Gross-Neveu Condensates, Nonlinear Dirac Equations and Minimal Surfaces

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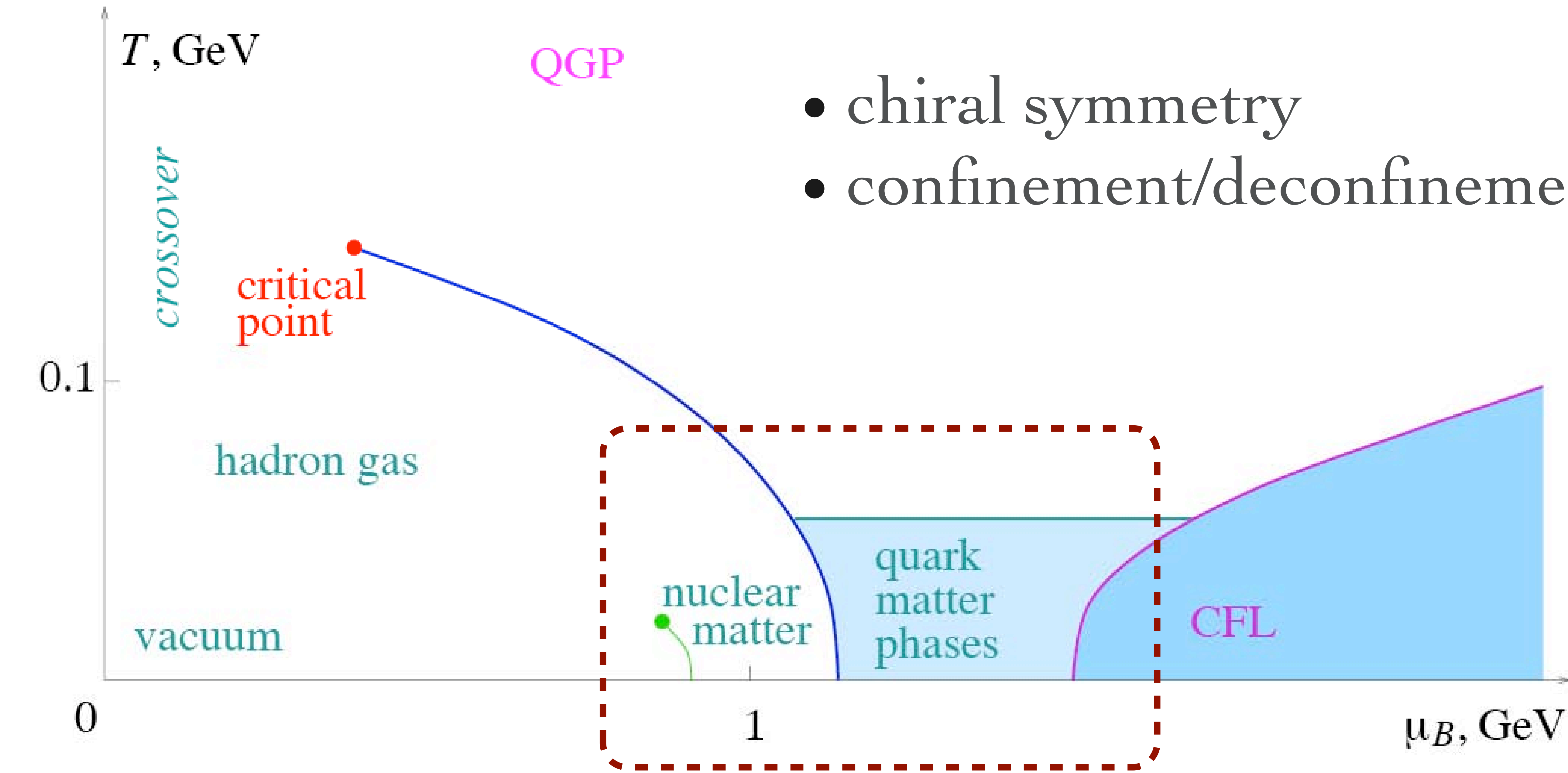
CAQCD 2011, Minnesota, May 2011

Başar, GD, Thies: PRD 2009, 0903.1868
Başar, GD: JHEP 2011, 1011.3835

motivation: phase diagram of QCD ?



- chiral symmetry
- confinement/deconfinement

Stephanov, 2009

Gross-Neveu Models

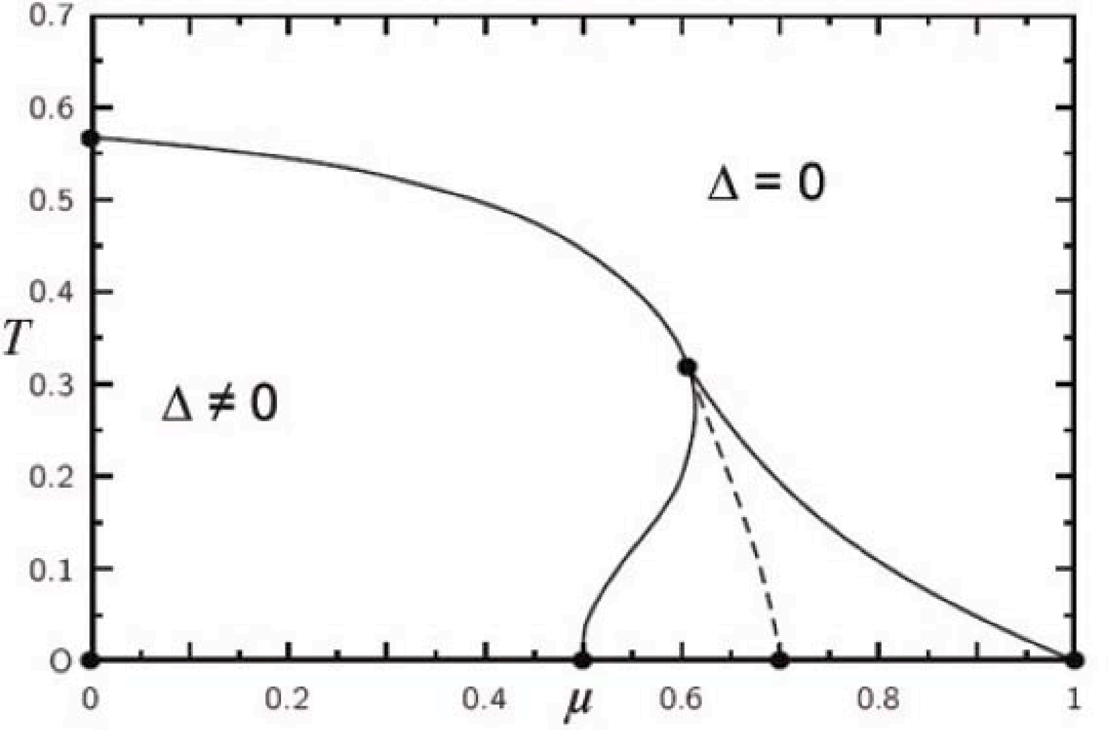
Gross/Neveu, 1974
Nambu/Jona-Lasinio, 1961

GN$_2$ $$\mathcal{L}_{\rm GN} = \bar\psi\, i\,\partial\!\!\!/\psi + \frac{g^2}{2}\left(\bar\psi\psi\right)^2 \qquad \psi \to \gamma^5\,\psi$$

χGN$_2$
NJL$_2$ $$\mathcal{L}_{\rm NJL} = \bar\psi\, i\,\partial\!\!\!/\psi + \frac{g^2}{2}\left[\left(\bar\psi\psi\right)^2 + \left(\bar\psi i\gamma^5\psi\right)^2\right] \qquad \psi \to e^{i\alpha\,\gamma^5}\,\psi$$

- renormalizable
- dynamical mass generation
- asymptotically free
- large N_f limit
- chiral symmetry breaking (discrete vs. continuous)

(T,μ) phase diagram?



