Hidden Beauty in QCD and $\mathcal{N} = 4$ Gauge Theory

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Based on work done in collaboration with
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Outline

Like essentially all of the talks in this parallel session, this talk will be about **gluon scattering amplitudes** in QCD and supersymmetric gauge theories.

Traditionally these have been notoriously difficult to compute, even at tree level, so this is fertile ground for new insights and methods.

Part 1: A brief review of some of the exciting progress at **tree-level**, including a partial solution to the on-shell recursion relation for gluons.

Part 2: Some recent ideas and techniques for exploring the structure of **multiloop** amplitudes.

Broad Goals of this Research Program

Explore the hidden mathematical structure in perturbative gauge theory, and

Exploit that structure to help make previously impossible calculations possible (in some cases, not just possible but trivial).

Generally, we begin with supersymmetric gauge theories, where the structure is simplest and new ideas are easiest to explore. Most of the techniques can be applied (see other talks), with some effort, to other theories, including honest QCD.

At tree-level there is no distinction: **tree-level gluon amplitudes in QCD are secretly supersymmetric**.

Tree Level

Why, in the 21st century, do we still find it useful to study tree amplitudes?

- Even just two years ago, **few** useful closed form expressions **were known**.
- **Compact explicit formulas** are better than having an algorithm can only be implemented numerically.
- Tree-level amplitudes form the basic building blocks of loop amplitudes through **unitarity** ,

$$\mathrm{Im}\, A^{1-\mathrm{loop}} \sim \sum \int A^{\mathrm{tree}} A^{\mathrm{tree}}.$$

(and, more importantly, **generalized unitarity**).

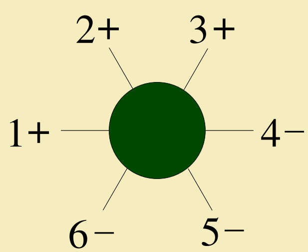
- A better understanding of the **mathematical structure** of tree-level amplitudes will guide us as we attack more complicated loop amplitudes.

Examples of Compact Forumlas

Consider the six-particle amplitude $A(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)$, originally calculated by summing 220 Feynman diagrams.

[Berends & Giele (1987)], [Mangano, Parke, Xu (1988)],

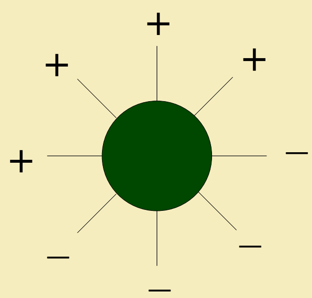
Today we know a very simple formula for this amplitude,



$$= \frac{\langle 1|2+3|4]^3}{(p_2+p_3+p_4)^2[2\,3][3\,4]\langle 5\,6\rangle\langle 6\,1\rangle[2|3+4|5\rangle} + \frac{[6|1+2|3\rangle^3}{(p_6+p_1+p_2)^2[2\,1][1\,6]\langle 5\,4\rangle\langle 4\,3\rangle[2|1+6|5\rangle}.$$

From [Roiban, M.S., Volovich (12/04)], based on [Bern, Del Duca, Dixon, Kosower (10/04)].

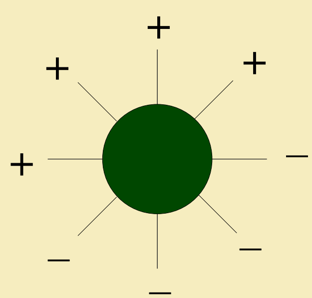
The eight-particle amplitude $A(1^-, 2^-, 3^-, 4^-, 5^+, 6^+, 7^+, 8^+)$ would require 34,300 Feynman diagrams (probably never seriously attempted), or 44 MHV diagrams:



$$= \frac{[\eta\,8]^3}{[8\,1][1\,2][2\,3][3\,\eta]}\,\frac{1}{(p_8+p_1+p_2+p_3)^2}\,\frac{\langle\eta\,4\rangle^3}{\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,7\rangle\langle 7\,\eta\rangle}$$

$$+\ 43\ \text{similar terms}$$

Also in this case there is a simpler formula



$$= \frac{[5|4+3+2|1\rangle^3}{(p_2+p_3+p_4+p_5)^2[2\,3][3\,4][4\,5]\langle 6\,7\rangle\langle 7\,8\rangle\langle 8\,1\rangle[2|3+4+5|6\rangle}$$

$$+\ 5\ \text{similar terms}$$

[Roiban, M.S., Volovich (12/04)].

