Mass dependence of fermion determinant in instanton background (& new results for determinants)

CAQCD 2006

Gerald Dunne

- determinants in quantum field theory
- Gel'fand & Yaglom technique
- higher dimensions and renormalization
- QCD instanton det, false vacuum decay

with: J. Hur & C. Lee (SNU), H. Min (Seoul), Q. Wang (UConn)

PRL, PRD 2005, 2006

general problem: determinant of a partial differential operator

Many applications in quantum field theory:

- effective action, partition function, free energy
- unquenching in lattice gauge theory
- tunneling rates, false vacuum decay, nucleation rate
- Faddeev-Popov determinants & gauge fixing

Effective action

Quantum field theory functional integral

$$Z = \int \mathcal{D}A \, exp \left[\int d^4x \, tr F^2 \right] det \left[i\not{D} - m \right]$$

Effective action : $S[A] = log \, det \left[i\not{D} - m \right]$

Exact results : covariantly constant $F_{\mu\nu}$

Instanton background

SU(2) single instanton

$$A_\mu(x) = A_\mu^a(x)\frac{\tau^a}{2} = \frac{\eta_{\mu\nu a}\tau^a x_\nu}{r^2+\rho^2}$$

scale

Self-duality ⇨ isospectrality

$$\boxed{\Gamma^F(A;m) = -2\,\Gamma^S(A;m) - \frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)}$$

⇨ compute scalar determinant instead of spinor determinant

Instanton background - asymptotics

function of ($m\rho$) only

Renormalized effective action :

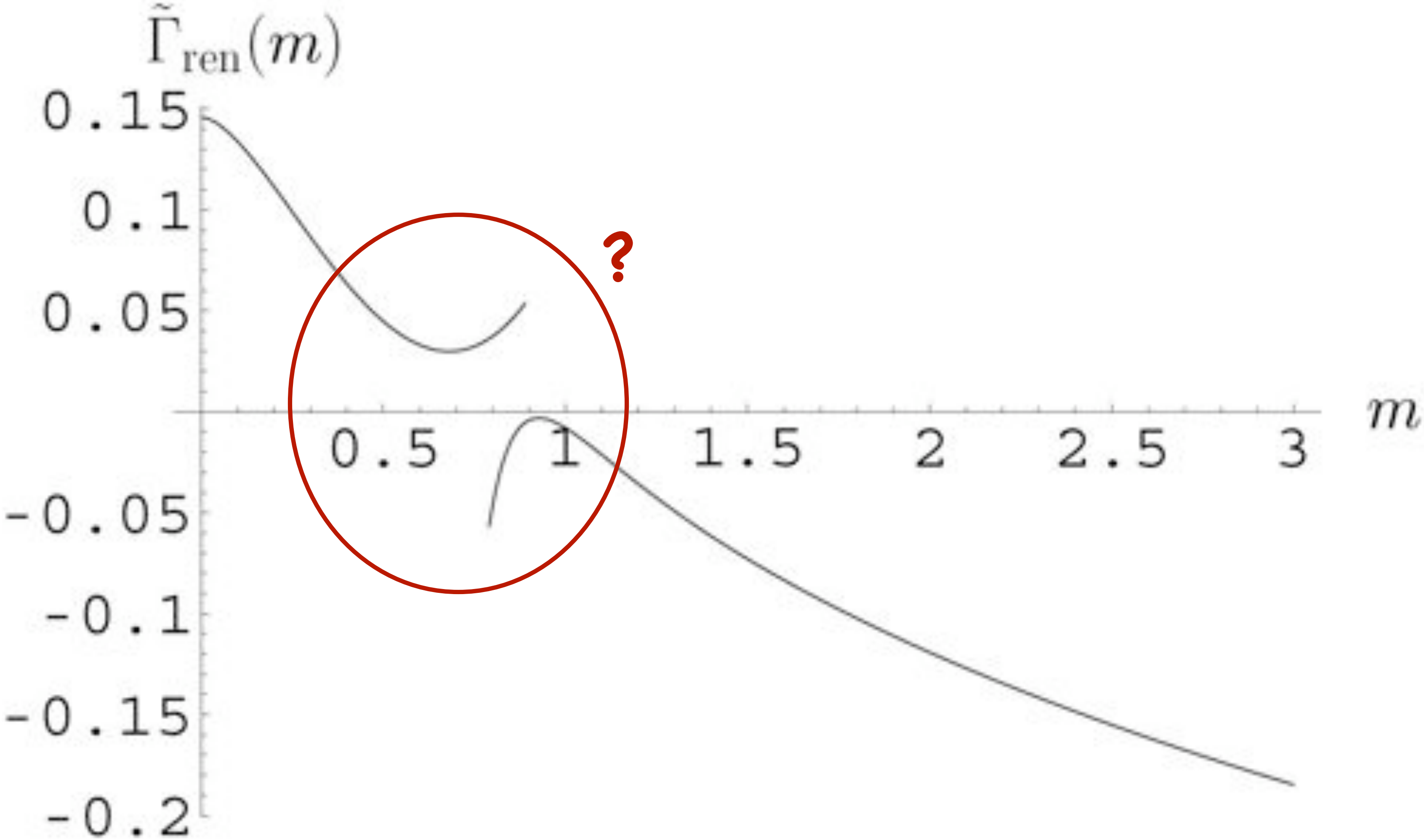
$$\Gamma^S_{\rm ren}(A;m) = \tilde{\Gamma}^S_{\rm ren}(m\rho) + \frac{1}{6}\ln(\mu\rho)$$

- Small m limit : exact massless Green's functions known
- Large m limit : from heat kernel expansion

$$\tilde{\Gamma}^S_{\rm ren}(m) \sim \begin{cases} \alpha\left(\frac{1}{2}\right) + \frac{1}{2}\left(\ln m + \gamma - \ln 2\right) m^2 + \ldots & \text{small m} \\ -\frac{\ln m}{6} - \frac{1}{75m^2} - \frac{17}{735m^4} + \frac{232}{2835m^6} - \frac{7916}{148225m^8} + \cdots & \text{large m} \end{cases}$$

$$\alpha\left(\frac{1}{2}\right) = -\frac{5}{72} - 2\zeta'(-1) - \frac{1}{6}\ln 2 \simeq 0.145873...$$

Instanton background



Question : how to connect large and small mass limits ?