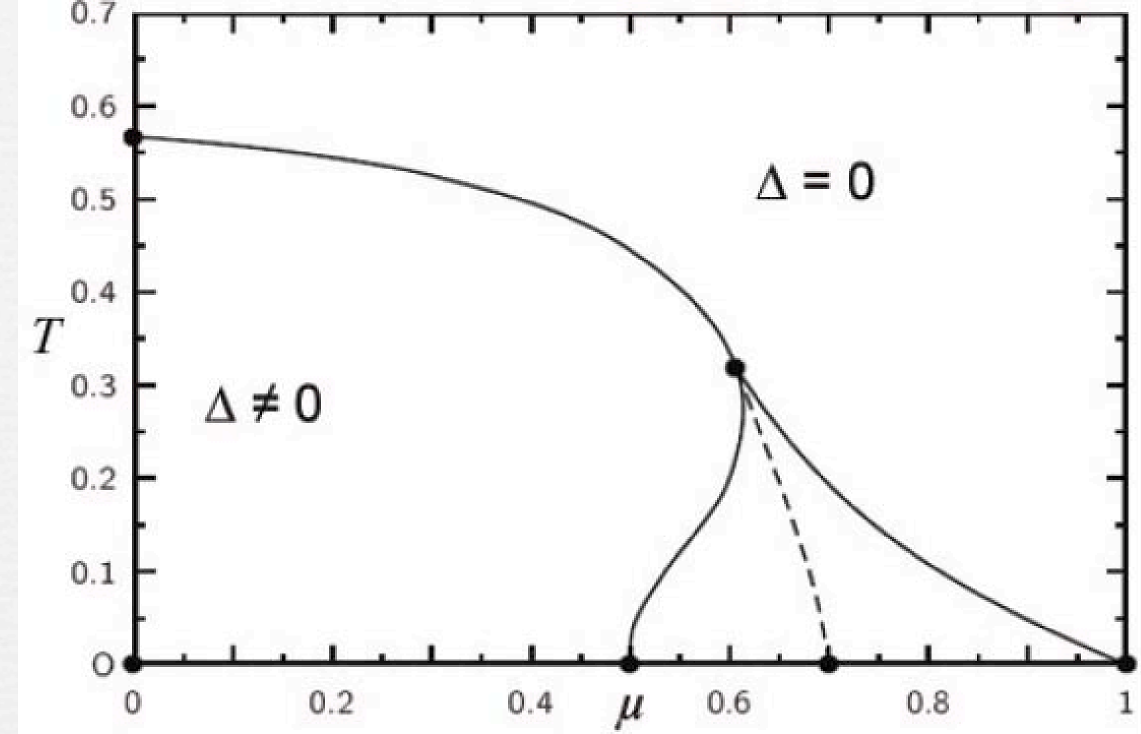
Wolff, 1985

uniform condensate

$$\sigma = \langle \bar{\psi}\psi \rangle$$

$$\pi = \langle \bar{\psi} i \gamma^5 \psi \rangle$$

$$\Delta = \sigma - i\pi$$

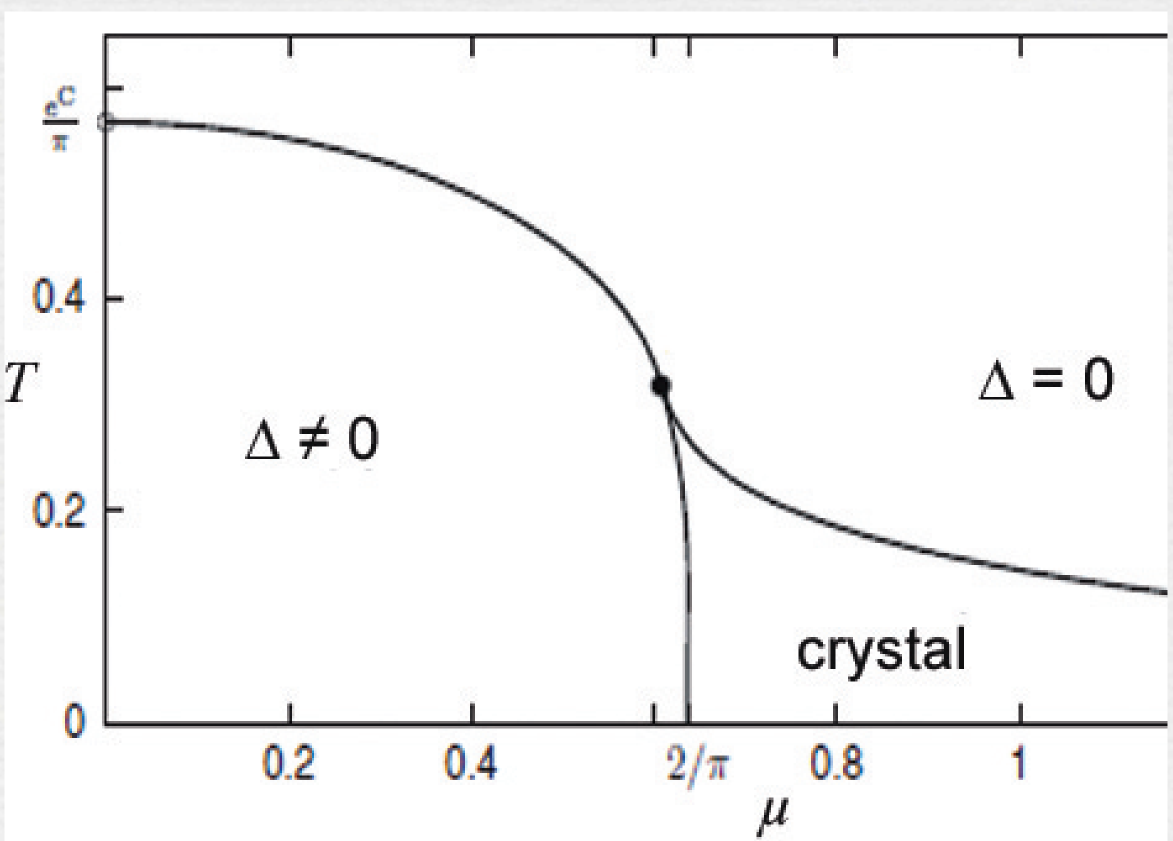


Barducci et al, 1995

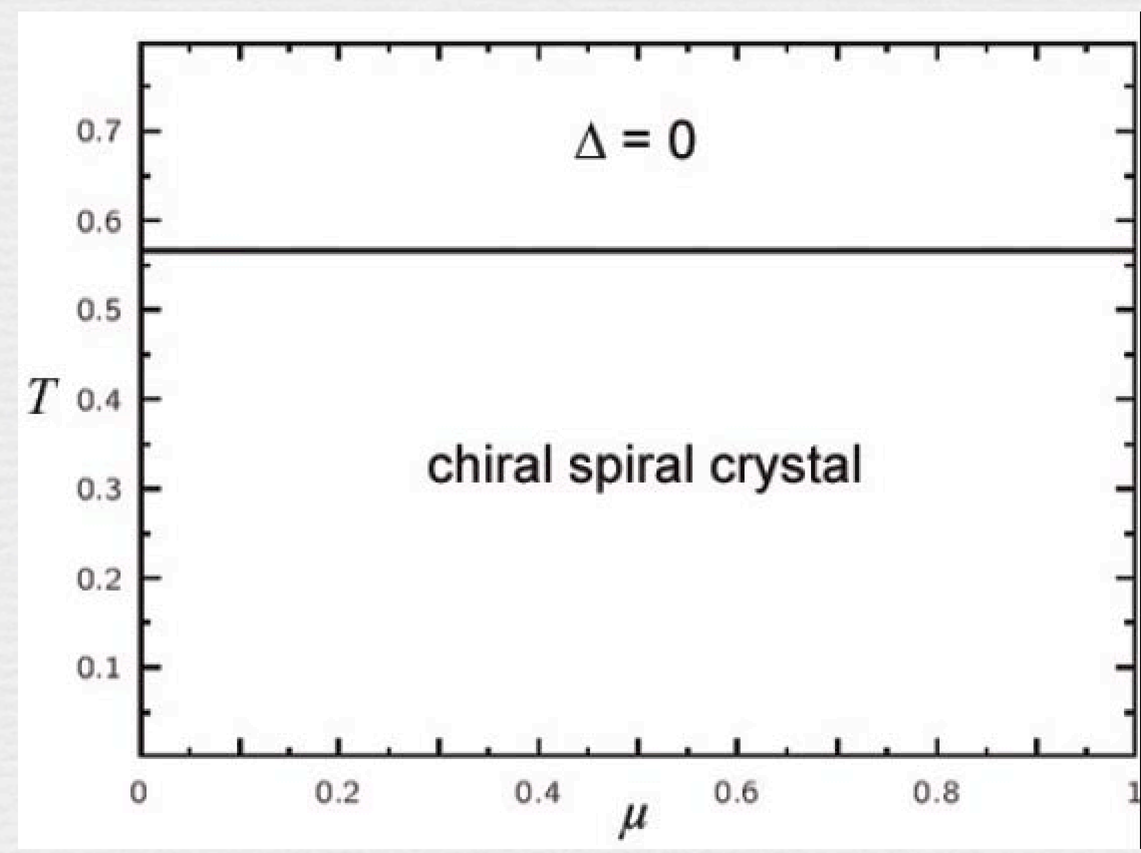
GN$_2$

NJL$_2$

inhomogeneous condensate



Thies & Urlichs, 2005



Başar, GD, Thies, 2009

Inhomogeneous Condensates

$$\sigma = \langle \bar{\psi}\psi \rangle$$

$$\pi = \langle \bar{\psi} i\gamma^5 \psi \rangle$$

$$\Delta = \sigma - i\pi$$

functional gap equation:

$$\sigma(x) = \frac{\delta}{\delta\sigma(x)} \ln\det\left(i\partial\!\!\!/ - \sigma(x)\right)$$

$$\Delta(x) = \frac{\delta}{\delta\Delta^*(x)} \ln\det\left(i\partial\!\!\!/ - \frac{1}{2}(1-\gamma^5)\Delta(x) - \frac{1}{2}(1+\gamma^5)\Delta^*(x)\right)$$

"inhomogeneous mean-field approximation"

Hartree-Fock : $\quad H\,\psi_E = E\,\psi_E$

$$H = \begin{pmatrix} -i\partial_x & \sigma(x) \\ \sigma(x) & i\partial_x \end{pmatrix} \quad \text{or} \quad H = \begin{pmatrix} -i\partial_x & \Delta(x) \\ \Delta^*(x) & i\partial_x \end{pmatrix}$$

such that

$$\sigma(x) = \mathrm{tr}_{E,T,\mu} \bar{\psi}_E(x)\psi_E(x)$$

$$\Delta(x) = \mathrm{tr}_{E,T,\mu} \left(\bar{\psi}_E(x)\psi_E(x) - i\bar{\psi}_E(x) i\gamma^5 \psi_E(x)\right)$$

zero temperature and density:

Dashen-Hasslacher-Neveu (1975): inverse scattering

single kink: $\sigma(x) = m \tanh(m\, x)$

"reflectionless" potential

Shei (1976): inverse scattering

twisted kink: $\Delta(x) = m \dfrac{\cosh\left(m \sin(\frac{\theta}{2})\, x - i\frac{\theta}{2}\right)}{\cosh\left(m \sin(\frac{\theta}{2})\, x\right)}$

"reflectionless" Dirac operator

nonzero temperature and finite density:

Thies, Urlichs (2005)

kink crystal: $\sigma(x) = m \sqrt{\nu}\, \mathrm{sn}(m\, x|\nu)$

"finite-gap" potential

Başar, GD (2008)

twisted kink crystal: $\Delta(x) = m \dfrac{\sigma(m\, x + i\mathbf{K}' - i\frac{\theta}{2})}{\sigma(m\, x + i\mathbf{K}')\, \sigma(i\frac{\theta}{2})} e^{iQx}$

"finite-gap" Dirac operator

solving the gap equation

$$\sigma = \langle \bar{\psi}\psi \rangle$$

$$\pi = \langle \bar{\psi} i \gamma^5 \psi \rangle$$

$$\Delta = \sigma - i\pi$$

$$\sigma(x) = \frac{\delta}{\delta\sigma(x)} \ln \det \left(i\partial\!\!\!/ - \sigma(x) \right)$$

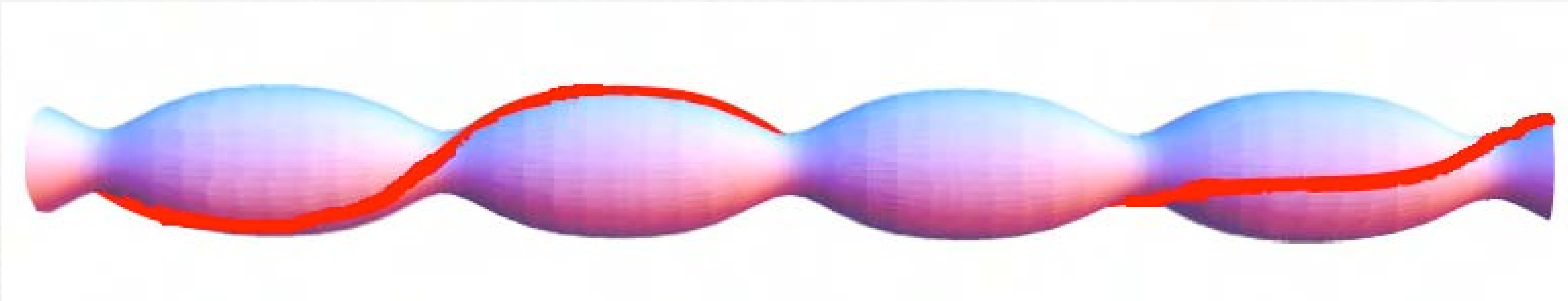
$$\Delta(x) = \frac{\delta}{\delta\Delta^*(x)} \ln \det \left(i\partial\!\!\!/ - \frac{1}{2}(1-\gamma^5)\Delta(x) - \frac{1}{2}(1+\gamma^5)\Delta^*(x) \right)$$

ansatz for Gorkov Green's function consistent with gap equation reduces gap equation to the (soluble) nonlinear Schrödinger equation:

$$\Delta'' - 2|\Delta|^2\,\Delta + i\,(b - 2E)\,\Delta' - 2\,(a - Eb)\,\Delta = 0$$

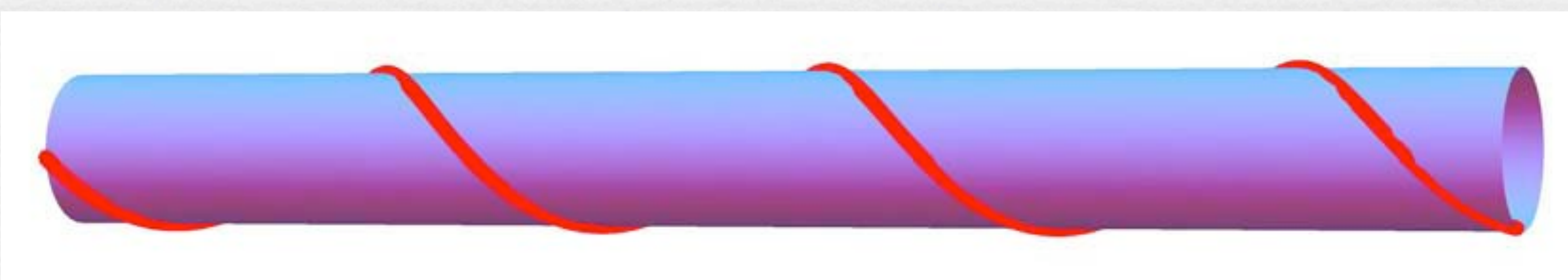
nonlinear Schrödinger equation (NLSE)

