From Trees to Loops and Back

by Andreas Brandhuber

Queen Mary, University of London

Continuous Advances in QCD 2006

based on hep-th/0510253 AB-Spence-Travaglini

and also
hep-th/0412108 Bedford-AB-Spence-Travaglini
hep-th/0410280 Bedford-AB-Spence-Travaglini
hep-th/0407214 AB-Spence-Travaglini

Goal

- The goal of this talk is to address the following question:

  Do MHV Diagrams provide a new, complete, perturbative expansion of Supersymmetric Yang-Mills ?

Evidence so far

- Tree Level Amplitudes ⟹ complete proof (Cachazo-Svrcek-Witten, Britto-Cachazo-Feng-Witten, Risager)
- One-Loop Amplitudes in (S)YM
  - MHV method at one-loop ⟹ Explicit calculation of MHV one-loop Amplitudes in N=4 SYM(AB-Spence-Travaglini)
  - Generalisation to N=1 SYM: MHV one-loop amplitudes (Bedford-AB-Spence-Travaglini, Quigley-Rozali)
  - Cut-Constructible Parts of MHV one-loop Amplitudes in pure Yang-Mills (Bedford-AB-Spence-Travaglini)
    New results for QCD from MHV diagrams!
    For rational parts other techniques needed
    (see Berger, Bern, Forde, Dixon, Kosower)

MHV Diagrams

$$A^{tree}_{MHV} = i g^{n-2} \frac{\langle i j\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

(Parke-Taylor, Berends-Giele)

- MHV amplitude (holomorphic) = line in twistor space = local interaction in Minkowski space (Witten, Cachazo-Svrcek-Witten)

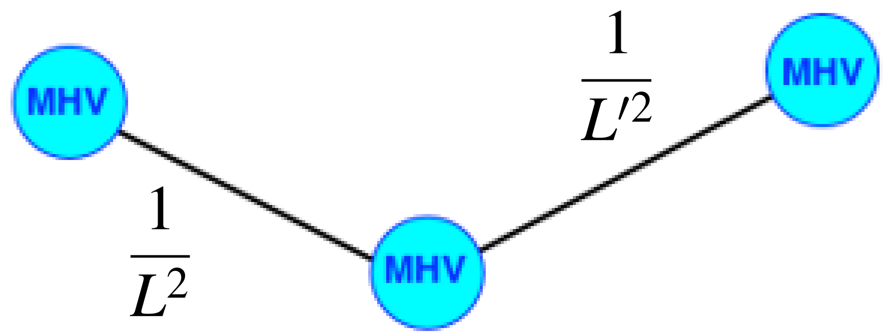
- CSW Rules (Cachazo-Svrcek-Witten)

  - MHV amplitudes promoted to local vertices using off-shell continuation: $L_\mu = l_\mu + z\,\eta_\mu\,,\ \lambda_a \sim L_{a\dot{a}}\tilde{\eta}^{\dot{a}}$

  - Connect MHV vertices with scalar propagators

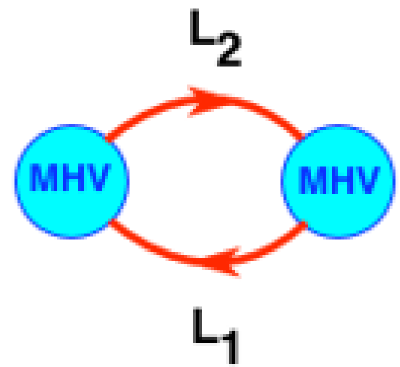
MHV Diagrams

Tree Level



MHV One Loop

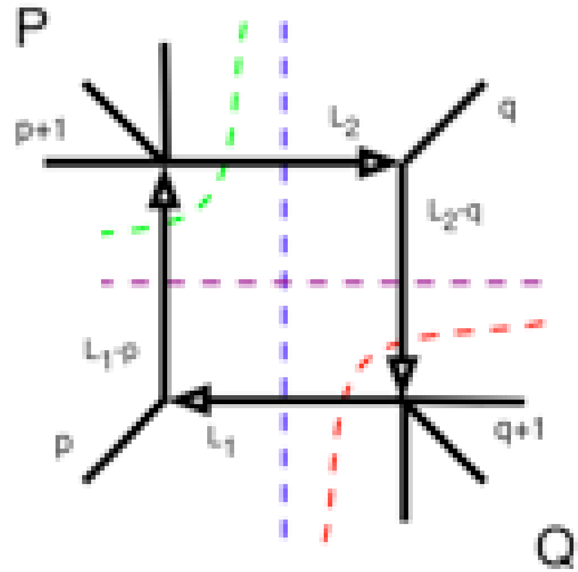
$$\int d\mathcal{M} \sum_{m_1, m_2, h}$$



- Covariance (= η−independence) is achieved after summing all MHV diagrams: Highly Non-Trivial Cancellations!

