Some results on scattering for log-subcritical and log-supercritical nonlinear wave equations

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Hsi-Wei Shih

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Abstract

We consider here two problems in the asymptotic behavior of semilinear second order wave equations.

First, we consider the $\dot{H}^1_x \times L^2_x$ scattering theory for the energy log-subcritical wave equation

$$\Box u = |u|^4 u g(|u|)$$

in $\mathbb{R}^{1+3}$, where g has logarithmic growth at 0. We discuss the solution with general (resp. spherically symmetric) initial data in the logarithmically weighted (resp. lower regularity) Sobolev space. We include also some observation about scattering in the energy subcritical case.

The second problem studied here involves the energy log-supercritical wave equation

$$\Box u = |u|^4 u \log^{\alpha}(2 + |u|^2), \qquad \text{for } 0 < \alpha \leq \frac{4}{3}$$

in $\mathbb{R}^{1+3}$. We prove the same results of global existence and $(\dot{H}^1_x \cap \dot{H}^2_x) \times H^1_x$ scattering for the equation with a slightly higher power of the logarithm factor in the nonlinearity than that allowed by [26].

Contents

Chapter 1

Introduction

Consider the semilinear wave equation

$$\begin{cases} \Box u := -\partial_t^2 u + \triangle u = f(u) & \text{on } \mathbb{R} \times \mathbb{R}^3 \\ u(t_0, x) = u_0(x) \\ \partial_t u(t_0, x) = u_1(x) \end{cases} \tag{1.1}$$

where f is a complex valued function. Let the potential function $F : \mathbb{C} \to \mathbb{R}$ be a real valued function such that

$$2F_{\bar{z}}(z) = f(z) \tag{1.2}$$

with $F(0) = 0$ and u be the solution to (1.1) with initial data $u_0 \in \dot{H}^1_x \cap \{\phi | \int_{\mathbb{R}^3} F(\phi) dx < \infty\}$ and $u_1 \in L^2_x$. We can easily verify that the equation has conserved energy (see theorem 3.1 and remark 3.3)

$$E(u)(t) := \int_{\mathbb{R}^3} \frac{1}{2}|u_t(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2 + F(u(t,x)) dx. \tag{1.3}$$

We recall quickly some common terminology associated to the scaling properties of the equation (1.1). For $1 < p <= \infty$, we consider $f(z) = |z|^{p-1}z$ and let u be the solution of (1.1). By scaling, $\lambda^{\frac{2}{1-p}} u(\frac{t}{\lambda}, \frac{x}{\lambda})$ is also a solution with initial data $\lambda^{\frac{2}{1-p}} u_0(\frac{t_0}{\lambda}, \frac{x}{\lambda})$ and $\lambda^{\frac{1+p}{1-p}} u_1(\frac{t_0}{\lambda}, \frac{x}{\lambda})$. Hence, the scaling of u preserves the homogeneous Sobolev norm $\| u_0 \|_{\dot{H}^{s_c}(\mathbb{R}^3)} + \| u_1 \|_{\dot{H}^{s_c - 1}(\mathbb{R}^3)}$ if $s_c := \frac{3}{2} - \frac{2}{p-1}$. (i.e. $p = 1 + \frac{4}{3 - 2s_c}$).

Definition 1.1. *For $1 < p < \infty$, $f(z) = |z|^{p-1}z$ and a given value s, we call (1.1) a $\dot{H}^s_x$-critical (subcritical, supercritical) nonlinear wave equation if $p = (<, >)\ 1 + \frac{4}{3-2s}$.*

In particular, when $s = 1$, *we call* (1.1) *an energy critical (subcritical, supercritical) nonlinear wave equation if* $p = (<, >)\ 5$.

The results of global existence and uniqueness for the energy-critical ($\Box u = |u|^4 u$) and energy-subcritical ($\Box u = |u|^{p-1}u$, where $p < 5$) wave equations are already established by [3], [25], [9], [10], [21], [12], [22] and [5]. It is natural to consider the decay of the solution, which we expect to behave linearly as $t \to \pm\infty$. The decay estimate and scattering theory (see chapter 3 for definition) of critical wave equation are shown in [2] (see also [1], [7] [19]). In [11] (see also [6]), the authors proved, by the property of conformal invariance, that the solutions for certain subcritical wave equations ($\frac{5}{2} < p \leq 3$) scatter in the weighted Sobolev space $\Sigma := X \times Y$, where

$$\begin{aligned} X &:= H^1_x(\mathbb{R}^3) \cap \{\phi :\ |x|\nabla\phi \in L^2_x(\mathbb{R}^3)\}, \\ Y &:= L^2_x(\mathbb{R}^3) \cap \{\phi :\ |x|\phi \in L^2_x(\mathbb{R}^3)\}. \end{aligned}$$

However, for energy subcritical wave equations, the $\dot{H}^1_x \times L^2_x$ scattering theory[1] still remains open. Hence, as a step toward this direction, it is naturally to consider the problem of scattering theory for the "slightly" subcritical wave equation. We will call this class of equations "log-subcritical" wave equations which are defined below.

In this dissertation, we study the $\dot{H}^1_x \times L^2_x$ scattering theory for log-subcritical wave equations with finite energy initial data, where the energy is defined by (1.3). The term *log-subcritical wave equation* refers to the equation (1.1) with f defined by

$$f(z) := \begin{cases} |z|^4 z g(|z|) & , |z| \neq 0 \\ 0 & , |z| = 0 \end{cases} \tag{1.4}$$

where $g : (0, \infty) \to \mathbb{R}$ is smooth and nonincreasing as well as satisfies

$$g(x) := \begin{cases} -\log(x) & , 0 < x < \frac{1}{3} \\ \sim 1 & , \frac{1}{3} \leq x < 1 \\ 1 & , x \geq 1. \end{cases} \tag{1.5}$$

We also prove global existence in the case of spherical symmetry for *log-supercritical wave equations*, by which we mean equations of the form,

$$\Box u = |u|^4 u \log^{\alpha}(2 + |u|^2) \tag{1.6}$$

[1] The $\dot{H}^1_x \times L^2_x$ scattering is defined in Definition 3.6.

In this dissertation, we will allow $0 < \alpha \le \frac{4}{3}$, extending the range $0 < \alpha \le 1$ allowed in [26].

Remark 1.2. *We can easily compute that the potential function of log-subcritical wave equations* (1.1)*,* (1.4) *and* (1.5) *is,*

$$F_{sub}(u) = \begin{cases} -\frac{1}{6}|u|^6\big(\log(|u|) - \frac{1}{6}\big) & , 0 < |u| < \frac{1}{3} \\ \sim \frac{1}{6}|u|^6 & , \frac{1}{3} \le |u| < 1 \\ \frac{1}{6}|u|^6 & , |u| \ge 1 \end{cases} \tag{1.7}$$

and the potential function of the log-supercritical wave equations (1.6) *is,*

$$F_{sup}(u) \sim |u|^6 \log^{\alpha}(2 + |u|^2). \tag{1.8}$$

The dissertation is organized as follows. In chapter 2, we introduce some notations and definitions which we use throughtout this dissertation. In chapter 3, we state some well-known results for general semilinear wave equations on $\mathbb{R}^{1+3}$. We also show certain spacetime bounds for spherically symmetric solutions to the log-subcritical and log-supercritical nonlinear wave equations by using modified radial Sobolev inequality and Morawetz inequality in this chapter.

In chapter 4, we consider the solutions to the log-subcritical wave equation (1.1), (1.4) and (1.5) with finite energy initial data. The global existence result is established in [9], [10], [12] and [19]. We will prove that the solutions with a class of initial data scatter in $\dot{H}^1_x \times L^2_x$. This class of data are contained in logarithmically weighted Sobolev spaces $X_1 \times Y_1$, where

$$\begin{cases} X_1 := \dot{H}^1_x(\mathbb{R}^3) \cap \{\phi : \ \log^{\gamma}(1 + |x|)\nabla\phi \in L^2_x(\mathbb{R}^3)\}, \\ Y_1 := L^2_x(\mathbb{R}^3) \cap \{\phi : \ \log^{\gamma}(1 + |x|)\phi \in L^2_x(\mathbb{R}^3)\} \end{cases} \tag{1.9}$$

for some $\gamma > \frac{1}{2}$. For initial data in these spaces, we show that the potential energy of the solution decays logarithmically for all large times. After dividing the time interval suitably, this decay helps us to control the key spacetime norm $\| \ f(u) \ \|_{L^1_t L^2_x}$. This spacetime bound implies scattering (we will sketch the proof in chapter 3 below, see also [2]). The proof of the spacetime bound here involves establishing a decay rate for certain constant-time norms of the solution and a bootstrap scheme motivated by that in [26]. We rely heavily on ideas from [2].

This chapter also considers the solution of log-subcritical wave equations with spherically symmetric data. We prove that the solution u with initial data in $X_2 \times Y_2$ scatters in $\dot{H}^1_x \times L^2_x$, where

$$\begin{cases} X_2 := \dot{H}^1_x(\mathbb{R}^3) \cap (\cup_{\delta>0} \dot{H}^{1-\delta}_x(\mathbb{R}^3)), \\ Y_2 := L^2_x(\mathbb{R}^3) \cap (\cup_{\delta>0} \dot{H}^{-\delta}_x(\mathbb{R}^3)). \end{cases} \tag{1.10}$$

The proof here again uses the ideas from [26] and the classical Morawetz inequality (see [18]). However, we need a slightly sharpened version of the bootstrap argument. We also give the remarks for some specific energy subcritical wave equations (see section 4.3).

Chapter 5 studies global existence for log-supercritical wave equations. The global regularity of energy supercritical wave equations ($\Box u = |u|^{p-1}u$, where $p > 5$) is still open. In [26], the author considered the log-supercritical wave equation

$$\Box u = u^5 \log^\alpha(2 + u^2) \tag{1.11}$$

with spherically symmetric initial data and established a global regularity result for $0 < \alpha \leq 1$. For general initial data, the same result for loglog-supercritical wave equations

$$\Box u = u^5 \log^c \left(\log(10 + u^2) \right)$$

with $0 < c < \frac{8}{225}$ is obtained in [20]. In the dissertation, we extend the result in [26] to the range $0 < \alpha \leq \frac{4}{3}$, again for spherically symmetric data. This improvement is attained by employing the potential energy bound in place of the kinetic energy bound used in [26] for pointwise control.

We will show some details of proofs in the appendix as chapter 6.

Chapter 2

Definitions and Notations

Throughout this dissertation, we use $M \lesssim N$ to denote the estimate $M \leq CN$ for some absolute constant C (which can vary from line to line).

We use $L_t^q L_x^r$ to denote the spacetime norm

$$\| u \|_{L_t^q L_x^r(I \times \mathbb{R}^3)} := \left(\int_I \left(\int_{\mathbb{R}^3} |u(t,x)|^r dx\right)^{q/r} dt\right)^{1/q}$$

with the usual modifications when q or r is equal to infinity.

Definition 2.1. *(Solution) Let u be a $\dot{H}_x^s \times \dot{H}_x^{s-1}$ solution to (1.1) on $[0,T)$ if $(u, \partial_t u) \in C_{t,loc}^0\big([0,T); \dot{H}_x^s(\mathbb{R}^3)\big) \times C_{t,loc}^1\big([0,T); \dot{H}_x^{s-1}(\mathbb{R}^3)\big)$. Furthermore, by Duhamel's formula, we can express u as*

$$u(t,x) = \cos(t\sqrt{-\triangle})u_0(x) + \frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}}u_1(x) - \int_0^t \frac{\sin((t-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau \quad (2.1)$$

where the operators $\cos(t\sqrt{-\triangle})$ and $\frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}}$ are defined by

$$\Big(\cos(t\sqrt{-\triangle})\phi\Big)^{\wedge}(\xi) = \cos(t|\xi|)\hat{\phi}(\xi)$$

and

$$\Big(\frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}}\phi\Big)^{\wedge}(\xi) = \frac{\sin(t|\xi|)}{|\xi|}\hat{\phi}(\xi).$$

Definition 2.2. *(admissible pair) We say that the pair (q,r) is admissible if $2 \leq q, r \leq \infty$, $(q,r) \neq (2,\infty)$ and*

$$\frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}.$$

We define the Strichartz space $S_\sigma(I)$ for any time interval I, as the closure of Schwartz function on $I \times \mathbb{R}^3$ under the norm

$$\| u \|_{S_\sigma(I)} := \sup_{(q,r):\ admissible} \| u \|_{L_t^q L_x^r(I \times \mathbb{R}^3)}$$

where (q, r) satisfies (3.3) below.

Chapter 3

Semilinear Wave Equations

In this chapter, we recall some well-known results of semilinear wave equations on $\mathbb{R}^{1+3}$.

We consider the semilinear wave equation (1.1)

$$\begin{cases} \Box u := -\partial_t^2 u + \triangle u = f(u) & \text{on } \mathbb{R} \times \mathbb{R}^3 \\ u(0,x) = u_0(x) \\ \partial_t u(0,x) = u_1(x) \end{cases}$$

where f is a complex valued function.

3.1 Results for general semilinear wave equations

3.1.1 Energy Conservation

Consider the potential function F as in (1.2). From equation (1.1), we get

$$\partial_t \left(\frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 + F(u) \right) - div(\nabla u \bar{u}_t) = 0 \tag{3.1}$$

where div denotes the divergence with respect to the spatial variables. It is easy to device the energy conservation law by equation (3.1).

Theorem 3.1. *(Energy Identity) Suppose $u \in C^2([0,T) \times \mathbb{R}^3)$ solves (1.1) and that $u(t,\cdot)$ is compactly supported for every t. Then*

$$E(u)(t) = E(u)(0)$$

for all $0 \leq t \leq T$.

Proof. Since u has compact support for all t, by divergence theorem and (3.1),

$$0 = \int_0^t \int_{\mathbb{R}^3} \partial_t \left(\frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 + F(u) \right) - div(\nabla u \bar{u}_t) \; dxdt = E(u)(t) - E(u)(0).$$

□

3.1.2 Local and Global Existence Results

Theorem 3.2. *(Local existence theory)(See [23]) Assume that* $f \in C^k$, $f(0) = 0$, *and that* $u_0 \in C_0^{k+1}(\mathbb{R}^3)$, $u_1 \in C_0^k(\mathbb{R}^3)$ *with* $k = 1, 2, \cdots$. *Then there is a* $T > 0$ *so that* (1.1) *has a unique solution* $u \in C^k([0,T] \times \mathbb{R}^3)$. *If the supremum,* T_*, *of such times* T *is finite then* $\sup_x |u(t,x)| \to \infty$ *as* $t \to T_*$.

