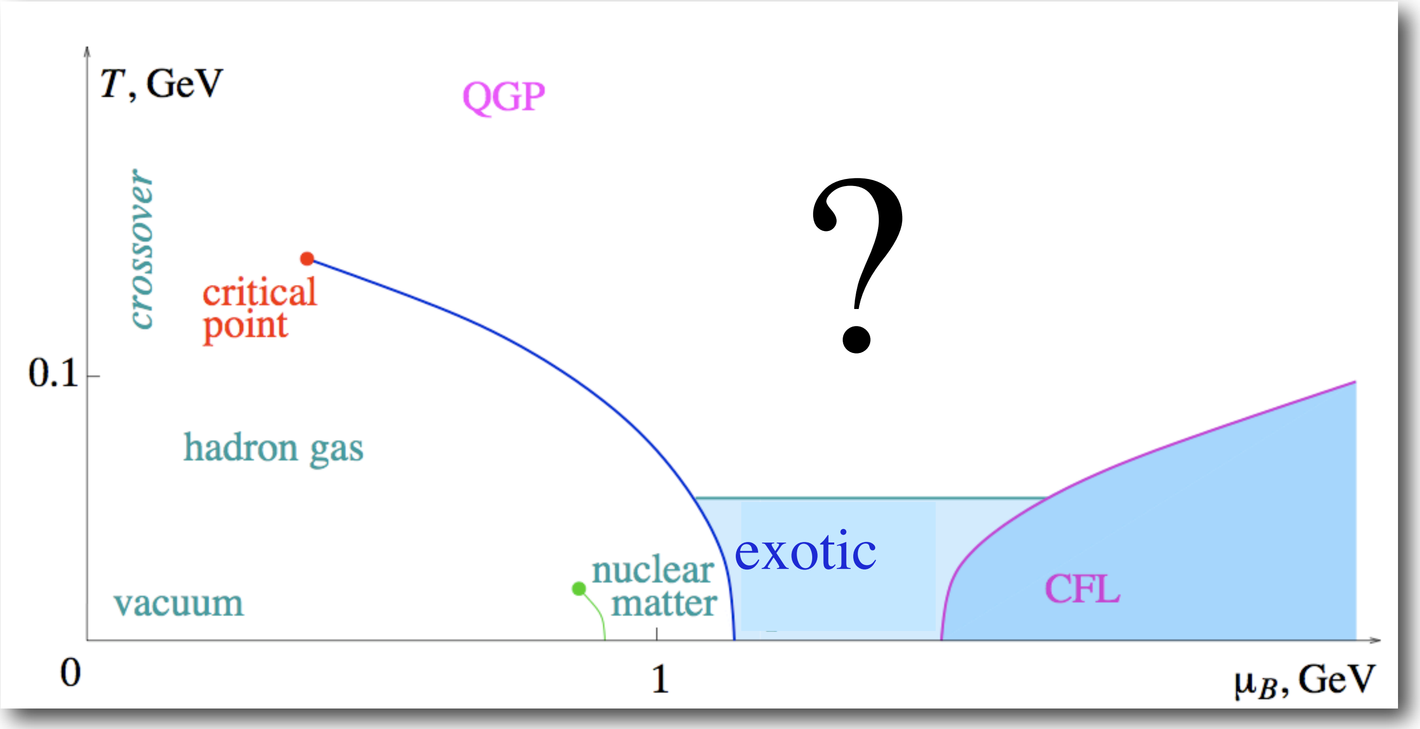
NOISE

A major challenge for physics: computing properties of strongly interacting dense matter from theory.
The obstacle: sign problems & noise in simulations.

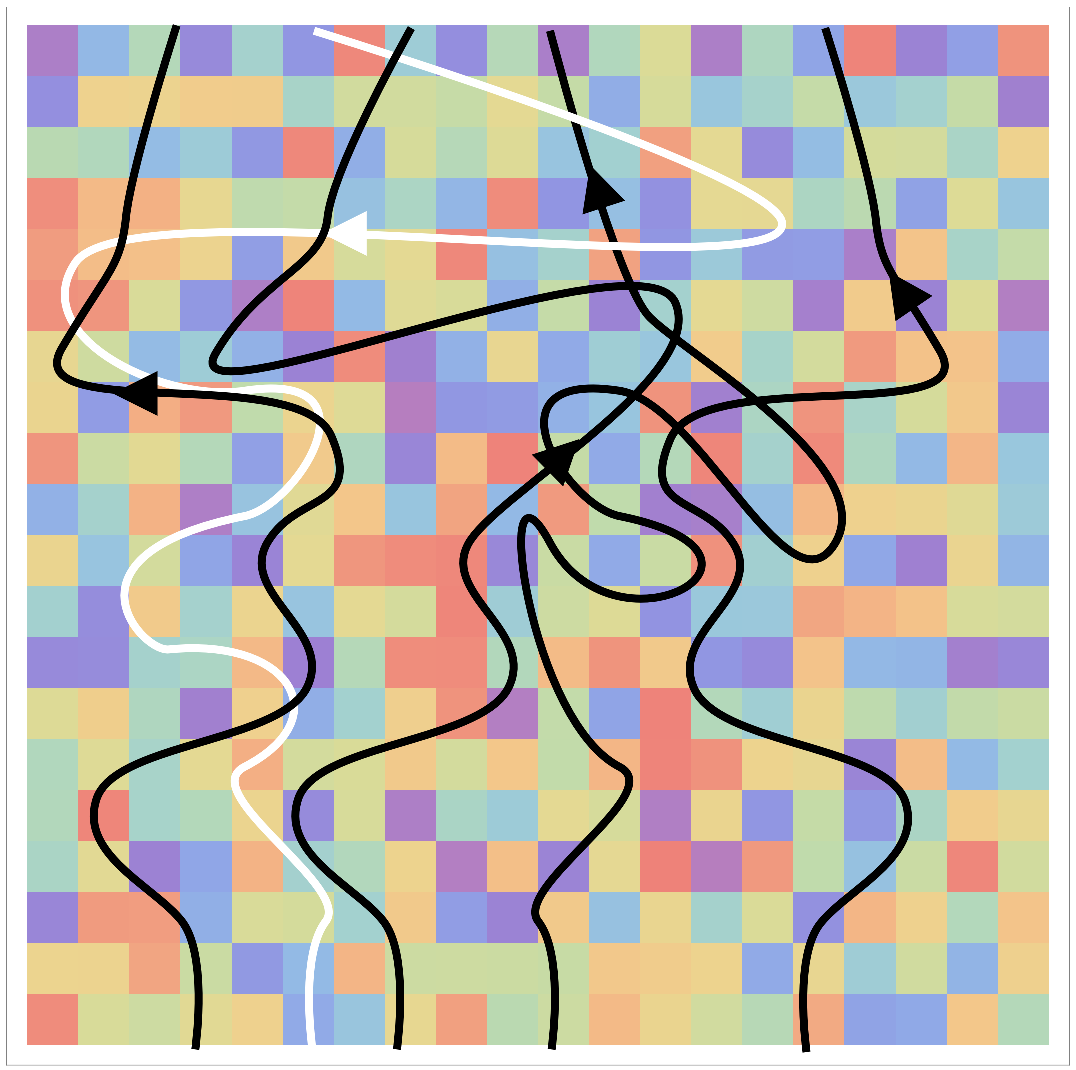
In QCD would like to map out phase diagram, but hit sign problem at nonzero μ_B...



...or noise problems in canonical simulations

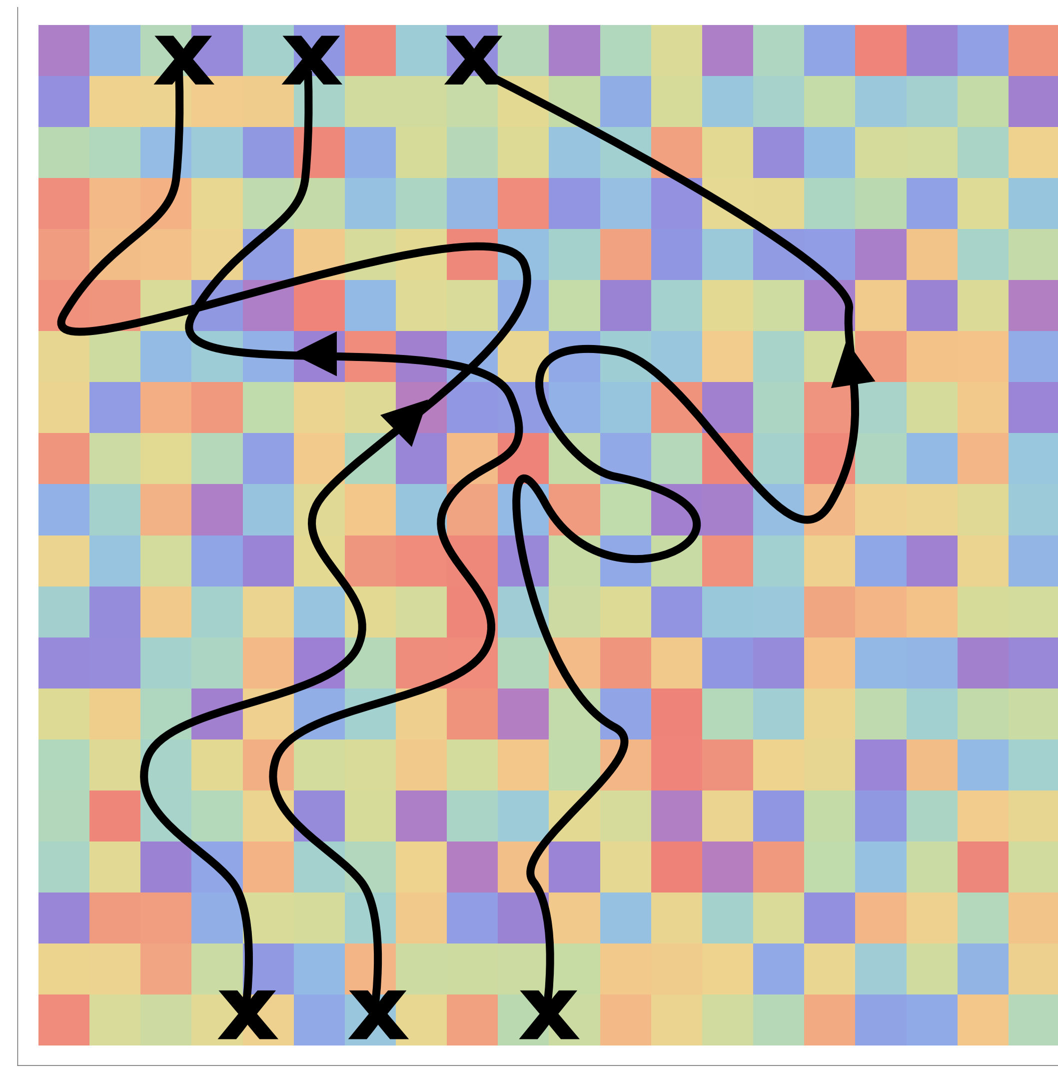
What is the sign/noise problem?

