AGT conjecture and non-perturbative beta-function in supersymmetric QCD

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based on:

Losev, Nekrasov, AM: "Small instantons, little strings and free fermions",
Ian Kogan memorial volume, From fields to strings: circumnavigating theoretical physics, hep-th/0302191;

Mironov, Morozov, AM: "Zamolodchikov asymptotic formula and instanton expansion in $\mathcal{N}=2$ SUSY $N_f = 2N_c$ QCD",
JHEP **11** (2009) 048, arXiv: 0909.3338 [hep-th];

AM: "On Gauge Theories as Matrix Models",
arXiv:1101.0676 [hep-th].

Renormalization (scale dependence, parameter dependence) of coupling constant in (supersymmetric, non-Abelian) gauge theory.

- Perturbation theory - asymptotic freedom (non-Abelian);

- Supersymmetric QCD: the NSVZ beta-function;

- Nonperturbative renormalization (by instanton configurations?), in particular - renormalization of the θ-parameter.

$\mathcal{N} = 2$ SUSY gauge theory: for the complexified coupling $\tau = \frac{\theta}{\pi} + i\frac{8\pi}{g^2}$ use of the known r.h.s. in the IR for

$$\tau(\mu)|_{\mu=a} = \tau(a;\Lambda)$$

Does it determine $\tau(\mu)$? At least - knowledge of $\tau(a;\Lambda)$ - can be seen as very nontrivial new application of 2d CFT in the context of 4d gauge theory!

Not known before to the specialists in 2d CFT and QCD ...

The $U(N_c)$ SQCD in UV

$$\mathcal{L}_0 = \mathrm{Tr} \int \left(\frac{1}{4g_0^2} \mathsf{F}_{\mu\nu}^2 - i \frac{\theta_0}{8\pi^2} F \wedge F \ldots \right) \tag{1}$$

flows to the IR Abelian effective theory:

$$\mathcal{L}_{\mathrm{eff}} \sim \mathrm{Im}\ T_{ij}(a)\ F_{\mu\nu}^i F_{\mu\nu}^j + \ldots \tag{2}$$

with holomorphic

$$T_{ij}(a) \underset{\text{weak coupling}}{\longrightarrow} \frac{i\beta}{4\pi} \log \frac{a_i - a_j}{\Lambda} + O\left(\left(\frac{\Lambda}{a} \right)^{\beta} \right)$$

The "tail" as summing over (only!) the instantons ...
$\beta = 2N_c - N_f$ is 1-loop (perturbatively exact - holomorphic!) beta function.

$\mathcal{N}=2$ SUSY: the holomorphic prepotential $T_{ij}=\frac{\partial^2\mathcal{F}}{\partial a_i\partial a_j}$

$$F_{UV}=\frac{1}{2}\tau_0\mathbf{a}^2 \;\longrightarrow\; F_{IR}=\mathcal{F}(\mathbf{a})\stackrel{?}{=}\mathcal{F}_{\mathsf{cl}}+\mathcal{F}_{\mathsf{pert}}+\mathcal{F}_{\mathsf{inst}} \qquad (3)$$

where $\mathcal{F}_{\mathsf{cl}}=F_{UV}$. Can be sometimes $\mathcal{F}(\mathbf{a})=\frac{1}{2}\tau\mathbf{a}^2$, *but* $\tau\neq\tau_0$, i.e. $\mathcal{F}_{\mathsf{pert}}\simeq 0$, *but* (!)

$$\mathcal{F}(\mathbf{a})=\mathcal{F}_{\mathsf{cl}}+\mathcal{F}_{\mathsf{inst}}\neq F_{UV}$$

Generally determined by *an integrable system*: family of curves Σ of genus=rank, with meromorphic dS_{SW} such that

$$\delta dS_{SW}\simeq \text{holomorphic} \qquad (4)$$

Nice geometric description of the *effective* IR theory ...
Direct relation to UV?

SW prepotentials as singular part of Nekrasov functions

$$Z(\mathbf{a},\mathbf{m},q_0;\epsilon_{1,2}) \underset{\epsilon_{1,2}\to 0}{\sim} \exp\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(\mathbf{a},\mathbf{m},q_0) + \ldots \tag{5}$$

Independent from SW "instanton" conjecture! Notations:

- Background deformation parameters $\frac{1}{\epsilon_1\epsilon_2} = \int_{\mathbb{R}^4} d^4x$;
- $\mathbf{a}$ is the N_c-vector of the complex scalar condensates;
- $\mathbf{m}$ is the N_f-vector of the fundamental (complex) masses;
- $q_0 = \exp(i\pi\tau_0)$ is the instanton-counting parameter.

What about the effective couplings?

