MHV Vertices and On-Shell Recursion Relations

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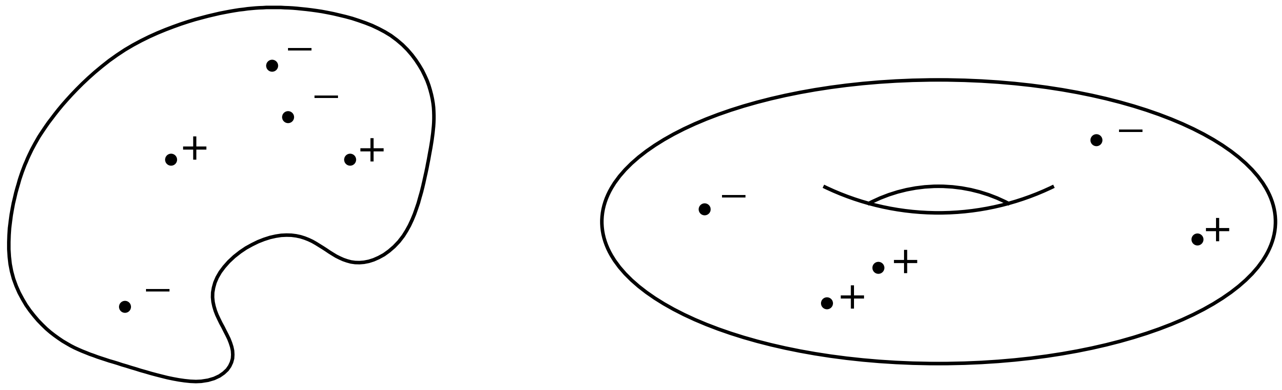
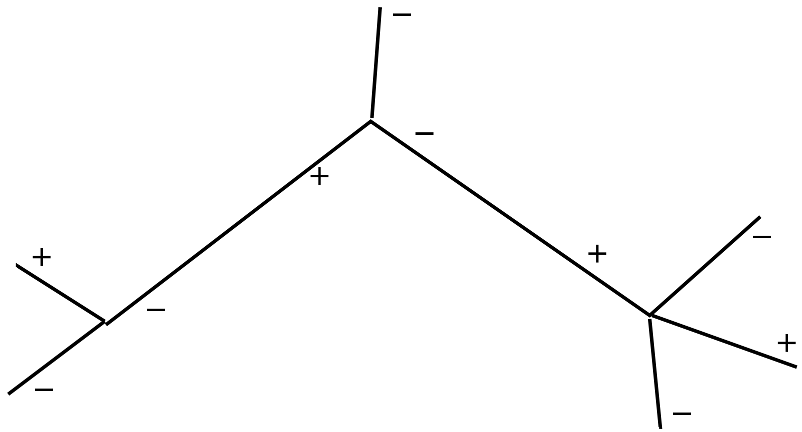
In December 2003, Witten proposed a string theory in twistor space dual to perturbative gauge theory. The scattering amplitudes of gluons get related to D-brane instanton corrections of the string theory.



This has inspired new developments in perturbative gauge theory, some of which we will review in this talk.

MHV diagrams

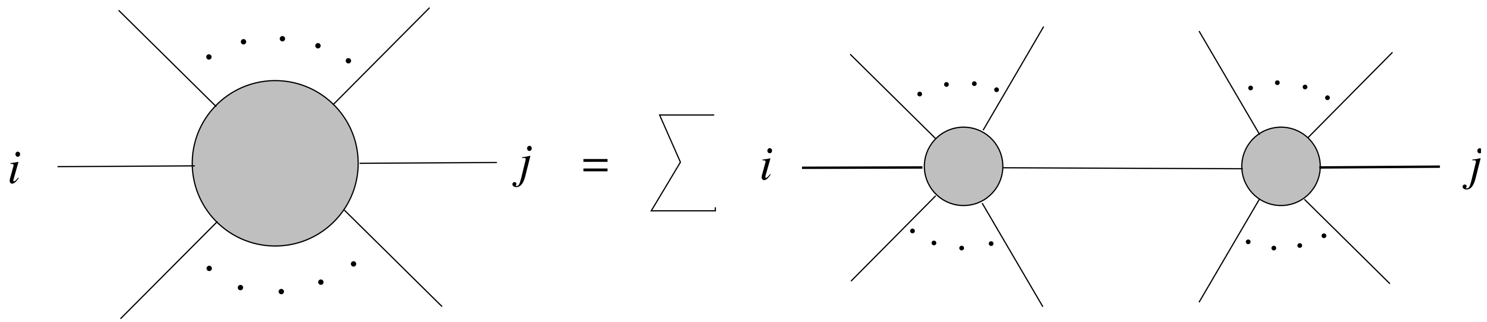
A new method to compute scattering amplitudes using Feynman diagrams with off-shell MHV vertices.



