Continuous Insights in the Wilson Dirac Operator

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Acknowledgments

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P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Microscopic Spectrum of the Wilson Dirac Operator, Phys. Rev. Lett. 105 (2010) 162002 [arXiv:1001.2937 [hep-th]].

G. Akemann, P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Wilson Fermions, Spectrum of the Wilson Dirac Operator at Finite Lattice Spacings, Submitted to Phys. Rev. $\mathbf{D}$ [arXiv:1012.0752[hep-lat]]; G. Akemann, P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Wilson Fermions, Random Matrix Theory and the Aoki Phase, PoS LATTICE2010 (2010 092 [[arXiv:1011.5118]]; Effect of Dynamical Quarks on the Spectrum of the Wilson Dirac Operator, PoS LATTICE2010 (2010) 079 [arXiv:1011.5121[hep-lat]].

K. Splittorff and J. J. M. Verbaarschot, The Physical Wilson Dirac Spectrum, to be published.

G. Akemann, P.H. Damgaard, M. Kieburg, K. Splittorff and J. J. M. Verbaarschot, Random Matrix Theory for the Wilson Dirac Operator and Orthogonal Polynomials, to be published.

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I. Introduction and Motivation

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Lattice Results

Wilson Dirac operator

In lattice QCD, the derivatives in the Dirac operator are replaced by discrete derivatives, but this leads to additional low energy modes at the other side of the Brillouin zone. These modes have to be eliminated to converge to the correct continuum limit, which solved by Wilson by introducing the so called Wilson term

$$D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{1}{2}a\nabla_\mu\nabla^*_\mu.$$

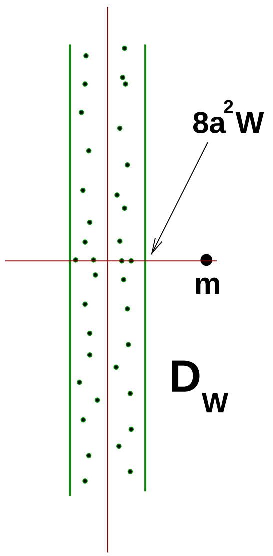
$$\{D_W, \gamma_5\} = a\gamma_5\nabla_\mu\nabla^*_\mu..$$

$$D_W^\dagger = \gamma_5 D_W \gamma_5.$$

Block structure

$$D_W = \begin{pmatrix} aA & id \\ id^\dagger & aB \end{pmatrix}$$

with $A^\dagger = A,\ B^\dagger = B.$



We will analyze the partition function

$$Z_{\rm QCD}(m_f; a) = \left\langle \prod_f \det(D_W + m_f) e^{-S_{\rm YM}} \right\rangle.$$

The Hermitian Wilson Dirac operator

Because

$$\det(D_W + m) = \det(\gamma_5(D_w + m)),$$

we also could have used the determinant of $D_5 \equiv \gamma_5(D_W + m)$ in the weight of the partition function.

This gives us two natural options to analyze the properties of the QCD partition function:

- √ In terms of of the eigenvalues λ_k^W of D_W . These are directly related to the chiral condensate

$$\frac{1}{VN_f}\partial_m \log Z = \left\langle \frac{1}{V}\sum_k \frac{1}{\lambda_k^W + m} \right\rangle .$$

- √ In terms of the eigenvalues of D_5 . Since $D_5^\dagger = D_5$ its eigenvalues are real and is computationally much simpler to work with.

Motivation

Lattice studies of the distribution of the smallest eigenvalue of the Wilson Dirac operator. Del Debbio-Giusti-Lüscher-Petronzio-Tantalo-2005

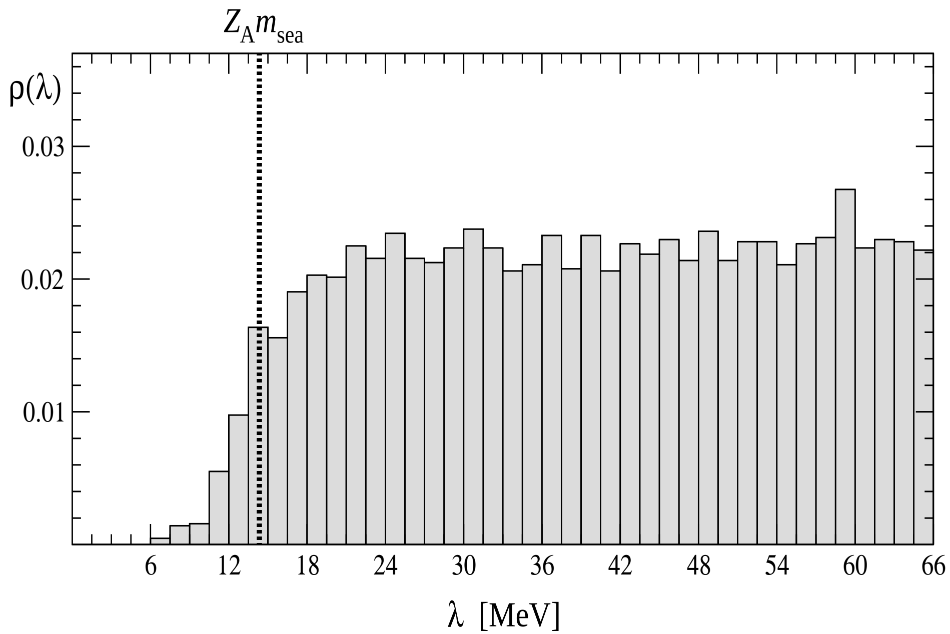
Mean fields studies of the spectrum of the Wilson Dirac operator based on chiral perturbation theory. Sharpe-2006

