Loop Amplitudes from MHV Diagrams

Gabriele Travaglini
Queen Mary, University of London

Brandhuber, Spence, GT hep-th/0407214
Bedford, Brandhuber, Spence, GT hep-th/0410280, 0412108
Andreas'talk: hep-th/0510253

Motivations

- Simplicity of scattering amplitudes unexplained by usual Feynman diagrams
  - Parke-Taylor formula for Maximally Helicity Violating amplitude of gluons (helicities are a permutation of $--++\,....+$)
- New methods account for this simplicity, and allow for much more efficient calculations
- LHC is coming !

The road to simplicity

- Colour decomposition (Berends, Giele; Mangano, Parke, Xu; Mangano; Bern, Kosower)
- Spinor helicity formalism (Berends, Kleiss, De Causmaecker, Gastmans, Wu; De Causmaecker, Gastmans, Troost, Wu; Kleiss, Stirling; Xu, Zhang, Chang; Gunion, Kunszt)

Colour decomposition

- Main idea: disentangle colour
- At tree level, Yang-Mills interactions are planar

$$\mathcal{A}^{tree}(\{p_i,\varepsilon_i\}) = \sum_{\sigma} Tr(T^{a_{\sigma_1}} \cdots T^{a_{\sigma_n}})\ \mathcal{A}(\sigma(p_1,\varepsilon_1),\ldots,\sigma(p_n,\varepsilon_n))$$

Colour-ordered partial amplitude

  - Include only diagrams with fixed cyclic ordering of gluons
  - Analytic structure is simpler

- At loop level: multi-trace contributions
  - subleading in 1/N

Spinor helicity formalism

- Consider a null vector p_μ
- Define $p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$ where $\sigma^\mu = (1, \vec{\sigma})$
- If $p^2 = 0$ then $det\ p = 0$
- Hence $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$
  - $\lambda\ (\tilde{\lambda})$ positive (negative) helicity spinors
- Inner products $\langle 12 \rangle := \varepsilon_{ab} \lambda_1^a \lambda_2^b$, $[12] := \varepsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$

Parke-Taylor formula

$$\mathcal{A}_{MHV}(1^+ \ldots i^- \ldots j^- \ldots n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

where s is the corresponding pole and z is the momentum fraction. The result for particles 2 and 3 nearly parallel, Eq. (5), is only simple because $\mathscr{M}_{4-1}(- + + + \cdots)$ is zero to this order in g so that there is no interference term and therefore azimuthal averaging is not required.

☞ The surprise about this result is that all denominators are simple dot products of two external momenta. The Feynman diagrams for n-gluon ($n > 5$) scattering contain propagators $(p_i + p_j + p_k)^2$, $(p_i + p_j + p_k + p_m)^2$, ... These propagators must cancel for Eq. (3) to be correct; this occurs for $n = 6$. Of course, Altarelli and Parisi have taught us that many cancellations are expected.

☞ We do not expect such a simple expression for the other helicity amplitudes. Also, we challenge the string theorists to prove more rigorously that Eq. (3) is correct.

Fermilab is operated by the Universities Research Association Inc. under contract with the United States Department of Energy.

[1] E. Eichten, I. Hinchliffe, K. Lane, and C. Quigg, Rev. Mod. Phys. **56**, 579 (1984).

[2] M. T. Grisaru, H. N. Pendleton, and P. van Nieuwenhuizen, Phys. Rev. D **15**, 997 (1977); M. T. Grisaru and H. N. Pendleton, Nucl. Phys. **B124**, 81 (1977).

[3] S. J. Parke and T. R. Taylor, Phys. Lett. **157B**, 81 (1985).

[4] T. Gottschalk and D. Sivers, Phys. Rev. D **21**, 102 (1980); F. A. Berends, R. Kleiss, P. de Causmaecker, R. Gastmans, and T. T. Wu, Phys. Lett. **103B**, 124 (1981).

