Perturbative Yang-Mills Theory from a String Theory in Twistor Space

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String theory: long list of field theory implications/techniques

- correlation functions (gauge/string duality)
- exact effective superpotentials (topological strings)
- new perturbative techniques for gauge theories and gravity

Earlier input:
- low energy limit of string amplitudes naturally implement gauge invariance
- KLT relations (gravity $\leftrightarrow$ gauge amplitudes)

String theory: long list of field theory implications/techniques

- correlation functions (gauge/string duality)
- exact effective superpotentials (topological strings)
- new perturbative techniques
  for gauge theories and gravity

Latest: Witten (2003)

The perturbative expansion of $\mathcal{N} = 4$ SYM is equivalent to the non-perturbative topological string field theory of the B-model on $\mathbb{P}^{3|4}$; a D1-instanton of genus g contributes to amplitudes with at most g loops and an instanton of degree d contributes to an amplitude with $d + L - 1$ negative helicity gluons.

Sunday: some of the details of how this came to be formulated and of the proposed string theory

- On-shell condition $p^2 = 0$ is imposed from the outset
- Natural variables – related to spinor helicity variables λ^α and $\tilde{\lambda}^{\dot{\alpha}}$
- Not the same however: $\bar{\lambda}^{\dot{\alpha}} \longrightarrow$ its Fourier-conjugate $\mu^{\dot{\alpha}}$

$$\tilde{\lambda}^{\dot{\alpha}} \qquad \longleftrightarrow \qquad \frac{\partial}{\partial \mu_{\dot{\alpha}}}$$

  – most natural signature $(++--)$; λ and $\tilde{\lambda}$ are independent

• gluon wave function:

$$\tilde{A}(\lambda, \mu, \pm) = \int d\tilde{\lambda}\, e^{i[\tilde{\lambda}, \mu]} A(\lambda, \tilde{\lambda}, \pm)$$

• each external gluon labeled by (λ, μ) or $(\lambda, \bar{\lambda})$ and choice of helicity

  – polarization vectors can be reconstructed

- Formal expression of string scattering amplitudes

$$\mathcal{A}_{tree} = \int d\mu_{\text{degree } d, \text{ genus } 0} \prod_{i=1}^{n} \Phi(Z(\sigma_i))\ d\sigma_i\ \langle J(\sigma_1) \ldots J(\sigma_n) \rangle_{g=0}$$

  ◇ Similar to Nair's presentation of MHV amplitudes

The Connected Prescription

(Spradlin, Volovich, RR)

– several different forms, useful for different purposes

- Color-ordered tree-level helicity amplitude:

$$\mathcal{A} = \sum_{q|H_r^A=0} \delta^4(H_1^A)\, J \frac{(\det F)^4}{\det(\partial \widehat{H}_r^A/\partial q_s)} \qquad\qquad J = \frac{1}{\prod \xi_i} \frac{J_0}{\prod(\sigma_i - \sigma_{i+1})}$$

where

$$H_r^A = \begin{cases} -\lambda_i^\alpha + \xi_i \sum_{k=0}^d \sigma_i^k a_k^\alpha \;, & A = \alpha = 1,2, \quad i = 1,\ldots,n \\ \sum_{i=1}^n \xi_i \sigma_i^k \bar{\lambda}_i^{\dot\alpha} \;, & A = \dot\alpha = 1,2, \quad k = 0,\ldots d \end{cases}$$

- $GL(2)$ symmetry $\longrightarrow$ there are 4 fake variables; determines J_0

$$q_s = (\sigma_i, \xi_i, a_k^\alpha) \qquad \# = (n + n + 2(d+1)) - 4$$

$$F_i{}^k = \xi_i \sigma_i^k \quad ; \quad i \text{ runs over negative hel. gluons}$$

- Slight sleight of hand in relating it to the twistor string

This formula is appealing for several of reasons

- as stringy as it gets
  - more of combinatorial type
  - suggests there should be a precise topological string interpretation
- uniformly covers all helicity orderings
  - the same solution determines all amplitudes with fixed n and n_-
- easily modified to accomodate scalar and fermion fields
  - if i, j are fermions then $(\det F\big|_{\hat{j}})^4 \to (\det F)^3 \det F\big|_{i\to j}$
- potential use in conjunction with unitarity method
  - Only numerator factor is sensitive to type of particles and helicity
  - numerator factor: $(\det F_1\big|_{\hat{i}} \det F_2\big|_{\hat{j}} - \det F_1\big|_{\hat{j}} \det F_2\big|_{\hat{i}})^4$

Explicit Tests and Properties

- $d = 1 \quad\quad \longrightarrow$ mostly $(+)$ MHV amplitudes

  essentially equivalent to Nair's and Witten's derivations

- $d = n - 3 \longrightarrow$ mostly $(-)$ MHV amplitudes

General? not analytically...

