Bulletin of the
University of Minnesota

ENGINEERING EXPERIMENT STATION

O. M. LELAND, Director

BULLETIN NO. 9

INFLUENCE LINES FOR ARCHES
WITH TABLES

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Vol. XXXVI No. 33 July 29, 1933

MINNEAPOLIS

Entered at the post-office in Minneapolis as second-class matter
Minneapolis, Minn.
Accepted for mailing at special rate of postage provided for in section 1103,
Act of October 3, 1917, authorized July 12, 1918

TABLE OF CONTENTS

TABLE AND PLATES

INFLUENCE LINES FOR ARCHES

INTRODUCTION

In the preliminary design of arches, either two-hinged or hingeless, the parabolic arch in which the moment of inertia varies with the secant of the slope is frequently employed. By the application of influence lines to this type of arch, the reactions and stresses at various points can readily be determined and will serve as a basis for modifying the preliminary design. This bulletin presents a tabulation of the ordinates for influence lines of this type of arch together with a tabulation of the areas under the influence lines, so arranged that it is not necessary to compute the ordinates of the influence lines. The only work required is that of multiplying the tabular values by certain ratios dependent upon the span and rise of the arch and the intensity of the load. These tables can be applied either to concentrated or to uniformly distributed loads over any part of the span, and the portions of the span to be loaded to obtain maximum stresses are readily determined. By the use of the tables, the work of carrying through the preliminary analysis is greatly expedited. Examples are shown which indicate clearly the rapidity with which the quantitative results may be obtained.

The basic assumptions made in the derivation of the tables is that the arch axis is a parabolic curve with vertex at the crown, and that the moment of inertia of each section varies as the secant of the slope of the arch axis at that point. These and the usual assumptions made in the common theory of flexure are the only ones made. The examples also show how the rib shortening and temperature stresses may be determined by the use of the tabulations.

THE TWO-HINGED ARCH

The two-hinged arch is a statically indeterminate structure. There are in general four components of reaction, one vertical and one horizontal at each end. If we consider the right end placed on rollers, so that the horizontal component of reaction at that end disappears, we render the structure statically determinate, and we shall call this statically determinate structure the "base structure." The right end, now unrestrained, will move horizontally. The horizontal component of end reaction in the two-hinged arch can be determined by finding the force H, required to move the end on rollers (in the base structure) back to its original position. This force is

$$H = -\frac{\int \frac{M' \, m \, ds}{E \, I}}{\int \frac{m^2 \, ds}{E \, I}} \quad \text{(1)*}$$

In equation (1),

H = Horizontal component of end reaction.
M' = Moment at any point in base structure due to external loads (positive moments producing tension in intrados).
m = Moment at same point in base structure due to unit horizontal load at end of arch.
ds = Differential length of arc.
E = Modulus of elasticity.
I = Moment of inertia at same point.

If we choose an origin of co-ordinates at the crown of the arch, X axis horizontal, Y axis vertical, the equation of the parabolic arch axis is

$$y = \frac{4h \, x^2}{l^2} \quad \text{(2)}$$

where

h = Rise of arch
l = Span length of arch

In accordance with our basic assumptions, I varies with the secant of α, the slope of the arch axis, or

$$I = I_c \sec \alpha \quad \text{(3)}$$

where I_c is the moment of inertia at the crown. Also from a well-known theorem in calculus,

$$ds = dx \sec \alpha \quad \text{(4)}$$

Hence substituting, (3) and (4) in (1) we get,

$$H = -\frac{\int \frac{M' \, m \, dx \sec \alpha}{E \, I_c \sec \alpha}}{\int \frac{m^2 \, dx \sec \alpha}{E \, I_c \sec \alpha}} = -\frac{\int M' \, m \, dx}{\int m^2 \, dx} \quad \text{(5)}$$

E and I_c being constants.

For any vertical downward load, P on the arch at a distance a to the right of the crown, the right reaction is

* See Johnson, Bryan, and Turneaure, *Modern Framed Structures,* Part II, p. 153, tenth edition; also Parcel and Maney, *Statically Indeterminate Stresses.*

$$R_R = \frac{P}{l}\left(\frac{l}{2} + a\right) \quad \ldots\ldots (6)$$

and is vertical, of course. The moment at any point x, to the right of load is

$$M' = \frac{P}{l}\left(\frac{l}{2} + a\right)\left(\frac{l}{2} - x\right) \quad \ldots\ldots (7)$$

and similarly, the moment at any point to left of load, is

$$M' = \frac{P}{l}\left(\frac{l}{2} - a\right)\left(\frac{l}{2} + x\right) \quad \ldots\ldots (8)$$

bearing in mind that abcissae, x, to the left of crown are negative.

For a unit horizontal load at right end of arch, the moment at any point is

$$m = -1\# \ (h - y) = -h\left(1 - \frac{4x^2}{l^2}\right) \quad \ldots\ldots (9)$$

the unit load being assumed to act toward the left.

Substituting these values in equation (5), we have

$$H = -\left[\frac{\int_{-l/2}^{+a} \frac{P}{l}\left(\frac{l}{2} - a\right)\left(\frac{l}{2} + x\right)\left(-h\right)\left(1 - \frac{4x^2}{l^2}\right)dx}{2\int_0^{l/2} h^2\left(1 - \frac{4x^2}{l^2}\right)^2 dx} + \frac{\int_{+a}^{+l/2} \frac{P}{l}\left(\frac{l}{2} + a\right)\left(\frac{l}{2} - x\right)\left(-h\right)\left(1 - \frac{4x^2}{l^2}\right)dx}{2\int_0^{l/2} h^2\left(1 - \frac{4x^2}{l^2}\right)^2 dx}\right] \quad \ldots\ldots (10)$$

Evaluating this and letting $\frac{a}{l} = Q$.

$$H = \frac{5}{8}\left(\frac{5}{16} - \frac{3}{2}Q^2 + Q^4\right)\frac{Pl}{h} = \frac{Pl}{h}\phi_1 \quad \ldots\ldots (11)$$

where ϕ_1 is a function of Q. This function of Q is the one given in column 2 of the tabulation, for values of Q varying by .01 from 0 to .50. The third column in the table gives $\Sigma\ \phi_1$, the summation of Q from the crown to each point tabulated. This is the equivalent of an area under the ϕ_1 curve. It is used when we have uniformly distributed load over part or all of the arch.

INFLUENCE LINES

An influence line is a graph of some stress or strain function of a structure, showing the variation of this function as a load traverses the structure. Since our stresses will be proportional to the load, we can choose a unit load for convenience in calculation. The influence line for H, for example, will be so constructed that every ordinate will give the value of H for a unit load placed on the structure at the point corresponding to this ordinate. Equation (11) is therefore the equation of this influence line, and column 2 of Table I, when multiplied by $\frac{l}{h}$ will give the ordinates of this influence line. If we consider a differential length of the structure, dx, with a load of uniform intensity, w, applied to it, we see that for a uniform load, w, applied to any portion of the structure, $H = w \frac{l}{h} \int \phi_1 \, dx$ which is the equivalent of the area under the influence line in the loaded region, multiplied by w. This is approximately equal to $w \frac{l}{h} \Sigma \phi_1$, replacing the integration by a summation.

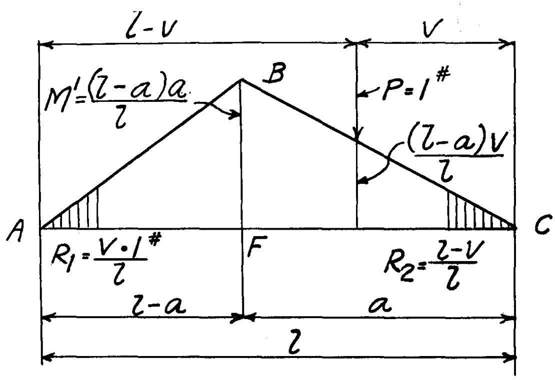


Fig. 1

In a simple beam the variation in moment at a particular point, such as F in Figure 1, may be evaluated for a concentrated load on the span at variable distances from the supports. For a unit load at any distance v, the moment at $F = R_1 \left(l - a\right) = \frac{(l - a)\,v}{l}$. It is zero when the load is at the supports, and as the variable is of first degree, increases linearly to $\frac{(l - a)\,a}{l}$ which is the maximum when the load is at F.

From this, it is evident that a diagram, such as Figure 1 may be drawn, with an ordinate equal to $\frac{(l-a)\,a}{l}$, at the point where the moment is required, and formed by lines AB and BC. The lines AB and BC are influence lines, and an ordinate under line BC has a value of $\frac{(l-a)\,v}{l}$, which is equal to the moment at F for a one-pound load at v. Similarly an ordinate under AB has a value of $\frac{a}{l}\left(l-v\right)$.

