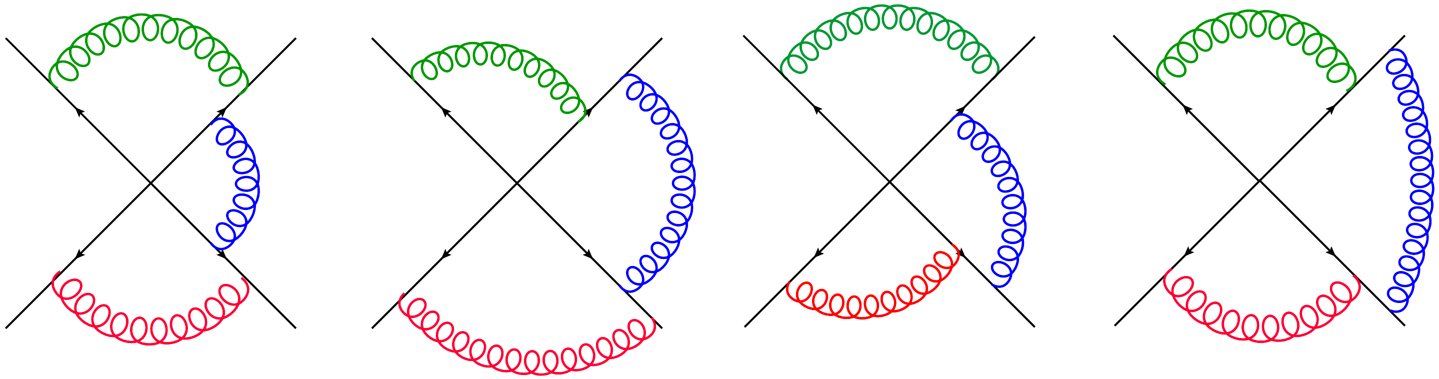
Continuous Advances in Quantum Chromodynamics

The Tait Institute

Soft gluon exponentiation in multiparton amplitudes

Einan Gardi (University of Edinburgh)



(a) (b) (c) (d)

- *Webs in multiparton scattering using the replica trick,* Gardi, Laenen, Stavenga and White, **JHEP 1011 (2010) 155**
- *General properties of multiparton webs: Proofs from combinatorics,* Gardi and White, **JHEP 1103 (2011) 079**

Exponentiation: motivation

multiple soft gluon emission $\simeq$ exp (single soft emission)

- Exponentiation is salient feature of gauge-theory scattering amplitudes:
  - ✓ All IR singularities exponentiate – relates to renormalization of Wilson line operators
  - ✓ In some cases finite parts exponentiate as well.
- Exponentiation is the key to resummation
- Exponentiation is only partially understood diagrammatically. Progress may shed light on multi-loop computations.

Diagrammatic soft-gluon exponentiation

Non-abelian, multiparton case (2010) :

Gardi-Laenen-Stavenga-White & Mitov-Sterman-Sung

Non-abelian, colour singlet case (1983) :

Gatheral, Frenkel-Taylor

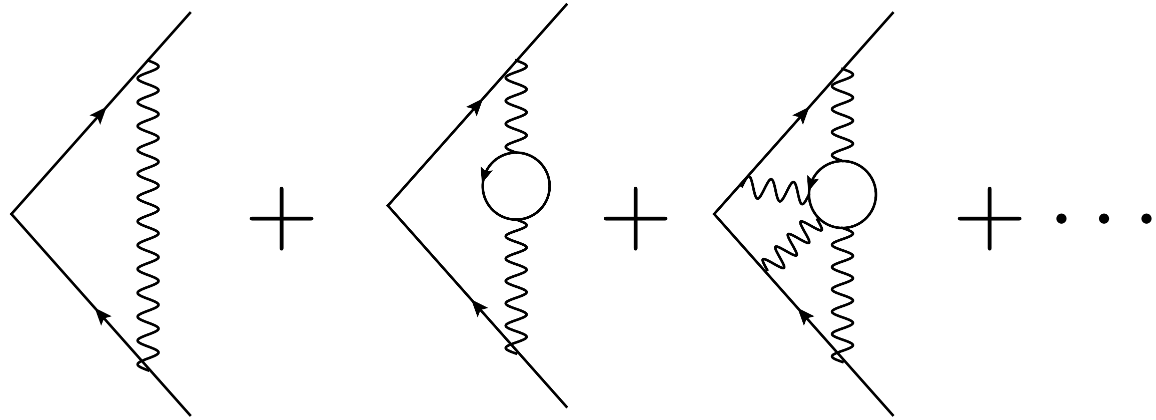
Abelian case (1961):

Yennie-Frautschi-Suura

Exponentiation in an Abelian theory

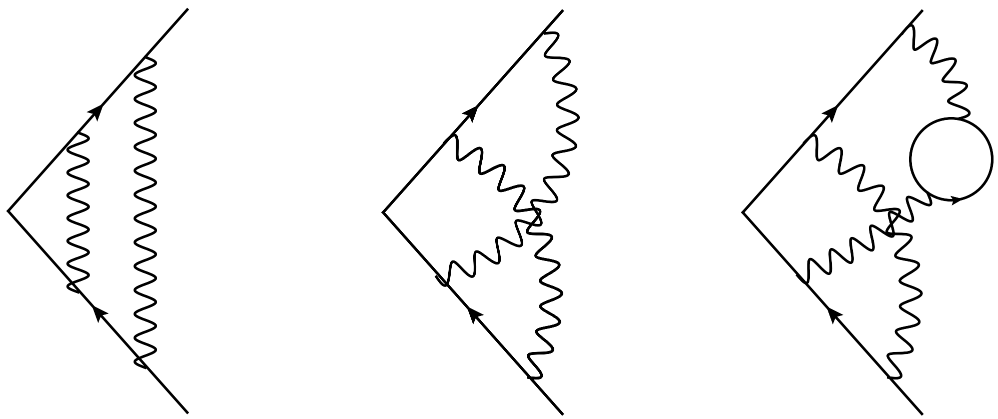
- The exponent only receives contributions from connected diagrams:

$$\mathcal{S} = \exp \Bigg\{$$



$$+ \cdots \Bigg\}$$

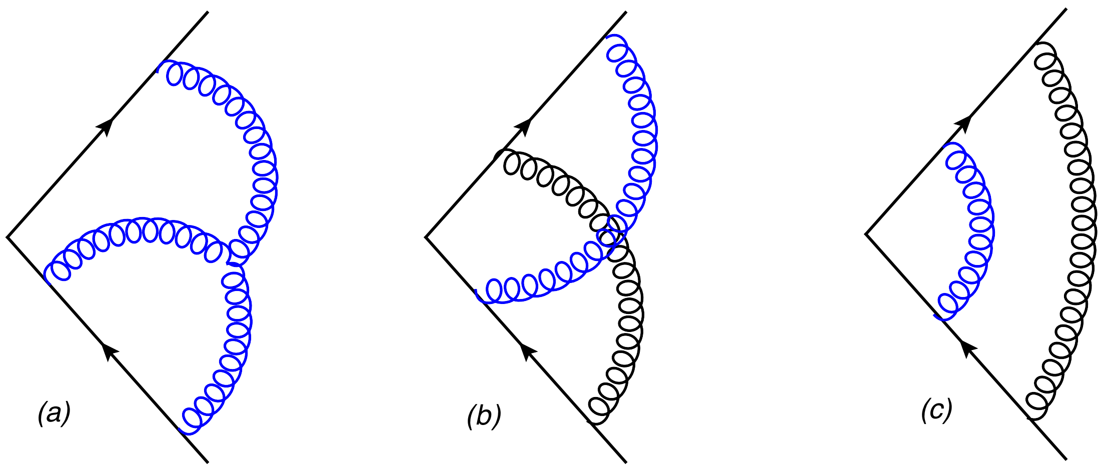
- Expanding the exponential exactly reproduces all disconnected diagrams:



Non-abelian exponentiation (colour-singlet case)

Two new features:

- 3 and 4 gluon vertices – more complicated connected diagrams
- Non-commuting generators for multiple emissions from a given Wilson line – colour-connected diagrams

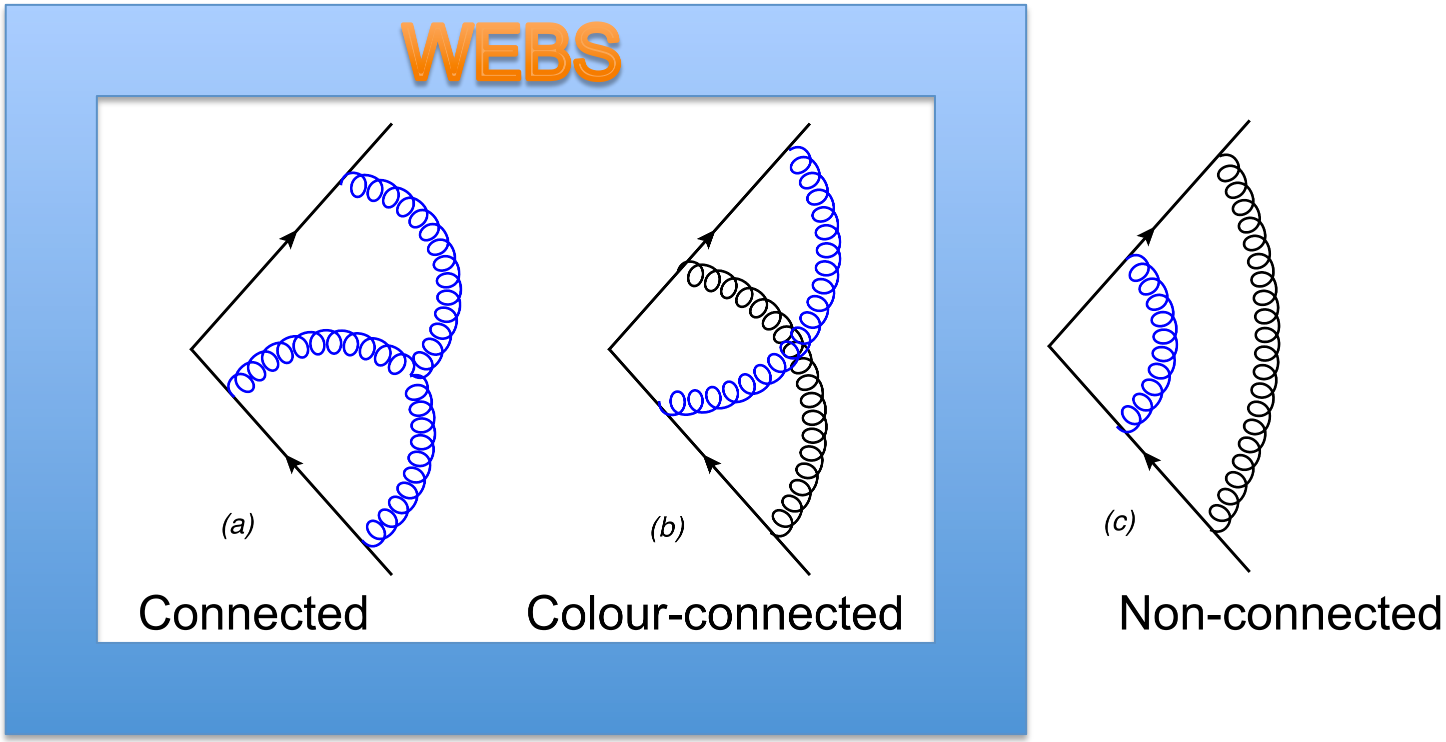


Connected   Colour-connected   Non-connected

Non-abelian exponentiation (colour-singlet case)

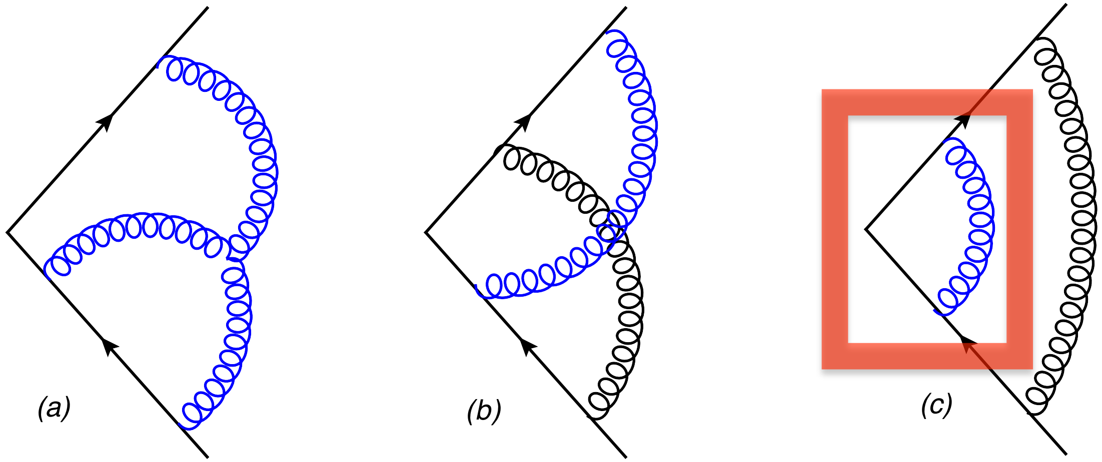
Two new features:

- 3 and 4 gluon vertices – more complicated connected diagrams
- Non-commuting generators for multiple emissions from a given Wilson line – colour-connected diagrams



Non-abelian exponentiation (colour-singlet case)

- *Reducible colour structure:* Diagram D is reducible if it can be decomposed into two subdiagrams such that $C(D) = C(H_1)C(H_2)$