[5] S. J. Parke and T. R. Taylor, Fermilab Report No. Pub-85/118-T, 1985 (to be published); Z. Kunszt, CERN Report No. TH-4319, 1985 (to be published).

[6] Another numerical fact worth mentioning is that to leading order in g but to all orders in N, the amplitude $|\mathscr{M}_{4+4}(- - + + + +)|^2$ is permutation symmetric apart from the factor $(p_1 \cdot p_2)^4$. This allows all permutations of this amplitude to be trivially calculated from one such permutation.

[7] G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).

- Colour decomposition and spinor helicity formalism make the simplicity manifest...

- ...but we still have to explain it !

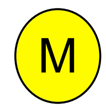
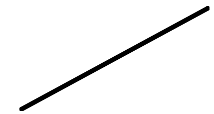
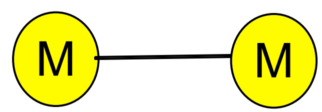
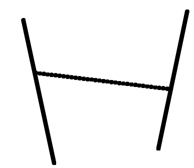
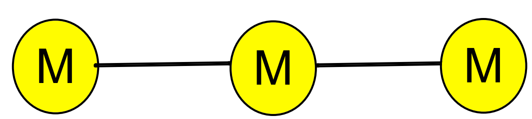
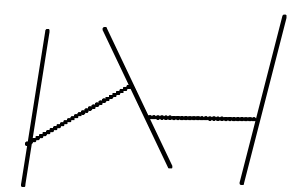
- Simple geometrical structure of the amplitudes in twistor space (Witten)

MHV Diagrams (Cachazo, Svrček, Witten)

- String theory on twistor space (Witten)
- Recursive structures in amplitudes
  (Britto, Cachazo, Feng + Witten)

Why MHV diagrams ?

- MHV amplitudes localise on complex lines in twistor space (Witten)
- A line in twistor space corresponds to a point in Minkowski space (Penrose) ➡
- An MHV amplitude can be thought of as a local interaction in spacetime ! (Cachazo, Svrček, Witten)
    - Locality manifest in light-cone formulation (Mansfield)

Amplitude	MHV diagrams	Twistor space structure
MHV		
nMHV		
nnMHV		

MHV Rules (tree level)

(Cachazo, Svrček, Witten)

- Off-shell continuation for internal (possibly loop) momenta needed:

- Internal momentum is off-shell
- Need to define spinor λ for an off-shell vector!

MHV amplitude ➡ MHV vertex

- Which propagators connect the MHV vertices ?

Off-shell continuation

- If $L^2 \neq 0$, we can write

$$L_{a\dot{a}} = l_{a\dot{a}} + z\eta_{a\dot{a}}$$

  - $\eta_{a\dot{a}} := \eta_a \tilde{\eta}_{\dot{a}}$ is a null reference vector
  - $z = L^2/2(L\cdot\eta)$ is a real number
  - $l_{a\dot{a}} := l_a \tilde{l}_{\dot{a}}$ is the off-shell continuation,
  - $l_a \Rightarrow L_{a\dot{a}}\tilde{\eta}^{\dot{a}}$ (equivalent to CSW's)

Internal propagators

- Just scalar propagators $\dfrac{i}{P^2 + i\varepsilon}$
- At loop level, the $i\varepsilon$ prescription is crucial in correctly determining the integration range

