MAPS OF AN INTERVAL

By

W.A. Coppel

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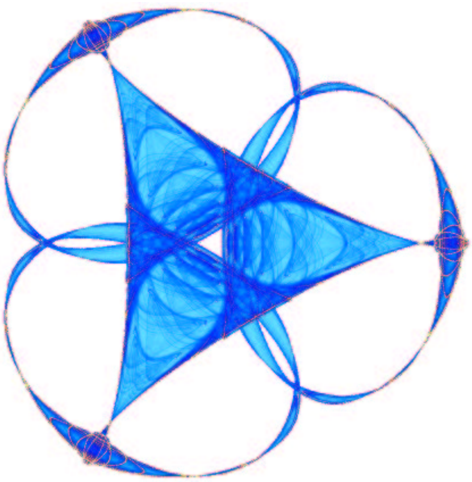

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PREFACE

The three expository lectures in this report were the James McKnight Memorial Lectures for 1983, given at the University of Miami, April 4-6. I am grateful to Professor Shair Ahmad for the invitation to give these lectures and for agreeing to their appearance in this form.

Some of the material here was prepared during a visit to the Institute for Mathematics and its Applications, University of Minnesota, and was presented in lectures there. I am grateful to Professor Hans Weinberger and Professor George Sell for making this visit possible.

I hope that by making the lectures available in this way they may reach a wider audience and initiate some young mathematicians into a rewarding area of research.

W.A. Coppel

Lecture 1

MOTIVATION AND ELEMENTARY PROPERTIES

Let I be a compact interval of the real line and $f: I \to I$ a continuous map of this interval into itself. Having performed the map f once, we can perform it again, and again, and again. That is, we consider the iterates f^n defined inductively by

$$f^1 = f, \; f^{n+1} = f \circ f^n \qquad (n \geqslant 1) \; .$$

We also take f^0 to be the identity map, defined by $f^0(x) = x$ for every $x \in I$. Evidently f^n is also a continuous map of I into itself. Broadly speaking, what we are interested in is the asymptotic behavior of the sequence $f^n(x)$ $(n \geqslant 0)$, i.e., the <u>orbit</u> of x, for arbitrary $x \in I$.

The problem seems so simple that one would expect the last word on the subject to have been said by Cauchy 150 years ago. Yet the astonishing fact is that until quite recently very little was known. However, there is now a great deal of activity in this area. In these lectures I will try to explain the reasons for this, and to give you some idea of what has been done and what remains to be done.

First, let me say something about the motives for considering this problem. One motive comes from ordinary differential equations. Let

$$x' = \phi(x) \tag{1}$$

be an autonomous system of differential equations, where $x \in \mathbb{R}^n$ and the prime denotes differentiation with respect to the independent variable t. We wish to know the asymptotic behavior of the solutions $x(t)$ for large t. If $n = 2$ the Poincaré-Bendixson theorem says that, under quite general conditions, any bounded solution is either asymptotically constant or asymptotically periodic. It has been known for some time that there is no such simple answer if $n = 3$. For example, take the van der Pol equation with harmonic forcing term:

$$y'' + \mu(y^2 - 1)\, y' + y = E \sin \omega t \, .$$

This can be written as a three-dimensional autonomous system

$$x_1' = \omega$$

$$x_2' = x_3$$

$$x_3' = E \sin x_1 - \mu(x_2^2 - 1)x_3 - x_2 .$$

The work of Cartwright and Littlewood, Levinson, and most recently Levi (1981), shows that solutions of systems of this type can behave very erratically, even though the systems themselves are analytic.

An example closer to our present interests is the system

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2 \\ x_2' &= r x_1 - x_2 - x_1 x_3 \\ x_3' &= -b x_3 + x_1 x_2 , \end{aligned} \tag{2}$$

where σ, r, b are positive constants, which was studied by the meteorologist Lorenz (1963). Suppose that a layer of fluid of uniform depth is heated so that the temperature difference between its upper and lower surfaces remains constant. The flow of the fluid under convection is described by a system of partial differential equations. By expanding the solutions in Fourier series these can be replaced by an infinite system of coupled ordinary differential equations. By truncating the Fourier series after three terms, Lorenz arrived at the system (2). He also derived some basic properties of this system. He showed that there is a bounded region in $\mathbb{R}^3$ such that every solution eventually enters this region and then never leaves it. Also, if one takes an arbitrary region in $\mathbb{R}^3$ then under the flow determined by the system (2) the volume V of this region tends to zero exponentially. In fact, if we regard (2) as having the form (1), then $\operatorname{div}\phi = -(\sigma + 1 + b)$ and hence, by a formula due to Liouville, $V' = -(\sigma + 1 + b)V$. Thus the flow squashes everything flat and we would expect that in some sense the system is asymptotically 2-dimensional.

Lorenz also integrated his differential equations numerically for the parameter values $\sigma = 10$, $b = 8/3$, $r = 28$. He found that the solutions in this case oscillated irregularly and observed that the n-th maximum m_n of the coordinate x_3 could be described approximately by an iteration sequence $m_{n+1} = f(m_n)$, where the function f has the form shown in Figure 1. He claimed

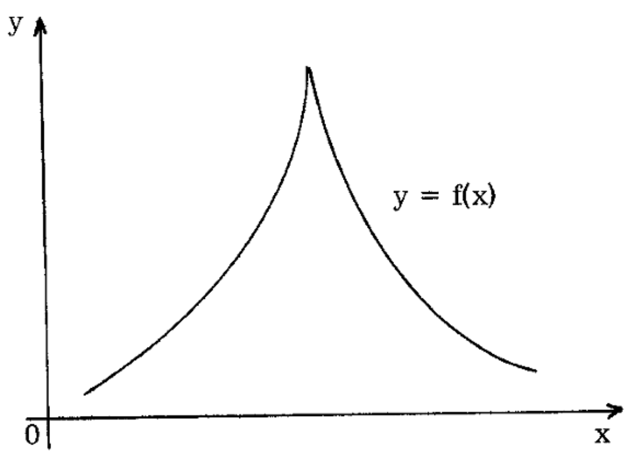

Figure 1

that the irregular behavior of the iteration sequence reflected the turbulent flow of the fluid.