MHV One-Loop N=4 SYM (AB-Spence-Travaglini)

$$A^{\text{tree}}_{\text{MHV}} \times$$



- are a linear combination of "2-mass easy" box functions
- Surprise: with a slight generalisation of BST, we find the box functions to all orders in ϵ (= dim'l regul. param.)

$$F^{2me}(s,t,P^2,Q^2) = -\frac{c_\Gamma}{\varepsilon^2}\left[\left(\frac{-s}{\mu^2}\right)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon,1-\varepsilon,as\right) + \left(\frac{-t}{\mu^2}\right)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon,1-\varepsilon,at\right)\right.$$

$$\left. - \left(\frac{-P^2}{\mu^2}\right)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon,1-\varepsilon,aP^2\right) - \left(\frac{-Q^2}{\mu^2}\right)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon,1-\varepsilon,aQ^2\right)\right] \quad a := \frac{2(pq)}{P^2Q^2 - st}$$

From Loops Back to Trees

via the Feynman Tree Theorem (FTT)

- We want to show that MHV diagrams are equivalent to Feynman diagrams for generic one-loop amplitudes in SYM
- Step 1: proof of covariance using FTT
- The FTT is based on the decomposition of the conventional Feynman propagator:

$$\Delta_F(P) = \Delta_R(P) + 2\pi\delta^{(-)}(P^2 - m^2)$$

$$\delta^{(-)}(P^2 - m^2) \equiv \delta(P^2 - m^2)\theta(-P_0)$$

FTT

- Assume we use Feynman rules with $\Delta_F(P)$ instead of $\Delta_R(P)$
- Since $\Delta_R(P)$ is a causal propagator (contrary to $\Delta_F(P)$) any loop integral

$$I_R = \int \prod_i d^4x_i \, \Delta_R(x_1 - x_2) V(x_2) \Delta_R(x_2 - x_3) V(x_3) \cdots \Delta_R(x_n - x_1) V(x_1) = 0$$

with local vertices has support for: $t_1 > t_2 > \cdots > t_n > t_1$

Since there are no closed time-like curves in Minkowski space this integral vanishes!

FTT cont'd

- Now use the decomposition of $\Delta_R(P)$ into $\Delta_F(P)$ and an on-shell delta-function

$$I_R := \int \frac{d^4L}{(2\pi)^4} f(L,\{K_i\}) \prod_i \left[\Delta_F(L+K_i) - 2\pi\delta^{(-)}\left((L+K_i)^2\right)\right] = 0$$

to find the FTT

$$I_F = -\int \frac{d^4L}{(2\pi)^4} f(L,\{K_i\}) \prod_i{}' \left[\Delta_F(L+K_i) - 2\pi\delta^{(-)}\left((L+K_i)^2\right)\right]$$

In a nutshell: the FTT reduces Loops to Trees! Or more precisely to the sum of all possible cuts.

$$I_F = I_{1-cut} + I_{2-cut} + I_{3-cut} + I_{4-cut}$$

FTT and MHV Diagrams

- MHV vertex = Point in Mink ⇒ apply FTT to MHV diagrams

  ☆ Simple proof of covariance for the sum of MHV diagrams contributing to generic (one-)loop amplitudes

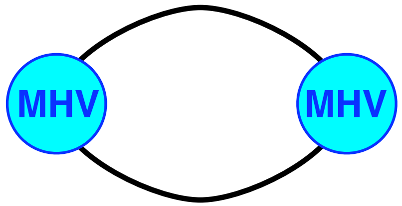
- Amplitude is a sum of terms in which at least one loop leg is cut

$$\mathcal{A} = \mathcal{A}_{1-cut} + \mathcal{A}_{2-cut} + \mathcal{A}_{3-cut} + \mathcal{A}_{4-cut}$$

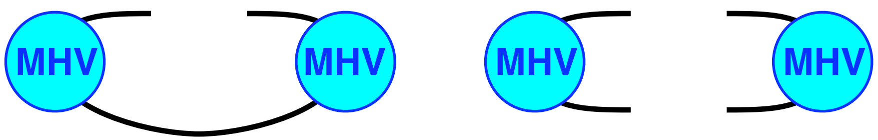
Key point: each set of p-particle cut diagrams sums to a covariant expression!