Obs: Constraint equations H_r^A depend only on n and d but not on the specific helicity assignment $\longrightarrow$ all amplitudes for fixed n and d emerge by summing over the same roots

- numerically: all 6-point NMHV amplitudes

General: More profitable to show that the connected prescription has the general properties of field theory amplitudes

- Cyclicity:

$$A(2,3,\dots,n,1) = A(1,2,\dots,n).$$

- Reflection:

$$A(n,n-1,\dots,1) = (-1)^n A(1,2,\dots,n).$$

- Dual Ward (or Sub-Cyclic) Identity:

$$\sum_{C(1,\dots,n-1)} A(1,2,3,\dots,n) = 0,$$

n – fixed; $C(1,\dots,n-1)$ set of cyclic permutations of $\{1,\dots,n-1\}$

- Generalized dual Ward identity:

$$\sum_{\mathrm{Perm}(i,j)} A(i_1,\dots,i_m,j_1,\dots,j_k,n+1) = 0, \qquad 1 \le m \le n-1, \qquad m+k=n,$$

where the sum is taken over permutations of the set $(i_1,\dots,i_m,j_1,\dots,j_k)$ which preserve the order of the $(i_1,\dots,i_m)$ and $(j_1,\dots,j_k)$ separately.

Consequences of the correlation function of vertex operators

$$J = \frac{1}{\prod \xi_i} \frac{J_0}{\prod(\sigma_i - \sigma_{i+1})}$$

- Conjugation/CPT: invariance under $+ \leftrightarrow -$ with simultaneous $\lambda \leftrightarrow \bar{\lambda}$.

$$A(\lambda_i, \bar{\lambda}_i, \eta_{iA}) = \int d^{4n}\psi \ \ \mathrm{exp}\left[i \sum_{i=1}^{n} \eta_{iA}\psi_i^A \right] A(\bar{\lambda}_i, \lambda_i, \psi_i^A).$$

- Obscured by treating λ and $\tilde{\lambda}$ nondemocratically

- Can be proved by elementary means (Spradlin, Volovich, RR) (Witten)

- Soft-Gluon Limit: in the limit $p_1 \to 0$ any amplitude behaves as

$$A(1^+, 2, \ldots, n) \longrightarrow \frac{\langle n\, 2 \rangle}{\langle n\, 1 \rangle \langle 1\, 2 \rangle} A(2, \ldots, n).$$

- Collinear Limit I: same helicity ($p_1 \to zp$ and $p_2 \to (1-z)p$ with $p^2 = 0$)

$$A(1^+, 2^+, 3, \ldots, n) \longrightarrow \frac{1}{\sqrt{z(1-z)}} \frac{1}{\langle 1\, 2 \rangle} A(p^+, 3, \ldots, n).$$

Important observation:

- The singular behaviour is captured by one (of the many) solution(s) of the constraint equations
- This solution has $\sigma_1 \simeq \sigma_2$

- Collinear Limit II: opposite helicity

  ($p_1 \to zp$ and $p_2 \to (1-z)p$ with $p^2 = 0$)

$$A(1^+, 2^-, 3, \ldots) \longrightarrow \frac{z^2}{\sqrt{z(1-z)}}\frac{1}{[1\,2]}A(p^+, 3, \ldots) + \frac{(1-z)^2}{\sqrt{z(1-z)}}\frac{1}{\langle 1\,2\rangle}A(p^-, 3, \ldots)$$

- Multi-particle Poles: Color-ordered amplitudes can only have poles in channels corresponding to a sum of cyclically adjacent momenta going on-shell.

$$A_n(1, \ldots, n) \longrightarrow \sum_{\chi=\pm} A_{m+1}(1, \ldots, m, p^\chi)\frac{i}{p_{1,m}^2}A_{n-m+1}(m+1, \ldots, n, p^{-\chi}).$$

  where $p_{1,m} = p_1 + p_2 + \cdots + p_m$ with $p_{1,m}^2 \to 0$

- Hard to prove in connected language

- Some arguments from relation with MHV vertices

Gukov, Motl, Nietzke

Property	connected presc.	gauge theory
$A(1,2,\ldots,n) = A(2,3,\ldots,1)$	manifest	non-manifest
$A(1,2,\ldots,n) = (-)^n A(n,n-1,\ldots,1)$	manifest	non-manifest
CPT	non-manifest	manifest
$\sum_{C(1,\ldots,n-1)} A(1,2,\ldots,n) = 0$	manifest	non-manifest
$\sum_{perm(i,j)} A(i_1,\ldots,i_m,j_1\ldots,j_k,n) = 0$	manifest	non-manifest
multi-particle poles	non-manifest	manifest

Connected vs. Disconnected?