Lorenz's work indicated to physicists that turbulence could arise in purely deterministic systems and need not have its origins in random fluctuations. It also provided mathematicians with a simple system of quadratic differential equations whose solutions behave in a surprisingly complicated way. However, in spite of considerable effort, little has been rigorously proved since Lorenz's original paper. The difficulty is that small changes in the initial conditions can radically alter the long-term behavior of the solutions.

One possible way of simplifying the problem is to consider diffeomorphisms of the plane rather than flows in 3-space. The motivation for this lies in Poincaré's method of "surfaces of section." Take the system (1), where $x \in \mathbb{R}^3$. Let S be a small piece of surface in $\mathbb{R}^3$ and suppose a solution $x(t)$, starting from a point $\xi \in S$, first meets the surface again at a point $\xi' \in S$ (see Figure 2). Then, under

POINCARÉ MAP

$$x' = \phi(x) \qquad x \in \mathbb{R}^3$$

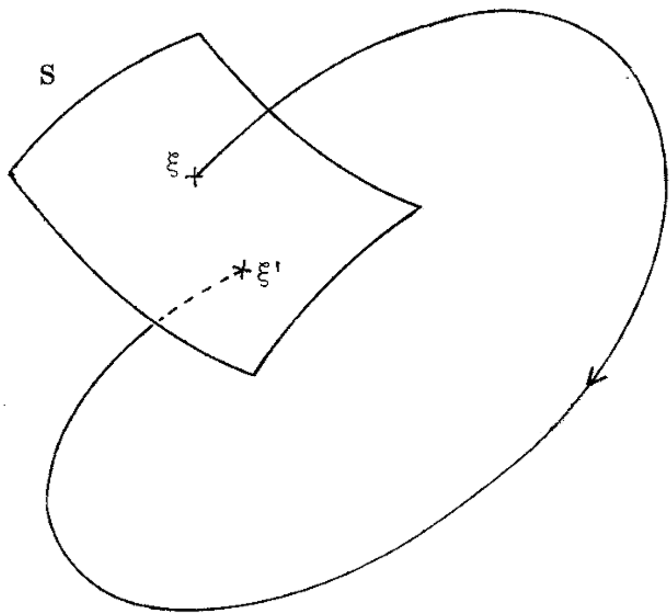

Figure 2

quite general conditions, the map $\xi \longrightarrow \xi'$ is a local diffeomorphism. A fixed point of the diffeomorphism corresponds to a periodic solution of the system of differential equations, and iteration of the diffeomorphism provides information about the asymptotic behavior of the solutions. The advantage of this approach lies in the reduction of dimension.

Actually the Poincaré map of the Lorenz system is not much easier to study than the Lorenz system itself, but it suggests the study of diffeomorphisms which behave in a similar way. Also, a computer can more easily and more accurately track a 2-dimensional diffeomorphism than a 3-dimensional flow. For these reasons Hénon (1976) was led to study the map

$$T(x,y) = (1 + y - ax^2, \ bx) \ .$$

This is a diffeomorphism of $\mathbb{R}^2$ onto itself, since

$$T^{-1}(x,y) = (b^{-1}y, \ -1 + x + ab^{-2}y^2) \ .$$

The map T has Jacobian $-b$, and Hénon showed that any quadratic map of $\mathbb{R}^2$ with constant Jacobian could be reduced to the above form by simple changes of coordinates. The Hénon map differs from the Lorenz system in that for some points (x,y) we have $T^n(x,y) \to (-\infty, -\infty)$ as $n \to \infty$. However, for $a > 0$ and $0 < b < 1$, Feit (1978) has characterized all such "divergent" points and has shown that there is a compact set K with the property that if (x,y) is not divergent then $T^n(x,y) \in K$ for all sufficiently large $n > 0$. Since T has Jacobian $-b$, the hypothesis $0 < b < 1$ means that T shrinks areas by a constant factor b. Thus we would expect that in some sense the behavior is asymptotically 1-dimensional.

Numerical studies suggested that for certain parameter values the Hénon map could possess a strange attractor, i.e., a compact invariant set which attracts neighboring orbits and on which the orbits behave erratically. Marotto (1979) has proved rigorously that there is a transversal homoclinic point for each $a > 1.55$ and all sufficiently small b. This implies that there is a compact invariant set on which the orbits behave erratically, but does not prove that it attracts neighboring orbits. Marotto's method is to treat the Hénon map as a perturbation of the 1-dimensional map $f(x) = 1 - ax^2$.

To summarize, the complicated behavior shown by some 3-dimensional flows can also be seen in 2-dimensional diffeomorphisms and in noninvertible 1-dimensional maps. By studying 1-dimensional maps we may hope to gain a better understanding of this behavior, since they are more amenable to mathematical analysis.

Another motive for studying 1-dimensional maps comes from mathematical biology. Many insects which inhabit temperate zones breed seasonally and have non-overlapping generations. In these circumstances if x_n denotes the total population in the n-th generation, it is plausible to write

$$x_{n+1} = f(x_n) \ .$$

Typically, the graph of f has the form shown in Figure 3, for which the analytic

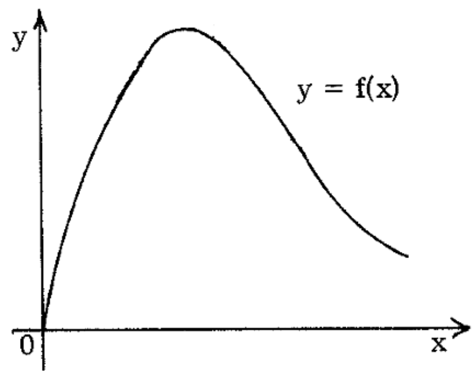

Figure 3

representation $f(x) = x\exp[a(1-x)]$ has been suggested. It is a pleasure for me to recall that models of this type, for voles rather than for insects, were proposed by Moran (1950), who retired from the Australian National University a few months ago after 30 years as head of the Department of Statistics. It might be supposed that these applications are somewhat fanciful, because of the many unpredictable factors which can influence the population and the difficulties of measurement. Recently, however, populations of bacteria have been studied in the laboratory. Here the environment is controlled, measurements are readily made, and the transition from one generation to the next is much more rapid.