Nekrasov functions *are* correlation functions: $N_c = 1$, $N_f = 0$ example ($[A, A^\dagger] = 1$, $t \simeq i\pi\tau_0 = \log\Lambda^2$, $\hbar = \epsilon_1 = -\epsilon_2 = 1$)

$$
\begin{aligned}
Z(a,t) &= e^{ta^2/2}\langle 0|e^{A}e^{tA^\dagger A}e^{A^\dagger}|0\rangle = \\
&= e^{ta^2/2}\sum_{n\geq 0}\frac{e^{tn}}{n!} = \exp\left(\tfrac{1}{2}ta^2 + e^t\right)
\end{aligned}
\tag{6}
$$

i.e. summation over the instantons in 4d is essentially the summation over random partitions (2d fermions and/or 2d bosons).

- Even for $N_c = 1$ the (non-commutative) instanton moduli space gives some contribution ($U(N_c)$ versus $SU(N_c)$);

- Holomorphic dependence in 4d theory is reproduced by the "holomorphic" (conformal) 2d field theory.

"Conformal 4d theory" ($N_c = 1$, $N_f = 2N_c = 2$; $\hbar = 1$)

$$\begin{gathered} Z_{\text{inst}}(a, \mathbf{m}, x = q_0) = \\ = \langle e^{i(a+m_2)\phi(\infty)} e^{-im_2\phi(1)} x^{L_0} e^{im_1\phi(1)} e^{-i(a+m_1)\phi(0)} \rangle = \\ = x^{a^2/2} (1-x)^{-m_1 m_2} \end{gathered} \tag{7}$$

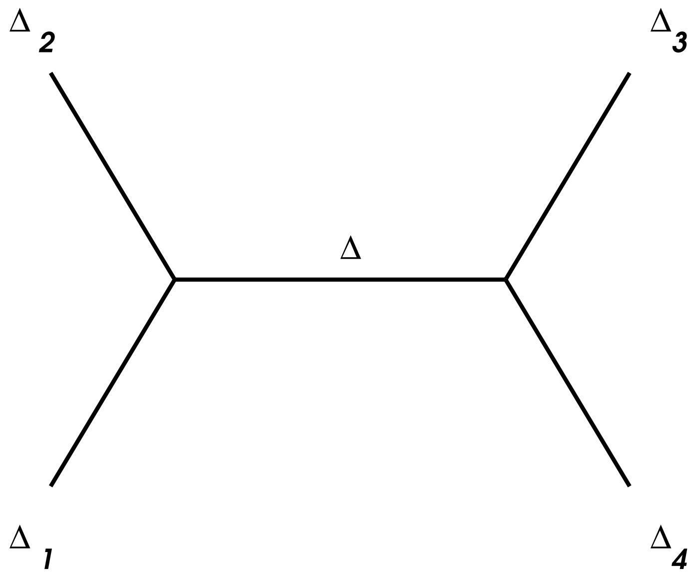
is a conformal block in *free* $c = 1$ 2d CFT (LMN-2003).

Asymptotically free case $m \to \infty$, $x \to 0$, $m^2 x = \Lambda^2$

$$e^{im\phi(1)} e^{-i(a+m)\phi(0)} |0\rangle \;\to\; e^{J_{-1}}|0\rangle = e^{A^\dagger}|0\rangle \tag{8}$$

with $J(z) = i\partial\phi(z) = \sum_{n\in\mathbb{Z}} \frac{J_n}{z^{n+1}}$, and $[J_n, J_m] = n\delta_{n+m,0}$ and $T(z) = -\frac{1}{2}(\partial\phi(z))^2 = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$.

The conformal block for $\Delta_f = \Delta_f(\mathbf{m}, \epsilon)$ and $\Delta = \Delta(\mathbf{a}, \epsilon)$



for the four fields ($f = 1, \ldots, 4$) at $\langle V_1(0) V_2(x) V_3(1) V_4(\infty)\rangle$; s-channel expansion at $x \to 0$: the weak coupling regime.

4d $N_c = 1$ or 2d $\widehat{U(1)}$ theory: only trivial $\mathbf{a}$-dependence (due to the $U(1)$ charge J_0 conservation). More generally - the AGT (Alday-Gaiotto-Tachikawa) conjecture:

4d $N_c = 2$ or 2d *Virasoro* theory: *non*-trivial $\mathbf{a}$-dependence, no constraints on the intermediate dimension $\Delta = \Delta(\mathbf{a}, \epsilon)$!