If in Figure 1 a uniform load of p pounds per foot is used, the load over any differential length to the right of F is p dv and the value of the ordinate for load p dv is p dv $\frac{(1-a)\,v}{l}$ which equals the moment at F. Summing these ordinates between limit of one foot apart gives

$$P\int_{v}^{v+1}\frac{(l-a)v}{l}\,dv = P\left(\frac{(l-a)v^2}{2l}\right)_{v}^{v+1} = \text{P multiplied by area of influence diagram per one foot of length.}$$

For a uniform load over any length of span the moment at F is equal to p times the area of influence diagram in the loaded region (using foot units).

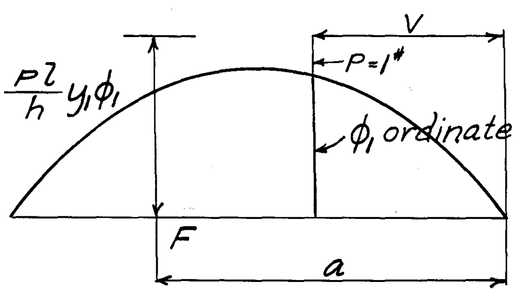


Fig. 2

If a two-hinged arch is selected, the moment at a particular point F, due to the horizontal reaction, is equal to $Hy_1 = 1\# \times \frac{l}{h}\,\phi_1 y_1$ from equation (11), where y_1 is the moment arm at F, and ϕ_1 is the ordinate under any point where a unit load is placed (see Fig. 2). Hence, each ϕ_1 value may be multiplied by the coefficient $y_1\frac{l}{h}$ and these values plotted and combined with the simple beam diagram to form a resultant influence line.

However, it is much more convenient to make a diagram directly from the ϕ_1 values in the table, and then upon this superimpose a simple beam diagram, and this procedure will be followed now.

As the ϕ_1 values are not the true moment values, and as the scale must be kept the same for the purpose of combining influence lines, it will be necessary also to divide all the simple beam ordinates by the coefficient of ϕ_1. This will be accomplished by dividing the simple beam ordinate $\frac{(l-a)a}{l}$ by $y_1\frac{l}{h}$, and then using lines AB and BC (Figure 1). After the diagrams are combined, any resulting areas used must in turn be multiplied by $y_1\frac{l}{h}$. This transformation will be algebraically presented on page 8.

As each ordinate represents a moment value, the maximum moment at a particular point F for live load is obtained when all the resultant areas of like sign are used. For this condition, the live load will be uniform over the areas used. For full uniform load, the total difference of resultant areas must be used. In Figure 3 the minus resultant areas ABCA and DEFD are those due to the greater effect of the horizontal force H over that of the vertical forces, and indicate compressive stress on intrados and tension on extrados.

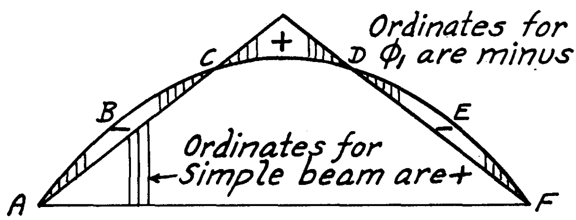


Fig. 3

If the load used is P pounds per lineal foot of span, then the length of the resultant areas must be expressed in feet, as the load P will be distributed over one foot of length.

The areas in Table I are based on length of span of unity, and must be multiplied by the length of span used in the design.

THE KERN POINT

For the purpose of determining the fiber stresses in the extreme fibers at any section, we use the kern points of that section. In Figure 4 we have a thrust N and a moment M applied at the centroid of the section. This is equivalent to a thrust N applied a distance e $\left(=\frac{M}{N}\right)$ from the centroidal axis as shown. The maximum fiber stresses are:

$$\left.\begin{aligned} f_1 &= \frac{N}{A} + \frac{Mc_1}{I} = \frac{N}{A} + \frac{Nec_1}{I} \\ f_2 &= \frac{N}{A} - \frac{Mc_2}{I} = \frac{N}{A} - \frac{Nec_2}{I} \end{aligned}\right\} \qquad (12)$$

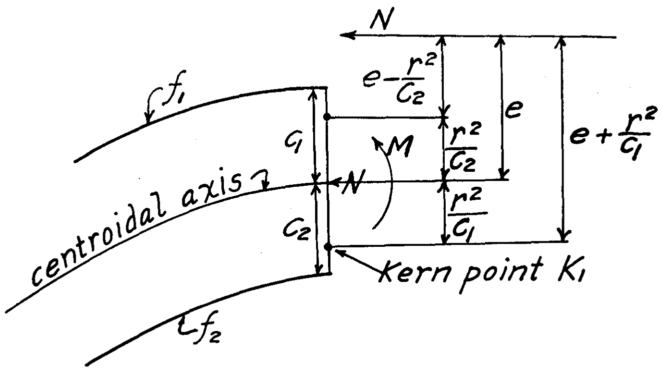


Fig. 4

Substituting $I = Ar^2$ where r is the radius of gyration of section,

$$\left.\begin{aligned} f_1 &= \frac{N}{A} + \frac{Nec_1}{Ar^2} = \frac{N}{A}\left(1 + \frac{ec_1}{r^2}\right) = \frac{N\left(\frac{r^2}{c_1} + e\right)c_1}{Ar^2} \\ f_2 &= \frac{N}{A} - \frac{Nec_2}{Ar^2} = \frac{N}{A}\left(1 - \frac{ec_2}{r^2}\right) = -\frac{N\left(e - \frac{r^2}{c_2}\right)c_2}{Ar^2} \end{aligned}\right\} \qquad (13)$$

If we call $N\left(\frac{r^2}{c_1} + e\right) = M_{k1}$ the moment about kern point k_1, and similarly $-N\left(e - \frac{r^2}{c_2}\right) = M_{k2}$ the moment about kern point k_2, where the kern point k_1 is located a distance $\frac{r^2}{c_1}$ below centroidal axis and the kern point k_2 is located a distance $\frac{r^2}{c_2}$ above centroidal axis, then

$$\left.\begin{aligned} f_1 &= \frac{c_1 M_{k1}}{I} \\ f_2 &= \frac{c_2 M_{k2}}{I} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Thus the expression for extreme fiber stress is reduced to a single term involving the moment of the resultant thrust at the section about the kern point. The total moment to use for determining fiber stress is composed of the moment about the kern point produced by the external forces in the base structure and the moment about the kern point produced by H, the horizontal component of end reaction. The influence line for fiber stress will be $\frac{c}{I}$ times the influence line for moment about the kern point, where c is distance from centroidal axis to the extreme fiber for which influence line is being constructed, the kern point corresponding to this extreme fiber being used, and I is the moment of inertia of the section.

ILLUSTRATIVE EXAMPLE FOR TWO-HINGED ARCH

Plate III (page 17) shows an arch axis which is used in this example. The arch is 400 feet in span with a rise of 60 feet. The moment at any point is

$$M = M' - Hy_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

where M' is the moment at this point in base structure due to external loads, H is the horizontal component of end reaction and y_1 is the vertical distance from springing line to this point. Substituting the value of H from equation 11, and taking P = 1 pound,

$$M = M' - \frac{l}{h}\phi_1 y_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

or

$$M = y_1 \frac{l}{h}\left(\frac{M' h}{y_1 l} - \phi_1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Thus $\frac{h}{y_1 l}$ is a scalar factor which reduces M' to the same scalar unit system as ϕ_1, and hence enables us to superimpose the influence lines corresponding to the two terms in parentheses in equation 17. If we draw an influence line representing the value $\frac{M' h}{y_1 l}$ and superimpose on it the influence line for ϕ_1, each ordinate included between these two influence lines multiplied by $y_1 \frac{l}{h}$ will be equal to the ordinate of the influence line for M. We will consider point (4) on the arch axis. A trussed rib arch, will be assumed, with kern points 5 feet on either side of central axis. Then, for the kern point k_1 at section (4) which corresponds to the extreme fiber in extrados, we find $y_1 = 40.2$ feet, and x_1 (distance from left end) $= 101.4$ feet. Placing a unit load at this point, the right reaction in the base structure will be $1 \times \frac{101.4}{400}$,

$$M' = 1 \times \frac{101.4}{400} \times 298.6 = 75.6951 \text{ ft.-lbs.}$$

and

$$\frac{M' h}{y_1 l} = \frac{75.6951 \times 60}{40.2 \times 400} = .28244$$

In similar fashion, $\frac{M' h}{y_1 l}$ equals

at point (4), intrados, $1 \times \frac{98.6}{400} \times 301.4 \times \frac{60}{49.8 \times 400} = .22378$

at point (1), intrados, $\frac{½\# \times 200}{65} \times \frac{60}{400} = .23077$

at point (1), extrados, $\frac{½\# \times 200}{55} \times \frac{60}{400} = .27272$

These are the ordinates of the vertices of triangular influence lines shown in Plate I superimposed on curve for ϕ_1. To find the fiber stress at center, extrados, the area A (Fig. 5) is first evaluated.