twisted kink crystal: general solution of NJL$_2$ gap equation



real kink crystal

spiral crystal

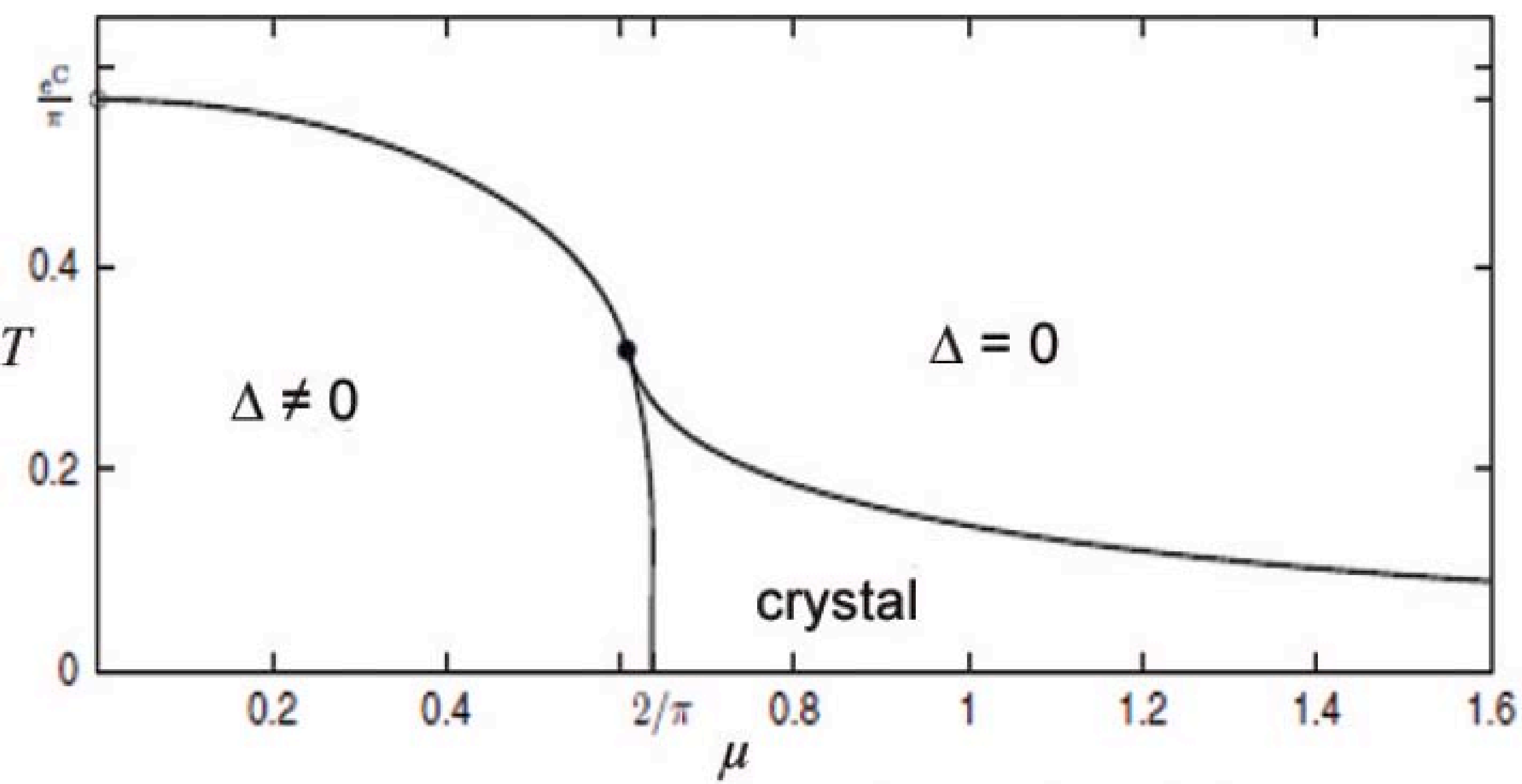


phase diagrams

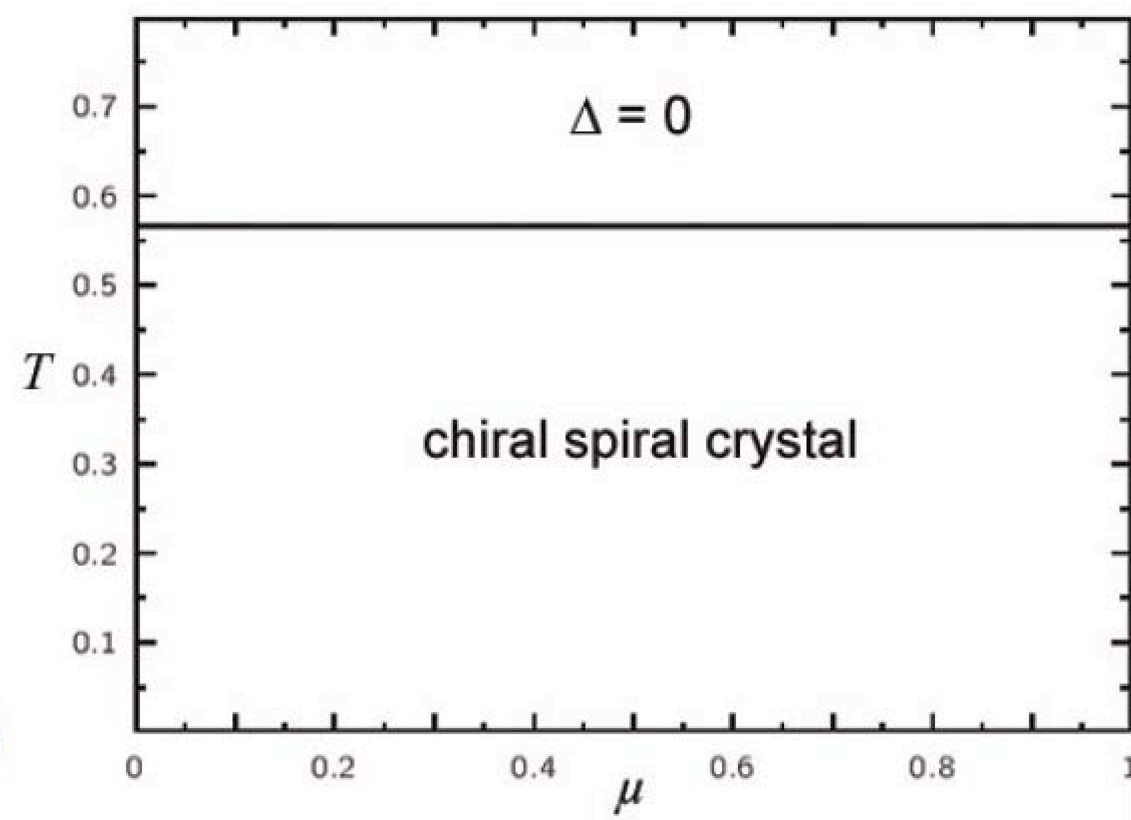
gap equation solution has 4 parameters

grand potential: $$\Psi = -\frac{1}{\beta} \int dE\, \rho(E)\, \ln\left(1 + e^{-\beta(E-\mu)}\right)$$