Gel'fand-Yaglom (1961)
computing ODE determinants efficiently

Theorem :

$$\boxed{\frac{\det \mathcal{M}_1}{\det \mathcal{M}_2} = \frac{\phi_1(L)}{\phi_2(L)}}$$

Solve related λ=0 initial value problem :

$$\mathcal{M}_i\, \phi_i = 0 \qquad\qquad \phi_i(0) = 0 \quad ; \quad \phi_i'(0) = 1$$

Comments :

- no detailed information needed about phase shift or bound state spectrum
- determinant determined by boundary values
- general b.c.'s similar

Instanton background in QCD

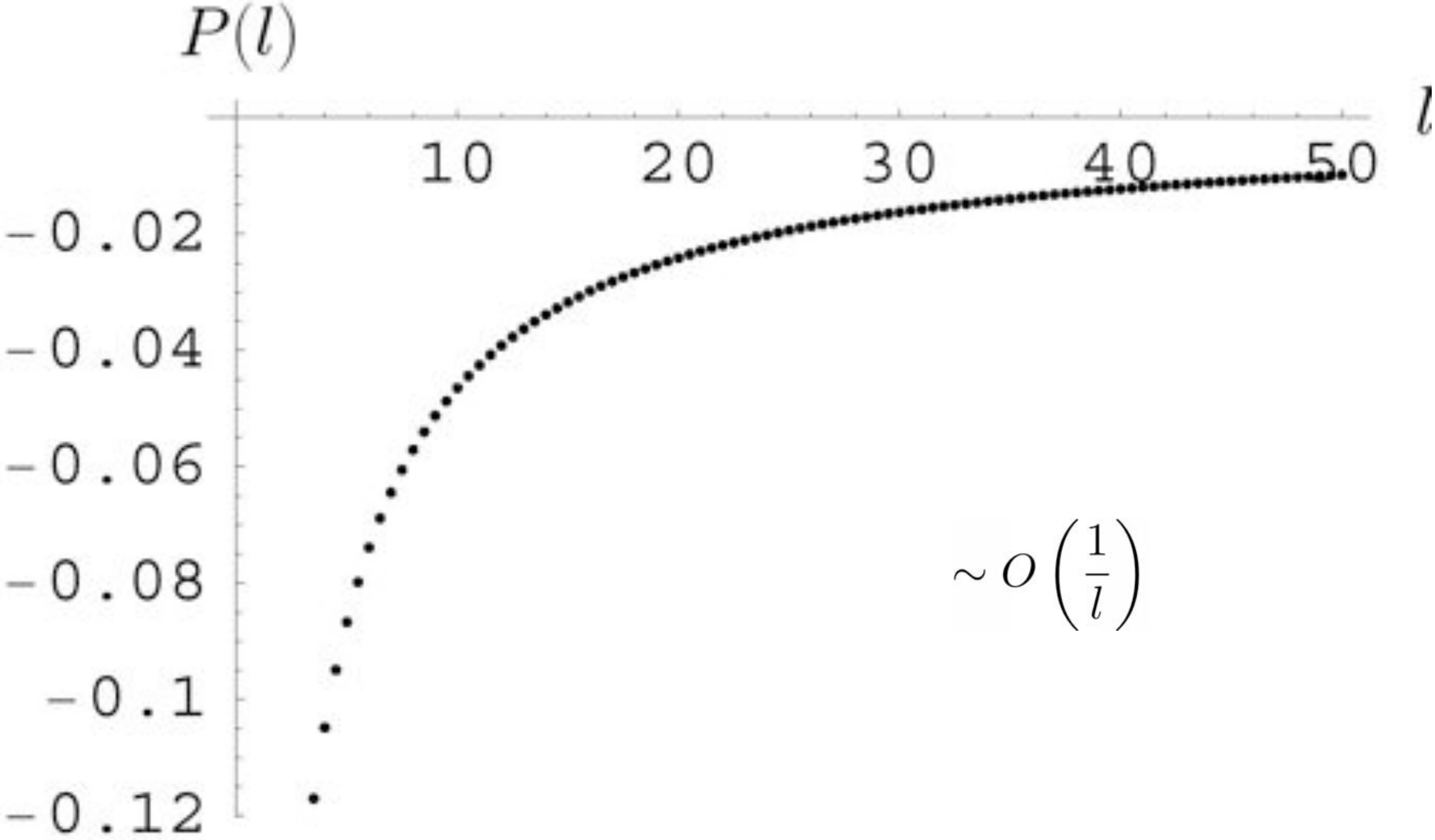
scalar (Klein-Gordon) determinant in an instanton background :

$$\Gamma^S(A;m) = \ln\left[\frac{\mathrm{Det}(-D^2+m^2)}{\mathrm{Det}(-\partial^2+m^2)}\right]$$

radial symmetry reduces problem to a sum over ODEs

$$\Gamma = \sum_{l=0,\frac{1}{2},\dots}(2l+1)(2l+2)\left\{\ln\left(\frac{\det\left[\mathcal{H}_l+m^2\right]}{\det\left[\mathcal{H}^{\mathrm{free}}_{(l)}+m^2\right]}\right)\right\}$$

l dependence of log det



"bad" news ?

$$\Gamma = \sum_{l=0,\frac{1}{2},1,\ldots} (2l+1)(2l+2)P(l)$$

quadratically divergent sum !!!