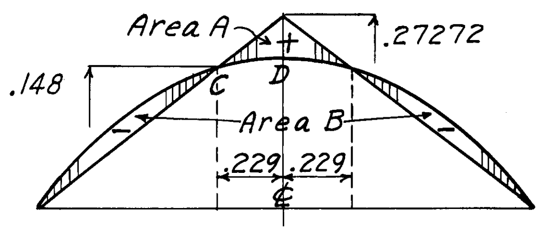


Fig. 5

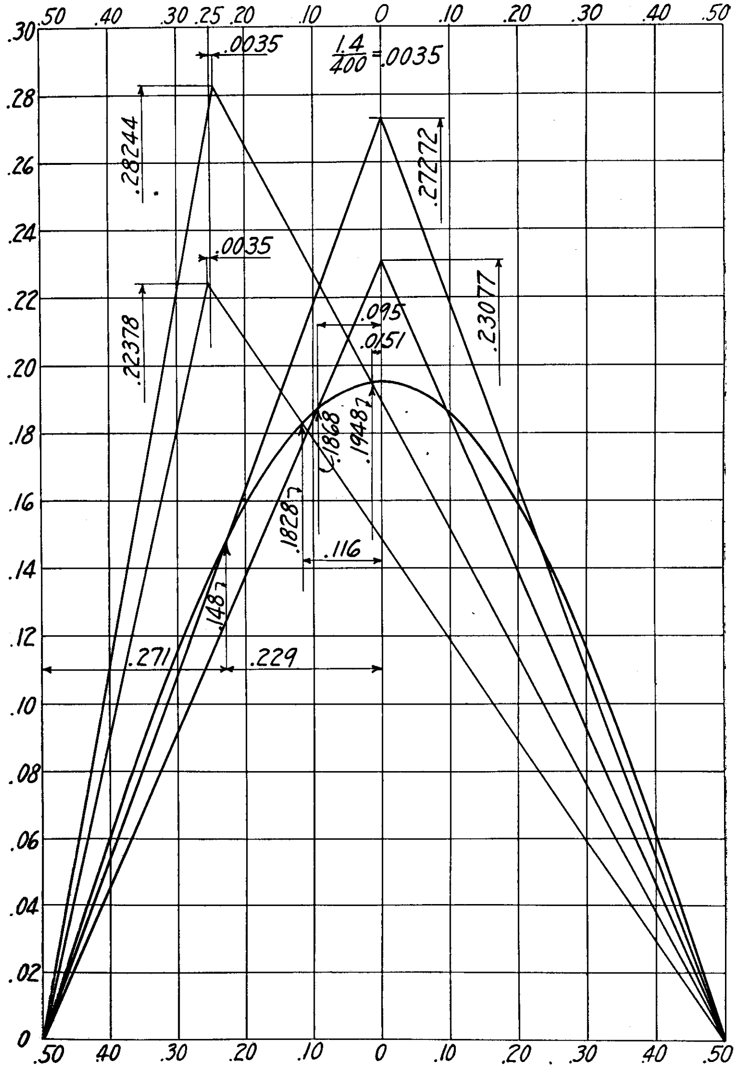
CALCULATION OF MOMENTS ABOUT KERN POINTS AT:

Center for Extrados

For live load compressive fiber stress

$$\text{Area A} = 2\left[\left(\frac{.27272 + .148}{2} \times .229\right) - \Sigma\, \phi_1\right]$$

where $\Sigma\, \phi_1$ is area under curve between C and D. (See Table I.)



INFLUENCE LINES FOR MOMENTS

PLATE I

$$\frac{.27272 + .148}{2} \times .229 = \qquad .048172$$

$$\begin{array}{lrr}
\Sigma\,\phi_1\ (\text{for } Q = .23) = & .0412001 & \\
\Sigma\,\phi_1\ (\text{for } Q = .22) = & .0397057 & \\
\hline
\text{Diff.} & .0014944 & \\
\Sigma\,\phi_1\ (\text{for } Q = .23) = & .041200 & \\
 & -.000149 & \\
\hline
\Sigma\,\phi_1\ (\text{for } Q = .229) = & .041051 = & .041051 \\
\hline
 & \text{Subtracting} & .007121
\end{array}$$

Multiplying by 2, Area A = .014242

For a uniform live load of 1200 pounds per foot, the moment about kern point for determining fiber stress in extrados (applying equation 17) is:

$$(.014242 \times 400) \times \frac{400}{60} \times 55 \times 1200$$

$$= .014242 \times 3{,}200{,}000 \times 55 = 2{,}506{,}590 \text{ ft.-lbs.}$$

In a similar fashion we can calculate the following values:

For live load tensile fiber stress

$$\text{Area B} = 2\left[(.0625 - .041051) - \left(\frac{.148 \times .271}{2}\right)\right] =$$

$$2\,(.021449 - .020054) = .002790$$

$$\text{Moment} = (.00279 \times 3{,}200{,}000 \times 55) = 491{,}000'\#$$

For dead load compressive fiber stress

For a uniform dead load of 2000

$$\text{Area } (A - B) = \frac{.27272}{2} - (.0625 \times 2) = .01136$$

$$\begin{array}{rl}
A = & .014242 \\
-B = & .002790 \quad \text{For check} \\
\hline
 & .011452
\end{array}$$

$$\text{Moment} = (.01136 \times 400)\left(\frac{20}{3} \times 55 \times 2000\right) =$$

$$\left(\frac{.01136}{3} \times 16{,}000{,}000 \times 55\right) = 3{,}332{,}560'\#$$

Center for Intrados

For live load tensile fiber stress, using Area A:
$\text{Moment} = (.003098 \times 3{,}200{,}000 \times 65) = 644{,}384'\#$

For live load compressive fiber stress, using Area B:
$\text{Moment} = (.012774 \times 3{,}200{,}000 \times 65) = 2{,}657{,}000'\#$

For dead load compressive fiber stress, using Area (B — A):

$$\text{Moment} = \left(\frac{.009615}{3} \times 16{,}000{,}000 \times 65\right) = 3{,}333{,}200'\#$$

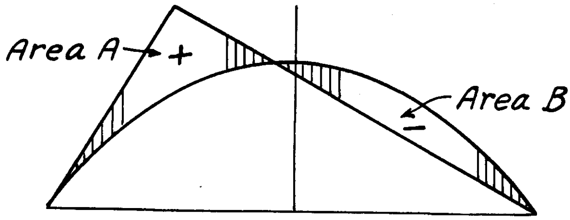


Fig. 6

Quarter Point for Extrados

For live load compressive fiber stress, using Area A:

$$\text{Moment} = (.031498 \times 3{,}200{,}000 \times 40.2) = 4{,}051{,}770'\#$$

For live load tensile fiber stress, using Area B:

$$\text{Moment} = (.015278 \times 3{,}200{,}000 \times 40.2) = 1{,}965{,}360'\#$$

For dead load compressive fiber stress, using Area (A — B):

$$\text{Moment} = \left(\frac{.01622}{3} \times 16{,}000{,}000 \times 40.2\right) = 3{,}477{,}780'\#$$

Quarter Point for Intrados

For live load tensile fiber stress, using Area A:

$$\text{Moment} = (.015255 \times 3{,}200{,}000 \times 49.8) = 2{,}431{,}040'\#$$

For live load compressive fiber stress, using Area B:

$$\text{Moment} = (.028365 \times 3{,}200{,}000 \times 49.8) = 4{,}520{,}250'\#$$

For dead load compressive fiber stress, using Area (B — A):

$$\text{Moment} = \left(\frac{.01311}{3} \times 16{,}000{,}000 \times 49.8\right) = 3{,}482{,}020'\#$$

SUMMING UP THE MOMENTS

	At Center	At Quarter Point
Total moment for compression on extrados	5,839,150′#	7,529,550′#
Total moment for compression on intrados	5,990,200′#	8,002,270′#

Thrust due to temperature

$$H_t{*} = \frac{15\, EI_c\, \omega t}{8\, h^2} = \frac{15 \times 29{,}000{,}000 \times .00039 \times I_c}{8 \times 60 \times 60 \times 12 \times 12} = .0409 \times I_c \text{ (60°}$$

variation in temperature.)

where

I_c = Moment of inertia at crown

ω = Coefficient of thermal expansion. For steel = .0000065

t = Variation in temperature

* Johnson, Bryan, and Turneaure, *Modern Framed Structures,* Part II, p. 155.