On-Shell Recursion

Where do these simple formulas come from? Their discoveries were 'accidents', but in hindsight we can observe that these compact formulas all seem to come out naturally from the on-shell recursion

$$A_n = \sum_{r=2}^{n-2} A_{r+1} \frac{1}{p_r^2} A_{n+1-r} \tag{1}$$

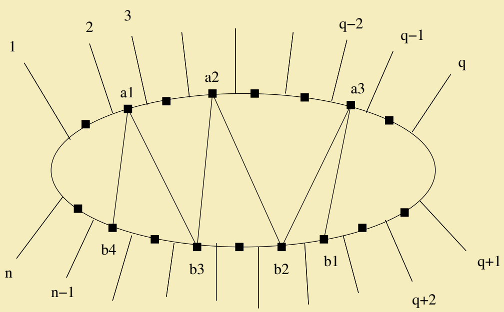
[Britto, Cachazo, Feng (12/04) & with Witten (01/05)].

The on-shell recursion is not particularly efficient when directly implemented as part of a numerical algorithm (see for example [Dinsdale, Ternick, Weinzierl (02/06)]) but, as the examples on the previous pages show, it is very well-suited for deriving compact analytical formulas.

On-shell recursions are great, but explicit solutions are even better!

Zigzag Diagrams

The on-shell recursion relation admits a closed form solution for all 'split helicity' amplitudes. The amplitude $A(1^-, \cdots, p^-, (q+1)^+, \cdots, n^+)$ is given by a sum over all 'zigzag diagrams', of an expression which is trivial to write down



$$= \frac{\langle q|P_{q,b_1} P_{b_1+1,a_1} P_{a_1+1,b_2} \cdots P_{b_4+1,1}|1\rangle^3}{P^2_{q,b_1} P^2_{b_1+1,a_1} P^2_{a_1+1,b_2} \cdots P^2_{b_4+1,1} [q-1|P_{q,b_1}|b_1\rangle \langle b_1+1|P_{b_1+1,a_1}|a_1]}$$

$$\times \frac{\langle b_1\, b_1+1\rangle \cdots \langle b_4\, b_4+1\rangle [a_1\, a_1+1] \cdots [a_3\, a_3+1]}{\langle q\, q+1\rangle \cdots \langle n\, 1\rangle [2\, 3] \cdots [q-2\, q-1]}$$

[Britto, Feng, Roiban, M.S., Volovich (03/05)].

$\Longrightarrow$ Amplitudes which were previously impossible to compute, or could only be evaluated numerically, can now be written down in closed form with no effort.

Tree Level — Solved?

At what point can we throw in the towel and declare that we have solved for the complete tree-level n-particle S-matrix of Yang-Mills theory?

I think that, even at tree-level, there is still some hidden beauty remaining in gluon amplitudes, so I propose...

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Exercise 1. Find a closed form expression for the $2n$-particle 'alternating helicity' amplitude

$$A(1^+, 2^-, 3^+, 4^-, \ldots, 2n^-).$$

Note that *any* amplitude may be obtained as some (multiple) collinear limit of an alternating helicity amplitude.

Onward to Loops

So far, everything I said applied to gluon amplitudes in any gauge theory, from QCD to $N = 4$ Yang-Mills.

