Supersymmetric two-dimensional QCD at finite temperature

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Outline

- light-cone coordinates
- (supersymmetric) discrete light-cone quantization
- $\mathcal{N}=1$ SQCD-CS$_{1+1}$
  [NBP **661**, 99 (2003); PRD **67**, 115005 (2003)]
- thermodynamics
  [PRD **70**, 065012 (2004)]
- Lanczos algorithm for density of states
- preliminary results
- future work

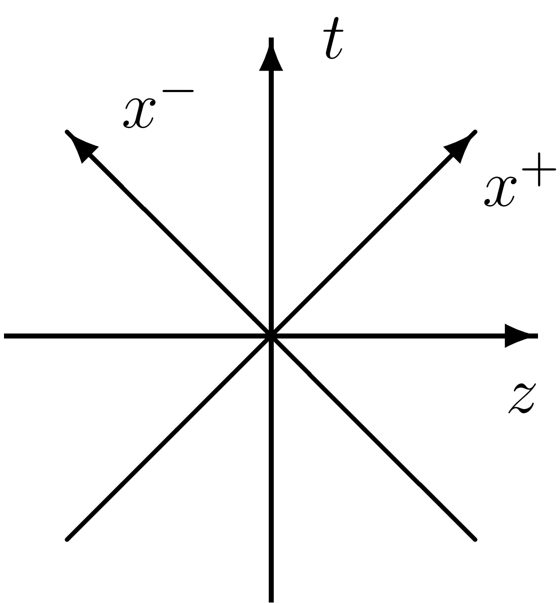
Acknowledgments

- work done in collaboration with
  - S. Pinsky, Y. Proestos, N. Salwen, and U. Trittmann.
- supported in part by
  - US Department of Energy.
  - Minnesota Supercomputing Institute.

Light-cone coordinates

Dirac, RMP **21**, 392 (1949).

- time: $x^+ = t + z$
- space: $\underline{x} = (x^-, \vec{x}_\perp), \;\; x^- \equiv t - z, \;\; \vec{x}_\perp = (x, y)$
- energy: $p^- = E - p_z$
- momentum: $\underline{p} = (p^+, \vec{p}_\perp), \;\; p^+ \equiv E + p_z, \;\; \vec{p}_\perp = (p_x, p_y)$
- mass-shell condition: $p^2 = m^2 \;\; \Rightarrow \;\; p^- = \frac{m^2 + p_\perp^2}{p^+}$



Discrete light-cone quantization (DLCQ)

Pauli and Brodsky, PRD **32**, 1993 (1985); 2001 (1985).
Brodsky, Pauli, and Pinsky, Phys. Rep. **301**, 299 (1997).

- light-cone box $-L < x^- < L$, $-L_\perp < x, y < L_\perp$ and periodic BC $\rightarrow$ discrete grid: $p_i^+ \rightarrow \frac{\pi}{L} n_i$, $\mathbf{p}_{i\perp} \rightarrow (\frac{\pi}{L_\perp} n_{ix}, \frac{\pi}{L_\perp} n_{iy})$.
- limit $L \rightarrow \infty$ exchanged for limit in terms of the integer *resolution* $K \equiv \frac{L}{\pi} P^+$ for fixed total momentum P^+.
- $n_i > 0 \;\Rightarrow$ # of particles $\leq K$.
- integrals replaced by discrete sums
$$\int dp^+ \int d^2 p_\perp f(p^+, \mathbf{p}_\perp) \simeq \frac{2\pi}{L} \left(\frac{\pi}{L_\perp} \right)^2 \sum_n \sum_{n_x, n_y = -N_\perp}^{N_\perp} f(n\pi/L, \mathbf{n}_\perp \pi/L_\perp).$$

Supersymmetric DLCQ (SDLCQ)

Matsumura, Sakai, and Sakai, PRD **52**, 2446 (1995).
Lunin and Pinsky, AIP Conf. Proc. **494**, 140 (1999).

- supersymmetry algebra: $\{Q^+, Q^+\} = 2\sqrt{2}P^+$, $\{Q^-, Q^-\} = 2\sqrt{2}P^-$, $\{Q^+, Q^-\} = -4P_\perp$.
- discretize supercharge Q^- and compute $P^-_{\rm SDLCQ} = \frac{1}{2\sqrt{2}}\{Q^-, Q^-\} \neq P^-_{\rm DLCQ}$
- preserves supersymmetry.
- for ordinary DLCQ, recover supersymmetry only in infinite resolution limit.
- primarily work in large-N_c approximation.