On-Shell Recursion Relations

The recursion relations express tree level amplitudes in terms of *on-shell* amplitudes with fewer particles:

$$\mathcal{A} = \sum_{L,R,h} \mathcal{A}_L^h \frac{1}{P_L^2} \mathcal{A}_R^{-h}.$$



We write on-shell momentum as a bispinor

$$P^{a\dot{a}} = \sigma_{\mu}^{a\dot{a}} P^{\mu} = \lambda^a \tilde{\lambda}^{\dot{a}},$$

and introduce the skew products

$$\langle \lambda, \lambda' \rangle = \lambda_a {\lambda'}_b \epsilon^{ab}$$

$$[\tilde{\lambda}, \tilde{\lambda}'] = \tilde{\lambda}_{\dot{a}} {\tilde{\lambda}'}_{\dot{b}} \epsilon^{\dot{a}\dot{b}}.$$

We take all gluons to be incoming and label them with the spinors $\lambda, \tilde{\lambda}$ and their helicities h. Then gluon polarization vectors are:

$$\epsilon^{-}_{a\dot{a}} = \frac{\lambda_a \rho_{\dot{a}}}{[\tilde{\lambda}, \rho]} \quad \epsilon^{+}_{a\dot{a}} = \frac{\kappa_a \tilde{\lambda}_{\dot{a}}}{\langle \kappa, \lambda \rangle}.$$

Instead of writing $\mathcal{A}(\epsilon_i)$ we use the helicity labels

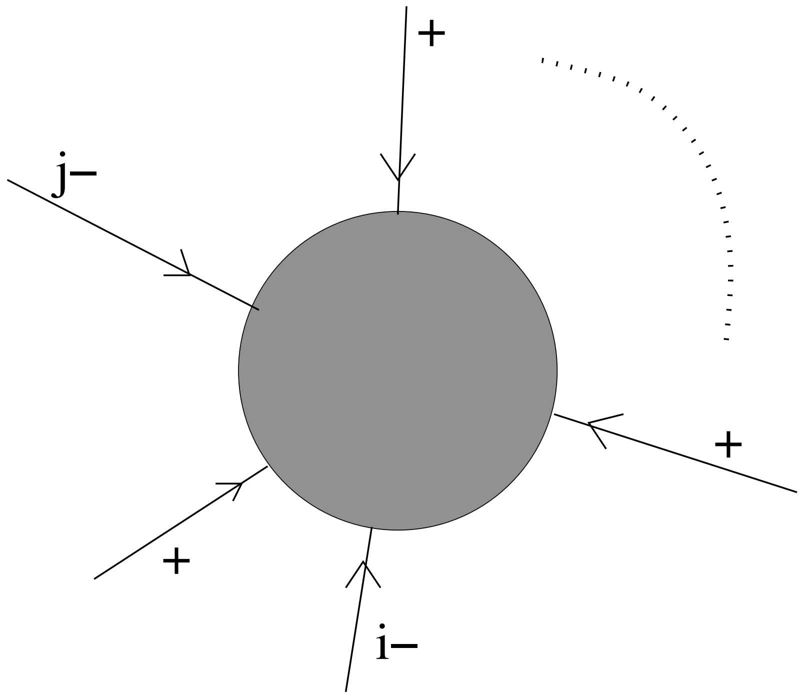
$$h_i = \pm.$$

We factor out delta function of momentum conservation and consider the color ordered amplitudes, that is we consider the ordered amplitude associated with $\mathrm{Tr}\, T_1 \ldots T_n$.

$$\hat{\mathcal{A}} = \delta^4 \left(\sum_i \lambda_i^a \tilde{\lambda}_i^{\dot{a}} \right) \sum_{S_n/Z_n} \mathrm{Tr}\, (T_{\sigma(1)} T_{\sigma(2)} \ldots T_{\sigma(n)}) \mathcal{A}(\lambda_{\sigma(i)}, \tilde{\lambda}_{\sigma(i)}, h_{\sigma(i)}).$$

Recall, that the color ordered MHV scattering amplitude with gluons i and j of negative helicity takes the simple form