FTT and MHV Diagrams cont'd

- MHV one-loop amplitudes



The MHV diagrams have the following 1-particle and 2-particle cuts

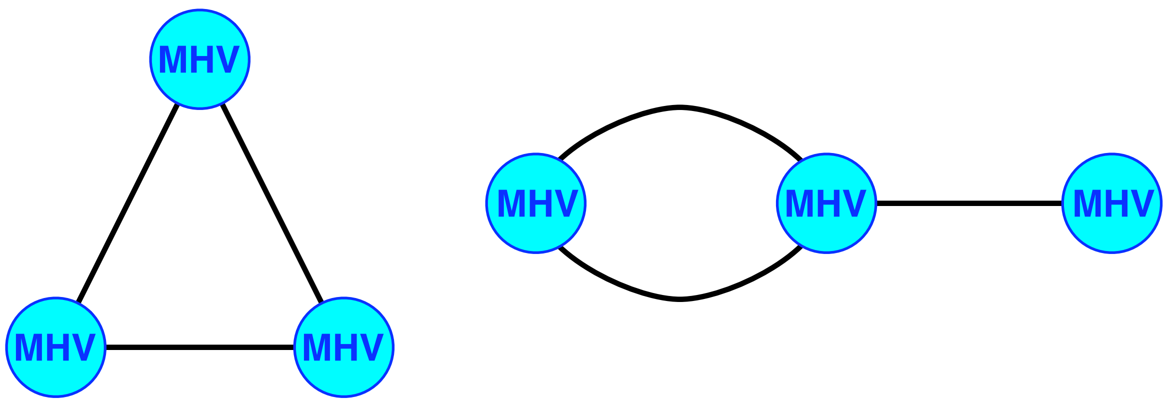


2-particle cuts give a phase space integral of a product of on-shell tree amplitudes and hence are covariant

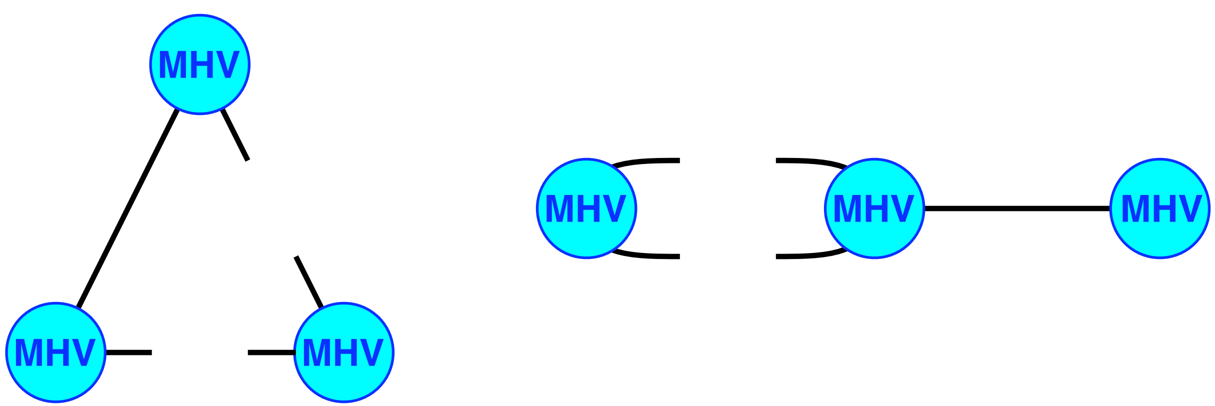
1-particle cuts also give a covariant contribution