Mesons and glueballs

- Either could be a boson or a fermion.
- Meson Fock state:

$$\bar{f}^\dagger_{i_1}(k_1)a^\dagger_{i_1 i_2}(k_2)\ldots b^\dagger_{i_n i_{n+1}}(k_{n-1})\ldots f^\dagger_{i_p}(k_n)|0\rangle$$

- Glueball Fock state: $\mathrm{Tr}[a^\dagger_{i_1 i_2}(k_1)\ldots b^\dagger_{i_n i_{n+1}}(k_n)]|0\rangle$

  $\bar{f}^\dagger_i$ and $f^\dagger_i$ $\longrightarrow$ fundamental partons

  $a^\dagger_{ij}$ and $b^\dagger_{ij}$ $\longrightarrow$ adjoint partons

- Work in large-N_c approximation.
  - no mixing, single-trace states only, baryons difficult
- SDLCQ can handle finite N_c with increased computing resources.

$\mathcal{N} = 1$ SQCD-CS$_{2+1}$

Action:

$$S = \int d^3x \mathrm{Tr} \Big\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \xi^\dagger D^\mu \xi + i\bar{\Psi} D_\mu \Gamma^\mu \Psi$$
$$-g \left[\bar{\Psi} \Lambda \xi + \xi^\dagger \bar{\Lambda} \Psi \right] + \frac{i}{2} \bar{\Lambda} \Gamma^\mu D_\mu \Lambda$$
$$+ \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} \left[A_\mu \partial_\nu A_\lambda + \frac{2i}{3} g A_\mu A_\nu A_\lambda \right] + \kappa \bar{\Lambda} \Lambda \Big\}$$

Adjoint fields: gauge boson A_μ (gluons)
Majorana fermion Λ (gluinos).

Fundamental fields: Dirac fermion Ψ (quarks)
complex scalar ξ (squarks).

Chern–Simons coupling: $\kappa \longrightarrow$ mass for adjoint fields.

Transformations

- Covariant derivatives:

$$\begin{aligned} D_\mu \Lambda &= \partial_\mu \Lambda + ig[A_\mu, \Lambda]\,, \qquad D_\mu \xi = \partial_\mu \xi + igA_\mu \xi\,, \\ D_\mu \Psi &= \partial_\mu \Psi + igA_\mu \Psi\,. \end{aligned}$$

- Supersymmetry transformations:

$$\delta A_\mu = \frac{i}{2}\bar{\varepsilon}\Gamma_\mu \Lambda\,, \qquad \delta\Lambda = \frac{1}{4}F_{\mu\nu}\Gamma^{\mu\nu}\varepsilon\,,$$

$$\delta\xi = \frac{i}{2}\bar{\varepsilon}\Psi\,, \qquad \delta\Psi = -\frac{1}{2}\Gamma^\mu \varepsilon D_\mu \xi\,.$$

- Reduce to 1+1 dimensions by assuming fields independent of the transverse coordinate x^2.

Supercharge

$$
\begin{aligned}
\bar{\varepsilon}Q \quad = \quad & \int dx^- dx^2 \left(\frac{i}{4}\bar{\varepsilon}\Gamma^{\alpha\beta}\Gamma^+ \mathsf{tr}\left(\Lambda F_{\alpha\beta}\right) \right. \\
& + \frac{i}{2}D_- \xi^\dagger \bar{\varepsilon}\Psi + \frac{i}{2}\xi^\dagger \bar{\varepsilon}\Gamma^{+\nu} D_\nu \Psi \\
& \left. - \frac{i}{2}\bar{\Psi}\varepsilon D^+ \xi + \frac{i}{2} D_\nu \bar{\Psi}\Gamma^{+\nu}\varepsilon\xi \right) \; .
\end{aligned}
$$

Dynamical fields

Fermion components:

$$\Lambda = \left(\lambda, \tilde{\lambda}\right)^T, \qquad \Psi = \left(\psi, \tilde{\psi}\right)^T, \qquad Q = \left(Q^+, Q^-\right)^T.$$