BUT :

bare expression, without regularization or renormalization

regularization & renormalization

Regularization : Pauli-Villars regulator mass Λ

$$\Gamma^S_\Lambda(A;m) = \ln\left[\frac{\mathrm{Det}(-D^2+m^2)}{\mathrm{Det}(-\partial^2+m^2)}\frac{\mathrm{Det}(-\partial^2+\Lambda^2)}{\mathrm{Det}(-D^2+\Lambda^2)}\right]$$

Renormalization : Minimal subtraction renormalization condition

$$\Gamma^S_{\mathrm{ren}}(A;m) = \lim_{\Lambda\to\infty}\left[\Gamma^S_\Lambda(A;m) - \frac{1}{12}\frac{1}{(4\pi)^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})\right]$$

$$= \lim_{\Lambda\to\infty}\left[\Gamma^S_\Lambda(A;m) - \frac{1}{6}\ln\left(\frac{\Lambda}{\mu}\right)\right]$$

Angular momentum cutoff regularization & renormalization

GD, Hur, Lee & Min, PRL & PRD 2005; hep-th/0410190; hep-th/0502087

split sum into 2 parts, with L large but finite

$$\Gamma^S_\Lambda(A;m) = \sum_{l=0,\frac{1}{2},\ldots}^{L} \Gamma^S_{(l)}(A;m) + \sum_{l=L+\frac{1}{2}}^{\infty} \Gamma^S_{\Lambda,(l)}(A;m)$$

evaluate analytically (large L)

evaluate numerically (large L)

large L behavior from WKB

analytic WKB (large l) computation : 2nd order radial WKB

$$\sum_{l=L+\frac{1}{2}}^{\infty} \Gamma^S_{\Lambda,(l)}(A;m) \sim \frac{1}{6}\ln\Lambda + 2L^2 + 4L - \left(\frac{1}{6} + \frac{m^2}{2}\right)\ln L$$

$$+ \left[\frac{127}{72} - \frac{1}{3}\ln 2 + \frac{m^2}{2} - m^2 \ln 2 + \frac{m^2}{2}\ln m\right] + O\left(\frac{1}{L}\right)$$

NOTE :

- $\ln\,\Lambda$ term exactly as required for renormalization
- quadratic, linear and log divergences, and finite part
- exactly cancel divergences from numerical sum in large L limit
- mass dependence in " subtraction" terms

Angular momentum cutoff technique

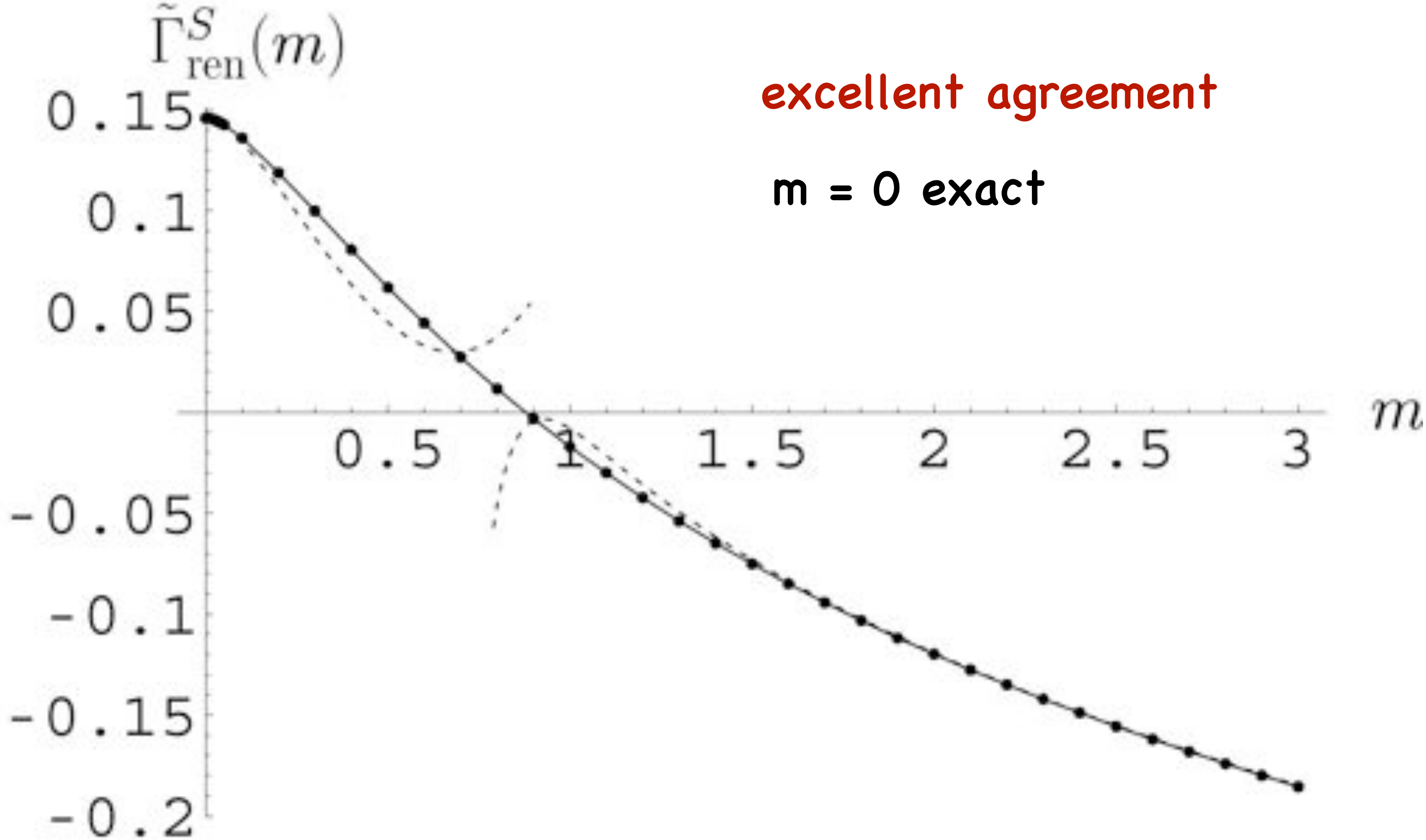
Finite and renormalized effective action :