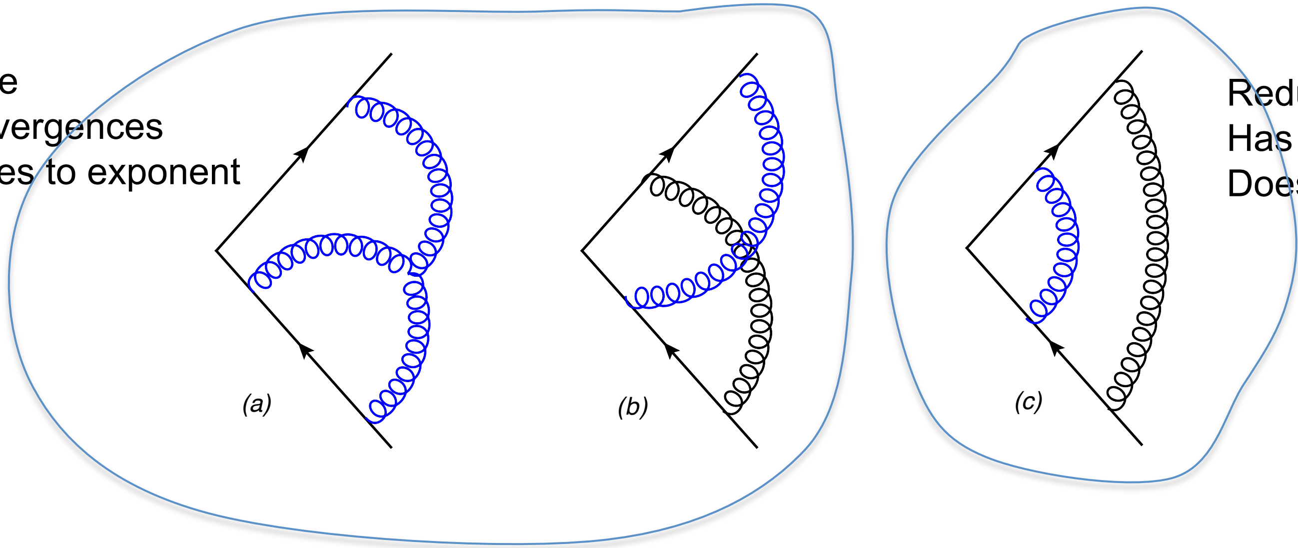
Irreducible Irreducible Reducible

- Reducible diagrams have subdivergences.
- Webs have irreducible colour structure; they have no subdivergences.

Non-abelian exponentiation (colour-singlet case)

- In the colour-singlet case diagrams fall into two classes

Irreducible
No subdivergences
Contributes to exponent

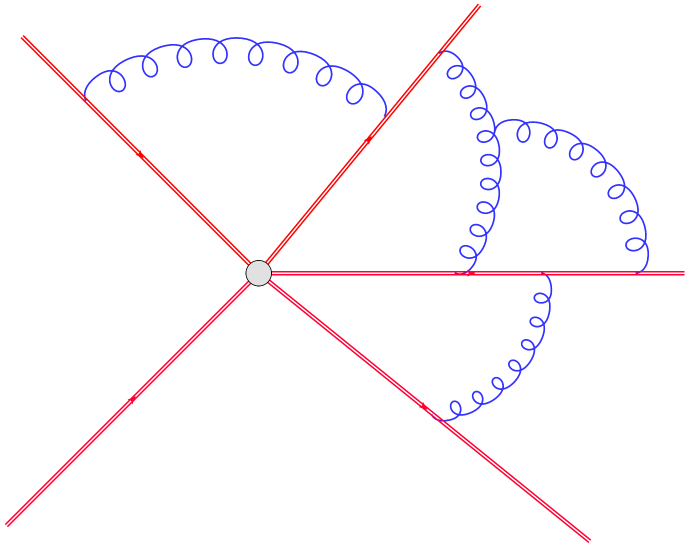


Reducible
Has a subdivergence
Does not contribute

- This goes hand-in-hand with the structure of renormalization: webs have a single overall divergence – generating an $\mathcal{O}(1/\epsilon)$ singularity at each order:

$$
\begin{aligned}
Z^{-1} &= \exp\left\{\zeta(\epsilon)\right\} \\
&= \exp\left\{-\frac{1}{2}\int_{\mu^2}^{\infty}\frac{d\lambda^2}{\lambda^2}\,\Gamma(\alpha_s(\lambda^2))\right\}
\end{aligned}
$$

Non-abelian exponentiation: the multi-leg case



- In the colour-singlet case diagrams fall into two classes: irreducible diagrams that contribute to the exponent, and reducible ones that do not.

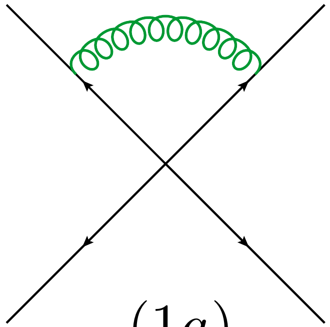
- In the multi-leg case this separation breaks down: *reducible diagrams do contribute…*

- How is this consistent with renormalizability?

- What is the proper generalization of WEBS to the multi-leg case? How can one compute directly the exponent?

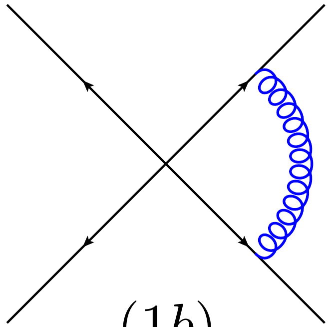
Non-abelian exponentiation: the multi-leg case

Exponentiating:



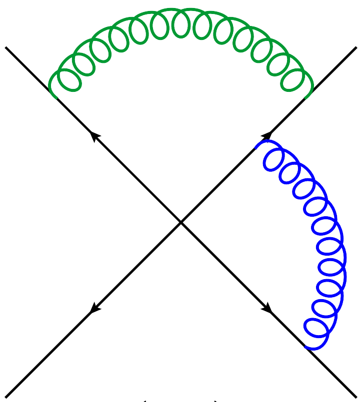
$(1a)$

+



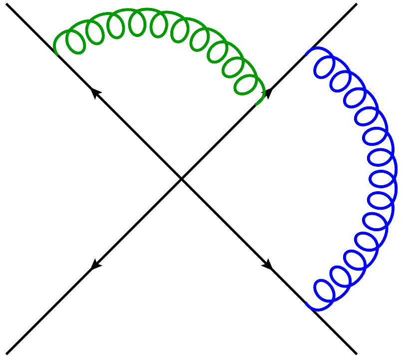
$(1b)$

At 2-loops, do we get



$(2a)$

+



$(2b)$

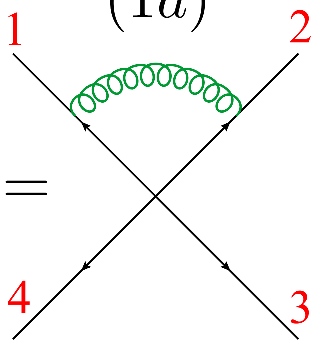
?

Non-abelian exponentiation: the multi-leg case

Exponentiating:

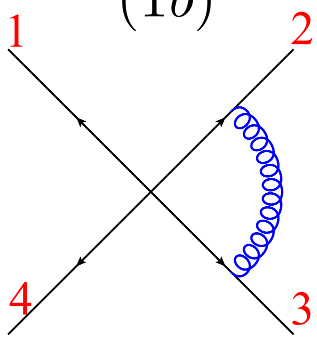
$(1a)$

$$D_{(1a)} + D_{(1b)} =$$



+

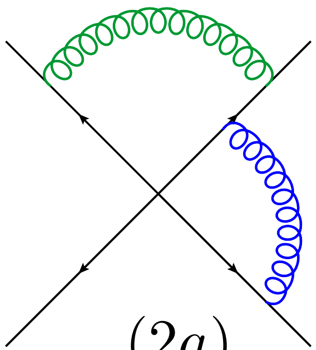
$(1b)$



$$= \mathcal{F}(1a)\,\mathrm{T}_1^{(a)}\mathrm{T}_2^{(a)} + \mathcal{F}(1b)\,\mathrm{T}_2^{(b)}\mathrm{T}_3^{(b)}$$