Lifshitz tail states in superconducting quantum dots with magnetic impurities. Lamacraft-Simons-1996

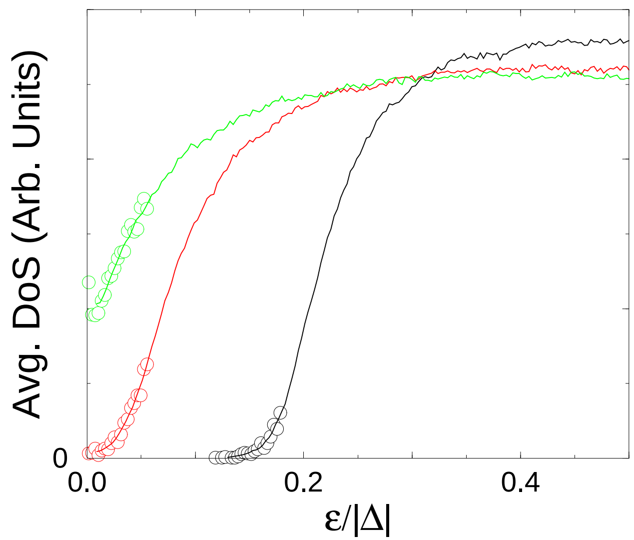
Discussions in the literature on the existence of the Aoki Phase.
Sharpe-Singleton-1998, Shindler-2009, Azcoiti-Di Carlo-Follana-Vaquero-2009

The existence of a gap in the spectrum of D_5 is important to evaluate its inverse efficiently.

Lattice Results for the Wilson Dirac Spectrum



Spectral density of $\gamma_5(D_W + m)$ on a 48×24^3 lattice.
Lüscher-2007

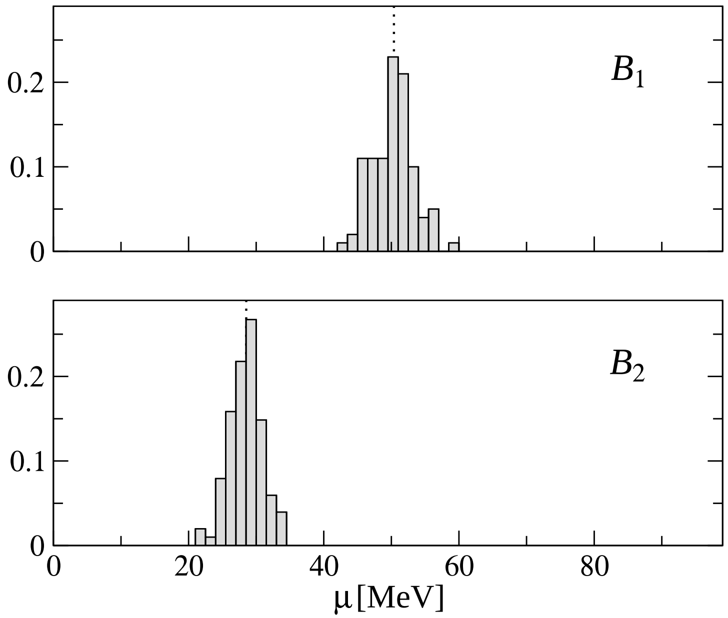


Spectral density of a Bogulubov Hamiltonian in the presence of magnetic impurities.

Lamacraft-Simons-2001

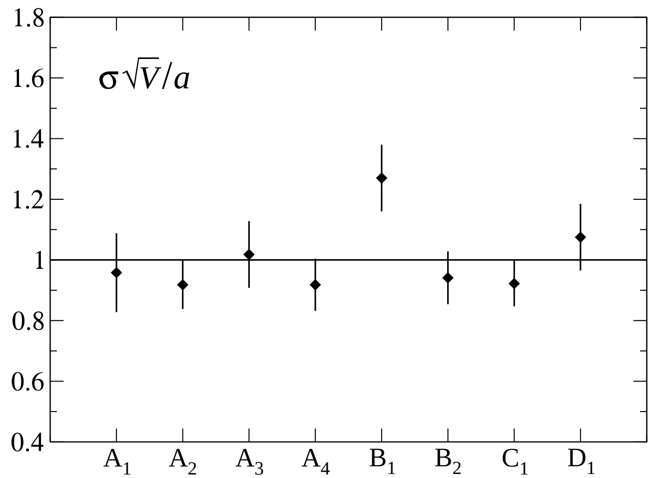
- ✓ Dirac spectrum has a gap.
- ✓ A Gaussian tail intrudes inside the gap.