FTT and MHV Diagrams cont'd

- more complicated examples can be treated in complete analogy
- NMHV Amplitudes



MHV diagrams



2-particle cut diagrams

Discontinuities

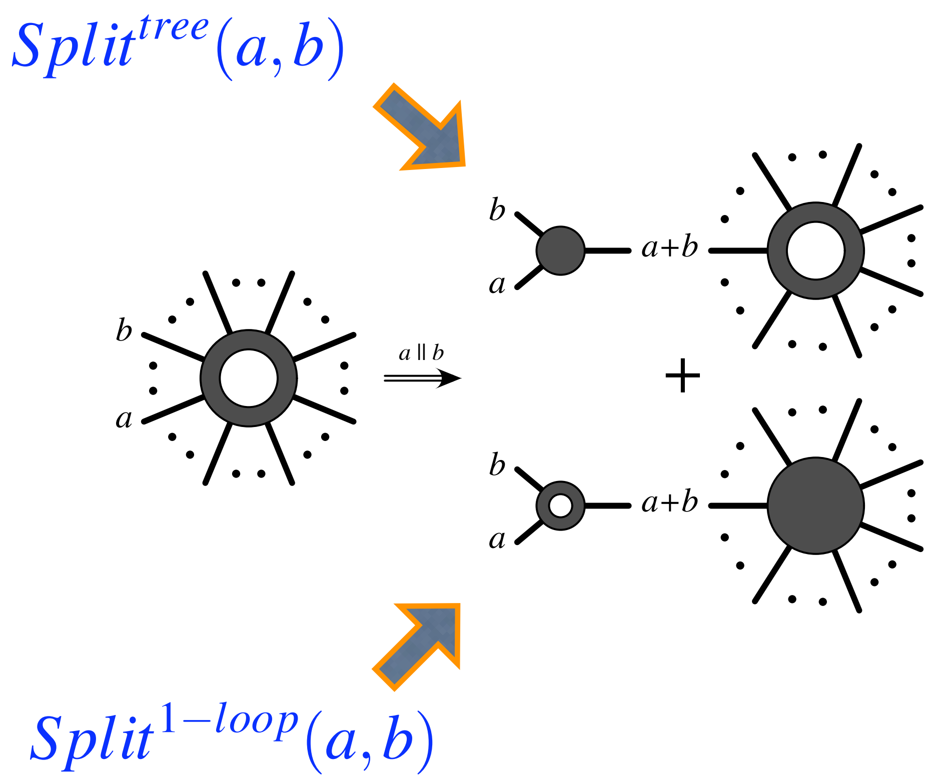
- One-loop MHV diagrams give covariant expressions
- Step 2: check unitarity cuts in all channels.
- Straightforward; diagrammatics is the same for Feynman 2-particle cuts in the FTT and unitarity 2-particle cuts.
- In a particular channel one fixes two propagators and replaces them by two on-shell delta functions. Summing all MHV diagrams sharing the same 2-particle cut, one obtains the full tree amplitudes on both sides of the cut. LIPS integration produces expected discontinuity.
- Works also for generalised (n-particle) cuts

Factorisation

- MHV one-loop diagrams give covariant formulas with all the correct (generalised) cuts
- Step 3: Check singularities in all kinematic limits. We'll check the universal collinear and some of the soft limits.
- Unphysical, η−dependent singularities (and cuts) are excluded by our proof of covariance
- The remaining ambiguity must be a polynomial term, which can be ruled out on dimensional grounds (as in Britto-Cachazo-Feng-Witten)

Universal collinear factorisation

- Consider a one-loop amplitude $\mathcal{A}_n^{1-loop}$ in the limit when momenta a and b become collinear (parallel)



Universal Collinear Factorisation

Involves tree and one-loop splitting functions:

$$Split_{-}^{tree}(a^{+},b^{+}) = \frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle a\,b\rangle}, \;\; Split_{+}^{tree}(a^{-},b^{-}) = -\frac{1}{\sqrt{z(1-z)}}\frac{1}{[a\,b]}$$

