Correlations and dynamics of 1d cold atoms

integrability in and out of equilibrium

Workshop on Low-dimensional quantum gases out of equilibrium
Minneapolis, 11 May 2012

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R. Konik, A. Shashi, A. Imambekov, ...

Plan of the talk

- Quick review: integrability for dynamics
- Interaction turnoff in Bose gas
- Numerical renormalization using integrability

  1d Bose gas in a trap

- *Interlude: generalized TBA and generalized Gibbs*

  Time-dependent dynamics after release

- Conclusions

Quick review: dynamics from integrability

July 2, 1906 – March 6, 2005

Bethe Ansatz (1931)

Integrable Hamiltonian:

$$H = \int_0^L dx\ \mathcal{H}(x)$$

'Reference state': vacuum, FM state,...

'Particles': atoms, down spins, ...

Exact many-body wavefunctions (in N-particle sector):

$$\Psi_N(\{x\}|\{\lambda\}) = \sum_P (-1)^{[P]} A_P(\{\lambda\}) e^{i x_j k(\lambda_{P_j})}$$

... parametrized by rapidities ...

... with specified relative amplitudes ...

... made up of free waves ...

... and obeying some form of Pauli principle

Imposing boundary conditions quantizes the allowable rapidities according to the Bethe equations

$$\boxed{\theta_{kin}(\lambda_j) + \frac{1}{L}\sum_k \theta_{scat}(\lambda_j - \lambda_k) = \frac{2\pi}{L} I_j}$$

Eigenstates: labeled by set of quantum numbers

Constructing all states in the Hilbert space $\Leftrightarrow$ Obtaining all solutions to the Bethe equations

The general idea, simply stated:

Start with your favourite quantum state
(expressed in terms of Bethe states)

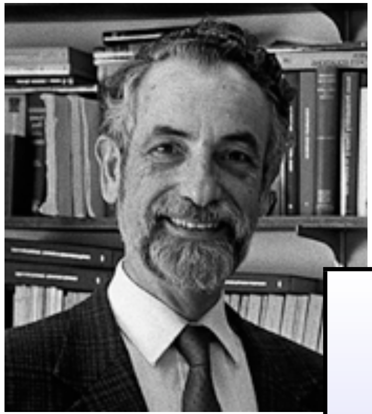
$$\mathcal{O} \rightarrow |\{\lambda\}\rangle$$

Apply some operator on it

Reexpress the result in the basis of Bethe states:

$$\mathcal{O}|\{\lambda\}\rangle = \sum_{\{\mu\}} F^{\mathcal{O}}_{\{\mu\},\{\lambda\}} |\{\mu\}\rangle$$

using 'matrix elements' $F^{\mathcal{O}}_{\{\mu\},\{\lambda\}} = \langle\{\mu\}|\mathcal{O}|\{\lambda\}\rangle$



The Lieb-Liniger model

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < l \leq N} \delta(x_j - x_l)$$

Interaction: repulsive (c > 0) or attractive (c < 0)

Second-quantized form:

$$H_0 = \int_0^L dx \left\{ \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) \right\}$$

Bethe equations

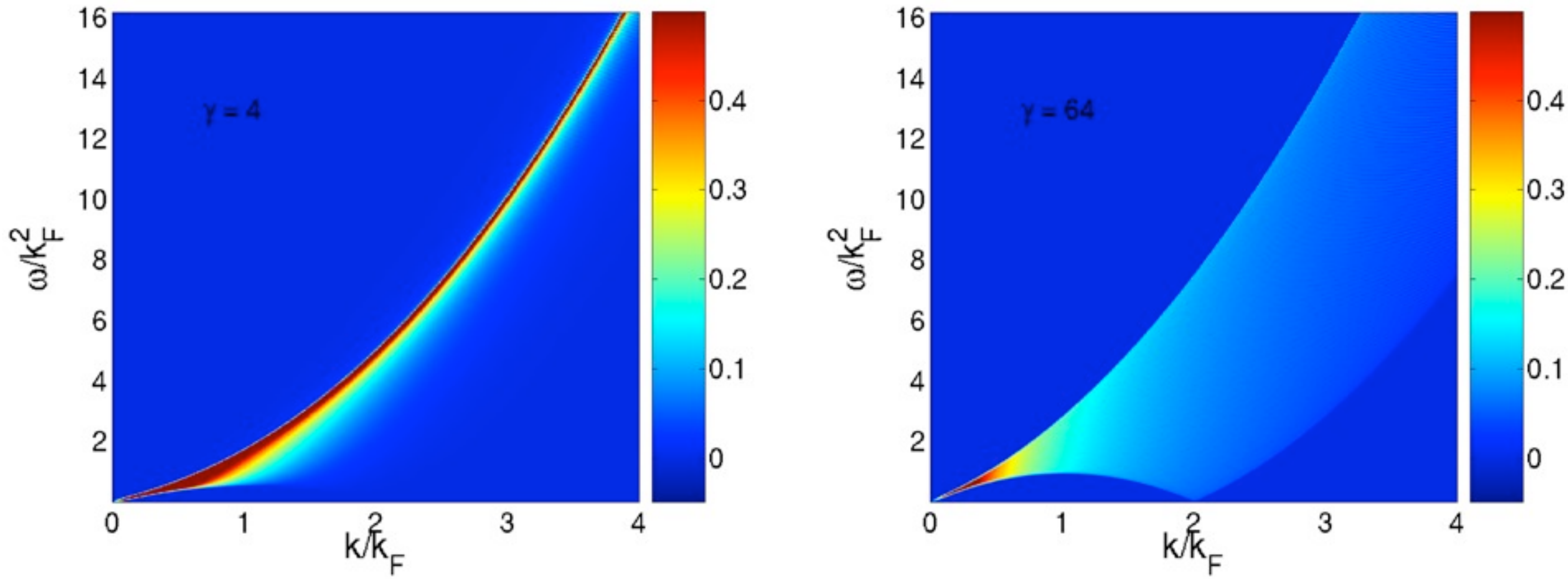
$$e^{i\lambda_j L} = \prod_{l \neq j} \frac{\lambda_j - \lambda_l + ic}{\lambda_j - \lambda_l - ic}, \qquad j = 1, \ldots, N$$

Repulsive Lieb-Liniger gas

Density-density (dynamical SF)

(J-S C & P Calabrese, PRA 2006)

$$S(k,\omega) = \frac{2\pi}{L} \sum_{\alpha} |\langle 0|\rho_k|\alpha\rangle|^2 \delta(\omega - E_\alpha + E_0)$$



Interaction turnoff in Lieb-Liniger

Interaction quench in Lieb-Liniger

J. Mossel and JSC, arxiv:1201.1885

For all $t < 0 : H = H_{c\neq 0}$ For all $t > 0 : H = H_{c=0}$

Simplest time-dependent correlation:

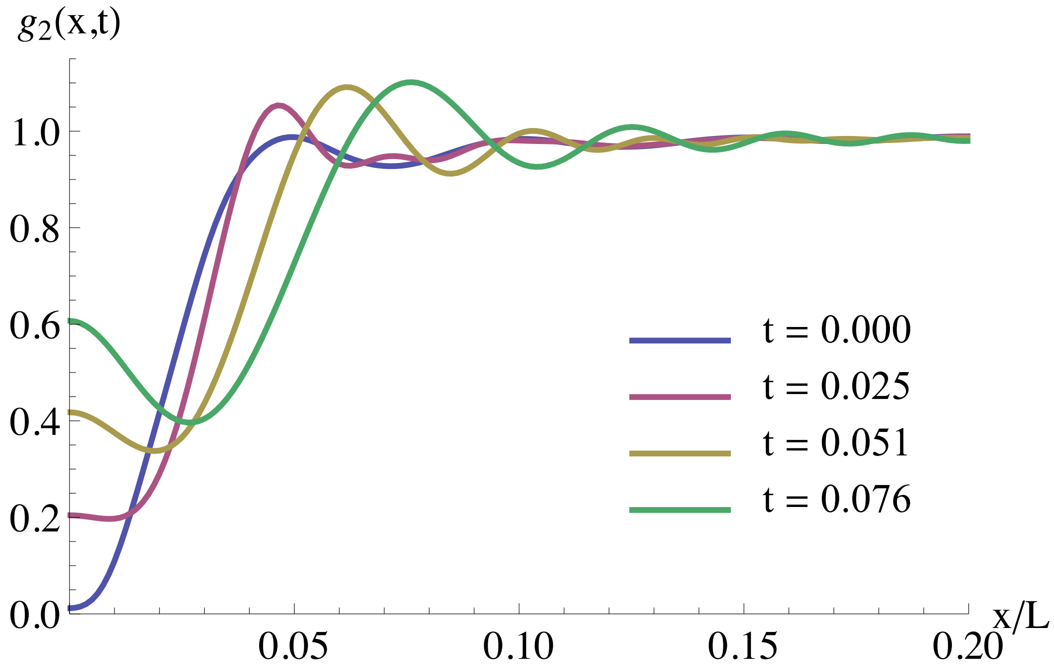
$$\boxed{g_2(x;t) = \frac{\langle\phi(t)|\Psi^\dagger(x)\Psi^\dagger(0)\Psi(x)\Psi(0)|\phi(t)\rangle}{\langle\phi(t)|\Psi^\dagger(0)\Psi(0)|\phi(t)\rangle^2}}$$

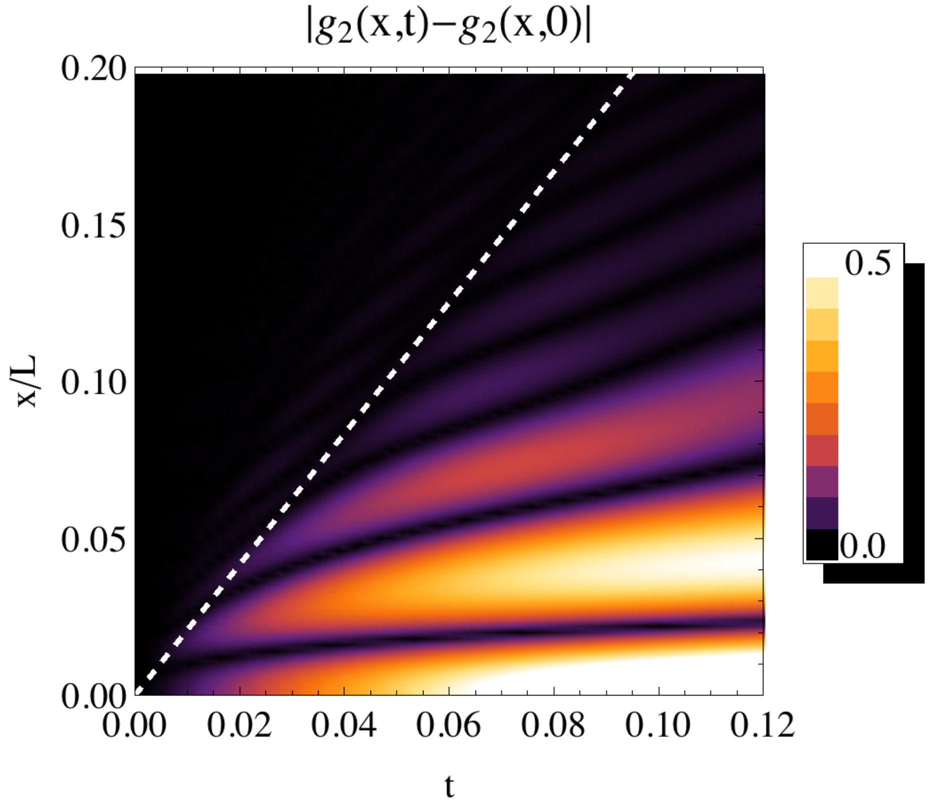
$$\langle\phi(t)|\Psi^\dagger(x)\Psi^\dagger(0)\Psi(x)\Psi(0)|\phi(t)\rangle$$
$$= \frac{1}{L^4}\sum_{k_1,k_2,k_3} e^{ik_{13}(x-2k_{23}t)}\langle\phi|\Psi^\dagger_{k_1}\Psi^\dagger_{k_2}\Psi_{k_3}\Psi_{k_1+k_{23}}|\phi\rangle$$

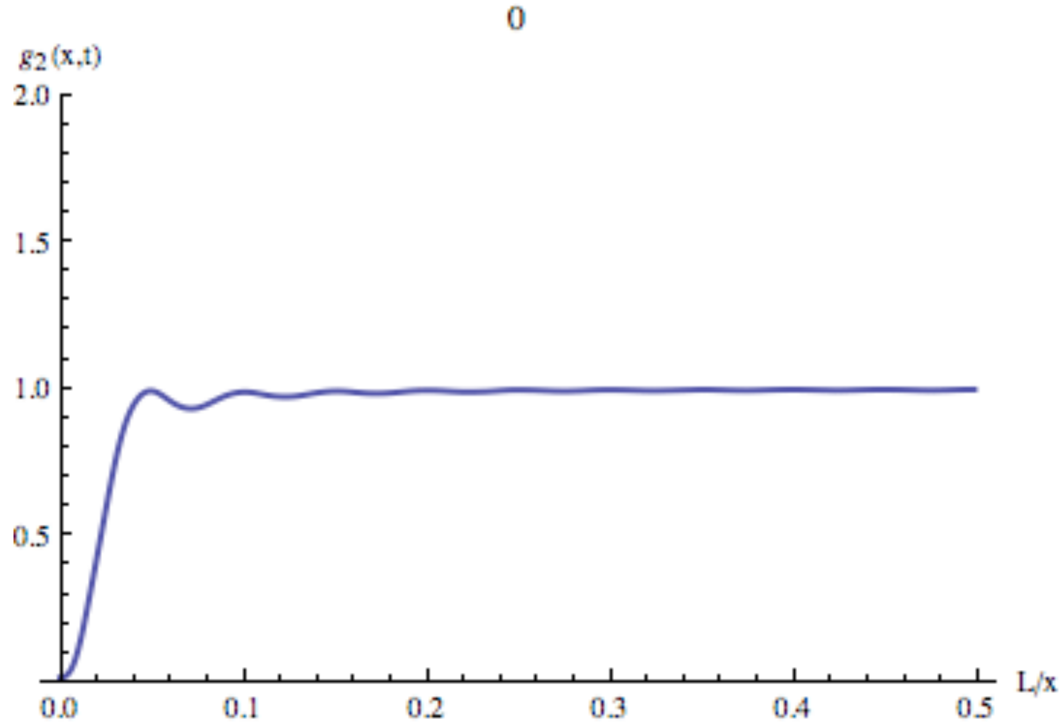
Expectation value: use triple resolution of identity

$$\sum_{n_1,n_2,n_3}\langle\phi|\Psi^\dagger_{k_1}|n_1\rangle\langle n_1|\Psi^\dagger_{k_2}|n_2\rangle\langle n_2|\Psi_{k_3}|n_3\rangle\langle n_3|\Psi_{k_1+k_2-k_3}|\phi\rangle$$