Distribution of the Smallest Eigenvalue



Distribution of the smallest eigenvalue of the Hermitian Wilson Dirac operator on a 64×32^3 lattice for two different values of the quark mass.

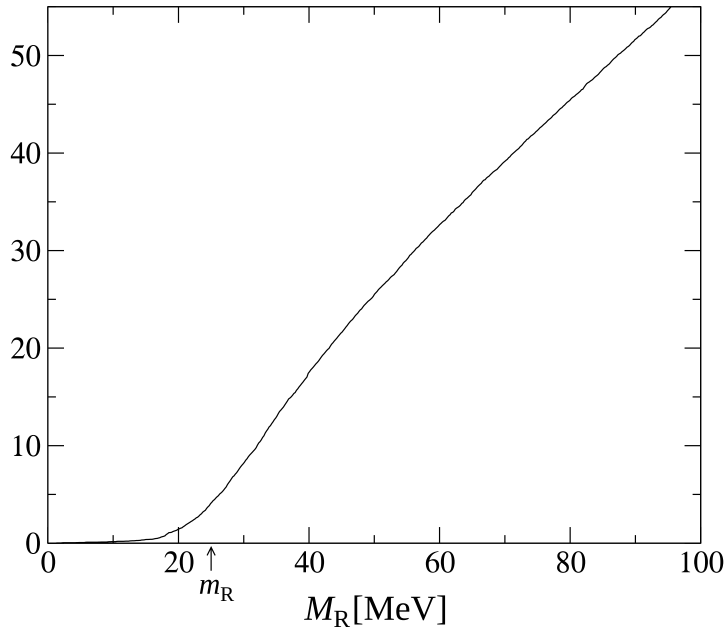
Del Debbio-Giusti-Lüscher-Petronzio-Tantalo-2005



Scaling of the width of the distribution.

Del Debbio-Giusti-Lüscher-Petronzio-Tantalo-2005

Role of Topology



Integrated eigenvalue density of the spectrum of $[\gamma_5(D_W + m)]^2$ on a $32^3 \times 64$ lattice with $a = 0.07$ Luescher-Palombo-2010

We will see that the bump is due to topology.

Questions

- Can we obtain analytical results for the spectrum of the Wilson Dirac operator?
  - Effect of topology?
  - Effect of the fermion determinant?
  - Distribution of the smallest eigenvalue?
- How does the probability to find more than ν real eigenvalues scale with the volume?
- What is the behavior of the Dirac spectrum in the approach to the continuum limit?
- Is it posible to interchange the chiral limit and the continuum limit?

General Discussion of Wilson Spectra

Wilson Dirac Spectra

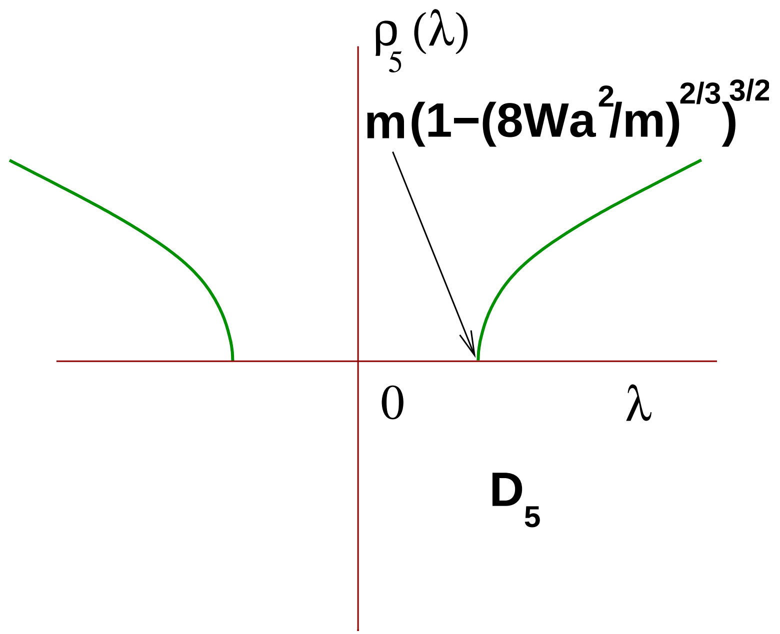
Spectral Flow and Index of Dirac Operator

Spectrum of D_5

Example: For $a = 0$ in a block diagonal representation of D_W we have for $\lambda_k \neq 0$

$$D_5 = \begin{pmatrix} m & i\lambda_k \\ -i\lambda_k & -m \end{pmatrix}.$$

Therefore the eigenvalues of D_5 are given by $\pm\sqrt{\lambda_k^2 + m^2}$ or by m for the topological zero modes.