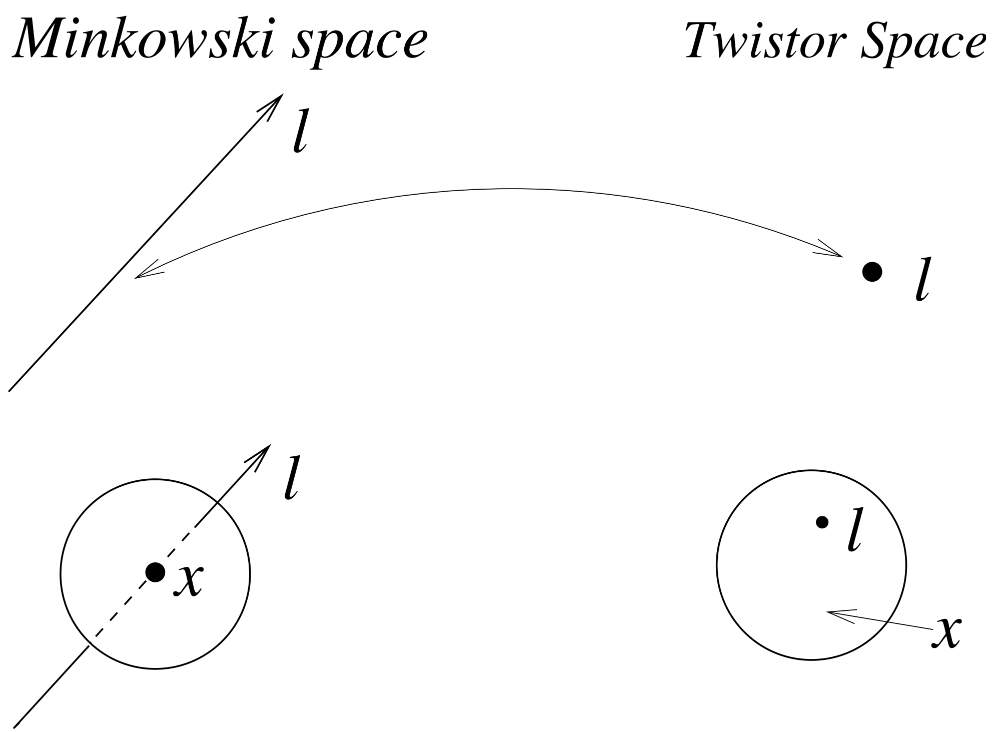
$$\mathcal{A} = \frac{\langle i,j\rangle^4}{\prod_{k=1}^{n}\langle k,k+1\rangle}\delta^4\left(\sum_k \lambda_k^a \tilde{\lambda}_k^{\dot{a}}\right).$$



We would like to understand why the MHV and other amplitudes have much simpler expressions than what we would expect from textbook recipes for computing them.

For that we need to venture into the twistor space.

Penrose's twistor space is the space of light rays in Minkowski space.



A point x in Minkowski space, maps to the sphere of light rays l through x.

Twistor space is the projective space $\mathbb{CP}^3$. Its homogeneous coordinates are

$$(\lambda^a, \mu^{\dot{a}}) \sim (c\lambda^a, c\mu^{\dot{a}}),$$

where $\mu^{\dot{a}}$ is conjugate to $\tilde{\lambda}^{\dot{a}}$

$$\mu \to -i\frac{\partial}{\partial\tilde{\lambda}.} \qquad \tilde{\lambda} \to i\frac{\partial}{\partial\mu}.$$

The conformal group $SU(2,2)$ is the group of motions of twistor space. It acts linearly on (λ, μ).

One gets twistor amplitudes by Fourier transform in $\tilde{\lambda}$ of momentum space amplitudes

$$\tilde{\mathcal{A}}(\lambda_i, \mu_i) = \int \prod_{i=1}^{n} d^2\tilde{\lambda}_i \, e^{i\tilde{\lambda}_i \mu_i} \mathcal{A}(\lambda_j, \tilde{\lambda}_j).$$

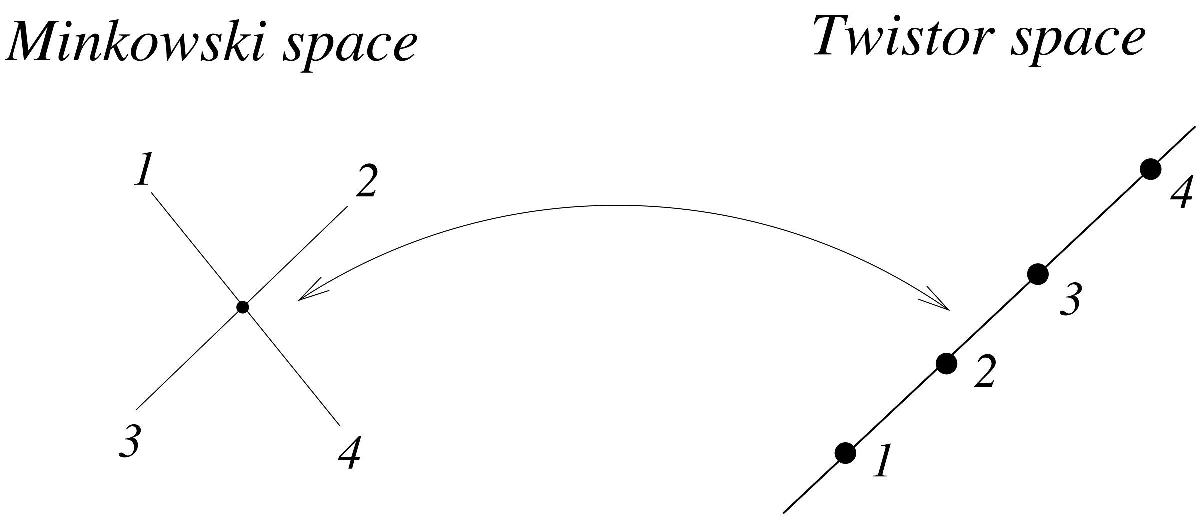
MHV amplitudes behave simply in twistor space because they are independent of $\tilde{\lambda}^{\dot{a}}$, they are *holomorphic.*

They vanish unless all gluons lie on a common complex line, $\mathbb{CP}^1$, in twistor space.