Sign Problem

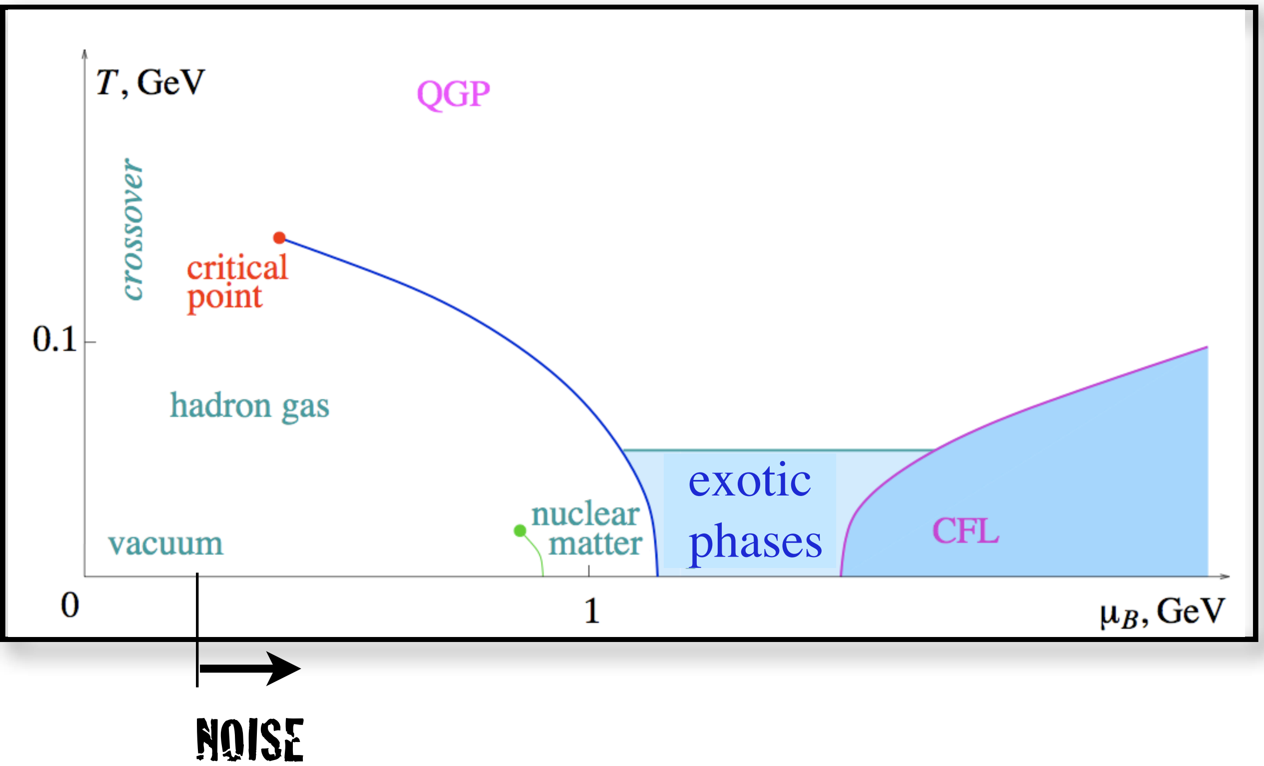


Grand canonical: exponentially difficult in volume V

Signal/Noise Problem

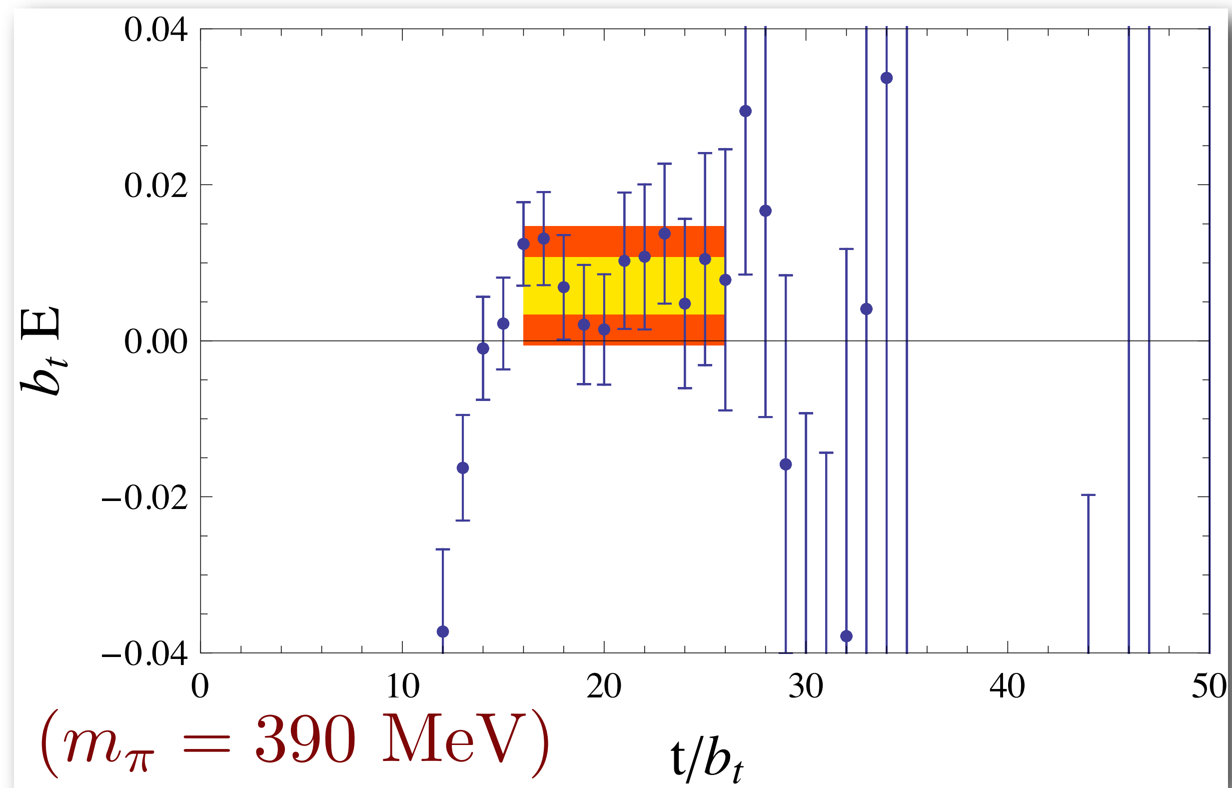


Canonical: exponentially bad S/N in Euclidian time t



Triton B.E.

S. R. Beane et al. (NPLQCD), Phys. Rev. D 80, 074501 (2009)



The obstacle:

Grand canonical - “fermion sign problem” hits you for $\mu_B > 3m_\pi/2$ (T=0)

Canonical (fixed B) - noise in nucleon correlator set by scale $3m_\pi/2$

This talk:

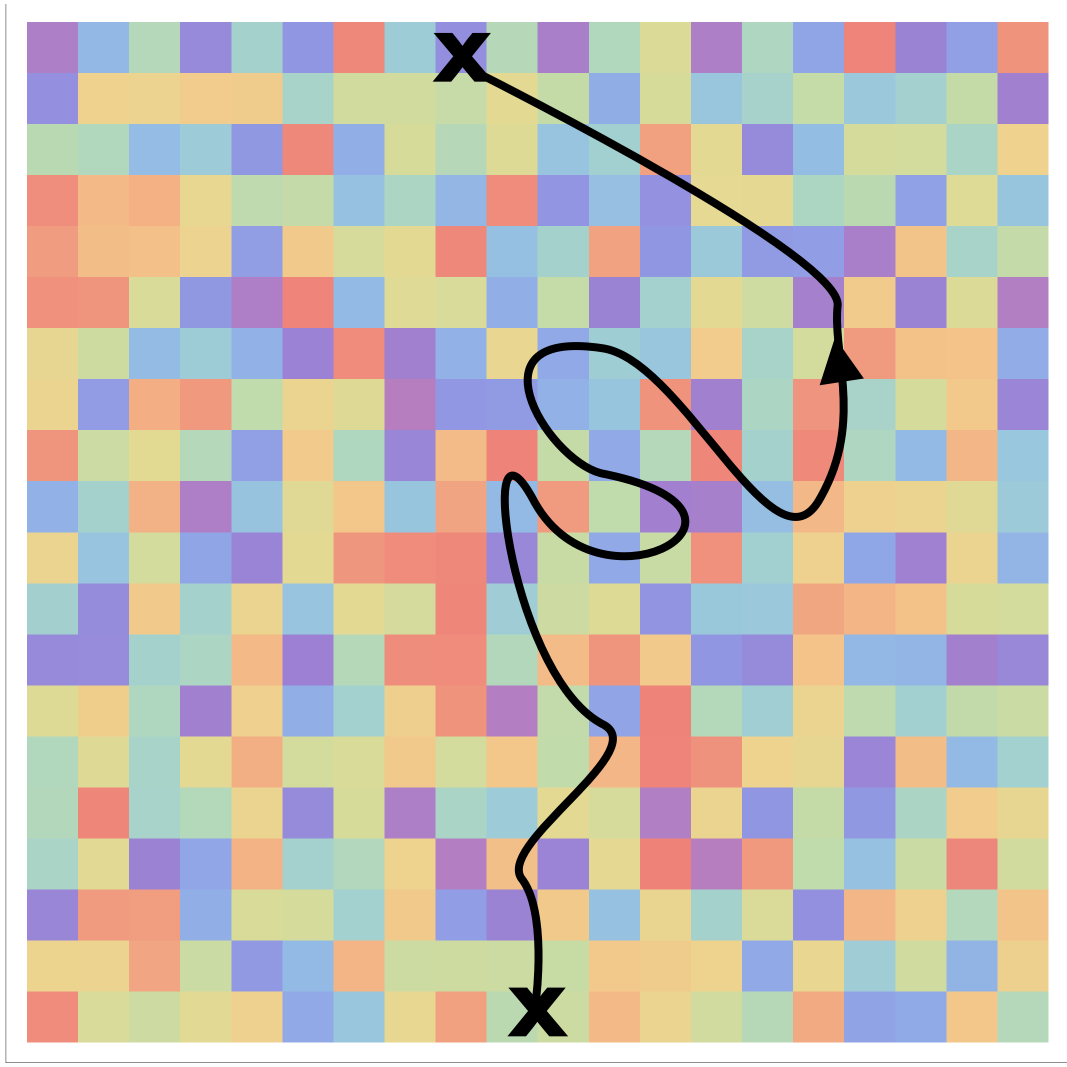
- What is the nature of noise in a noisy simulation?
- Could there be more effective ways to deal with it?
- Connections between noise, renormalization group and properties of disordered media

with:

Michael Endres
Jong-Wan Lee
Amy Nicholson

S/N ~ sign problem:

In background gauge field, quarks don't know about each other.



Is the quark in a pion?

⇒ big correlation after long time

Is the quark in a baryon?

⇒ exponentially smaller correlation

(quarks "weigh more" in a baryon)

What is a quark to do??

S/N ~ sign problem:

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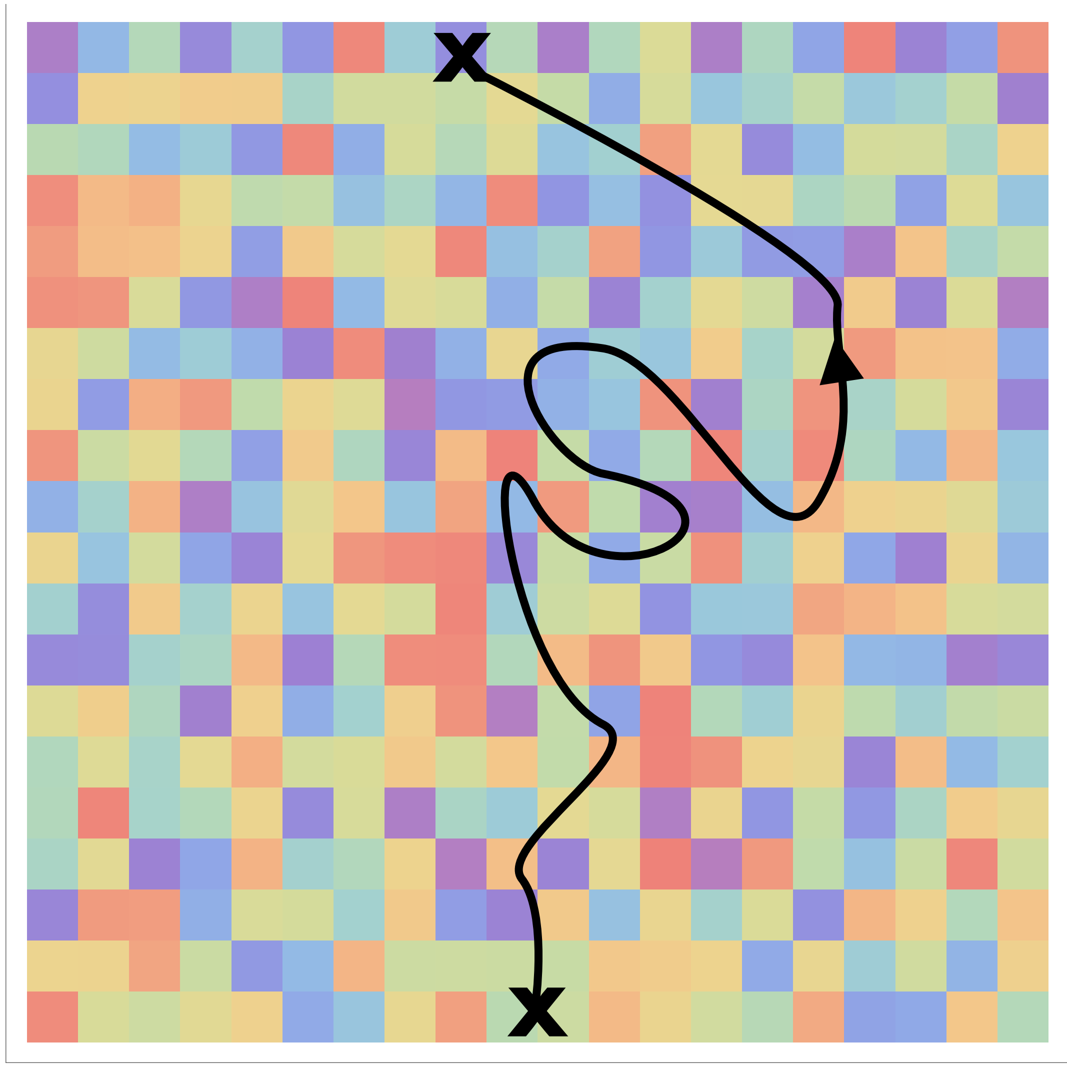

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What is a quark to do??

