AdS/CFT duality, spin chains and 2d effective actions

R. Roiban, A. Tirziu and A. A. Tseytlin

“Asymptotic Bethe ansatz S-matrix and Landau-Lifshitz type effective 2-d actions”, hep-th/0604199

also talks by Belitsky, Korchemsky, and Klebanov

AdS/CFT

$\mathcal{N} = 4$ SYM at $N = \infty$

dual to type IIB superstrings in $AdS_5 \times S^5$

Parameters:

$\lambda = g^2_{YM} N$ related to string tension

$2\pi T = \frac{R^2}{\alpha'} = \sqrt{\lambda}$

$g_s = \frac{\lambda}{4\pi N} \to 0$

string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, J, m, ...) = \Delta(\lambda, J, m, ...)$$

J - global charges of $SO(2,4) \times SO(6)$:

spins $S_1, S_2;\ J_1, J_2, J_3$

m - windings, folds, cusps, oscillation numbers, ...

Operators: Tr$(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3} D_+^{S_1} D_\perp^{S_2} ... F_{mn} ... \Psi ...)$

Solve SYM/string theory $\to$

compute $E = \Delta$ for any λ (and J,m)

Solve non-trivial max. susy 4-d CFT = string in curved R-R background

Remarkable well-defined (but hard !) problem of mathematical physics hope to learn more about less susy theories, e.g., role of integrability and string picture in perturbative / large energy QCD

Perturbative expansions are opposite:

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative planar gauge theory

“Constructive” approach:

use perturbative results on both sides and other properties (integrability, susy,...) as guides to guesses of exact answers (Bethe ansatz,...)

Remarkable recent progress:

– “semiclassical” states with large quantum numbers ($J \gg 1$)

dual to “long” gauge operators

$E = \Delta$ – same dependence on J, m

coefficients = interpolating functions of λ

– connection to spectrum of integrable spin chain

– advances in uncovering of the structure of underlying Bethe ansatz

Particular sector: string states in S^5

e.g., strings moving in S^3 of S^5 – operators built of SYM scalars

$\Phi_1 = \phi_1 + i\phi_2, \quad \Phi_2 = \phi_3 + i\phi_4$

dimensions = eigenvalues of spin chain Hamiltonian

"$SU(2)$ sector": Tr$(\Phi^{J_1}\Phi^{J_2}) + ...,\quad J = J_1 + J_2$

planar 1-loop dilatation operator of $\mathcal{N} = 4$ SYM

= Hamiltonian of ferromagnetic Heisenberg XXX$_{1/2}$ spin chain
(Minahan, Zarembo, 2002):

$$H_1 = \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{J}(I - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1})$$

Higher orders (Beisert,Kristjansen,Staudacher, 2003; Beisert, 2004):

$$H_2 = \frac{\lambda^2}{(4\pi)^4}\sum_{l=1}^{J}(-3 + 4\vec{\sigma}_l \cdot \vec{\sigma}_{l+1} - \vec{\sigma}_l \cdot \vec{\sigma}_{l+2})$$

H_3 contains $\vec{\sigma}_l \cdot \vec{\sigma}_{l+3}$ but also $(\vec{\sigma}_l \cdot \vec{\sigma}_{l+1})(\vec{\sigma}_{l+2} \cdot \vec{\sigma}_{l+3})$, etc.

"long-range" ferromagnetic spin chain with "multi-spin" interactions

but (at least to 3 loop order) H= effective Hamiltonian for Hubbard model

(Rej, Serban, Staudacher, 2005)

Integrability $\longrightarrow$ Exact Bethe ansatz $\longrightarrow$ Spectrum ?

Expect spectrum to have qualitatively same structure at any λ:

smooth change with λ (at least for large J_1, J_2)

indeed, remarkable correspondence between string and gauge states

Structure of Spectrum at large J:

1-loop: Heisenberg model $E = J + \lambda E_1 + O(\lambda^2)$

$E_1 = 0$: ferromagnetic vacuum

– BPS operator Tr Φ^J (point-like string)

$E_1 \sim \frac{1}{J^2}$: magnons

– BMN states Tr$([\Phi_1....\Phi_1]\Phi_2[\Phi_1....\Phi_1]\Phi_2...)$

("short" fast strings with c.o.m. along S^5 geodesic $J_1 \gg J_2$)