minimize Ψ w.r.t. parameters, as function of T and μ



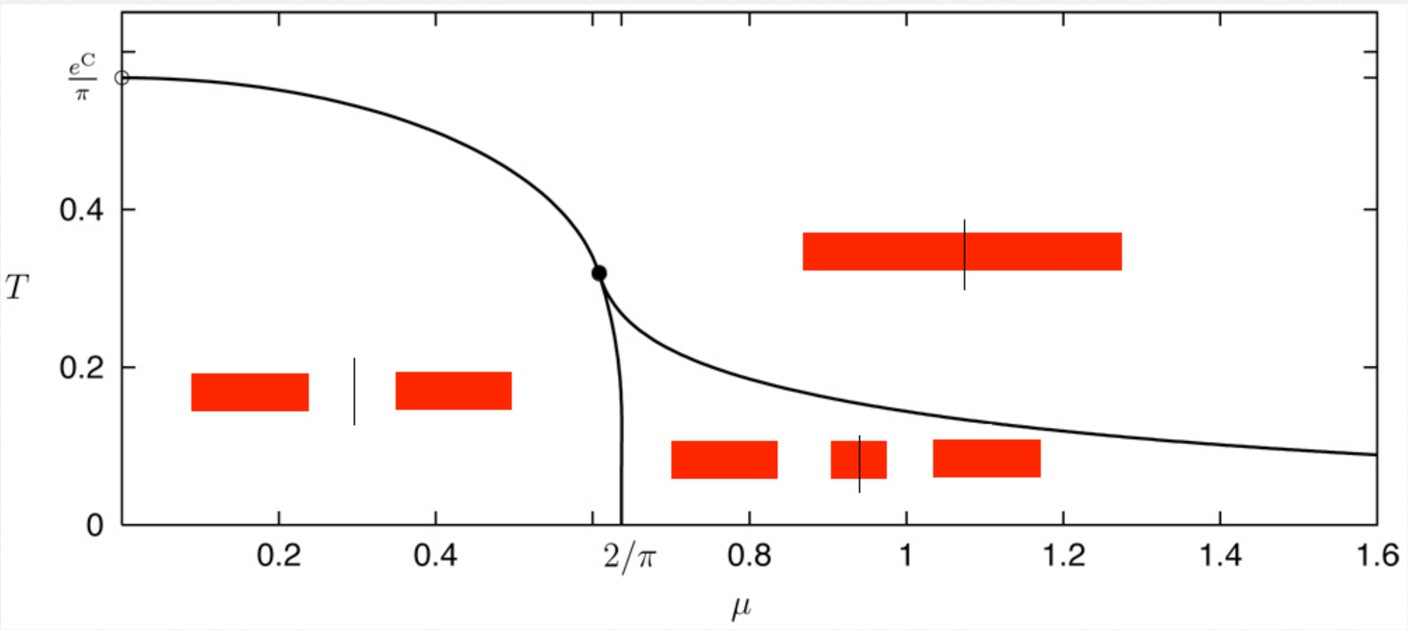
Thies & Urlichs, 2005



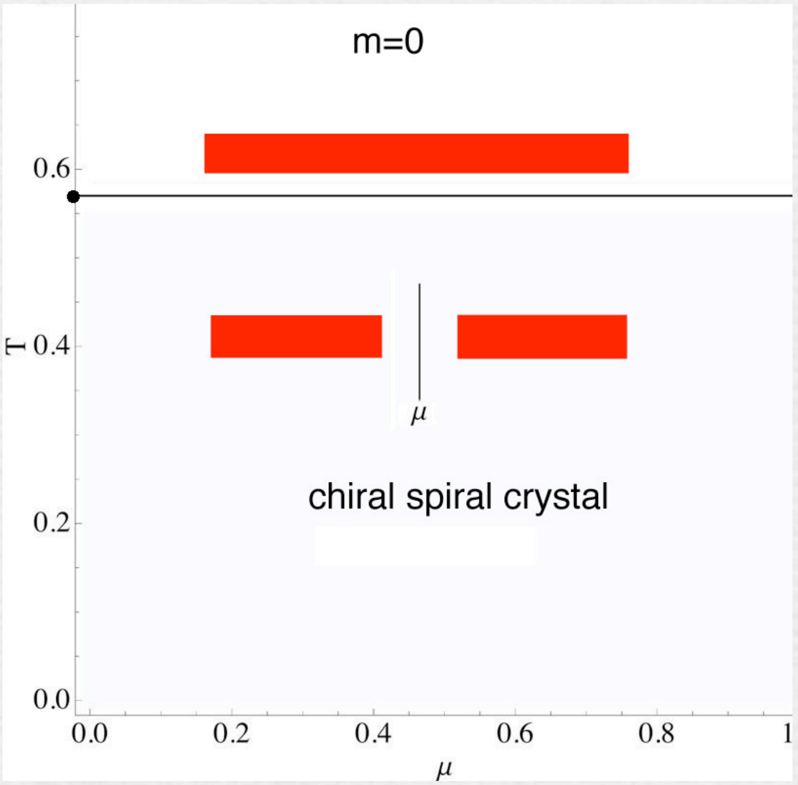
Başar, GD, Thies, 2009

physics: Peierls instability and chiral symmetry

Peierls: 1d system prefers to open gap at Fermi level



GN: discrete chiral symmetry
symmetric spectrum



NJL: continuous chiral symmetry
non-symmetric spectrum

all-orders Ginzburg-Landau expansion

$$\ln\det\left(i\partial\!\!\!/ - \sigma(x)\right) = \sum_n \alpha_n(T,\mu)\, a_n[\sigma(x)]$$

$$\ln\det\left(i\partial\!\!\!/ - \frac{1}{2}(1-\gamma^5)\Delta(x) - \frac{1}{2}(1+\gamma^5)\Delta^*(x)\right) = \sum_n \alpha_n(T,\mu)\, b_n[\Delta(x)]$$

Ginzburg-Landau expansion is an expansion
in the conserved quantities of the
mKdV (GN_2) and AKNS (NJL_2) integrable hierarchy

"magic" of integrable hierarchies:

Başar, GD, Thies, 2009

Correa, GD, Plyushchay, 2009

$$\frac{\delta a_n[\sigma]}{\delta\sigma(x)} = \gamma_n\,\sigma(x)$$

$$\frac{\delta b_n[\Delta]}{\delta\Delta^*(x)} = \tilde{\gamma}_n\,\Delta(x) + \tilde{\beta}_n\,\Delta'(x) \qquad \forall\ n$$

extension to higher dimensions?

“vortex crystal”, “Skyrme crystal”, “baryon crystal”, ...

1+1 dim. physics = Peierls instability
+ chiral symmetry breaking

1+1 dim. gap equation solutions are special:

1. "finite gap" ("periodic reflectionless")
2. Ginzburg-Landau = integrable hierarchies

extension to higher dimensions?

"vortex crystal", "Skyrme crystal", "baryon crystal", ...

a new perspective

1+1 dim. Gross-Neveu models and string theory

Kink dynamics, sinh-Gordon solitons and strings in AdS_3 from the Gross-Neveu model

Andreas Klotzek* and Michael Thies†

Hartree-Fock: $(i\partial\!\!\!/ - \sigma(x))\,\psi_k = 0$

$$\sigma(x) = \sum_k \bar{\psi}_k(x)\psi_k(x)$$

amazingly, “mode-by-mode” : $\bar{\psi}_k(x)\psi_k(x) = f(k)\,\sigma(x) \qquad \forall\, k$

$$\Downarrow$$

nonlinear Dirac equation : $(i\partial\!\!\!/ - l(k)\bar{\psi}_k(x)\psi_k(x))\,\psi_k = 0$

Neveu-Papanicolaou 1977,
Lund 1977

nonlinear Dirac equation :

$$\left(i\partial\!\!\!/ - l\,\bar{\psi}(x)\psi(x)\right)\psi = 0$$

bilinear : $\sigma(x) = \bar{\psi}(x)\psi(x)$

$$\sigma\sigma'' - (\sigma')^2 - \sigma^4 = -1$$

$\sigma = \bar{\psi}\psi = e^{\theta/2}$ $\Rightarrow$ 1 dim. Sinh-Gordon $-\theta'' + 4\sinh\theta = 0$

$$\Leftrightarrow$$

$\sigma'' - 2\sigma^3 + \sigma = 0$ NLSE

exact solutions to time-dependent Hartree-Fock

Hartree-Fock: $(i\partial\!\!\!/ - \sigma(x,t))\,\psi_k = 0$

$$\sigma(x,t) = \sum_k \bar{\psi}_k(x,t)\psi_k(x,t)$$

amazingly, "mode-by-mode" :

$$\bar{\psi}_k(x,t)\psi_k(x,t) = f(k)\,\sigma(x,t) \qquad \forall\, k$$

nonlinear Dirac equation :

$$\left(i\partial\!\!\!/ - l(k)\bar{\psi}_k(x,t)\psi_k(x,t)\right)\psi_k(x,t) = 0$$

$\sigma = \bar{\psi}\psi \equiv e^{\theta/2}$ $\quad\Rightarrow\quad$ 2 dim. Sinh-Gordon