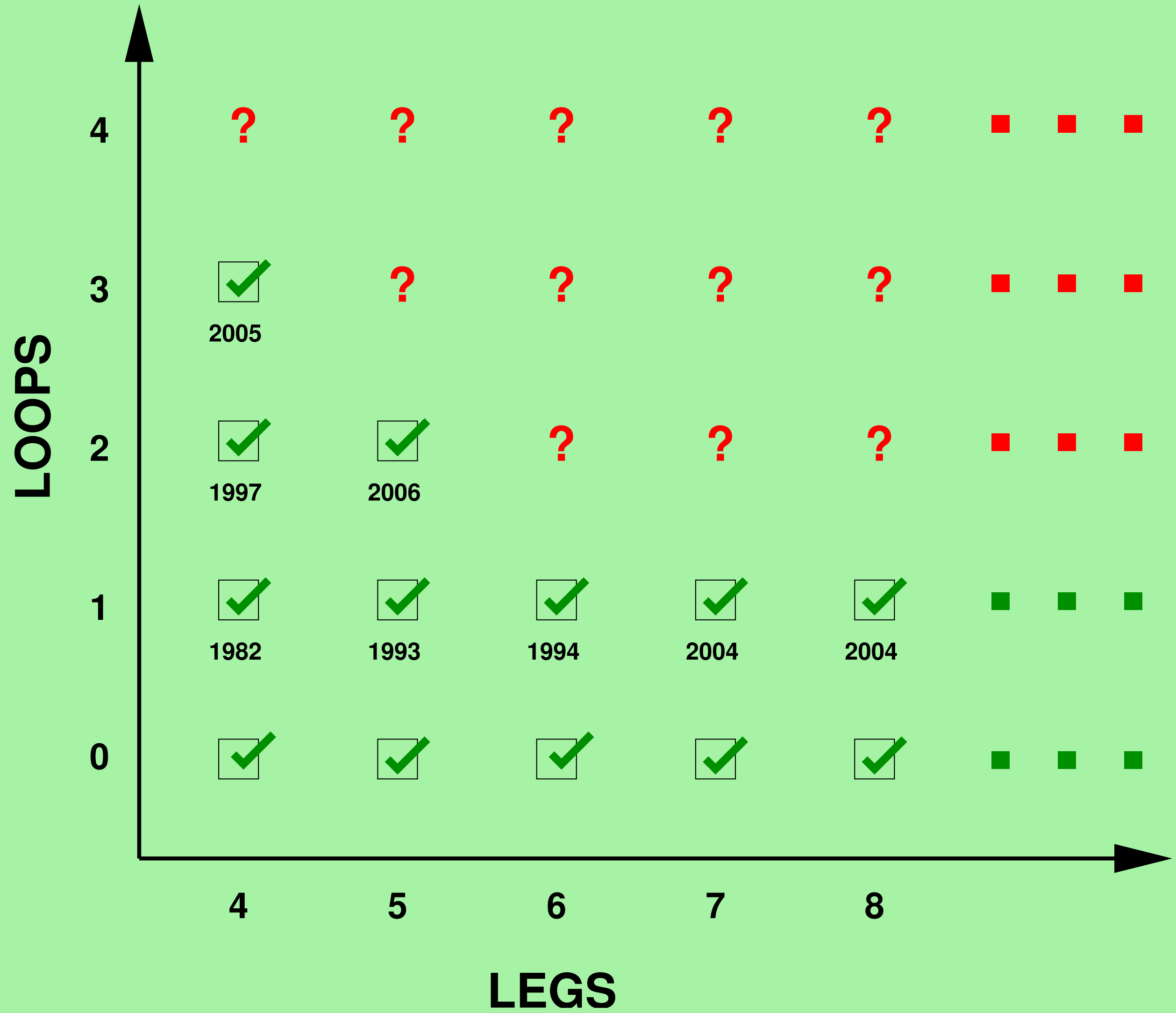
One-loop gluon amplitudes admit the decomposition

$$\mathcal{A}^{\rm QCD} = \mathcal{A}^{\mathcal{N}=4} - 4\mathcal{A}^{\mathcal{N}=1}_{\rm chiral} + \mathcal{A}^{\rm scalar}.$$

At higher loops there is no nice decomposition like this, so I will focus on $N = 4$ Yang-Mills theory.

As at tree-level, the strategy is to first find the hidden beauty in multiloop amplitudes where it is easiest to find, and then root it out of the more complicated, less supersymmetric amplitudes.