Constraints in light-cone gauge ($A^+ = 0$):

$$\begin{aligned}
\partial_- \tilde{\lambda} &= -\frac{ig}{\sqrt{2}}\left([A^2, \lambda] + i\xi\psi^\dagger - i\psi\xi^\dagger\right), \\
\partial_- \tilde{\psi} &= -\frac{ig}{\sqrt{2}} A^2\psi + \frac{g}{\sqrt{2}}\lambda\xi - \kappa\lambda/\sqrt{2}, \;\; \partial_-^2 A^- = gJ,
\end{aligned}$$

with

$$J \equiv i[A^2, \partial_- A^2] + \frac{1}{\sqrt{2}}\{\lambda, \lambda\} + \kappa\partial_- A^2 - ih\partial_-\xi\xi^\dagger + i\xi\partial_-\xi^\dagger + \sqrt{2}\psi\psi^\dagger.$$

Reduced supercharge

$$\begin{aligned} Q^- &= g\int dx^- \left\{ 2^{3/4}\left(\mathrm{i}[A^2, \partial_- A^2] - \kappa \partial_- A^2 + \frac{1}{\sqrt{2}}\{\lambda, \lambda\} \right) \frac{1}{\partial_-}\lambda \right. \\ & -\frac{1}{\sqrt{2}}\left(i\sqrt{2}\xi\partial_-\xi^\dagger - i\sqrt{2}\partial_-\xi\xi^\dagger + 2\psi\psi^\dagger \right) \frac{1}{\partial_-}\lambda \\ & \left. -2\left(\xi^\dagger A^2 \psi + \psi^\dagger A^2 \xi \right) \right\} . \end{aligned}$$

Mode expansions

$$
\begin{aligned}
A^2_{ij}(0,x^-) &= \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left(a_{ij}(k) e^{-ik\pi x^-/L} + a^\dagger_{ji}(k) e^{ik\pi x^-/L} \right) , \\
\lambda_{ij}(0,x^-) &= \frac{1}{2^{\frac{1}{4}}\sqrt{2L}} \sum_{k=1}^{\infty} \left(b_{ij}(k) e^{-ik\pi x^-/L} + b^\dagger_{ji}(k) e^{ik\pi x^-/L} \right) , \\
\xi_i(0,x^-) &= \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left(c_i(k) e^{-ik\pi x^-/L} + \tilde{c}^\dagger_i(k) e^{ik\pi x^-/L} \right) , \\
\psi_i(0,x^-) &= \frac{1}{2^{\frac{1}{4}}\sqrt{2L}} \sum_{k=1}^{\infty} \left(d_i(k) e^{-ik\pi x^-/L} + \tilde{d}^\dagger_i(k) e^{ik\pi x^-/L} \right) .
\end{aligned}
$$

Commutation relations

For finite N_c:

$$\left[a_{ij}, a^\dagger_{kl}\right] = \left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) \,,$$

$$\left\{b_{ij}, b^\dagger_{kl}\right\} = \left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) \,,$$

$$\left[c_i, c^\dagger_j\right] = \delta_{ij} \,, \qquad \left[\tilde{c}_i, \tilde{c}^\dagger_j\right] = \delta_{ij} \,,$$

$$\left\{d_i, d^\dagger_j\right\} = \delta_{ij} \qquad \left\{\tilde{d}_i, \tilde{d}^\dagger_j\right\} = \delta_{ij} \,.$$

Additional (Z_2) symmetry

Kutasov, NPB **414**, 33 (1994).

$$a_{ij}(k, n^\perp) \to -a_{ji}(k, n^\perp) \quad b_{ij}(k, n^\perp) \to -b_{ji}(k, n^\perp)$$

Divides between even and odd numbers of gluons.
Diagonalize in each sector.

Finite temperature

- partition function: $Z = \sum e^{-p_0/T}$
- one-dimensional bosonic free energy

$$\mathcal{F}_B = \frac{VT}{\pi} \sum_{n=1}^{\infty} \int_{M_n}^{\infty} dp_0 \frac{p_0}{\sqrt{p_0^2 - M_n^2}} \ln\left(1 - e^{-p_0/T}\right).$$

- fermionic free energy

$$\mathcal{F}_F = -\frac{VT}{\pi} \sum_{n=1}^{\infty} \int_{M_n}^{\infty} dp_0 \frac{p_0}{\sqrt{p_0^2 - M_n^2}} \ln\left(1 + e^{-p_0/T}\right).$$

Total free energy

- expand logs and integrate over p_0

$$\mathcal{F}(T,V) = -\frac{(K-1)\pi}{4} V T^2 - \frac{2VT}{\pi} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} M_n \frac{K_1\left((2l+1)\frac{M_n}{T}\right)}{(2l+1)}$$

- the sum over l is well approximated by the first few terms.
- represent the sum over n as an integral over a density of states: $\sum_n \to \int \rho(M) dM$
- approximate ρ by a continuous function.
- compute $\int dM$ by standard numerical techniques.