Nonlocal density correlation as a function of time

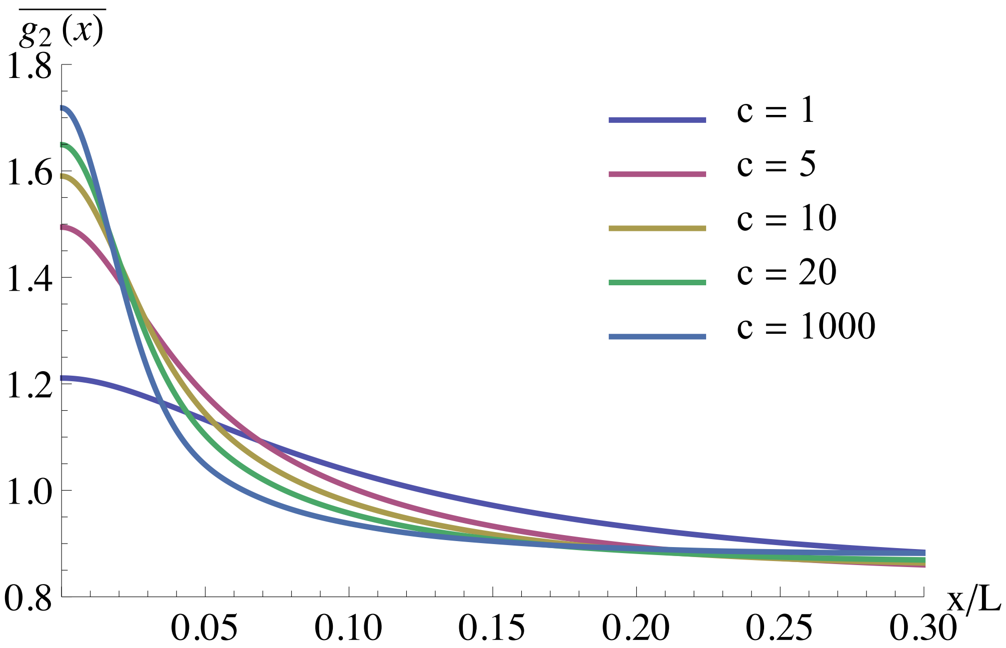


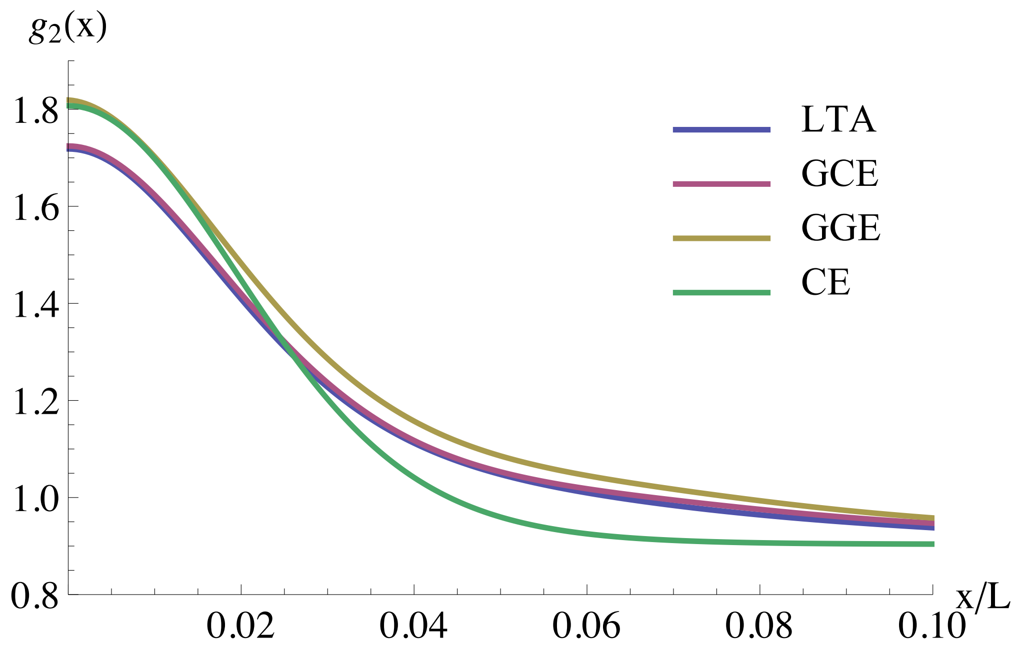




Long-time average of static density correlation

$$\overline{g_2(x;c)} = \lim_{T \to T_{LTA}} \frac{1}{T} \int_0^T g_2(x,t;c)\,dt$$





Compared to various ensembles (c = 1000)

GGE doesn't work here, but GenCanE does (finiteness?)

Numerical renormalization using integrability

Renormalization from integrability
(nonrelativistic BA solvable models)

JSC & R. M. Konik 1203.0901

$$H = H_{int} + H_{pert}$$

Exactly solvable theory

Some perturbation (with operator whose FF can be calculated)

Implementation of a 'fifth way' for RG (supplementing Wilson's original 'four ways')

Numerical renormalization

Summary of procedure:

- Choose a first set of N + dN states
- Compute H and diagonalize it
- Toss away highest-lying dN states
- Include next dN states
- Recompute H and diagonalize
- ...

States basis: usually, free states or from exact diag

Here: exact BA eigenstates

Include states in order of expected importance

Usually: chosen only according to energy

Here: must use a more refined measure, exploiting knowledge of matrix elements

Example: Lieb-Liniger in a trap

Hamiltonian:

$$\boxed{H = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + 2c\sum_{i<j}\delta(x_i - x_j) + \sum_i V(x_i)}$$

with simple harmonic trap $V(x) = \frac{m}{2}\omega^2 x^2$

The perturbation can be written in terms of the density operator

$$H_{pert} = \int_0^L dx\, V(x)\rho(x)$$

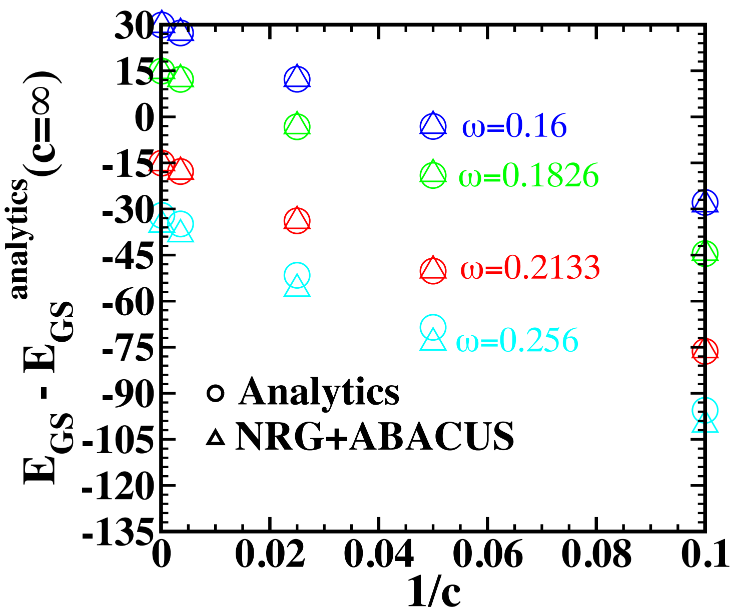
whose matrix element is available as a function of rapidities in bra and ket states