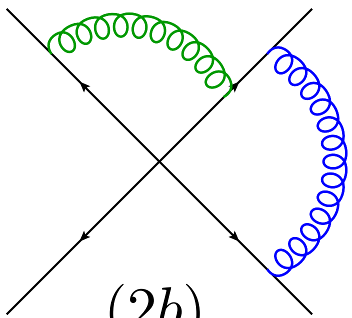
At 2-loops we get:
$$\frac{1}{2}\left[D_{(1a)} + D_{(1b)}\right]^2 = \frac{1}{2}\left[D_{(1a)}D_{(1b)} + D_{(1b)}D_{(1a)} + \cdots\right]$$
$$= \frac{1}{2}\mathcal{F}(1a)\mathcal{F}(1b)\ \mathrm{T}_1^{(a)}\left[\mathrm{T}_2^{(a)}\mathrm{T}_2^{(b)} + \mathrm{T}_2^{(b)}\mathrm{T}_2^{(a)}\right]\mathrm{T}_3^{(b)}$$
$$= \frac{1}{2}\Big[\mathcal{F}(2a) + \mathcal{F}(2b)\Big]\quad\Big[C(2a) + C(2b)\Big]$$

While



$(2a)$

+



$(2b)$

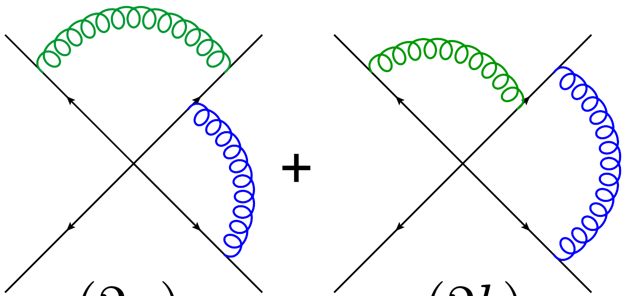
$$= \mathcal{F}(2a)\,C(2a) + \mathcal{F}(2b)\,C(2b)$$

Non-abelian exponentiation: the multi-leg case

Exponentiating 1-loop diagrams yields:

$$\frac{1}{2}\left[D_{(1a)}+D_{(1b)}\right]^2=\frac{1}{2}\Big[\mathcal{F}(2a)+\mathcal{F}(2b)\Big]\quad\Big[C(2a)+C(2b)\Big]$$

While the 2-loop amplitude is:



$(2a)$ + $(2b)$

$$=\mathcal{F}(2a)\,C(2a)+\mathcal{F}(2b)\,C(2b)$$

The 2-loop contribution to the exponent is therefore:

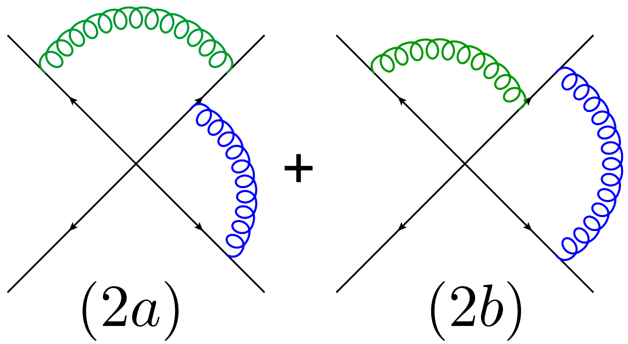
$$\frac{1}{2}\Big[\mathcal{F}(2a)-\mathcal{F}(2b)\Big]\quad\Big[C(2a)-C(2b)\Big]$$

In the multi-leg case, *reducible diagrams do contribute to the exponent*!

Non-abelian exponentiation: the multi-leg case

The 2-loop contribution to the exponent

is the *anti-symmetric part* of



$$= \frac{1}{2} \underbrace{\Big[\mathcal{F}(2a) - \mathcal{F}(2b)\Big]}_{\mathcal{O}(1/\epsilon)} \underbrace{\Big[C(2a) - C(2b)\Big]}_{\mathrm{i} f^{abc} \mathrm{T}_1^{(a)} \mathrm{T}_3^{(b)} \mathrm{T}_2^{(c)}}$$

These properties (single pole, maximally non-Abelian colour structure) are familiar from the colour singlet case.

Non-abelian exponentiation: the multi-leg case

- In contrast to the colour-singlet case, reducible diagrams enter the exponent.
- As in the colour-singlet case, each diagram enters the exponent with a *modified colour factor* $\widetilde{C}(D)$.
- While individual diagrams do not have "web properties", only particular linear combinations that do enter the exponent. We're led to define webs W_i as sums:

$$\mathcal{S} = \exp\left[\sum_i W_i\right] \qquad W_i = \sum_{\{D\}_i} \mathcal{F}(D)\,\widetilde{C}(D)$$

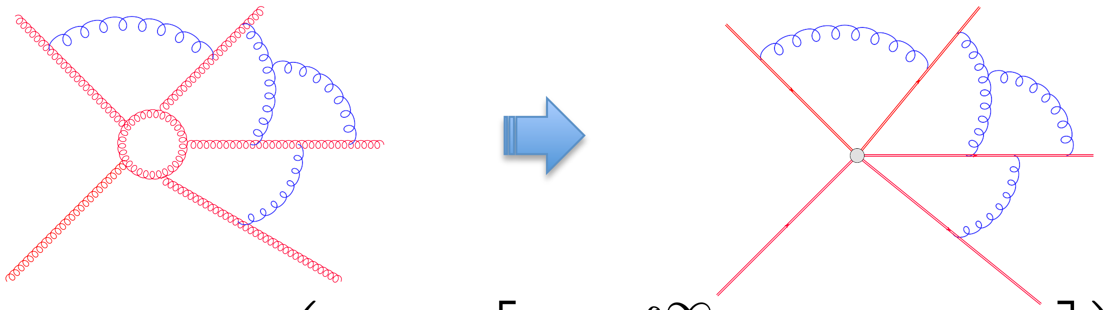
- *modified colour factors* $\widetilde{C}(D)$ are linear combination of (ordinary) colour factors of diagrams that are obtained by permuting attachments to the Wilson lines, so:

$$W_i = \sum_D \mathcal{F}(D) \underbrace{\sum_{D'} R_{DD'}\, C(D')}_{\widetilde{C}(D)} = \mathcal{F}^T R C$$

- R is the web mixing matrix.