But a line in twistor space corresponds to a point in Minkowski space.

$$\mu_i^{\dot{a}} + x^{a\dot{a}}\lambda_{ia} = 0$$

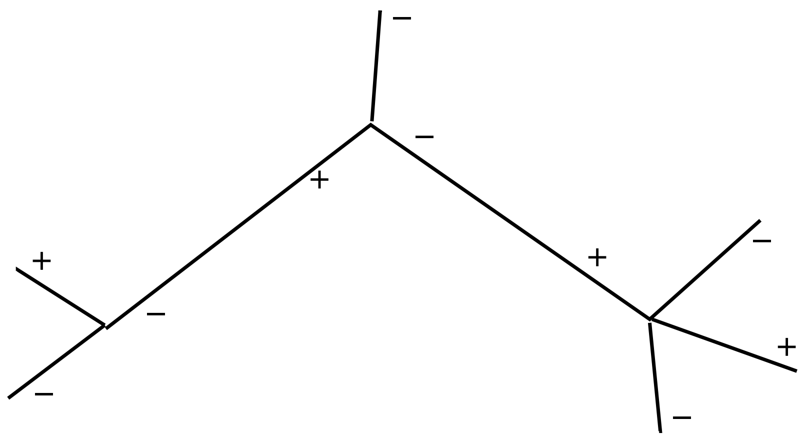
So MHV amplitudes behave like *local interaction vertices.*



MHV diagrams

These twistor considerations inspired a new method to compute scattering amplitudes using Feynman diagrams with off-shell MHV vertices.

[Cachazo, P.S., Witten]



Off-shell continuation of MHV amplitudes

To use MHV amplitudes in Feynman diagrams, we need to define λ^a for an off-shell momentum.

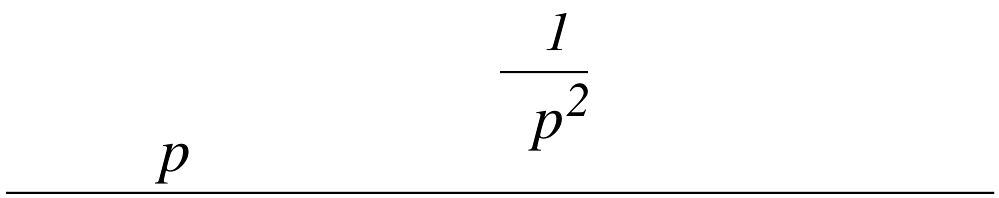
For on-shell momentum $P^{a\dot{a}} = \lambda^a \tilde{\lambda}^{\dot{a}}$, we have

$$\lambda^a = \frac{P^{a\dot{a}}\eta_{\dot{a}}}{[\tilde{\lambda}, \eta]}.$$

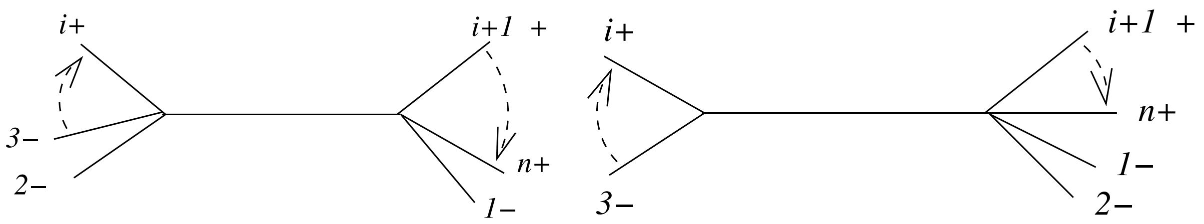
For an off-shell line, we do not know what $[\tilde{\lambda}, \eta]$ is. Fortunately, $[\tilde{\lambda}, \eta]$ scales out of Feynman diagrams, so we take

$$\lambda^a = P^{a\dot{a}}\eta_{\dot{a}}.$$

The propagator connecting MHV vertices is the scalar propagator



It is easy to compute complicated amplitudes analytically. For example, the n gluon $---++\cdots++$ amplitude is a sum of $2n-3$ MHV diagrams.



$$\begin{aligned} A = \quad & \sum_{i=3}^{n-1} \frac{\langle 1\lambda_{2,i}\rangle^3}{\langle \lambda_{2,i}\, i{+}1\rangle\langle i{+}1\, i{+}2\rangle\ldots\langle n1\rangle} \frac{1}{q_{2i}^2} \frac{\langle 2{,}3\rangle^3}{\langle \lambda_{2,i}2\rangle\langle 34\rangle\ldots\langle i\lambda_{2,i}\rangle} \\ + \quad & \sum_{i=4}^{n} \frac{\langle 12\rangle^3}{\langle 2\lambda_{3,i}\rangle\langle \lambda_{3,i}\, i{+}1\rangle\ldots\langle n1\rangle} \frac{1}{q_{3i}^2} \frac{\langle \lambda_{3,i}3\rangle^3}{\langle 3{,}4\rangle\ldots\langle i{-}1\, i\rangle\langle i\lambda_{3,i}\rangle}, \end{aligned} \tag{1}$$

MHV diagrams construction has a surprisingly simple gauge theory proof.