Remark 3.3. *The assumptions of compact support and smoothness of* u *in the above theorems can be removed by approximation arguments.*

(i) If u *is the solution to* (1.1) *with initial data* $u_0 \in \dot{H}_x^1 \cap \{\phi | \int_{\mathbb{R}^3} F(\phi)dx < \infty\}$ *and* $u_1 \in L_x^2$, *the conclusion* $E(u)(t) = E(u)(0)$ *also hold for all* t.

(ii) If u *is the solution to* (1.1) *such that* $\sup_{x \in \mathbb{R}^3, 0<t<T} |u(t,x)| < \infty$ *for all* $0 < T < \infty$, *then* (1.1) *has a unique global solution on* $[0, \infty) \times \mathbb{R}^3$.

3.1.3 Strichartz Inequality

Theorem 3.4. *(Strichartz estimates for wave equation [24], [13], [8], [17], [15].) Let* I *be a time interval, and let* $u : I \times \mathbb{R}^3 \to \mathbb{C}$ *be a Schwartz solution to the wave equation* $\Box u = G$ *with initial data* $u(t_0) = u_0$, $\partial_t u(t_0) = u_1$ *for some* $t_0 \in I$. *Then we have the estimates*

$$\begin{aligned} \| u \|_{L_t^q L_x^r(I \times \mathbb{R}^3)} + \| u \|_{C_t^0 \dot{H}_x^\sigma(I \times \mathbb{R}^3)} + \| \partial_t u \|_{C_t^0 \dot{H}_x^{\sigma-1}(I \times \mathbb{R}^3)} \\ \lesssim \| u_0 \|_{\dot{H}_x^\sigma(\mathbb{R}^3)} + \| u_1 \|_{\dot{H}_x^{\sigma-1}(\mathbb{R}^3)} + \| G \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I \times \mathbb{R}^3)} \end{aligned} \tag{3.2}$$

where (q, r) *and* $(\tilde{q}, \tilde{r})$ *are admissible pairs as in definition 2.2 and obey the scaling condition*

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - \sigma = \frac{1}{\tilde{q}'} + \frac{3}{\tilde{r}'} - 2 \tag{3.3}$$

and where $\tilde{q}'$ *and* $\tilde{r}'$ *are conjugate to* $\tilde{q}$ *and* $\tilde{r}$ *respectively. In addition, if* u *is a spherically symmetric solution, we allow* $(q, r) = (2, \infty)$.

3.1.4 Morawetz Inequality

Theorem 3.5. *(Morawetz Inequality)(See [18]) Let I be any time interval and $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be the solution to* (1.1) *with finite energy E. Let F be the potential function as in* (1.2)*. Then*

$$\int_I \int_{\mathbb{R}^3} \frac{G(u)}{|x|} dxdt + \int_I \int_{\mathbb{R}^3} \frac{|\nabla_\omega u|^2}{|x|} dxdt \lesssim E \tag{3.4}$$

where $G(u) := 2F(u) - 2Re\big[\bar{u}F_{\bar{z}}(u)\big]$ *and* $|\nabla_\omega u|^2 := |\nabla u|^2 - u_r^2 = |\nabla u|^2 - |\frac{x}{|x|} \cdot \nabla u|^2$.

Proof. We will show the details of the proof in Appendix. □

3.1.5 Scattering

Definition 3.6. *Let X and Y be two function spaces. We say that a global solution $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ to* (1.1) *scatters in $X \times Y$ (or $X \times Y$ scattering) as $t \to +\infty$ $(-\infty)$ if there exists a linear solution v^+ (v^-) with initial data in $X \times Y$ such that*

$$\| u(t,x) - v^+(t,x) \|_{X\times Y} \longrightarrow 0 \text{ as } t \to +\infty$$

$$(\| u(t,x) - v^-(t,x) \|_{X\times Y} \longrightarrow 0 \text{ as } t \to -\infty).$$

Proposition 3.7. *Let u be the solution to* (1.1) *with initial data $(u_0, u_1) \in \dot{H}^1_x \times L^2_x$. Then the* spacetime bound,

$$\| f(u) \|_{L^1_t L^2_x([t_0,\infty)\times\mathbb{R}^3)} < \infty \tag{3.5}$$

for some $t_0 > 0$, implies the $\dot{H}^1_x \times L^2_x$ scattering (as $t \to \infty$). Let $u \in C^1_t(\mathbb{R}, \dot{H}^1_x(\mathbb{R}^3)) \cap C^0_t(\mathbb{R}, L^2_x(\mathbb{R}^3))$ be the solution to (1.1) *and v satisfy $\Box v = 0$ with initial data $v_0 \in \dot{H}^1_x(\mathbb{R}^3)$, $v_1 \in L^2_x(\mathbb{R}^3)$ (to be chosen shortly). By Duhamel's formula* (2.1),

$$u(t,x) = \cos(t\sqrt{-\triangle})u_0(x) + \frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}} u_1(x) - \int_0^t \frac{\sin((t-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau \tag{3.6}$$

and

$$v(t,x) = \cos(t\sqrt{-\triangle})v_0(x) + \frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}} v_1(x) \tag{3.7}$$

Hence, that the solution u scatters and asymptotically approaches v in $\dot{H}^1_x \times L^2_x$ means

$$\| \cos(t\sqrt{-\triangle})(u_0 - v_0) + \frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}}(u_1 - v_1) - \int_0^t \frac{\sin((t-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau \|_{\dot{H}^1_x \times L^2_x} \longrightarrow 0 \tag{3.8}$$

as $t \to \infty$. From basic trigonometric identities, we can verify that (3.8) is implied by

$$\| (u_0 - v_0) + \int_0^t \frac{\sin((-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau \|_{\dot{H}^1_x} \longrightarrow 0$$

and

$$\| (u_1 - v_1) + \int_0^t \cos((-\tau)\sqrt{-\triangle}) f(u(\tau))d\tau \|_{L^2_x} \longrightarrow 0$$

as $t \to \infty$. Therefore, if

$$\Big(\int_0^t \frac{\sin((-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau, \int_0^t \cos((-\tau)\sqrt{-\triangle}) f(u(\tau))d\tau \Big) \tag{3.9}$$

converges in $\dot{H}^1_x \times L^2_x$ as $t \to \infty$, then taking v_0, v_1 in (3.7) as follows,

$$v_0(x) := u_0(x) - \int_0^\infty \frac{\sin((-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau$$

and

$$v_1(x) := u_1(x) - \int_0^\infty \cos((-\tau)\sqrt{-\triangle}) f(u(\tau))d\tau,$$

we would have, by (3.6), (3.7) and elementary trigonometric formulas,

$$\begin{aligned} &\| u - v \|_{\dot{H}^1_x \times L^2_x} \\ &= \| - \int_0^t \frac{\sin((t-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau \\ &\qquad + \int_0^\infty \frac{\cos(t\sqrt{-\triangle})\sin((-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau \\ &\qquad + \int_0^\infty \frac{\sin(t\sqrt{-\triangle})\cos((-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau \|_{\dot{H}^1_x \times L^2_x} \\ &= \| \int_t^\infty \frac{\sin(t-\tau)\sqrt{-\triangle}}{\sqrt{-\triangle}} f(u(\tau))d\tau \|_{\dot{H}^1_x \times L^2_x} . \end{aligned} \tag{3.10}$$

It remains to show two things. (i) First, that our initial data v_0, v_1 are well-defined. That is, that (3.9) does indeed converge in $\dot{H}^1_x \times L^2_x$. (ii) Second, we must show that the right side of (3.10) converges to 0 as $t \to \infty$.

The claim (i) here can be shown by several ways, e.g. by showing that

$$\lim_{N\to\infty} \| \int_N^\infty \frac{\sin((-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau))d\tau \|_{\dot{H}^1_x} = 0$$

and

$$\lim_{N\to\infty} \| \int_N^\infty \cos((-\tau)\sqrt{-\triangle}) f(u(\tau))d\tau \|_{L^2_x} = 0$$

where $N \in \mathbb{N}$. These two equalities follow from the dominated convergence theorem once we show that

$$\int_0^\infty \| \frac{\sin((-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(u(\tau)) \|_{\dot{H}^1_x} (\tau)d\tau < \infty \tag{3.11}$$

and

$$\int_0^\infty \| \cos((-\tau)\sqrt{-\triangle}) f(u(\tau)) \|_{L^2_x} (\tau)d\tau < \infty. \tag{3.12}$$

But (3.11) *and* (3.12) *follow quickly from* (3.5) *and the Plancherel theorem. The claim (ii) has now been established already in the discussion of claim (i). This concludes the argument that the finiteness of* (3.5) *implies scattering.*

3.2 Spacetime bounds of spherically symmetric solutions

In the last part of this chapter, we assume that u is the spherically symmetric solution to the log-subcritical (1.4), (1.5) (or log-supercritical (1.6)) wave equation with finite initial data and F is the corresponding potential function. We can obtain the following a priori estimate for the solution.

Lemma 3.8. *(Pointwise estimate for spherically symmetric solution,(see [4], [26])) Let I be any time interval and $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be the spherically symmetric solution to the log-subcritical* (1.4), (1.5) *(or log-supercritical* (1.6)*) wave equation with finite energy E and vanish at ∞. Let F be the potential function, then for any $t \in I$,*

$$|x|^2\Big(F(u)^{\frac{1}{2}}|u|\Big)(t,x) \lesssim E \tag{3.13}$$

Proof. The proof for log-supercritical case is similar and easier than that for log-subcritical case. Thus, we just prove the lemma for the log-subcritical case here. Define $\phi(z) := (F(z))^{1/2}z$ and $r := |x|$. From (1.7), we can compute that, for fixed t,

$$|\partial_r\big(\phi(u(t,x))\big)| \lesssim |u|^3|\partial_r u|(t,x)\chi_{\{|u|\geq\frac{1}{3}\}}(x) + |u|^3(-\log(|u|))^{\frac{1}{2}}|\partial_r u|(t,x)\chi_{\{|u|<\frac{1}{3}\}}(x)$$

where χ is the characteristic function on $\mathbb{R}^3$. Then, by Fundamental theorem of Calculus, Hölder's inequality and energy conservation,

$$\begin{aligned}
\big|\phi(u(t,x))\big| &\lesssim \left| \int_r^\infty \Big[|u|^3 |\partial_r u| \chi_{\{|u|\ge \frac{1}{3}\}} + |u|^3 (-\log(|u|))^{\frac{1}{2}} |\partial_r u| \chi_{\{|u|<\frac{1}{3}\}} \Big](t,s)\ ds \right|, \\
&\lesssim \Big(\int_r^\infty \frac{|u|^6}{s^2} s^2 \chi^2_{\{|u|\ge\frac{1}{3}\}} ds \Big)^{1/2} \Big(\int_r^\infty \frac{|\partial_r u|^2}{s^2} \chi_{\{|u|\ge\frac{1}{3}\}} s^2\ ds \Big)^{1/2} \\
&\quad + \Big(\int_r^\infty \frac{|u|^6(-\log(|u|))}{s^2} s^2 \chi^2_{\{|u|<\frac{1}{3}\}} ds \Big)^{1/2} \Big(\int_r^\infty \frac{|\partial_r u|^2}{s^2} \chi_{\{|u|<\frac{1}{3}\}} s^2\ ds \Big)^{1/2}, \\
&\lesssim \frac{1}{r^2} \Big(\int_{\mathbb{R}^3} F(u) dx \Big)^{1/2} E^{1/2} \lesssim \frac{1}{r^2} E.
\end{aligned}$$

□

Inserting (3.13) into (3.4), we obtain that, for any time interval I,

$$\int_I \int_{\mathbb{R}^3} F^{\frac{5}{4}}(u) |u|^{\frac{1}{2}} dxdt \le \int_I \int_{\mathbb{R}^3} \frac{F(u)}{|x|} \cdot \sup_{x\in\mathbb{R}^3} \big(|x| F^{\frac{1}{4}}(u) |u|^{\frac{1}{2}} \big) dxdt \lesssim E^{3/2}. \tag{3.14}$$

This implies

$$\int_I \int_{\{|u|\le\frac{1}{3}\}} |u|^8 (-\log(|u|))^{\frac{5}{4}} dxdt + \int_I \int_{\{|u|>\frac{1}{3}\}} |u|^8 dxdt \lesssim E^{3/2} \qquad \text{(log-subcritical case)} \tag{3.15}$$

and

$$\int_I \int_{\mathbb{R}^3} |u|^8 \log^{\frac{5\alpha}{4}} (2+|u|^2) dxdt \lesssim E^{3/2} \qquad \text{(log-supercritical case)}. \tag{3.16}$$

Chapter 4

Log-subcritical Wave Equations

In this chapter, we consider two cases of the scattering theory for log-subcritical wave equations under different initial data. We can take advantage of time reversal symmetry and it suffices to prove that the solution u scatters in $\dot{H}^1_x \times L^2_x$ as $t \to \infty$.

Before we study the scattering property, we should take care of the global wellposedness of the solutions. Heuristically, the nonlinearity (1.4), $f(u) \sim |u|^4 u$ when $|u|$ is far from 0 and $|u|^5 \lesssim |f(u)| < |u|^p$ for any $0 < p < 5$ when $|u| \sim 0$. Hence, (1.1) is slightly into the subcritical regime. It is reasonable to believe that the solution is globally wellposed. Precisely speaking, we can follow the idea of the proof of global wellposedness of energy critical NLW on (1.1), (1.4) and (1.5) (see [9], [10], [21], [14] and [23]). The main ingredient of the proof of global wellposedness is nonconcentration of the potential energy. For completeness, we will include the proof in the appendix and continue to discuss the scattering theory here.

Throughout this chapter, we denote

$$A = \{(t,x) \in (0,\infty) \times \mathbb{R}^3 : \ |u| < \frac{1}{3}\}, \ B = \{(t,x) \in (0,\infty) \times \mathbb{R}^3 : \ |u| \geq \frac{1}{3}\}$$

and for any interval I,

$$A_I = A \cap (I \times \mathbb{R}^3), \ B_I = B \cap (I \times \mathbb{R}^3) \tag{4.1}$$

4.1 General Initial Data In Log-weighted Sobolev Spaces

Theorem 4.1. *Let $\gamma > \frac{1}{2}$ and u be the solution to the log-subcritical wave equation* (1.1)*,* (1.4) *and* (1.5) *with initial data*

$$\begin{cases} u_0(x) \in X_1 \\ u_1(x) \in Y_1 \end{cases} \tag{4.2}$$

where X_1 and Y_1 are defined by (1.9)*, then u scatters in $\dot{H}_x^1 \times L_x^2$.*

Proof. To prove theorem 4.1, we need some decay estimates for the equation with initial data satisfying (4.2).