How lattice QCD solves this problem:

- Every quark has long correlation in a background gauge field

  "M_q" ~ $M_\pi/2$

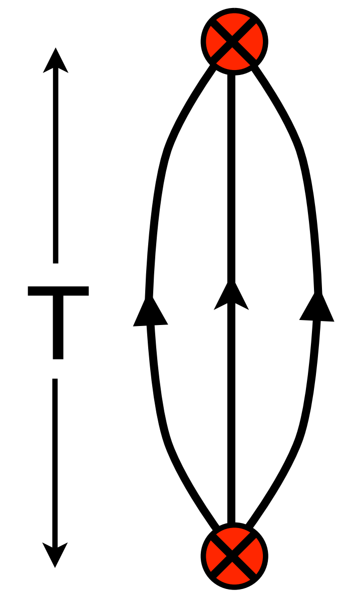
- If the quark is in a baryon, EXPONENTIAL CANCELLATIONS when averaging gauge fields (since $M_B > 3\, M_\pi/2$)

Lepage argument for the signal/noise problem

e.g: measuring the nucleon mass in LQCD:

$$\langle C \rangle = \frac{1}{N} \sum_{\{A\}} C(A) \propto e^{-\boxed{M}T} + \dots$$

nucleon:
lightest 3q state



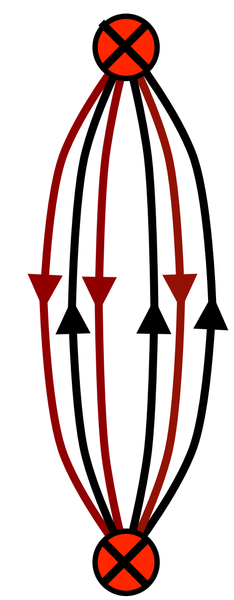
$C(A)$

Dispersion in measurement:

3 anti-quarks
3 quarks

$$\sigma^2 = \langle C^\dagger C \rangle = \frac{1}{N} \sum_{\{A\}} C^\dagger(A) C(A) \propto e^{-\boxed{3m_\pi}T} + \dots$$

3π: lightest 3q+ 3q* state

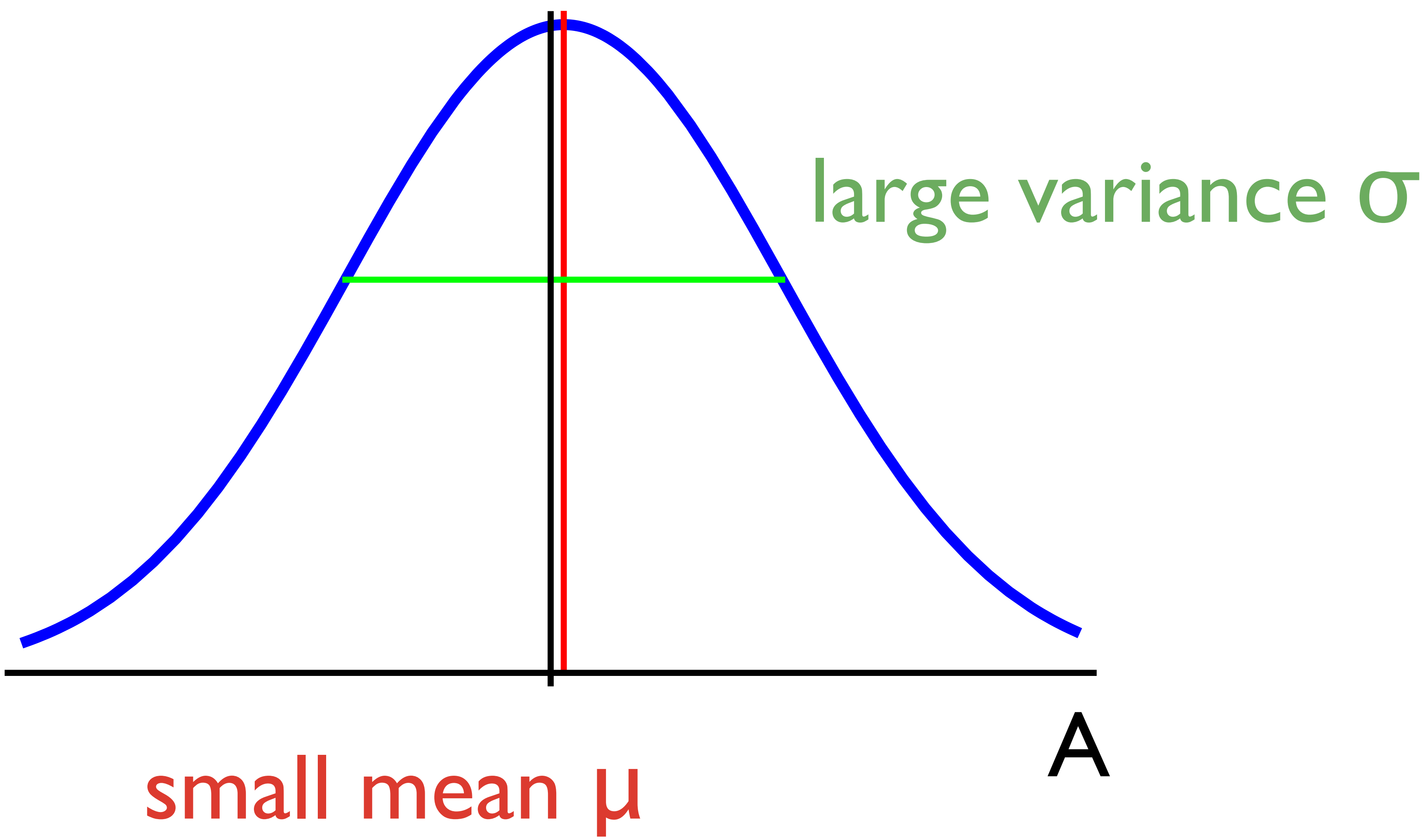


$C^\dagger(A)C(A)$

$$\frac{\text{signal}}{\text{noise}} \propto \frac{1}{\sqrt{N}} e^{-\left(M - \frac{3}{2} m_\pi\right)T}$$

Basically same picture, with caveats:
"N"? Gaussian distribution?

Lepage picture for distribution of correlator for large baryon number, $C_B(t, A)$

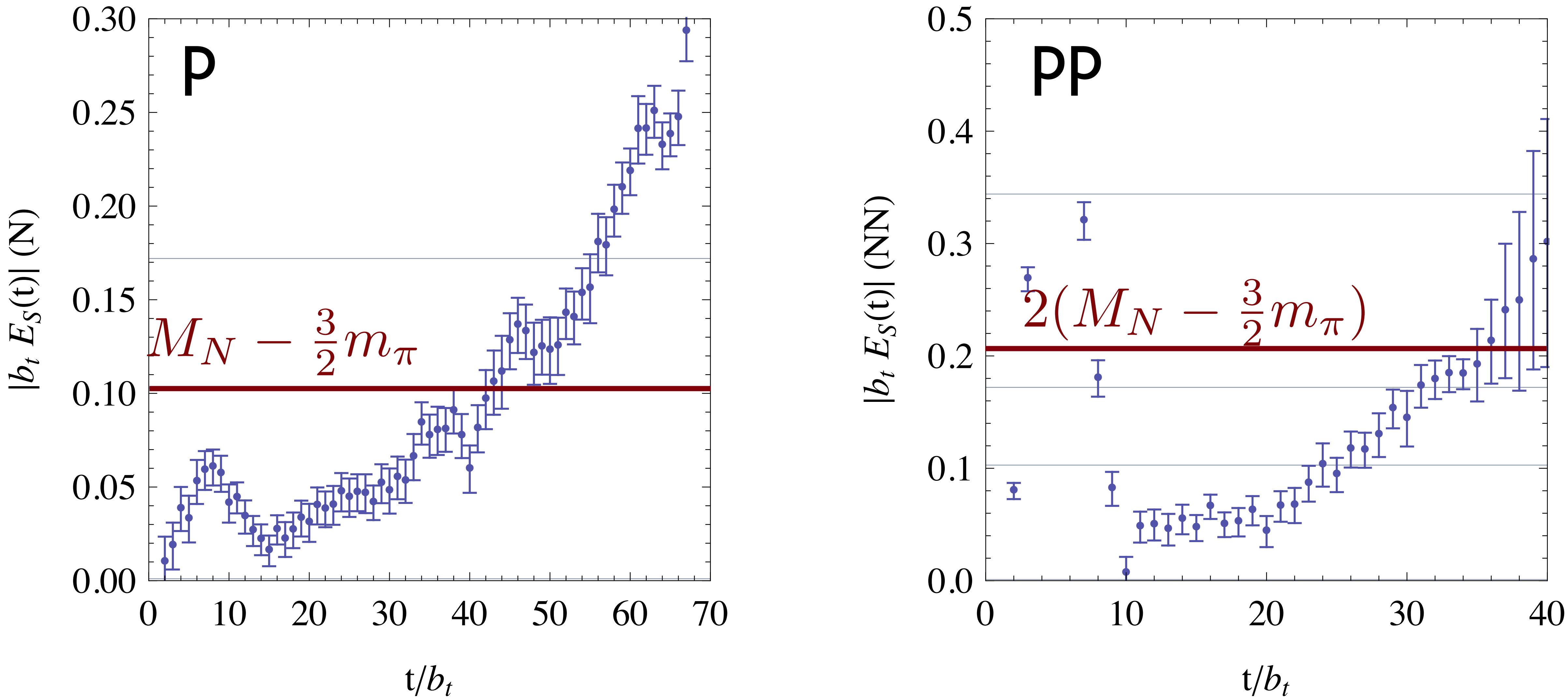


Eg, we're screwed

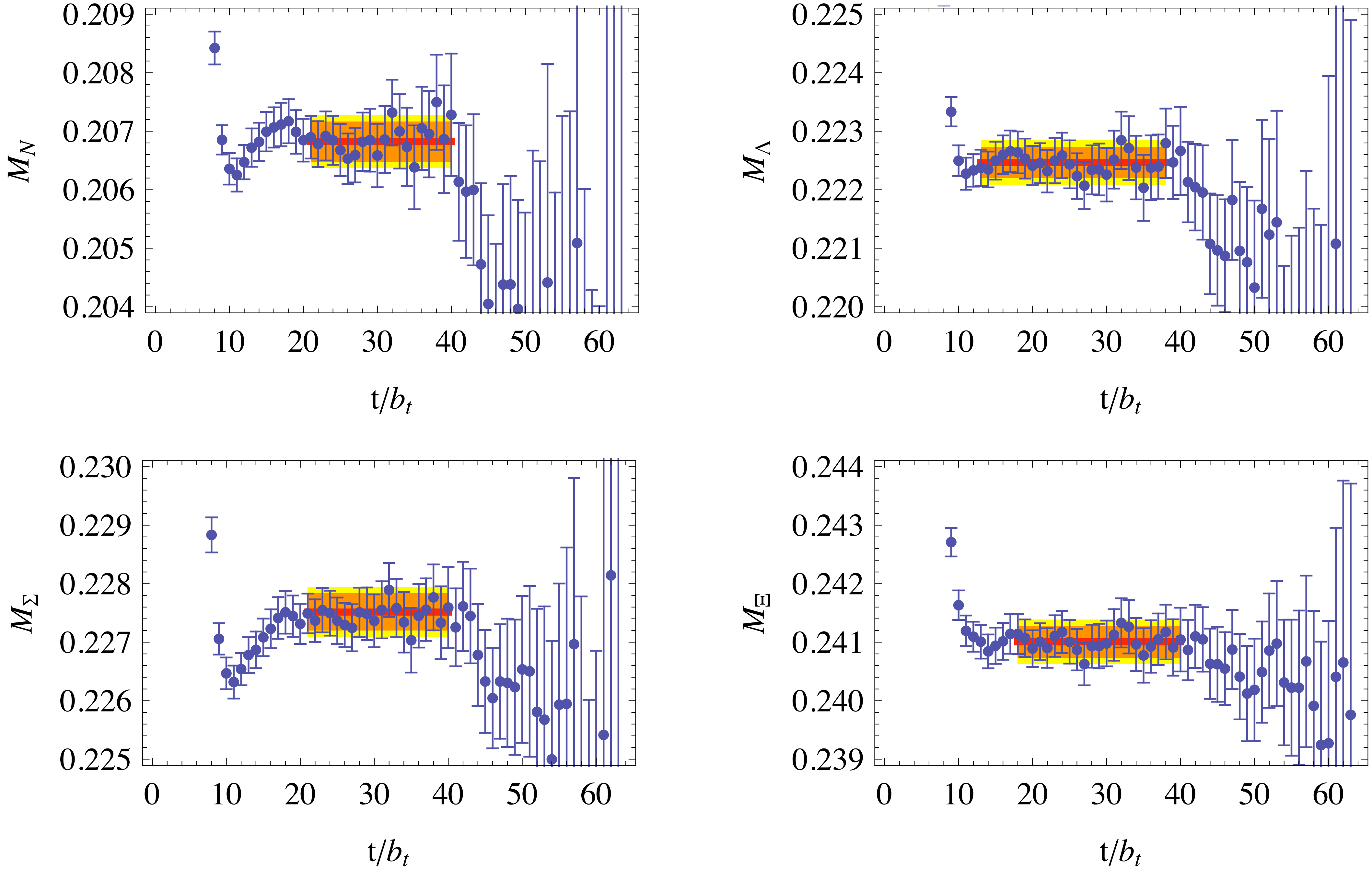
Actual data

S. R. Beane et al. (NPLQCD), Prog.Part.Nucl.Phys. 66 (2011) 1-40

Noise/signal: $\mathcal{S}(t) = \dfrac{\sigma(t)}{\bar{x}(t)}$ $\qquad E_s = -\dfrac{1}{t}\ln \mathcal{S}(t)$



Lepage argument give *rough* idea of noise problem



Example of high precision baryon octet effective mass plots

S. R. Beane et al., (NPLQCD) Phys. Rev. D 79, 114502 (2009)

Simpler system to analyze: "unitary fermions"

(Endres, Lee, Nicholson, DK, to appear)

What are unitary fermions?