One-dimensional maps have also been used to model many other phenomena, from enzyme reactions to oil-drilling. I don't think it is worthwhile providing a list, but there is one other motive for considering such maps which should be mentioned here, although I will have more to say about it in my third lecture. This is the study of critical point phenomena in physics. In fact the renormalization procedures of statistical mechanics have suggested new results and new techniques in the study of maps of an interval.

I have been using the terms "1-dimensional map" and "map of an interval" interchangeably, and perhaps this is the place to add a word of explanation. It is certainly of interest to consider not only maps of an interval into itself, but also maps of a circle into itself. The problems are very similar, but for maps of a circle the notion of topological degree enters and sometimes complicates the statements of results. It is also of interest to consider analytic maps of one complex dimension, i.e., rational functions on the Riemann sphere. This got off to an earlier start with the work of Julia (1918) and Fatou (1919/20). Some very interesting results have recently been announced by Sullivan (1982) and Douady and Hubbard (1982). The techniques here are somewhat different, and I mention this work mainly to give you a broader perspective.

Thus "1-dimensional map" will indeed mean "map of an interval" in these lectures. Moreover it will be convenient to restrict attention to compact intervals, even if this restriction is not always necessary.

Let, then, $f: I \to I$ be once more a continuous map of a compact interval into itself. There is a very simple graphical procedure for following the orbits of f. In the (x, y)-plane draw the curve $y = f(x)$ and the straight line $y = x$. To obtain the orbit with initial point x_0 we go vertically to $y = f(x)$, then horizontally to $y = x$. This gives x_1, and the process is repeated <u>ad infinitum</u> (see Figure 4).

A point $c \in I$ is said to be a <u>fixed point</u> of f if $f(c) = c$. Thus the fixed points are given by the intersections of the curve $y = f(x)$ and the straight line $y = x$. The interval I necessarily contains at least one fixed point. For, if $I = [a, b]$, we have

$$f(a) - a \geqslant 0 \geqslant f(b) - b \ ,$$

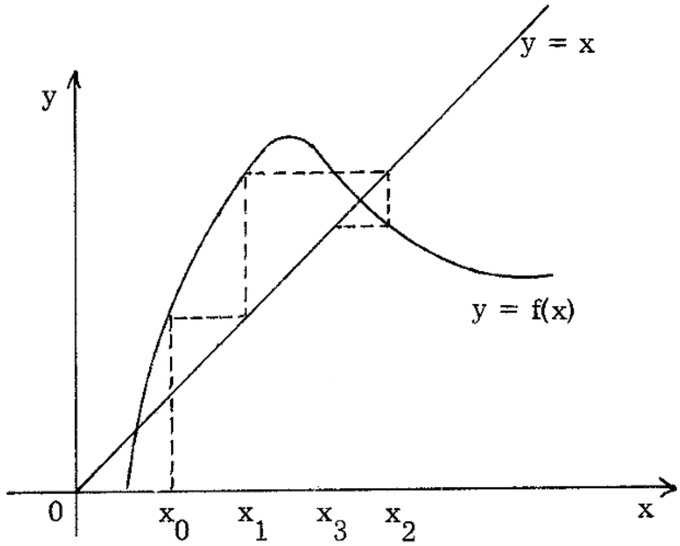

Figure 4

and so the assertion follows from the intermediate value theorem for continuous functions. If f is differentiable the fixed point c is said to be attractive if $|f'(c)| < 1$ and repulsive if $|f'(c)| > 1$. In the first case there is an open interval G containing c such that $f^n(x) \to c$ as $n \to \infty$ uniformly for $x \in G$. In the second case there is an open interval G containing c such that no orbit $\{f^n(x)\}$ is entirely contained in G except the trivial one corresponding to $x = c$.

A point $c \in I$ is said to be a periodic point of f with period m if

$$f^m(c) = c\,, \qquad f^k(c) \neq c \quad \text{for} \quad 1 \leqslant k < m\,.$$

The orbit of c then consists of the points $c, f(c), \ldots, f^{m-1}(c)$ repeated periodically. If f is differentiable the periodic orbit is said to be attractive or repulsive according as $g = f^m$ satisfies $|g'(c)| < 1$ or $|g'(c)| > 1$. Since, by the chain rule for differentiation,

$$g'(c) = f'(c)f'[f(c)]\dots f'[f^{m-1}(c)] \quad ,$$

the definition depends only on the periodic orbit and not on the initial point c. If the periodic orbit is attractive there is an open interval G containing c such that $f^n(x) - f^n(c) \to 0$ as $n \to \infty$ uniformly for $x \in G$. Of course, a fixed point is a periodic point of period 1.

In this introductory lecture I have simply tried to set the scene. In the next two lectures there will be a bit more action.

Lecture 2

PERIODIC ORBITS, TOPOLOGICAL DYNAMICS, CHAOS

Let I be a compact interval and $f: I \to I$ a continuous map of this interval into itself. I recall that the sequence of iterates f^n is defined by

$$f^0 = \mathrm{id}, \quad f^{n+1} = f \circ f^n \qquad (n \geq 0) .$$

A point $x \in I$ is a fixed point of f if $f(x) = x$ and a periodic point of period m if

$$f^m(x) = x , \qquad f^k(x) \neq x \quad \text{for} \quad 1 \leq k < m .$$

The orbit $\{f^n(x)\}$ of x under f is then periodic.