The universal formulas for the expansion of conformal block (BPZ, 1984; Alesha Zamolodchikov, later 80-s)

$$Z(\boldsymbol{\Delta}(\mathbf{a}), \boldsymbol{\Delta}_f(\mathbf{m}), x) = x^{\Delta - \Delta_1 - \Delta_2} \cdot \sum_{|Y|=|Y'|\geq 0} x^{|Y|} \gamma_{12}(\boldsymbol{\Delta}; Y) Q_{\boldsymbol{\Delta}}(Y, Y')^{-1} \gamma_{34}(\boldsymbol{\Delta}; Y') \tag{9}$$

where $Q_{\boldsymbol{\Delta}}(Y, Y') = \langle \Delta | L_Y L_{-Y'} | \Delta \rangle$ is the scalar product of the Virasoro descendants (analogs of Fock basis $\frac{(A^\dagger)^n}{\sqrt{n!}}|0\rangle$)

$$|\Delta, Y\rangle \equiv L_{-Y}|\Delta\rangle = L_{-k_1} \dots L_{-k_l}|\Delta\rangle$$

($k_1 \geq k_2 \geq \ldots \geq k_l > 0$), and

$$\gamma_{\mathsf{ff'}}(\boldsymbol{\Delta}, Y) = \prod_i \left(\Delta + k_i(Y)\Delta_{\mathsf{f}} - \Delta_{\mathsf{f'}} + \sum_{j<i} k_j(Y) \right) \tag{10}$$

The hidden ϵ-dependence due to the central charge $c = 1 + \frac{6(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$, and, for the $SU(2)$-theory

$$\mathsf{a} = a_1 - a_2 \tag{11}$$

$\Delta \sim \frac{\mathsf{a}^2}{\epsilon_1\epsilon_2} + \ldots,\ \Delta_f \sim \frac{m_f^2}{\epsilon_1\epsilon_2} + \ldots,\ f = 1, \ldots, N_f = 4.$

From each level of the Virasoro module $L_{-Y}|\Delta\rangle$ with $|Y| = k$

$$\sum_{|Y|=|Y'|=k} Z_{Y,Y'}(\Delta; \{\Delta_\mathsf{f}\}) \sim Z_k(\mathsf{a}, \mathbf{m}) \tag{12}$$

one gets the k-instanton contribution into Nekrasov function (“volume” of k-instanton moduli space in ϵ-deformed theory).

What does it give rise to?

The Zamolodchikov asymptotic formula (for $\Delta \gg \Delta_f$, only trivial part of it)

$$Z(\Delta, x) = x^{\Delta}(1 + Z_1 x + Z_2 x^2 + \dots) \underset{\Delta\to\infty}{\sim} (16q)^{\Delta} \tag{13}$$

where

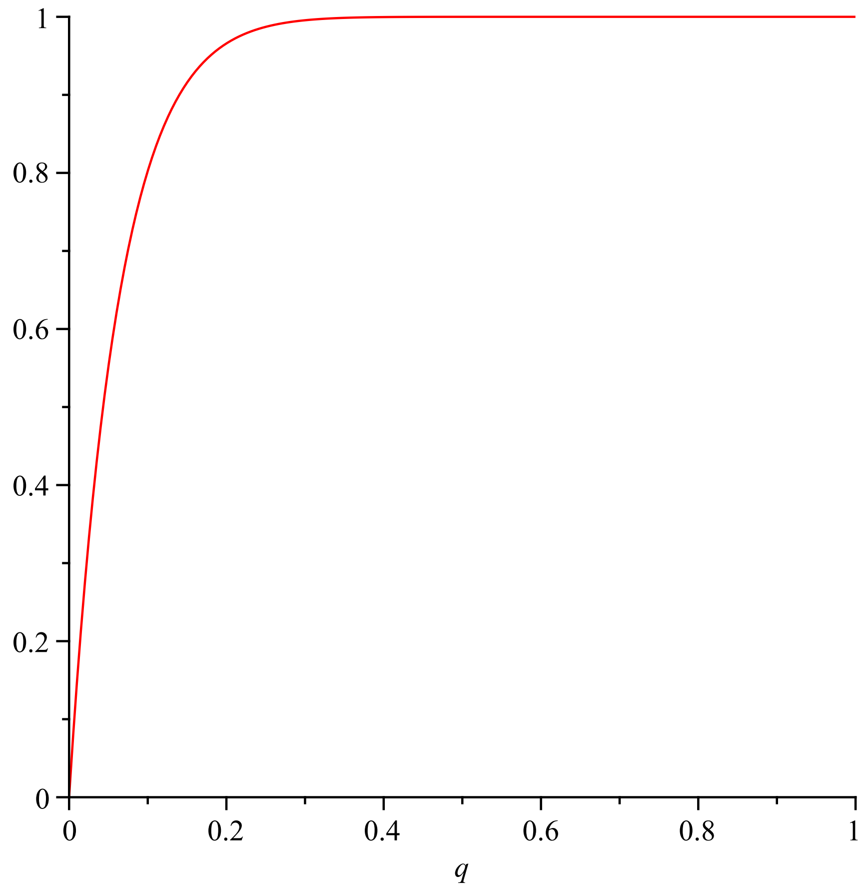
$$x = \frac{\theta_{10}^4(q)}{\theta_{00}^4(q)} = 16q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8 \tag{14}$$