$$\partial_\mu\partial^\mu\theta + 4\sinh\theta = 0$$

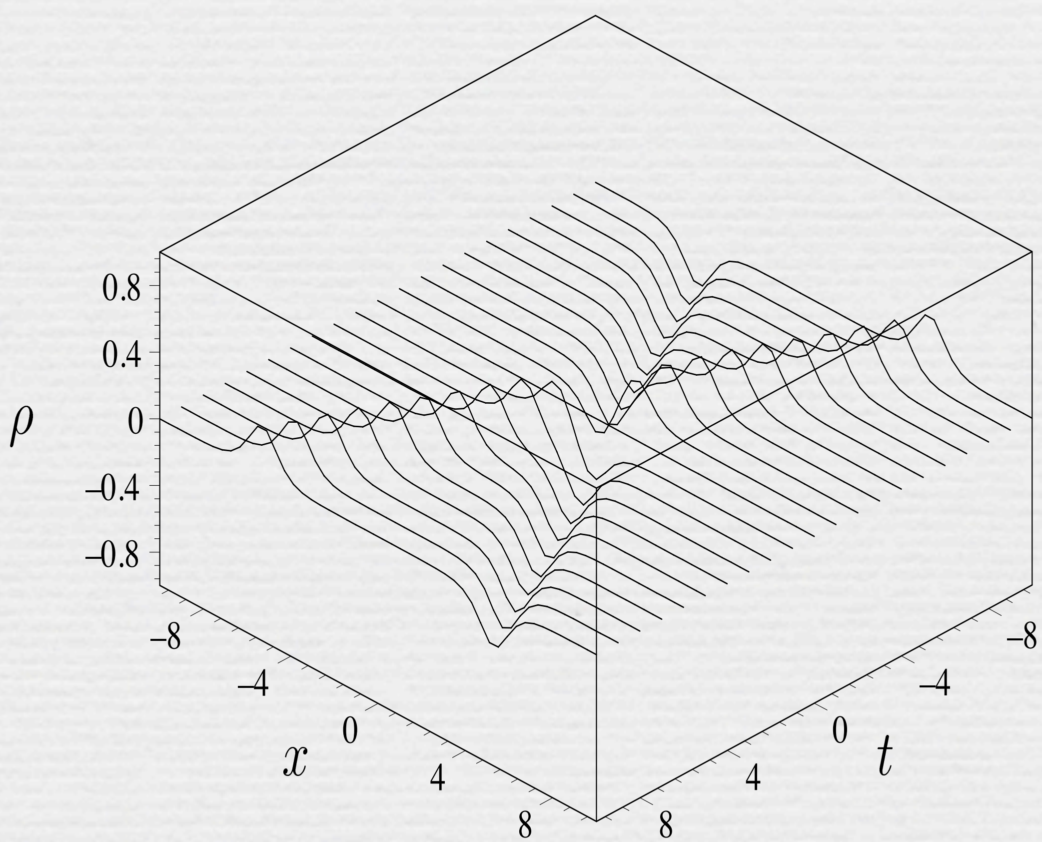
boosted solution: $\sigma(x) \to \sigma\left(\dfrac{x - vt}{\sqrt{1-v^2}}\right)$

breather solution: not real

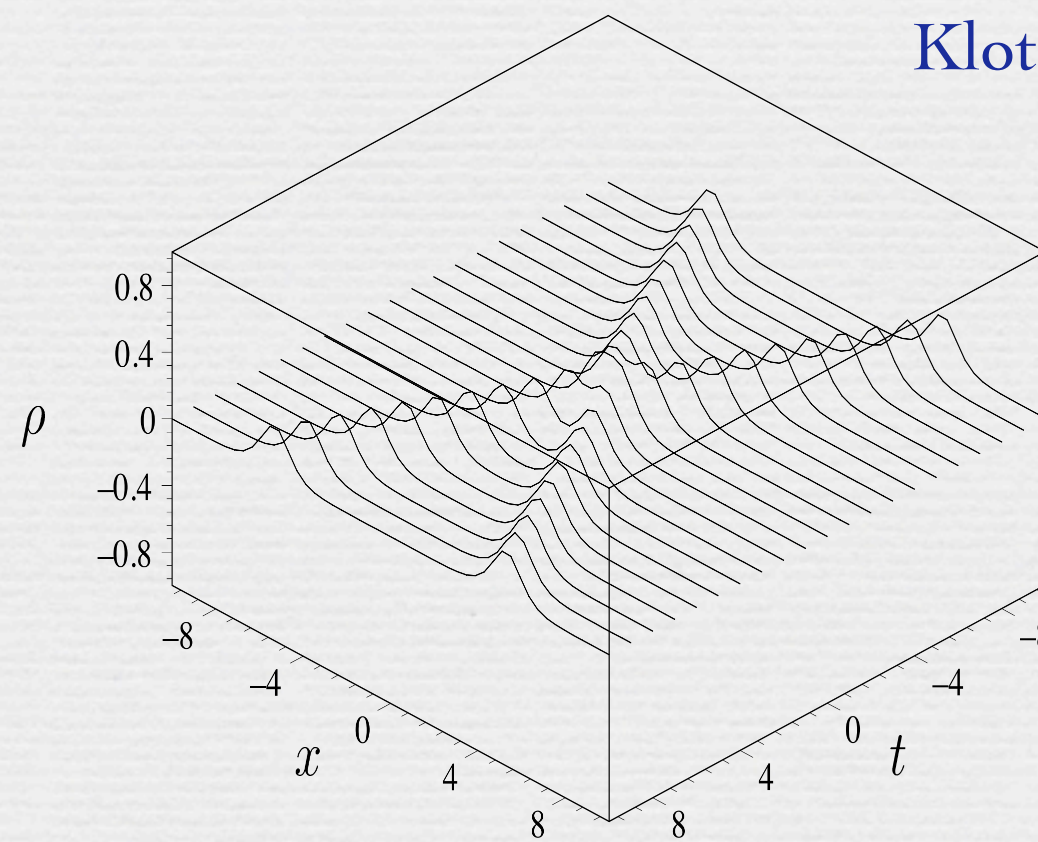
scattering solution :

$$\sigma(x,t) = \frac{v\cosh(2x/\sqrt{1-v^2}) - \cosh(2vt/\sqrt{1-v^2})}{v\cosh(2x/\sqrt{1-v^2}) + \cosh(2vt/\sqrt{1-v^2})}$$

Klotzek, Thies, 2010



baryon-antibaryon



baryon-baryon

this means we have a solution to the gap equation:

$$\sigma(x,t) = \frac{\delta}{\delta\sigma(x,t)} \ln\det\left(i\partial\!\!\!/ - \sigma(x,t)\right)$$

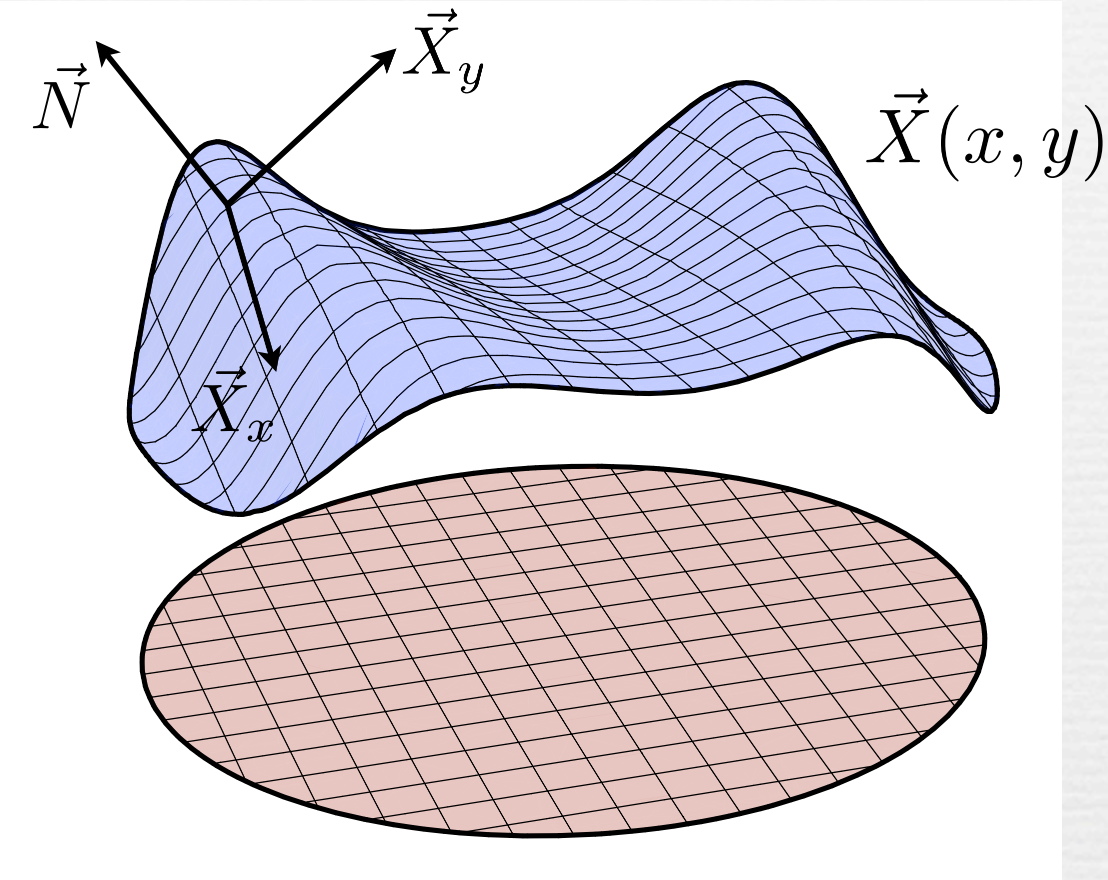
suggests we can find a solution to :

$$\sigma(x,y) = \frac{\delta}{\delta\sigma(x,y)} \ln\det\left(i\partial\!\!\!/ - \sigma(x,y)\right)$$

could represent a static 2d inhomogeneous condensate of the 2+1 dimensional Gross-Neveu model a geometric perspective ...