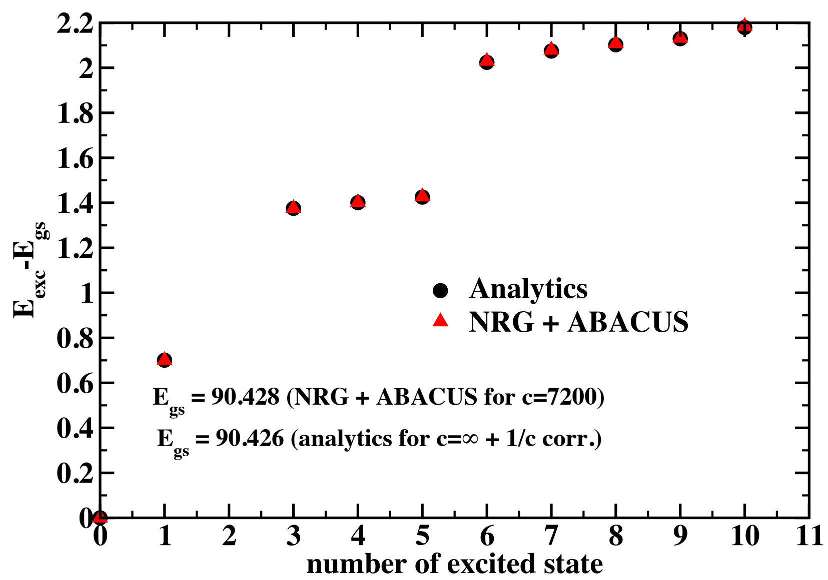
Lieb-Liniger in a trap: static properties

Ground state in a trap obtained as linear combination of Bethe states

$$|\psi\rangle_{GS} = \sum_s c_s |s\rangle$$

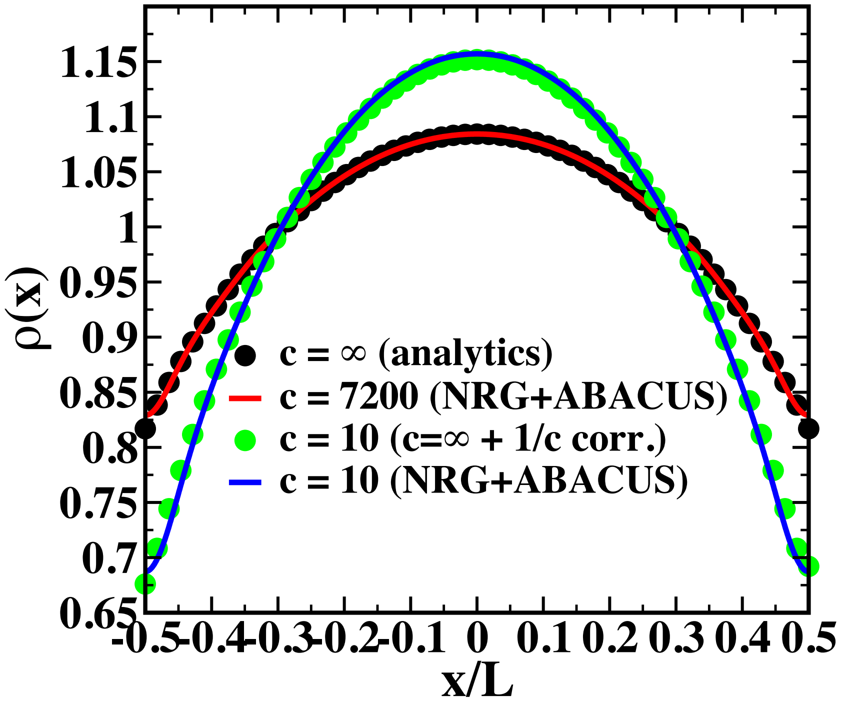
Energies of ground and low-lying states can be tested at large c by comparing to analytical results

