathrm{d}y_t \\ &+\hat{\pi}_t^w[\frac{1}{2}x^2(2W_t - W_t^2 + 1) + x(V_t - W_tV_t) + (\frac{1}{2}W_t - \frac{1}{2}V_t^2 - 1)]\,\mathrm{d}t \\ &-(x\hat{\pi}_t^w)\,\mathrm{d}y_t + \frac{1}{2}x^2\hat{\pi}_t^w\,\mathrm{d}t - x^2\hat{\pi}_t^2\,\mathrm{d}t \\ &= [\frac{1}{2}\hat{\pi}_{txx}^w - (Q_{tx} - x)\hat{\pi}_{tx}]\,\mathrm{d}t,\end{aligned}$$

with initial data

$$\hat{\pi}_0^w = e^{\frac{1}{2}x^2}\pi_0.$$

Introduce

$$\mathrm{d}\tilde{x}_t = \mathrm{d}\tilde{w}_t - (Q_{T-t,x}(\tilde{x}_t) - \tilde{x}_t)\,\mathrm{d}t,$$

with initial data

$$\tilde{x}_0 = x,$$

where $\tilde{w}_t$ is independent of $\mathcal{F}_t$.

To solve the above stochastic differential equation, we write it as

$$\mathrm{d}\tilde{x}_t + (W_{T-t} - 1)\tilde{x}_t\,\mathrm{d}t = \mathrm{d}\tilde{w}_t - V_{T-t}\,\mathrm{d}t.$$

We multiply both sides by $e^{\int_0^t (W_s - 1)\,\mathrm{d}s}$, and by using Itô's formula, we obtain

$$\mathrm{d}[e^{\int_0^t (W_{T-s}-1)\,\mathrm{d}s} \cdot \tilde{x}_t] = e^{\int_0^t (W_{T-s}-1)\,\mathrm{d}s} \cdot (\,\mathrm{d}\tilde{w}_t - V_{T-t}\,\mathrm{d}t).$$

Therefore,

$$\tilde{x}_t = e^{-\int_0^t (W_{T-s}-1)\,\mathrm{d}s} \cdot [x + \int_0^t e^{\int_0^s (W_{T-r}-1)\,\mathrm{d}r}\,\mathrm{d}\tilde{w}_s - \int_0^t e^{\int_0^s (W_{T-r}-1)\,\mathrm{d}r} V_{T-s}\,\mathrm{d}s],$$

and

$$\tilde{x}_T = e^{-\int_0^T (W_t-1)\,\mathrm{d}t} \cdot x - e^{-\int_0^T (W_t-1)\,\mathrm{d}t} \cdot \int_0^T e^{\int_t^T (W_s-1)\,\mathrm{d}s} V_t\,\mathrm{d}t$$
$$+ e^{-\int_0^T (W_t-1)\,\mathrm{d}t} \cdot \int_0^T e^{\int_0^s (W_{T-r}-1)\,\mathrm{d}r}\,\mathrm{d}\tilde{w}_s.$$

Denote

$$\tilde{E}\tilde{x}_T := e^{-\int_0^T (W_t-1)\,\mathrm{d}t} \cdot x - f_T,$$
$$\tilde{\mathrm{Var}}[\tilde{x}_T] := g_T^2.$$

Therefore, we have

$$f_T = \int_0^T e^{-\int_0^t (W_s-1)\,\mathrm{d}r} V_t\,\mathrm{d}s,$$
$$g_T = \int_0^T e^{-2\int_0^t (W_s-1)\,\mathrm{d}s}\,\mathrm{d}t.$$

Now let us apply Itô's formula to $\hat{\pi}^w_{T-t}(\tilde{x}_t)$:

$$\mathrm{d}\hat{\pi}^w_{T-t}(\tilde{x}_t) = \frac{\partial}{\partial t}\hat{\pi}^w_{T-t}(\tilde{x}_t)\,\mathrm{d}t + \hat{\pi}^w_{T-t,x}(\tilde{x}_t)\,\mathrm{d}\tilde{x}_t + \frac{1}{2}\hat{\pi}^w_{T-t,xx}(\tilde{x}_t)\,\mathrm{d}\tilde{x}_t\,\mathrm{d}\tilde{x}_t$$
$$= [-\frac{1}{2}\hat{\pi}^w_{T-t,xx}(\tilde{x}_t) + \hat{\pi}^w_{T-t,x}(\tilde{x}_t)(Q_{T-t,x}(\tilde{x}_t) + \tilde{x}_t)]\,\mathrm{d}t + \hat{\pi}^w_{T-t,x}(\tilde{x}_t)\,\mathrm{d}\tilde{w}_t$$
$$-\hat{\pi}^w_{T-t,x}(\tilde{x}_t)(Q_{T-t,x}(\tilde{x}_t) + \tilde{x}_t)\,\mathrm{d}t + \frac{1}{2}\hat{\pi}^w_{T-t,xx}(\tilde{x}_t)\,\mathrm{d}t$$
$$= \hat{\pi}^w_{T-t,x}(\tilde{x}_t)\,\mathrm{d}\tilde{w}_t.$$