$$\Gamma_{\rm ren} = \lim_{L\to\infty} \left\{ \sum_{l=0,\frac{1}{2},\ldots}^{L} \Gamma_{(l)} + 2L^2 + 4L - \left(\frac{1}{6} + \frac{m^2}{2}\right)\ln L + \left[\frac{127}{72} - \frac{1}{3}\ln 2 + \frac{m^2}{2} - m^2 \ln 2 + \frac{m^2}{2}\ln m\right]\right\}$$

evaluated numerically (large L)

evaluated analytically (large L)

comparison with asymptotic results

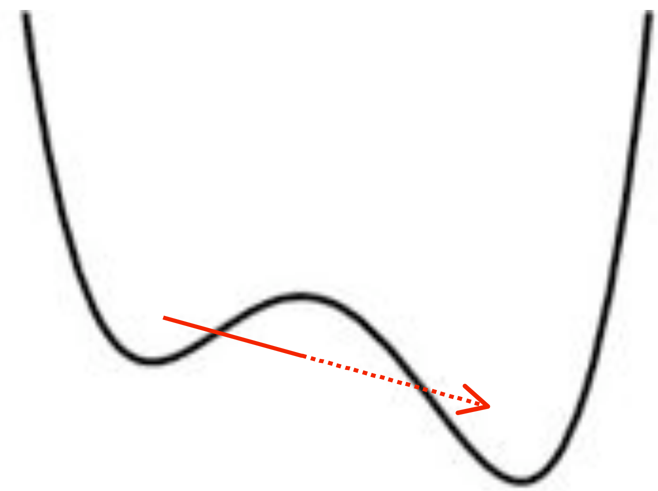


excellent agreement

m = 0 exact

False vacuum decay

$$S_{\rm cl}[\phi] = \int d^4x \left(\frac{1}{2} (\partial_\mu \phi)^2 + U(\phi) \right)$$

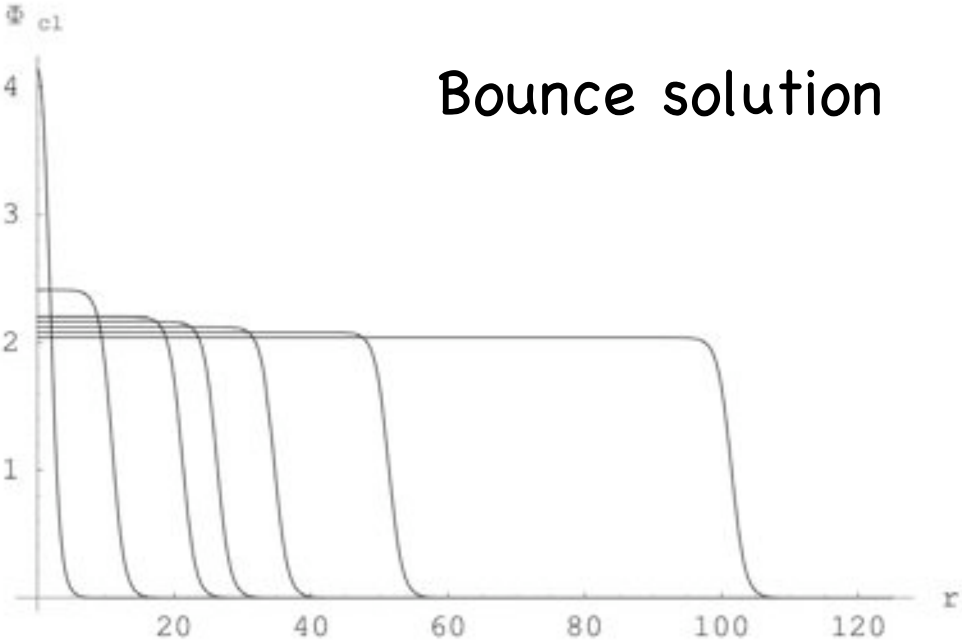


$$\gamma = \left(\frac{S_{\rm cl}[\phi_{\rm cl}]}{2\pi} \right)^2 \left(\frac{\det' \left(-\nabla^2 + U''(\phi_{\rm cl}) \right)}{\det \left(-\nabla^2 + U''(\Phi_-) \right)} \right)^{-1/2} e^{-S_{\rm cl}[\phi_{\rm cl}] - S_{\rm ct}[\phi_{\rm cl}]}$$