In my second year as an undergraduate at Melbourne University I proved the following result, although it was not published until some time later (Coppel, 1955):

The sequence $\{f^n(x)\}$ converges for every $x \in I$ if and only if the map f has no periodic point of period 2. This implies, as a corollary, that if f has a periodic point of period $n > 1$ then it also has a periodic point of period 2. Šarkovskii (1964a) obtained a remarkable extension of this corollary. Let the positive integers be totally ordered in the following way:

$$3 \vdash 5 \vdash 7 \vdash 9 \vdash \ldots \vdash 2.3 \vdash 2.5 \vdash \ldots \vdash 2^2.3 \vdash 2^2.5 \vdash \ldots \vdash 2^3 \vdash 2^2 \vdash 2 \vdash 1 .$$

If a continuous map $f: I \to I$ has a periodic point of period n and if $n \vdash m$ then f also has a periodic point of period m. Moreover this result is the best possible. That is, for each positive integer n there exists a map f which has a periodic point of period n but no periodic point of period m for any $m \vdash n$. Also, there exists a map f which has periodic points of period 2^d, for every $d \geq 0$, but of no other periods.

Šarkovskii's proof of his theorem was quite complicated, but recently a much simpler approach has been found, which uses directed graphs. Let $x_1 < \ldots < x_n$ be the points of a periodic orbit of period n and set $I_j = [x_j, x_{j+1}]$ $(1 \leq j < n)$. The vertices of the directed graph are the subintervals $I_1, \ldots, I_{n-1}$

and there is an arrow $I_j \to I_k$ if I_k is contained in the interval $\{f(x_j), f(x_{j+1})\}$ with endpoints $f(x_j)$ and $f(x_{j+1})$. The basic observation, due to Straffin (1978), is that f also has a periodic orbit of period m if the digraph contains a primitive cycle $J_1 \to J_2 \to \ldots \to J_m \to J_1$ of length m. Here a cycle is said to be primitive if it does not consist entirely of a cycle of smaller length described several times.

For example, suppose x is a point of period 3 with $f(x) < x < f^2(x)$. The directed graph has two vertices, namely the intervals $I_1 = [f(x), x]$ and $I_2 = [x, f^2(x)]$, connected in the following way:

$$\circlearrowleft I_1 \rightleftarrows I_2$$

Corresponding to the loop $I_1 \to I_1$ there is a fixed point of f and corresponding to the primitive cycle $I_1 \to I_2 \to I_1$ there is a point of period 2. Moreover, for any positive integer $m > 2$ there is a point of period m, corresponding to the primitive cycle $I_1 \to I_2 \to I_1 \to I_1 \to \ldots \to I_1$ of length m. Thus there are orbits of period n for every $n \geqslant 1$, in agreement with Šarkovskii's theorem. This special case was later rediscovered by Li and Yorke (1975).

L. Block (1981) has proved the stability of the Šarkovskii ordering: if the continuous map $f: I \to I$ has a periodic orbit of period $n > 1$ then there exists a $\delta > 0$ such that any continuous map $g: I \to I$ with $|g(x) - f(x)| \leqslant \delta$ for every $x \in I$ has a periodic orbit of period m for all m satisfying $n \vdash m$.

Šarkovskii's theorem is by now quite widely known. It is not so well known that at about the same time he discovered some interesting results connected with the topological dynamics of maps of an interval.

Topological dynamics began with the work of Poincaré and G. D. Birkhoff, and there are today sizeable books on the subject. Let me first recall some of its concepts and the elementary relationships between them.

Let X be a compact metric space and $f: X \to X$ a continuous map of X into itself. Note that f is not required to be a homeomorphism of X onto itself, as is often assumed. If $x \in X$ the orbit of x is the set $\gamma(x) = \bigcup_{n \geqslant 0} f^n(x)$ and the limit set of x is the set

$$\omega(x) = \bigcap_{m \geqslant 0} \overline{\bigcup_{n \geqslant m} f^n(x)} .$$

Equivalently, $y \in \omega(x)$ if and only if $f^{n_k}(x) \to y$ for some sequence of integers $n_k \to \infty$. The limit set $\omega(x)$ is closed and strongly invariant, i.e.,

$$f\omega(x) = \omega(x) = \overline{\omega(x)} .$$

It may be shown that the set $\omega(x)$ is finite if and only if x is asymptotically periodic, i.e., there exists a periodic point y such that $d[f^n(x), f^n(y)] \to 0$ as $n \to \infty$.

A point $x \in I$ is said to be recurrent if $x \in \omega(x)$. Some authors refer to this as Poisson stability and then use recurrence in a somewhat stronger sense. If we denote by P the set of all periodic points and by R the set of all recurrent points then

$$f(P) = P \subseteq R = f(R) .$$

The closure $\bar{R}$ is known as the center of f. By continuity we have

$$f(\bar{P}) = \bar{P} \subseteq \bar{R} = f(\bar{R}) .$$

Moreover, the inclusion in the middle may be strict.

A point $x \in I$ is said to be nonwandering if every open set which contains x contains at least two points of some orbit. Equivalently, x is nonwandering if $f^{n_k}(x_k) \to x$ for some sequence of points $x_k \to x$ and some sequence of integers $n_k \to \infty$. If we denote by Ω the set of all nonwandering points then

$$f(\Omega) \subseteq \Omega = \bar{\Omega} ,$$

where the inclusion on the left may be strict. It follows that the kernel $K = \bigcap_{n \geqslant 0} f^n(\Omega)$ satisfies

$$f(K) = K = \bar{K} .$$

Moreover, any limit set is contained in K, i.e.,

$$\bigcup_{x \in X} \omega(x) \subseteq K .$$

A theorem of Birkhoff says that, if f is a homeomorphism of X onto itself, then for any open set G which contains Ω there is a positive integer $m = m(G)$ such that at most m points of any orbit $\{f^n(x)\}$ lie outside G.