Therefore, we conclude that

$$\hat{\pi}^w_T(x) = \tilde{E}\hat{\pi}_0(\tilde{x}_T),$$

that is,

$$\hat{\pi}^w_T(x) = \frac{1}{\sqrt{2\pi g_T^2}} \int_{-\infty}^{\infty} e^{\frac{-(e^{-\int_0^T (W_t-1)\,\mathrm{d}t}\cdot x - f_T - y)^2}{2g_T^2}} e^{\frac{1}{2}y^2} \pi_0(y)\,\mathrm{d}y,$$

and moreover

$$\bar{\pi}_T^w(x) = \frac{1}{\sqrt{2\pi g_T^2}} e^{-Q_T(x)} \int_{-\infty}^{\infty} e^{\frac{-(e^{-\int_0^T (W_t-1)\,\mathrm{d}t}\cdot x - f_T - y)^2}{2g_T^2}} e^{\frac{1}{2}y^2} \pi_0(y)\,\mathrm{d}y, \tag{8.8}$$

We are going to find $(\bar{\pi}_T^w, 1)_{\mathbb{R}}$ now. Denote $P_T = e^{-\int_0^T (W_t-1)\,\mathrm{d}t}$,

$$(\bar{\pi}_T^w, 1)_{\mathbb{R}} = \frac{1}{\sqrt{2\pi g_T^2}} \int_{-\infty}^{\infty} e^{\frac{1}{2}y^2} \pi_0(y) \int_{-\infty}^{\infty} e^{-(\frac{1}{2}W_T x^2 + V_T x + U_T)} e^{\frac{-(P_T\cdot x - y - f_T)^2}{2g_T^2}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{\sqrt{2\pi g_T^2}} \int_{-\infty}^{\infty} e^{\frac{1}{2}y^2} \pi_0(y) \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}(W_T + \frac{P_T^2}{g_T^2})x^2 + (-V_T + \frac{y+f_T}{g_T^2}P_T)x - (U_T + \frac{(y+f_T)^2}{2g_T^2})}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{\sqrt{g_T^2 \cdot W_T + P_T^2}} \int_{-\infty}^{\infty} e^{\frac{1}{2}y^2} \pi_0(y) e^{\frac{(-V_T + \frac{y+f_T}{g_T^2}P_T)^2}{2(W_T + \frac{P_T^2}{g_T^2})} - (U_T + \frac{(y+f_T)^2}{2g_T^2})}\,\mathrm{d}y$$

$$= \frac{e^{-U_T + \frac{V_T^2}{W_T}}}{\sqrt{g_T^2 \cdot W_T + P_T^2}} \int_{-\infty}^{\infty} e^{\frac{1}{2}y^2} \pi_0(y) e^{\frac{-W_T(y + f_T + \frac{V_T P_T}{W_T})^2}{2(W_T g_T^2 + P_T^2)}}\,\mathrm{d}y.$$

Solving (8.4) is very similar as above, only need to notice our processes start at $t = t_0$ and initial data is $\bar{\pi}_{t_0}$, therefore we can set

$$Q_{t_0,0}(x) = \frac{1}{2}x^2 W_{t_0} + xV_{t_0} + U_{t_0},$$

and

$$Q_{t_0,t} = Q_{t_0+t} = \frac{1}{2}x^2 W_{t_0+t} + xV_{t_0+t} + U_{t_0+t}.$$

Similarly, we can deduce an evolutionary equation with random coefficients from (8.4):

$$\mathrm{d}\hat{\pi}_{t_0,t}^w = [\frac{1}{2}\hat{\pi}_{t_0,txx}^w - (Q_{t_0,tx} - x)\hat{\pi}_{t_0,tx}]\,\mathrm{d}t,$$

with initial condition

$$\hat{\pi}_{t_0,0}^w = e^{Q_{t_0,0}}\pi_0.$$

Introduce

$$\mathrm{d}\tilde{x}_{t_0,t} = \mathrm{d}\tilde{w}_{t_0+t} - (Q_{T-t,x}(\tilde{x}_{t_0,t}) - \tilde{x}_{t_0,t})\,\mathrm{d}t,$$

with initial data

$$\tilde{x}_{t_0,0} = x.$$

After solving the SDE above, we obtain that

$$\tilde{x}_{t_0,t} = e^{-\int_0^t (W_{T-s}-1)\,\mathrm{d}s} \cdot [x + \int_0^t e^{\int_0^s (W_{T-r}-1)\,\mathrm{d}r}\,\mathrm{d}\tilde{w}_{t_0+s} - \int_0^t e^{\int_0^s (W_{T-r}-1)\,\mathrm{d}r} V_{T-s}\,\mathrm{d}s],$$

and

$$\tilde{x}_{t_0,T-t_0} = e^{-\int_{t_0}^T (W_t-1)\,\mathrm{d}t} \cdot x - e^{-\int_{t_0}^T (W_t-1)\,\mathrm{d}s} \cdot \int_{t_0}^T e^{\int_t^T (W_s-1)\,\mathrm{d}s} V_t\,\mathrm{d}t$$
$$+ e^{-\int_{t_0}^T (W_t-1)\,\mathrm{d}s} \cdot \int_0^{T-t_0} e^{\int_0^s (W_{T-r}-1)\,\mathrm{d}r}\,\mathrm{d}\tilde{w}_{t_0+s}.$$