$E_1 \sim \frac{1}{J}$: low-energy spin waves

"Thermodynamic" limit ($J_1 \sim J_2 \gg 1$): long "locally BPS" operators

Tr$([\Phi_1....\Phi_1][\Phi_2...\Phi_2][\Phi_1....\Phi_1][\Phi_2...\Phi_2]...)$

– (long fast strings, nearly-null world surface) (Frolov, AT, 2002)

$E_1 \sim 1$: "intermediate" states

– long rotating slow-moving strings $J_2 \gg J_1 \gg 1$

(Gubser, Klebanov, Polyakov, 2002; Beisert, Frolov, Staudacher, AT, 2003; Hofman, Maldacena, 2006; Dorey, 2006)

$E_1 \sim J$: anti-ferromagnetic state (and near-by spinons)

– long slowly-rotating string $J_1 = J_2$ (and fluctuations)

(Roiban, Tirziu, AT, 2006)

Different limits:

perturbative semiclassical string side: $\lambda \gg 1,\ \frac{J}{\sqrt{\lambda}}$=fixed

perturbative gauge side: $\lambda \ll 1,\ \ J \gg 1$

still, in some cases few leading coefficients match exactly (for BMN and fast strings): susy protection

general pattern: strong-weak coupling interpolation

Example: single-spin (GKP) string at rest

$E = J + f(\lambda) + O(\frac{1}{J})$

$f(\lambda \to 0) = c_1\lambda + c_2\lambda^2 + ...$

$f(\lambda \to \infty) = a_1\sqrt{\lambda} + a_2 + \frac{a_3}{\sqrt{\lambda}} + ...$

$f(\lambda) = 2\sqrt{1 + \frac{\lambda}{\pi^2}}$?

as for two “giant” magnons (Hofman, Maldacena)

Low-energy states: fast 2-spin strings

perturbative string:

classical + quantum $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections

first large λ or large J at fixed $\tilde{\lambda} \equiv \frac{\lambda}{J^2}$, then expand in $\tilde{\lambda}$

perturbative SYM:

first small λ, then expand in large J

get same structure and same coefficients at first two orders

(Frolov, AT, 2003; Beisert,Minahan,Staudacher,Zarembo, 2003)

interpolating function of λ starting from "3-loop" order

quantum string expansion near fast strings contains "non-analytic" terms with explicit factors of $\sqrt{\lambda}$ (Beisert, AT, 2005)

$$E = J\Big[1 + \tilde{\lambda}(a_0 + \frac{a_1}{J} + ...) + \tilde{\lambda}^2(b_0 + \frac{b_1}{J} + ...) + \tilde{\lambda}^3(f(\lambda) + ...) + ...\Big], \qquad \tilde{\lambda} \equiv \frac{\lambda}{J^2}$$

interpolating function:

$$f(\lambda \gg 1) = c_0 + \frac{c_1}{\sqrt{\lambda}} + ... , \qquad f(\lambda \ll 1) = d_1 + d_2\lambda + ... , \qquad c_0 \neq d_1$$

All-order Bethe ansatz

Strong indications of integrability of both string and gauge theory:

expect Bethe ansatz description of spectrum for any λ

$$e^{ip_k J} = \prod_{j\neq k}^{M} S(p_k, p_j; \lambda)$$

$$S(p_k, p_j; \lambda) = S_1(p_k, p_j; \lambda)\, e^{i\theta(p_k, p_j; \lambda)}$$

$$S_1 = \frac{u_k - u_j + i}{u_k - u_j - i}\,.$$

scattering of elementary excitations (magnons) with 1-d momenta p_j and rapidities u_j (Beisert, Dippel, Staudacher, 2004)

$$u_j = \tfrac{1}{2}\cot\tfrac{p_j}{2}\,\sqrt{1+\tfrac{\lambda}{\pi^2}\sin^2\tfrac{p_j}{2}}$$

dispersion relation justified on supersymmetry grounds

(Beisert, 2005; Hofman, Maldacena, 2006)

S = phase shift due to magnon scattering (Staudacher, 2005)

Find p_j for bound states with $\sum_{k=1}^{M} p_k = 2\pi m$ (cyclicity of the trace)

Then energies of states:

$$E = \sum_{j=1}^{M}\left(\sqrt{1+\tfrac{\lambda}{\pi^2}\sin^2\tfrac{p_j}{2}} - 1\right).$$

“Asymptotic ansatz” (BDS):

$J \to \infty$, up to λ^J order: $S = S_1, \ \theta \to 0$

But to match string theory results need non-trivial phase θ

(Arutyunov, Frolov, Staudacher, 2004)