$\mathcal{N} = 4$ Yang-Mills Status Report



Legs Versus Loops

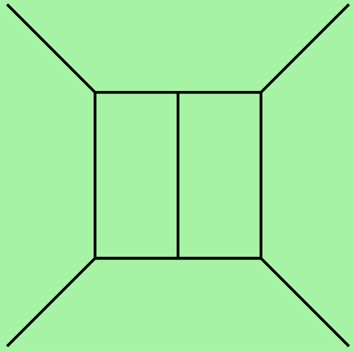
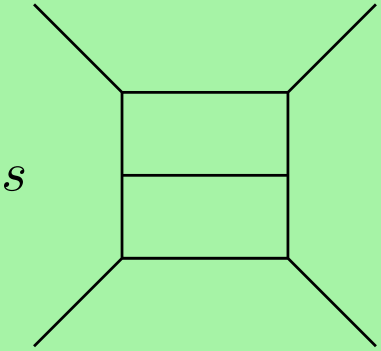
It is clear that the twistor revolution of ’03 and subsequent developments have been much kinder to legs than to loops.

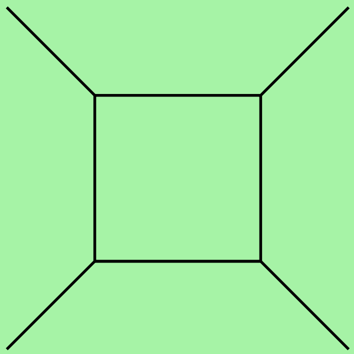
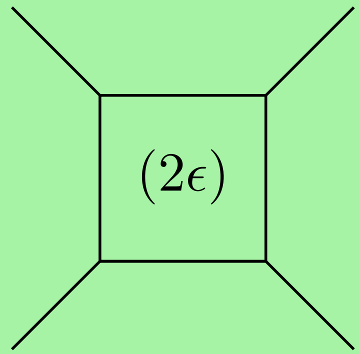
In part, this is because application of twistor methods really relies on the four-dimensional helicity decomposition $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$ whereas $N = 4$ loop amplitudes don’t really exist in four-dimensions, but rather are most commonly expressed in $D = 4 - 2\epsilon$ dimensions.

Moreover, loop integrals are in general difficult to evaluate.

Some new insights are needed if we want to start blasting up the vertical axis of the loops versus legs plot.

An intriguing idea in this direction is that MHV amplitudes satisfy **iteration relation**. For example, at two loops it has been shown that...

$M_4^{(2)}(\epsilon) = \frac{1}{4} s^2 t$  $+ \quad \frac{1}{4} t^2 s$ 

$= \frac{1}{8} s^2 t^2 \Big($  $\Big)^2 - \frac{1}{2} s t f(\epsilon)$ 

$- \frac{\pi^4}{72} + \mathcal{O}(\epsilon)$

$f(\epsilon) = (\psi(1-\epsilon) - \psi(1))/\epsilon$ [Anastasiou, Bern, Dixon, Kosower (2003)]

This identity is purely a property of Feynman loop integrals in scalar ϕ^3 theory—a property which happens to have a very nice application to Yang-Mills theory!

Iterative Structures

It has been conjectured that similar iterative relations hold to all loop orders,

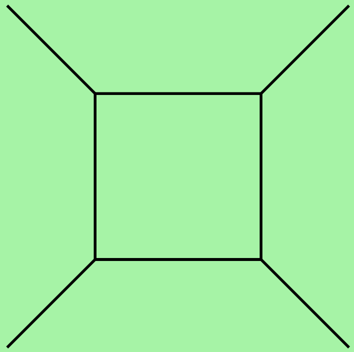
$$\boxed{M_n^{(L)}(\epsilon) = P^{(L)}(M_n^{(1)}(\epsilon), \ldots, M_n^{(L-1)}(\epsilon)) + f^{(L)}(\epsilon) M_n^{(1)}(L\epsilon) + C^{(L)} + \mathcal{O}(\epsilon)}$$

but they have only been explicitly verified for $(L, n) = (2, 4), (3, 4), (2, 5)$. [Bern, Dixon, Smirnov (2005)], [Cachazo, M.S., Volovich (2006)] & [Bern, Czakon, Kosower, Roiban, Smirnov (2006)].