Lemma 4.2. *(decay of potential energy) Let γ and u satisfy the assumptions in theorem 4.1. There exists $T = T(\| u_0 \|_{X_1}, \| u_1 \|_{Y_1}, \gamma) \gg 1$ such that for $\tau > T$,*

$$\int_{\mathbb{R}^3} F\big(u(\tau, x)\big) dx \lesssim \frac{1}{\log^{2\gamma} \tau} \tag{4.3}$$

where $F(z) = F_{sub}(z)$ is defined by (1.7)*.*

Proof. The proof of the lemma essentially follows the proof of Lemma 2.1 in [2] and we will make some modifications here. Define

$$e[u](t, x) := \frac{1}{2}|\partial_t u(t, x)|^2 + \frac{1}{2}|\nabla u(t, x)|^2 + F(u(t, x)). \tag{4.4}$$

We claim that there exists $C_\gamma = C_\gamma(\| u_0 \|_{X_1}, \| u_1 \|_{Y_1}, \gamma) \gg 1$ such that for $s > C_\gamma$,

$$\int_{|x|>s} e[u](0, x) dx \lesssim \frac{1}{\log^{2\gamma} s}. \tag{4.5}$$

We will prove the claim in the appendix and continue the proof of this lemma here. Choose T such that $T > \max(C_\gamma^2, \log^{4\gamma} T)$. We aim to show that (4.3) holds for all $\tau > T$.

Define the truncated forward light cone by

$$K_a^b(c) := \{(t, x) | a \le t \le b,\ |x| \le t + c,\ 0 \le a < b \le \infty\}$$

and the boundary of the truncated cone by

$$M_a^b(c) := \partial K_a^b(c) = \{(t, x) | a \le t \le b,\ |x| = t + c,\ 0 \le a < b \le \infty\}$$

Fix $\tau > T$, let $s = \sqrt{\tau} > C_\gamma$. For any $t_1 > 0$, the energy conservation law on the exterior of the truncated forward light cone $K_0^{t_1}(s)$ implies that

$$\int_{|x|>s+t_1} e[u](t_1)dx + \frac{1}{\sqrt{2}}\text{flux}(0,t_1,s) = \int_{|x|>s} e[u](0)dx \lesssim \frac{1}{\log^{2\gamma} s} \tag{4.6}$$

where

$$\text{flux}(a,b,c) := \int_{M_a^b(c)} \Big\{\frac{1}{2}\Big|u_t + \frac{x\cdot\nabla u}{|x|}\Big|^2 + F(u)\Big\}d\sigma.$$

Hence,

$$\int_{|x|>s+\tau} F(u(\tau))dx \le \int_{|x|>s+\tau} e[u](\tau)dx \lesssim \frac{1}{\log^{2\gamma} s} \lesssim \frac{1}{\log^{2\gamma}\tau}. \tag{4.7}$$

and it suffices to show that

$$\int_{|x|\le s+\tau} F(u(\tau))dx \lesssim \frac{1}{\log^{2\gamma}\tau}. \tag{4.8}$$

Define $w(t,x) = u(t-s,x)$. The bound (4.8) is equivalent to

$$\int_{|x|\le s+\tau} F(w(s+\tau))dx \lesssim \frac{1}{\log^{2\gamma}\tau}.$$

Denote $w_t := \partial_t w$. Multiplying the equation $f(w) - \Box w = 0$ by $tw_t + x\cdot\nabla w + w$, we get

$$\partial_t(tQ_0 + w_t w) - div(tP_0) + R_0 = 0 \tag{4.9}$$

where

$$\begin{aligned}
Q_0 &= e[w] + w_t\big(\frac{x}{t}\cdot\nabla w\big),\\
P_0 &= \frac{x}{t}\Big(\frac{w_t^2 - |\nabla w|^2}{2} - F(w)\Big) + \nabla w\big(w_t + \frac{x}{t}\cdot\nabla w + \frac{w}{t}\big),\\
R_0 &= |w|^6 g(|w|) - 4F(w)
\end{aligned}$$

where g is defined by (1.5). Define the horizontal sections of the forward solid cone by

$$D(t) := \{|x| \in \mathbb{R}^3 : |x| \le t\}.$$

Fix $0 < T_1 < T_2$, we integrate (4.9) on $K_{T_1}^{T_2}(0)$. By divergence theorem, we have

$$\begin{aligned}
&\int_{D(T_2)} (T_2 Q_0 + w_t w)dx - \int_{D(T_1)} (T_1 Q_0 + w_t w)dx \\
&\quad - \frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}(0)} (tQ_0 + w_t w + tP_0\frac{x}{|x|})d\sigma + \int_{K_{T_1}^{T_2}(0)} R_0 \, dxdt \\
&\quad =: L_1 + L_2 + L_3 + L_4 = 0.
\end{aligned} \tag{4.10}$$

Now, following the same steps in [2], we define $v(y) := w(|y|, y)$. Since L_3 is the integral on $M_{T_1}^{T_2}(0)$, using spherical coordinates, we obtain that

$$L_3 = -\int_{T_1}^{T_2}\int_{S^2} r\Big(v_r + \frac{v}{r}\Big)^2 r^2 dr d\omega + \frac{1}{2}\int_{S^2} T_2^2 v^2(T_2\omega)d\omega - \frac{1}{2}\int_{S^2} T_1^2 v^2(T_1)\omega d\omega, \tag{4.11}$$

$$\begin{aligned} L_1 = &\int_{D(T_2)} \Big\{T_2\Big(\frac{|w_t|^2}{2} + \frac{1}{2}\Big(w_r + \frac{1}{r}w\Big)^2 + \frac{1}{2r^2}|\nabla_\omega w|^2 + F(w)\Big) \\ &+ r\Big(w_r + \frac{1}{r}w\Big)w_t\Big\}dx - \frac{1}{2}\int_{S^2} T_2^2 v^2(T_2\omega)d\omega \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} L_2 = -&\int_{D(T_1)} \Big\{T_1\Big(\frac{|w_t|^2}{2} + \frac{1}{2}\Big(w_r + \frac{1}{r}w\Big)^2 + \frac{1}{2r^2}|\nabla_\omega w|^2 + F(w)\Big) \\ &+ r\Big(w_r + \frac{1}{r}w\Big)w_t\Big\}dx + \frac{1}{2}\int_{S^2} T_1^2 v^2(T_1\omega)d\omega. \end{aligned} \tag{4.13}$$

Since $L_4 \geq 0$, plugging (4.11), (4.12) and (4.13) into (4.10) , we deduce that

$$T_2\int_{D(T_2)} F(w)dx \leq CT_1E + \int_{T_1}^{T_2}\int_{S^2} T_2\big(v_r + \frac{v}{r}\big)^2 r^2 dr d\omega$$

where C is a constant and E is the energy. Therefore,

$$\int_{D(T_2)} F(w(T_2))dx \leq C\frac{T_1}{T_2}E + \int_{T_1}^{T_2}\int_{S^2} \big(v_r + \frac{v}{r}\big)^2 r^2 dr d\omega. \tag{4.14}$$

For any $T_1 \geq s$, by (4.6), the second term in the RHS of (4.14) is controlled by

$$\begin{aligned} \int_{T_1}^{T_2}\int_{S^2} \big(v_r + \frac{v}{r}\big)^2 r^2 dr d\omega &\lesssim \int_{M_{T_1}^{T_2}(0)} \Big\{\frac{1}{2}\Big|w_t + \frac{x\cdot\nabla w}{|x|}\Big|^2\Big\}d\sigma \\ &\lesssim \frac{1}{\log^{2\gamma} s} \lesssim \frac{1}{\log^{2\gamma}\tau}. \end{aligned}$$

Now, choosing $T_2 = \tau + s$ and $T_1 = \frac{\tau+s}{\log^{2\gamma}\tau} > \sqrt{\tau} = s$, (4.14) implies

$$\int_{D(\tau+s)} F(w(\tau+s,x))dx \lesssim \frac{1}{\log^{2\gamma}\tau}. \tag{4.15}$$

Combining (4.7) and (4.15), the lemma is proved. $\square$

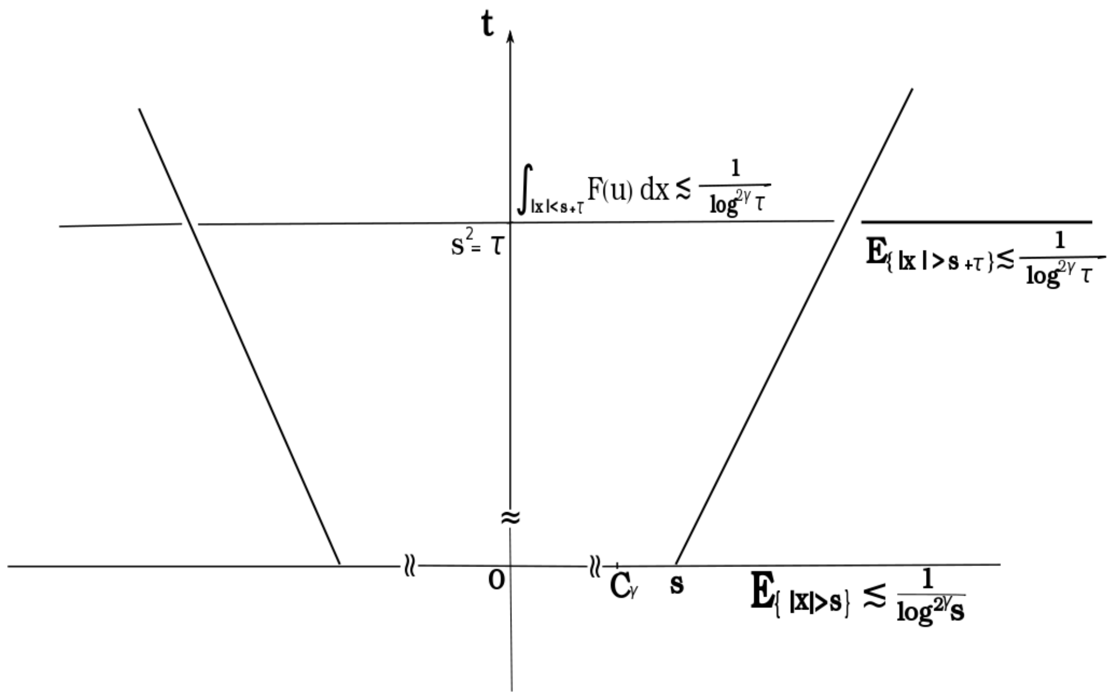

Figure 4.1: Since the initial data decays logarithmically, using finite speed of propagation and similar estimates in [2], we can prove that the potential energy logarithmically decays in time.

Before we prove theorem 4.1, let's observe the following fact. Let I be any time interval with length $3 < |I| < \infty$. By Hölder's inequality, we have that, for $0 < \delta < 2$,

$$\begin{aligned}
&\| |u|^4 u(-\log(|u|)) \|_{L_t^1 L_x^2(I\times\mathbb{R}^3)} \\
&\le \| u^{3-\delta}(-\log(|u|))^{\frac{3-\delta}{6}} \|_{L_t^\infty L_x^{\frac{6}{3-\delta}}(I\times\mathbb{R}^3)} \| u^2 \|_{L_t^{\frac{2}{2-\delta}} L_x^{\frac{6}{\delta}}(I\times\mathbb{R}^3)} \| u^\delta(-\log(|u|))^{\frac{3+\delta}{6}} \|_{L_t^{\frac{2}{\delta}} L_x^\infty(I\times\mathbb{R}^3)} \\
&= \| u(-\log(|u|))^{\frac{1}{6}} \|^{3-\delta}_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \| u \|^2_{L_t^{\frac{4}{2-\delta}} L_x^{\frac{12}{\delta}}(I\times\mathbb{R}^3)} \| u^\delta(-\log(|u|))^{\frac{3+\delta}{6}} \|_{L_t^{\frac{2}{\delta}} L_x^\infty(I\times\mathbb{R}^3)} \\
&\le \| u(-\log(|u|))^{\frac{1}{6}} \|^{3-\delta}_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \| u \|^2_{L_t^{\frac{4}{2-\delta}} L_x^{\frac{12}{\delta}}(I\times\mathbb{R}^3)} \| u^\delta(-\log(|u|))^{\frac{3+\delta}{6}} \|_{L_t^\infty L_x^\infty(I\times\mathbb{R}^3)} |I|^{\frac{\delta}{2}}
\end{aligned}$$

If $|u| \le \frac{1}{3}$, we can estimate that $\| u^\delta(-\log(|u|))^{\frac{3+\delta}{6}} \|_{L_t^\infty L_x^\infty(I\times\mathbb{R}^3)} \lesssim \left(\frac{1}{\delta}\right)^{1/2+\delta/6}$. Let

$\delta = \frac{2}{\log|I|}$, we obtain

$$\| \, |u|^4 u(-\log(|u|)) \, \|_{L_t^1 L_x^2(I\times\mathbb{R}^3)} \lesssim \| \, u(-\log(|u|))^{\frac{1}{6}} \, \|^{3-\delta}_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \| \, u \, \|^2_{L_t^{\frac{4}{2-\delta}} L_x^{\frac{12}{\delta}}(I\times\mathbb{R}^3)} \log^{1/2}|I|. \tag{4.16}$$

To complete the proof of theorem 4.1, by remark 3.7, it suffices to show that

$$\| \, f(u) \, \|_{L_t^1 L_x^2([T,\infty)\times\mathbb{R}^3)} < \infty \quad \text{for some } T < \infty.$$

Let $J = (3^i, \infty)$, where i is sufficiently large and to be determined later. Then

$$\begin{aligned} \| \, f(u) \, \|_{L_t^1 L_x^2(J\times\mathbb{R}^3)} &\lesssim \| \, |u|^4 u(-\log(|u|)) \, \|_{L_t^1 L_x^2(A_J)} \\ &\quad + \| \, |u|^4 u \, \|_{L_t^1 L_x^2(B_J)} =: M_1 + M_2. \end{aligned}$$