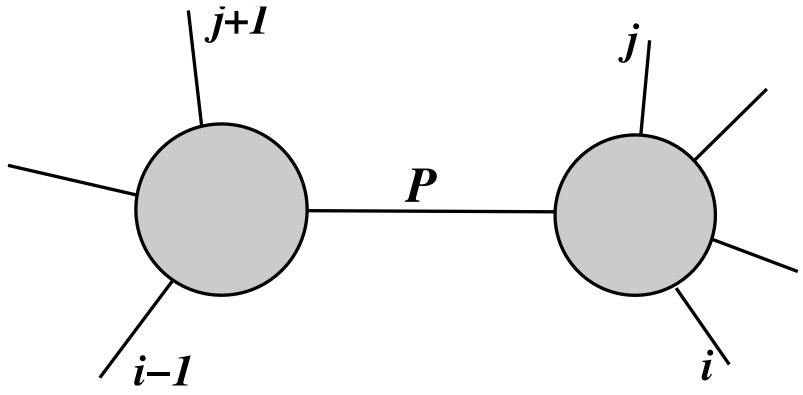
Essentially it reduces to verifying that, that the answer obtained from MHV diagrams is Lorentz invariant (independent of the auxiliary spinor η) and has the correct singularity structure:

- multiparticle singularities $(p_i + p_{i+1} + \cdots + p_j)^2 \longrightarrow 0$
- collinear singularities $(p_i + p_{i+1})^2 \longrightarrow 0$
- soft singularities $p_i^2 \longrightarrow 0$

Multiparticle singularity in the channel $P^2 \to 0$ is $1/P^2$ times the product of the two sub-amplitudes

$$\mathcal{A} \longrightarrow \mathcal{A}_L \frac{1}{P^2} \mathcal{A}_R.$$

It arises from MHV diagrams with off-shell P line going on-shell. The MHV diagrams factorize into $\mathcal{A}_L$ and $\mathcal{A}_R$.



MHV diagrams give a function $\mathcal{A}_{MHV}$ which has the same pole locations and residues as the scattering amplitude $\mathcal{A}$ (the residues are determined in terms of lower point amplitudes and universal soft and collinear factors).

Since tree-level amplitudes are rational functions of the spinors $\lambda, \tilde{\lambda}$, this is enough to conclude that

$$\mathcal{A}_{MHV} = \mathcal{A}.$$

New proofs of MHV diagrams:

new twistor actions [Mason]

a field redefinition of the light-cone Lagrangian [Mansfield]

Twistor String Theory

The MHV diagrams have been motivated from twistor space. One may ask whether there is a string theory in twistor space that leads to MHV diagrams.

The answer seems to be yes. So far, at least for tree amplitudes in $\mathcal{N} = 4$ gauge theory. [however, see second Spradlin's talk]

In order to define B-model topological strings on twistor space, one has to make it into a Calabi-Yau manifold. This is accomplished by introducing fermionic coordinates

$$\psi^A \qquad A = 1 \ldots 4$$

$$(\lambda Z^I, \lambda\psi^A) \sim (Z^I, \psi^A).$$

We get super-twistor space $\mathbb{CP}^{3|4}$ with holomorphic $(3|4)$ form

$$\Omega = \epsilon_{IJKL}\epsilon_{ABCD}\, Z^I dZ^J dZ^K dZ^L\, d\psi^A d\psi^B d\psi^C d\psi^D$$

The group of motions of super-twistor space is the $\mathcal{N} = 4$ super-conformal group $SU(2,2|4)$.

The topological open strings are described by a twistor field $\mathcal{A}$ that is a $(0,1)$ form on the twistor space. Expanding it in ψ^A we get

$$\mathcal{A}(Z,\psi) = A + \chi_A \psi^A + \cdots + G(\psi)^4.$$

Under Penrose's twistor transform, the twistor fields are related to gauge fields:

$A \to h = +1$ gluon,
$\chi_A \to h = +1/2$ fermion,
$G \to h = -1$ gluon.