With $k_a := zk_P,\; k_b := (1-z)k_P,\; k_P^2 \to 0$

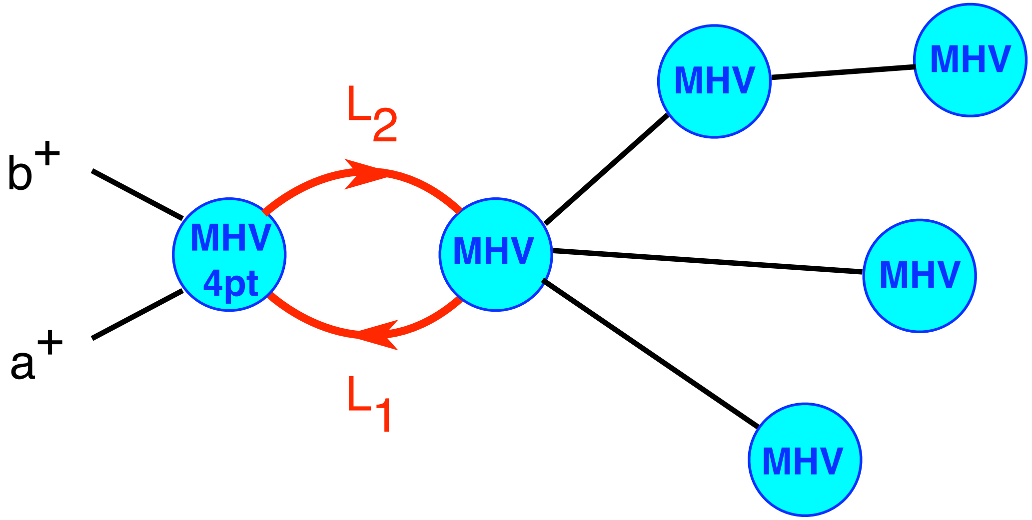
The one-loop splitting function is

$$Split^{1-loop}(a,b) = Split^{tree}(a,b) \times r(z)$$

- All order in ϵ expression for $r(z)$ was calculated by (Kosower-Uwer, Bern-Del Duca-Kilgore-Schmidt)

Collinear Limits from MHV Diagrams

- At tree level, collinear limits come out as expected (CSW)
  - Legs a and b belong to the same MHV vertex.
- At one-loop consider "singular channel" (Kosower) and "non-singular channel" MHV diagrams
  - "non-singular channel": Tree splitting function
  - "singular channel": One-loop splitting function



"singular channel" MHV diagram

One-loop splitting functions from MHV diagrams

- all order in ϵ one-loop splitting function from generic "singular channel" MHV diagram shown before (have to work to all orders in LIPS integration)
- the result agrees with known result

$$r(z) := \frac{c_\Gamma}{\varepsilon^2}\left(\frac{-s_{ab}}{\mu^2}\right)^{-\varepsilon}\left[1 - {}_2F_1\left(1,-\varepsilon,1-\varepsilon,\frac{z-1}{z}\right) - {}_2F_1\left(1,-\varepsilon,1-\varepsilon,\frac{z}{z-1}\right)\right]$$

Soft Limits

- Behaviour of one-loop amplitudes when one leg s becomes soft is given by:

$$\mathcal{A}_n^{1-loop}(1,\ldots,a,s,b,\ldots,n) \xrightarrow{k_s \to 0}$$

$$Soft^{tree}(a,s,b)\;\mathcal{A}_{n-1}^{1-loop}(1,\ldots,a,b,\ldots,n)$$

$$+\;Soft^{1-loop}(a,s,b)\;\mathcal{A}_{n-1}^{tree}(1,\ldots,a,b\ldots,n)$$

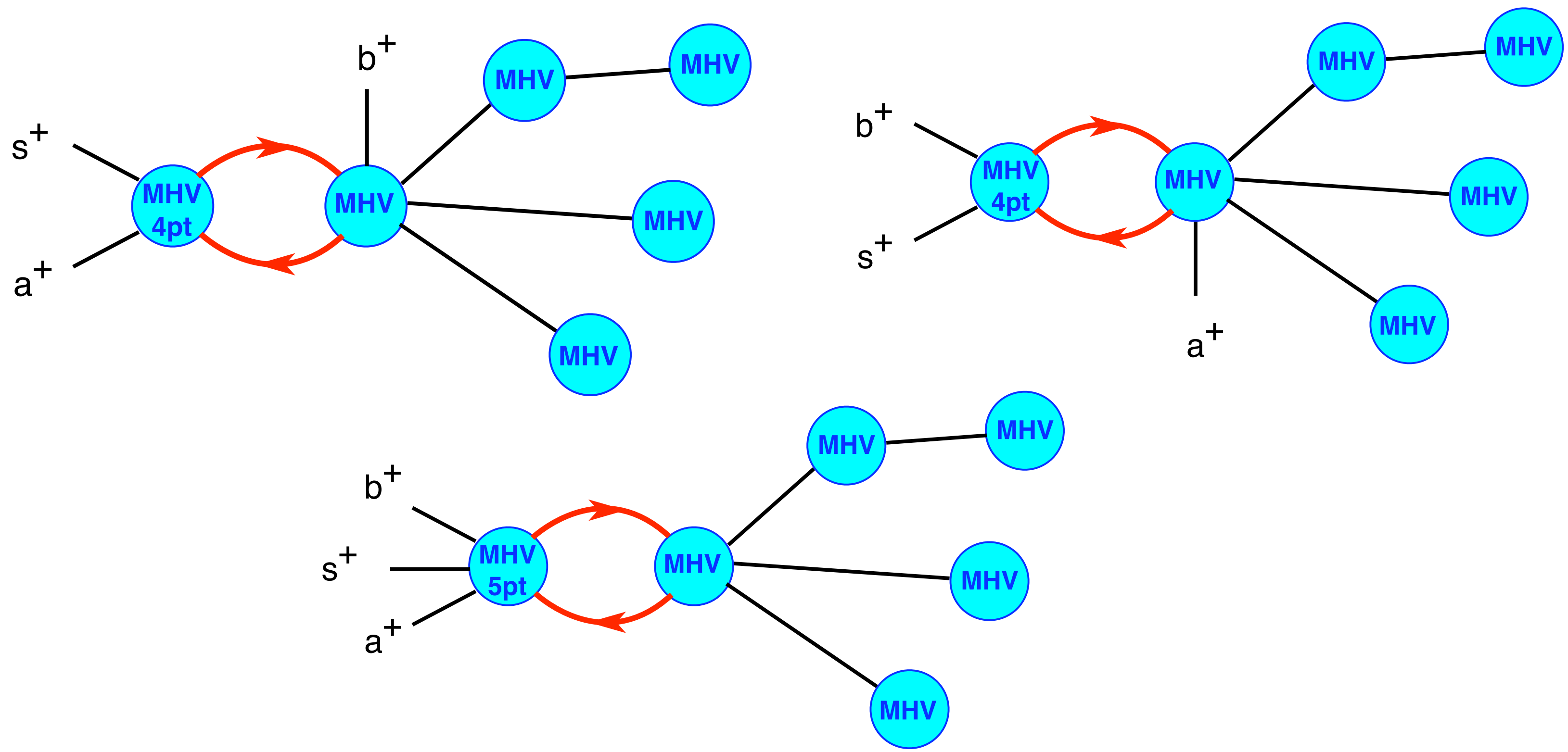
with

$$Soft^{tree}(a,s^+,b) \;=\; \frac{\langle a\,b\rangle}{\langle a\,s\rangle\,\langle s\,b\rangle}$$

$$Soft^{1-loop}(a,s,b) \;=\; Soft^{tree}(a,s,b)\left(-\frac{c_\Gamma}{\varepsilon^2}\frac{\pi\varepsilon}{\sin(\pi\varepsilon)}\right)\left(-\frac{s_{ab}}{s_{as}s_{sb}}\mu^2\right)^{\varepsilon}$$

Soft Limits from MHV Diagrams

- For concreteness we consider the limit: $a^+ s^+ b^+ \longrightarrow a^+ b^+$
- Three MHV diagrams contribute in this case
- Again the MHV diagrams reproduce the all order in ϵ, one-loop soft function



Conclusions

- MHV diagrams at one-loop in (S)YM
  - covariance (from FTT)
  - correct cuts (by construction)
  - correct soft and collinear limits (to all orders in ϵ)
    - Multiparticle poles?
- Further applications of FTT
  - rederivation of MHV 1-loop measure $d\mathcal{M}$
  - FTT applies also to massive/non-susy theories

Conclusions cont'd

- MHV diagrams work better than expected
  - all order in ϵ splitting one-loop function and 4-point one-loop amplitude in N=4 SYM
- Should work for higher loops (work in progress)
  - Connections with integrability (Minahan-Zarembo ...) and higher loop recursion relations (Anastasiou-Bern-Dixon-Kosower, Bern-Dixon-Smirnov, Cachazo-Spradlin-Volovich, Bern-Czakon-Kosower-Roiban-Smirnov) ?