Since $(2+\delta, \frac{6(2+\delta)}{\delta})$ is an admissible pair satisfying (3.3) for $\sigma = 1$, from Hölder's inequality and lemma 4.2,

$$\begin{aligned} M_2 &\leq \| \, u \, \|^{3-\delta}_{L_t^\infty L_x^6(B_J)} \| \, u \, \|^{2+\delta}_{L_t^{2+\delta} L_x^{\frac{6(2+\delta)}{\delta}}(B_J)} \\ &\lesssim \frac{1}{\big(\log(3^i)\big)^{\frac{3-\delta}{3}\gamma}} \| \, u \, \|^{2+\delta}_{S_1(J)} . \end{aligned} \tag{4.17}$$

On the other hand, define interval J_k by subdividing J according to $J = \cup_{k=1}^{\infty}(3^{2^{k-1}i}, 3^{2^k i}) =: \cup_{k=1}^{\infty} J_k$. Define $\delta_k := \frac{2}{\log|J_k|}$. By (4.16), lemma 4.2 and the fact that the admissible pairs $(\frac{4}{2-\delta_k}, \frac{12}{\delta_k})$ satisfying (3.3) for $\sigma = 1$, we have

$$\begin{aligned} M_1 &\leq \sum_{k=1}^{\infty} \| \, u^5(-\log(|u|)) \, \|_{L_t^1 L_x^2(J_k\times\mathbb{R}^3)} \\ &\lesssim \sum_{k=1}^{\infty} \Big[\frac{1}{(\log 3^{2^{k-1}i})^{\frac{3-\delta_k}{3}\gamma}} (\log 3^{2^k i})^{1/2} \Big] \| \, u \, \|^2_{L_t^{\frac{4}{2-\delta_k}} L_x^{\frac{12}{\delta_k}}(J_k\times\mathbb{R}^3)} \\ &\lesssim \Big[\sum_{k=1}^{\infty} i^{\frac{1}{2}-\frac{3-\delta_k}{3}\gamma} \cdot 2^{(k-1)(\frac{1}{2}-\frac{3-\delta_k}{3})\gamma} \Big] \| \, u \, \|^2_{S_1(J)} . \end{aligned}$$

Since $\gamma > \frac{1}{2}$, we can choose i sufficiently large such that $(\frac{3-\delta_k}{3}\gamma - \frac{1}{2}) > c > 0$ for all k. Hence,

$$M_1 \lesssim i^{-c} \sum_{k=1}^{\infty} 2^{-(k-1)c} \| \, u \, \|^2_{S_1(J)} . \tag{4.18}$$

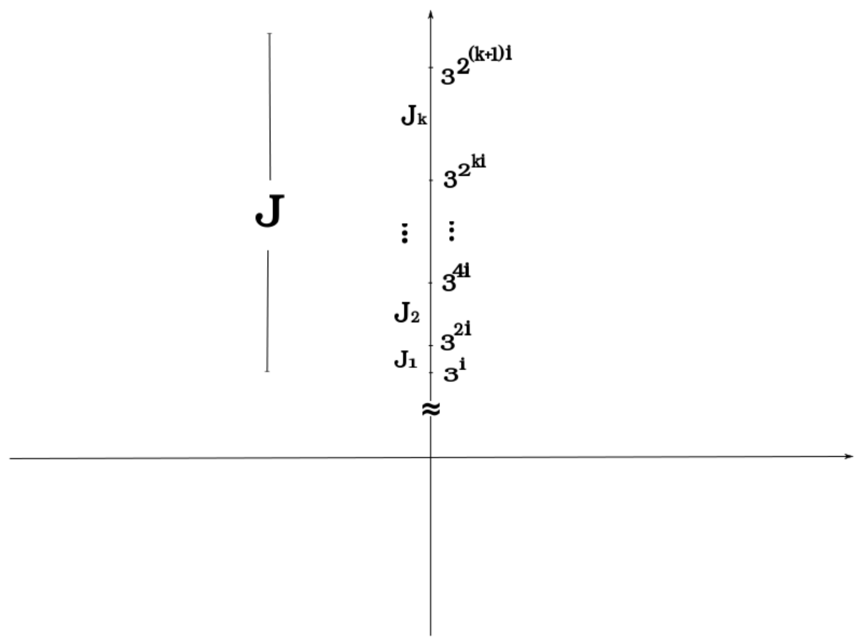

Figure 4.2: Since the potential energy logarithmically decays in time, for i sufficiently large, we can divide the time interval $(3^i, +\infty)$ suitably such that $\| f(u) \|_{L_t^1 L_x^2(J_k)}$ logarithmically decays in time.

Combining (4.17) and (4.18) together, for $\epsilon_0 > 0$ sufficiently small, we can choose i sufficiently large such that

$$\| f(u) \|_{L_t^1 L_x^2(J\times\mathbb{R}^3)} \leq \epsilon_0 \big(\| u \|_{S_1(J)}^2 + \| u \|_{S_1(J)}^{2+\delta} \big).$$

By the Strichartz estimate (3.2), we have

$$\| u \|_{S_1(J)} \leq CE^{1/2} + \epsilon_0 \big(\| u \|_{S_1(J)}^2 + \| u \|_{S_1(J)}^{2+\delta} \big).$$

From a continuity argument, we conclude that

$$\| u \|_{S_1(J)} \leq 2CE^{1/2}.$$

This implies that

$$\| f(u) \|_{L_t^1 L_x^2(J\times\mathbb{R}^3)} < \infty.$$

$\square$

4.2 Spherically Symmetric Initial Data In Lower Regularity Sobolev Spaces

In this section, we consider the solutions to the log-subcritical wave equations with spherically symmetric initial data. If the finite energy initial data are in any lower regularity Sobolev spaces, we obtain the $\dot{H}_x^1 \times L_x^2$ scattering. The spirit of the proof follows from [26] and a slightly sharpened bootstrap argument in lemma 4.5 and 4.6.

Throughout this section, we denote that, for given $\delta > 0$,

$$Z(t) := \| u(t,x) \|_{\dot{H}_x^{1-\delta}(\mathbb{R}^3)} + \| \partial_t u(t,x) \|_{\dot{H}_x^{-\delta}(\mathbb{R}^3)} \tag{4.19}$$

It is easy to show that $Z(t) > 0$ for any time t.[1]

Theorem 4.3. *Let u be the solution to the log-subcritical wave equation* (1.1), (1.4), (1.5) *with spherically symmetric initial data*

$$\begin{cases} u_0(x) \in X_2 \\ u_1(x) \in Y_2 \end{cases} \tag{4.20}$$

where X_2 and Y_2 are defined by (1.10). *Then u scatters in $\dot{H}_x^1 \times L_x^2$.*

The prove theorem 4.3, we need some intermediate lemmas.

Lemma 4.4. *Let $I = [a, b]$ be any interval where $0 \leq a < b \leq \infty$ and u be the solution to the log-subcritical wave equation* (1.1), (1.4), (1.5) *with spherically symmetric initial data*

$$\begin{cases} u(a,x) = u_0(x) \in \dot{H}_x^1 \cap \dot{H}_x^{1-\delta} \\ \partial_t u(a,x) = u_1(x) \in L_x^2 \cap \dot{H}_x^{-\delta} \end{cases}$$

[1] If $Z(t_0) = 0$ for some t_0, it is easy to prove that the solution u has energy $E(t_0) = 0$ and, hence, $E(t) = 0$ for any time t, by energy conservation. This implies the solution $u(t,x) \equiv 0$ for all t.

for some fixed $0<\delta<\frac{1}{2}$. Then there exists $0<\epsilon(\delta)\ll 1$ such that for $0<\epsilon<\epsilon(\delta)$,

$$\begin{aligned}\| u \|_{S_{1-\delta}(I)} \lesssim Z(a) + \Big(\| u \|_{S_{1-\delta}(I)}^{1+\frac{\epsilon}{2\delta}} + \| u \|_{S_{1-\delta}(I)} \Big)\times \\ \Big(\| |u|(-\log(|u|))^{\frac{5}{32}} \|_{L^8_{t,x}(A_I)}^{4-\frac{\epsilon}{2\delta}-\epsilon} + \| u \|_{L^8_{t,x}(B_I)}^4 \Big)\Big(\frac{1}{\epsilon}\Big)^{\frac{7}{16}}\end{aligned} \tag{4.21}$$

where the constant hidden in (4.21) is independent of the interval I and ϵ.

Proof. By the Strichartz estimate (3.2),

$$\| u \|_{S_{1-\delta}(I)} \lesssim Z(a) + \| f(u) \|_{L_t^{\frac{2}{2-\delta}} L_x^{\frac{2}{1+\delta}}\left(I\times\mathbb{R}^3\right)} . \tag{4.22}$$

Consider that

$$\begin{aligned}\| f(u) \|_{L_t^{\frac{2}{2-\delta}} L_x^{\frac{2}{1+\delta}}\left(I\times\mathbb{R}^3\right)} &\lesssim \| -|u|^4u(\log(|u|)) \|_{L_t^{\frac{2}{2-\delta}} L_x^{\frac{2}{1+\delta}}(A_I)} + \| |u|^4u \|_{L_t^{\frac{2}{2-\delta}} L_x^{\frac{2}{1+\delta}}(B_I)} \\ &=: N_1 + N_2\end{aligned}$$

with A_I and B_I as in (4.1). By Hölder's inequality,

$$N_2 \leq \| u \|_{L_t^{\frac{2}{1-\delta}} L_x^{\frac{2}{\delta}}(B_I)} \| u \|_{L^8_{t,x}(B_I)}^4 \leq \| u \|_{S_{1-\delta}(I)} \| u \|_{L^8_{t,x}(B_I)}^4 . \tag{4.23}$$

On the other hand, choose $\epsilon(\delta)$ sufficiently small such that for $0<\epsilon<\epsilon(\delta)$,

$$\begin{aligned} 0<\frac{1}{p} &:= \frac{8\delta+\epsilon-8\delta^2+2\epsilon\delta}{8(2\delta+\epsilon)} \leq \frac{1}{2}, \\ 0<\frac{1}{q} &:= \frac{\delta}{2}+\frac{\epsilon(1-2\delta)}{8(2\delta+\epsilon)} \leq \frac{1}{2} \\ \text{and} \qquad \frac{3}{8} &\approx \frac{12+\frac{5\epsilon}{2\delta}+5\epsilon}{32} < \frac{7}{16}. \end{aligned}$$

It is clear that (p,q) is an admissible pair satisfying (3.3) for $\sigma=1-\delta$. By Hölder's inequality and interpolation theory, we can estimate that

$$\begin{aligned} N_1 &\leq \| |u|^{5-\epsilon}(-\log(|u|))^{\frac{5(4-\frac{\epsilon}{2\delta}-\epsilon)}{32}} \|_{L_t^{\frac{2}{2-\delta}} L_x^{\frac{2}{1+\delta}}(A_I)} \| |u|^{\epsilon}(-\log(|u|))^{\frac{12+\frac{5\epsilon}{2\delta}+5\epsilon}{32}} \|_{L^\infty_{t,x}(A_I)} \\ &\leq \| u \|_{L_t^pL_x^q(A_I)}^{1+\frac{\epsilon}{2\delta}} \| u(-\log(|u|))^{\frac{5}{32}} \|_{L^8_{t,x}(A_I)}^{4-\frac{\epsilon}{2\delta}-\epsilon} \| |u|^{\epsilon}(-\log(|u|))^{\frac{12+\frac{5\epsilon}{2\delta}+5\epsilon}{32}} \|_{L^\infty_{t,x}(A_I)} \end{aligned} \tag{4.24}$$

$$\lesssim \| u \|_{L_t^pL_x^q(A_I)}^{1+\frac{\epsilon}{2\delta}} \| |u|(-\log(|u|))^{\frac{5}{32}} \|_{L^8_{t,x}(A_I)}^{4-\frac{\epsilon}{2\delta}-\epsilon} \left(\frac{1}{\epsilon}\right)^{\frac{12+\frac{5\epsilon}{2\delta}+5\epsilon}{32}} . \tag{4.25}$$

The last factor of (4.25) comes from maximizing the last factor on the right of (4.24) using calculus. We note that the constant hidden in the last inequality is independent of ϵ. By (4.23) and (4.25), we have

$$\begin{aligned}\| f(u) \|_{L_t^{\frac{2}{2-\delta}} L_x^{\frac{2}{1+\delta}}(I\times\mathbb{R}^3)} &\lesssim \| u \|_{S_{1-\delta}(I)}^{1+\frac{\epsilon}{2\delta}} \| \, |u|(-\log(|u|))^{\frac{5}{32}} \|_{L^8_{t,x}(A_I)}^{4-\frac{\epsilon}{2\delta}-\epsilon} \left(\frac{1}{\epsilon}\right)^{\frac{7}{16}} \\ &\quad + \| u \|_{S_{1-\delta}(I)} \| u \|_{L^8_{t,x}(B_I)}^4 .\end{aligned}$$

From (4.22),

$$\begin{aligned}\| u \|_{S_{1-\delta}(I)} &\lesssim Z(a) + \| u \|_{S_{1-\delta}(I)}^{1+\frac{\epsilon}{2\delta}} \| \, |u|(-\log(|u|))^{\frac{5}{32}} \|_{L^8_{t,x}(A_I)}^{4-\frac{\epsilon}{2\delta}-\epsilon} \left(\frac{1}{\epsilon}\right)^{\frac{7}{16}} \\ &\quad + \| u \|_{S_{1-\delta}(I)} \| u \|_{L^8_{t,x}(B_I)}^4 \\ &\lesssim \text{(RHS) of } (4.21)\end{aligned}$$

We can observe that all constants hidden in the above inequalities are independent of the interval I and ϵ. Hence, lemma 4.4 is proved. $\square$

Lemma 4.5. *(Continuity argument) Let $I(=[a,b])$ and u satisfy the assumptions of lemma 4.4, C be the constant hidden in (4.21) and $0 < \epsilon(\delta)$ be chosen in lemma 4.4. Let $\epsilon_0 = \frac{1}{100C}$ and $0 < \epsilon < \epsilon(\delta)$ such that $Z(a)^{\frac{\epsilon}{2\delta}} \geq \frac{1}{2}$ and $(2C)^{\frac{\epsilon}{2\delta}} \leq 2$. We define*

$$Q(I) := \Big(\| \, |u|(-\log(|u|))^{\frac{5}{32}} \|_{L^8_{t,x}(A_I)}^{4-\frac{\epsilon}{2\delta}-\epsilon} + \| u \|_{L^8_{t,x}(B_I)}^{4-\frac{\epsilon}{2\delta}-\epsilon} \Big).$$