This conjecture doesn't come completely out of the blue—it is based upon similar iterative structures which have been shown to hold for the infrared and collinear singularities of multiloop amplitudes [Catani (1998)], [Sterman, Tejeda-Yeomans (2003)].

Brute force analysis of these conjectures is tough, because explicitly evaluating these integrals is an exceedingly difficult task. The simplest integral is:

$$-\frac{1}{2}st$$

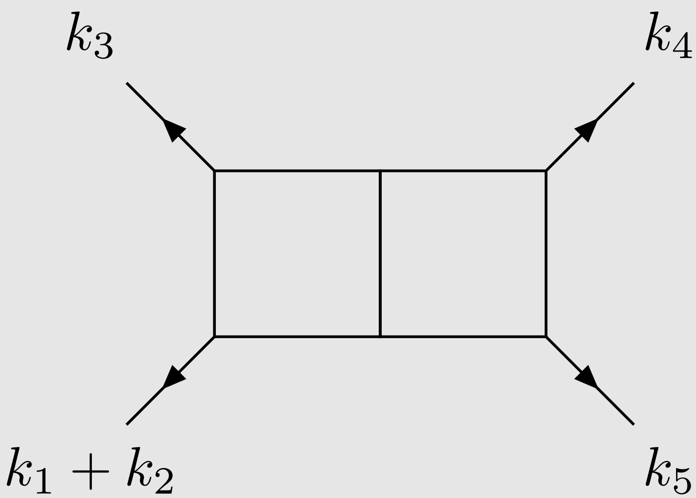


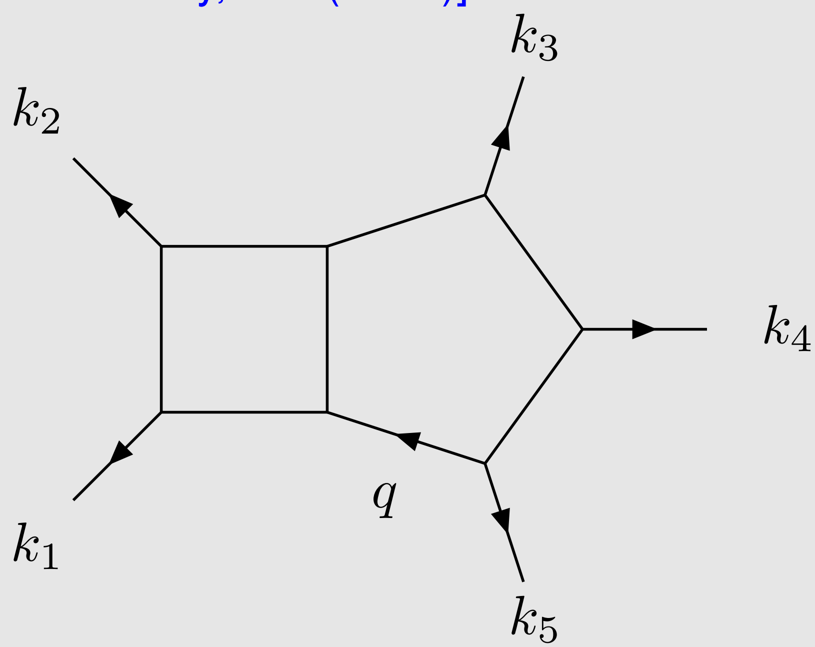
$$= -\frac{2}{\epsilon^2} + \frac{1}{4}L^2 + \frac{2\pi^2}{3} + \epsilon\Big[-H_{001}(x) - LH_{01}(x)$$
$$-\frac{1}{2}L^2H_1(x) - \frac{\pi^2}{2}H_1(x) + \frac{11\pi^2}{12}L + \frac{17\zeta(3)}{3}\Big]$$
$$+\epsilon^2\Big[H_{0001}(x) + H_{0011}(x) + H_{0101}(x) + H_{1001}(x) - \frac{1}{2}LH_{001}(x)$$
$$-LH_{011}(x) - LH_{101}(x) + \frac{1}{2}L^2H_{11}(x) + \frac{\pi^2}{2}H_{11}(x) + \frac{1}{12}L^3H_1(x)$$
$$-\zeta(3)H_1(x) + \frac{\pi^2}{4}LH_1(x) + \frac{1}{64}L^4 + \frac{\pi^2}{24}L^2 - \frac{\zeta(3)}{2}L + \frac{41\pi^4}{720}\Big]$$
$$+\epsilon^3\,[37 \text{ terms}] + \epsilon^4\,[79 \text{ terms}] + \mathcal{O}(\epsilon^5)$$