At center $\frac{5{,}990{,}000}{10} = F_s$ (F_s is maximum axial stress in chord)

For intrados $\frac{599{,}000}{14{,}000} = 42.79$ sq. in. area required at 14,000 lbs. fiber stress.

$I = \bar{I} + ad^2$, and neglecting $\bar{I}$ of chord about chord center,

$Ad^2 = [42.79 \times 60^2]\ 2 = 308{,}090 = I_c$ approximately.

Thrust for temperature $= (.0409 \times 303{,}090) = 12{,}600\#$ approximately.

$(12{,}600 \times 65) \div (10 \times 14{,}000) = 5.85$ sq. in., area required for H_t

By taking area required for M and adding area required for H_t above, we may find a closer approximation for I_c.

$$\begin{array}{r} 42.79 \text{ sq. in.} \\ 5.85 \phantom{\text{ sq. in.}} \\ \hline 48.64 \text{ sq. in.} \end{array} \quad \text{Say 50 sq. in.}$$

$(50 \times 60^2) \times 2 = 360{,}000 = I_c$

$360{,}000 \times .0409 = 14{,}724\# = H_t$

$(14{,}724 \times 65) \div (10 \times 14{,}000) = 6.90$ sq. in. $=$ New area for H_t

$42.79 + 6.90 = 49.69$ sq. in. $=$ Area required for intrados at center.

Area required at center for extrados $\left(\frac{5{,}839{,}000}{10 \times 14{,}000} + \frac{14{,}724 \times 55}{10 \times 14{,}000}\right) = 47.49$ sq. in.

Area required at quarter for extrados $\left(\frac{7{,}529{,}550}{10 \times 14{,}000} + \frac{14{,}724 \times 40.2}{10 \times 14{,}000}\right) = 58.06$ sq. in.

Area required at quarter for intrados $\left(\frac{8{,}002{,}270}{10 \times 14{,}000} + \frac{14.724 \times 49.8}{10 \times 14{,}000}\right) = 62.43$ sq. in.

COMPUTATION FOR SHEAR AT QUARTER POINT

For illustration, the shear will be determined by the use of influence lines for a simple beam. The shear at the left end in a simple beam for a concentrated unit load moving from right to left is zero when the load is at the right reaction and increases uniformly to unity when the load reaches the left reaction where the shear is a maximum (Fig. 7).

For the load approaching any particular section (a) the shear at (a) increases until it is equal to the ordinate under the load. When the load moving from the right approaches infinitely close to the section, the ordinate at (a) is a maximum. When the load passes the section, ordinates change sign and diminish as shown.

Employing a similar method for using ϕ_1 in shear diagram, as was done in the moment diagrams, and considering a load on the right of any section taken, we have for shear on the arch (see Fig. 8):

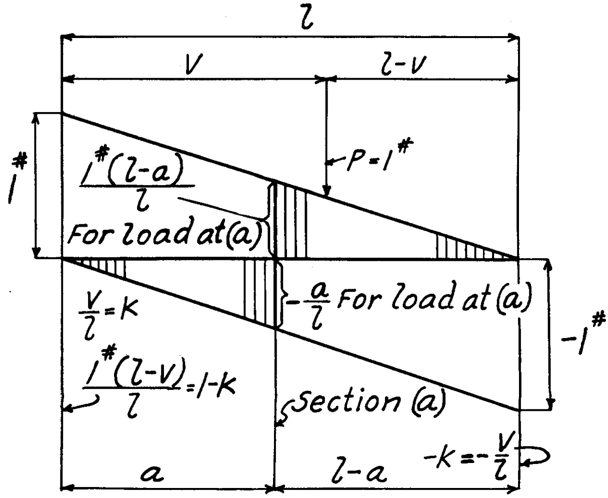


Fig. 7

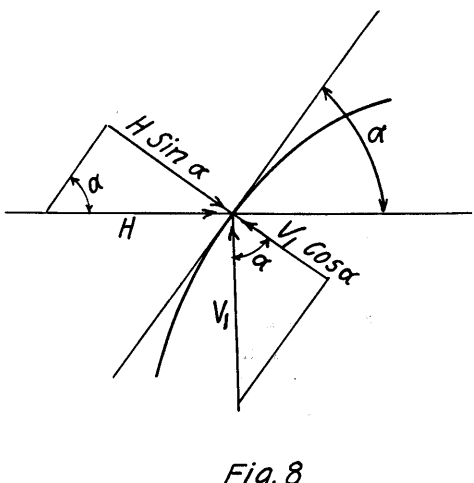


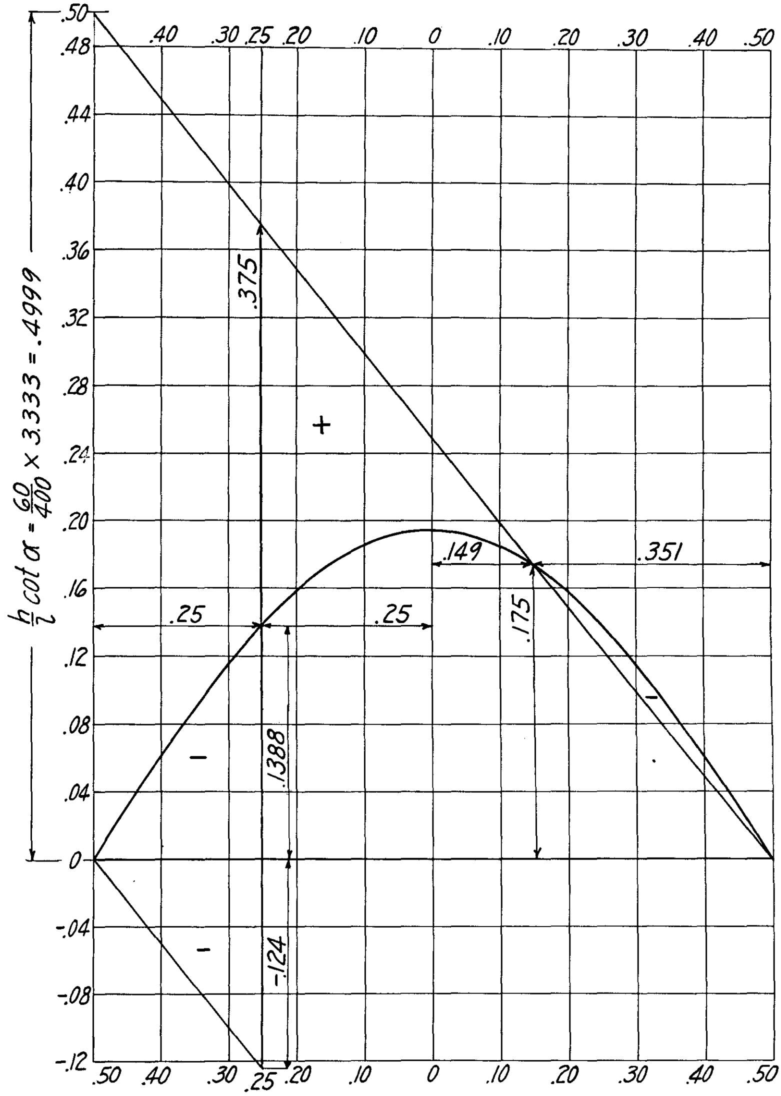
Fig. 8

$$V_1 \cos \alpha - H \sin \alpha = (1 - k) \cos \alpha - \frac{Pl}{h}\phi_1 \sin \alpha =$$

$$\left[\left(1 - k\right)\frac{h}{l}\cot \alpha - \phi_1\right]\frac{l}{h}\sin \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

where

$$H = \frac{Pl}{h}\phi_1, \quad P = 1\#, \quad \frac{\cos \alpha}{\sin \alpha} = \cot \alpha$$



INFLUENCE LINES FOR SHEAR AT QUARTER POINT

PLATE II

The term (1—k) is the same as for the ordinate value shown for a simple beam (see Fig. 7), and α is the slope of arch axis at the point for which the shear is to be found.

The term (1—k) $\frac{h}{l}$ cot α becomes equal to 1# $\left(\frac{h}{l} \cot \alpha\right)$, when the load is at the left reaction, where k = 0. This is the fictitious ordinate at the end which serves to locate the upper branch of the influence line.

The ordinate at the left reaction $\frac{h}{l}$ cot α may be laid off at the reaction point which will give the influence line for shear to the same scale as the ϕ_1 diagram. (See Plate II and Fig. 9.)