θ – common to all sectors, structure fixed by symmetries (Beisert, 2005)

$$\theta(p', p; \lambda) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{rs}(\lambda) \Big[q_s(p) q_r(p') - q_s(p') q_r(p) \Big]$$

$$q_{r+1}(p) = \frac{2}{r} \sin \tfrac{rp}{2} \left(\frac{\sqrt{1 + 4\bar{\lambda} \sin^2 \frac{p}{2}} - 1}{\bar{\lambda} \sin \frac{p}{2}} \right)^r , \qquad \bar{\lambda} = \frac{\lambda}{(4\pi)^2}$$

$$c_{rs}(\lambda) = \bar{\lambda}^{\frac{r+s-1}{2}} \Big[\delta_{r,s-1} + \frac{1}{\sqrt{\bar{\lambda}}} a_{rs} + \frac{1}{(\sqrt{\bar{\lambda}})^2} b_{rs} + ... \Big]$$

At small λ expect $c_{rs} \to \lambda^2 + ...$

Matching to classical string: $(c_{rs})_{\lambda \to \infty} \to \lambda^{\frac{r+s-1}{2}} \delta_{r,s-1}$ (AFS)

$AdS_5 \times S^5$ superstring (Metsaev, AT, 98) $\longrightarrow$ 1-loop corrections to spinning string energies (Frolov, AT, 2003)

imply $a_{rs} \neq 0$ (Beisert, AT, 2005) $\longrightarrow$

e.g., presence of interpolating functions in E_{string} starting with “3-loop” order

1-loop string results translate to (Hernandez, Lopez, 2006)

$$a_{rs} = \frac{4}{\pi} \frac{(r-1)(s-1)}{(r-1)^2 - (s-1)^2}$$

Consistent (Arutyunov, Frolov, 2006) with crossing (Janik, 2006)

Outstanding problem: compute θ, i.e. interpolating functions $c_{rs}(\lambda)$

From some basic principles (crossing,...)?

Compute directly from string theory ?

String sigma model: suggests to relate S to scattering matrix of integrable 2d field theory whose fundamental excitations correspond to spin chain magnons (Polchinski, Mann, 2005; Gromov, Kazakov, Sakai, Viera, 2006; Kloze, Zarembo, 2006)

But:

S= scattering of magnons with “non-relativistic” dispersion relation

identify and compute same object on string side

effective S-matrix of “positive-energy” parts of BMN-type string modes (Roiban, Tirziu, AT, 2006)

Key role played by non-relativistic “Landau-Lifshitz” type effective action

Effective field theory approach:

two microscopically consistent theories

– spin chain and superstring –

lower part of the spectrum approximated by low-energy 2d effective actions: slow modes at large J

lead to non-relativistic "Landau-Lifshitz" 2d action

(Kruczenski, 2003; Kruczenski, Ryzhov, AT, 2004)

$\lambda \gg 1$ to $\lambda \ll 1$ interpolation between "string" and "gauge" effective actions and corresponding "spin chains"

Coherent-state action for low-energy excitations of spin chain (determined by H = dilatation operator)

and "fast-string" limit of string action

$\vec{n}$ – transverse position of string in S^3

or spin coherent state $U^\dagger \vec{\sigma} U = \vec{n},\ \vec{n}^2 = 1$

Continuum limit:

Classical Landau-Lifshitz equations of motion

$$\partial_t n_i = \frac{1}{2}\tilde{\lambda}\epsilon_{ijk} n_j \partial_\sigma^2 n_k\ , \qquad \tilde{\lambda} \equiv \frac{\lambda}{J^2}$$

describing lower part of spectrum with energies $\sim J\tilde{\lambda} = \frac{\lambda}{J}$

LL action beyond leading order

effective actions from gauge-theory spin chain and string theory:

$$S = \int dt \int_0^J d\sigma \, L \, , \qquad \bar{\lambda} = \frac{\lambda}{(2\pi)^2}$$

$$L = \vec{C}(n) \cdot \partial_t \vec{n} - \frac{1}{4}\vec{n} \left(\sqrt{1 - \bar{\lambda}\partial_\sigma^2} - 1\right) \vec{n} \, - \frac{3\bar{\lambda}^2}{128}(\partial_\sigma \vec{n})^4$$

$$- \frac{\bar{\lambda}^3}{64}\left[-\frac{7}{4} \, (\partial_\sigma \vec{n})^2 (\partial_\sigma^2 \vec{n})^2 \, + \, \mathrm{b}(\lambda) \, (\partial_\sigma \vec{n} \partial_\sigma^2 \vec{n})^2 \, + \, \mathrm{c}(\lambda) \, (\partial_\sigma \vec{n})^6 \right] + ...$$

quadratic part is exact: reproduces the BMN dispersion relation for small ("magnon") fluctuations near BPS vacuum $\vec{n} = (0, 0, 1)$.