Suppose now that $X = I$ is a compact interval. It is an interesting, nontrivial, and only recently discovered fact that in this case there are several additional relationships between these concepts. First of all we have

$$\bar{P} = \bar{R} .$$

This was announced by Šarkovskii (1964b) with a sketch of the proof, and was rediscovered by Coven and Hedlund (1980). Secondly, we have

$$\bar{R} = \Omega(f|\Omega) .$$

This says that $\bar{R}$ is the nonwandering set of f restricted to the compact invariant set Ω. The significance of this is that Birkhoff originally defined the center as the nonwandering set of the nonwandering set of the nonwandering set..., continued by transfinite induction until one gets nothing smaller. It was then a theorem that this was the same as the closure of the set of recurrent points. The above result shows that in the case of a compact interval one need make at most one step past Ω itself. This result is also due to Šarkovskii (1964b) and was reproved by Nitecki (1980) for piecewise monotone maps. Actually both this result and the previous one follow at once from the following simple, but general, lemma:

Let $f: I \to I$ be a continuous map of a compact interval into itself and let J be an open subinterval which contains no periodic point of f. Then, for each $x \in I$, the points of the orbit $\{f^n(x)\}$ which lie in J form a strictly monotonic, finite or infinite, sequence.

Coven and Nitecki (1981) have shown that $x \in \Omega$ if and only if $f^{n_k}(x_k) = x$ for some sequence $x_k \to x$ and some sequence $n_k \to \infty$. A. Bloh (1982a) has announced that, in the case $X = I$,

$$\bigcup_{x \in I} \omega(x) = K \; .$$

Šarkovskii (1967) had already proved that $\bigcup_{x \in I} \omega(x)$ is the set of all points y such that every open interval which contains y contains at least three points of some orbit, and also that

$$\bigcup_{x \in I} \omega(x) = \Omega_- \cup \Omega_+ \; ,$$

where Ω_+, resp. Ω_-, denotes the set of all points y such that every open interval with y as left, resp. right, endpoint contains at least two points of some orbit. This enabled him to obtain a strengthening of Birkhoff's theorem: if $f: I \to I$ is a continuous map of a compact interval into itself, then for any open set G which contains $\bigcup_{x \in I} \omega(x)$ there is a positive integer $m = m(G)$ such that at most m points of any orbit $\{f^n(x)\}$ lie outside G. It is easily seen that $\bigcup_{x \in I} \omega(x)$ is the smallest closed set with this property.

I want to say something now about chaos. This catchy word was introduced into our subject by Li and Yorke (1975), but their use of it was rather ad hoc. Instead, I will define a continuous map $f: I \to I$ of a compact interval into itself to be chaotic if f has a periodic point whose period is not a power of 2. To justify this definition it is necessary to show thatsuch maps share a number of properties not possessed by other maps and that in some sense they behave more wildly. This is the content of the following

THEOREM 1. Let $f: I \to I$ be a continuous map of a compact interval into itself. Then the following statements are equivalent:

(i) f has a periodic point whose period is not a power of 2,

(ii) there exist disjoint closed subintervals J, K and a positive integer m such that

$$J \cup K \subseteq f^m(J) \cap f^m(K) \; ,$$

(iii) f has a special homoclinic point,

Here $x \in I$ is a special homoclinic point if there exists a periodic point $z \neq x$, of period n, say, such that $f^{mn}(x) = z$ for some $m > 0$ and $x = f^{m_k n}(y_k)$ for some sequence $y_k \to z$ and some sequence $m_k \to \infty$. The equivalence of (i), (ii) and (iii) was proved by L. Block (1978). Actually it is possible to replace (iii) by

(iii)' f has a homoclinic point,

Here $x \in I$ is a homoclinic point if there exists a periodic point z, not containing x in its orbit, such that $z \in \omega(x)$ and $x = f^{n_k}(y_k)$ for some sequence $y_k \to z$ and some sequence $n_k \to \infty$.

(iv) there exists a point $x \in I$ such that

$$f^{qm}(x) \leqslant x < f^m(x) \qquad \text{or} \qquad f^{qm}(x) \geqslant x > f^m(x)$$

for some $m \geqslant 1$ and some odd q (Li et al., 1982),

(v) there exists a point $x \in I$ such that the limit set $\omega(x)$ is infinite and contains a periodic orbit (Šarkovskii, 1966),

(vi) $\bar{R} \neq R$ (A. Bloh, 1982a),

(vii) there exists a closed set $X \subseteq I$, strongly invariant under f^m for some $m > 0$, and a continuous map h of X onto Σ such that each point of Σ is the image of at most 2 points of X and

$$h \circ f^m(x) = \sigma \circ h(x) \quad \text{for every } x \in X .$$

Here Σ is the set of all infinite sequences $\alpha = (a_1, a_2, \ldots)$, where $a_k = 0$ or 1. If we define the distance between two elements α, β of Σ by $d(\alpha,\beta) = 0$ if $\alpha = \beta$ and $= 2^{-k}$ if $\alpha \neq \beta$ and k is the least integer such that $a_k \neq b_k$ then Σ becomes a compact metric space. The shift σ is the continuous 2 - 1 map of Σ onto itself defined by $\sigma(a_1, a_2, \ldots) = (a_2, a_3, \ldots)$.

The virtue of (vii), which sharpens a result of L. Block (1978), is that it enables us to read off properties of the map f from properties of the universal map σ.

(viii) f has positive topological entropy (Misiurewicz, 1979).

Topological entropy can be defined in the following way. Let $f: X \to X$ be a continuous map of a compact metric space into itself. For any $\epsilon > 0$ and any positive integer n let $N = N(n, \epsilon)$ denote the least positive integer for which there exist points $x_1, \ldots, x_N$ with the property that for any $x \in X$ there is a point x_j $(1 \leqslant j \leqslant N)$ such that $d\left[f^k(x), f^k(x_j)\right] < \epsilon$ for $0 \leqslant k < n$. Evidently $N(n, \epsilon)$ is nondecreasing for decreasing ϵ. The topological entropy $h(f)$ is defined by

$$h(f) = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log N(n, \epsilon) \quad .$$

For continuous maps $f: X \to X$ of an arbitrary compact metric space into itself the properties (i) - (viii) are no longer equivalent. For example, if X is the closed unit disc in $\mathbb{R}^2$ and f a rotation of X through 120°, then every point except the origin has period 3. Thus (i) holds, but not (v). In fact the map f cannot be regarded as chaotic in any reasonable sense. The most natural definition of chaos in the general case is (viii), but it is rather difficult to work with.