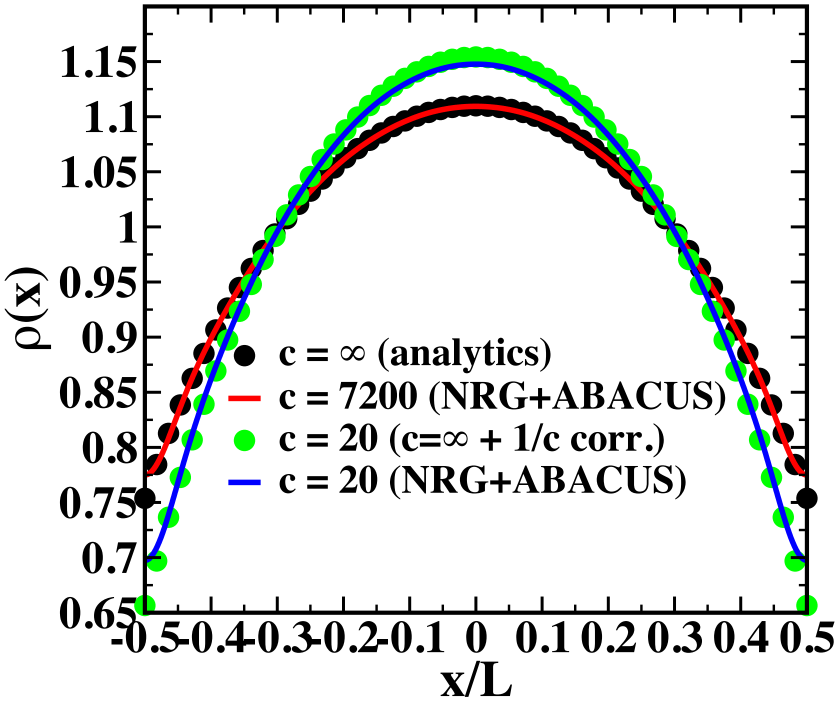




Lieb-Liniger in a trap: static properties beyond energetics

Density profile in ground state of trapped gas





$\omega = 0.16$ $N = 56$ $\omega = 0.183$

Time evolution after trap release

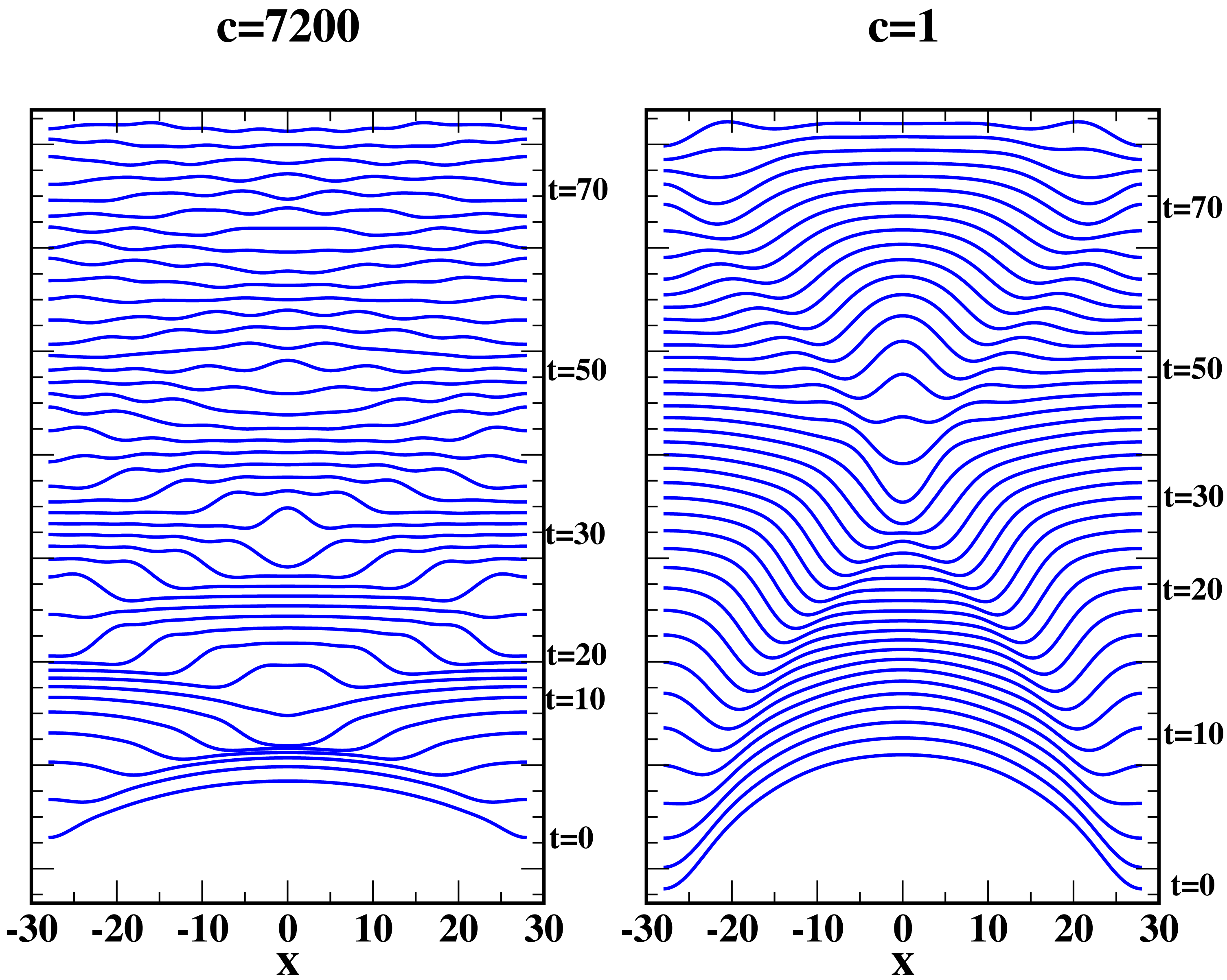
The time evolution is trivially given by

$$|\psi(t)\rangle_{GS} = \sum_s c_s e^{-iE_s t} |s\rangle$$

Advantage of using integrable basis:
if post-quench evolution is via integrable H,
energies are known exactly, and thus
arbitrary t is accessible without loss of accuracy!

(note however that we always use
periodic boundary conditions)

Local density as function of time

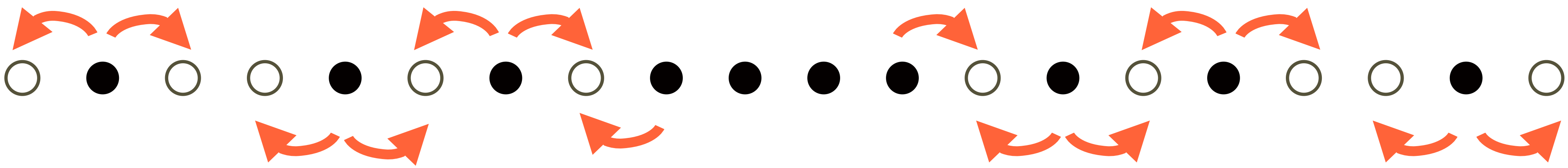


Correlations long after release

(in other words: in the diagonal ensemble)

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle_{DE} \equiv \sum_s |c_s|^2 \langle s | \mathcal{O}^\dagger \mathcal{O} | s \rangle$$

Implementation of 'ABACUS for arbitrary states'



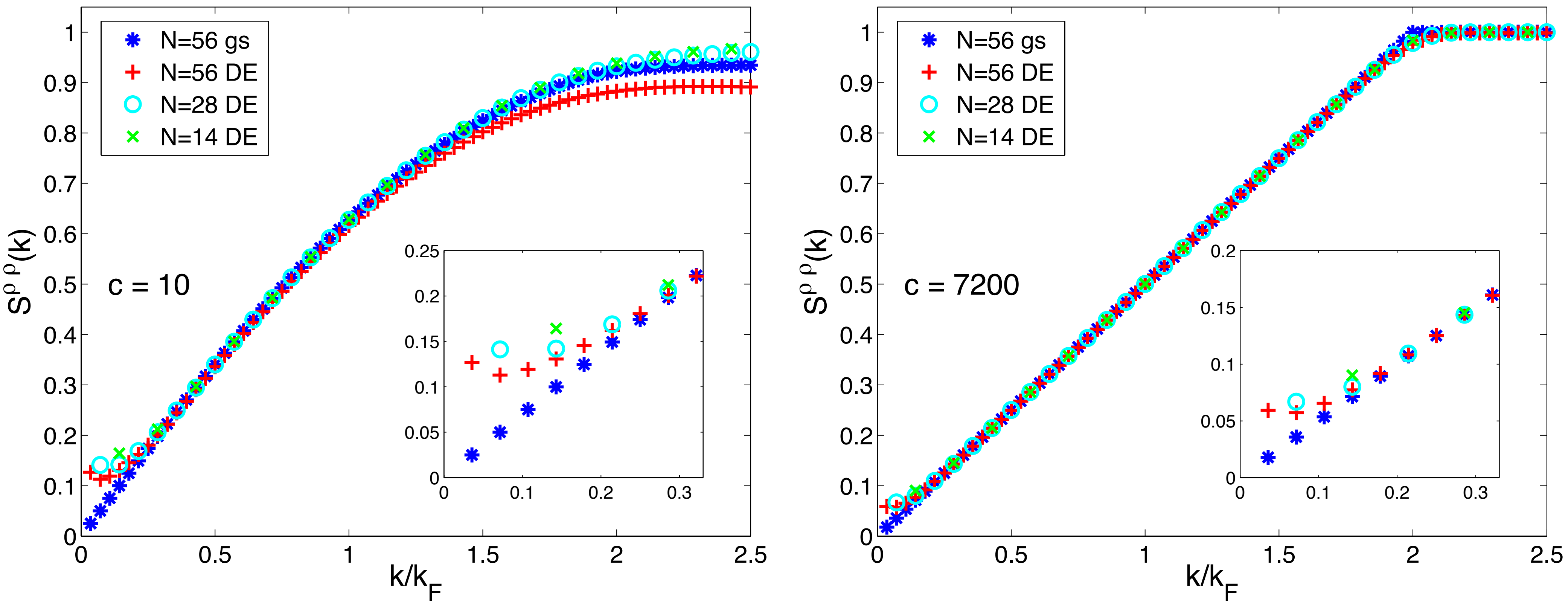
Matrix elements: are not suppressed as 'rapidly' as around the ground state. Large summation.