Lanczos algorithm for density of states

- Density of states: $\rho(M^2) = \sum_n d_n \delta(M^2 - M_n^2)$ where d_n is the degeneracy of the mass eigenvalue M_n.
- Cumulative distribution function: $N(M^2) = \int^{M^2} dM^2 \rho(M^2)$.
- Density written in form of a trace over $e^{-iP^- x^+}$:

$$
\begin{aligned}
\rho(M^2) &= \frac{1}{2P^+} \sum_n d_n \delta(M^2/2P^+ - P_n^-) \\
&= \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+/2P^+} \sum_n d_n e^{-iP_n^- x^+} dx^+ \\
&= \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+/2P^+} \mathrm{Tr} e^{-iP^- x^+} dx^+ .
\end{aligned}
$$

Estimation of the trace

- Approximate the trace as an average over a random sample of vectors
  R. Alben, M. Blume, H. Krakauer, and L. Schwartz, PRB **12**, 4090 (1975).
  A. Hams and H. De Raedt, PRE **62**, 4365 (2000).
- Define a local density for a single vector $|s\rangle$ as

$$\rho_s(M^2) = \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+/2P^+} \langle s| e^{-iP^- x^+} |s\rangle dx^+,$$

so that the average can be written

$$\rho(M^2) \simeq \frac{1}{S} \sum_{s=1}^{S} \rho_s(M^2).$$

Random phase vectors

T. Iitaka and T. Ebisuzaki, PRE **69**, 057701 (2004).

- The sample eigenstates $|s\rangle$ chosen as random phase vectors.
- The coefficient of each Fock state in the basis is a random number of modulus one.

Approximation of matrix element

J. Jaklič and P. Prelovšek, PRB **49**, 5065(R) (1994);
M. Aichhorn, M. Daghofer, H.G. Evertz, and W. von der Linden, PRB **67**, 161103(R) (2003).

- Approximate $\langle s|e^{-iP^- x^+}|s\rangle$ by Lanczos iteration.
- Let D be the length of $|s\rangle$, and define $|u_1\rangle = \frac{1}{\sqrt{D}}|s\rangle$ as the initial Lanczos vector. Then

$$\rho_s(M^2) = \frac{D}{4\pi P^+} \int e^{iM^2 x^+/2P^+} \langle u_1|e^{-iP^- x^+}|u_1\rangle dx^+.$$

- $\langle u_1|e^{-iP^- x^+}|u_1\rangle$ can be approximated by the $(1,1)$ element of the exponentiation of the Lanczos tridiagonalization of P^-.

Exponentiation

- Let P_s^- be the tridiagonal Lanczos matrix and solve

$$P_s^- \vec{c}_n^{\,s} = \frac{M_{sn}^2}{2P^+} \vec{c}_n^{\,s}.$$

- A diagonal matrix $\Lambda_{ij} = \delta_{ij} \frac{M_{sn}^2}{2P^+}$ is related by the usual similarity transformation $P_s^- = U \Lambda U^{-1}$, where $U_{ij} = (c_j^s)_i$.
- The $(1,1)$ element is given by

$$\left(e^{-iP_s^- x^+} \right)_{11} = \sum_n |(c_n^s)_1|^2 e^{-iM_{sn}^2 x^+ / 2P^+}.$$

Estimation of local density

- $w_{sn} \equiv D|(c_n^s)_1|^2$, weight of each Lanczos eigenvalue.
- The local density is

$$\begin{aligned}
\rho_s(M^2) &\simeq \frac{D}{4\pi P^+}\int e^{iM^2x^+/2P^+}\sum_n |(c_n^s)_1|^2 e^{-iM_{sn}^2x^+/2P^+}dx^+ \\
&\simeq \frac{D}{4\pi P^+}\sum_n |(c_n^s)_1|^2 2\pi\delta(M^2/2P^+ - M_{sn}^2/2P^+) \\
&\simeq \sum_n w_{sn}\delta(M^2 - M_{sn}^2),
\end{aligned}$$

- Only the extreme Lanczos eigenvalues are good approximations to eigenvalues of the original P^-.
- The other Lanczos eigenvalues provide a smeared representation of the full spectrum.