Nonrelativistic 2-particle scattering:

$$\mathcal{A} = \frac{4\pi}{M} \frac{1}{p\cot\delta - ip} \qquad p\cot\delta = -\frac{1}{a} + \frac{1}{2} r_0 p^2 + O(p^4)$$

phase shift

$$p = \sqrt{ME}$$

scattering length

effective range

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phase shift

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scattering length

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"unitary" fermions: $p\cot\delta = 0$

$$\delta = 90^\circ$$

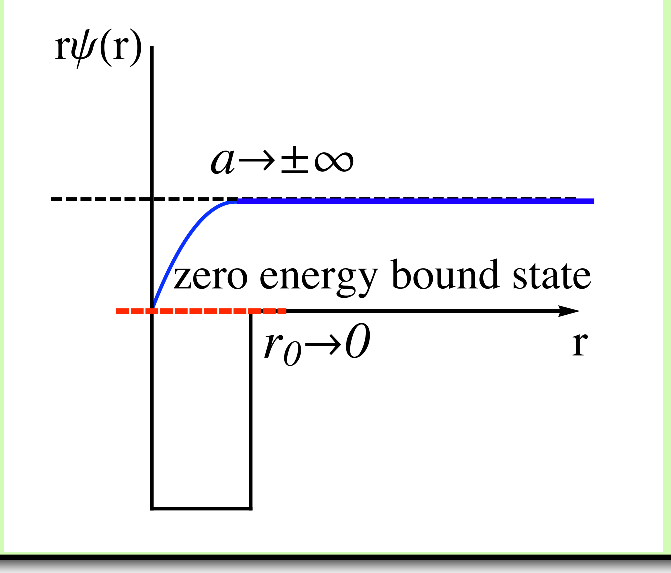
$$a = \infty$$

$$r_0 = 0\ldots$$

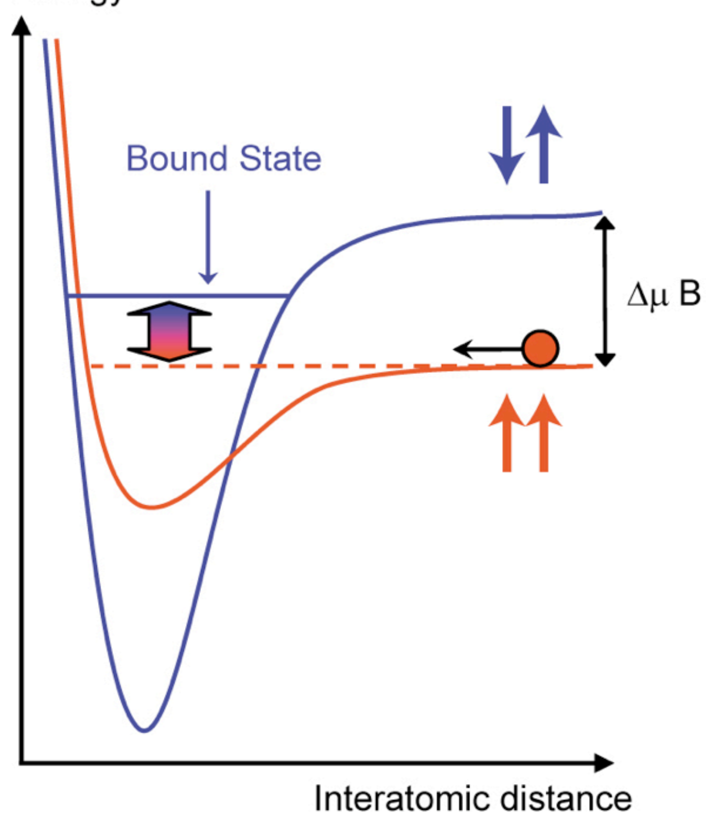
A strongly-coupled conformal system

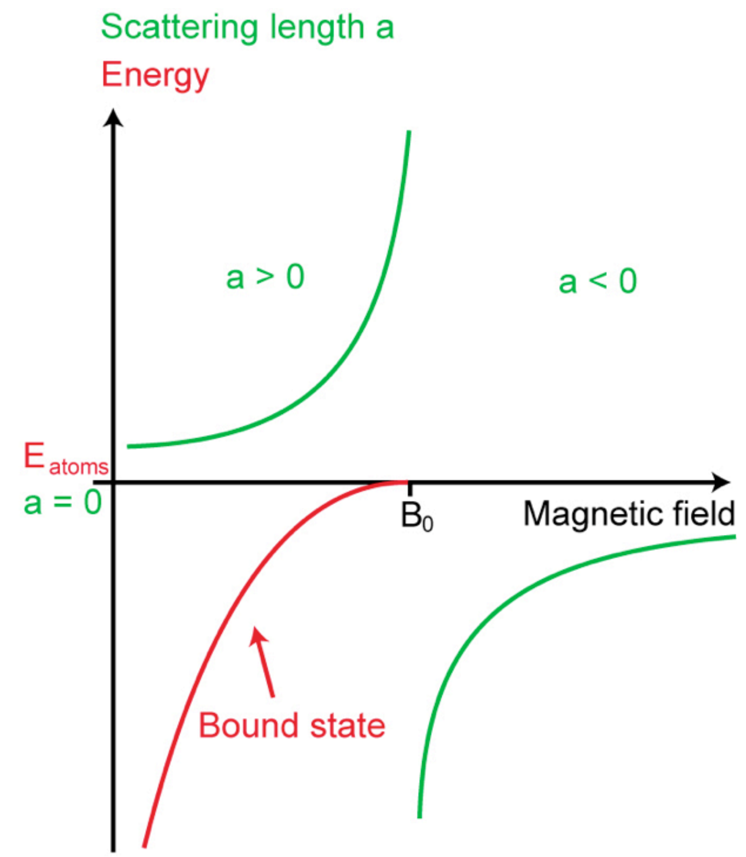
Zero-range potential

Zero-energy bound state



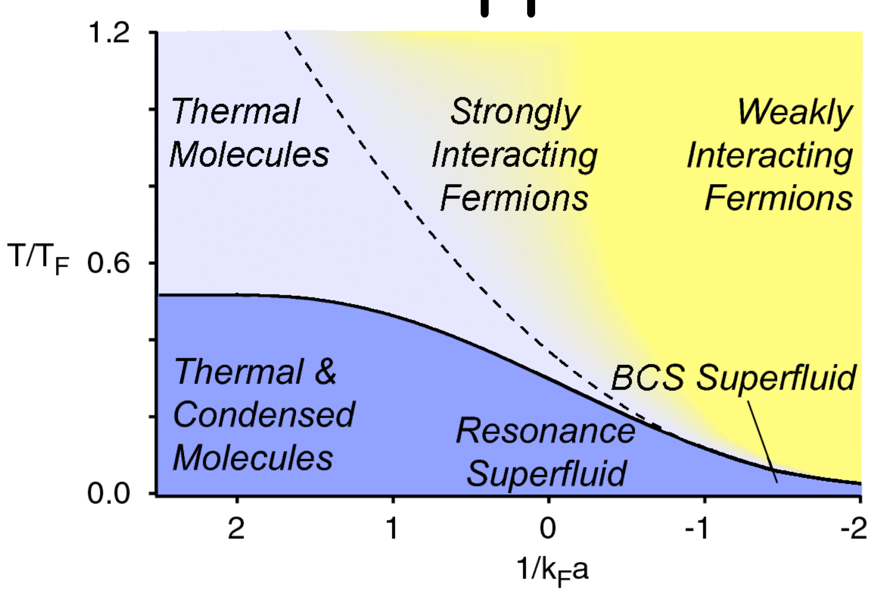
Can study universal properties of unitary fermions experimentally with trapped atoms (JILA, MIT, Innsbruck)



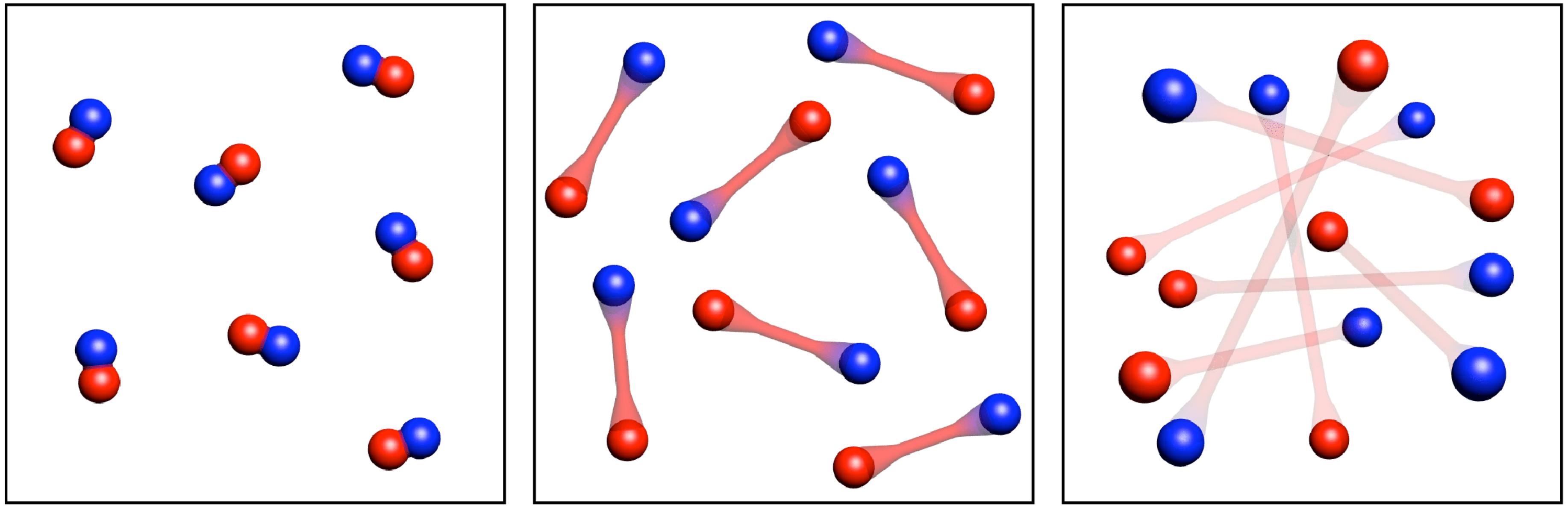


From:
Ketterle Lab web page

Feshbach resonance with trapped atoms: tune to unitarity



Unitary fermions



BEC of Molecules

Crossover Superfluid

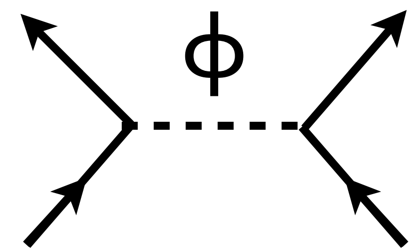
BCS state

$\frac{1}{a} \to +\infty$ $\qquad$ $\frac{1}{a} = 0$ $\qquad$ $\frac{1}{a} \to -\infty$

- Less severe sign problem; no gauge symmetry; nonrelativistic (quenched)
- Have simulated up to 66 fermions on 14^3 x 64 lattice
- 1-2% accuracy in energies

Simulation:

2-particle interaction induced by auxiliary scalar ϕ exchange, tuned to unitarity



ϕ lives on time links
(couples to $\psi^*\psi$)

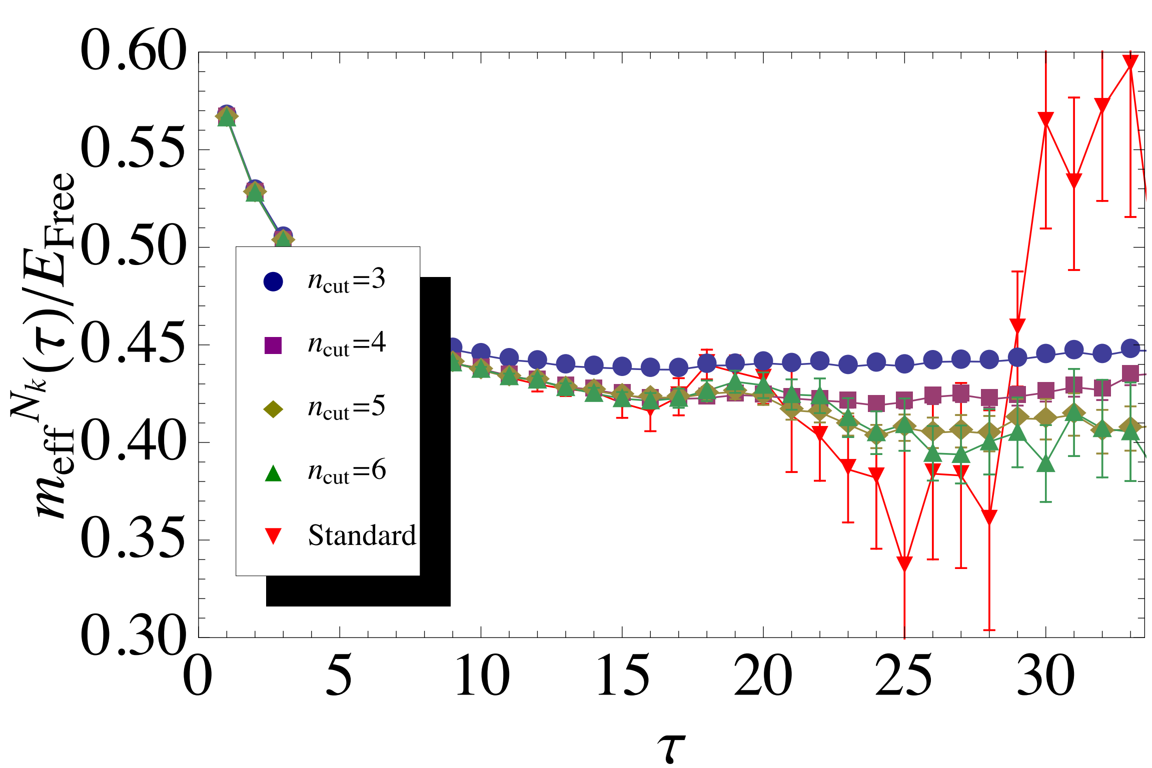
Generate an ensemble of random ϕ fields, compute average of an N-particle correlator $C_N(T;\phi)$ from t=0 to t=T