cf. $\partial_t^2 - \partial_\sigma^2 + m^2 \to (i\partial_t - \sqrt{m^2 - \partial_\sigma^2})(-i\partial_t - \sqrt{m^2 - \partial_\sigma^2})$

Orders $\bar{\lambda}$ and $\bar{\lambda}^2$:

agreement between two effective actions $\longrightarrow$ explains observed agreement of energies of states, integrable structures, etc.

"3-loop" coefficients in the string and gauge theory expressions are interpolating functions:

$$\lambda \gg 1 : \qquad \mathrm{b} = -\frac{25}{2} + O(\frac{1}{\sqrt{\lambda}}) \, , \qquad \mathrm{c} = \frac{13}{16} + O(\frac{1}{\sqrt{\lambda}}) \, ,$$

$$\lambda \ll 1 : \qquad \mathrm{b} = -\frac{23}{2} + O(\lambda) \, , \qquad \mathrm{c} = \frac{12}{16} + O(\lambda)$$

Field theory S-matrix for "magnons"

LL action in terms of complex scalar ϕ:

$$n_s = 2\sqrt{1-z^2}\, z_s\ , \qquad \phi \equiv z_1 + iz_2\ ,$$

$$S = \int dt \int_0^J dx\, \left\{ \phi^* \left[i\partial_t - (\sqrt{1-\bar\lambda\partial_x^2} - 1)\right]\phi \ -\ V(\phi,\phi^*)\right\}$$

$$V = V_4 + V_6 + ...\ , \qquad V_{2n} \sim \sum_{k=1}^{\infty} \bar\lambda^k \partial_x^{2k} (\phi^*\phi)^n$$

$$\begin{aligned} V_4 \quad &= \quad \frac{\bar\lambda}{4}(\phi^{*2}\phi'^2 + c.c) \\ &- \quad \frac{\bar\lambda^2}{8}\left[\frac{1}{2}|\phi|^2(\phi''''\phi^* + c.c.) + 6|\phi''|^2|\phi|^2 - 3|\phi'|^4\right] + ... \end{aligned}$$

generalizing 2-partile scattering in LL model (Kloze, Zarembo, 2006)

$$\langle kk'|\hat S|pp'\rangle = \langle kk'|Te^{-i\int dt dx V_4}|pp'\rangle\ .$$

$$\delta(\omega_p+\omega_{p'}-\omega_k-\omega_{k'})\delta(p+p'-k-k') = K(p,p')\,\delta_+(p,p',k,k')\ ,$$

$$\delta_+ \equiv \delta(p-k)\delta(p'-k') + \delta(p-k')\delta(p'-k)\ ,$$

$$K(p,p') = \ \frac{e(p)\, e(p')}{p\, e(p') - p'\, e(p)}\ ,$$

$$\omega_p = e(p) - 1\ , \qquad e(p) \equiv \sqrt{1+\bar\lambda p^2}$$

Bethe ansatz S-matrix vs LL effective action

$$p \to 0\,, \qquad u(p) \to \frac{e(p)}{p} = \frac{1}{p}\sqrt{1+\bar\lambda p^2}$$

$$S_{\rm BDS} = \frac{u(p')-u(p)+i}{u(p')-u(p)-i} \qquad \to \qquad \tilde S_{\rm BDS} = \frac{1+\frac{ipp'}{pe(p')-p'e(p)}}{1-\frac{ipp'}{pe(p')-p'e(p)}}$$

Tree level:

$$(\tilde S_{\rm BDS})_{\rm tree} = \frac{2ipp'}{pe(p')-p'e(p)}$$

Similarly:

$$(\tilde S_{\rm AFS})_{\rm tree} = \frac{2iF(p,p')}{p\, e(p')-p'\, e(p)}\ ,$$

$$F(p,p') = pp' + \frac{1}{2}[p\, e(p')-p'\, e(p)]\ \tilde\theta_{\rm AFS}$$

$$= \big(pp' - \bar\lambda^{-1}[e(p)-1][e(p')-1]\big)\left[1+\frac{1}{4}\big(\bar\lambda pp' - [e(p)-1][e(p')-1]\big)\right]$$