Let

$$\tilde{E}[\tilde{x}_{t_0,T-t_0}] := e^{-\int_0^{T-t_0} (W_t-1)\,\mathrm{d}t} \cdot x - f_{t_0,T-t_0},$$
$$\tilde{\mathrm{Var}}[\tilde{x}_{t_0,T-t_0}] := g^2_{t_0,T-t_0},$$

where

$$f_{t_0,T-t_0} = e^{-\int_{t_0}^T (W_t-1)\,\mathrm{d}s} \cdot \int_{t_0}^T e^{\int_t^T (W_s-1)\,\mathrm{d}s} V_t\,\mathrm{d}t$$
$$g^2_{t_0,T-t_0} = e^{-2\int_{t_0}^T (W_t-1)\,\mathrm{d}s} \cdot \int_{t_0}^T e^{2\int_t^T (W_s-1)\,\mathrm{d}s}\,\mathrm{d}t.$$

By using Itô's formula to $\hat{\pi}^w_{t_0,t}(\tilde{x}_{t_0,T-t_0-t})$, we have

$$\hat{\pi}^w_{t_0,T-t_0}(x) = \tilde{E}[\hat{\pi}_{t_0}(\tilde{x}_{t_0,T-t_0})].$$

Denote $e^{-\int_{t_0}^T (W_t-1)\,\mathrm{d}s}$ by $P_{t_0,T-t_0}$. Therefore, we conclude

$$\hat{\pi}^w_{t_0,T-t_0}(x) = \frac{1}{\sqrt{2\pi g^2_{t_0,T-t_0}}} \int_{-\infty}^{\infty} e^{\frac{-(P_{t_0,T-t_0}\cdot x - f_{t_0,T-t_0} - y)^2}{2g^2_{t_0,T-t_0}}} e^{Q_{t_0}} \bar{\pi}_{t_0}(y)\,\mathrm{d}y.$$

and moreover

$$\bar{\pi}^w_{t_0,T-t_0}(x) = \frac{1}{\sqrt{2\pi g^2_{t_0,T-t_0}}} e^{-Q_T(x)} \int_{-\infty}^{\infty} e^{\frac{-(P_{t_0,T-t_0}\cdot x - f_{t_0,T-t_0} - y)^2}{2g^2_{t_0,T-t_0}}} e^{Q_{t_0}} \bar{\pi}_{t_0}(y)\,\mathrm{d}y. \quad (8.9)$$

And finally, we have that

$$(\bar{\pi}^w_{t_0,T-t_0}, 1)_{\mathbb{R}} = \frac{e^{-U_T + \frac{V_T^2}{2W_T}}}{\sqrt{g^2_{t_0,T-t_0} \cdot W_T + P^2_{t_0,T-t_0}}} \int_{-\infty}^{\infty} e^{Q_{t_0}(y)} \bar{\pi}_{t_0}(y) e^{-\frac{W_T(y + f_{t_0,T-t_0} + P_{t_0,T-t_0}\frac{V_T}{W_T})^2}{2(W_T g^2_{t_0,T-t_0} + P^2_{t_0,T-t_0})}}\,\mathrm{d}y.$$

Now we are ready to use Theorem 6.3 to obtain $P(I_{\tau \geq t_0} \mid \mathcal{F}_T^y)$ equals to

$$\frac{\sqrt{g_T^2 \cdot W_T + P_T^2}}{\sqrt{g_{t_0,T-t_0}^2 \cdot W_T + P_{t_0,T-t_0}^2}} \cdot \frac{\int_{-\infty}^{\infty} e^{Q_{t_0}(y)} \bar{\pi}_{t_0}(y) e^{-\frac{W_T(y+f_{t_0,T-t_0}+P_{t_0,T-t_0}\frac{V_T}{W_T})^2}{2(W_T g_{t_0,T-t_0}^2 + P_{t_0,T-t_0}^2)}} \mathrm{d}y}{\int_{-\infty}^{\infty} e^{\frac{1}{2}y^2} \pi_0(y) e^{\frac{-W_T(y+f_T+\frac{V_T P_T}{W_T})^2}{2(W_T g_T^2 + P_T^2)}} \mathrm{d}y} \tag{8.10}$$

Remark 8.1. It is interesting to note here as $T \to \infty$, all of f_T, g_T, $f_{t_0,T-t_0}$, $g_{t_0,T-t_0}$, P_T, $P_{t_0,T-t_0}$, W_T and V_T converge.

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