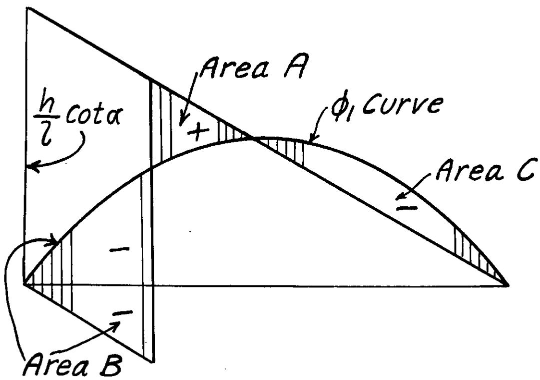


Fig. 9

Referring to Plate III at quarter point,

tan α = .3

cot α = 1 ÷ tan α = 3.333

sin α = .2874

From Plate II and Figure 9

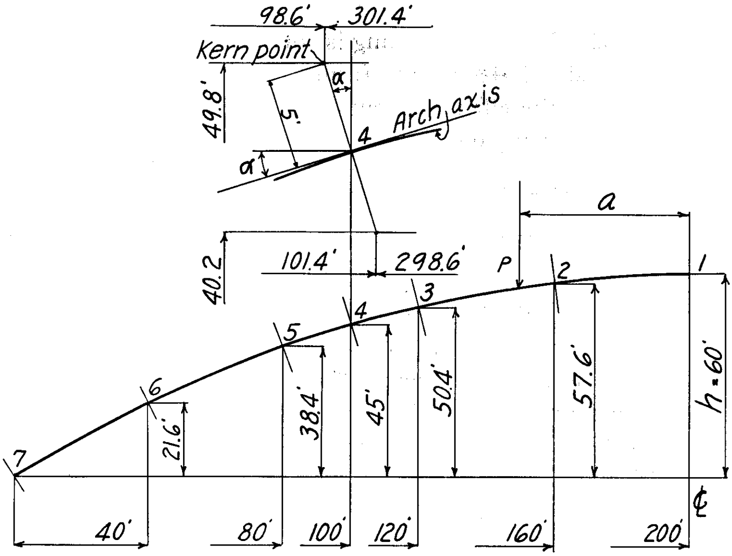
Area B = ½ (.25 × .124) + (.0625 — .04407) = .03393

Area A = .03759 Area C = .00372

Areas inside of curve = B + C = .03765

For LL: Applying equation (18)

$$(.03759 \times 400) \times \frac{400}{60} \times .2374 \times 1200 = 34{,}570\# = \text{LL shear}$$



$$y = \frac{4hx^2}{l^2} = \frac{4 \times 60x^2}{160000} = .0015\ x^2$$

$$\tan\alpha = \frac{dy}{dx} = \frac{8hx}{l^2} = \frac{8 \times 60x}{160000} = .003x$$

VALUES OF Y AND α

point	Y	tan α	cot α	sin α
1	0	0	∞	0
2	2.4'	.12		
3	9.6'	.24		
4	15.0'	.30	3.3333	.2874
5	21.6'	.36		
6	38.4'	.48		
7	60.0'	.60		

EXAMPLE OF TWO-HINGED ARCH

PLATE III

For DL:

Area (B + C − A) = .00006. Applying equation (18), DL shear = 104#

The theoretical DL shear if loading is considered uniform, should be equal to 0. The value 104# results from inaccuracies in finding areas.

The shear due to the effects of temperature = $H_t \sin \alpha$, and should be accounted for. From page 13:

$H_t = 14{,}724$. $H_t \sin \alpha = 14{,}724 \times .2874 = 4{,}230\#$

THE HINGELESS ARCH

As in the case of the two-hinged arch, the origin of co-ordinates is taken at the crown for the hingeless arch. The same basic assumptions are made, particularly, the axis is parabolic and I varies with the secant of α. The arch is assumed symmetrical about a vertical line through the vertex. The fundamental formulae have been derived and are given in Johnson, Bryan, and Turneaure, Part II, Page 188, equations (16), (17), and (18).

In the hingeless arch we have three redundant forces, and these will be taken as the moment, shear, and thrust at the crown. The tables give the ordinates of influence lines for these redundant forces, and also a summation of areas under these influence lines proceeding from the crown. The influence line for moment about the kern point at any section will be a composite of the influence line for the base structure (two cantilever arches formed by cutting arch at crown) and the influence lines due to H, V, and M, the redundant forces.

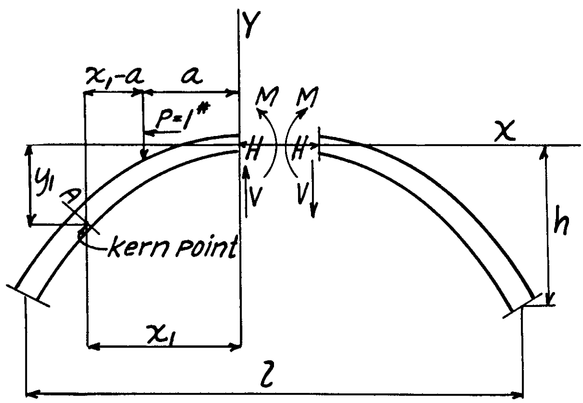


Fig. 10

Figure 10 shows diagrammatically a hingeless arch, with a load P = 1 lb. at a distance "a" from the vertex.

$$M = \phi_2 l, \; V = \phi_3, \; H = \frac{l}{h} \phi_4 \qquad (19)$$

where

ϕ_2, ϕ_3, ϕ_4 are functions describing the respective influence lines and are given in Table I.

The moment about the kern point is

$$M_{kA} = M + Vx_1 + Hy_1 - P\,(x_1 - a) \qquad (20)$$

as long as the load P lies between the vertex and the kern point kA. When the load P is to the left of this kern point, the last term in equation 20 is to be omitted. When the load P is on the right half of the arch, the last term is also omitted, and we note that the term containing V changes sign, so that

$$M_{kA} = M - Vx_1 + Hy_1 \qquad (21)$$

From these results, we can form the composite influence line for M_{kA}. The ordinate at any point will be found by adding the tabular value for M multiplied by l, the tabular value for V multiplied by x_1, the tabular value for H multiplied by $y_1 \frac{l}{h}$ and subtracting the value $(x_1 - a)$, for load between vertex and kA. The influence line can be constructed for an arch of unit length, if desired, the results being multiplied by l finally. In finding the area under this composite influence line, the work can be greatly expedited by noting that it is only necessary to sum up the areas given in Table I ($\Sigma\phi$) multiplying each by the same factor as was used above.

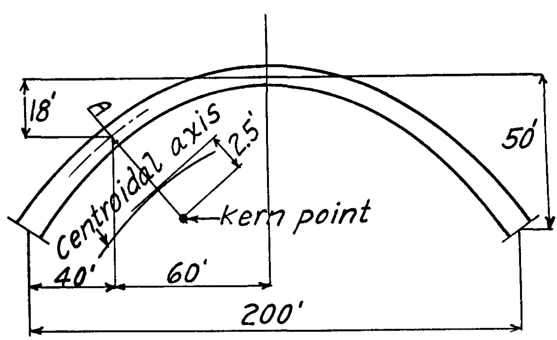


Fig. 11

ILLUSTRATIVE EXAMPLE FOR HINGELESS ARCH

To illustrate the preceding discussion, the arch shown in Figure 11 will be analyzed to determine the moment about kern point for fiber stress in extrados at A.

$$y = \frac{4hx^2}{l^2}$$

$$\frac{dy}{dx} = \frac{8h}{l^2} x = \frac{8 \times 50}{40{,}000} \times 60 = .6 = \tan \alpha \qquad \sin \alpha = .515 \qquad \cos \alpha = .8572$$

$$x_1 = 60 - 2.5' \sin \alpha = 58.71' \qquad \frac{x_1}{l} = .29355$$

$$y_1 = 18 + 2.5' \cos \alpha = 20.14' \qquad \frac{y_1}{h} = .4028$$

TABLE II

Points x	Q	ϕ_2	$\frac{x_1}{l}\phi_3$	$\frac{y_1}{h}\phi_4$	$\frac{x_1 - a}{l}$	$\phi_2 + \frac{x_1}{l}\phi_3 + \frac{y_1}{h}\phi_4 - \frac{x_1 - a}{l}$
+100	.5	0	0	0		0
+ 80	.4	—.00513	+.0082194	+.0122371		+.0153265
+ 60	.3	—.01200	+.0305292	+.0386688		+.0571980
+ 40	.2	—.01013	+.0634068	+.0666151	—.09355	+.0263419
+ 20	.1	+.00800	+.1033296	+.0870048	—.19355	+.0047844
0	0	+.04688	+.1467750	+.0944042	—.29355	—.0054908
— 20	.1	+.00800	—.1033296	+.0870048		—.0083248
— 40	.2	—.01013	—.0634068	+.0666151		—.0069217
— 60	.3	—.01200	—.0305292	+.0386688		—.0038604
— 80	.4	—.00513	—.0082194	+.0122371		—.0011123
—100	.5	0	0	0		0

$$M_{KA} = P\left[l\phi_2 + x_1\phi_3 + \frac{l\,y_1}{h}\phi_4 - (x_1 - a)\right]$$ multiplied by actual length of span.