Loop MHV diagrams

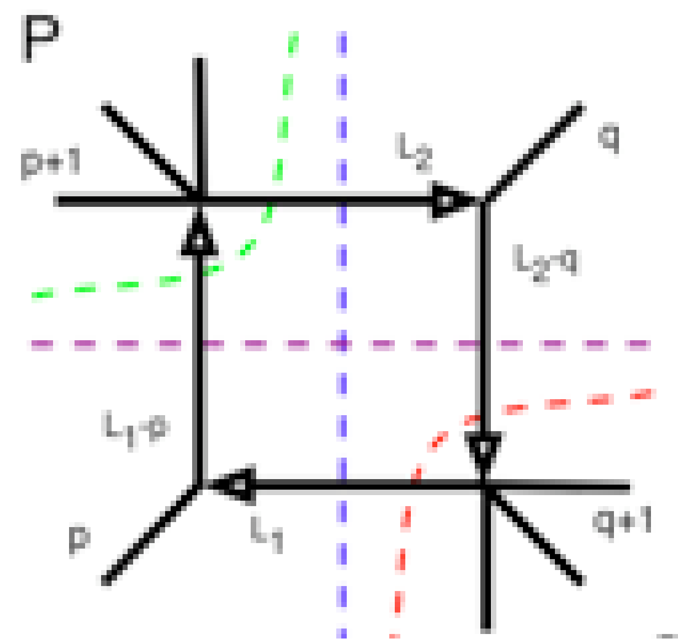
(Brandhuber, Spence, GT)

- Initial prognosis poor...
  - Twistor string theory dual to conformal supergravity (not Yang-Mills) at the quantum level
- Try anyway !

- Simplest amplitude: 1-loop MHV amplitude in N=4 super Yang-Mills

- Computed in 1994 by Bern, Dixon, Dunbar, Kosower

$$\mathcal{A}^{1-loop} = \mathcal{A}^{tree} \sum$$



Two-mass easy box function

In general

- Sew d MHV vertices

- $d = q - 1 + l$

  q = # negative helicity gluons,

  l = # loops

- As at tree level, we use

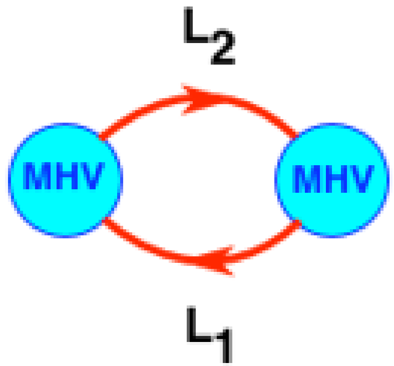
  a. CSW off-shell continuation

  b. Scalar propagators

- MHV, 1-loop: $d = 2$

Calculation from MHV diagrams

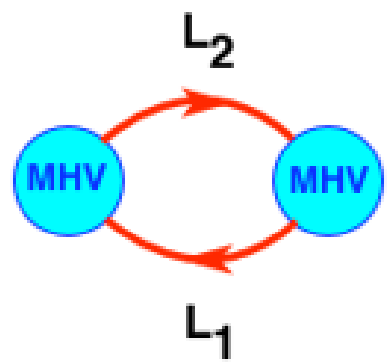
$$\sum \int d\mathcal{M}$$



- The sum is over
  - all possible MHV diagrams
  - internal particle species (g, f, s) and helicities
  - different from unitarity-based approach of BDDK
- We have to find the measure...

The integration measure

- P_L is the momentum on the left



$$d\mathcal{M} := \frac{d^4L_1}{L_1^2 + i\varepsilon}\frac{d^4L_2}{L_2^2 + i\varepsilon}\,\delta^{(4)}(L_2 - L_1 + P_L)$$

- Use $L = l + z\eta$, and $L \to (l, z)$

$$\Rightarrow \quad \frac{d^4L}{L^2 + i\varepsilon} = \frac{dz}{z + i\,\mathrm{sgn}\,(l_0\eta_0)\varepsilon}\,\frac{d^3l}{2l_0}$$

dispersive measure X phase-space measure (Nair measure)

Loop integral becomes:

(Dispersion integral) X (2-particle LIPS integral)

- LIPS integral:
  - computes the cut of the amplitude
  - regularise IR divergences: $4-2\varepsilon$ dimensions
- Dispersion integral reconstructs the amplitude from its cuts

Comments

- Final result is covariant (η-dependence drops out) and agrees with BDDK
  - Proof of covariance for generic amplitudes: Andreas'talk
- Result expressed as:

  (Dispersion integral) x (Phase space integral)
- Dispersion integrals are simple - no subtractions needed (van Neerven)
- The return of the analytic S-matrix !