If $\| u \|_{L^8_{t,x}(B_I)} \leq 1$ and $Q(I) \leq \epsilon_0 \Big(\frac{\epsilon^{\frac{7}{16}}}{Z(a)^{\frac{\epsilon}{2\delta}}} \Big)$, then we have

$$\| u \|_{S_{1-\delta}(I)} \leq 2CZ(a).$$

Proof. We will prove this lemma by contradiction. For $0 \leq t \leq b-a$, from Dominated Convergence Theorem, we have that the function $\Phi(t) := \| u \|_{S_{1-\delta}([a,a+t])}$ is nondecreasing and continuous in $[0, b-a]$ and $\Phi(0) = 0$. By the hypothesis and (4.21), we have

$$\Phi(t) \leq CZ(a) + \frac{1}{100}\Big(\Phi(t)^{1+\frac{\epsilon}{2\delta}} + \Phi(t)\Big)\Big(\frac{1}{Z(a)^{\frac{\epsilon}{2\delta}}}\Big) \tag{4.26}$$

for all $t \in [0, b-a]$. Assume for contradiction that there exists $t_0 \in [0, b-a)$ such that $\Phi(t_0) = 2CZ(a)$. If $2CZ(a) < 1$, (4.26) implies that

$$2CZ(a) = \Phi(t_0) \leq CZ(a) + \frac{1}{50}(2CZ(a))\Big(\frac{1}{Z(a)^{\frac{\epsilon}{2\delta}}}\Big) \leq \frac{11}{10}CZ(a)$$

On the other hand, if $2CZ(a) \geq 1$, (4.26) implies that

$$2CZ(a) = \Phi(t_0) \leq CZ(a) + \frac{1}{50}(2CZ(a))^{1+\frac{\epsilon}{2\delta}}\Big(\frac{1}{Z(a)^{\frac{\epsilon}{2\delta}}}\Big) \leq \frac{11}{10}CZ(a)$$

We get contradictions in both of the above situations and the lemma is proved. $\square$

Lemma 4.6. *(Finite division) Let $I(=[a,b])$ and u satisfy the assumptions of lemma 4.4 and C be the constant hidden in (4.21). We denote $Z_i = (2C)^i Z(a)$ where $i = 0, 1, 2, \cdots$. For any $\epsilon_0 > 0$, we can choose $\epsilon \ll 1$ and finitely many numbers $a = T_0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = b$, where $N = N(\epsilon_0, \epsilon, \delta, E, Z_0, C)$, such that for $I_j := [T_j, T_{j+1}]$,*

$$Q(I_j) = \epsilon_0\Big(\frac{\epsilon^{\frac{7}{16}}}{Z_j^{\frac{\epsilon}{2\delta}}}\Big) \tag{4.27}$$

for $0 \leq j \leq N-1$ and $Q(I_N) \leq \epsilon_0\Big(\frac{\epsilon^{\frac{7}{16}}}{Z_N^{\frac{\epsilon}{2\delta}}}\Big)$.

Proof. We observe that

$$\begin{aligned}\sum_{i=0}^{\infty}\Big[\epsilon_0\Big(\frac{\epsilon^{\frac{7}{16}}}{Z_i^{\frac{\epsilon}{2\delta}}}\Big)\Big]^{\frac{8}{4-\frac{\epsilon}{2\delta}-\epsilon}} &\gtrsim_{\epsilon_0, Z_0} \Big\{\epsilon^{\frac{7}{8-\frac{\epsilon}{\delta}-2\epsilon}}\sum_{i=0}^{\infty}\frac{1}{(2C)^{\frac{8i\epsilon}{8\delta-\epsilon-2\delta\epsilon}}}\Big\} \\ &\longrightarrow \infty \quad \text{as } \epsilon \to 0.\end{aligned}$$

Therefore, by (3.15), we can choose ϵ sufficiently small such that

$$3\Big(\int\int_A |u|^8(-\log(|u|))^{\frac{5}{4}}dxdt + \int\int_B |u|^8 dxdt\Big) < \sum_{i=0}^{K}\Big[\epsilon_0\Big(\frac{\epsilon^{\frac{7}{16}}}{Z_i^{\frac{\epsilon}{2\delta}}}\Big)\Big]^{\frac{8}{4-\frac{\epsilon}{2\delta}-\epsilon}} \tag{4.28}$$

for some $K = K(\epsilon_0, \epsilon, \delta, E, Z_0, C)$.

Fix this ϵ, if $Q(I) < \epsilon_0\Big(\frac{\epsilon^{\frac{7}{16}}}{Z_0^{\frac{\epsilon}{2\delta}}}\Big)$, then we say $T_1 = b$ and the lemma is proved. Otherwise, we can choose $0 < T_1 < b$ such that (4.27) holds for $j = 0$. Again, if $Q([T_1, b) < \epsilon_0\Big(\frac{\epsilon^{\frac{7}{16}}}{Z_1^{\frac{\epsilon}{2\delta}}}\Big)$, then we say $T_2 = b$. Otherwise, we can choose $T_1 < T_2 < b$ such that (4.27) holds for $j = 1$. By continuing this process, we can choose $a < T_1 < T_2 < \cdots$ such that (4.27) holds for $j = 0, 1, \cdots$. It suffices to show that this process will stop in at most $K+1$ steps. Indeed, assume that there are more than $K+1$ subintervals satisfying (4.27). Since

$$Q(I_j)^{\frac{8}{4-\frac{\epsilon}{2\delta}-\epsilon}} \leq 3\Big(\int\int_{A_{I_j}} |u|^8(-\log(|u|))^{\frac{5}{4}}dxdt + \int\int_{B_{I_j}} |u|^8 dxdt\Big),$$

for $j = 0, 1, \cdots$, by our construction of I_j, we have

$$\begin{aligned}
\sum_{j=0}^{K+1} \epsilon_0 \Big(\frac{\epsilon^{\frac{7}{16}}}{Z_j^{\frac{\epsilon}{2\delta}}} \Big) &= \sum_{j=0}^{K+1} Q(I_j)^{\frac{8}{4-\frac{\epsilon}{2\delta}-\epsilon}} \\
&\leq \sum_{i=0}^{K+1} 3 \Big(\int \int_{A_{I_j}} |u|^8 (-\log(|u|))^{\frac{5}{4}} dxdt + \int \int_{B_{I_j}} |u|^8 dxdt \Big) \\
&\leq 3 \Big(\int \int_{A} |u|^8 (-\log(|u|))^{\frac{5}{4}} dxdt + \int \int_{B} |u|^8 dxdt \Big).
\end{aligned}$$

This contradicts (4.28) and the lemma is proved. □

Corollary 4.7. *Let I and u satisfy the assumptions of lemma 4.4 and C be the constant hidden in* (4.21)*. If $\| u \|_{L^8_{t,x}(B_I)} \leq 1$, then $u \in L_{t,x}^{\frac{8}{1+2\delta}}(I \times \mathbb{R}^3)$.*

Proof. Let $\epsilon(\delta)$ be chosen in lemma 4.4 and $0 < \epsilon < \epsilon(\delta)$ satisfy lemma 4.6, $Z(a)^{\frac{\epsilon}{2\delta}} \geq \frac{1}{2}$ and $(2C)^{\frac{\epsilon}{2\delta}} \leq 2$. Let $\{I_j\}_{j=0}^N$ be the subintervals constructed by lemma 4.6 such that (4.27) holds for $0 \leq j \leq N$.

We claim that

$$\| u \|_{S_{1-\delta}(I_j)} \leq 2CZ_j \quad \text{for } 0 \leq j \leq N. \tag{4.29}$$

where $Z_j = (2C)^j Z(a)$. Indeed, by lemma 4.5, (4.29) holds for $j = 0$. Again, if (4.29) holds for $j = k-1$, we have $Z(T_k) \leq \| u \|_{S_{1-\delta}(I_{k-1})} \leq Z_k$. Since $Z_k^{\frac{\epsilon}{2\delta}} \geq Z(a)^{\frac{\epsilon}{2\delta}} \geq \frac{1}{2}$, applying lemma 4.5 on the interval I_k, we obtain (4.29) for $j = k$. By induction on j, the claim is proved and this implies

$$\| u \|_{L_{t,x}^{\frac{8}{1+2\delta}}(I \times \mathbb{R}^3)} \leq \sum_{j=0}^{N+1} \| u \|_{S_{1-\delta}(I_j)} \leq \sum_{j=0}^{N+1} (2C)^j Z_0 < \infty.$$

□

Corollary 4.8. *Let u be the solution to the log-subcritical wave equation* (1.1), (1.4), (1.5) *with spherically symmetric initial data*

$$\begin{cases} u(0,x) = u_0(x) \in \dot{H}_x^1 \cap \dot{H}_x^{1-\delta} \\ \partial_t u(0,x) = u_1(x) \in L_x^2 \cap \dot{H}_x^{-\delta} \end{cases}$$

for some fixed $0 < \delta < \frac{1}{2}$. Then $u \in L_{t,x}^{\frac{8}{1+2\delta}}(\mathbb{R}_+ \times \mathbb{R}^3)$.

Proof. By (3.15), we can choose finitely many numbers $0 = S_0 < S_1 < \cdots < S_{M-1} < S_M = \infty$ such that $\| u \|_{L^8_{t,x}(B_{[S_k,S_{k+1}]})} \leq 1$ for $0 \leq k \leq M$. By corollary 4.7 and energy conservation, we have $(u(S_k,x), \partial_t u(S_k,x)) \in \left(\dot{H}^1_x \cap \dot{H}^{1-\delta}_x\right) \times \left(L^2_x \cap \dot{H}^{-\delta}_x\right)$ and $\| u \|_{L^{\frac{8}{1+2\delta}}_{t,x}([S_k,S_{k+1}]\times\mathbb{R}^3)} < \infty$ for $0 \leq k \leq M$. Hence,

$$\| u \|_{L^{\frac{8}{1+2\delta}}_{t,x}(\mathbb{R}_+\times\mathbb{R}^3)} \leq \sum_{k=0}^{M} \| u \|_{L^{\frac{8}{1+2\delta}}_{t,x}([S_k,S_{k+1}]\times\mathbb{R}^3)} < \infty.$$

□

To finish the proof of theorem 4.3, by remark 3.7, it suffices to show that $\| f(u) \|_{L^1_t L^2_x((T,\infty)\times\mathbb{R}^3)} < \infty$ for some $0 < T < \infty$. Since the initial data satisfy (4.20), we can choose some $0 < \delta < \frac{1}{2}$ such that $u_0 \in \dot{H}^1_x(\mathbb{R}^3) \cap \dot{H}^{1-\delta}_x(\mathbb{R}^3)$ and $u_1 \in L^2_x(\mathbb{R}^3) \cap \dot{H}^{-\delta}_x(\mathbb{R}^3)$. Observe that

$$\begin{aligned}
&\| f(u) \|_{L^1_t L^2_x((T,\infty)\times\mathbb{R}^3)} \\
&\quad \lesssim \| |u|^5(\log(|u|)) \|_{L^1_t L^2_x(A_T)} + \| |u|^5 \|_{L^1_t L^2_x(B_T)} \\
&\quad \lesssim \| u \|^{\frac{4}{1+2\delta}}_{L^{\frac{8}{1+2\delta}}_{t,x}(A_T)} \| u \|_{L^2_t L^\infty_x(A_T)} \| u^{\frac{8\delta}{1+2\delta}}(\log(|u|)) \|_{L^\infty_{t,x}(A_T)} \\
&\qquad + \| u \|^4_{L^8_{t,x}(B_T)} \| u \|_{L^2_t L^\infty_x(B_T)} \\
&\quad \lesssim \| u \|_{L^2_t L^\infty_x((T,\infty)\times\mathbb{R}^3)} \left[\left(\frac{1+2\delta}{8\delta}\right) \| u \|^{\frac{4}{1+2\delta}}_{L^{\frac{8}{1+2\delta}}_{t,x}(A_T)} + \| u \|^4_{L^8_{t,x}(B_T)} \right]
\end{aligned}$$

where $A_T := A \cap \left((T,\infty)\times\mathbb{R}^3\right)$ and $B_T := B \cap \left((T,\infty)\times\mathbb{R}^3\right)$. The last inequality above is from the fact that $|u^{\frac{8\delta}{1+2\delta}}(\log(|u|))| \lesssim \frac{1+2\delta}{8\delta}$ for $|u| \leq \frac{1}{3}$. By corollary 4.8 and (3.15), for sufficiently small $\epsilon > 0$, we can choose $T = T(\epsilon)$ sufficiently large such that $\left(\frac{1+2\delta}{8\delta}\right) \| u \|^{\frac{4}{1+2\delta}}_{L^{\frac{8}{1+2\delta}}_{t,x}(A_T)} + \| u \|^4_{L^8_{t,x}(B_T)} < \epsilon$. Hence, by the Strichartz inequality ([16]),

$$\| u \|_{L^2_t L^\infty_x((T,\infty)\times\mathbb{R}^3)} \leq CE^{1/2} + \epsilon C \| u \|_{L^2_t L^\infty_x((T,\infty)\times\mathbb{R}^3)} .$$

Again for $\epsilon < \frac{1}{2C}$, we have $\| u \|_{L^2_t L^\infty_x((T,\infty)\times\mathbb{R}^3)} < 2CE^{1/2}$ and this implies $\| f(u) \|_{L^1_t L^2_x((T,\infty)\times\mathbb{R}^3)} < \infty$.

4.3 Energy Subcritical NLWs with Specific Spherically Symmetric Initial Data

In the last part of this chapter, we will discuss an observation, for energy subcritical NLW, inspired by the proof of theorem 4.3. For given $0 < \delta < \frac{1}{2}$, let $(u_0, u_1) \in \left(\dot{H}^1_x(\mathbb{R}^3) \cap \dot{H}^{1-\delta}_x(\mathbb{R}^3)\right) \times \left(L^2_x(\mathbb{R}^3) \cap \dot{H}^{-\delta}_x(\mathbb{R}^3)\right)$ be spherically symmetric functions. In this, we consider the energy-subcritical NLW:

$$\begin{cases} \Box u = |u|^{4-\epsilon} u \\ u(0,x) = u_0(x) \\ \partial_t u(0,x) = u_1(x) \end{cases} \tag{4.30}$$

where we will allow ϵ to depend on the given data (u_0, u_1). That is, we find a relation (R) (see definition 4.10) among ϵ, the energy E and $Z(0)$ as in (4.19), the lower regularity norm of the initial data, for which the solution scatters. We remark that relation (R) holds for data large in both the energy and $\dot{H}^{1-\delta}$ norms provided that ϵ is taken sufficiently small (depending on the size of these norms). In [17], scattering was established in $\dot{H}^{1-\delta}$ for the $\dot{H}^{1-\delta}$ critical NLW from small data. Our remarks here are related to that work, e.g. relation (R) quantifies the extent to which large data can be allowed. Also, we will prove scattering in $\dot{H}^1$, rather than $\dot{H}^{1-\delta}$.