where $L = \ln(x)$ and $x = t/s$. Adapted from [Bern, Dixon, Smirnov (05/05)].

A Very Recent Example: Two Loops, Five Particles

A decade-old conjecture for the two-loop five-particle amplitude involved the (at that time unevaluated) integrals [Bern, Rozowsky, Yan (1997)]





The first of these was evaluated by [Smirnov (2000)].

Evaluating loop integrals such as these is a serious affair, and until recently was no place for amateurs.

Some New Loop Technology

The ‘empowerment of the loop proletariat’ commenced last fall when M. Czakon released a user-friendly Mathematica program which greatly facilitates these kinds of calculations. For the pentagon-box integral, one finds

$$
\begin{aligned}
I(\epsilon) = {} & \frac{(s_2 s_5)^{-2\epsilon/3}(s_3 s_4)^{-1-\epsilon/3}}{s_1}\Bigg[-\frac{3}{\epsilon^4} + \frac{1}{\epsilon^2}\Bigg(-3A\left(\frac{s_1}{s_3}\right) - 2A\left(\frac{s_4}{s_2}\right) - A\left(\frac{s_3}{s_1}\right) - A\left(\frac{s_4}{s_1}\right) \\
& -3A\left(\frac{s_1}{s_4}\right) - 2A\left(\frac{s_3}{s_5}\right) - \frac{4}{3}\ln^2\left(\frac{s_2}{s_5}\right) + \frac{1}{6}\ln^2\left(\frac{s_3}{s_4}\right) - \frac{4}{3}(L_2L_3 + L_4L_5) + \frac{2}{3}(L_2 + L_3)(L_4 + L_5) \\
& -\frac{\pi^2}{6}\Bigg) + \frac{1}{\epsilon}\Bigg(2F\left(\frac{s_1}{s_3}, \frac{s_5}{s_3}\right) + 2F\left(\frac{s_3}{s_1}, \frac{s_4}{s_1}\right) + 2F\left(\frac{s_2}{s_4}, \frac{s_1}{s_4}\right) + 6A_1\left(\frac{s_1}{s_3}\right) + 2A_1\left(\frac{s_3}{s_1}\right) \\
& -A_1\left(\frac{s_1}{s_4}\right) + A_1\left(\frac{s_4}{s_1}\right) + A_2\left(\frac{s_1}{s_3}\right) - A_2\left(\frac{s_3}{s_1}\right) + 6A_2\left(\frac{s_1}{s_4}\right) - 2A_2\left(\frac{s_4}{s_1}\right) - 2A_2\left(\frac{s_4}{s_2}\right) \\
& -2A_2\left(\frac{s_3}{s_5}\right) + 3A_3\left(\frac{s_1}{s_3}\right) + A_3\left(\frac{s_3}{s_1}\right) + A_3\left(\frac{s_4}{s_1}\right) + 2A_3\left(\frac{s_4}{s_2}\right) + 2A_3\left(\frac{s_3}{s_5}\right) - 4A_3\left(\frac{s_5}{s_2}\right) \\
& +(3L_1 - 2L_2 - 4L_3 + L_4 + 2L_5)A\left(\frac{s_1}{s_3}\right) - \frac{1}{3}(3L_1 + 2L_2 - 2L_3 + L_4 - 4L_5)A\left(\frac{s_3}{s_1}\right) \\
& -\frac{1}{3}(3L_1 - 4L_2 + L_3 - 2L_4 + 2L_5)A\left(\frac{s_4}{s_1}\right) + (2L_2 + L_3 - L_4 - 2L_5)A\left(\frac{s_1}{s_4}\right)
\end{aligned}
$$