Cumulative distribution function

- The contribution to the cumulative distribution function is

$$N_s(M^2) \equiv \int^{M^2} dM^2 \rho(M^2) \simeq \sum_n w_{sn} \theta(M^2 - M_{sn}^2).$$

- The full CDF is then approximated by the average

$$N(M^2) \simeq \frac{1}{S} \sum_s N_s(M^2).$$

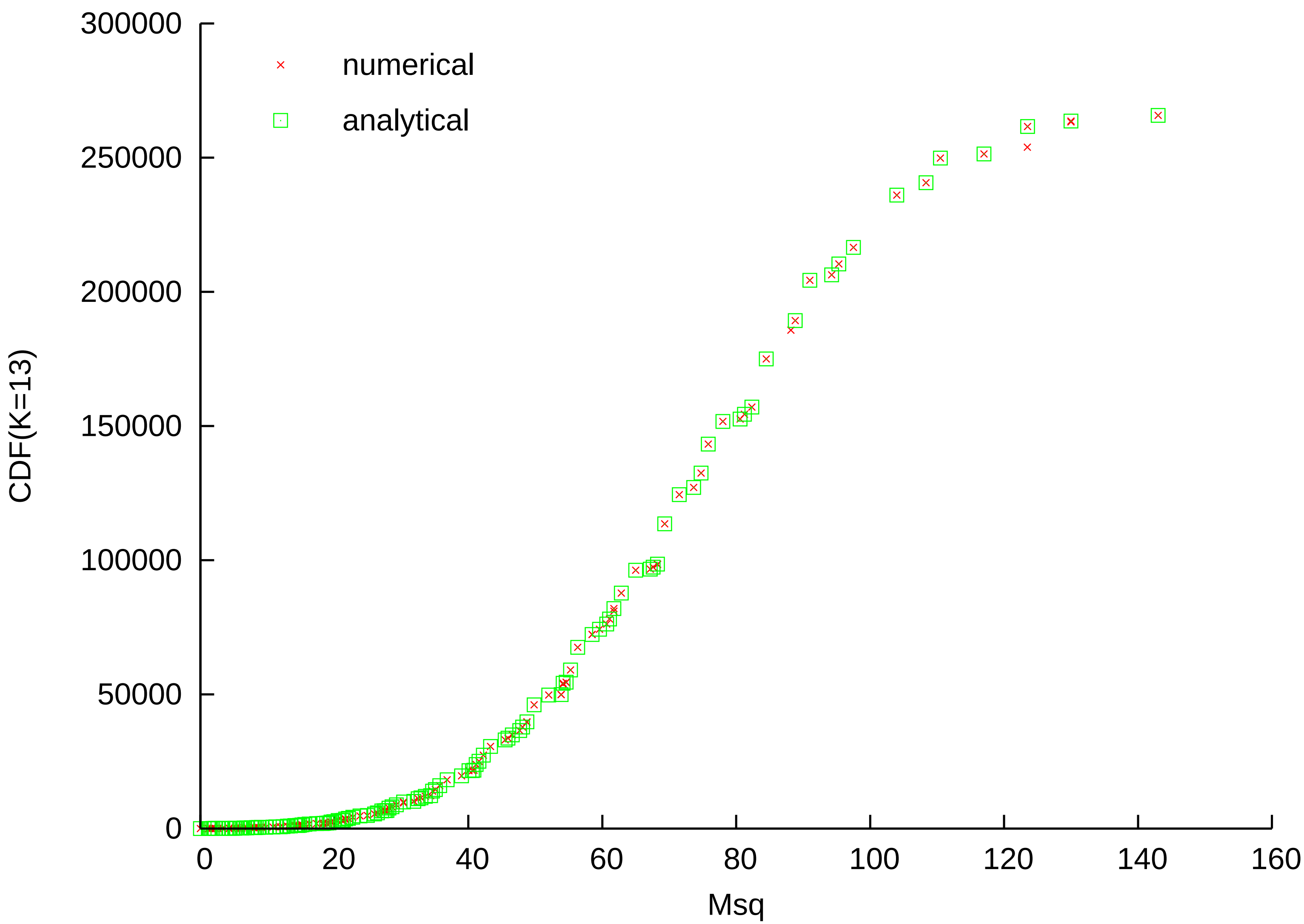
Averaging theta functions

- Use first sample run as a template for values M^2_{1n} at which to evaluate N.
- The contributions of the other samples to N are estimated at these values by linear interpolation in cases where the Lanczos eigenvalues M^2_{sn} are not the same as those in the first set.
- In cases where duplicate eigenvalues are generated by the Lanczos iterations, only one is included in the template and the associated weights are added together.