In order to prove that u scatters in $\dot{H}^1_x \times L^2_x$, It suffices to show that $\| u^{5-\epsilon} \|_{L^1_t L^2_x([T,\infty)\times\mathbb{R}^3)} < \infty$ for some $T < \infty$. By the Strichartz estimate and Hölder's inequality,

$$\begin{aligned} \| u \|_{L^2_t L^\infty_x\left([T,\infty)\times\mathbb{R}^3\right)} &\le CE^{\frac{1}{2}} + C \| u^{5-\epsilon} \|_{L^1_t L^2_x\left([T,\infty)\times\mathbb{R}^3\right)} \\ &\le CE^{\frac{1}{2}} + C \| u \|_{L^2_t L^\infty_x\left([T,\infty)\times\mathbb{R}^3\right)} \| u \|^{4-\epsilon}_{L^{8-2\epsilon}_{t,x}\left([T,\infty)\times\mathbb{R}^3\right)} \end{aligned}$$

Following similar arguments as in the proof of theorem 4.3, we only need to show that $\| u \|_{L^{8-2\epsilon}_{t,x}\left([T,\infty)\times\mathbb{R}^3\right)} < \infty$ for some $T < \infty$. Let $\epsilon_0(\delta) := \frac{8\delta}{1+2\delta}$ (so that $\dot{H}^{1-\delta}$ is the scale invariant norm for (4.30) with $\epsilon = \epsilon_0(\delta)$). We restrict to the case $0 < \epsilon < \epsilon_0(\delta)$.

In this case, (4.30) is $\dot{H}^{1-\delta}$-supercritical NLW. We denote

$$\gamma_\epsilon = \frac{3\epsilon}{16\delta - \frac{5}{2}\delta\epsilon - \frac{5}{4}\epsilon},$$
$$\kappa_\epsilon = \frac{8 - \frac{5}{4}\epsilon}{4 - \gamma_\epsilon - \epsilon},$$
$$\frac{1}{\alpha_\epsilon} = \frac{1+2\delta}{8} + \frac{3(1-2\delta)}{8(1+\gamma_\epsilon)}$$

and

$$\frac{1}{\beta_\epsilon} = \frac{1+2\delta}{8} - \frac{1-2\delta}{8(1+\gamma_\epsilon)}.$$

Note that

(i) As $\epsilon \to \epsilon_0(\delta)$, then $\gamma_\epsilon \to 4 - \epsilon$ and $\kappa_\epsilon \to \infty$.

(ii) $(\alpha_\epsilon, \beta_\epsilon)$ is an admissible pair satisfying (3.3) for $\sigma = 1 - \delta$.

Remark 4.9. Let u be the spherically symmetric solution to the energy-subcritical NLW (4.30) with energy E. We observe that lemma 3.8 holds for u. Hence, for any interval $I = [a, b]$ where $0 \le a < b \le \infty$, (3.14) implies,

$$\int_I \int_{\mathbb{R}^3} |u(t,x)|^{8-\frac{5\epsilon}{4}} dxdt \le C_1 E^{3/2} \tag{4.31}$$

where we can choose the constant C_1 which is independent of ϵ. Moreover, by the Strichartz estimate,

$$\| u \|_{S_{1-\delta}(I)} \le CZ(a) + C \| u^{5-\epsilon} \|_{L_t^{\frac{2}{2-\delta}} L_x^{\frac{2}{1+\delta}}(I\times\mathbb{R}^3)}. \tag{4.32}$$

Definition 4.10. *Given $0 < \delta < \frac{1}{2}$, let $0 < \epsilon < \epsilon_0(\delta)$, u be the solution to (4.30) with energy E and lower regularity norm $Z(0) > 0$. We say that the triple $(E, Z(0), \epsilon)$ satisfies the relation (R) if*

$$C_1 E^{3/2} \le \Big(\frac{1}{2(2C)^{1+\gamma_\epsilon} Z(0)^{\gamma_\epsilon}}\Big)^{\kappa_\epsilon} \frac{1}{1 - (2C)^{-\gamma_\epsilon \kappa_\epsilon}}$$

Lemma 4.11. *Given $0 < \delta < \frac{1}{2}$ and $0 < \epsilon < \epsilon_0(\delta)$. Let $(u_0, u_1) \in \big(\dot{H}_x^1(\mathbb{R}^3) \cap \dot{H}_x^{1-\delta}(\mathbb{R}^3)\big) \times \big(L_x^2(\mathbb{R}^3) \cap \dot{H}_x^{-\delta}(\mathbb{R}^3)\big)$ be spherically symmetric functions such that the triple $(E, Z(0), \epsilon)$ satisfies (R) and u be the solution to (4.30), then $u \in L_{t,x}^{\frac{8}{1+2\delta}}(\mathbb{R}_+ \times \mathbb{R}^3)$.*

Proof. Since $(E, Z(0), \epsilon)$ satisfies (R), by (4.31) and an argument similar to that in proof of lemma 4.6, we can choose finitely many numbers $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = \infty$ such that

$$\| u \|^{4-\gamma_\epsilon-\epsilon}_{L^{8-\frac{5\epsilon}{4}}_{t,x}([T_i,T_{i+1}]\times\mathbb{R}^3)} = \frac{1}{2(2C)^{1+\gamma_\epsilon}\big((2C)^i Z(0)\big)^{\gamma_\epsilon}} \tag{4.33}$$

for $0 \leq i \leq N-1$ and $\| u \|^{4-\gamma_\epsilon-\epsilon}_{L^{8-\frac{5\epsilon}{4}}_{t,x}([T_N,T_{N+1}]\times\mathbb{R}^3)} \leq \frac{1}{2(2C)^{1+\gamma_\epsilon}\big((2C)^N Z(0)\big)^{\gamma_\epsilon}}$.

We claim that

$$Z(T_i) < (2C)^i Z(0) \tag{4.34}$$

and

$$\| u \|_{S_{1-\delta}([T_i,T_{i+1}])} < (2C)^{i+1} Z(0) \tag{4.35}$$

for $0 \leq i \leq N$.

Observe that (4.34) is clearly true for $i = 0$ and $Z(T_i) \leq \| u \|_{S_{1-\delta}([T_{i-1},T_i])}$ for $1 \leq i \leq N$. Hence, it suffices to show that (4.35) holds and then (4.34) is automatically true.

A similar proof to that of lemma 4.5 applies here. Assume (4.35) is true for $i \leq j-1$. We aim to prove (4.35) for $i = j$. (Note that (4.34) follows from our assumption when $i = j$.) Let $\phi(t) = \| u \|_{S_{1-\delta}([T_j,T_j+t])}$, then ϕ is continuous and nondecreasing function on $[0, T_{j+1}-T_j]$ and $\phi(0) = 0$. Assume for contradiction that there exists $t_0 \in [0, T_{j+1}-T_j]$ such that $\phi(t_0) = (2C)^{j+1}Z(0)$. By Hölder's inequality, (4.32), (4.33) and (4.34), we have

$$\begin{aligned}
(2C)^{j+1}Z(0) = \phi(t_0) &\leq CZ(T_j) + C \| u^{5-\epsilon} \|_{L_t^{\frac{2}{2-\delta}} L_x^{\frac{2}{1+\delta}}([T_j,T_j+t_0]\times\mathbb{R}^3)} \\
&\leq CZ(T_j) + C \| u \|^{1+\gamma_\epsilon}_{L_t^{\alpha_\epsilon} L_x^{\beta_\epsilon}([T_j,T_j+t_0]\times\mathbb{R}^3)} \| u \|^{4-\gamma_\epsilon-\epsilon}_{L^{8-\frac{5\epsilon}{4}}_{t,x}([T_j,T_j+t_0]\times\mathbb{R}^3)} \\
&\leq CZ(T_j) + C \| u \|^{1+\gamma_\epsilon}_{S_{1-\delta}([T_j,T_j+t_0])} \| u \|^{4-\gamma_\epsilon-\epsilon}_{L^{8-\frac{5\epsilon}{4}}_{t,x}([T_j,T_j+t_0]\times\mathbb{R}^3)} \\
&\leq C(2C)^j Z(0) + \frac{1}{4\big[(2C)^{j+1}Z(0)\big]^{\gamma_\epsilon}} \| u \|^{1+\gamma_\epsilon}_{S_{1-\delta}([T_j,T_j+t_0])} \\
&< \frac{1}{2}(2C)^{j+1}Z(0) + \frac{1}{4\big[(2C)^{j+1}Z(0)\big]^{\gamma_\epsilon}} \times \Big[(2C)^{j+1}Z(0)\Big]^{1+\gamma_\epsilon} \\
&= \frac{3}{4}(2C)^{j+1}Z(0)
\end{aligned}$$

The contradiction implies that (4.35) holds for $i = j$. By inductive argument on i, the claim is proved. To finish proving this lemma, we have

$$\| u \|_{L_{t,x}^{\frac{8}{1+2\delta}}(\mathbb{R}_+ \times \mathbb{R}^3)} \leq \sum_{i=0}^{N+1} \| u \|_{S_{1-\delta}([T_i, T_{i+1}])} \leq \sum_{i=0}^{N+1} (2C)^i Z(0) < \infty$$

□

Corollary 4.12. *Let $\delta, \epsilon, u_0, u_1$ and u satisfy the assumptions of lemma 4.11, then u scatters in $\dot{H}_x^1 \times L_x^2$.*

Proof. By the above discussion, it suffices to show $\| u \|_{L_{t,x}^{8-2\epsilon}\left([T,\infty) \times \mathbb{R}^3\right)} < \infty$ for some $T < \infty$. Since $0 < \epsilon < \epsilon_0(\delta)$ is equivalent to $\frac{8}{1+2\delta} < 8 - 2\epsilon$. The proof of $L_{t,x}^{8-2\epsilon}$ spacetime bound is straightforward by (4.31), lemma 4.11 and interpolation theory. □

Chapter 5

Log-supercritical Wave Equations

For spherically symmetric log-supercritical NLWs (1.1), (1.6) with finite energy E, we observe that the potential energy bound provides slightly better pointwise control, (3.16), of the solution than the one from the kinetic energy bound[1] (see [4], [26]). In this section, we consider a slightly more supercritical wave equation than the equation in [26] and prove the same global regularity result by using (3.16).

Theorem 5.1. *Define $\tilde{H}_x^2(\mathbb{R}^3) := \dot{H}_x^1(\mathbb{R}^3) \cap \dot{H}_x^2(\mathbb{R}^3)$. Let $0 < \alpha \leq \frac{4}{3}$ and (u_0, u_1) be smooth, compactly supported and spherically symmetric initial data with energy E, then there exists a global smooth solution to*

$$\begin{cases} \Box u = |u|^4 u \log^\alpha(2 + |u|^2) \\ u(0,x) = u_0(x) \\ \partial_t u(0,x) = u_1(x) \end{cases} \tag{5.1}$$

Furthermore, we have universal bound of $\tilde{H}_x^2 \times H_x^1$ norm , which depends on the energy E and $\tilde{H}_x^2 \times H_x^1$ norm of the initial data, of the solution u; this implies that the solution u scatters in $\tilde{H}_x^2(\mathbb{R}^3) \times H_x^1(\mathbb{R}^3)$.[2]

Remark 5.2. The above theorem was proved in [26] for $\alpha = 1$ and it is easy to get the same result for $\alpha < 1$ from that argument. We take advantage of (3.16) to extend the range of α up to $\frac{4}{3}$. In the remainder of this section, we will essentially follow Tao's

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[1] The kinetic energy bound can only provide $\int_I \int_{\mathbb{R}^3} |u|^8 \log^\alpha \left(2 + |u|^2\right) dxdt \lesssim E^{3/2}$.

[2] *The definition of the $\tilde{H}_x^2 \times H_x^1$ scattering for the solution u is similar as the definition 3.6 and in place of $\dot{H}_x^1 \times L_x^2$- norm by $\tilde{H}_x^2 \times H_x^1$-norm.*

argument to prove theorem 5.1 using (3.16) and sketch the proof of $\tilde{H}_x^2 \times H_x^1$ scattering. We will skip the argument providing an explicit $\tilde{H}_x^2 \times H_x^1$ universal bound here (see the details in [26]).

Remark 5.3. We state a known global continuation result here, which will be used in our proof (see e.g. [23]).

[Classical Existence Theory] *Let* $I = [0, T]$ *and* $u : I \times \mathbb{R}^3 \to \mathbb{C}$ *be the classical solution*[3] *to* (5.1) *provided*

$$\| u \|_{L_t^\infty L_x^\infty([0,T]\times\mathbb{R}^3)} < \infty,$$

then there is $\delta > 0$ *such that one can extend the solution* u *on* $[0, T+\delta] \times \mathbb{R}^3$.

Proof. By time reversal symmetry, it suffices to consider the global existence and scattering theory of u on $\mathbb{R}_+ \times \mathbb{R}^3$.

By Sobolev embedding theorem, for a classical solution u to (5.1) on $[0, T] \times \mathbb{R}^3$, we have

$$\| u \|_{L_t^\infty L_x^\infty([0,T]\times\mathbb{R}^3)} \lesssim \sum_{j=1}^{2} \| \nabla_x^j u \|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^3)} . \tag{5.2}$$

Hence, applying the classical existence theory in remark 5.3, in order to show the global existence, it suffices to prove that for any fixed $0 < T \leq \infty$, we have

$$\sum_{j=1}^{2} \| \nabla_x^j u \|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^3)} < \infty,$$

provided that u is the classical solution to (5.1) on $[0, T] \times \mathbb{R}^3$.

Let $I = [a, b] \subseteq [0, T]$ be any interval. We define

$$M_I := \int_I \int_{\mathbb{R}^3} |u(t,x)|^8 \log^{\frac{5\alpha}{4}}(2 + |u(t,x)|^2)\,dxdt$$

$$N_I := \sum_{j=0}^{1} \| \nabla_x^j u \|_{L_t^2 L_x^\infty(I\times\mathbb{R}^3)} + \| \nabla_{t,x} \nabla_x^j u \|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}$$

$$D_I := \| \nabla_{t,x} u(a) \|_{H_x^1(\mathbb{R}^3)} .$$

In addition, we say $D = \| \nabla_{t,x} u(0) \|_{H_x^1(\mathbb{R}^3)}$.