Among the nonchaotic maps of an interval there is a subclass whose behavior is especially regular. This subclass is characterized in the following

THEOREM 2. Let $f: I \to I$ be a continuous map of a compact interval into itself. Then the following statements are equivalent:

(i) every point of I is asymptotically periodic,

(ii) $\bar{P} = P$,

(iii) $R = P$,

(iv) $\Omega = P$.

The equivalence of (ii) with (i), (iii), and (iv) respectively is due to Šarkovskii (1965), A. Bloh (1982a), and Xiong Jin-Cheng (1981) and Nitecki (1982).

Lecture 3

QUADRATIC MAPS. QUALITATIVE AND QUANTITATIVE UNIVERSALITY

In the last lecture we studied the behavior under iteration of an arbitrary continuous map f. To begin today, I want to consider one particular example.

Iteration of a linear map $f(x) = ax + b$ is trivial, since the recurrence $x_{n+1} = ax_n + b$ has the explicit solution

$$x_n = \begin{cases} a^n x_0 + b(1 - a^n)/(1 - a) & \text{if } a \neq 1 , \\ x_0 + nb & \text{if } a = 1 . \end{cases}$$

One might suppose that the next simplest case is that of a quadratic map

$$f(x) = ax^2 + 2bx + c .$$

Without loss of generality we can assume $a = 1$ and $b = 0$, since the linear change of variables $y = ax + b$ transforms the recurrence $x_{n+1} = f(x_n)$ into the recurrence $y_{n+1} = y_n^2 + \gamma$, where $\gamma = ac - b^2 + b$. However, there is another canonical form which is just as popular. The map f has a real fixed point β if and only if $(2b - 1)^2 \geqslant 4ac$. If this condition is satisfied and if we set $z = a(\beta - x)/\lambda$, where $\lambda = 2(b + a\beta)$, the recurrence $x_{n+1} = f(x_n)$ is transformed into the recurrence $z_{n+1} = \lambda z_n(1 - z_n)$.

Thus we are going to study the map f defined by

$$f(x) = \lambda x(1 - x) .$$

The unit interval $I = [0, 1]$ is mapped into itself by f if $0 \leqslant \lambda \leqslant 4$. Moreover the interval I contains at most one attractive periodic orbit for a given value of λ. This follows either from a complex variable theorem of Julia (1918) or from a real variable theorem of Singer (1978).

For $\lambda = 4$ the orbits can be described completely. In fact, if $x_0 = \sin^2 \pi\theta/2$ $(0 \leqslant \theta \leqslant 1)$, then

$$x_1 = 4 \sin^2 \pi\theta/2 \cos^2 \pi\theta/2 = \sin^2 \pi\theta$$

and in general $x_n = \sin^2(2^{n-1}\pi\theta)$. The asymptotic behavior of the orbit through x_0 is determined by the binary expansion of θ.

At the other end of the range, it is easily seen that for $0 < \lambda < 1$ there is a unique fixed point in I, namely $x = 0$. Since $f'(x) = \lambda(1-2x)$, the fixed point at the origin is attractive for $0 < \lambda < 1$ and actually $f^n(x) \to 0$ as $n \to \infty$ for every $x \in I$. For $\lambda > 1$ the fixed point at the origin is repulsive and the interval I contains a second fixed point at $\tilde{x} = 1 - \lambda^{-1}$. For $1 < \lambda < 3$ this fixed point is attractive and actually $f^n(x) \to \tilde{x}$ as $n \to \infty$ for every $x \in I$, except the two end-points. (The convergence is monotonic after the first term if $1 < \lambda < 2$ and ultimately oscillatory if $2 < \lambda < 3$.) At $\lambda = 3$ the fixed point $\tilde{x}$ loses its stability, because $f'(\tilde{x}) = -1$, and there is a bifurcation. For $\lambda > 3$ the fixed point $\tilde{x}$ is repulsive and there is an orbit of period 2, whose points are located at the roots of the quadratic equation

$$\lambda^2 x^2 - \lambda(\lambda+1)x + \lambda + 1 = 0 .$$

For $3 < \lambda < \lambda_2 = 1 + \sqrt{6}$ this orbit of period 2 is attractive. In fact, every orbit is asymptotic to it, except the countably many orbits which terminate in a fixed point after a finite number of steps.

At $\lambda = \lambda_2$ the derivative of f^2 takes the value -1 along the orbit of period 2, and there is a bifurcation. For $\lambda > \lambda_2$ the orbit of period 2 is repulsive and there is an orbit of period 4. In fact the Finnish mathematician Myrberg (1963) showed that there is an infinite sequence of such period-doubling bifurcations. Thus there is an increasing sequence of parameter values $\lambda_1 < \lambda_2 < \ldots$ such that a periodic orbit of period 2^k exists for $\lambda > \lambda_k$ but not for $\lambda < \lambda_k$. This orbit is attractive for $\lambda_k < \lambda < \lambda_{k+1}$, but at $\lambda = \lambda_{k+1}$ the derivative of f^{2^k} takes the value -1 along the orbit and then the orbit of period 2^{k+1} appears. The numerical values are

$$\lambda_1 = 3 , \quad \lambda_2 = 3.449 , \quad \lambda_3 = 3.544 , \quad \lambda_4 = 3.564 , \ldots$$

and $\lambda_k \to \lambda^* = 3.570\ldots$ as $k \to \infty$.