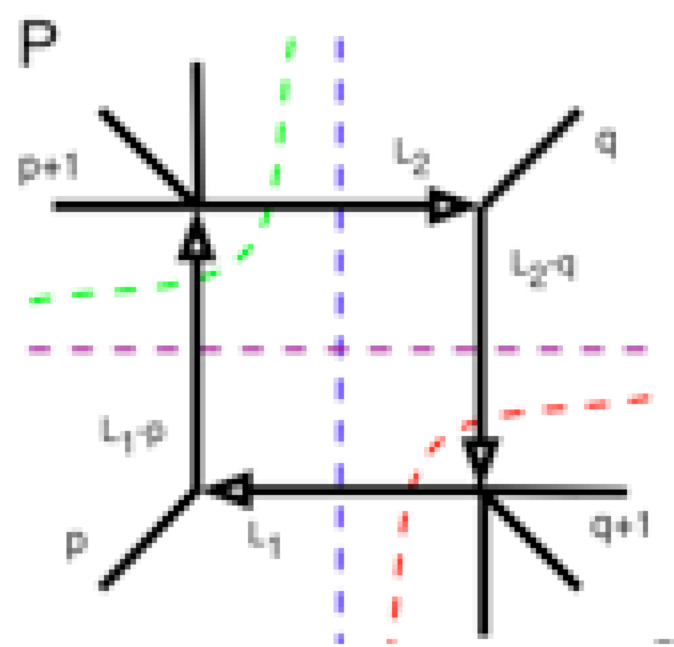
New form of the 2-mass easy box function

$$F(s,t,P^2,Q^2) = -\frac{1}{\varepsilon^2}\left[(-s)^{-\varepsilon} + (-t)^{-\varepsilon} - (-P^2)^{-\varepsilon} - (-Q^2)^{-}\right] + B(s,t,P^2,Q^2)$$

$$B(s,t,P^2,Q^2) = Li_2(1-aP^2) + Li_2(1-aQ^2) - Li_2(1-as) - Li_2(1-at)$$

$$a = \frac{P^2+Q^2-s-t}{P^2Q^2-st}$$

$$s = (P+p)^2 \qquad t = (P+q)^2$$



- More compact than usual expression
- Simpler analytic continuation

Further applications

- One-loop MHV amplitudes in N=1 super Yang-Mills (Bedford, Brandhuber, Spence, GT; Quigley, Rozali)
  - Result expressed in terms of finite boxes, and triangles
  - Agreement with BDDK
  - No twistor string theory for N=1 Super Yang-Mills...
  - ...nevertheless MHV diagram method works !

- Cut-constructible part of 1-loop MHV amplitudes in non-supersymmetric Yang-Mills (Bedford, Brandhuber, Spence, GT)
    - Extends 5-pt and adjacent negative helicity cases of BDK and BDDK
    - First new result at 1-loop in pure YM
- Non-supersymmetric amplitudes are not cut-constructible in 4 dimensions
    - rational terms

- Supersymmetric decomposition:

$$\mathcal{A}_g = \underbrace{(\mathcal{A}_g + 4\mathcal{A}_f + 3\mathcal{A}_s)}_{\text{N=4 susy amplitude}} - \underbrace{4(\mathcal{A}_f + \mathcal{A}_s)}_{\text{N=1 amplitude}} + \mathcal{A}_s$$

- Compute $\mathcal{A}_s$ (simpler than $\mathcal{A}_g$)

- MHV method calculates cut-constructible part

Summary

- MHV diagrams provide a new diagrammatic method to calculate scattering amplitudes at tree and one-loop level
- Proof for generic one-loop amplitudes: Andreas'talk
  - Feynman Tree Theorem

- Higher loops ?

- Lagrangian derivation, tree level (Mansfield)

- Twistor action derivation (Boels, Mason, Skinner)

- New method to calculate Green's functions ?