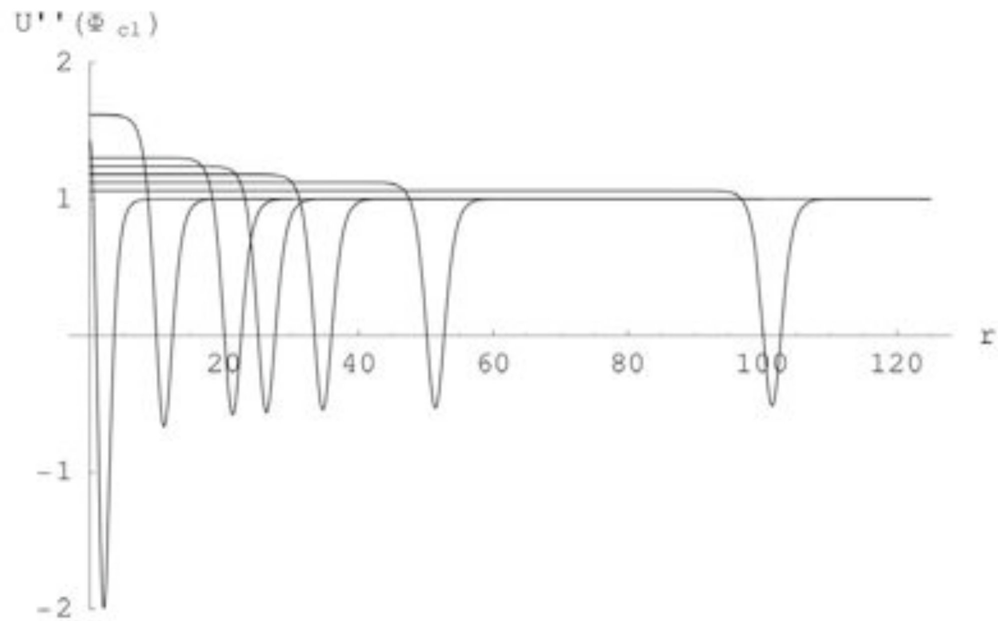
Analytical results in thin-wall limit

Away from thin-wall limit: radial fluctuation problem

Bounce solution

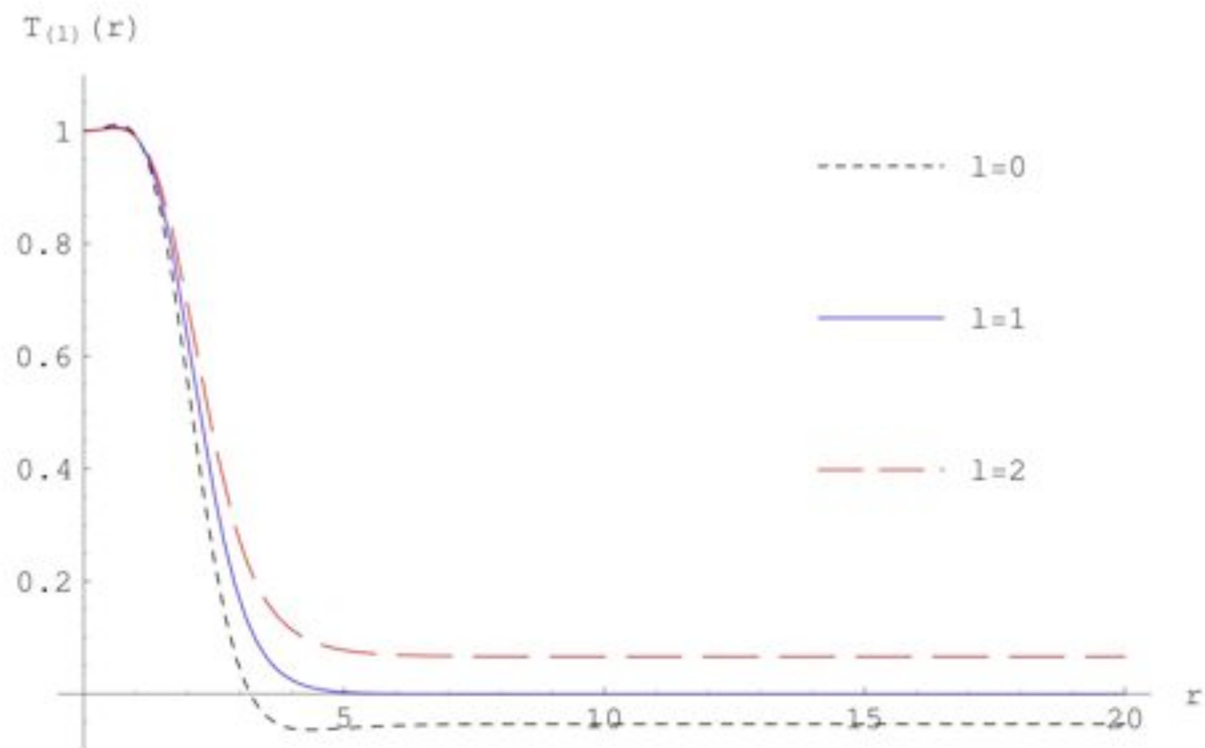


Fluctuation potential



Fluctuation operator modes

- l=0 : one negative mode
- l=1 : 4 zero modes
- l>1 : positive modes



contributions to log det

GD & H. Min, hep-th/0511156

- l=0

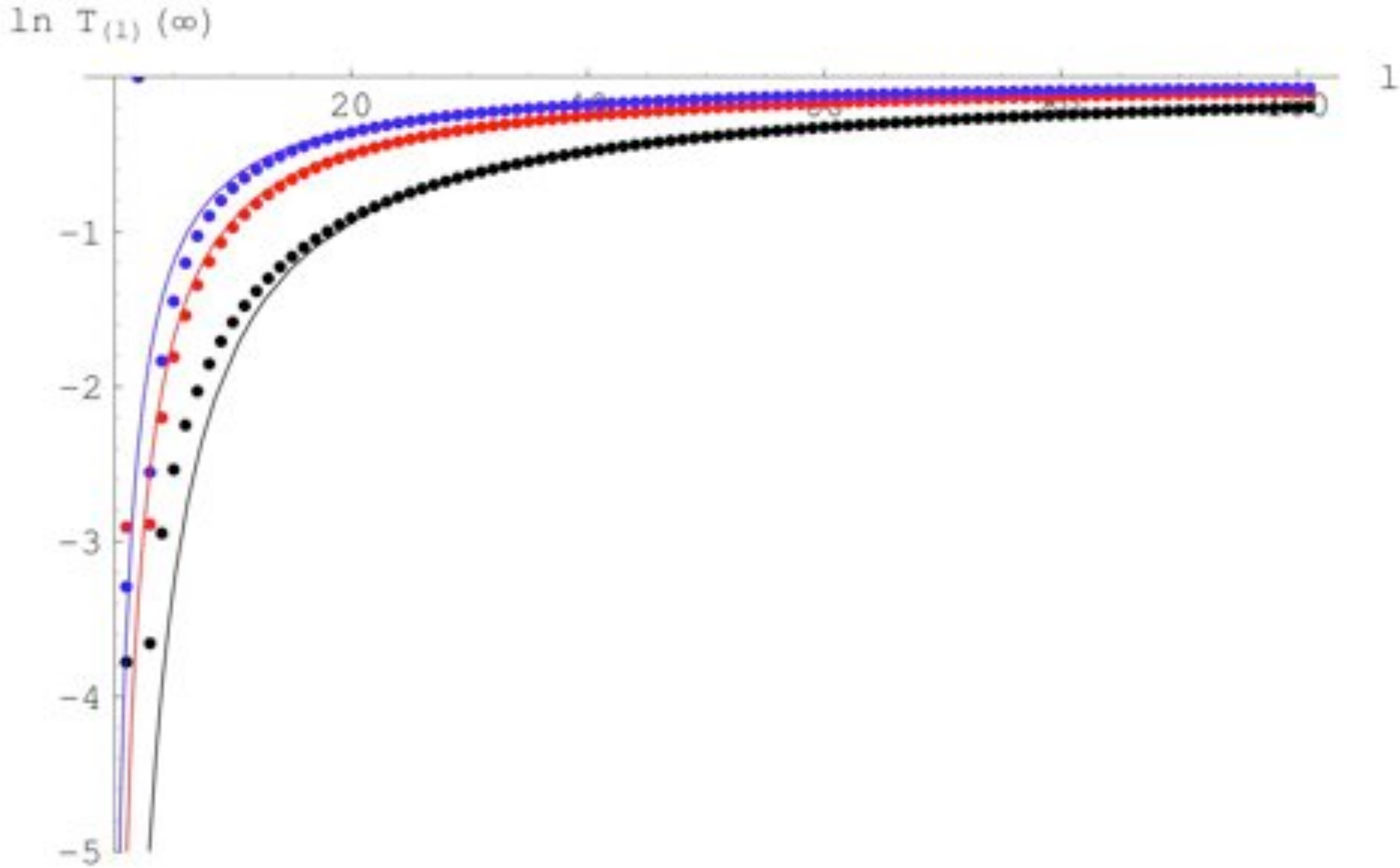
$$\left|\frac{\det \mathcal{M}_{(l=0)}}{\det \mathcal{M}^{\rm free}_{(l=0)}}\right|^{-1/2} = \left|T_{(0)}(\infty)\right|^{-1/2}$$