Correlations long after release

(in other words: in the diagonal ensemble)

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle_{DE} \equiv \sum_s |c_s|^2 \langle s | \mathcal{O}^\dagger \mathcal{O} | s \rangle$$

1) Static structure factor



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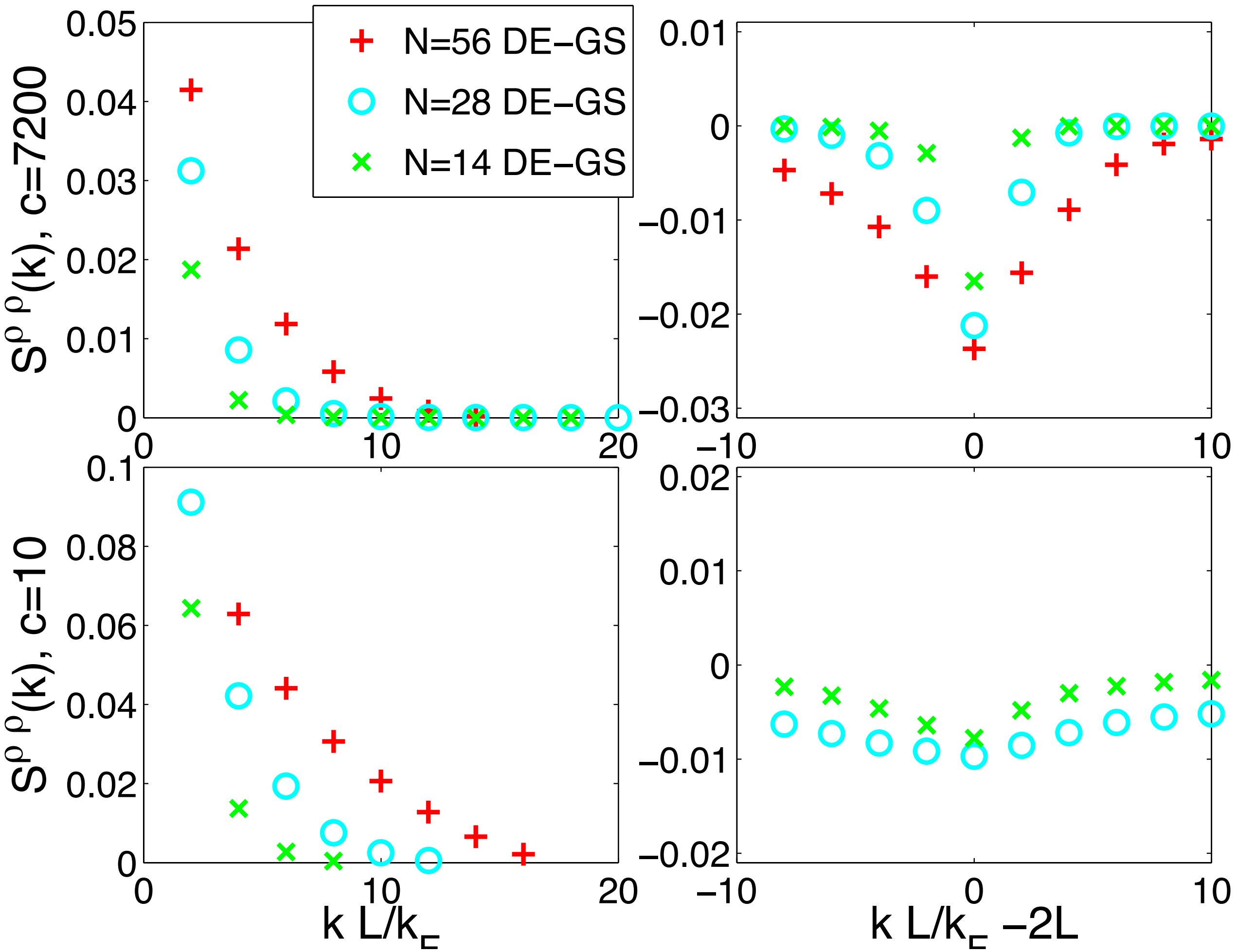
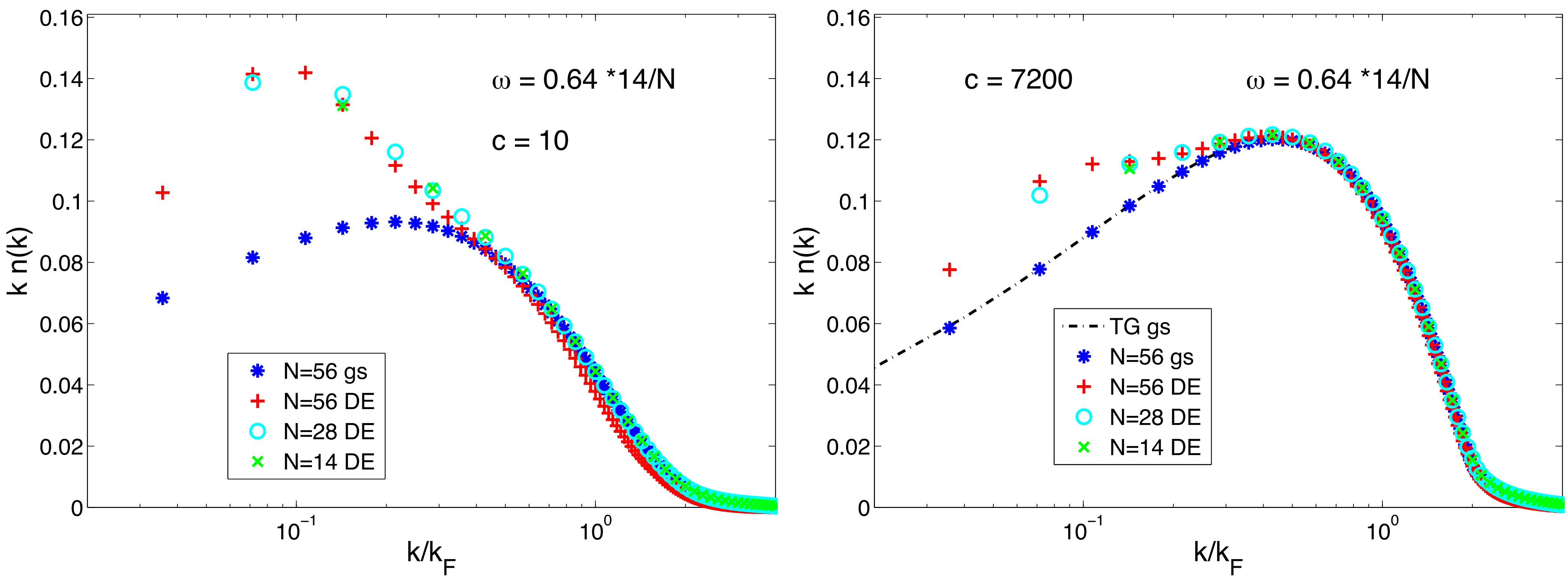


Correlations long after release

(in other words: in the diagonal ensemble)

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle_{DE} \equiv \sum_s |c_s|^2 \langle s | \mathcal{O}^\dagger \mathcal{O} | s \rangle$$

2) Momentum distribution function



Interlude:
Generalized TBA

Generalized TBA and generalized Gibbs

J. Mossel and JSC, arxiv:1203.1305

Integrable Hamiltonian H_0 and its conserved charges $\hat{Q}_n$

$$\left[H_0, \hat{Q}_n\right] = 0, \qquad \left[\hat{Q}_n, \hat{Q}_m\right] = 0, \quad \forall\ n, m$$