is the relation between the modulus $q = e^{i\pi\tau}$ and the branch point of the $N_c = 2$, $N_f = 4$ SW elliptic curve

$$\eta^2 = \xi(\xi - 1)(\xi - x) \tag{15}$$

with a holomorphic (at vanishing masses!)

$$dS_{SW} \sim \mathsf{a}\frac{d\xi}{\eta} \tag{16}$$



Real slice of the Zamolodchikov dependence $x = x(q)$.
At weak coupling both $g \simeq g_0$ and $\theta \simeq \theta_0$ with high accuracy.

This gives the *quadratic* prepotential

$$\mathcal{F}(\mathsf{a}) = \frac{1}{2\pi i}\mathsf{a}^2 \log q = \tfrac{1}{2}\tau \mathsf{a}^2 \tag{17}$$

but with the

$$\tau = \tau_0 + \sum_{n \geq 0} c_n x^n, \quad c_n \neq 0 \tag{18}$$

instanton-renormalized coupling, where the coefficients can be easily extracted from the Zamolodchikov formula.

Explicit *non-perturbative* beta function in 4d theory! Perturbatively: $\mathcal{F}_{\mathrm{pert}} = \sum_{f=1}^{N_f} \sum_{j=1}^{N} F(a_j + m_f) - \sum_{i \neq j}^{N} F(a_i - a_j)$ with $F(a) \sim \frac{a^2}{2} \log a$ gives rise to a constant redefinition of τ_0.

What is known from 2d CFT for the $U(N_c)$ theory?

Pure gauge theory with $N_c = N$ and $N_f = 0$: degenerate conformal block (matrix element) for W_N-algebra:

$$\begin{gathered}\mathcal{W}_1^{(N)}|\Psi\rangle = \Lambda^N|\Psi\rangle \\ \mathcal{W}_n^{(N)}|\Psi\rangle = 0,\ n>1, \qquad \mathcal{W}_n^{(K)}|\Psi\rangle = 0,\ n>0,\ K<N\end{gathered} \tag{19}$$

("coherent" Whittaker states - direct analogs of $\frac{(A^\dagger)^n}{\sqrt{n!}}|0\rangle$) and

$$\mathcal{W}_0^{(K)}|\Psi\rangle = \sum_{j=1}^{N}\left(\frac{a_j^K}{\epsilon_1\epsilon_2} + \ldots\right)|\Psi\rangle, \qquad K = 1,\ldots,N \tag{20}$$

Then the SW curve is ($D|\Psi\rangle = \frac{z}{w}|\Psi\rangle$) described as

$$\begin{gathered}\langle\Psi|\mathcal{D}_N|\Psi\rangle = 0 \\ \mathcal{D}_N \equiv D^N - T(w)D^{N-2} - \ldots - \mathcal{W}^{(N)}(w)\end{gathered} \tag{21}$$

with the differential $dS_{SW} \sim z\frac{dw}{w}$.

Nontrivial normalization of $|\Psi\rangle = |\Psi_{\mathbf{a}}\rangle$

$$\langle \Psi_{\mathbf{a}} | \Psi_{\mathbf{a}} \rangle |_{\Lambda=0} \sim \prod_{i,j} \Gamma_2(a_i - a_j | \epsilon_1, \epsilon_2) \tag{22}$$

in terms of the Barnes double-gamma function

$$\Gamma_2(x|\epsilon_1, \epsilon_2) \sim \prod_{n,m \geq 0} \frac{1}{x + n\epsilon_1 + m\epsilon_2} \tag{23}$$

so that

$$Z(\mathbf{a}, \Lambda; \epsilon) = \langle \Psi_{\mathbf{a}} | \Psi_{\mathbf{a}} \rangle \sim \exp \frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}(\mathbf{a}, \Lambda) + \ldots \tag{24}$$

Still not much clear - representation theory of W-algebras, but the 2d CFT representation of the ingredients of 4d $\mathcal{N} = 2$ SUSY gauge theory exists!

Conclusions

- New emergence of 2d CFT in 4d context: allows to get *nontrivial* results;

- Still lack of reasonable physical interpretation: too exotic theories ...
  For more realistic the holomorphic couplings do not mean too much ...;

- Anyhow - 2d is always easier than 4d! It may help more ...