- l=1

$$\left(\frac{S_{\rm cl}[\varphi_{\rm cl}]}{2\pi}\right)^2 \left(\frac{\det' \mathcal{M}_{(l=1)}}{\det \mathcal{M}^{\rm free}_{(l=1)}}\right)^{-1/2} = \left[\frac{\pi}{4}\Phi_\infty\left(\Phi_0 - \frac{3}{2}\Phi_0^2 + \frac{\alpha}{2}\Phi_0^3\right)\right]^2$$

- l>=2

$$\ln\left(\frac{\det \mathcal{M}_{(l)}}{\det \mathcal{M}^{\rm free}_{(l)}}\right) \sim \frac{\frac{1}{2}\int_0^\infty dr\, r\, V(r)}{(l+1)} - \frac{\frac{1}{8}\int_0^\infty dr\, r^3\, V(V+2)}{(l+1)^3} + \dots$$



ln T(1)(∞)
20
1
-1
-2
-3
-4
-5

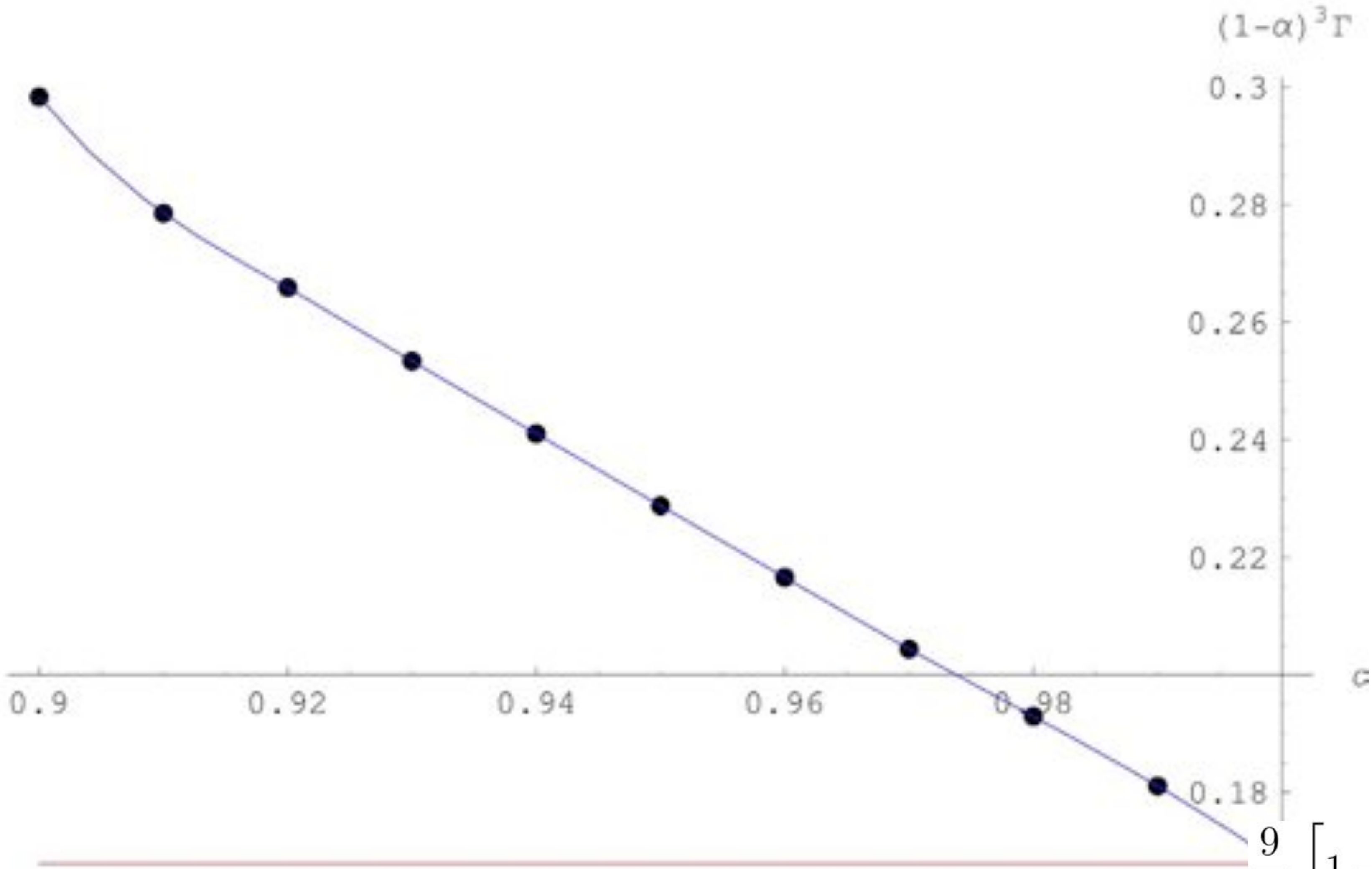
Finite, renormalized log det