Generic Hamiltonian: $H(\{\beta\}) = \sum_n \beta_n \hat{Q}_n$

shares wavefunctions with $H_0, \forall \{\beta_n\} \in \mathbb{R}$

Density matrix: generalized Gibbs ensemble (GGE)

$$\hat{\rho}_{GGE} = e^{-\sum_n \beta_n \hat{Q}_n}$$

Conjecture: after a quantum quench, GGE describes state at asymptotically large time

Rigol, Dunjko, Yurovsky, Olshanii, PRL 2007

Jaynes, Phys. Rev. 1957

Generalized inverse temperatures are set by initial conditions

$$\boxed{\langle\hat{Q}_m\rangle = \mathrm{Tr}\left\{\hat{Q}_m e^{-\sum_n \beta_n \hat{Q}_n}\right\}/\mathcal{Z}_{GGE} \quad m = 0, 1, 2, \ldots}$$

where $\mathcal{Z}_{GGE} = \mathrm{Tr} e^{-\sum_n \beta_n \hat{Q}_n}$

In reality, two major difficulties:

Conserved charges are generically nontrivial

Self-consistency problem difficult to solve

In practice: only implemented for free theories

Generalized TBA for Lieb-Liniger

Eigenvalues of conserved charges trivial on Bethe states

$$\hat{Q}_n|\{\lambda\}\rangle = Q_n|\{\lambda\}\rangle, \qquad Q_n = \sum_j \lambda_j^n$$

(although charges themselves are nontrivial *Davies & Korepin*)

The generalized Hamiltonian is trivially diagonalized by the Bethe states

$$H(\{\beta\})|\{\lambda\}\rangle = E(\{\beta\})|\{\lambda\}\rangle,$$

$$\boxed{E(\{\beta\}|\{\lambda\}) = \sum_{n=0}^{\infty}\sum_{j=1}^{N}\beta_n\lambda_j^n \equiv \sum_{j=1}^{N}\varepsilon_0(\lambda_j)}$$

$$\varepsilon_0(\lambda) \equiv \sum_{n=0}^{\infty}\beta_n\lambda^n$$ 'Driving' term for generalized TBA

Thermodynamic limit: root density distributions

$$\rho(\lambda) + \rho_h(\lambda) = \frac{1}{2\pi} + a_2 * \rho(\lambda), \qquad a_2(\lambda) \equiv \frac{1}{\pi}\frac{c}{\lambda^2 + c^2}$$

Eigenvalue of conserved charges simply given by

$$Q_n = L \int_{-\infty}^{\infty} d\lambda \lambda^n \rho(\lambda)$$

Knowing the root density distribution
is thus equivalent to
knowing the eigenvalues of all conserved charges

Solving the generalized TBA

Simply repeat logic of Yang & Yang

$$\mathcal{Z} = \int \mathcal{D}[\rho] e^{-G[\rho,\rho_h[\rho]]} \qquad G[\rho,\rho_h] = \sum_{n=0} \beta_n Q_n - S[\rho,\rho_h]$$

$$S = L \int_{-\infty}^{\infty} d\lambda \left[(\rho + \rho_h) \ln(\rho + \rho_h) - \rho \ln \rho - \rho_h \ln \rho_h \right]$$

Usual saddle-point reasoning gives

$$\boxed{\varepsilon(\lambda) + a_2 * \ln(1 + e^{-\varepsilon(\lambda)}) = \varepsilon_0(\lambda)} \qquad \varepsilon(\lambda) = \ln \frac{\rho_h(\lambda)}{\rho(\lambda)},$$

Solution exists and is unique, density obtained from

$$\rho(\lambda) = \vartheta(\lambda) \left(\frac{1}{2\pi} + a_2 * \rho(\lambda) \right) \qquad \vartheta(\lambda) = \frac{\rho(\lambda)}{\rho(\lambda) + \rho_h(\lambda)} = \frac{1}{1 + e^{\varepsilon(\lambda)}}$$

Thus: two different ways of proceeding in practice:

1) given (by some 'beta-oracle') a set $\{\beta_n\}$

solve for $\varepsilon(\lambda)$ from which $\rho(\lambda)$ can be obtained

2) given (by some 'rho-oracle') a $\rho(\lambda)$

solve for $\varepsilon(\lambda)$ from which $\{\beta_n\}$ can be obtained

$$\beta_n = \frac{\partial^n}{\partial \lambda^n} \left(\varepsilon(\lambda) + a_2 * \ln[1 + e^{-\varepsilon(\lambda)}] \right) \Big|_{\lambda=0}$$

Constructing the GGE explicitly

JSC & R. M. Konik 1203.0901

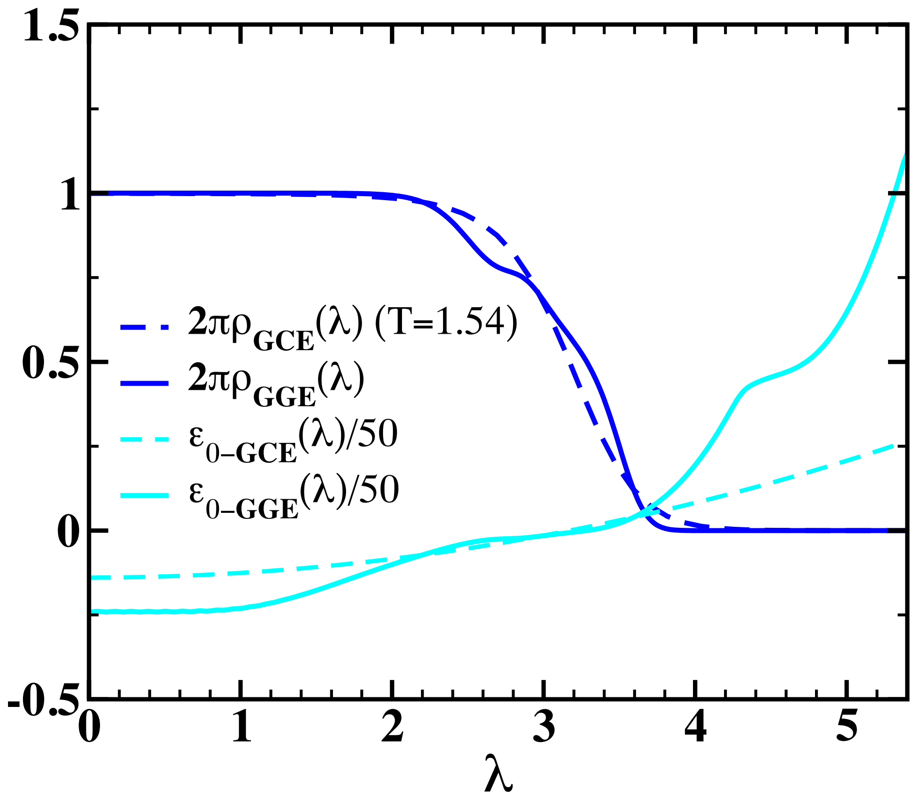
Lieb-Liniger in a trap, long after release