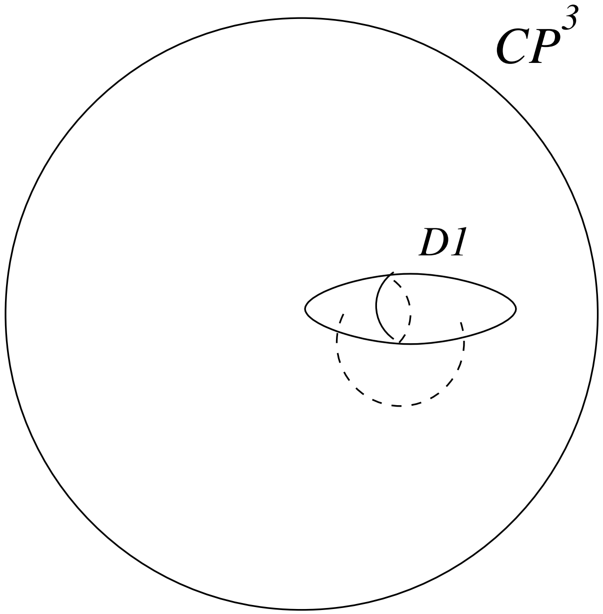
We get the entire spectrum of $\mathcal{N} = 4$ SYM!

$$\mathcal{S} = \int \Omega \wedge \left(\mathcal{A} \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right)$$

$\mathcal{S}$ is an effective action for an open topological string theory, the B-model, in twistor space.

The cubic vertex $\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}$ gives the interactions of the self-dual $\mathcal{N} = 4$ gauge theory. So, we have the correct spectrum, but we do not yet have the interactions...

The interactions of the full gauge theory come from *D1-brane instantons* on which the open strings can end.



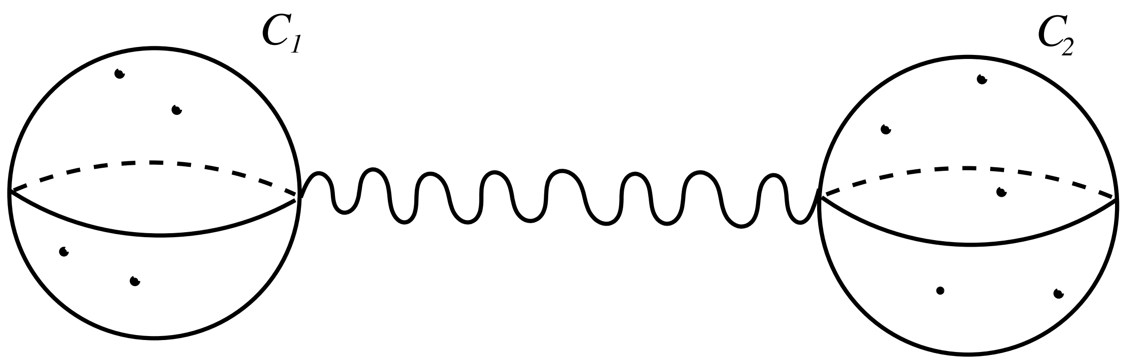
Open string with wavefunction $\psi = \mathcal{A}_I d\bar{z}^I$ couples to D1-instanton via

$$\int_C J \wedge \psi = \int_C J_z \mathcal{A}_{\bar{z}}\, dz \wedge d\bar{z}$$

Two disconnected D1-instantons can be connected with an open string with propagator

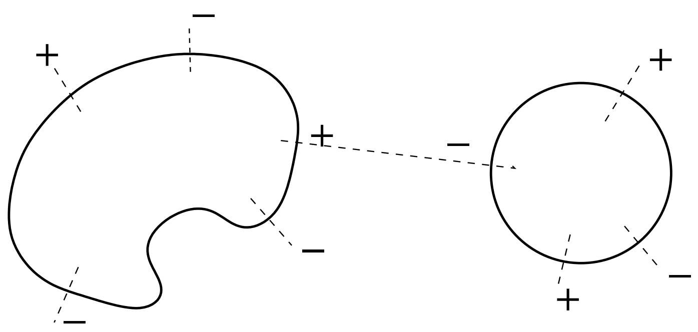
$$\bar{\partial} G = \bar{\delta}^3(Z'^I - Z^I)\delta^4(\psi'^A - \psi^A).$$

$$\bar{\delta}(z) = d\bar{z}\delta(z)\delta(\bar{z})$$

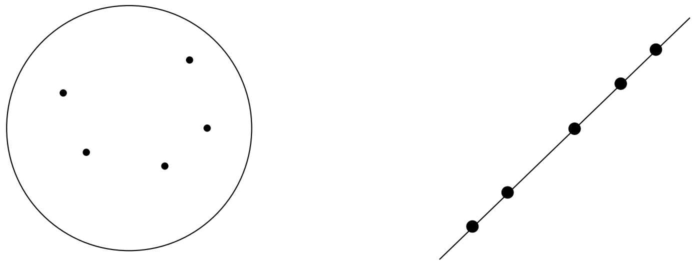


A degree d instanton is in the homology class of d times the class of complex line. Its area is $2\pi d$; it is quantized.