$$+\frac{2}{3}(L_2+2L_3+2L_4-5L_5)A\left(\frac{s_4}{s_2}\right)-\frac{2}{3}(5L_2-2L_3-2L_4-L_5)A\left(\frac{s_3}{s_5}\right)$$
$$+\frac{2}{9}\pi^2(L_2+2L_3+2L_4-5L_5)+\frac{1}{27}(2L_2+L_3+L_4-2L_5)^3-\frac{2}{3}(L_2L_3^2+L_2^2L_4+L_2L_3L_4+L_3L_5^2)$$
$$\left.+\frac{10}{9}L_2(L_3+L_4)L_5+\frac{4}{9}L_3^2L_5+\frac{2}{9}(L_3L_4-L_4^2-L_2^2)L_5-\frac{2}{27}L_5^3+14\zeta(3)\right)+\cdots$$

[Cachazo, M.S., Volovich (2006)] where $s_i = -(p_i + p_{i+1})^2$, $L_i = \ln(s_i)$, and the A's are some reasonably simple expressions in terms of polylogarithms.

Armed with this result, we were able to verify (mostly analytically, but one part numerically) that the parity-even part of the two-loop, five-particle amplitude satisfies the ABDK iteration

$$M_5^{(2)}(\epsilon) = \frac{1}{2}\left(M_5^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)M_5^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon).$$

The more complicated parity-odd part of the amplitude was constructed and checked numerically by [Bern, Czakon, Kosower, Roiban, Smirnov (2006)].

Some New Loop Technology

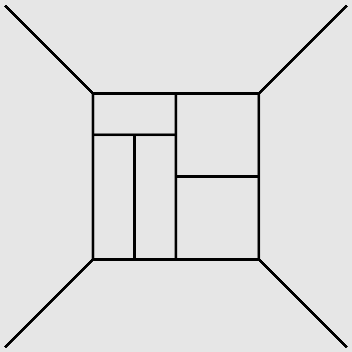
Brute force evaluation is clearly a losing strategy in the long run, and new technology is needed to explore these kinds of iterative structures more easily at higher loops...

I will now explain a method which allows these relations to be studied **without the need to fully evaluate any loop integrals** [Cachazo, M.S., Volovich (01/06, and to appear)].

Some New Loop Technology

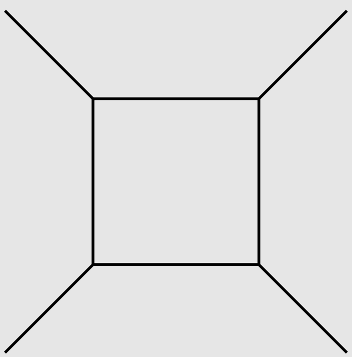
STEP 1. We observe that any dimensionally regulated L-loop four-particle Feynman integral can be written in the form (Mellin-Barnes representation)

[Smirnov, Tausk, Czakon, ...]



$$= \int_{-i\infty}^{+i\infty} dy\, x^y F(y,\epsilon), \qquad \text{where } x = t/s,$$

for some function $F(y,\epsilon)$, which is relatively easy to determine (a few seconds in Mathematica). For example,



$$\Longrightarrow F(y,\epsilon) = \Gamma(1+\tfrac{1}{2}\epsilon+y)\Gamma^2(y-\tfrac{1}{2}\epsilon)\Gamma^2(-y-\tfrac{1}{2}\epsilon)\Gamma(1-\tfrac{1}{2}\epsilon-y).$$

The final integral over y is the really nasty one.

Some New Loop Technology

STEP 1. Any four-particle integral $= \int dy\, x^y F(y,\epsilon)$.