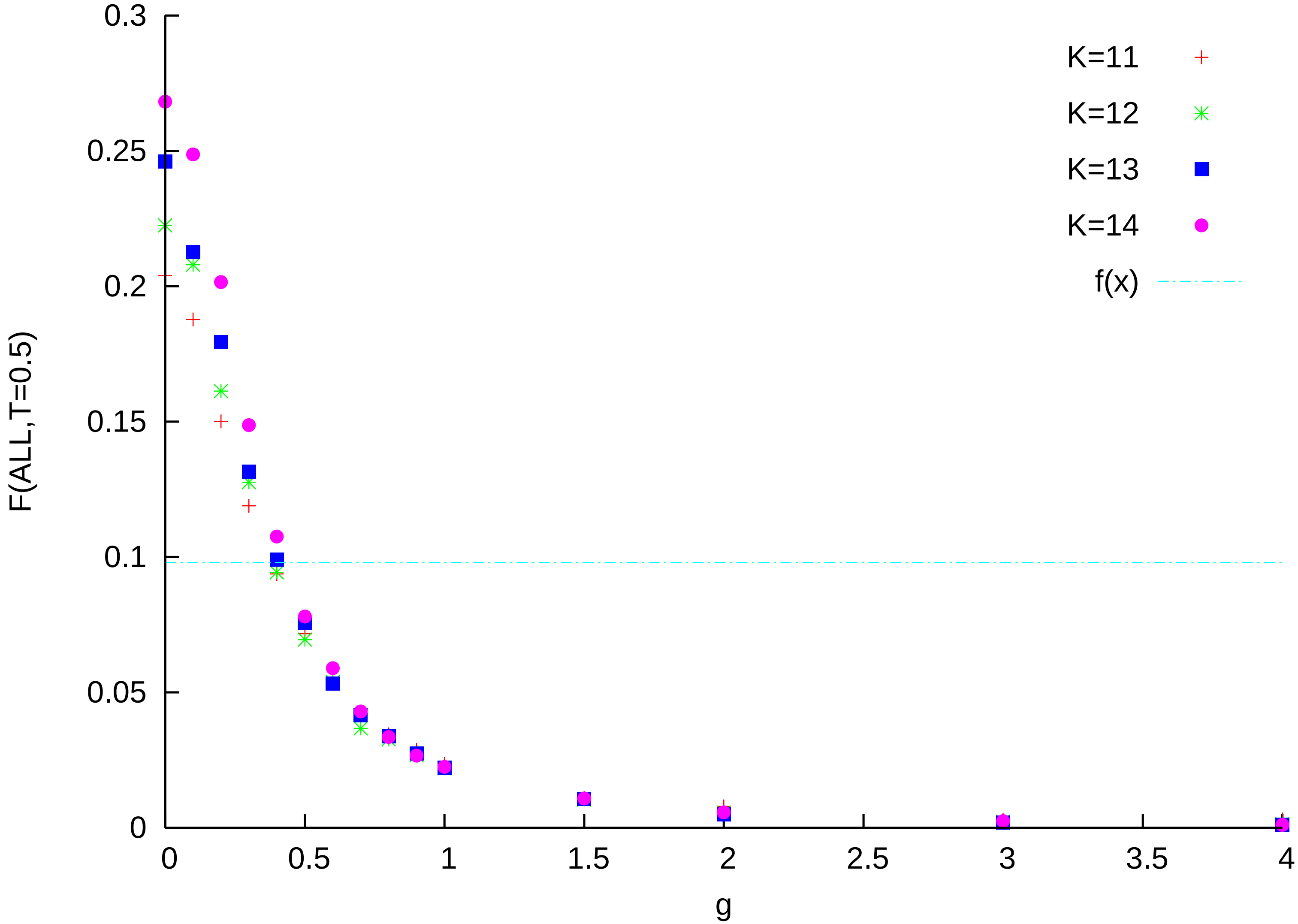
Convergence

- The convergence of the approximation is dependent on the number of Lanczos iterations per sample, as well as the number S of samples.
- Test runs indicate that the recommended value of 20 samples is sufficient.
- The number of Lanczos iterations needs to be on the order of 1000 per sample; using only 100 leaves errors on the order of 1-2%.