[3] *We call* u *classical solution to* (1.1) *if* u *solves* (1.1) *and is smooth and compactly supported for each time.*

From the Strichartz inequality, Hölder's inequality and (5.2),

$$
\begin{aligned}
N_I &\le C \parallel \nabla_{t,x} u(a) \parallel_{H^1_x(\mathbb{R}^3)} + C\sum_{j=0}^{1} \parallel \nabla_x^j \big(|u|^4 u \log^{\alpha}(2+|u|^2)\big) \parallel_{L^1_t L^2_x(I\times\mathbb{R}^3)} \\
&\le CD_I + C\sum_{j=0}^{1} \parallel |u|^4 |\nabla_x^j u| \log(2+|u|^2) \parallel_{L^1_t L^2_x(I\times\mathbb{R}^3)} \\
&\le CD_I + C \parallel |u|^4 \log^{\frac{5\alpha}{8}}(2+|u|^2) \parallel_{L^2_t L^2_x(I\times\mathbb{R}^3)} \times \\
&\qquad \left[\sum_{j=0}^{1} \parallel \nabla_x^j u \parallel_{L^2_t L^\infty_x(I\times\mathbb{R}^3)} \parallel \log^{\frac{3\alpha}{8}}(2+|u|^2) \parallel_{L^\infty_t L^\infty_x(I\times\mathbb{R}^3)}\right] \\
&\le CD_I + C \parallel u \log^{\frac{5\alpha}{32}}(2+|u|^2) \parallel^4_{L^8_t L^8_x(I\times\mathbb{R}^3)} \times \\
&\qquad \left[\sum_{j=0}^{1} \parallel \nabla_x^j u \parallel_{L^2_t L^\infty_x(I\times\mathbb{R}^3)} \parallel \log(2+|u|^2) \parallel^{\frac{3\alpha}{8}}_{L^\infty_t L^\infty_x(I\times\mathbb{R}^3)}\right] \\
&\le CD_I + CM_I^{\frac{1}{2}} N_I \log^{\frac{3\alpha}{8}}\left(2 + \parallel u \parallel^2_{L^\infty_t L^2_x(I\times\mathbb{R}^3)}\right) \\
&\le CD_I + CM_I^{\frac{1}{2}} N_I \log^{\frac{1}{2}}\left(2 + N_I^2\right).
\end{aligned}
$$

From the result in [26][4] , for any $\epsilon_0 > 0$,

$$
\sum_{i=0}^{k} \frac{\epsilon_0}{\log(2+(2C)^i D)} \to \infty \quad as\ k \to \infty.
$$

Hence, for any fixed ϵ_0, the finiteness of $M_{[0,T]}$ from (3.16) implies that we can choose finitely many numbers $0 = T_0 < T_1 < \cdots < T_K < T_{K+1} = T$, where K depending on D, E and ϵ_0, such that

$$
M_i := \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} |u(t,x)|^8 \log^{\frac{5\alpha}{4}}(2+|u(t,x)|^2)\,dxdt = \frac{\epsilon_0}{\log(2+(2C)^i D)}
$$

for $0 \le i \le K-1$ and $M_K \le \frac{\epsilon_0}{\log(2+(2C)^K D)}$.

Choosing $\epsilon_0 = \frac{1}{(100C)^2}$, by iteration and continuity arguments, we claim that $N_{[T_i,T_{i+1}]} < (2C)^{i+1} D$ for $0 \le i \le K$.[5] Indeed, assume that this claim is false for some $i = j$. Then there exists $t_0 \in (T_j, T_{j+1})$ such that $N_{[T_j,t_0]} = (2C)^{j+1} D$. We have

$$
(2C)^{j+1} D \le C(2C)^j D + CM_j^{\frac{1}{2}} N_{[T_j,t_0]} \log^{\frac{1}{2}}\left(2 + N^2_{[T_j,t_0]}\right)
$$

[4] Corollary 3.2 in [26]

[5] See the similar arguments in lemma 4.5 and corollary 4.8 or proposition 3.1 in [26].

$$
\begin{aligned}
&\leq \frac{1}{2}(2C)^{j+1}D + \frac{\log^{\frac{1}{2}}\left(2+(2C)^{j+1}D\right)}{100\log^{\frac{1}{2}}\left(2+(2C)^{j}D\right)} \times (2C)^{j+1}D \\
&\leq \frac{3}{4}(2C)^{j+1}D
\end{aligned}
$$

Thus the claim is proved by contradiction. This implies

$$
\sum_{j=1}^{2} \| \nabla_x^j u \|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^3)} \leq N_{[0,T]} \leq \sum_{i=0}^{K} N_{[T_i,T_{i+1}]} < \sum_{i=0}^{K}(2C)^{i+1}D < \infty.
$$

The universal bound only depends on D and E[6] , indicating the global existence.

Now, we sketch the proof of $\tilde{H}_x^2 \times H_x^1$ scattering here. From a similar argument as the one discussed in remark 2, in order to prove $\tilde{H}_x^2(\mathbb{R}^3) \times H_x^1(\mathbb{R}^3)$ scattering, it suffices to show that

$$
\| |u|^4 u \log^\alpha(2+|u|^2) \|_{L_t^1 H_x^1(\mathbb{R}_+\times\mathbb{R}^3)} < \infty. \tag{5.3}
$$

By the above discussion, the universal bound is independent of T. Hence, we have $N_{\mathbb{R}_+} < \infty$. By Hölder's inequality,

$$
\| |u|^4 u \log^\alpha(2+|u|^2) \|_{L_t^1 H_x^1(\mathbb{R}_+\times\mathbb{R}^3)} \lesssim M_{\mathbb{R}_+}^{1/2} N_{\mathbb{R}_+} \log^{1/2}(2+N_{\mathbb{R}_+}^2) < \infty.
$$

Then (5.3) is proved. □

[6] In fact, from corollary 3.2 in [26], we have $N_{\mathbb{R}_+} \lesssim (2+D)^{(2+D)^{O(E)}}$.

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Chapter 6

Appendix

6.1 Proof of Theorem 3.5

Without loss of generality, we may assume that the solution u is continuous and let $I = [0, T]$. It suffices to show that

$$2\pi \int_0^T |u(0,t)|^2 dt + \int_0^T \int_{\mathbb{R}^3} \frac{G(u)}{|x|} dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla_\omega u|^2}{|x|} dxdt \lesssim E. \tag{6.1}$$

Proof. Let $r = |x|$. Compute that

$$\partial_t \Big(2r^{-1} Re\big[(x \cdot \nabla u + u)\bar{u}_t\big]\Big) = 2r^{-1} Re\Big((x \cdot \nabla u_t)\bar{u}_t + |u_t|^2 + (x \cdot \nabla \bar{u})u_{tt} + \bar{u}u_{tt}\Big) \tag{6.2}$$

and consider

$$div \Bigg\{ r^{-1} \Big[-|u_t|^2 x - 2Re\big[(x \cdot \nabla u)\nabla \bar{u}\big] + |\nabla u|^2 x - 2Re\big[u \nabla \bar{u}\big] - r^{-2}|u|^2 x + 2F(u)x \Big] \Bigg\} \tag{6.3}$$

Since

$$
\begin{aligned}
I &= \Big(\frac{|u_t|^2 x}{|x|}\Big) = \frac{2Re\big[(\nabla u_t \cdot x)\bar{u}_t\big] + 2|u_t|^2}{|x|}, \\
II &= div\Big(\frac{2Re\big[(x\cdot\nabla u)\nabla\bar{u}\big]}{|x|}\Big) = \frac{2Re\big[\nabla(x\cdot\nabla u)\cdot\nabla\bar{u} + (x\cdot\nabla\bar{u})\triangle u\big]}{|x|} - \frac{2|x\cdot\nabla u|^2}{|x|^3}, \\
III &= div\Big(\frac{|\nabla u|^2 x}{|x|}\Big) = \frac{2Re\big[\nabla(x\cdot\nabla u)\nabla\bar{u}\big]}{|x|}, \\
IV &= div\Big(\frac{2Re\big[u\nabla\bar{u}\big]}{|x|}\Big) = \frac{2|\nabla u|^2 + 2Re\big[\bar{u}\triangle u\big]}{|x|} - \frac{2Re\big[u(x\cdot\nabla\bar{u})\big]}{|x|^3}, \\
V &= div\Big(\frac{|u|^2 x}{|x|^3}\Big) = \frac{2Re\big[u(\nabla\bar{u}\cdot x)\big]}{|x|^3}
\end{aligned}
$$

and

$$
VI = div\Big(\frac{2F(u)x}{|x|}\Big) = \frac{4Re\big[F_{\bar{z}}(u)(\nabla\bar{u}\cdot x)\big] + 4F(u)}{|x|},
$$

we have

$$
\begin{aligned}
(6.3) &= -I - II + III - IV - V + VI \\
&= 2r^{-1}\Bigg[- |u_t|^2 - Re\big[\bar{u}_t(x\cdot\nabla u_t)\big] - Re\big[(x\cdot\nabla\bar{u})\triangle u\big] - Re\big[\bar{u}\triangle u\big] - |\nabla u|^2 \\
&\quad + 2Re\big[F_{\bar{z}}(u)(\nabla\bar{u}\cdot x)\big] + 2F(u) + \frac{|x\cdot\nabla u|^2}{|x|^2}\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_t\Big(r^{-1}2Re\big[(x\cdot\nabla u + u)\bar{u}_t\big]\Big) \\
&\qquad + div\Big\{r^{-1}\Big[- |u_t|^2 x - 2Re\big[(x\cdot\nabla u)\nabla\bar{u}\big] + |\nabla u|^2 x - 2Re\big[u\nabla\bar{u}\big] - r^{-2}|u|^2 x + 2F(u)x\Big]\Big\} \\
&= \frac{2Re\big[(\nabla\bar{u}\cdot x + \bar{u})(-\triangle u + u_{tt} + 2F_{\bar{z}}(u))\big]}{|x|} + \frac{2}{|x|}\Big[\frac{|x\cdot\nabla u|^2}{|x|^2} - |\nabla u|^2 + 2F(u) - 2Re\big[\bar{u}F_{\bar{z}}(u)\big]\Big] \\
&= \frac{2}{|x|}\Big[\frac{|x\cdot\nabla u|^2}{|x|^2} - |\nabla u|^2 + 2F(u) - 2Re\big[\bar{u}F_{\bar{z}}(u)\big]\Big] \qquad by\ (1.1).
\end{aligned}
$$

By divergence theorem, integrating the above equality on the exterior to the cylinder

$[0,T] \times B(0,\delta)$, we have

$$\Big[\int_{\mathbb{R}^3\setminus B(0,\delta)} 2r^{-1} Re\big[(x\cdot\nabla u + u)\bar{u}_t\big]dx\Big]_0^T$$
$$+\frac{1}{\delta}\int_0^T\int_{\partial B(0,\delta)} \frac{|u_t|^2|x|^2 + 2|x\cdot\nabla u|^2 - |\nabla u|^2|x|^2 + 2Re\big[u(x\cdot\nabla\bar{u})\big] + |u|^2 - 2F(u)|x|^2}{|x|} dS(x)dt$$
$$= \int_0^T\int_{\mathbb{R}^3\setminus B(0,\delta)} \frac{2}{|x|}\Big[\frac{|x\cdot\nabla u|^2}{|x|^2} - |\nabla u|^2 + 2F(u) - 2Re\big[\bar{u}F_{\bar{z}}(u)\big]\Big]dx.$$

Since $\frac{(x\cdot\nabla u)}{|x|} = u_r$, we can simplify the above equation

$$\Big[\int_{\mathbb{R}^3\setminus B(0,\delta)} 2Re\big[(u_r + \frac{u}{|x|})\bar{u}_t\big]dx\Big]_0^T$$
$$+\int_0^T\int_{\partial B(0,\delta)} |u_t|^2 + 2|u_r|^2 - |\nabla u|^2 + \frac{2Re\big[u\bar{u}_r\big]}{|x|} + \frac{|u|^2}{|x|^2} - 2F(u)dS(x)dt$$
$$= \int_0^T\int_{\mathbb{R}^3\setminus B(0,\delta)} \frac{2}{|x|}\Big[(|u_r|^2 - |\nabla u|^2 + 2F(u) - 2Re\big[\bar{u}F_{\bar{z}}(u)\big]\Big]dxdt \tag{6.4}$$

Consider the second integral of the LHS above, by energy conservation and Hardy's inequality,

$$\int_0^T\int_0^\infty\int_{\partial B(0,\delta)} |u_t|^2 + 2|u_r|^2 - |\nabla u|^2 + \frac{2Re\big[u\bar{u}_r\big]}{|x|} - 2F(u)dS(x)d\delta dt \lesssim E.$$

By the assumption that u is continuous, we have that

$$\lim_{\delta\to 0}\int_{\partial B(0,\delta)} \frac{|u|^2}{|x|^2} dS(x) = 4\pi u(t,0).$$

Hence, as $\delta \to 0$, (6.4) becomes

$$2\pi\int_0^T u^2(t,0)dt + \int_0^T\int_{\mathbb{R}^3} \frac{|\nabla u|^2 - |u_r|^2 + (2F(u) - 2Re\big[\bar{u}F_{\bar{z}}(u)\big])}{|x|}dxdt = -\Big[\int_{\mathbb{R}^3} Re\big[(u_r + \frac{u}{|x|})\bar{u}_t dx\big]\Big]_0^T$$

By Hölder's inequality, Hardy's inequality and Energy conservation, the RHS of above equation is less than CE.