$+$ = Compression on extrados (counter-clockwise moments)

$-$ = Tension on extrados (clockwise moments)

Plate V shows the composite influence line.

Taking from Table I (see Plate IV), the areas under the influence diagram

Uniform Partial Live Load: Left Side of Span,

$l\,\Sigma\,\phi_2 = 200 \times .00203154 \qquad = - \ .406308$

$x_1\,\Sigma\,\phi_3 = 58.71 \times .0656374 \qquad = + \ 3.853572$

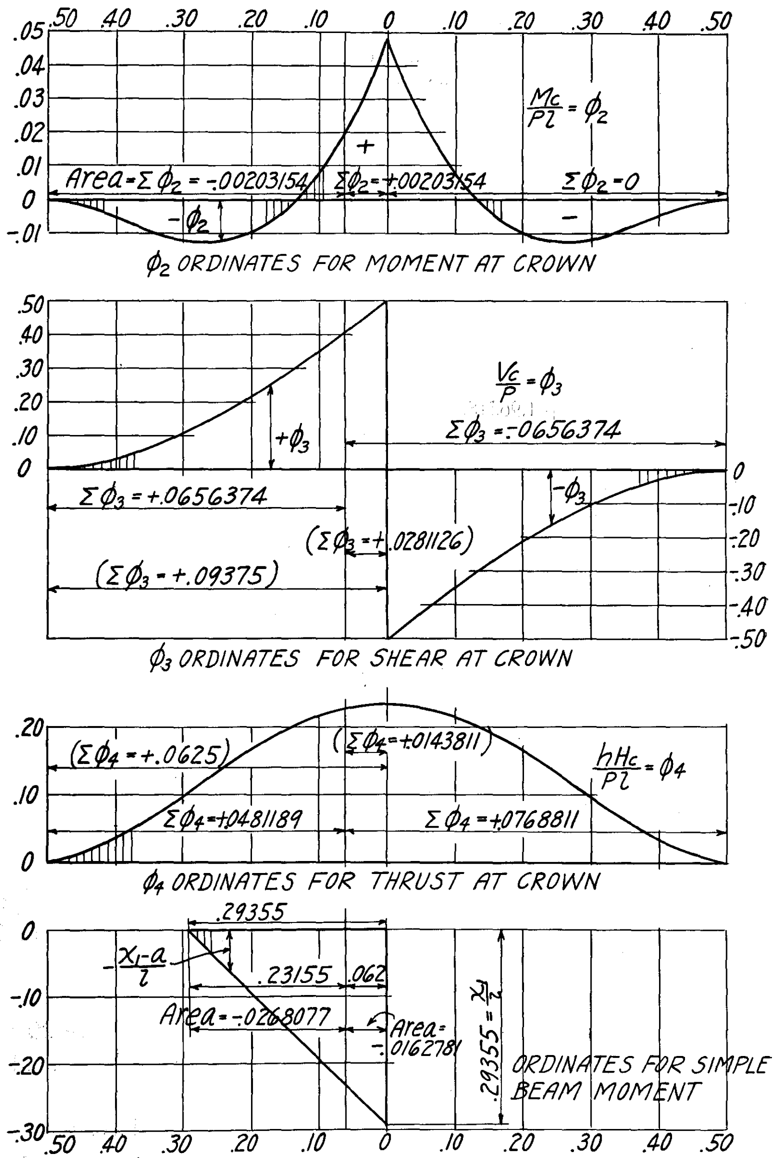
$\frac{l\,y_1}{h}\,\Sigma\,\phi_4 = 80.56 \times .0481189 \qquad = + \ 3.876459$

$\left[\text{Area under } \frac{(x_1 - a)}{l}\right] l = 200 \times .0268077 \qquad = - \ 5.361540$

$+ \ 7.730031$
$- \ 5.767848$

$+ \ 1.962183$

$+ \ 1{,}962183 \times 200 \times 1200 = + \ 470{,}924'\#$ Moment for compression on extrados.



INFLUENCE LINES FOR HINGELESS ARCH

PLATE IV

Uniform Partial Live Load: Right Side of Span,

$l\,\Sigma\,\phi_2 = 200 \times .00203154$ (See above) $= + .406308$

$x_1\Sigma\,\phi_3 = 58.71 \times .0656374$ " " $= - 3.853572$

$\frac{l\,y_1}{h}\,\Sigma\,\phi_4 = 80.56 \times .0768811$ $= + 6.193541$

$\left[\frac{(x_1 - a)}{l}\right] l = 200 \times .0162781$ $= - 3.255620$

$$\begin{array}{r} -\ 7.109192 \\ +\ 6.599849 \\ \hline -\ .509343 \end{array}$$

$.509343 \times 200 \times 1200 = 122{,}242'\#$ Moment for tension on extrados.

Uniform Full Dead Load,

$$\begin{array}{r} +\ 1.962183 \\ -\ .509343 \\ \hline +\ 1.452840 \end{array}$$

$1.45284 \times 200 \times 6000 = + 1{,}743{,}408'\#$ Moment for compression on extrados.

LOADS CONCENTRATED AT SPANDREL POSTS

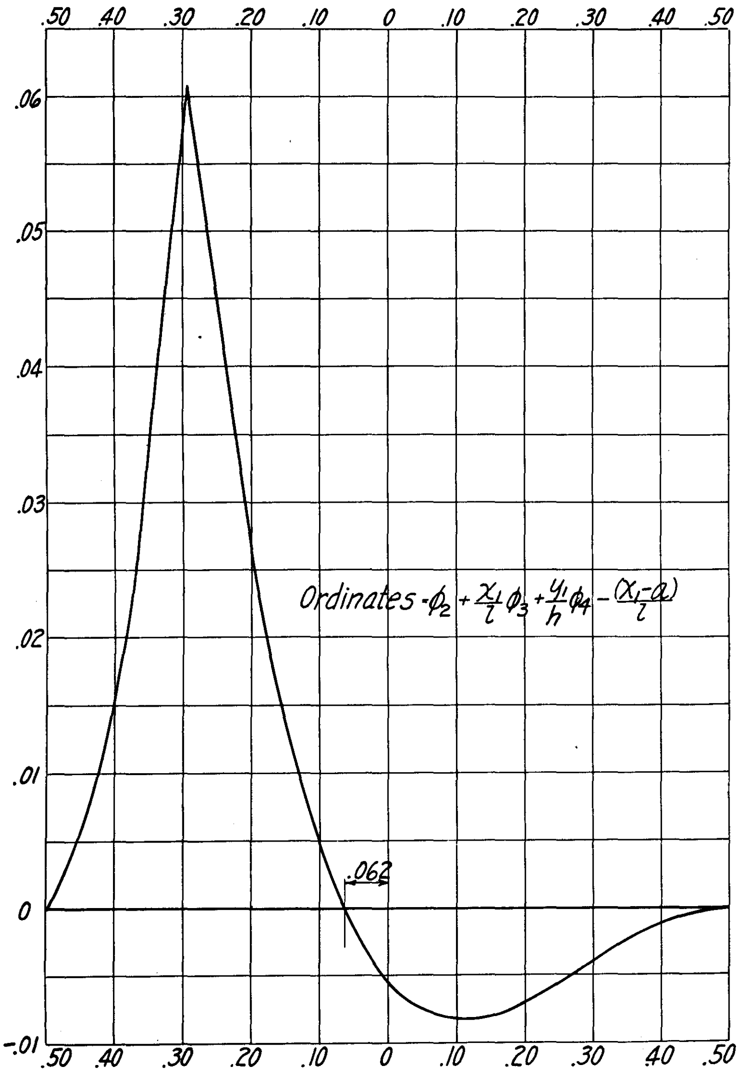
For the case of spandrel posts supported by the arch, a diagram of combined ordinates similar to diagram shown on Plate V may be made to determine the portions of span to be loaded. A study of this diagram will show that maximum concentrated loads should be placed over the maximum ordinates. The distribution of the loads to the posts may then be determined and the true moment ordinates at the position of the posts calculated.