What happens for $\lambda^* < \lambda < 4$? In discussing this I will not distinguish between what has been rigorously proved and what has been numerically observed. First of all, λ^* is also the limit of a decreasing sequence $\tilde{\lambda}_1 > \tilde{\lambda}_2 > \ldots$ such that for $\tilde{\lambda}_{k-1} > \lambda > \tilde{\lambda}_k$ there are 2^k disjoint subintervals, which are permuted cyclically by f, such that f^{2^k} is chaotic on each subinterval. The subintervals coalesce in pairs at $\lambda = \tilde{\lambda}_{k-1}$.

Secondly, in the vivid terminology of May (1976), the region $\lambda^* < \lambda < 4$ contains narrow "windows," i.e., maximal subintervals within which there is an attractive periodic orbit of fixed period m. As in the case m = 2 there is then an infinite sequence of period-doubling bifurcations. Thus there are increasing sequences $\lambda_1(m) < \lambda_2(m) < \ldots$ such that for $\lambda_k(m) < \lambda < \lambda_{k+1}(m)$ there is an attractive periodic orbit of period $2^{k-1}m$, and $\lambda_k(m) \to \lambda^*(m)$ as $k \to \infty$.

For m = 3 there is just one window and we have

$$\lambda_1(3) = 3.8284\ , \qquad \lambda_2(3) = 3.8415, \ldots\ , \qquad \lambda^*(3) = 3.8496\ .$$

The orbit of period 3 is "born" when two conjugate complex fixed points of f^3 coalesce and then separate into a pair of real fixed points, one attractive and one repulsive. At the transition point the derivative of f^3 takes the value +1. This is known as a saddle-node bifurcation. For many values of λ just less than $\lambda_1(3)$ there is a type of chaotic behavior called intermittency. In this case there is a small interval in which $f^3(x) - x$ is very close to zero but does not vanish. Orbits which start in this interval behave for a long time like an orbit of period 3, but then behave erratically until they are returned to the interval and the process begins again.

For $m > 3$ the situation is similar, except that there is more than one window containing an attractive periodic orbit of period m. In fact, if we denote by $\rho(m)$ the number of such windows then $\rho(m) \to \infty$ as $m \to \infty$.

Each window $\lambda_1(m) < \lambda < \lambda_2(m)$ contains a unique parameter value $\bar{\lambda}(m)$ for which the attractive periodic orbit of period m is <u>superstable</u>, i.e., the (unique) critical point $\bar{x} = 1/2$ belongs to the orbit. Then $f^{m'}(\bar{x})$ not only has absolute value less than 1, but actually vanishes. This superstable orbit can be represented by a finite sequence $(c_1, \ldots, c_{m-1})$, where $c_j = L$ (left) or R (right) according as $f^j(\bar{x}) \lessgtr \bar{x}$. We always have $c_1 = R$. For if $f(\bar{x}) < \bar{x}$, then $f^k(\bar{x}) < f^{k-1}(\bar{x})$ for all $k \geqslant 1$, because f is an increasing function on $[0, \bar{x}]$, and hence the orbit through $\bar{x}$ cannot be periodic. A superstable fixed point $(m = 1)$ is represented by the empty sequence $\emptyset$.

Two questions arise. What sequences of L's and R's actually occur for some superstable orbit? We will call such sequences <u>realizable</u>. Secondly, given two realizable sequences, which occurs for the smaller value of the parameter λ? The answers to these questions are as follows.

Let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ be any two finite sequences of L's and R's and let $\ell - 1$ be the length of their largest common initial sequence. We define $A < B$ if either (i) there is an even number of R's among $a_1, \ldots, a_{\ell-1}$ and $a_\ell = L$ or $b_\ell = R$, or (ii) there is an odd number of R's among $a_1, \ldots, a_{\ell-1}$ and $a_\ell = R$ or $b_\ell = L$. This is a total ordering of the set of all finite sequences. A sequence is realizable if and only if it is preceded, in this sense, by each of its terminal sequences. Moreover one realizable sequence occurs for a smaller parameter value than another realizable sequence if and only if the first precedes the second.

The thirteen realizable sequences corresponding to superstable orbits of period less than 7 are

$$\emptyset < R < RLR < RLR^3 < RLR^2 < RL < RL^2RL < RL^2R$$
$$< RL^2R^2 < RL^2 < RL^3R < RL^3 < RL^4 .$$

The set of all realizable sequences has some remarkable algebraic properties. Again let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ be any two finite sequences of L's and R's. We define

$$A*B = \begin{cases} Ab_1Ab_2\ldots Ab_mA & \text{if } A \text{ contains an even number of R's ,} \\ Ab'_1Ab'_2\ldots Ab'_mA & \text{if } A \text{ contains an odd number of R's ,} \end{cases}$$

where $b'_j = L$ or R according as $b_j = R$ or L. We also set $A*B = A$ if $B = \emptyset$. This operation is associative:

$$A_1*(A_2*A_3) = (A_1*A_2)*A_3 \; .$$

If A and B are realizable, then so also is $A*B$. If A, B_1 and B_2 are realizable and $B_1 < B_2$ then $A*B_1 < A*B_2$. Moreover, if C is realizable and $A*B_1 < C < A*B_2$ then $C = A*B$ for some B such that $B_1 < B < B_2$. This shows that the ordered set of all realizable sequences contains segments which differ from it only by a constant factor. These segments have the same property, and so on.

If we set $H(A) = A*R$ then, for any realizable sequence A, H(A) is also realizable, $A < H(A)$, and there is no realizable sequence between A and H(A). Thus H(A) is the "harmonic" of A, describing the superstable orbit of double the period in the window adjacent to the window containing the superstable orbit described by A.

These remarkable properties of the superstable orbits were observed by Metropolis, Stein and Stein (1973) and Derrida, Gervois and Pomeau (1978). Moreover, they claimed that these properties hold not only for the quadratic map $x \rightarrow \lambda x(1-x)$, but also for a large class of maps whose graphs have the same shape. In particular they should hold for the class $\mathcal{S}$ of all maps $x \rightarrow \lambda f(x)$ with the following properties:

(i) f is C^3 and $f(0) = f(1) = 0$,

(ii) f has a unique critical point $\bar{x}$ and $f''(\bar{x}) < 0$,

(iii) the Schwarzian derivative

$$Sf(x) = f'''(x)/f'(x) - \frac{3}{2}\left\{f''(x)/f'(x)\right\}^2$$

is negative for every $x \neq \bar{x}$.