Non-abelian exponentiation using the replica trick

- Each hard parton, $l=1..L$, is represented by a Wilson line ray:



$$\Phi^{(l)}_{a_l b_l} \equiv \left(\mathcal{P}\exp\left[\mathrm{i}g_s\int_0^\infty dt\beta_l\cdot A(t\beta_l)\right]\right)_{a_l b_l}$$

- Generating functional for the Eikonal amplitude:
  Wilson lines are sources for the soft gluon field A_s^μ

$$\mathcal{M}_{b_1\ldots b_L}(p_1,\ldots,p_L) = H_{a_1\ldots a_L}\mathcal{Z}_{a_1\ldots a_L,b_1\ldots b_L}$$

$$\mathcal{Z} = \int [\mathcal{D}A_s^\mu]\, e^{\mathrm{i}S[A_s^\mu]}\left[\Phi^{(1)}\otimes\Phi^{(2)}\otimes\cdots\otimes\Phi^{(L)}\right]$$

- We want to write $\mathcal{Z} = \exp\{\cdots\}$: we wish to compute $\ln\mathcal{Z}$ directly

Non-abelian exponentiation using the replica trick

- Replicate the theory N times, such that different replicas do not interact:

$$S[A_\mu] \longrightarrow \sum_{i=1}^{N} S[A_\mu^i]$$

- The generating functional for the replicated theory is:

$$\mathcal{Z}^N = \int [\mathcal{D}A_\mu^1] \dots [\mathcal{D}A_\mu^N] \; e^{\mathrm{i}\sum_i S[A_\mu^i]} \left[(\Phi_1^{(1)}\Phi_2^{(1)} \dots \Phi_N^{(1)}) \otimes (\Phi_1^{(2)}\Phi_2^{(2)} \dots \Phi_N^{(2)}) \otimes \dots\dots \otimes (\Phi_1^{(L)}\Phi_2^{(L)} \dots \Phi_N^{(L)}) \right]$$

- The order of attachments to the Wilson lines is important:

$$\left[\Phi_1{}^{(l)} \Phi_2{}^{(l)} \dots \Phi_N{}^{(l)} \right]_{a_1 b_1} = \left(\mathcal{P} \exp \left[\mathrm{i}g_s \int dt \, \beta_l^\mu A_\mu^1 \right] \right)_{a_1 c_2} \dots \left(\mathcal{P} \exp \left[\mathrm{i}g_s \int dt \, \beta_l^\mu A_\mu^N \right] \right)_{c_N b_1}$$

- So

$$\left[\Phi_1^{(l)} \Phi_2^{(l)} \dots \Phi_N^{(l)} \right]_{a_l b_l} \neq \left(\mathcal{P} \exp \left[\mathrm{i}g_s \sum_{i=1}^{N} \int dt \, \beta_l^\mu A_\mu^i \right] \right)_{a_l b_l}$$

Non-abelian exponentiation using the replica trick

- Replicate the theory N times, such that different replicas do not interact:

$$S[A_\mu] \longrightarrow \sum_{i=1}^{N} S[A_\mu^i]$$

- The generating functional for the replicated theory is:

$$\mathcal{Z}^N = \int [\mathcal{D}A_\mu^1] \ldots [\mathcal{D}A_\mu^N] \; e^{\mathrm{i}\sum_i S[A_\mu^i]} \left[(\Phi_1^{(1)} \Phi_2^{(1)} \ldots \Phi_N^{(1)}) \otimes (\Phi_1^{(2)} \Phi_2^{(2)} \ldots \Phi_N^{(2)}) \otimes \cdots\cdots \otimes (\Phi_1^{(L)} \Phi_2^{(L)} \ldots \Phi_N^{(L)}) \right]$$

- The order of attachments to the Wilson lines is important:

$$\left[\Phi_1{}^{(l)} \Phi_2{}^{(l)} \ldots \Phi_N{}^{(l)} \right]_{a_1 b_1} = \left(\mathcal{P} \exp \left[\mathrm{i} g_s \int dt \, \beta_l^\mu A_\mu^1 \right] \right)_{a_1 c_2} \cdots \left(\mathcal{P} \exp \left[\mathrm{i} g_s \int dt \, \beta_l^\mu A_\mu^N \right] \right)_{c_N b_1}$$

- Instead $$\left[\Phi_1^{(l)} \Phi_2^{(l)} \ldots \Phi_N^{(l)} \right]_{a_l b_l} = \left(\mathcal{R} \mathcal{P} \exp \left[\mathrm{i} g_s \sum_{i=1}^{N} \int dt \, \beta_l^\mu A_\mu^i \right] \right)_{a_l b_l}$$

- $\mathcal{R}$ is a Replica-ordering operator

Non-abelian exponentiation using the replica trick

- The generating functional for the replicated theory can be written as:

$$\mathcal{Z}^N = \int [\mathcal{D}A^1_\mu] \dots [\mathcal{D}A^N_\mu] \; e^{\mathrm{i}\sum_i S[A^i_\mu]} \, \mathcal{R} \left\{ \mathcal{P}\exp\left[\mathrm{i}g_s \sum_{i=1}^N \int dt\, \beta_1^\mu A^i_\mu\right] \otimes \dots \otimes \mathcal{P}\exp\left[\mathrm{i}g_s \sum_{i=1}^N \int dt\, \beta_L^\mu A^i_\mu\right]\right\}$$

- Diagram D computed in this theory will have kinematic dependence $\mathcal{F}(D)$ as in the original theory, but colour factor $C_N(D)$ which differ from $C(D)$ due to the action of $\mathcal{R}$.

- Now expand in powers of N

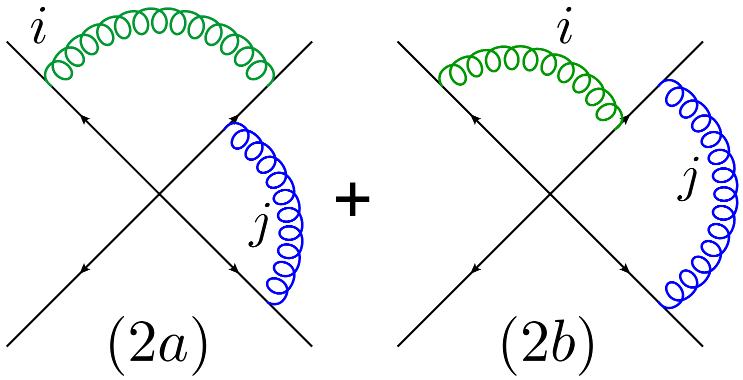
$$\mathcal{Z}^N = 1 + N \ln \mathcal{Z} + \mathcal{O}(N^2)$$

- Contribution of a given diagram D to $\ln \mathcal{Z}$ can be readily determined as the coefficient of N^1 in the expansion of $\mathcal{F}(D)\, C_N(D)$.

- That's it! Here is an algorithm to compute the exponent directly: *diagram D contributes with a modified colour factor $\widetilde{C}(D)$, which is $\mathcal{O}(N^1)$ term in the expansion its colour factor in the replicated theory.*

The replica trick: two-loop example

The 2-loop contribution *to the exponent* is:



$$= \mathcal{F}(2a)\,\widetilde{C}(2a) + \mathcal{F}(2b)\,\widetilde{C}(2b)$$

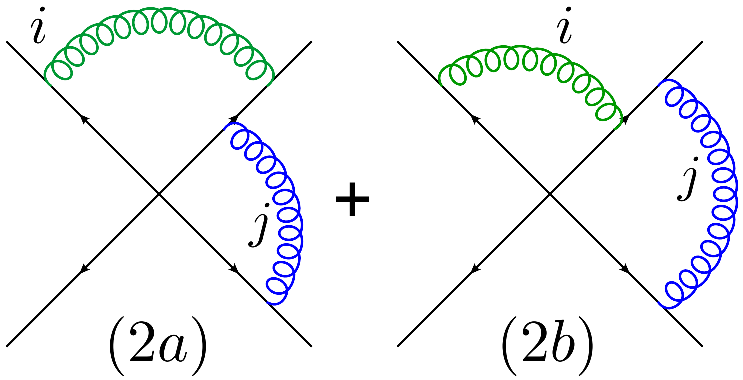
To compute $\widetilde{C}(2a)$ and $\widetilde{C}(2b)$ replicate the theory: $i = 1..N$ and $j = 1..N$

h hierarchy	$\mathcal{R}\left[C(2a)\right]$	$\mathcal{R}\left[C(2b)\right]$	$M_N(h)$
$i = j$	$C(2a)$	$C(2b)$	N
$i < j$	$C(2b)$	$C(2b)$	$N(N-1)/2$
$i > j$	$C(2a)$	$C(2a)$	$N(N-1)/2$