$$\Gamma_{\overline{\mathrm{MS}}} = \frac{1}{2}\ln\left|T_{(0)}(\infty)\right| - 2\ln\left[\frac{\pi}{2}\Phi_\infty\left(\Phi_0 - \frac{3}{2}\Phi_0^2 + \frac{\alpha}{2}\Phi_0^3\right)\right]$$

$$+\frac{1}{2}\sum_{l=2}^{\infty}(l+1)^2\left\{\ln\left(T_{(l)}(\infty)\right) - \frac{\frac{1}{2}\int_0^\infty dr\, r\, V(r)}{(l+1)} + \frac{\frac{1}{8}\int_0^\infty dr\, r^3\, V(V+2)}{(l+1)^3}\right\}$$

$$-\frac{3}{4}\int_0^\infty dr\, r\, V(r) + \frac{1}{16}\int_0^\infty dr\, r^3\, V(V+2)\left(\frac{1}{2} - \gamma_E - \ln\frac{r}{2}\right)$$

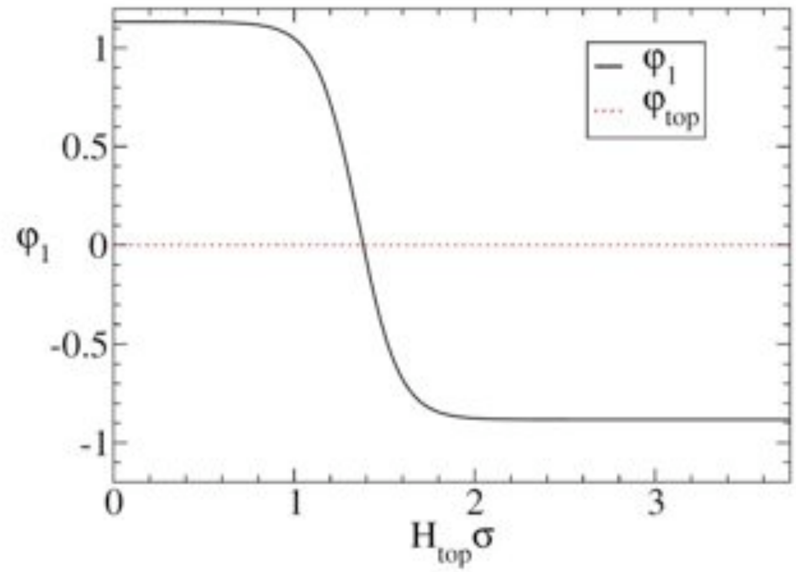
comparison with analytic thin-wall result

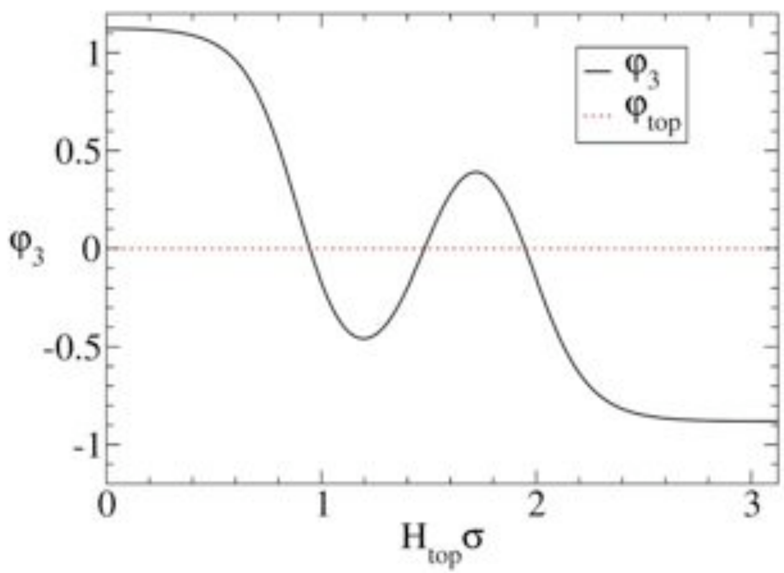


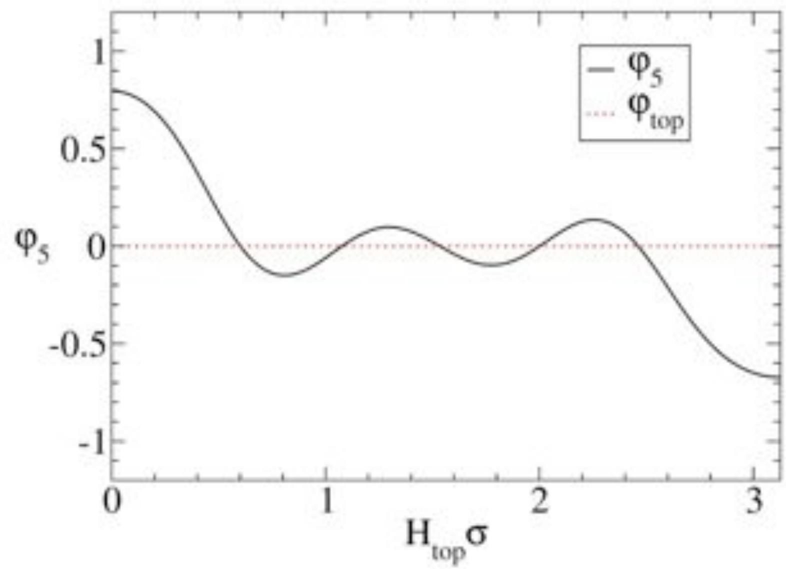
$$\frac{9}{32}\left[1-\frac{2\pi}{9\sqrt{3}}\right]$$