It should be noted that when the loads are applied through spandrel posts to the arch, the influence lines would become straight lines between points of spandrel post load concentrations, and this would produce an appreciable difference in the determination of areas under influence lines as well as in determining the portion of the span to be loaded with live load.

For example, in the preceding problem, if spandrel posts are 20 feet apart (at tenth points of span), we would get a spandrel post load of $1200 \times 20 = 24{,}000$ pounds on each post.

At the post 20 feet to the left of the crown, we would have a load of $\frac{.15 - .053437}{.1} \times 24{,}000 = 23{,}175$ pounds, because the loading would extend only over the interval .15 l to .053437 l. The last figure is obtained by linear interpolation between the ordinates (Table II) $+.0047844$ ($Q = .1$) and $-.0054908$ ($Q = 0$) giving, as the point of zero ordinate

$$\frac{.0054908}{.0047844 + .0054908} \times .1\,l = .053437\,l$$



COMPOSITE INFLUENCE LINE FOR MOMENT

PLATE V

Taking each load multiplied by its ordinate (which is the same as the sum of the ordinates multiplied by the load on one post, but modifying the last ordinate in the ratio $\frac{23175}{24000}$), we obtain from the tabulation on page 20:

$$
\begin{array}{lr}
 & .0153265 \\
 & .0571980 \\
 & .0263419 \\
.0047844 \times .96563 = .0046200 & .0046200 \\
\hline
 & .1034864
\end{array}
$$

Multiplying by l (= 200) and by P (= 24,000) we get

$M_{kA} = 496{,}735$ ft.-lbs.

This value is appreciably greater than the one based on the area of the curved influence diagram, and is the proper value to be used for loads concentrated at spandrel posts.

DERIVATION OF FORMULAE FOR THE HINGELESS ARCH

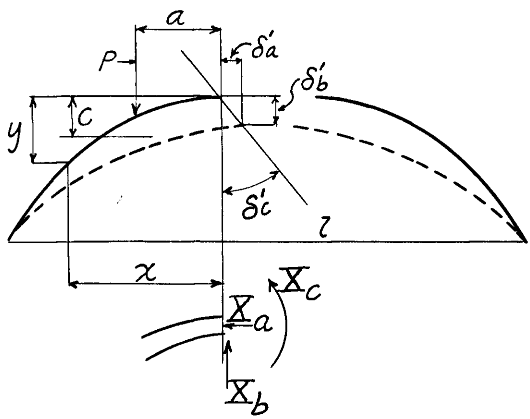


Fig. 12

For the symmetrical hingeless arch shown in Figure 12, the origin of co-ordinates may be conveniently taken at the crown. If we imagine the arch cut at this point, it becomes two cantilevers which are considered as the base structure. If a load is applied on the left side of the arch, there will be components of deflections δ'_a, δ'_b, and δ'_c at the crown. The magnitude of the reactive forces X_a, X_b, X_c must be sufficient to restore the arch to its original position, and may be determined by equating the deflections of the base structure to those of the reactive forces.

We may assume that a positive moment is one which produces tension in the intrados, and use the fundamental formulae:

$$\left.\begin{aligned} +\delta'_a + X_a\,\delta_{aa} + X_b\,\delta_{ab} + X_c\,\delta_{ac} &= 0 \\ +\delta'_b + X_a\,\delta_{ba} + X_b\,\delta_{bb} + X_c\,\delta_{bc} &= 0 \\ +\delta'_c + X_a\,\delta_{ca} + X_b\,\delta_{cb} + X_c\,\delta_{cc} &= 0 \end{aligned}\right\}^{*}$$

in which,

δ'_a = Horizontal deflection of base structure
δ'_b = Vertical deflection of base structure
δ'_c = Angular deflection of base structure
δ'_{ab} = Deflection *a* due to a unit load at b
δ'_{ac} = Deflection *a* due to a unit couple c

Now considering both sides of the arch, the relative deflections $\delta_{ab} = 0$ and $\delta_{bc} = 0$. Also δ_{ca} will be equal to zero if the center of co-ordinates is taken at a point called the elastic center of the arch. The distance from the crown to the elastic center, which is shown as c in Figure 12 must be such that the angular deflection at crown caused by a one-pound load applied horizontally at the elastic center will be equal and opposite to the deflection δ_{ca}. Using the general formula for deflection,

$$\delta_{ca} = \int \frac{M\,m_c\,ds}{E\,I} = \int \frac{(y-c)\,ds}{E\,I} = 0$$

where

$M = 1\#\ (y - c)$

m_c = unit couple on cantilever = 1

Using $ds = dx \sec \alpha$ and $I = I_c \sec \alpha$

$$\frac{1}{E\,I_c}\int (y-c)\,dx = 0$$

$$\int_0^{l/2} y\,dx = \int_0^{l/2} c\,dx = \frac{c\,l}{2}$$

and

$$c = \int_0^{l/2} \frac{y\,dx}{\frac{l}{2}} = \frac{\displaystyle\int_0^{l/2} \frac{4h\,x^2\,dx}{l^2}}{\frac{l}{2}} = \frac{1}{3}\,h$$

From Maxwell's theorem $\delta_{ab} = \delta_{ba}$, $\delta_{bc} = \delta_{cb}$, and $\delta_{ca} = \delta_{ac}$ and these may be omitted from the general equation, leaving the following equations with center of co-ordinates at the elastic center.

* See Parcel and Maney. *Statically Indeterminate Stresses.*

$$+\delta'_a + X_a\,\delta_{aa} = 0 \qquad X_a = -\frac{\delta'_a}{\delta_{aa}}$$

$$+\delta'_b + X_b\,\delta_{bb} = 0 \qquad X_b = -\frac{\delta'_b}{\delta_{bb}}$$

$$+\delta'_c + X_c\,\delta_{cc} = 0 \qquad X_c = -\frac{\delta'_c}{\delta_{cc}}$$

EVALUATION OF X_a, X_b, AND X_c

Evaluating X_a, the horizontal force

$$X_a = -\frac{\delta'_a}{\delta_{aa}} = -\frac{\displaystyle\int_a^{l/2} \frac{M'\,m_a\,dx}{E\,I_c}}{\displaystyle 2\int_0^{l/2} \frac{m_a^2\,dx}{E\,I_c}} = -\frac{\displaystyle\int_a^{l/2} -P\,(x-a)\,(y-c)dx}{\displaystyle 2\int_0^{l/2} (y-c)^2dx}$$

where

$$c = \frac{1}{3}h$$

$$y = \frac{4hx^2}{l^2}$$

M' = Moment in base structure

m_a = Moment due to 1# load applied horizontally at elastic center

Integrating and placing $a/l = Q$

$$X_a = \frac{P\,l}{h}\,\frac{15}{4}\left(\frac{1}{16} - \frac{1}{2}\,Q^2 + Q^4\right)$$

Evaluating the vertical force X_b

$$X_b = -\frac{\delta'_b}{\delta_{bb}} = -\frac{\displaystyle\int_a^{l/2} \frac{M'\,m_b\,dx}{E\,I_c}}{\displaystyle 2\int_0^{l/2} \frac{m_b^2\,dx}{E\,I_c}} = -\frac{\displaystyle\int_a^{l/2} -P\,(x-a)\,x\,dx}{\displaystyle 2\int_0^{l/2} x^2\,dx}$$

$$X_b = P\left(\frac{1}{2} - \frac{3}{2}\,Q + 2Q^3\right)$$

Evaluating the moment X_c

$$X_c = -\frac{\delta'_c}{\delta_{cc}} = -\frac{\displaystyle\int_a^{l/2} \frac{M\,m_c\,dx}{E\,I_c}}{\displaystyle 2\int_0^{l/2} \frac{m_c^2\,dx}{E\,I_c}} = -\frac{\displaystyle\int_a^{l/2} -P\,(x-a)\,dx}{\displaystyle 2\int_0^{l/2} dx}$$

$X_c = Pl\,(1/8 - 1/2\,Q + 1/2\,Q^2)$ at elastic center

or

$M_c = Pl\,(1/8 - 1/2\,Q + 1/2\,Q^2) - (h/3\,X_a)$ at crown

$= Pl\,(1/8 - 1/2\,Q + 1/2\,Q^2) - 5/4\,Pl\,(1/16 - 1/2\,Q^2 + Q^4)$

For the foregoing equations, the stresses due to shortening of arch rib from thrust have not been included, and these may be calculated by the formula:

$$H_s = -\frac{45\,H\,I_0\,L\cos\alpha_1}{4h^2\,l\,A_0}\,*$$

$=$ Thrust due to rib shortening

$H =$ Thrust due to given loading

$L =$ Length of arch rib

$I_0 =$ Moment of inertia at crown

$A_0 =$ Area at crown

$\alpha_1 =$ Angle of arch at springing line

Temperature stresses should also be determined, and are given by the following formula:

$$H_t = \frac{45\,EI_0\,\omega t}{4h^2}\,*$$

$=$ Thrust at crown due to temperature

$\omega =$ Coefficient of thermal expansion

$t =$ Variation in temperature

CONCLUSION

From the illustrative examples herein given, the application of the tabulated influence diagram and the tabulated areas under this diagram is clearly explained. In the preliminary design of any arch, the designer can determine the arch sections required rapidly and with a fair degree of accuracy, so that a final design should not require much change in the sections. It is true that dead load is not uniform, but tends in most cases, to increase toward the springing line. However, it can easily be shown that the use of a mean value of the dead load as a uniformly distributed load does not lead to any large errors. If greater accuracy is desired, the spandrel post loads can be determined separately, and the ordinates corresponding to these spandrel posts can be used. For a trial design, it is not even necessary to select a definite spacing of spandrel posts, as the assumption that the load is uniformly distributed along the horizontal produces fairly accurate results.