STEP 2. It we want to study some iterative equation, it is clearly tempting to try to collect all of the terms appearing in some relation inside one y integral, and then expand through $\mathcal{O}(\epsilon)$ under the y integral.

This looks impossible, because $F(y,\epsilon)$ has poles which collide with the integration contour $\mathrm{Re}(y) = 0$ at $\epsilon = 0$, e.g.

$$F(y,\epsilon) = \Gamma(1 + \tfrac{1}{2}\epsilon + y)\Gamma^2(y - \tfrac{1}{2}\epsilon)\Gamma^2(-y - \tfrac{1}{2}\epsilon)\Gamma(1 - \tfrac{1}{2}\epsilon - y).$$

This signals that expanding in ϵ and performing the y integral do not commute—we are not allowed to expand in ϵ under the integral. We call these annoying poles obstructions.

Some New Loop Technology

STEP 1. Any four-particle integral $= \int dy\, x^y F(y, \epsilon)$.

STEP 2. Collect all integrals under a single y integral.

STEP 3. Kill all obstructions by means of a suitably chosen linear differential operator, e.g.

$$\left(\left(x\frac{d}{dx}\right)^2 - \frac{\epsilon^2}{4}\right)^2 \int_{-i\infty}^{+i\infty} dy\, x^y F(y,\epsilon) = \int_{-i\infty}^{+i\infty} dy\, x^y (y^2 - \frac{\epsilon^2}{4})^2 F(y,\epsilon)$$

Now the poles at $y = \pm\frac{1}{2}\epsilon$ are completely removed, and it is safe to expand in ϵ under the y integral.

Different integrals can have different numbers of obstructions; choose a differential operator $\mathcal{L}$ which kills all of them...

Some New Loop Technology

STEP 1. Any four-particle integral $= \int dy\, x^y F(y, \epsilon)$.

STEP 2. Collect all integrals under a single y integral.

STEP 3. Kill all obstructions with a differential operator $\mathcal{L}$.

STEP 4. We must fix the ambiguity introduced by acting with $\mathcal{L}$. Fortunately, the types of operators which we need to use have very simple kernels.

For example, at two loops, the only ambiguity is

$$\frac{a_1}{\epsilon^4} + \frac{a_2}{\epsilon^2} \ln^2 x + a_3 \ln^4 x + \mathcal{O}(\epsilon^2).$$

Three numbers are a very small price to pay in exchange for being able to unambiguously fix all polylog functions. In any case these numbers can be fixed using infrared and collinear limits.

Some Caveats

I think the idea that one can isolate the all of the dependence on the kinematic variables through an inverse Mellin transform with respect to $x = t/s$ is pretty neat, and it clearly tells part of the story of these seemingly very complicated integrals, but it is also clear that they do not tell the whole story...

For one thing, the pattern of obstructions in the three-loop amplitude of [Bern, Dixon, Smirnov (05/05)] does not seem to hint at any nice pattern.

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For one thing, the pattern of obstructions in the three-loop amplitude of [Bern, Dixon, Smirnov (05/05)] does not seem to hint at any nice pattern.

Moreover, there is the important question of how to obtain a formula for the L-loop amplitude $M_4^{(L)}(\epsilon)$ on which one might try similar analyses.

So far, the best ansatz we have comes from the 'rung rule' according to which higher-loop diagrams can be built up out of lower-loop amplitudes by adding 'rungs' in all possible locations [Bern, Rozowsky, Yan (1997)], but this rule could break down starting at four loops, leaving us in the dark.

Exercise 2. Expose additional hidden beauty buried deep inside the structure of multiloop amplitudes. Extra credit for unlocking any all-loop information!

Summary

Recent insights into the mathematical structure of perturbative gauge theory has helped to make previously impossible calculations possible, and sometimes even simple.

Within a period of little over a year, tree-level Yang-Mills theory was (more or less) completely solved, as was supersymmetric Yang-Mills theory at one loop.

Prospects are good for progress on multiloop amplitudes, especially in the $\mathcal{N} = 4$ supersymmetric theory.

To infinity and beyond!