Başar, GD: JHEP 2011, 1011.3835

immersion of a surface in 3 dimensions



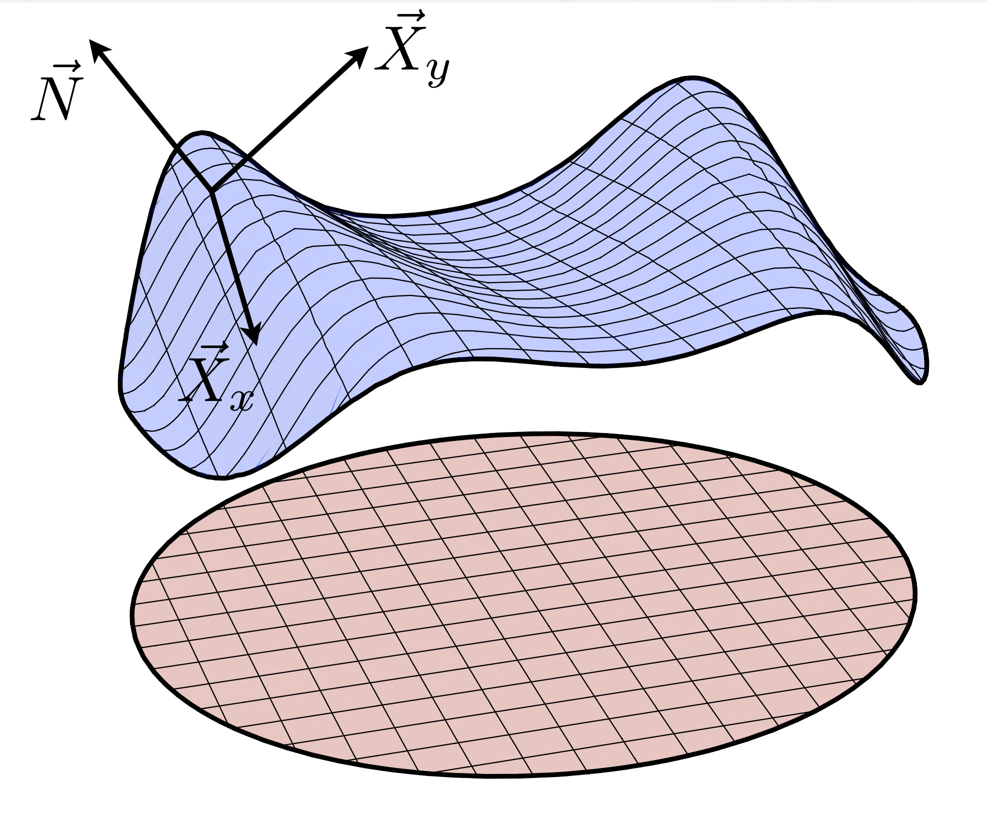
$$ds^2 = f^2(x_+, x_-)\, dx_+\, dx_-$$

H: mean curvature

Gauss-Codazzi equations

$$\begin{aligned} f f_{+-} - f_+ f_- - \frac{1}{4} H^2 f^4 &= -Q^{(+)} Q^{(-)} \\ Q_-^{(+)} &= \frac{1}{2} f^2 H_+ \\ Q_+^{(-)} &= \frac{1}{2} f^2 H_- \end{aligned}$$

spinor representation of surfaces



(Weierstrass, Enneper, Eisenhart, Hopf,
Neveu-Papanicolaou, Lund,
Bobenko, Gel'fand, Konopelchenko, ...)

$$SO(1,2) \sim SU(1,1)$$

$$\vec{X} = (X_1, X_2, X_3) \qquad \leftrightarrow \qquad X = -i \begin{pmatrix} X_3 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 \end{pmatrix}$$

$$\vec{X}_+, \quad \vec{X}_-, \quad \vec{N} \qquad \Rightarrow \qquad SU(1,1) \text{ spinors } \psi$$

Gauss-Codazzi equations

$$\begin{aligned} f f_{+-} - f_+ f_- - \frac{1}{4} H^2 f^4 &= -Q^{(+)} Q^{(-)} \\ Q^{(+)}_- &= \frac{1}{2} f^2 H_+ \\ Q^{(-)}_+ &= \frac{1}{2} f^2 H_- \end{aligned}$$

spinor representation:

Dirac equation: $(i\partial\!\!\!/ - S)\psi = 0$

induced metric factor: $f = \bar{\psi}\psi$

mean curvature: $S = H\,\bar{\psi}\psi$

Hopf differentials: $Q^{(+)} = -i(\psi_1^* \psi_{1,+} - \psi_{1,+}^* \psi_1)$

$Q^{(-)} = i(\psi_2^* \psi_{2,-} - \psi_{2,-}^* \psi_2)$

constant mean curvature: nonlinear Dirac equation

$$\left(i\partial\!\!\!/ - l\,\bar{\psi}(x)\psi(x)\right)\psi = 0$$

$f^2 = e^\theta$ $\Rightarrow$ Sinh-Gordon $\partial_\mu \partial^\mu \theta + 4\sinh\theta = 0$

constant mean curvature in flat $\mathbb{R}^{2,1}$ space

$\Leftrightarrow$

zero mean curvature in AdS_3 space

Pohlmeyer, Lund/Regge,
Barbashov et al, ...

explicit map between time-dependent solutions
to Gross-Neveu gap equation
&
classical string solutions in AdS_3

Klotzek, Thies, 2010

Başar, GD, 2010

suggests new geometrical approach to search for
inhomogeneous solutions to Gross-Neveu gap equations

geometric meaning of static inhomogeneous condensates in 1+1 dimensions for GN$_2$

immersion of curves into 3 dimensional space

Da Rios (1906),
student of Levi-Civita

"vortex filament equations"

SUL MOTO D'UN LIQUIDO INDEFINITO CON UN FILETTO VORTICOSO

che, per le (18) e (21), diventano:

$$-\frac{d\tau}{dt} - \left(\frac{c''}{c} - \tau^2\right)' = cc',$$

$$c'' = c'' - c\tau^2 + c\tau^2,$$

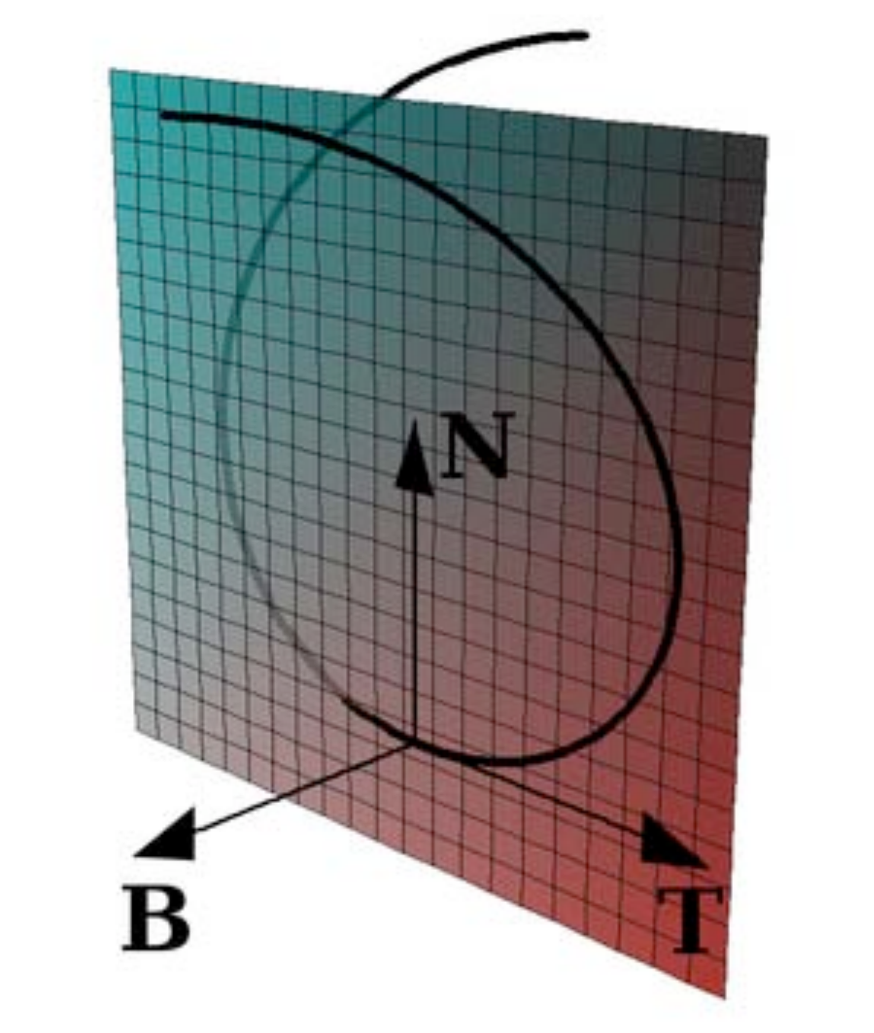
$$\frac{dc}{dt} = c\tau' + 2c'\tau.$$

Abbiamo quindi finalmente le equazioni cercate:

$$(22)\qquad \begin{cases} \dfrac{dc}{dt} = c\tau' + 2c'\tau, \\ \dfrac{d\tau}{dt} = -cc' + \left(\tau^2 - \dfrac{c''}{c}\right)'. \end{cases}$$

Il teorema di esistenza, applicato a questo sistema di equazioni a
-mette di asserire con tutto rigore che le funzioni $c(s, t)$, $\tau(s, t)$
regolarità) univocamente definite dai valori inizi

The intrinsic equations (22) as they were presented by Da Rios in his first paper published in 1906; c and τ stand for curvature and torsion of the vortex filament, respectively.



Frenet-Serret equations
for moving frame of curve
immersion can be written
as a Dirac equation

$$\frac{d}{ds}\begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

“potential” satisfies NLSE (equiv Sinh-Gordon in 1+1)

(spectral) deformations of these curves : mKdV hierarchy

mKdV governs thermodynamics of 1+1 GN model

proposal/conjecture :

Başar, GD, 2010

Gauss-Codazzi equations for surface embedding
can be written as a Dirac equation

solutions satisfy Sinh-Gordon

(spectral) deformations of these surfaces :
(m) Novikov-Veselov hierarchy

question: does mNV govern the thermodynamics
of 2+1 dimensional GN model ?

lowest nontrivial equation of mKdV :

$$\Delta'' - 2|\Delta|^2\Delta = \nu\,\Delta$$

lowest nontrivial equation of mNV :

$$\nabla^2\Delta - \left[\left(\frac{\partial}{\bar{\partial}} + \frac{\bar{\partial}}{\partial}\right)|\Delta|^2\right]\Delta = \nu\,\Delta$$

TWO-DIMENSIONAL SCHRÖDINGER OPERATOR: INVERSE SCATTERING TRANSFORM AND EVOLUTIONAL EQUATIONS

1986

S. P. NOVIKOV AND A. P. VESELOV

VESELOV–NOVIKOV EQUATION AS A NATURAL TWO-DIMENSIONAL GENERALIZATION OF THE KORTEWEG–DE VRIES EQUATION

1987

L. V. Bogdanov

The Miura transformation between KdV and MKdV solutions is generalized to the two-dimensional case. An integrable equation associated with the two-dimensional Dirac operator — the modified Veselov–Novikov equation — is introduced.

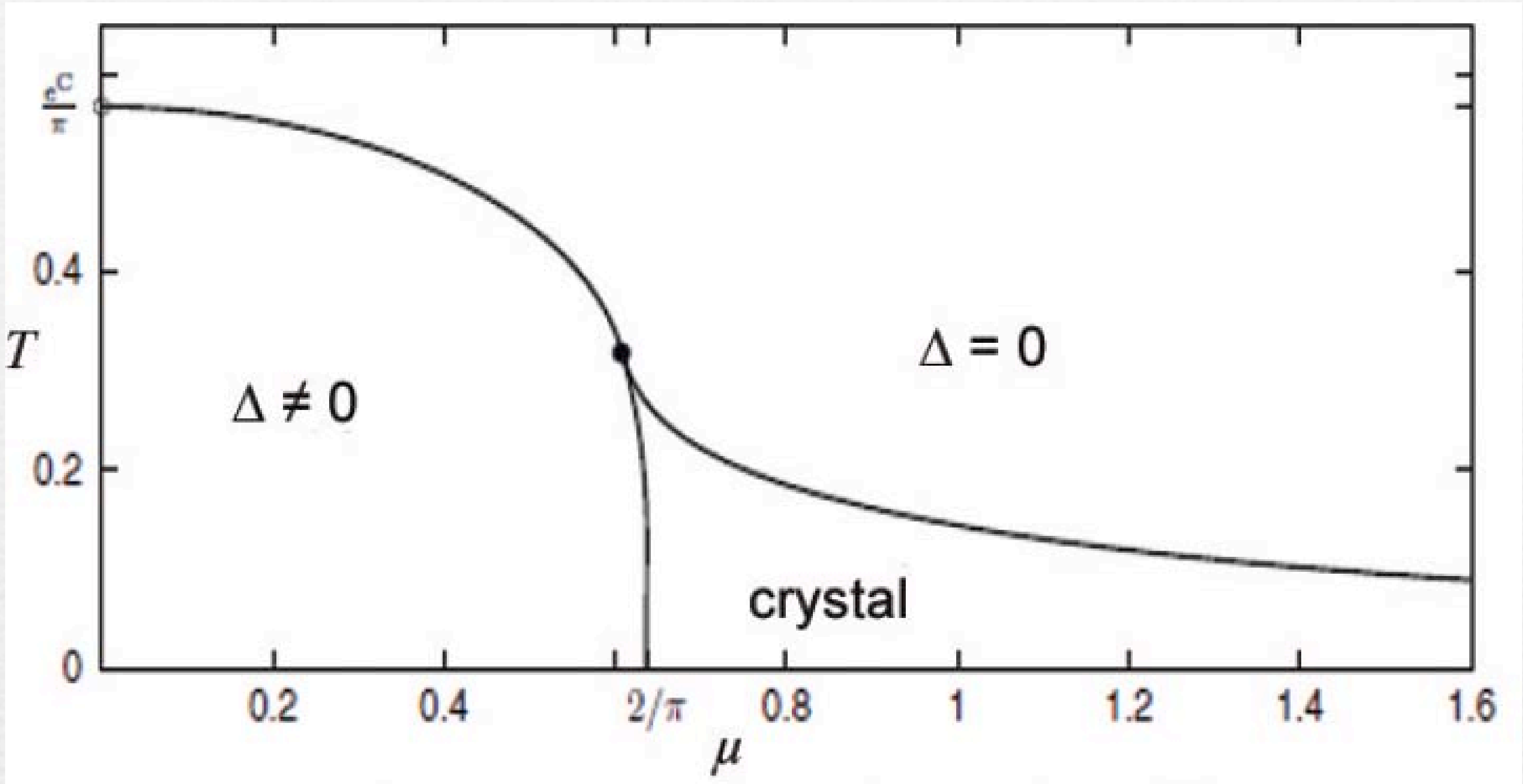
localized
“dromion”
solutions

Conclusions

- gap equation for chiral GN/NJL$_2$
- full, exact, thermodynamics & phase diagrams
- physics = Peierls instability + chiral symmetry breaking
- Ginzburg-Landau expansion = mKdV or AKNS hierarchy
- geometric picture = curve and surface embedding
- higher dimensional models : Novikov-Veselov hierarchy ?

  Peierls instability ?

  surface emeddings ?



$\frac{e^C}{\pi}$
0.4
0.2
0
T
$\Delta \neq 0$
$\Delta = 0$
crystal
0.2
0.4
$2/\pi$
0.8
1
1.2
1.4
1.6
μ