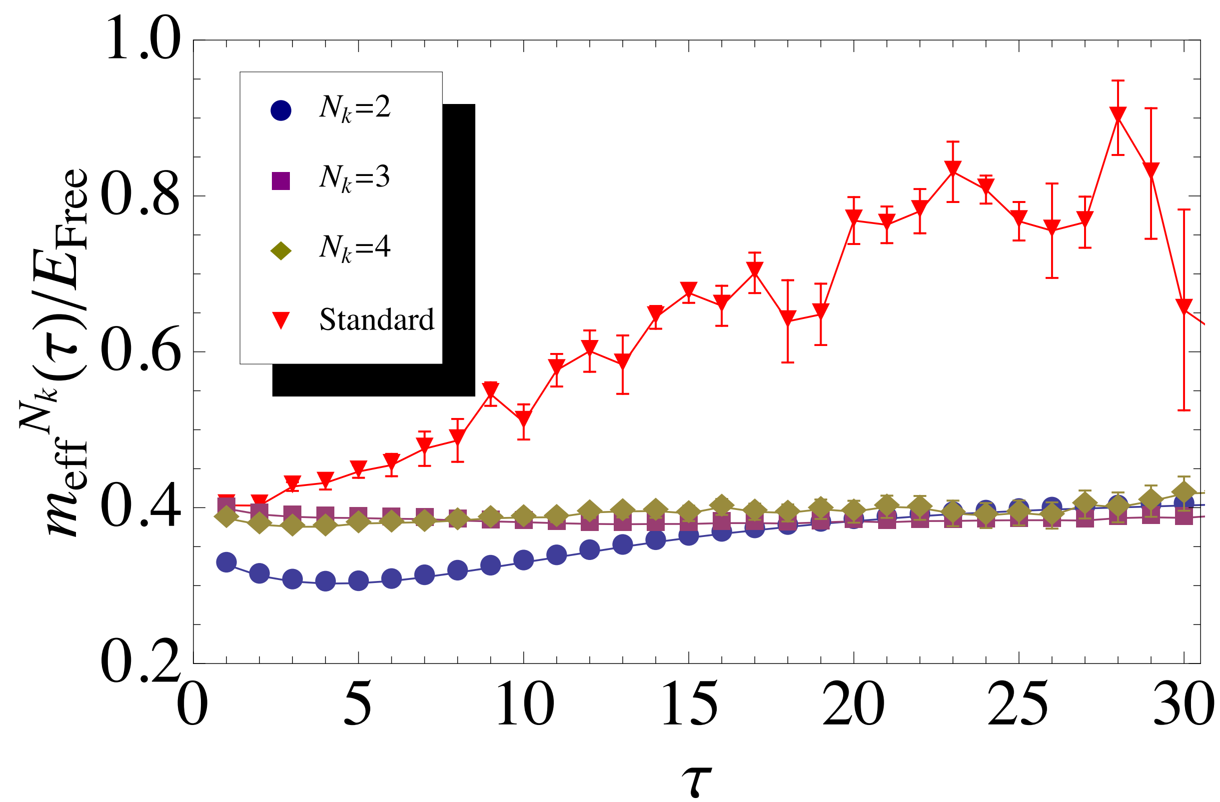
Extract ground state energy:

$$E_N = \lim_{T\to\infty}\left[-\frac{1}{T}\ln\langle C_N(T,\phi)\rangle_\phi\right]$$

Plot $-\frac{1}{T}\ln\langle C_N(T,\phi)\rangle_\phi$ vs T, look for "plateau"

Result of conventional calculation in **RED**





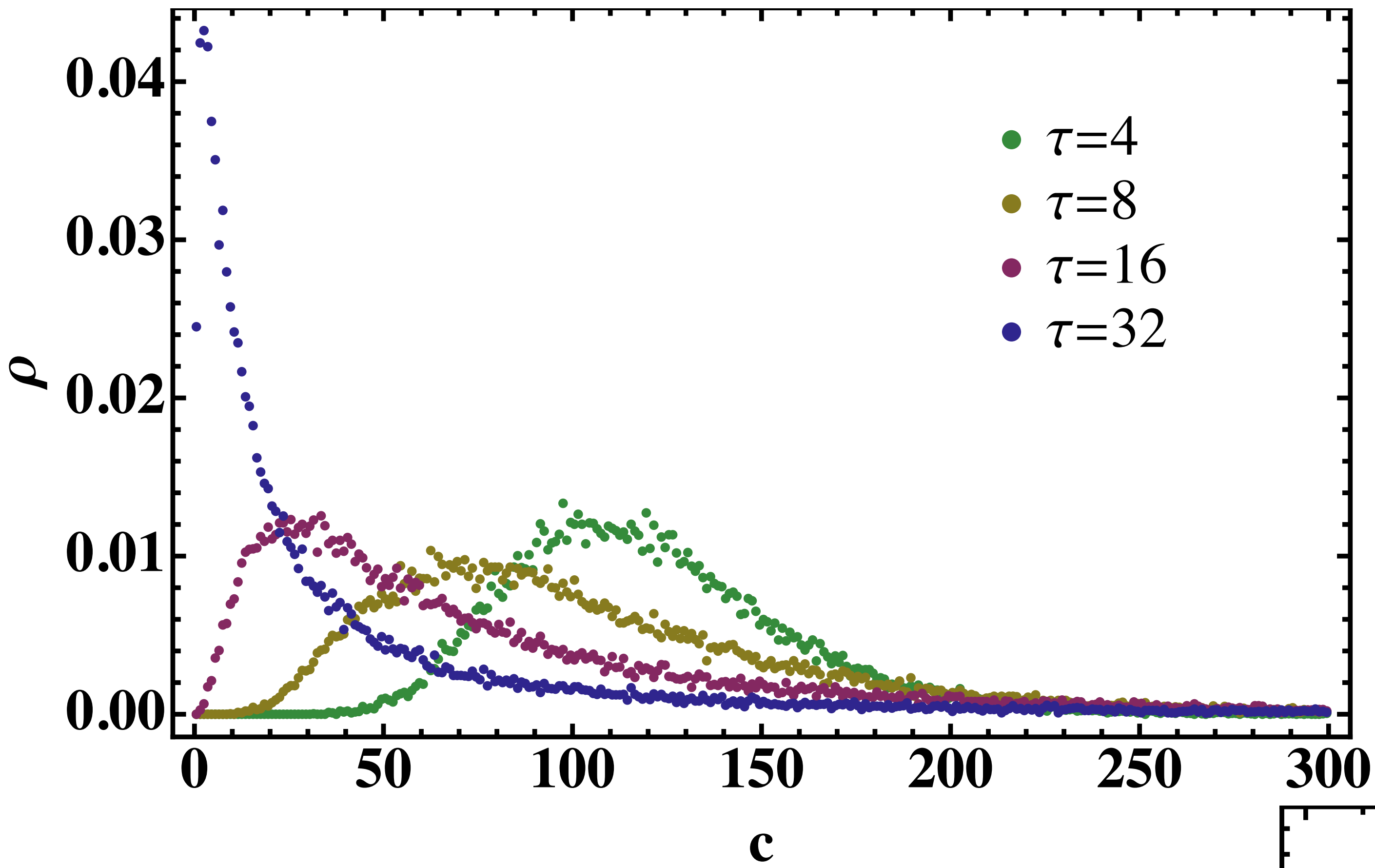
N= 16 fermions

N= 46 fermions

Conventional calculation shows:

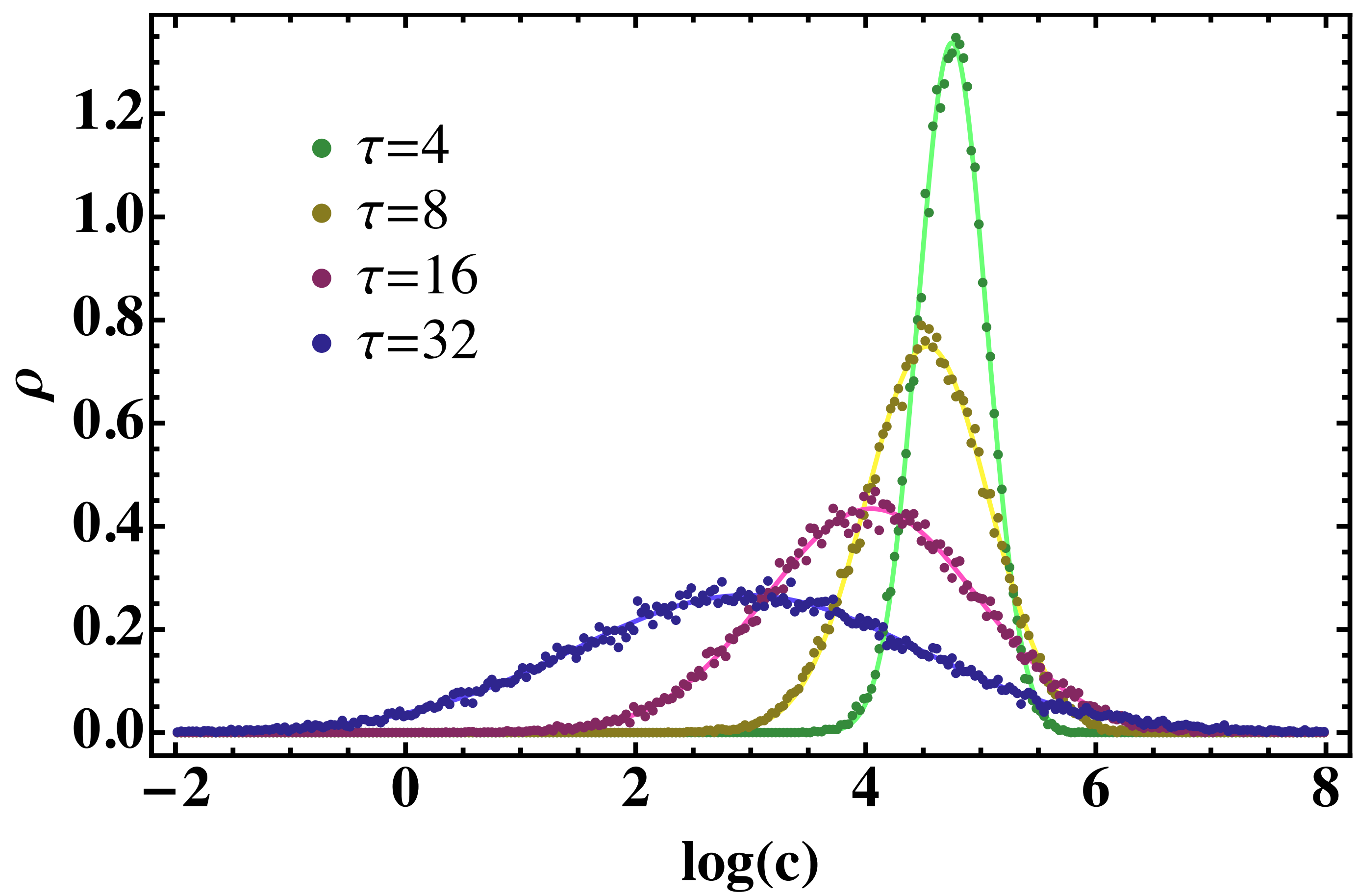
- noise
- drift
- worse for large N

Look at raw correlator distributions: Log-normal?!



N=4 correlator distribution at different times (τ)

Distribution of Log[correlator] with Gaussian fit



Distributions close to Log-normal are ubiquitous: arise from products of positive random numbers

$$y = \prod_{i=1}^{T}\left(1 + \tfrac{1}{T}x_i\right) \xrightarrow{T\to\infty} \exp\left[\underbrace{\tfrac{1}{T}\textstyle\sum_{i=1}^{T} x_i}\right]$$

random

Normal distributed
(CLT)

P(y) is Log-Normal distribution as T→∞

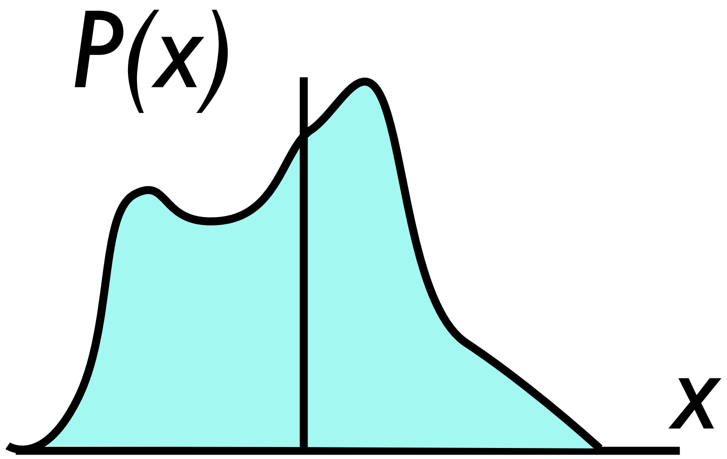
Aside: flow of statistics to "attractor distributions"

Example: central limit theorem

Suppose *x* has some distribution *P(x)*; specify by cumulants after shift and rescale so that μ= <x> =0, σ=1

$$P(x) \Longrightarrow P(0, 1, \kappa_3, \kappa_4, \ldots)$$

κ_n = n^{th} cumulant
κ_1 = μ =0
κ_2 = σ^2 =1



Average: $x \rightarrow \dfrac{x_1 + x_2}{\sqrt{2}}$ $\qquad \kappa_n \rightarrow 2^{1-n/2}\kappa_n$

κ_1=μ=relevant
κ_2=σ^2=marginal
$\kappa_{3\ldots}$=irrelevant

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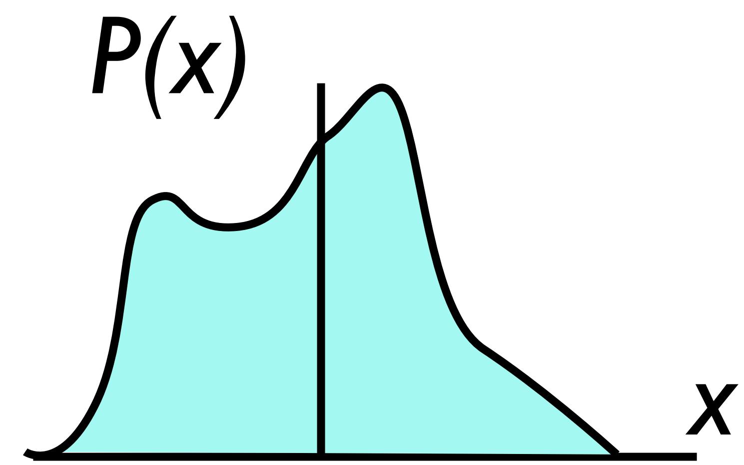

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Repeat averaging: $P(0,1,\kappa_3,\kappa_4,\ldots) \rightarrow P(0,1,0,0,\ldots)$

Normal distribution

Aside: flow of statistics to "attractor distributions"