The instanton contributes to amplitudes with $d + 1$ negative helicity gluons and an arbitrary number of positive helicity gluons. For example $d = 3$:



$d = 1$ instantons are complex lines:



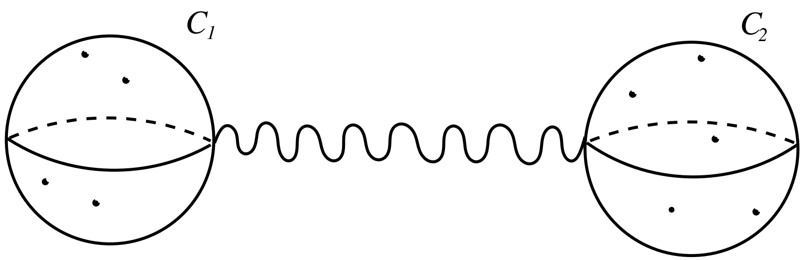
Complex line *Real cross−section*

These give MHV amplitudes

$$\mathcal{A}(i^-, j^-) = \frac{\langle i, j\rangle^4}{\prod_{k=1}^n \langle k, k+1\rangle} \delta^4(\sum_i \lambda_i \tilde{\lambda}_i).$$

$\delta^4(\sum_i \lambda_i \tilde{\lambda}_i) \ldots$ from integral over moduli $x^{a\dot{a}}$ of lines in $\mathbb{CP}^{3|4}$.
$\langle i, j\rangle^4 \ldots$ from integral over fermionic moduli.
$1/\prod_k \langle \lambda_k, \lambda_{k+1}\rangle \ldots$ from the current correlator $\langle J_1 J_2 \ldots J_n\rangle$.

Next is the $d = 2$ amplitude. Consider the contribution from two degree one instantons C and C' that are connected by an open string.



The instantons are described by the equations

$$\mu^{\dot{a}} = x_i^{a\dot{a}} \lambda_a \qquad \psi^A = \theta_i^{Aa} \lambda_a$$

$x_i^{a\dot{a}}$ ... bosonic moduli of the i^{th} instanton.
θ_i^{Aa} ... fermionic moduli.

The amplitude can be evaluated using residues:

$$A = \sum \frac{1}{P^2} A_L A_R(\lambda = P\eta).$$

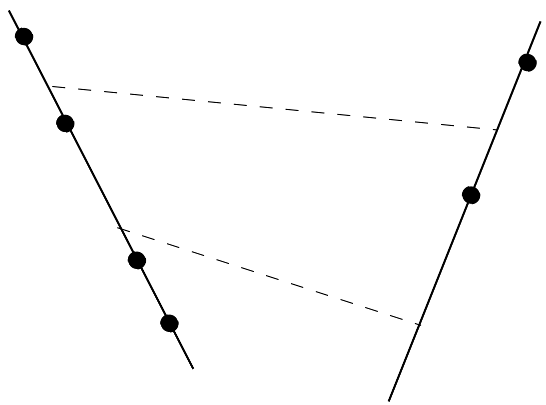
But this is exactly the sum over MHV diagrams!

Similarly, d degree one instantons connected by twistor propagators lead to MHV diagrams with d MHV vertices.

Hence, the MHV vertices correspond to $d = 1$ D-instantons and the scalar $1/k^2$ propagators correspond to open string twistor propagators.

One can alternatively compute the amplitude from *connected* instantons. [see Roiban's talk]

There is mounting evidence that MHV diagrams correctly compute YM loop amplitudes [see Travaglini's and Brandhuber's talks]

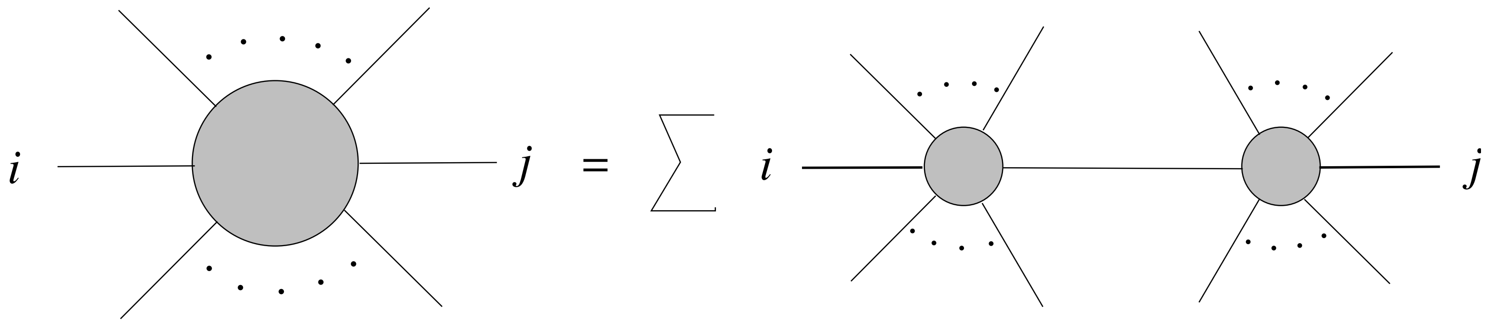


Unfortunately we have not yet learned how to derive them from twistor string theory.