* For further study, see Johnson, Bryan, and Turneaure, Part 2, pp. 189-190.

FORMULAE FOR TABLE I

Column (1) Value of $Q = \frac{a}{l}$

Column (2) Ordinates (ϕ_1) for the two-hinged arch

$$\phi_1 = \frac{5}{8}\left(\frac{5}{16} - \frac{3}{2} Q^2 + Q^4\right)$$

Column (3) Summation of ordinates (ϕ_1) beginning at center of arch for one half of arch

Column (4) Ordinates (ϕ_2) for the hingeless arch

$$\phi_2 = \left(\frac{1}{8} - \frac{1}{2} Q + \frac{1}{2} Q^2\right) - \frac{5}{4}\left(\frac{1}{16} - \frac{1}{2} Q^2 + Q^4\right)$$

Column (5) Summation of ordinates (ϕ_2) beginning at center of arch for one half of arch

Column (6) Ordinates (ϕ_3) for the hingeless arch

$$\phi_3 = \left(\frac{1}{2} - \frac{3}{2} Q + 2Q^3\right)$$

Column (7) Summation of ordinates (ϕ_3) beginning at center of arch for one half of arch

Column (8) Ordinates (ϕ_4) for the hingeless arch

$$\phi_4 = \frac{15}{4}\left(\frac{1}{16} - \frac{1}{2} Q^2 + Q^4\right)$$

Column (9) Summation of ordinates (ϕ_4) beginning at center of arch for one half of arch

TABLE I

TABLE OF ORDINATES FOR INFLUENCE LINES

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
.00	.19531	.0000000	.04688	.0000000	.50000	.0000000	.23437	.0000000
.01	.19522	.0019527	.04199	.0004442	.48500	.0049250	.23419	.0023431
.02	.19494	.0039038	.03733	.0008406	.47002	.0097001	.23362	.0046825
.03	.19447	.0058510	.03289	.0011914	.45505	.0143254	.23269	.0070144
.04	.19382	.0077925	.02867	.0014990	.44013	.0188013	.23138	.0093351
.05	.19297	.0097265	.02468	.0017655	.42525	.0231281	.22971	.0116409
.06	.19195	.0116513	.02091	.0019933	.41043	.0273065	.22767	.0139281
.07	.19073	.0135649	.01736	.0021845	.39569	.0313370	.22528	.0161931
.08	.18934	.0154654	.01402	.0023412	.38102	.0352205	.22253	.0184325
.09	.18776	.0173511	.01090	.0024657	.36646	.0389578	.21944	.0206425
.10	.18600	.0192200	.00800	.0025600	.35200	.0425500	.21600	.0228200
.11	.18406	.0210705	.00530	.0026263	.33766	.0459982	.21224	.0249615
.12	.18194	.0229006	.00282	.0026668	.32346	.0493037	.20815	.0270637
.13	.17964	.0247087	.00053	.0026833	.30939	.0524678	.20376	.0291235
.14	.17718	.0264929	—.00156	.0026781	.29549	.0554921	.19907	.0311378
.15	.17454	.0282517	—.00344	.0026529	.28175	.0583781	.19408	.0331038
.16	.17172	.0299831	—.00514	.0026098	.26819	.0611277	.18883	.0350187
.17	.16874	.0316856	—.00666	.0025506	.25483	.0637426	.18332	.0368796
.18	.16559	.0333574	—.00799	.0024773	.24166	.0662249	.17756	.0386842
.19	.16228	.0349969	—.00914	.0023915	.22872	.0685766	.17158	.0404301
.20	.15881	.0366025	—.01013	.0022950	.21600	.0708000	.16538	.0421150
.21	.15518	.0381726	—.01094	.0021895	.20352	.0728974	.15898	.0437369
.22	.15141	.0397057	—.01160	.0020767	.19130	.0748713	.15241	.0452940
.23	.14747	.0412001	—.01211	.0019579	.17933	.0767242	.14568	.0467846
.24	.14339	.0426545	—.01247	.0018349	.16765	.0784590	.13882	.0482072
.25	.13916	.0440674	—.01270	.0017090	.15625	.0800781	.13184	.0495605
.26	.13479	.0454373	—.01279	.0015815	.14515	.0815849	.12476	.0508436
.27	.13029	.0467628	—.01276	.0014536	.13437	.0829822	.11762	.0520556
.28	.12565	.0480426	—.01261	.0013267	.12390	.0842733	.11043	.0531958
.29	.12089	.0492754	—.01235	.0012019	.11378	.0854614	.10321	.0542639
.30	.11600	.0504600	—.01200	.0010800	.10400	.0865500	.09600	.0552600
.31	.11099	.0515951	—.01156	.0009622	.09458	.0875426	.08882	.0561841
.32	.10587	.0526795	—.01103	.0008492	.08554	.0884429	.08170	.0570366
.33	.10063	.0537120	—.01044	.0007417	.07687	.0892546	.07466	.0578183
.34	.09529	.0546917	—.00978	.0006407	.06861	.0899817	.06774	.0585301
.35	.08985	.0556175	—.00907	.0005464	.06075	.0906281	.06096	.0591735
.36	.08431	.0564883	—.00832	.0004593	.05331	.0911981	.05436	.0597500
.37	.07868	.0573034	—.00754	.0003801	.04631	.0916958	.04797	.0602614
.38	.07297	.0580617	—.00674	.0003087	.03974	.0921257	.04182	.0607101
.39	.06718	.0587624	—.00593	.0002453	.03364	.0924922	.03594	.0610987
.40	.06131	.0594050	—.00513	.0001900	.02800	.0928000	.03038	.0614300
.41	.05538	.0599885	—.00434	.0001427	.02284	.0930538	.02516	.0617074
.42	.04938	.0605124	—.00357	.0001032	.01818	.0932585	.02031	.0619343
.43	.04333	.0609760	—.00285	.0000712	.01401	.0934190	.01589	.0621150
.44	.03724	.0613789	—.00218	.0000461	.01037	.0935405	.01193	.0622537
.45	.03110	.0617206	—.00157	.0000274	.00725	.0936281	.00846	.0623552
.46	.02492	.0620008	—.00104	.0000144	.00467	.0936873	.00553	.0624247
.47	.01872	.0622190	—.00061	.0000063	.00265	.0937234	.00318	.0624677
.48	.01250	.0623751	—.00028	.0000019	.00118	.0937421	.00144	.0624903
.49	.00625	.0624688	—.00007	.0000003	.00030	.0937490	.00037	.0624988
.50	.00000	.0625000	—.00000	.0000000	.00000	.0937500	.00000	.0625000

BIBLIOGRAPHY

JOHNSON, BRYAN, and TURNEAURE. Modern framed structures, Part II. New York: Wiley. 1931.

MCALLISTER, E. H. Thrusts, moments and shears in elastic arches. *University of Oregon Bulletin*, 1925.

MCCULLOUGH, C. B., and THAYER, E. S. Elastic arch bridges. New York: Wiley. 1931.

PARCEL, J. I., and MANEY, G. A. Statically indeterminate stresses. New York: Wiley. 1926.

STEINMAN, D. B. Translation of J. Melan's Theory of arches and suspension bridges. New York: Wiley. 1913.

TURNEAURE, F. E., and MAURER, E. R. Reinforced concrete construction. New York: Wiley. 1932.

VAN DEN BROEK, J. A. Elastic energy theory. New York: Wiley. 1931.

WEINER, B. L. Design of a reinforced concrete skew arch. *Proc. Am. Soc. of C. E.*, p. 1212. 1932.

WHITNEY, C. S. Design of symmetrical concrete arches. *Trans. Am. Soc. of C. E.*, p. 360. 1925.

End