Detailed matching to LL effective field theory S-matrix with corresponding coefficients

In BDS case matching extends to quantum LL S-matrix

AFS S-matrix from string theory:

Comparing spin chain phase shift for magnons near ferromagnetic vacuum

should re-organize the string-theory side S-matrix for BMN-type modes

into the S-matrix for an effective field theory of the positive-energy modes

classical string action on $R \times S^3$

$$ds^2 = -dt^2 + [d\alpha + C(n)]^2 + d\vec{n}d\vec{n}$$

in "uniform" gauge: $t = \tau, \ \ \tilde{\alpha} = \frac{J}{\sqrt{\lambda}}\sigma$, i.e. $p_\alpha = \frac{J}{\sqrt{\lambda}}$=const (Kruczenski, Ryzhov, AT, 2004)

$$L = C_t - \sqrt{[1 - \frac{1}{4}(\partial_t \vec{n})^2][1 + \frac{\bar{\lambda}}{4}(\partial_x \vec{n})^2] + \frac{\bar{\lambda}}{16}(\partial_t \vec{n} \cdot \partial_x \vec{n})^2} \ .$$

Expanding near $\vec{n} = (0, 0, 1)$

$$\begin{aligned} L \ &= \ i\phi^* \partial_t \phi - \frac{1}{2}\phi^*(\partial_t^2 - \bar{\lambda}\partial_x^2)\phi + \frac{1}{4}\left[\phi^{*2}(\dot{\phi}^2 - \bar{\lambda}\phi'^2) + c.c.\right] \\ &+ \ \frac{1}{8}\left(\dot{\phi}^{*2} - \bar{\lambda}\phi'^{*2}\right)\left(\dot{\phi}^2 - \bar{\lambda}\phi'^2\right) + O(\phi^6) \end{aligned}$$

"integrate out" negative-energy modes

Start with relativistic scalar field theory:

$$\partial_t^2 - \partial_x^2 + m^2 = -D_+ \, D_-$$

$$D_\pm(\partial) \equiv i\partial_t \mp e(i\partial_x) \,, \qquad e(i\partial_x) \equiv \sqrt{m^2 - \partial_x^2}$$

consider diagrams where only ϕ_+ and ϕ_+^* appear on external lines:

which field theory with 1-st order kinetic term D_- reproduces same S-matrix: field redefinition

$$\hat{\phi}_\pm = \sqrt{D_\mp(\partial)} \, \phi_\pm$$

$$\begin{aligned} L &= \hat{\phi}_+^* \left(i\partial_t - \sqrt{m^2 - \partial_x^2} \right) \hat{\phi}_+ \\ &- \frac{V_4(\partial^{(i)}) \hat{\phi}_+^*(z_1) \hat{\phi}_+^*(z_2) \hat{\phi}_+(z_3) \hat{\phi}_+(z_4)}{\sqrt{D_-(\partial^{(1)}) \, D_-(\partial^{(2)}) \, D_-(\partial^{(3)}) \, D_-(\partial^{(4)})}} \\ &+ \text{terms containing } \hat{\phi}_-, \hat{\phi}_-^* \,. \end{aligned}$$

Result is same as Bethe ansatz AFS one:

$$(S_{\text{string}}^{\mathrm{SU}(2)})_{\text{tree}} = \frac{2iF(p,p')}{pe(p') - p'e(p)}$$

corresponding non-relativistic effective Lagrangian

$$L = \phi^* \Big[i\partial_t - (\sqrt{1 - \bar{\lambda} \partial_x^2} - 1) \Big] \phi - V_4(\phi) + O(\phi^6) \,,$$

$$V_4 = \frac{1}{4} \big\{ \big(\frac{1}{\sqrt{e(i\partial_x)}} \phi^* \big)^2 \big[\big(\frac{e(i\partial_x) - 1}{\sqrt{e(i\partial_x)}} \phi \big)^2$$

agrees with expansion of LL action

Some conclusions

- Matching between gauge and string states near and far from BPS limit
- Presence of non-analytic in λ terms in quantum string semiclassical expansion implies non-trivial interpolation functions in Bethe ansatz phase, string energies, LL action
- Landau-Lifshitz type action: low-energy effective action for relevant string/spin chain modes
- S-matrix in Bethe ansatz – effective S-matrix for positive energy part of string modes
- Direct confirmation of AFS phase
- Compute directly quantum string 1-loop S-matrix
- Find/Guess exact form of the phase?!