Example: central limit theorem

Suppose x has some distribution $P(x)$; specify by cumulants after shift and rescale so that $\mu = \langle x \rangle = 0$, $\sigma = 1$

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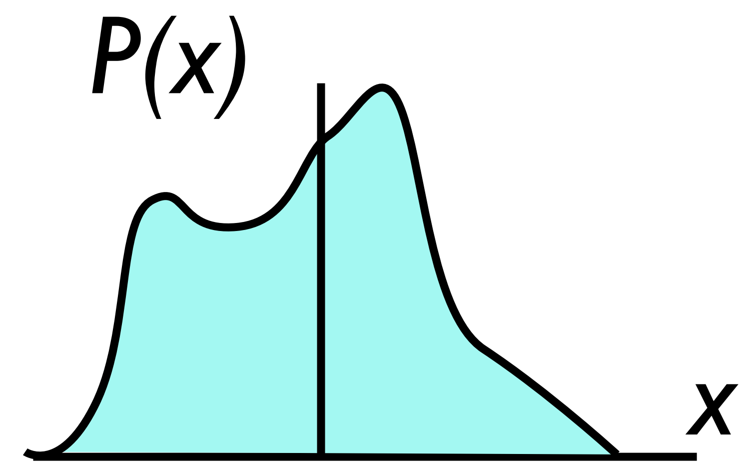

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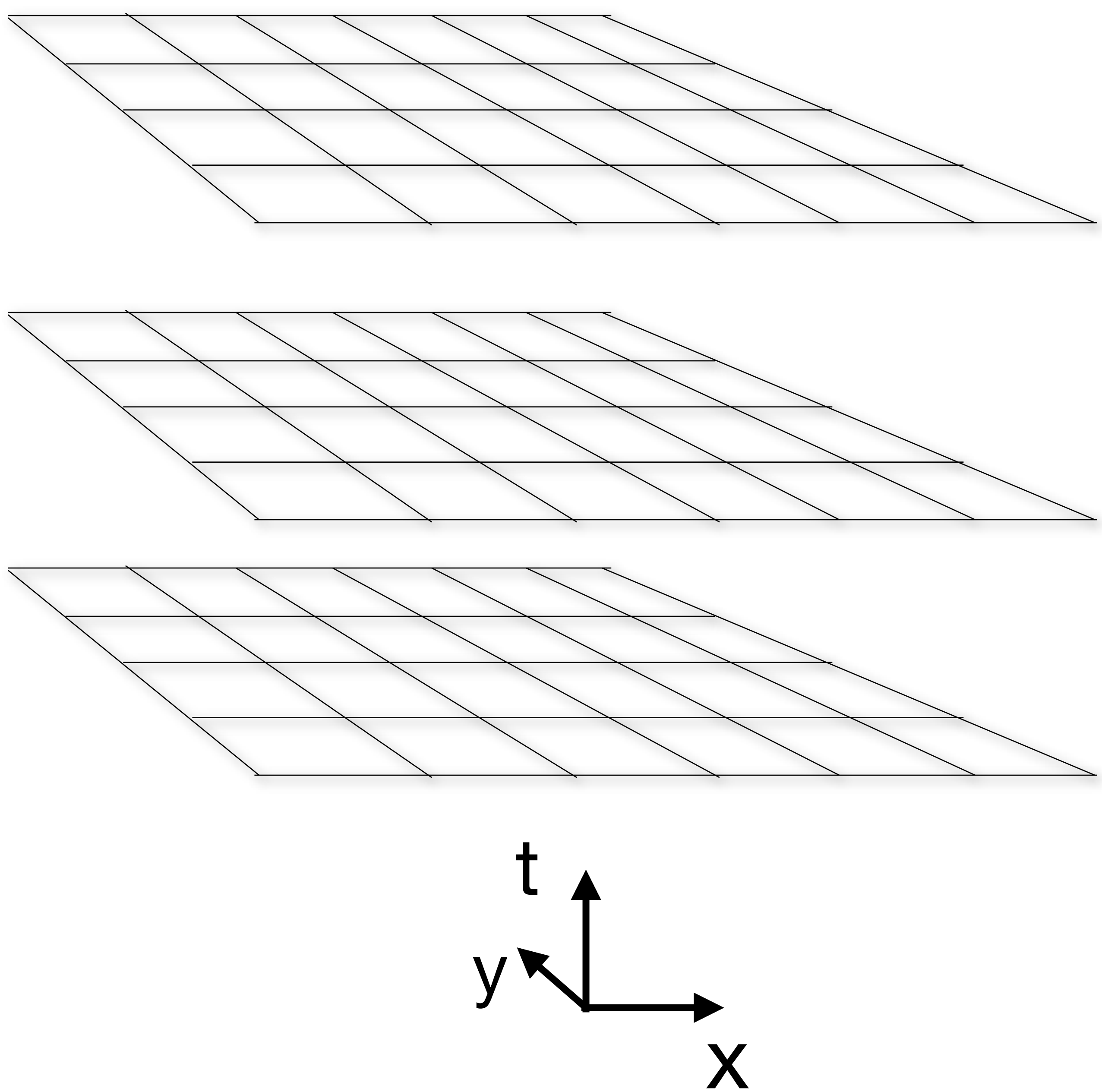
$\kappa_1 = \mu$ = relevant
$\kappa_2 = \sigma^2$ = marginal
$\kappa_{3\ldots}$ = irrelevant

Repeat averaging: $$P(0, 1, \kappa_3, \kappa_4, \ldots) \to P(0, 1, 0, 0, \ldots)$$

Normal distribution

Just like renormalization group; κ_n *like dim 2n operators in* φ *QFT*

Fermion correlators are products of random matrices (transfer matrices)



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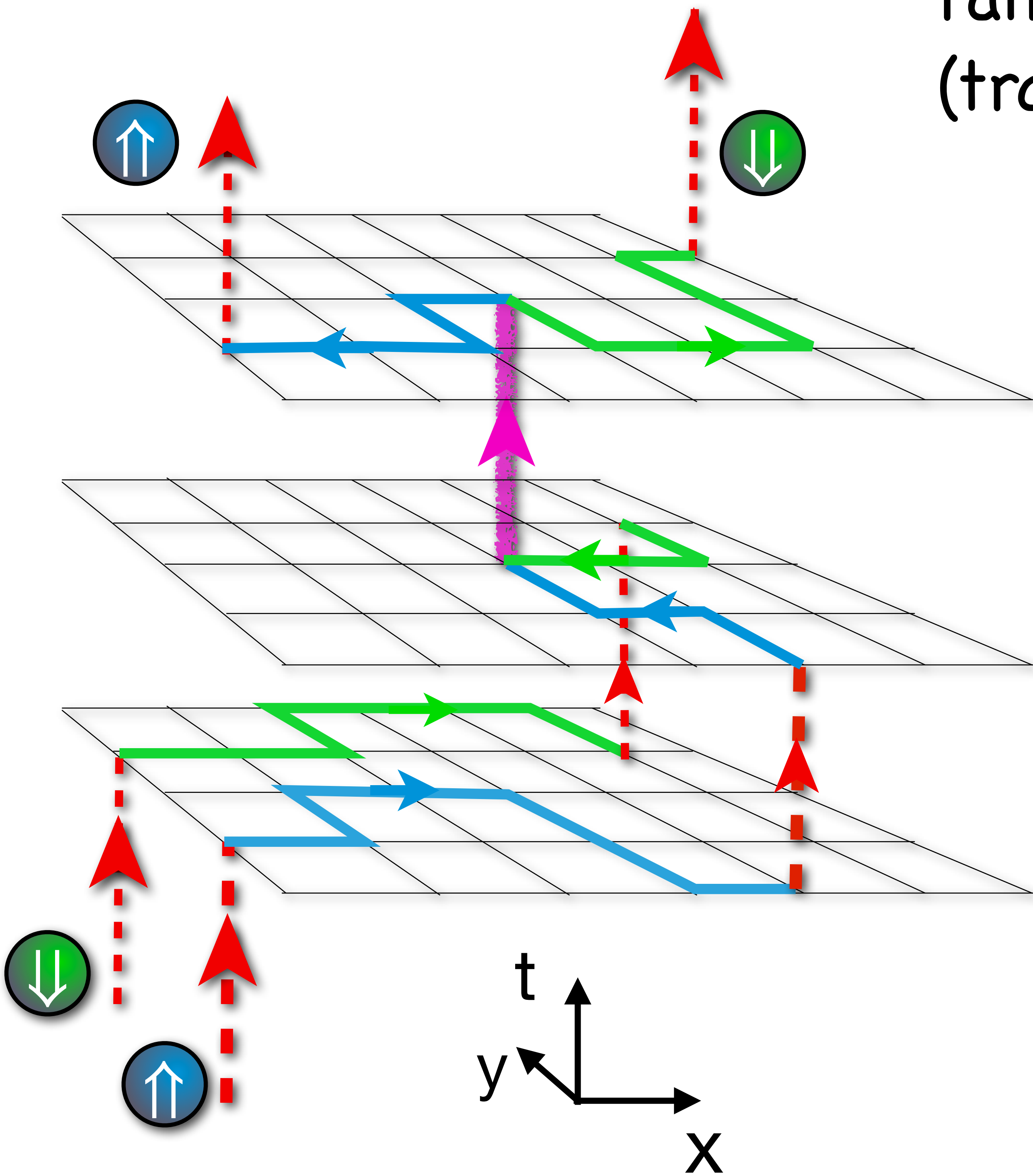

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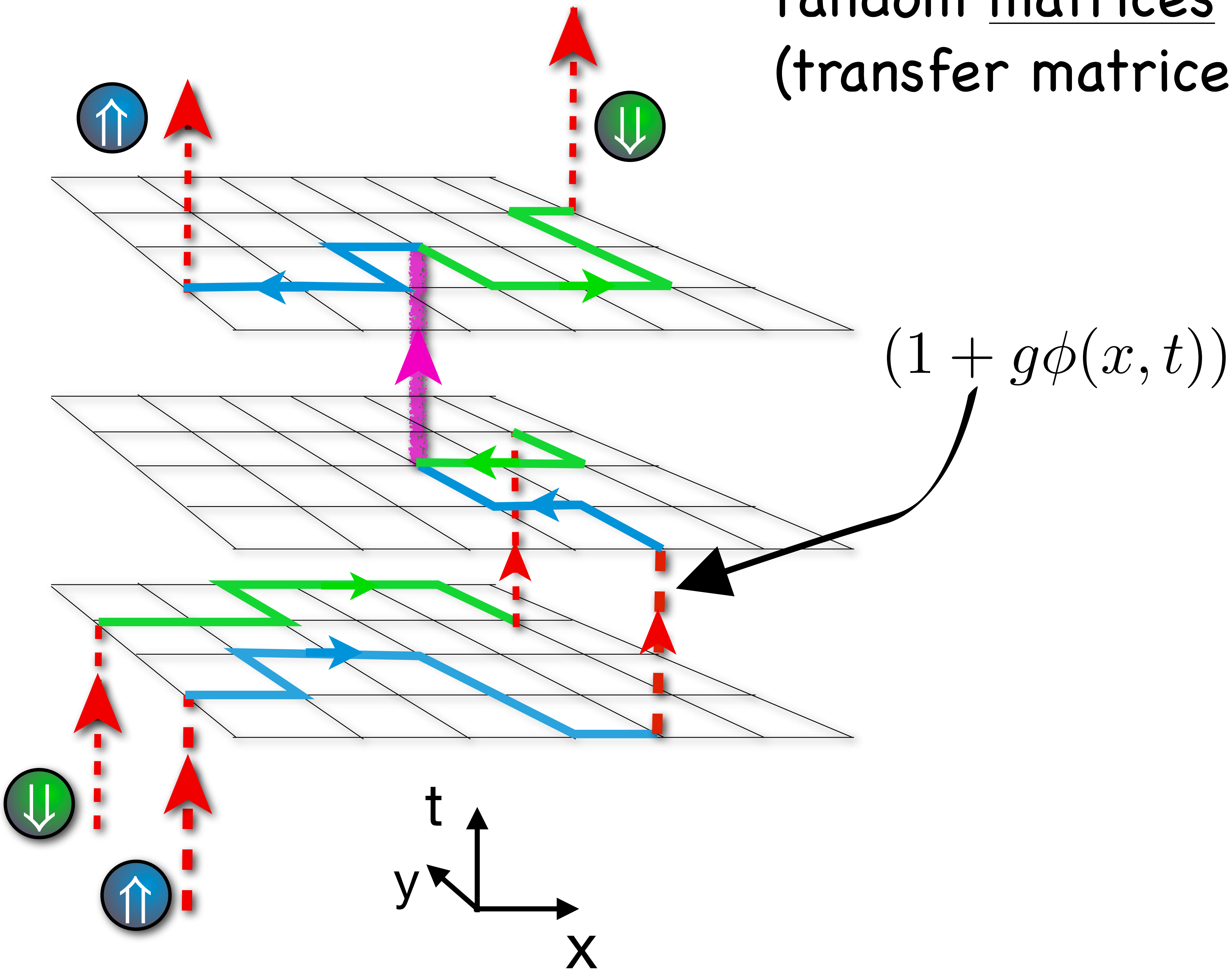