GGE root density distribution is simply

$$\rho_{GGE}(\lambda) = \sum_s |c_s|^2 \rho_s(\lambda)$$

which is obtained from NRG+ABACUS

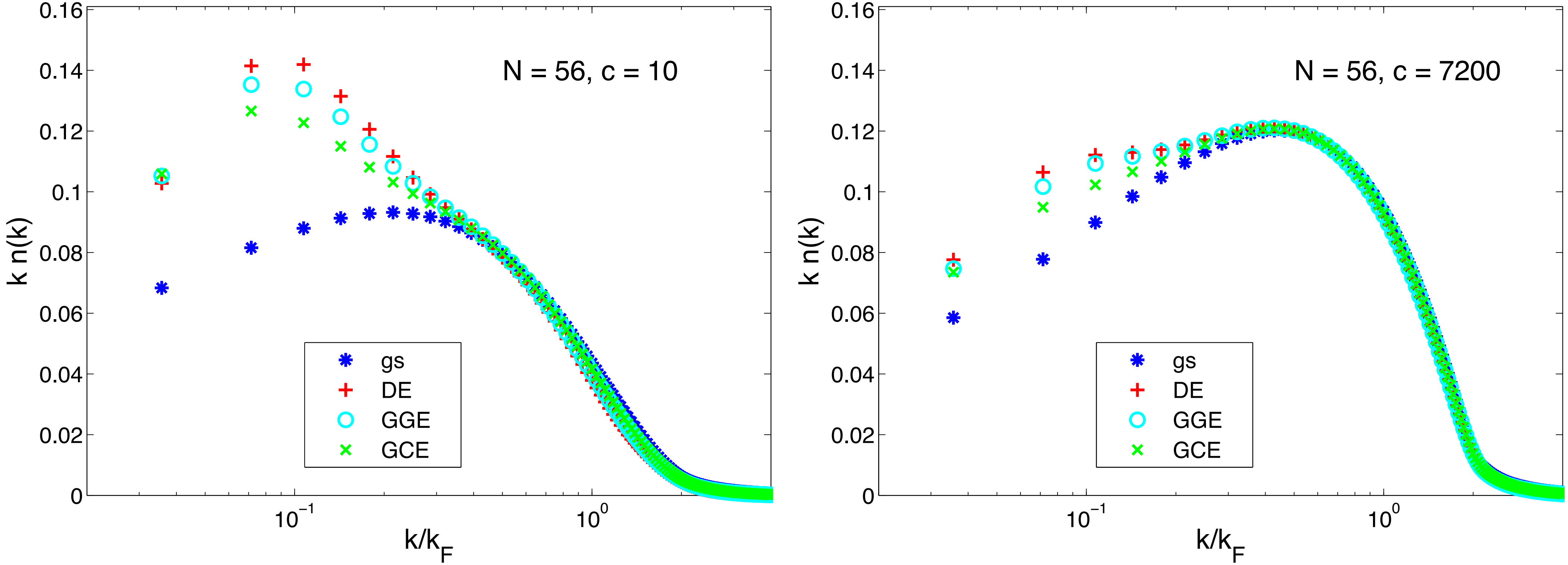
This root density is non-thermal



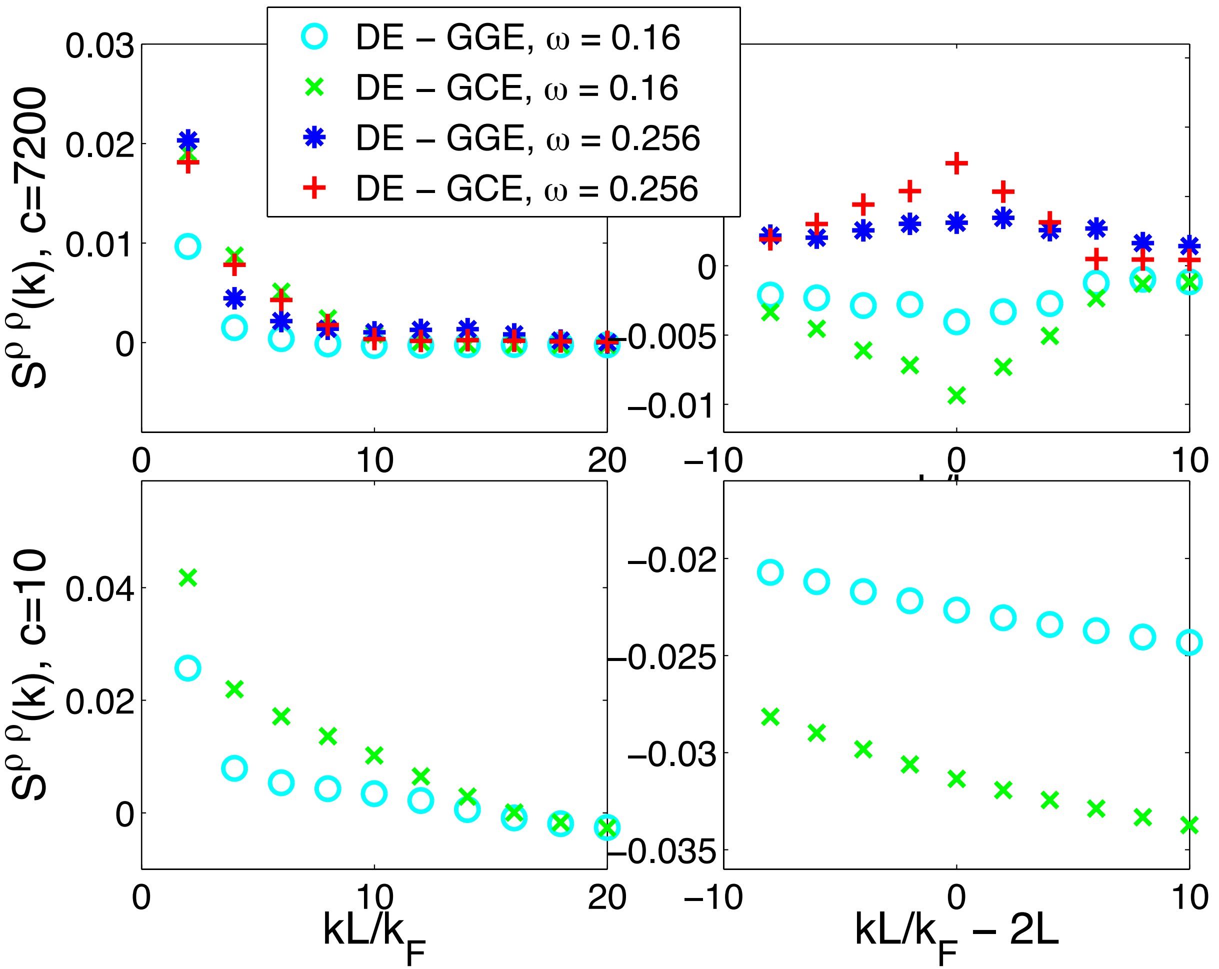
We are now in position to compare three different ensembles:

1) diagonal
2) grand canonical
3) generalized Gibbs

Momentum distribution function



Static structure factor



- *both GCE and GGE disagree with DE*
- *disagreement grows with trap strength*
- *disagreement grows as interactions decrease*
- *GGE considerably better match to DE than GCE*

Conclusions

- Interaction turnoff in Lieb-Liniger
  - *simple explicitly solvable case*
  - *in this finite system, GenCan better then GenGibbs*
- NRG + ABACUS
  - *now able to tackle 'deformed' integrable models*
  - *method gives access to time evolution after quench*
- Not discussed here (for private discussions):
  - *correlation prefactors*
  - *thermal correlations*
  - *correlations on Moses states*