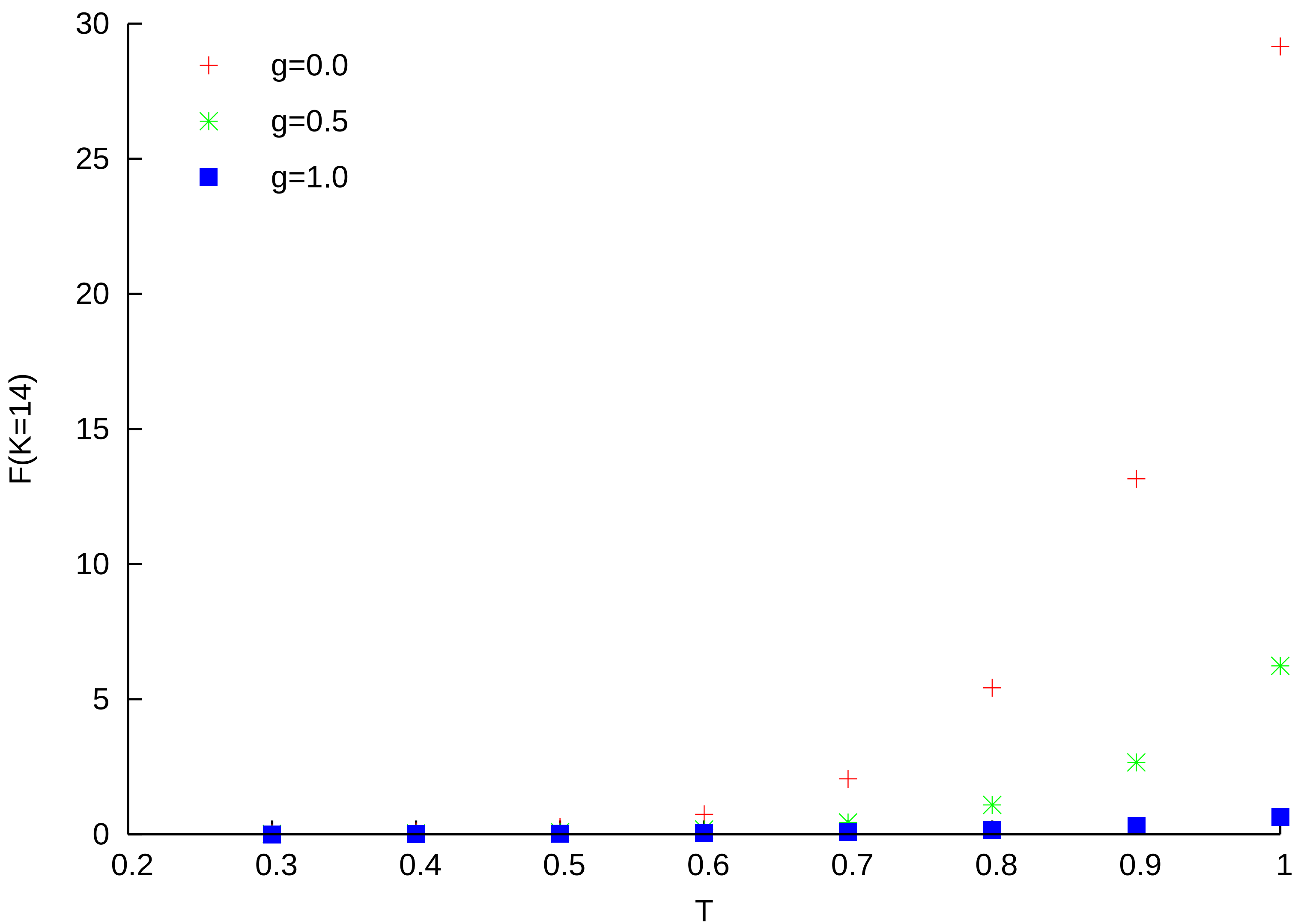
Cumulative distribution function



Free energy at fixed T



Free energy at fixed g



Future work

- SYM: higher resolution.
- SQCD:
  - 2+1
  - finite-N_c effects,
    eg meson/glueball mixing, baryon states.
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-
-
-
- QCD