Summing up:
$$C_N(a) = \frac{N}{2}\left[C(a) - C(b)\right] + \frac{N^2}{2}\left[C(a) + C(b)\right]$$
$$C_N(b) = \frac{N}{2}\left[C(b) - C(a)\right] + \frac{N^2}{2}\left[C(a) + C(b)\right]$$

The replica trick: two-loop example

The 2-loop contribution *to the exponent* is:



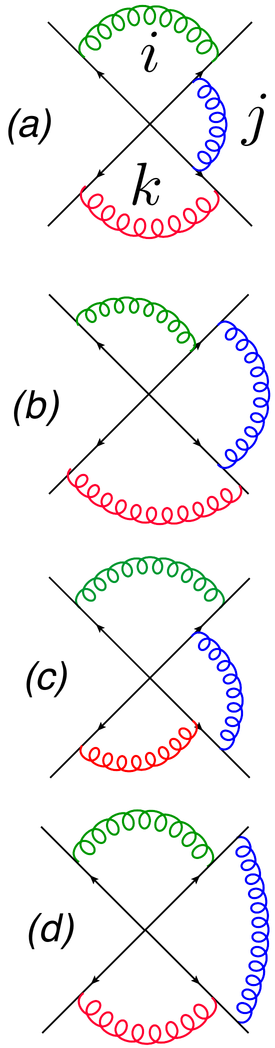
$$= \mathcal{F}(2a)\,\widetilde{C}(2a) + \mathcal{F}(2b)\,\widetilde{C}(2b)$$
$$= \frac{1}{2}\Big(C(a) - C(b)\Big)\Big(\mathcal{F}(a) - \mathcal{F}(b)\Big)$$

To compute $\widetilde{C}(2a)$ and $\widetilde{C}(2b)$ replicate the theory: $i = 1..N$ and $j = 1..N$

h hierarchy	$\mathcal{R}\left[C(2a)\right]$	$\mathcal{R}\left[C(2b)\right]$	$M_N(h)$
$i = j$	$C(2a)$	$C(2b)$	N
$i < j$	$C(2b)$	$C(2b)$	$N(N-1)/2$
$i > j$	$C(2a)$	$C(2a)$	$N(N-1)/2$

Summing up:
$$C_N(a) = \frac{N}{2}\left[C(a) - C(b)\right] + \frac{N^2}{2}\left[C(a) + C(b)\right]$$
$$C_N(b) = \frac{N}{2}\left[C(b) - C(a)\right] + \frac{N^2}{2}\left[C(a) + C(b)\right]$$

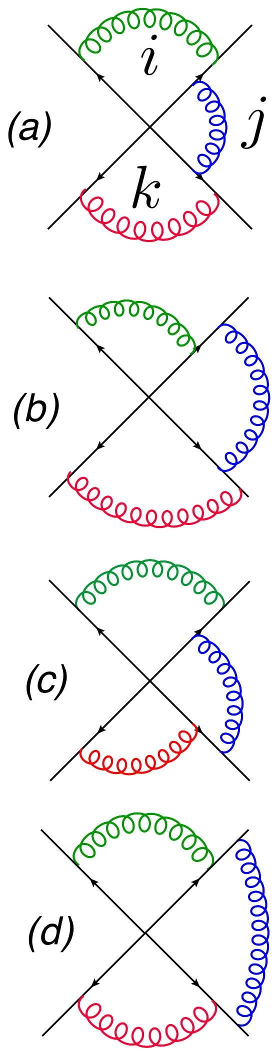
The replica trick: three-loop example



h	$\mathcal{R}\left[C(a)\right]$	$M_N(h)$
$i = j = k$	$C(a)$	N
$i = j < k$	$C(a)$	$N(N-1)/2$
$k < i = j$	$C(c)$	$N(N-1)/2$
$j = k < i$	$C(a)$	$N(N-1)/2$
$i < j = k$	$C(b)$	$N(N-1)/2$
$j < i = k$	$C(a)$	$N(N-1)/2$
$i = k < j$	$C(d)$	$N(N-1)/2$
$i < j < k$	$C(b)$	$N(N-1)(N-2)/6$
$j < i < k$	$C(a)$	$N(N-1)(N-2)/6$
$i < k < j$	$C(d)$	$N(N-1)(N-2)/6$
$k < i < j$	$C(d)$	$N(N-1)(N-2)/6$
$k < j < i$	$C(c)$	$N(N-1)(N-2)/6$
$j < k < i$	$C(a)$	$N(N-1)(N-2)/6$

$$C_N(a) = \frac{1}{6}\Big(2C(a) + C(c) + C(b) + 2C(d)\Big)N^3 + \frac{1}{2}\Big(C(a) - C(d)\Big)N^2 + \frac{1}{6}\Big(C(a) - C(c) - C(b) + C(d)\Big)N$$

The replica trick: three-loop example



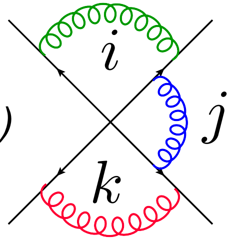
h	$\mathcal{R}\,[C(a)]$	$M_N(h)$
$i = j = k$	$C(a)$	N
$i = j < k$	$C(a)$	$N(N-1)/2$
$k < i = j$	$C(c)$	$N(N-1)/2$
$j = k < i$	$C(a)$	$N(N-1)/2$
$i < j = k$	$C(b)$	$N(N-1)/2$
$j < i = k$	$C(a)$	$N(N-1)/2$
$i = k < j$	$C(d)$	$N(N-1)/2$
$i < j < k$	$C(b)$	$N(N-1)(N-2)/6$
$j < i < k$	$C(a)$	$N(N-1)(N-2)/6$
$i < k < j$	$C(d)$	$N(N-1)(N-2)/6$
$k < i < j$	$C(d)$	$N(N-1)(N-2)/6$
$k < j < i$	$C(c)$	$N(N-1)(N-2)/6$
$j < k < i$	$C(a)$	$N(N-1)(N-2)/6$

modified colour factor: $\widetilde{C}(a) = \frac{1}{6}\Big(C(a) - C(c) - C(b) + C(d)\Big)$

The replica trick: three-loop example

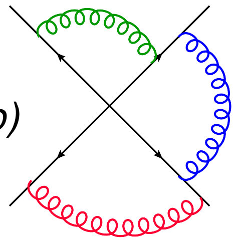
modified colour factors:

(a)



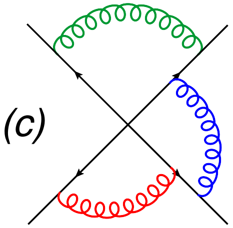
$$\widetilde{C}(a) = \frac{1}{6}\Big(C(a) - C(b) - C(c) + C(d)\Big)$$

(b)



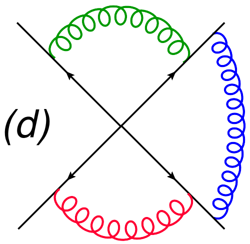
$$\widetilde{C}(b) = -\frac{1}{3}\Big(C(a) - C(b) - C(c) + C(d)\Big)$$