On-Shell Recursion Relations

The recursion relations express tree level amplitudes in terms of *on-shell* amplitudes with fewer external particles, which leads to dramatic simplifications. [Britto, Cachazo, Feng]

$$\mathcal{A} = \sum_{L,R,h} \mathcal{A}_L^h \frac{1}{P_L^2} \mathcal{A}_R^{-h}.$$



A Brief Derivation

Consider a tree level amplitude $\mathcal{A}(1, 2, \ldots, n)$ of massless particles with momenta $p_i = \lambda_i \tilde{\lambda}_i$. We shift the momenta of two particles n, k

$$p_k(z) = p_k - z\lambda_k \tilde{\lambda}_n, \qquad p_n(z) = p_n + z\lambda_k \tilde{\lambda}_n$$

while preserving momentum conservation

$$\sum_i p_i(z) = 0.$$

This is equivalent to

$$\tilde{\lambda}_k \to \tilde{\lambda}_k - z\tilde{\lambda}_n, \qquad \lambda_n \to \lambda_n + z\lambda_k$$

hence, it keeps the particles on-shell.

The function $\mathcal{A}(z) = \mathcal{A}(p_1, \ldots, p_k(z), \ldots, p_n(z), \ldots, p_n)$ is a physical on-shell amplitude with complex momenta for all z.
The amplitude $\mathcal{A}(z)$:

- is a rational function of z (since it is rational in the spinors $\lambda_i, \tilde{\lambda}_i$)

- has only simple poles from multiparticle singularities $1/P_{ij}^2$

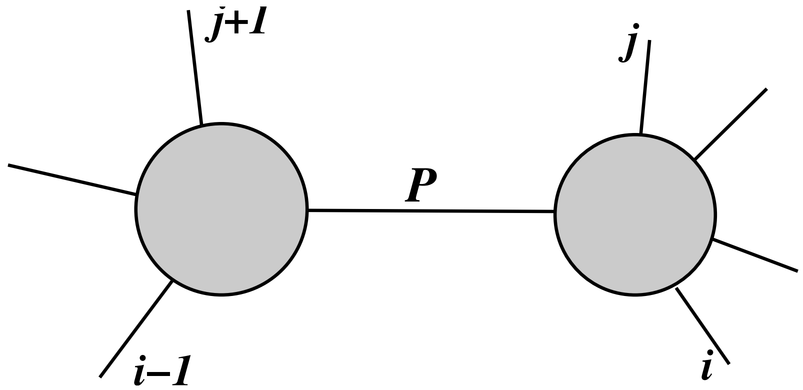
Hence it is determined by its residues

$$\mathcal{A} = \oint_0 \frac{\mathcal{A}(z)}{z} = -\sum_i \mathrm{Res}\left(\frac{\mathcal{A}(z_i)}{z_i}\right) - \mathrm{Res}\left(\frac{\mathcal{A}(z)}{z}\right)\Big|_\infty .$$

In favorable situations, the residue from infinity vanishes (i.e. for tree-level YM amplitudes).

The poles of $\mathcal{A}(z)$ come from multiparticle singularities where internal propagators $1/(p_i + \cdots + p_j)^2$ that separate particles k, n go on-shell. These are simple poles because $(p_i + \cdots + p_j)^2$ is at most linear in z. The residues are universal; they are products of the two on-shell sub-amplitudes

$$\mathcal{A} \longrightarrow \mathcal{A}_L \frac{1}{P_{ij}^2} \mathcal{A}_R.$$



Evaluating the residues leads to the BCFW recursion relations

[Britto, Cachazo, Feng, Witten]

$$\mathcal{A}(1,2,\ldots,n) = \sum_{ij}\sum_{h} \frac{\mathcal{A}_L^{-h}(z_{ij})\mathcal{A}_R^{h}(z_{ij})}{P_{ij}^2}$$

where z_{ij} is the location of the multiparticle singularities in the complex z plane.

$$P_{ij}^2(z_{ij}) = P_{ij}^2 - z\langle\lambda_k|P_{ij}|\tilde{\lambda}_n] = 0$$

The recursion relations express on-shell amplitudes in terms of lower-point amplitudes, which leads to dramatic reduction in computational complexity.

Clearly the proof applies to general perturbative amplitudes that are rational functions of the massless spinors $\lambda_i, \tilde{\lambda}_j$. They can also be generalized to massive amplitudes. [Badger, Glover, Khoze, P.S.]

Possible complication occurs if the residue at infinity $\mathrm{Res}\,(\mathcal{A}(z)/z)_\infty$ is nonzero, because the residue does not have an obvious recursive evaluation.

The most notable generalization is to the computation of the finite rational part of one-loop QCD amplitudes. [see Berger's and Forde's talks]

New proofs of recursion relations: [see Vaman's talk]

Summary

- MHV vertices give a new way to compute scattering amplitudes with off-shell MHV vertices
- at YM tree-level, they have been related to an instanton expansion of twistor string theory
- the on-shell recursion relations give a new powerful way to compute amplitudes in terms of lower-point on-shell amplitudes
- the relations have been successfully applied both to general tree level amplitudes and to rational parts of one-loop QCD amplitudes