The Schwarzian derivative occurred already in work of Lagrange and it plays an important role in complex analysis. Geometrically, the condition $Sf < 0$ means that the cross-ratio of any four points of the interval I is increased under the map f. The significance of this condition for maps of an interval was first pointed out by Allwright (1978) and Singer (1978). In the first place, the chain rule for Schwarzian derivatives shows that $Sf < 0$ implies $Sf^n < 0$ for all $n \geqslant 1$. The condition $Sf < 0$ also implies that any attractive periodic orbit is the limit set of some critical point or endpoint of I. It follows that, for any map in the class $\mathcal{S}$, almost all points (in the sense of Lebesgue measure on I) have the same limit set as the critical point $\bar{x}$. In this sense we can restrict attention to the orbit through $\bar{x}$.

If $\bar{x}$ is not a periodic point we can associate with it an infinite sequence $(c_1, c_2, \ldots)$, where $c_j = L$ or R according as $f^j(\bar{x}) \lesseqgtr \bar{x}$. This is the <u>kneading sequence</u> of Milnor and Thurston (1977). The terminating sequences which we considered previously for $\bar{x}$ periodic may be regarded as a special case. The questions of realizability and ordering can be answered also in the general case. Guckenheimer (1979) has shown that a map in $\mathcal{S}$ is almost determined by its kneading sequence. If $f, g \in \mathcal{S}$ have the same kneading sequence and if f has no attractive periodic orbit then f and g are topologically equivalent. On the other hand, all maps in $\mathcal{S}$ with the same kneading sequence and with an attractive periodic orbit belong to at most two topological equivalence classes.

One outstanding conjecture is that the parameter values λ for which there exists an attractive periodic orbit form an open, dense set. This is not known even for the quadratic map. However, for the quadratic map Jakobson (1981) has shown that the complement of this set has positive Lebesgue measure. In unpublished work Sullivan, and Milnor and Thurston, have shown that the topological entropy of the quadratic map is a monotonic function of the parameter. This had been conjectured by Milnor some years before. These results have not yet been extended to arbitrary maps in the class $\mathcal{S}$.

The theory of kneading sequences for maps in $\mathcal{S}$ expresses a kind of qualitative universality. Feigenbaum (1978) has discovered a kind of quantitative universality. Namely, if $\bar{\lambda}_k$ is the unique value of λ in the interval $\lambda_k < \lambda < \lambda_{k+1}$ for which there is a superstable orbit of period 2^k then

$$(\bar{\lambda}_k - \bar{\lambda}_{k-1})/(\bar{\lambda}_{k+1} - \bar{\lambda}_k) \to \delta \quad \text{as } k \to \infty ,$$

where the constant $\delta = 4.6692\ldots$ has the same value for all maps in $\mathcal{S}$. Moreover, if we set

$$\xi_k = (\bar{\lambda}_k f)^{2^{k-1}} (\bar{x}) ,$$

which implies

$$\bar{x} = (\bar{\lambda}_k f)^{2^{k-1}} (\xi_k) ,$$

then

$$(\xi_k - \bar{x})/(\bar{x} - \xi_{k+1}) \to \alpha \quad \text{as } k \to \infty ,$$

where the constant $\alpha = 2.5029\ldots$ has the same value for all maps in $\mathcal{S}$. Also, we obtain the same limit δ if we replace $\bar{\lambda}_k$ by $\bar{\lambda}_k(m)$, where $\bar{\lambda}_k(m)$ is the unique value of λ in the window $\lambda_k(m) < \lambda < \lambda_{k+1}(m)$ for which there is a superstable orbit of period 2^{k-1} m. Feigenbaum summed up this phenomenon by saying that "the local structure of highly bifurcated stable cycles is universal."

In a second paper, Feigenbaum (1979) gave a more conceptual interpretation of the constants α, δ. It is convenient to transform the interval $[0, 1]$ to the interval $[-1, 1]$ by a linear change of variable, to restrict attention to even functions so that the unique critical point is now $\bar{x} = 0$, and to scale the parameter λ so that $f(0) = 1$. The corresponding class $\mathcal{F}$ is mapped into itself by the nonlinear operator $\mathfrak{J}$ defined by

$$\mathfrak{J}f(x) = \beta^{-1} f[f(\beta x)] , \quad \text{where } \beta = f(1).$$

A heuristic argument, supported by numerical calculations, led Feigenbaum to claim that

(i) the map $\mathcal{J}$ has a fixed point g,

(ii) the universal constant $\alpha = -[g(1)]^{-1}$,

(iii) the universal constant δ is a simple eigenvalue of the derivative $D\mathcal{J}(g)$ and is the only point of the spectrum of this linear operator which does not lie inside the unit circle.

The existence of a fixed point $g \in \tilde{\mathcal{S}}$ has now been rigorously established by Campanino and Epstein (1981) and by Lanford (1982), who also proves (iii). The proofs rely on computations to obtain first an approximate fixed point. Recently Vul and Khanin (1982) have proposed a stable method of obtaining an approximate fixed point, which they claim also leads to a proof of the Feigenbaum conjectures.

It would be impossible in three lectures to give you a complete picture of the subject of one-dimensional maps. In particular I have said nothing about relationships with ergodic theory. However, I hope I have said enough to convince you that it offers many interesting problems, which are varied in nature and in degree of difficulty.

Some of the excitement at present lies in the hope that it will lead to an understanding of the nature of turbulence. Perhaps this is expecting too much. I think that it may be possible to obtain rather complete results for one-dimensional maps, but for maps in more than one dimension we may have to be content with a framework of general concepts and a successful analysis of some special cases. However, I do not regard this as a reason for not studying one-dimensional maps.

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