(c)



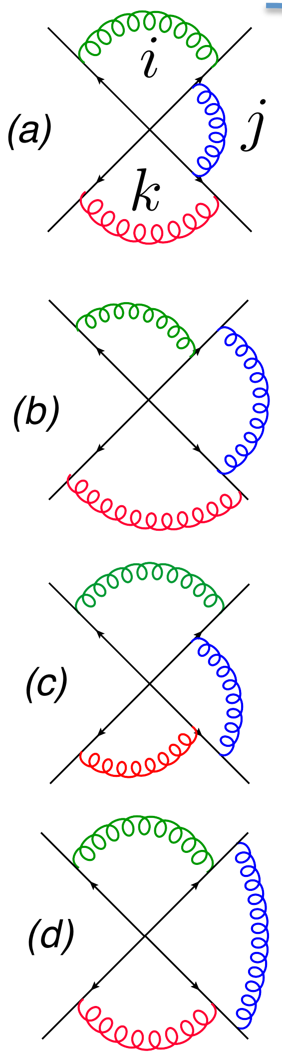
$$\widetilde{C}(c) = -\frac{1}{3}\Big(C(a) - C(b) - C(c) + C(d)\Big)$$

(d)



$$\widetilde{C}(d) = \frac{1}{6}\Big(C(a) - C(b) - C(c) + C(d)\Big)$$

The replica trick: three-loop example



The entire web contributes:

$$W_{(1,2,2,1)} = \frac{1}{6}\Big(\mathcal{F}(3a) - 2\mathcal{F}(3b) - 2\mathcal{F}(3c) + \mathcal{F}(3d)\Big) \times \Big(C(3a) - C(3b) - C(3c) + C(3d)\Big)$$

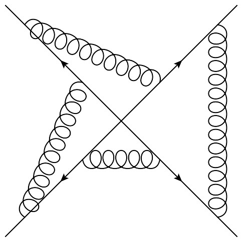
$$= \begin{pmatrix} \mathcal{F}(3a) \\ \mathcal{F}(3b) \\ \mathcal{F}(3c) \\ \mathcal{F}(3d) \end{pmatrix}^T \frac{1}{6} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} C(3a) \\ C(3b) \\ C(3c) \\ C(3d) \end{pmatrix}$$

Kinematics

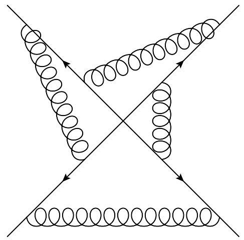
Web mixing matrix R

Colour

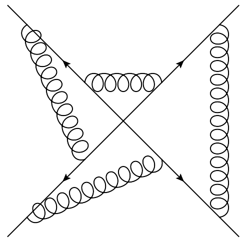
Mixing matrices: four-loop example



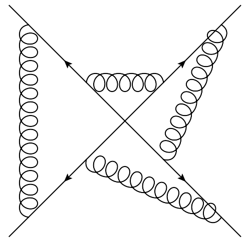
[[1,2],[3,1],[3,4],[2,4]]



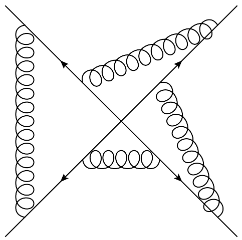
[[1,2],[2,3],[4,3],[4,1]]



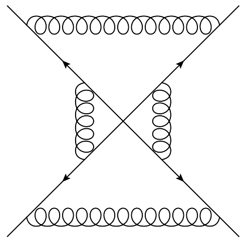
[[1,2],[3,2],[3,4],[4,1]]



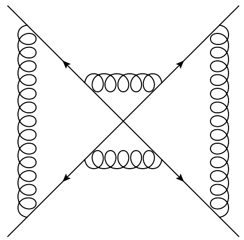
[[1,2],[2,3],[3,4],[1,4]]



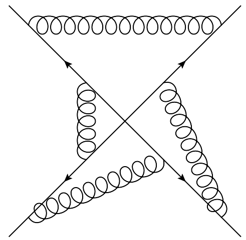
[[1,2],[3,2],[4,3],[1,4]]



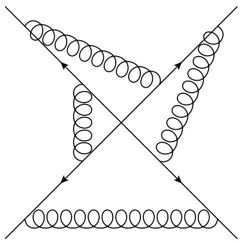
[[1,2],[1,3],[4,3],[4,2]]



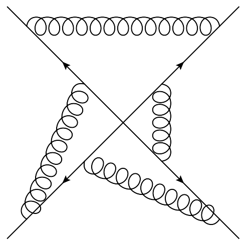
[[1,2],[3,2],[3,4],[1,4]]



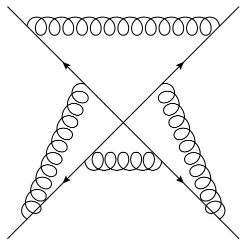
[[1,2],[1,3],[3,4],[4,2]]



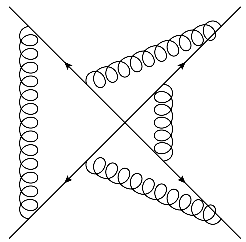
[[1,2],[3,1],[4,3],[4,2]]



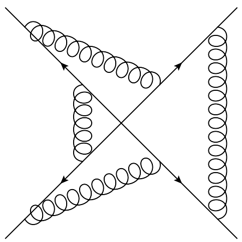
[[1,2],[1,3],[4,3],[2,4]]



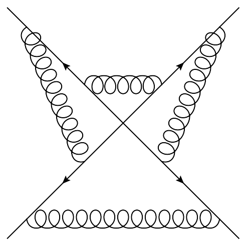
[[1,2],[1,3],[3,4],[2,4]]



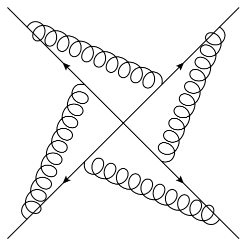
[[1,2],[2,3],[4,3],[1,4]]



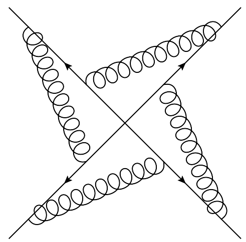
[[1,2],[3,1],[3,4],[4,2]]



[[1,2],[3,2],[4,3],[4,1]]



[[1,2],[3,1],[4,3],[2,4]]



[[1,2],[2,3],[3,4],[4,1]]