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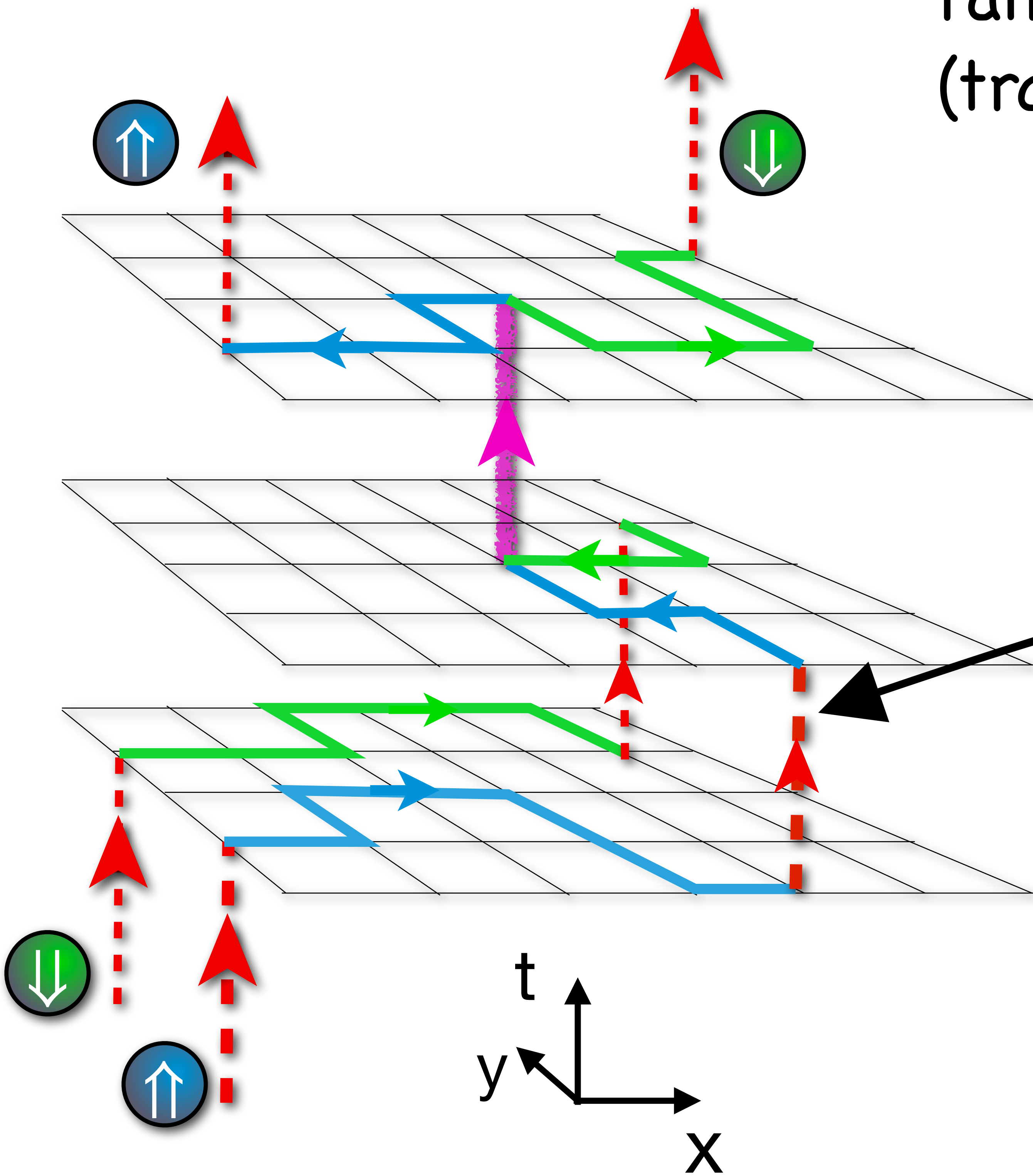


$$(1 + g\phi(x,t))$$

Toy model:

$$C(T) = \prod_{i=1}^{T} (1 + g\phi_i)$$

$$\phi_i \in [-1, 1], \quad g = \frac{1}{2}$$

Toy model:

$$C(T) = \prod_{i=1}^{T} (1 + g\phi_i)$$

$\phi_i \in [-1, 1]$ uniform dist.

Exact answer for the "energy":

$$E(T) \equiv -\frac{1}{T} \ln \langle C(T) \rangle_\phi = 0$$

Compare with simulation (finite sample size), g=1/2

Two strategies:

- Conventional: $$E \to -\frac{1}{T} \ln \left[\frac{1}{N} \sum_{i=1}^{N} C(T, \phi_i) \right]$$
- New: use identity

$$\ln\langle C \rangle = \sum_n \frac{\kappa_n}{n!} \qquad \kappa_n = \text{ cumulants of } \ln C(T, \phi)$$

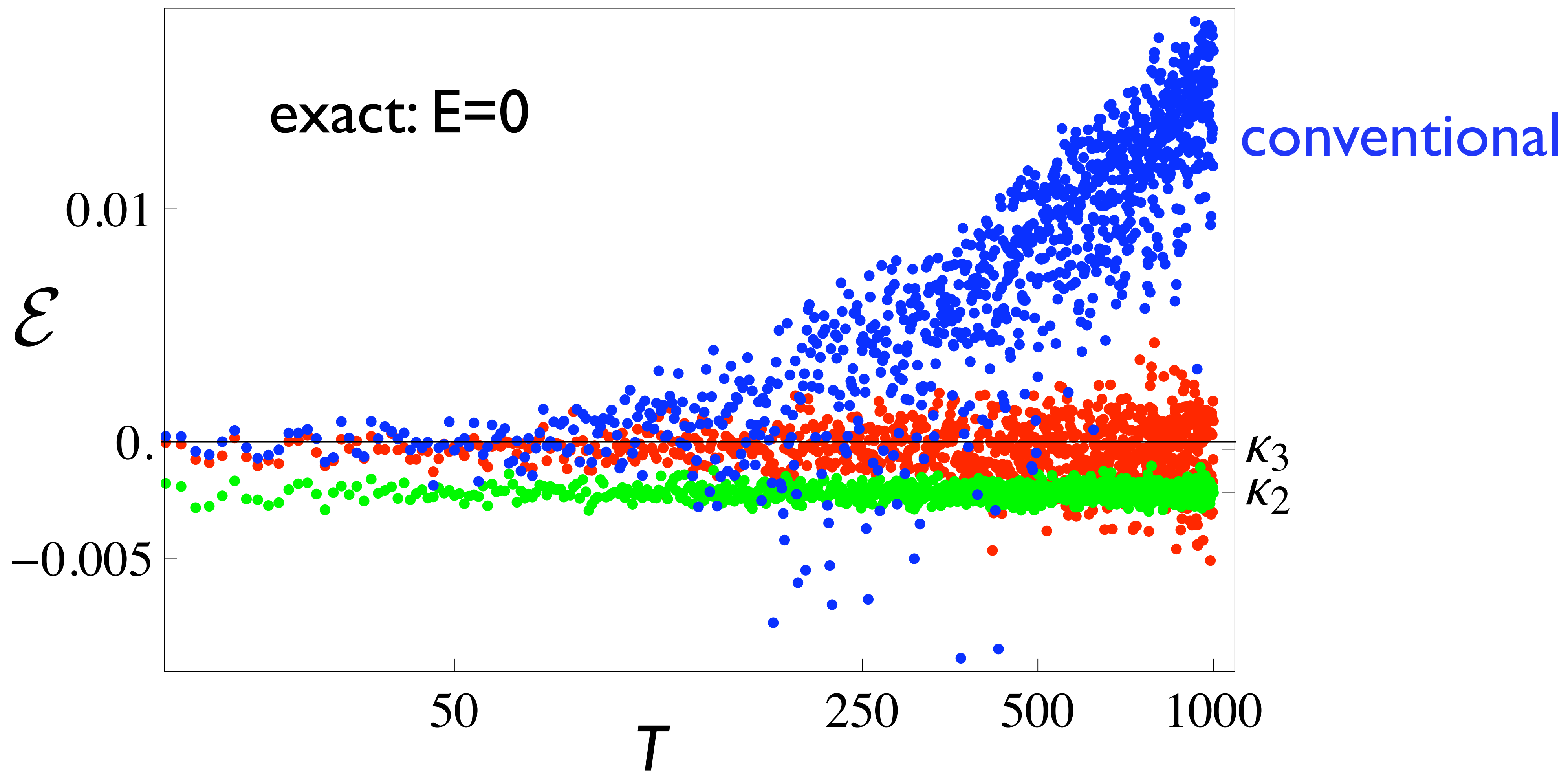
and estimate κ_n from sample for low n.

Why?

- If C distribution Log-Normal, $\kappa_{n>2}=0$;
- κ_n expansion like EFT expansion

Toy model simulation

Sample size
N=50,000



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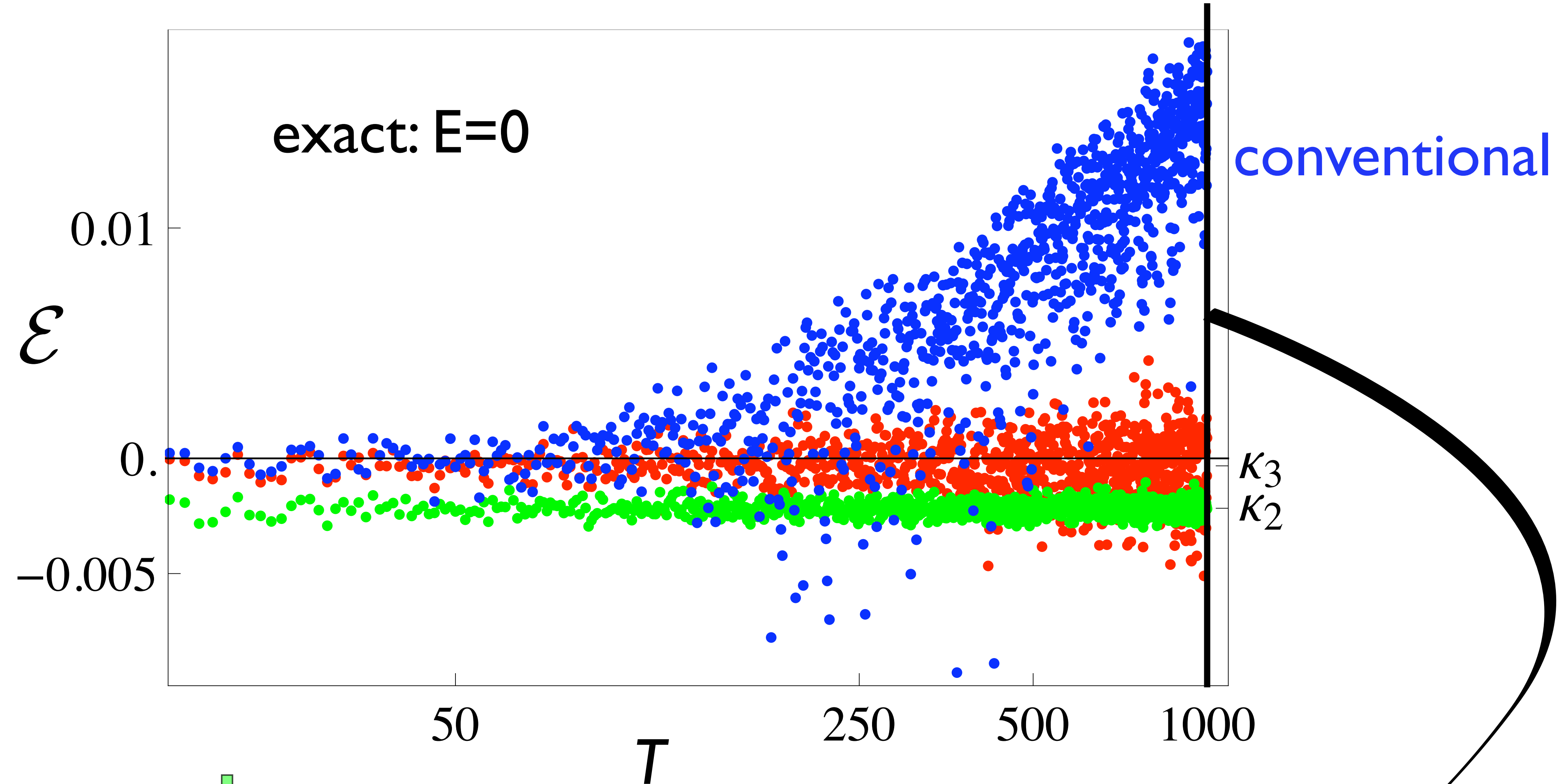
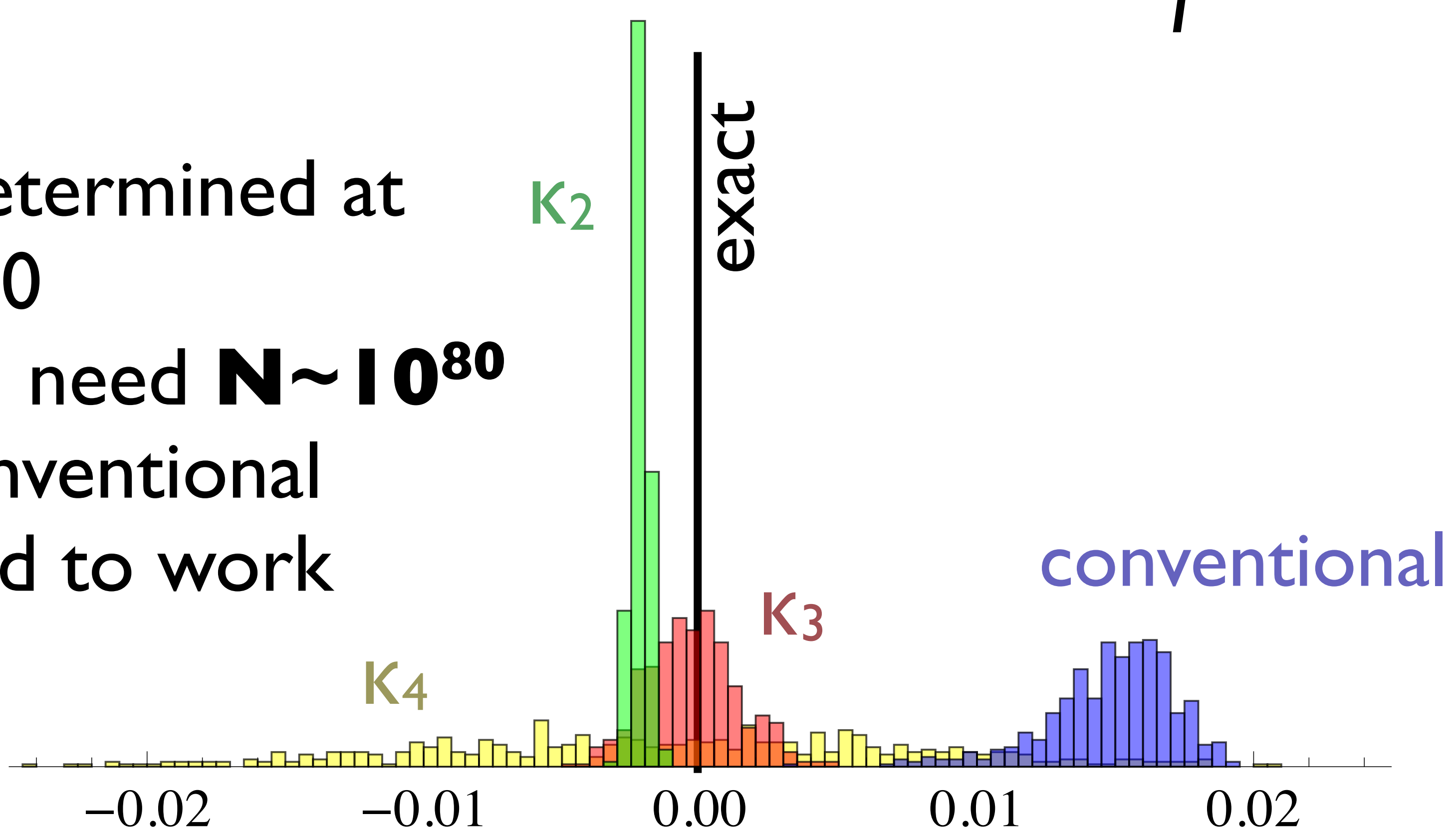
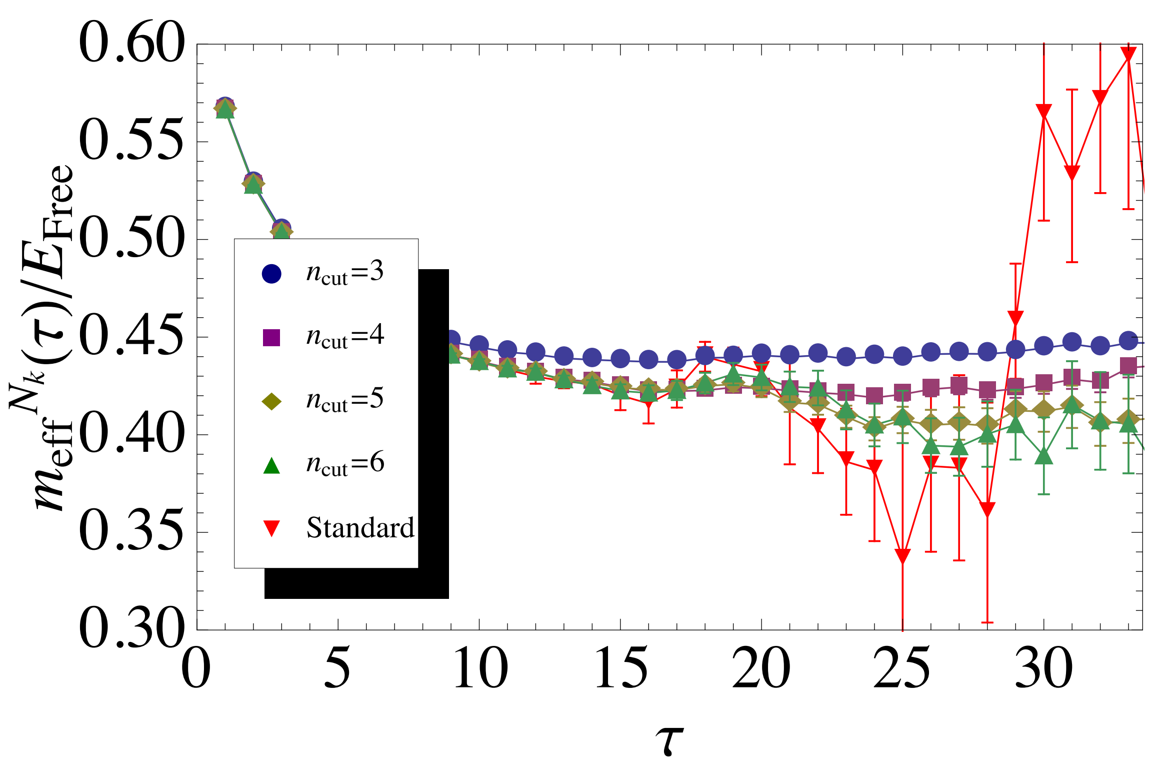