- Connected instantons appear to give the YM amplitudes
- Cachazo, Svrcek, Witten: disconnected instantons do the same

$$T = \int d^4x_A^{a\dot{a}} d^4x_B^{b\dot{b}} \int \frac{dl'}{2\pi} d^2m_1'^{\dot{a}} d^2m_2'^{\dot{b}} V(\lambda_A, l') V(\lambda_B, l') I_\eta(x_A - x_B) G(m'_{12})$$
$$\times \prod_{i\in A} \delta^2(\mu_{Ai}^{\dot{a}} - x_A^{a\dot{a}}\lambda_{Ai\,a}) \delta^2(m_1'^{\dot{a}} - x_A^{a\dot{a}} l'_a) \prod_{i\in B} \delta^2(\mu_{Bi}^{\dot{b}} - x_B^{b\dot{b}}\lambda_{Bi\,b}) \delta^2(m_2'^{\dot{a}} - x_A^{a\dot{a}} l'_a)$$

- Bena, Bern, Kosower: result from partly connected instantons
- Gukov, Motl, Nietzke: All prescriptions localize on the intersection of the various components of the moduli space of curves
- Witten: Residue theorem: concise way of phrasing this

(details to be worked out)

$$\sum_{\{f_j(z_i)=0\}} \frac{1}{\det \partial f_j / \partial z_i} = 0$$

Agreement : roots at fin. dist.
Degenerate curves : roots at infinity

Loops

Formally:

$$A = \int d\mu_{\text{curves of degree } d \text{ and genus } g \leq L} \prod_{i=1}^{n} \Phi(Z(\sigma_i))\, d\sigma_i \, \langle J(\sigma_1) \ldots J(\sigma_n) \rangle_g$$

- Modification of earlier discussion:

  - $$\prod_i \frac{1}{\sigma_i - \sigma_{i+1}} \longrightarrow \prod_i f(\theta(\sigma_i - \sigma_{i+1}), \text{ aux. gauge field})$$
  - Integration measure complicated, but computable/guessable
  - Analysis of resulting expression : challenging

- What about partly connected?

- use the form of connected prescription inspired by string theory
  - $Z^A = (\lambda, \mu, \psi^A)$
- start with $(0^h, 1^-, 2^-, 3^+, \ldots, n^+, (n+1)^{-h})$

$$\mathcal{A}_{\text{tree}} = \int [d\mu_{d=2\ curves}] \prod_{i=0}^{n+1} \delta^7\left(\frac{Z_i^A}{Z_i^1} - \frac{Z^A(\sigma_i)}{Z^1(\sigma_i)}\right) \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}$$

$$\mathcal{A}_{1-\text{loop}} = \int dZ_0^A dZ_{n+1}^A G_1(Z_0^A, Z_{n+1}^A) \times$$
$$\times \int [d\mu_{d=2\ curves}] \prod_{i=0}^{n+1} \delta^7\left(\frac{Z_i^A}{Z_i^1} - \frac{Z^A(\sigma_i)}{Z^1(\sigma_i)}\right) \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}$$

- Can be massaged into a more useful form by suitably using symmetries and changing the integration variables

$$Z^A(\sigma_i) = a_0^A + a_1^A\sigma_i + a_2^A\sigma_i^2 \quad ; \quad Z^1(\sigma) = \sigma \quad ; \quad \frac{a_0^2}{a_2^2} = 1 \quad ; \quad C = \sqrt{a_0^2 a_2^2}$$

$$\mathcal{A}_{1-\mathrm{loop}} = \int dZ_0^A dZ_{n+1}^A G_1(Z_0^A, Z_{n+1}^A) \times$$
$$\times \int \frac{dC}{C^3} \prod_{A\neq 1} da_1^A \prod_{A\neq 1,2} d\left(\frac{a_0^A}{a_0^2}\right) d\left(\frac{a_2^A}{a_2^2}\right) \prod_{A,i} \delta^7\left(\frac{Z_i^A}{Z_i^1} - \frac{Z^A(\sigma_i)}{Z^1(\sigma_i)}\right) \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}$$

- The curve in these variables:

$$a_0^2\left(\frac{a_0^A}{a_0^2}\right)\sigma^{-1} + a_1^A + a_2^2\left(\frac{a_2^A}{a_2^2}\sigma\right) \longrightarrow a_1^A + C\left(\frac{a_0^A}{a_0^2}\sigma^{-1} + \frac{a_2^A}{a_2^2}\sigma\right)$$

- Further change of variables:

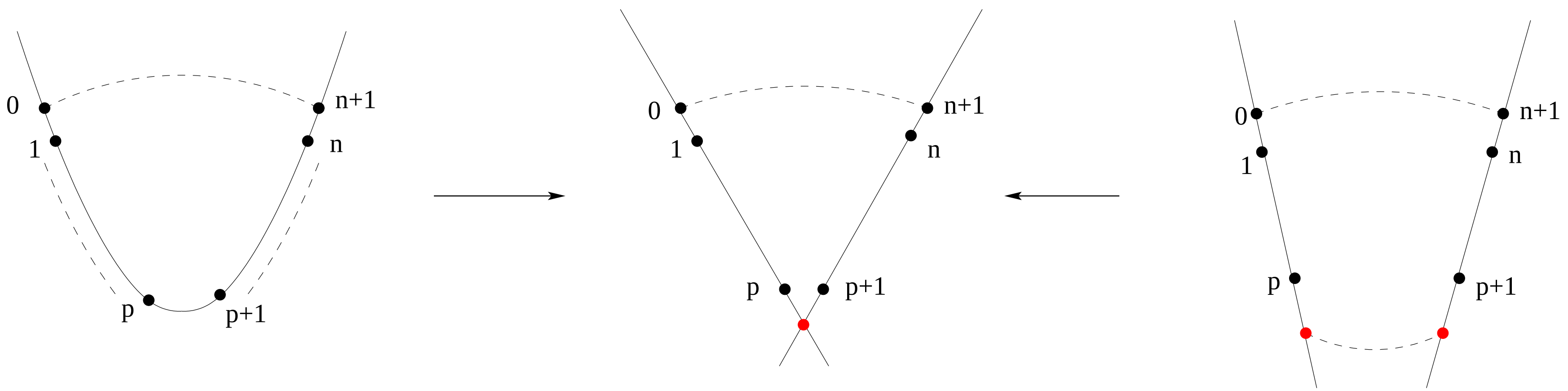
$$\sigma_i \rightarrow \begin{cases} \frac{\sigma_i}{a_2{}^2} & i \in (p\ldots 1) \quad a_1^A + \frac{a_2^A}{a_2^2}\sigma + C\frac{a_0^A}{a_0^2}\sigma^{-1} \\ \frac{\sigma_i}{a_2{}^2} & i \in (p\ldots 1) \quad a_1^A + \frac{a_2^A}{a_2^2}\sigma + C\frac{a_0^A}{a_0^2}\sigma^{-1} \end{cases}$$

Many asymptotics, depending on the choice of p

$$\frac{1}{C^2}\prod_i \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}} \to$$
$$\to \frac{d\sigma_p}{\sigma_p - \sigma_{p+1}} \cdots \frac{d\sigma_n}{\sigma_n - \sigma_1}\frac{d\sigma_1}{\sigma_1\sigma_2 - C}\frac{d\sigma_2}{\sigma_2 - \sigma_3} \cdots \frac{d\sigma_{p-2}}{\sigma_{p-2} - \sigma_{p-1}}\frac{d\sigma_{p-1}}{\sigma_{p-1}\sigma_p - C}$$

The moral:

- Integrate C using the pole at infinity $\to$ set $C = 0$



$$\frac{1}{C^2}\prod_i \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}} \to$$
$$\to \frac{d\sigma_p}{\sigma_p - \sigma_{p+1}} \cdots \frac{d\sigma_n}{\sigma_n - \sigma_1}\frac{d\sigma_1}{\sigma_1\sigma_2 - C}\frac{d\sigma_2}{\sigma_2 - \sigma_3} \cdots \frac{d\sigma_{p-2}}{\sigma_{p-2} - \sigma_{p-1}}\frac{d\sigma_{p-1}}{\sigma_{p-1}\sigma_p - C}$$

The moral:

- Integrate C using the pole at infinity $\to$ set $C = 0$

$\to$ same as gluing two MHV vertices and doing one momentum integral using residue theorem for the pole at infinity

- Transform to momentum space
  - generically convergent momentum integral
  - divergence if one vertex contains only two external lines

• Same structure of divergences as in gluing MHV vertices into loops

(Bena, Bern, Kosower, RR)

Summary

- reviewed the connected prescription of the twistor string, its formulation at tree-level, features and advantages and briefly touched on some of its features at loop level

- though from string theory perspective it appears to be a nontrivial integral, the connected prescription is a collection of algebraic operations;

- described some properties of the resulting expression and why we believe it is correct

- outlined how this may be turned into the calculation of loop amplitudes and why the result is correct by connecting it to MHV vertices

- while at the moment it cannot compete in efficiency with other methods, it may have lessons to teach us at least as far as formal developments are concerned