Konoplich & Rubin 1986

With gravity: new bounces

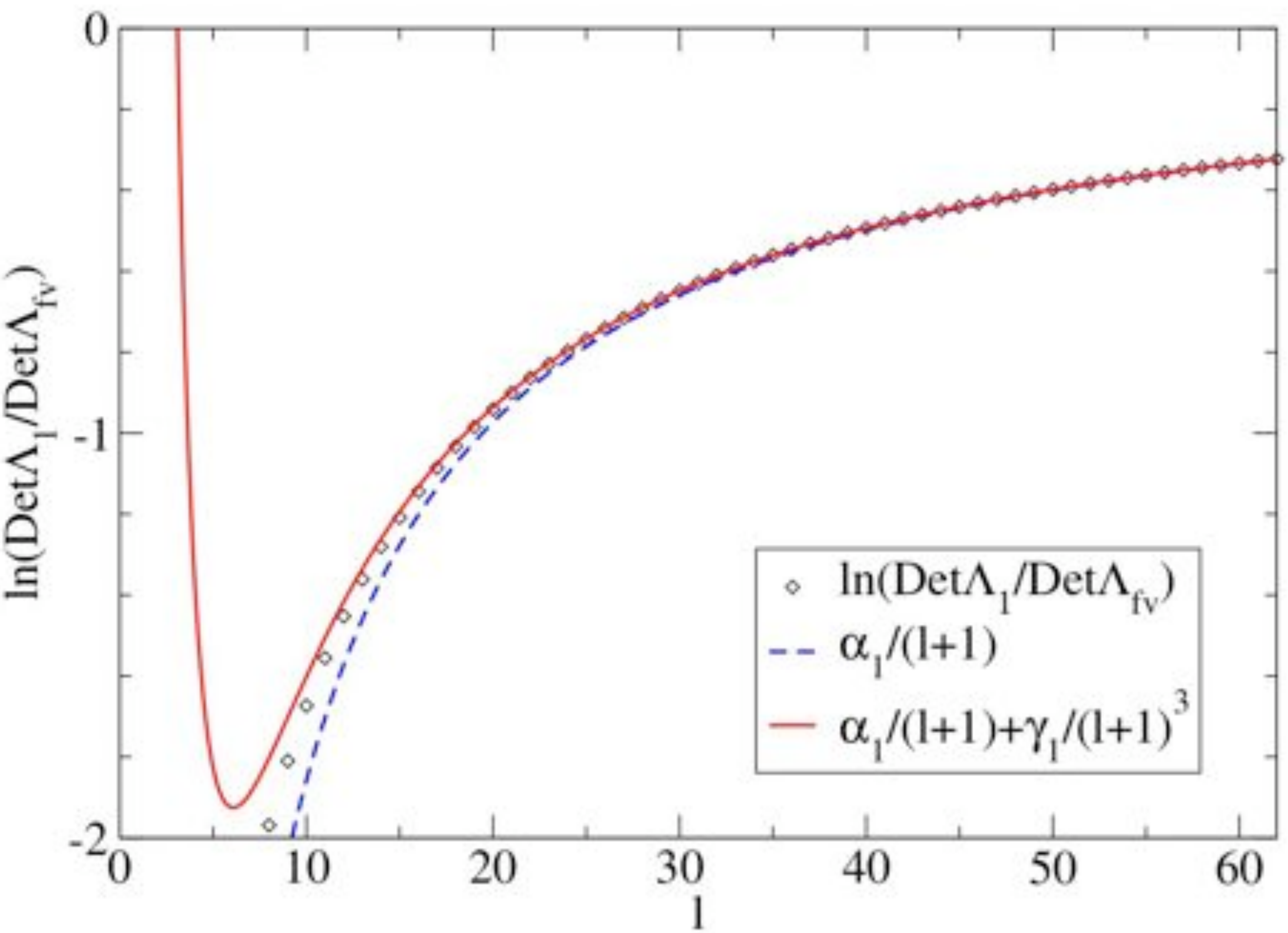






Large l log determinant

GD & Q. Wang, hep-th/0605176



Goal : enlarge class of computable determinants

1 dimension : Gel'fand-Yaglom 1961

$$\ln\left(\frac{\det\mathcal{M}}{\det\mathcal{M}^{\rm free}}\right) = \ln\left(\frac{\psi(L)}{\psi^{\rm free}(L)}\right)$$

Higher dimension : radial

$$\ln\left(\frac{\det\mathcal{M}}{\det\mathcal{M}^{\rm free}}\right) = \sum_{l=0}^{\infty} \deg(l)\,\ln\left(\frac{\psi_l(R)}{\psi_l^{\rm free}(R)}\right)$$

Divergent !

Forman 1992

Angular momentum cutoff technique :

$$\ln\left(\frac{\det\mathcal{M}}{\det\mathcal{M}^{\rm free}}\right) = \sum_{l=0}^{\infty} \deg(l)\left\{\ln\left(\frac{\psi_l(R)}{\psi_l^{\rm free}(R)}\right) - \frac{a}{l+1} - \frac{b}{(l+1)^3}\right\} - c$$

GD & H. Min, 2005

Finite & renormalized

Conclusions

- Method ideal for radially separable problems

- Coupled equations: fermions & nonabelian

- instanton determinant : interpolates smoothly between chiral and heavy quark limits (PRL 94, 2005)

- False vacuum decay : flat space (GD & H. Min, PRD 72, 2005)
  curved space (GD & Q.Wang, hepth/0605176)

- Accessible problems :

  Monopole, sphaleron, skyrmion, vortex, BPS, domain wall backgrounds