- E as determined at T=1000
- Would need **N~10^{80}** for conventional method to work



Back to data from unitary fermion simulations



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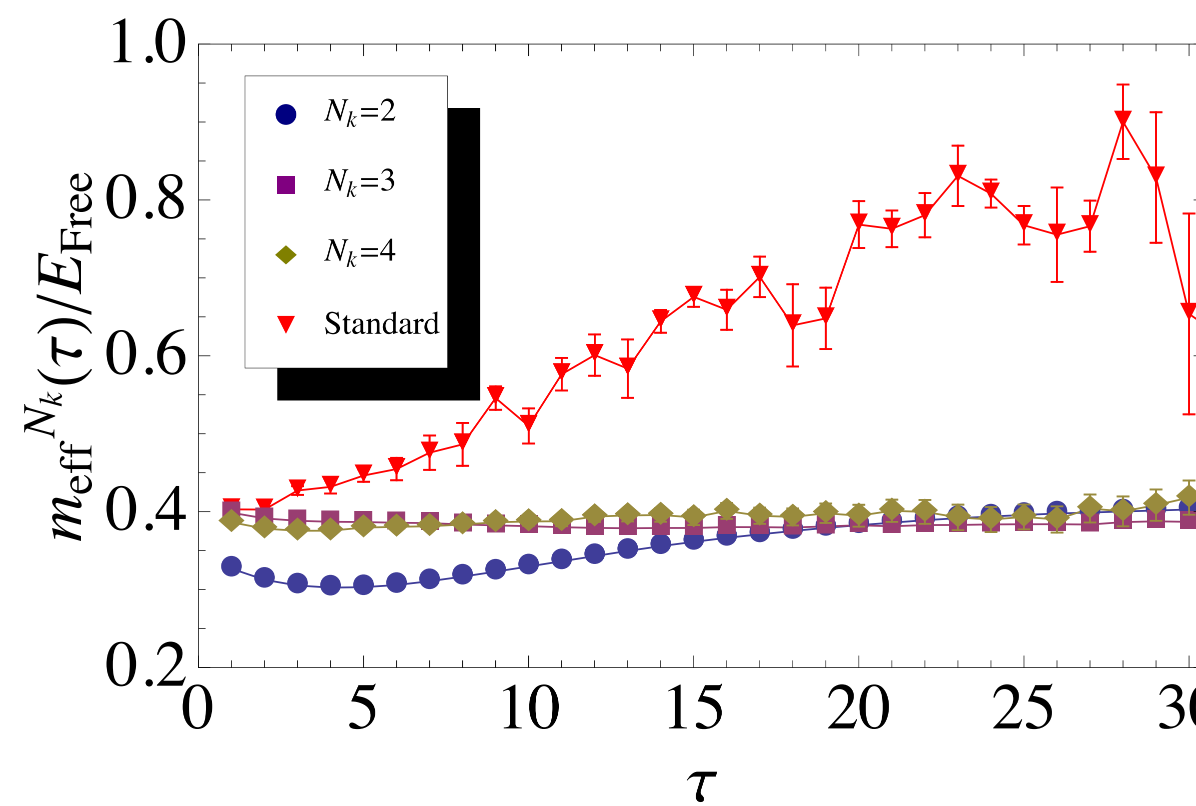
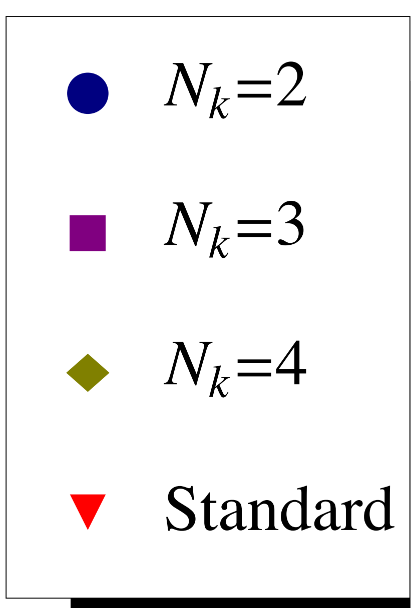


N= 46 fermions

Conventional calculation: poor results

Cumulant expansion works much better

-- especially for large N



Toy model too simplistic;

Determining distributions of random matrix products too hard...

....but mean field analysis used for disordered media may provide connection between microphysics and correlator distributions in lattice gauge theory

Mean field argument: distribution for y = Log[C_N(phi,T)]

$$
\begin{aligned}
P(y) &= \int [D\phi] e^{-\int d^4x \frac{1}{2} m^2 \phi^2} \delta(\ln C_N[\phi] - y) \\
&= \int \frac{ds}{2\pi} \int [D\phi] e^{is(\ln C_N - y) - \int d^4x \, \frac{1}{2} m^2 \phi^2} \\
&\equiv \int \frac{ds}{2\pi} \int [D\phi] e^{-S}
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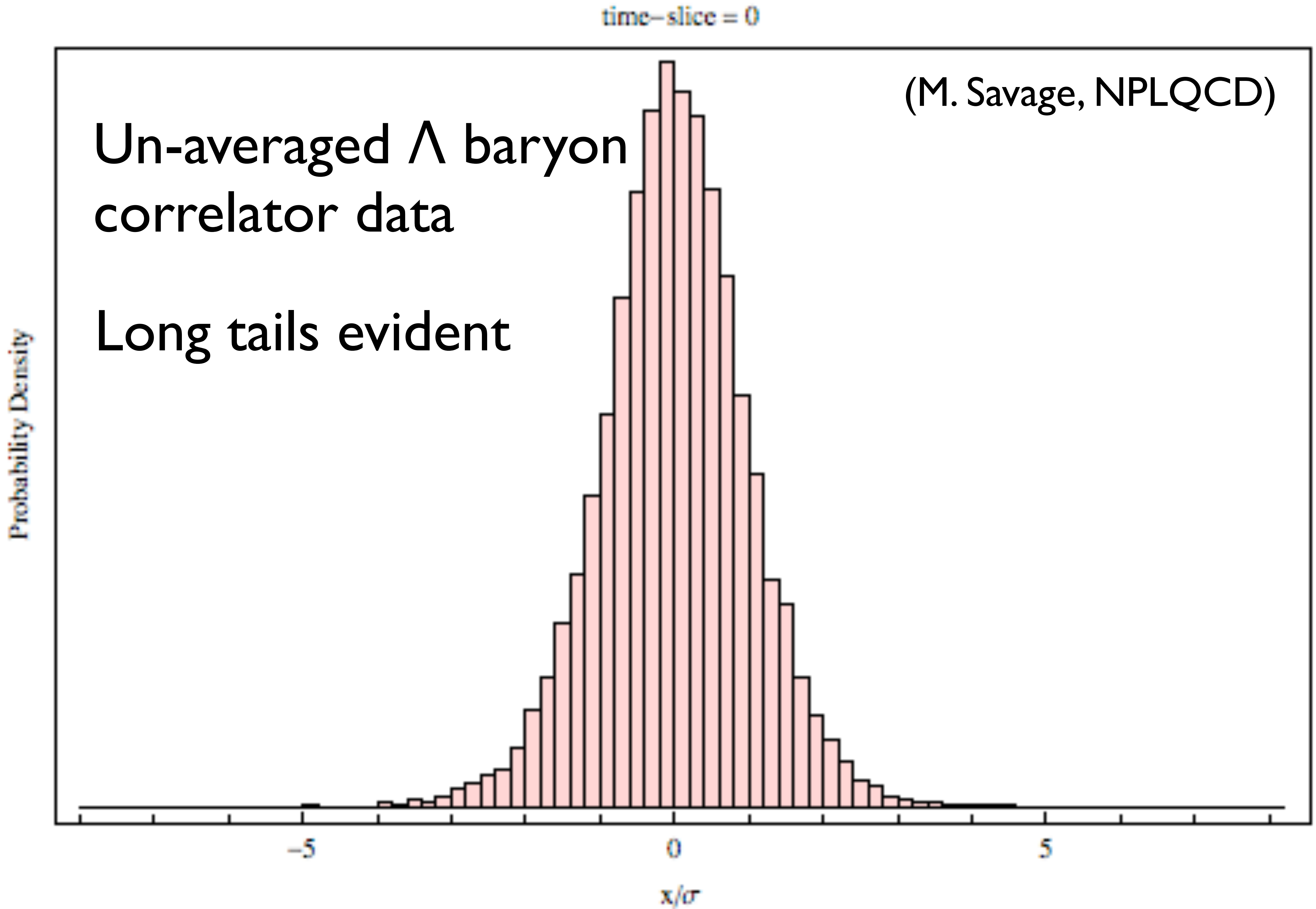
Result: Log Normal distribution (with corrections);
μ, σ^2 scale with N and T as seen in data

Do heavy-tailed non-gaussian distributions occur in lattice QCD? Probably, especially large baryon number



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- Possibly amenable to analysis similar to EFT -- characterized by hierarchy of correlations
- By better understanding the noise, we might better hear the underlying message
- Stay optimistic and don't pay too much attention to no-go theorems!