Mixing matrices: four-loop example

The resulting mixing matrix:

$$\widetilde{C} = RC = \frac{1}{24}\begin{pmatrix}
6 & -6 & 2 & 2 & -2 & 4 & -4 & 2 & -2 & -2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-6 & 6 & -2 & -2 & 2 & -4 & 4 & -2 & 2 & 2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 6 & -2 & 2 & 4 & -4 & -2 & 2 & -6 & 4 & 4 & -4 & -4 & 0 & 0 \\
2 & -2 & -2 & 6 & 2 & 4 & -4 & -2 & -6 & 2 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & 2 & 6 & 4 & -4 & -6 & -2 & -2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 2 & 2 & 2 & 4 & -4 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -2 & -2 & -2 & -4 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -2 & -2 & -6 & -4 & 4 & 6 & 2 & 2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -6 & -2 & -4 & 4 & 2 & 6 & -2 & 4 & 4 & -4 & -4 & 0 & 0 \\
-2 & 2 & -6 & 2 & -2 & -4 & 4 & 2 & -2 & 6 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -2 & 2 & 0 & 0 & -2 & 2 & -2 & 4 & 0 & 0 & -4 & 0 & 0 \\
2 & -2 & 2 & -2 & -2 & 0 & 0 & 2 & 2 & -2 & 0 & 4 & -4 & 0 & 0 & 0 \\
-2 & 2 & -2 & 2 & 2 & 0 & 0 & -2 & -2 & 2 & 0 & -4 & 4 & 0 & 0 & 0 \\
2 & -2 & -2 & 2 & -2 & 0 & 0 & 2 & -2 & 2 & -4 & 0 & 0 & 4 & 0 & 0 \\
-18 & -6 & -6 & -6 & -18 & 12 & 12 & -6 & -18 & -18 & 12 & 12 & 12 & 12 & 24 & 0 \\
-6 & -18 & -18 & -18 & -6 & 12 & 12 & -18 & -6 & -6 & 12 & 12 & 12 & 12 & 0 & 24
\end{pmatrix}
\begin{pmatrix}
C[[1,2],[3,1],[3,4],[2,4]] \\
C[[1,2],[2,3],[4,3],[4,1]] \\
C[[1,2],[3,2],[3,4],[4,1]] \\
C[[1,2],[2,3],[3,4],[1,4]] \\
C[[1,2],[3,2],[4,3],[1,4]] \\
C[[1,2],[1,3],[4,3],[4,2]] \\
C[[1,2],[3,2],[3,4],[1,4]] \\
C[[1,2],[1,3],[3,4],[4,2]] \\
C[[1,2],[3,1],[4,3],[4,2]] \\
C[[1,2],[1,3],[4,3],[2,4]] \\
C[[1,2],[1,3],[3,4],[2,4]] \\
C[[1,2],[2,3],[4,3],[1,4]] \\
C[[1,2],[3,1],[3,4],[4,2]] \\
C[[1,2],[3,2],[4,3],[4,1]] \\
C[[1,2],[3,1],[4,3],[2,4]] \\
C[[1,2],[2,3],[3,4],[4,1]]
\end{pmatrix}$$

Staircase diagrams cannot be made
out of the decomposition of others.
They are free of subdivergences.

Properties of web mixing matrices

Any web mixing matrix R admits:

A. It is *idempotent:*

$$R^2 = R$$

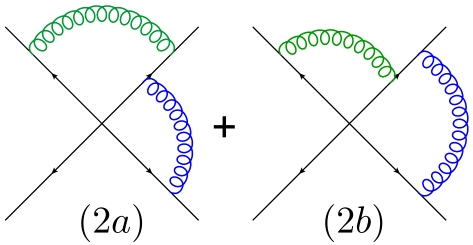
R is diagonalisable, with all its eigenvalues 0 or 1.

B. Its rows sum to zero:

$$\sum_{D'} R_{DD'} = 0$$

The zero-sum row property

We have seen that the exponent is made out of the anti-symmetric part (maximally non-Abelian) of the colour factor, for example:



$$\frac{1}{2}\Big[\mathcal{F}(2a) - \mathcal{F}(2b)\Big]\ \Big[C(2a) - C(2b)\Big]$$

How does this generalise?

In general contributions to the exponent take the form:

$$W = \sum_{D,D'} \mathcal{F}(D) R_{DD'} C(D')$$

The fully symmetric part is removed by the zero-sum property: $\displaystyle\sum_{D'} R_{DD'} = 0$

Web mixing matrix as a projection operator

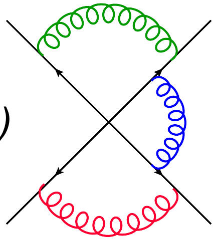
R is idempotent and therefore diagonalisable, with all its eigenvalues:
1 with degeneracy r (rank)
0 with degeneracy $d - r$ (dimension minus rank)

$$
\begin{aligned}
W &= \mathcal{F}^{\mathrm{T}}\,\widetilde{C} = \mathcal{F}^{\mathrm{T}}\,R\,C \\
&= \left(\mathcal{F}^{\mathrm{T}}Y^{-1}\right)Y\,R\,Y^{-1}\left(Y\,C\right) \\
&= \left(\mathcal{F}^{\mathrm{T}}Y^{-1}\right)\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_d)\left(Y\,C\right) \\
&= \sum_{H=1}^{r}\left(\mathcal{F}^{\mathrm{T}}Y^{-1}\right)_H\left(Y\,C\right)_H
\end{aligned}
$$

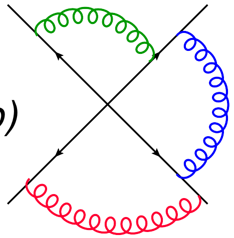
Only certain linear combinations of kinematic functions $\mathcal{F}$
and corresponding linear combinations of colour factors C
enter the exponent: *the eigenvectors corresponding to eigenvalue 1*.

Cancellation of subdivergences: example

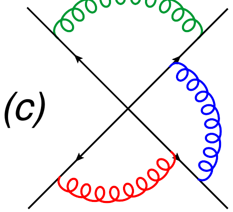
(a)



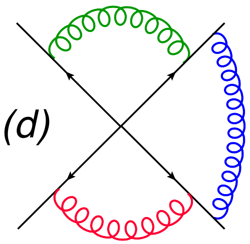
(b)



(c)



(d)



Having diagonalised the mixing matrix, only eigenvectors corresponding to eigenvalue 1 enter the exponent.

<u>Example</u>:

$$\mathcal{F}^{\mathrm{T}} = \sum_{G,E} \mathcal{F}(G)\, Y^{-1}_{GE} Y_{ED}$$

$$= \frac{1}{6} \begin{pmatrix} \mathcal{F}(a) - 2\mathcal{F}(b) - 2\mathcal{F}(c) + \mathcal{F}(d) \\ -\mathcal{F}(a) + 2\mathcal{F}(b) + 2\mathcal{F}(c) + 5\mathcal{F}(d) \\ \mathcal{F}(a) - 2\mathcal{F}(b) + 4\mathcal{F}(c) + \mathcal{F}(d) \\ \mathcal{F}(a) + 4\mathcal{F}(b) - 2\mathcal{F}(c) + \mathcal{F}(d) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

This mixing matrix has rank 1:
only the first linear combination enters the exponent.

<u>Cancellation of subdivergences</u>:
the leading, $\mathcal{O}(\epsilon^{-3})$ singularity, cancels in this linear combination: the ratio of coefficients is 2:1:1:2 as dictated by the number of ways of sequential shrinking.

Conclusions

- Diagrammatic exponentiation has been extended to the multi-parton case.
- Webs are formed by sets of (reducible) diagrams, related by permutations. Contributions to the exponent appear through mixing between kinematic and colour factors of the diagrams in the set.
- A general formula for the mixing matrices was derived using the replica trick.
- Mixing matrices have interesting mathematical properties: they are idempotent and have zero sum rows.
- Multi-parton webs involve particular linear combinations of kinematic dependence from different diagrams
    - ✓ In these certain subdivergences cancel, as dictated by renormalization of the multi-eikonal vertex.
    - ✓ The required cancellations provide important checks on multi-loop computations. The are being studied.