□

6.2 Proof of (4.5)

Since $(u_0, u_1) \in X_1 \times Y_1$, defined by (1.9), we have

$$\| u_0 \|_{X_1}^2 \geq \int_{\mathbb{R}^3} |\nabla u_0|^2 \log^{2\gamma}(1+|x|)dx \gtrsim (\log^{2\gamma} s)\int_{|x|>s} |\nabla u_0|^2 dx.$$

Hence,

$$\int_{|x|>s} |\nabla u_0|^2 dx \lesssim \frac{\| u_0 \|_{X_1}^2}{\log^{2\gamma} s}. \tag{6.5}$$

Similarly,

$$\int_{|x|>s} |u_1|^2 dx \lesssim \frac{\| u_1 \|_{Y_1}^2}{\log^{2\gamma} s}. \tag{6.6}$$

Now, consider

$$\begin{aligned}\int_{|x|>s} F(u_0(x))dx &= \int_{\{|x|>s\}\cap\{|u_0|<1/3\}} F(u_0(x))dx + \int_{\{|x|>s\}\cap\{|u_0|\geq 1/3\}} F(u_0(x))dx \\ &\lesssim \int_{\{|x|>s\}\cap\{|u_0|<1/3\}} |u_0|^6(-\log(|u_0|))dx + \int_{\{|x|>s\}\cap\{|u_0|\geq 1/3\}} |u_0|^6 dx \\ &=: I + II.\end{aligned}$$

Let

$$\begin{aligned}I = &\int_{\{|x|>s\}\cap\{|u_0(x)|<\frac{1}{|x|^{2/3}}\}} |u_0|^6(-\log(|u_0|))dx + \int_{\{|x|>s\}\cap\{\frac{1}{|x|^{2/3}}\leq |u_0(x)|\leq \frac{1}{3}\}} |u_0|^6(-\log(|u_0|))dx \\ &=: I_1 + I_2.\end{aligned}$$

When s is sufficiently large,

$$\begin{aligned}I_1 &\lesssim \int_{\{|x|>s\}\cap\{|u_0|<\frac{1}{|x|^{2/3}}\}} |u_0|^{\frac{11}{2}}\big(\sup_{|u_0|<s^{-2/3}} |u_0|^{\frac{1}{2}}(-\log|u_0|)\big)dx \\ &\lesssim \int_{|x|>s} |x|^{-11/3}dx \lesssim s^{-2/3} \lesssim \frac{1}{\log^{2\gamma} s}.\end{aligned} \tag{6.7}$$

Now, we aim to prove

$$I_2 + II \lesssim \frac{1}{\log^{2\gamma} s} \quad \text{for } s \text{ sufficiently large.}$$

For $\alpha \in \mathbb{R}$, define $Q(\alpha) := \int_{\mathbb{R}^3} \left|\frac{u_0 \log^{\alpha}(2+|x|)}{2+|x|}\right|^2 dx$. We claim that, for $\alpha \leq \gamma$,

$$Q(\alpha) \leq C(\| u_0 \|_{X_1}, E, \alpha)$$

where E is the energy. Indeed, if $\alpha \leq 0$, by Hölder's and Hardy's inequality,

$$\begin{aligned} Q(\alpha) &= \int_{|x|<3} \Big| \frac{u_0 \log^{\alpha}(2+|x|)}{2+|x|} \Big|^2 dx + \int_{|x|\geq 3} \Big| \frac{u_0 \log^{\alpha}(2+|x|)}{2+|x|} \Big|^2 dx \\ &\lesssim_\alpha \int_{|x|<3} |u_0|^2 dx + \int_{\mathbb{R}^3} \Big| \frac{u_0}{|x|} \Big|^2 dx \\ &\lesssim \| u_0 \|^2_{\dot{H}^1_x(\mathbb{R}^3)} + \Big(\int_{\mathbb{R}^3} F(u_0) dx \Big)^{\frac{1}{3}} \\ &\leq C(E, \alpha). \end{aligned} \tag{6.8}$$

Again, if $0 < \alpha \leq \gamma$,

$$\begin{aligned} Q(\alpha) &= \int_{\mathbb{R}^3} \Big| \frac{u_0 \log^{\alpha}(2+|x|)}{2+|x|} \Big|^2 dx \\ &\lesssim_\alpha \int_{|x|<3} |u_0|^2 dx + \int_{\mathbb{R}^3} \Big| \frac{u_0 \log^{\alpha}(2+|x|)}{|x|} \Big|^2 dx \\ &\lesssim \Big(\int_{|x|<3} |u_0|^6 dx \Big)^{\frac{1}{3}} + \int_{\mathbb{R}^3} \Big| \nabla \big(u_0 \log^{\alpha}(2+|x|) \big) \Big|^2 dx \\ &\lesssim_\alpha \Big(\int_{\mathbb{R}^3} F(u) dx \Big)^{\frac{1}{3}} + \int_{\mathbb{R}^3} \Big| \nabla u_0 \log^{\alpha}(2+|x|) \Big|^2 dx + \int_{\mathbb{R}^3} \Big| \frac{u_0 \log^{\alpha-1}(2+|x|)}{2+|x|} \Big|^2 dx \\ &\lesssim E^{\frac{1}{3}} + \int_{|x|<3} |\nabla u_0|^2 dx + \int_{|x|\geq 3} \Big| \nabla u_0 \log^{\gamma}(1+|x|) \Big|^2 dx + Q(\alpha-1) \\ &\lesssim E^{\frac{1}{3}} + E + \| u_0 \|_{X_1} + Q(\alpha-1). \end{aligned} \tag{6.9}$$

By inductive argument and (6.8), the claim is proved.

Fix $s \gg 1$. Let χ be the smooth radial function which equals 1 on $\{|x| > s\}$, 0 on $\{|x| < \frac{s}{2}\}$, $0 \leq \chi \leq 1$ and $|\nabla \chi| \lesssim \frac{1}{s}$. Hence, we have $|\nabla \chi| \lesssim \frac{1}{|x|}$. By Sobolev embedding theorem and Hardy's inequality,

$$\begin{aligned}
\log^{6\gamma} s \int_{|x|>s} |u_0|^6 dx &\le \int_{|x|>s} |u_0|^6 \log^{6\gamma}(|x|) dx \\
&\le \int_{\mathbb{R}^3} \big(\chi |u_0| \log^\gamma(2+|x|)\big)^6 dx \\
&\lesssim \Big[\int_{\mathbb{R}^3} \Big| \nabla\big(\chi u_0 \log^\gamma(2+|x|)\big)\Big|^2 dx \Big]^3 \\
&\lesssim_\gamma \Bigg\{ \int_{\mathbb{R}^3} \Big| \nabla\chi u_0 \log^\gamma(2+|x|)\Big|^2 dx + \int_{\mathbb{R}^3} \Big| \chi \nabla u_0 \log^\gamma(2+|x|)\Big|^2 dx \\
&\qquad\qquad + \int_{\mathbb{R}^3} \Big| \frac{\chi u_0 \log^{\gamma-1}(2+|x|)}{2+|x|} \Big|^2 dx \Bigg\}^3 \\
&=: \big(J_1 + J_2 + J_3\big)^3.
\end{aligned}$$

We can compute that

$$J_2 \lesssim_\gamma \int_{\mathbb{R}^3} \Big| \nabla u_0 \log^\gamma(1+|x|)\Big|^2 dx + \int_{|x|<3} |\nabla u_0|^2 dx \le \| u_0 \|_{X_1}^2 + E$$

and

$$J_3 \lesssim C(\| u_0 \|_{X_1}, E, \gamma) \qquad \text{(by above claim).}$$

Since $\nabla \chi \lesssim \frac{1}{|x|}$,

$$\begin{aligned}
J_1 &\lesssim \int_{|x|>\frac{s}{2}} \Big| \frac{u_0 \log^\gamma(2+|x|)}{|x|} \Big|^2 dx \lesssim \int_{|x|>\frac{s}{2}} \Big| \frac{u_0 \log^\gamma(2+|x|)}{2+|x|} \Big|^2 dx \\
&\lesssim C(\| u_0 \|_{X_1}, E, \gamma).
\end{aligned}$$

Hence, $\log^{6\gamma} s \int_{|x|>s} |u_0|^6 dx \le C(\| u_0 \|_{X_1}, E, \gamma)$ for sufficiently large s. Then we deduce

$$II \le \int_{|x|>s} |u_0|^6 dx \lesssim \frac{1}{\log^{6\gamma} s} \le \frac{1}{\log^{2\gamma} s}. \tag{6.10}$$

Similarly,

$$\begin{aligned}\log^{6\gamma-1} s \int_{\{|x|>s\}\cap\{\frac{1}{|x|^{2/3}}\leq|u_0|\leq\frac{1}{3}\}} & |u_0|^6(-\log|u_0|)dx \\ & \lesssim \log^{6\gamma-1} s \int_{|x|>s} |u_0|^6 \log(|x|)dx \\ & \lesssim \int_{|x|>s} |u_0|^6 \log^{6\gamma}(|x|)dx \\ & \lesssim C(\| u_0 \|_{X_1}, E, \gamma).\end{aligned}$$

Therefore,

$$I_2 \lesssim \frac{1}{\log^{6\gamma-1} s} \leq \frac{1}{\log^{2\gamma} s}. \tag{6.11}$$

Combining (6.5), (6.6), (6.7), (6.10) and (6.11), we obtain (4.5).

6.3 Proof of the nonconcentation of the potential energy for log-subcritical NLW

We recall the energy log-subcritical NLW (1.1) here.

$$\begin{cases} \Box u := -\partial_t^2 u + \triangle u = f(u) & \text{on } \mathbb{R}\times\mathbb{R}^3 \\ u(t_0,x) = u_0(x) \\ \partial_t u(t_0,x) = u_1(x) \end{cases}$$

where f is defined by

$$f(z) := \begin{cases} |z|^4 z g(|z|) & ,|z| \neq 0 \\ 0 & ,|z| = 0 \end{cases}$$

and $g : (0,\infty) \to \mathbb{R}$ is smooth and nonincreasing as well as satisfies

$$g(x) := \begin{cases} -\log(x) & ,0 < x < \frac{1}{3} \\ \sim 1 & ,\frac{1}{3} \leq x < 1 \\ 1 & ,x \geq 1. \end{cases}$$

The potential energy is

$$F(u) = F_{sub}(u) = \begin{cases} -\frac{1}{6}|u|^6\big(\log(|u|) - \frac{1}{6}\big) & ,0 < |u| < \frac{1}{3} \\ \sim \frac{1}{6}|u|^6 & ,\frac{1}{3} \leq |u| < 1 \\ \frac{1}{6}|u|^6 & ,|u| \geq 1 \end{cases}$$

Without loss of generality, we only need to consider the classical solution here.

Proposition 6.1. *Let u be a classical solution to* (1.1), (1.4) *and* (1.5). *Let* $[0,T)$ *be the maximal lifespan of u. Then, for any $x_0 \in \mathbb{R}^3$,*

$$\limsup_{t \nearrow T} \int_{|x-x_0|<T-t} F(u)dx = 0$$

Proof. To simplify the notation, it is convenient to shift (T, x_0) to $(0,0)$. Hence, we hope to show that

$$\limsup_{t \nearrow 0} \int_{|x|<|t|} F(u)dx = 0 \tag{6.12}$$

We adopt the notations in (4.4) and (4.9) here.

$$\partial_t(tQ_0 + u_t u) - div(tP_0) + R_0 = 0 \tag{6.13}$$

where

$$\begin{aligned}
Q_0 &= e[u] + u_t\big(\frac{x}{t} \cdot \nabla u\big), \\
P_0 &= \frac{x}{t}\Big(\frac{u_t^2 - |\nabla u|^2}{2} - F(u)\Big) + \nabla u\big(u_t + \frac{x}{t} \cdot \nabla u + \frac{u}{t}\big), \\
R_0 &= |u|^6 g(|u|) - 4F(u).
\end{aligned}$$

For $T_2 < T_1 < 0$, consider the truncated backward light cone with the apex at $(0,0)$

$$\bar{K}_{T_2}^{T_1} := \{(t,x)|T_2 \le t \le T_1,\ |x| \le |t|\},$$

the boundary of the truncated cone

$$\bar{M}_{T_2}^{T_1} := \partial \bar{K}_{T_2}^{T_1} = \{(t,x)|T_2 \le t \le T_1,\ |x| = |t|\}$$

and the horizontal sections of the backward solid cone

$$\bar{D}(t) := \{(t,x)\big||x| \le |t|\}.$$

It is easy to see that

$$\int_{D(T_2)} e[u](t)dx = \int_{D(T_1)} e[u](t)dx + \text{flux}(u, \bar{M}_{T_2}^{T_1}) \tag{6.14}$$

where $\text{flux}(u, \bar{M}^{T_1}_{T_2}) := \frac{1}{2\sqrt{2}} \int_{\bar{M}^{T_1}_{T_2}} \left|\frac{x}{|x|}\partial_t u - \nabla u\right|^2 + F(u)dx$. We integrate (6.13) on $\bar{K}^{T_1}_{T_2}$. By divergence theorem, we have

$$\begin{aligned} &\int_{\bar{D}(T_1)} (T_1 Q_0 + u u_t u)dx - \int_{\bar{D}(T_2)} (T_2 Q_0 + u_t u)dx \\ &\qquad + \frac{1}{\sqrt{2}} \int_{\bar{M}^{T_1}_{T_2}(0)} (tQ_0 + u_t u + tP_0 \frac{x}{|x|})d\sigma + \int_{\bar{K}^{T_1}_{T_2}(0)} R_0 \ dxdt. \end{aligned}$$

Using the similar estimates (4.10)-(4.14), we obtain

$$|T_2| \int_{\bar{D}(T_2)} F(u)dx \le CT_1 E + |T_2| \int_{T_2}^{T_1} \int_{S^2} \left(v_r + \frac{v}{r}\right)^2 r^2 dr d\omega + \int_{\bar{K}^{T_1}_{T_2}} R_0 \ dxdt$$

where $v(y) := u(-|y|, y)$. Now let $T_1 \nearrow 0$ and divide the above inequality by $|T_2|$ on both sides, we get

$$\int_{\bar{D}(T_2)} F(u)dx \le \int_{T_2}^{0} \int_{S^2} \left(v_r + \frac{v}{r}\right)^2 r^2 dr d\omega + \frac{1}{|T_2|} \int_{\bar{K}^{0}_{T_2}} R_0 \ dxdt. \tag{6.15}$$

It suffices to show that the RHS of (6.15) approaches 0 as $T_2 \nearrow 0$.

We observe the first term on the RHS of (6.15) that

$$\int_{T_2}^{0} \int_{S^2} (v_r + \frac{v}{r})^2 r^2 dr d\omega \le 2\text{flux}(u, \bar{M}^{0}_{T_2}).$$

Since the LHS of (6.14) is bounded by the energy, by Dominated Convergence Theorem,

$$\text{flux}(u, \bar{M}^{0}_{T_2}) \to 0 \quad \text{as} \quad T_2 \nearrow 0.$$

Now, we consider the last term is (6.15). Since u is the classical solution, when $|T_2| \approx 0$, we have

$$\int_{\bar{K}^{0}_{T_2}} R_0 \ dxdt \le CT_2^4.$$

Hence,

$$\begin{aligned} \int_{\bar{D}(T_2)} F(u)dx &\le 2\text{flux}(u, u, \bar{M}^{0}_{T_2}) + CT_2^3 \\ &\to 0 \quad \text{as} \quad T_2 \nearrow 0. \end{aligned}$$

and (6.12) is proved.

$\square$