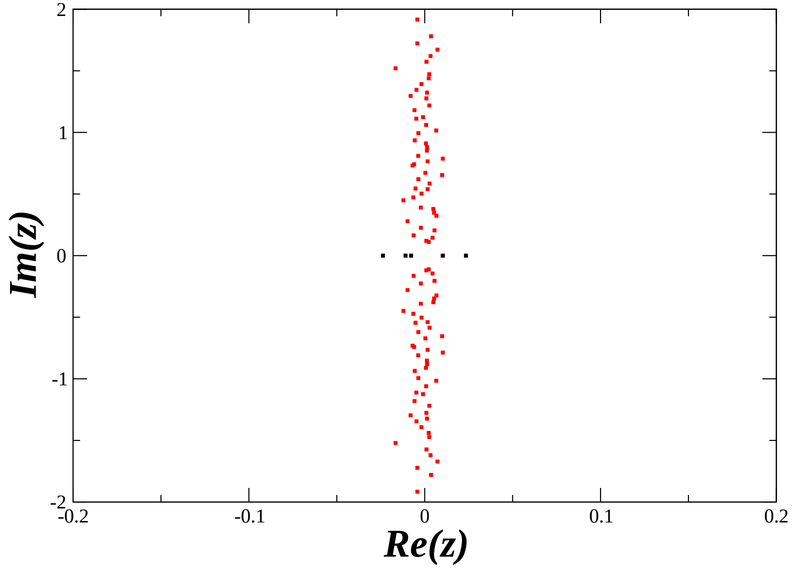


Spectrum of D_5

The partition function will have a singularity for $a^2 = m/8W$. The corresponding phase for larger values of a is known as the Aoki phase.

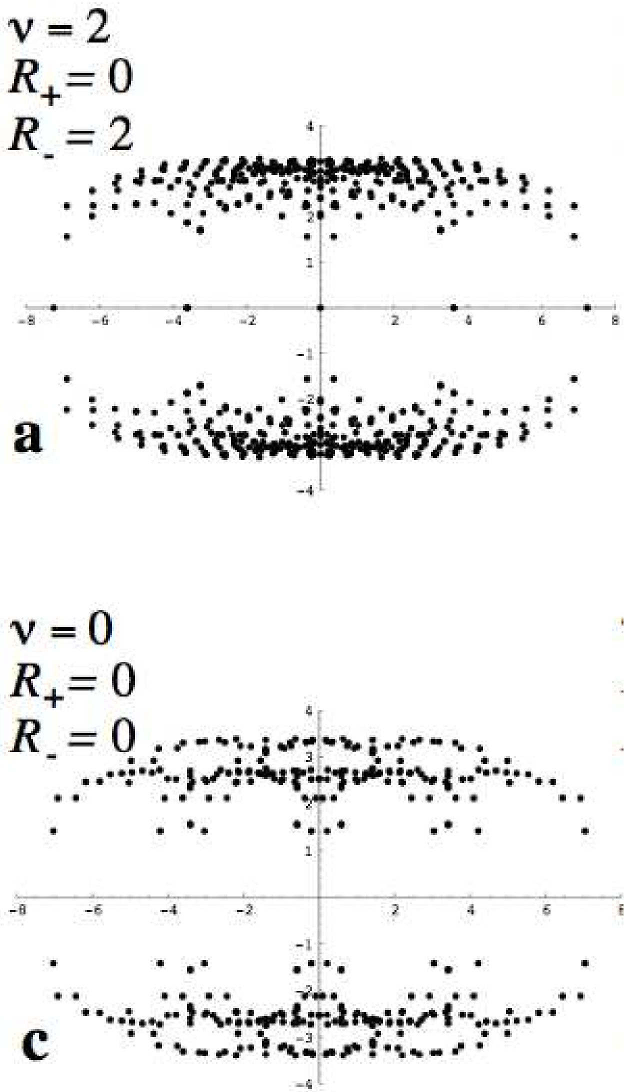
Since a plays the role of a regulator, it can also be chosen complex. Once choice is to take it purely imaginary. Then D_W remains anti-hermitian. This choice is known as twisted mass Wilson fermions. However, the fermion determinant is only positive definite for degenerate pairs of flavors.

Typical Spectrum of D_W



Spectrum of the Wilson Dirac operator for topological charge equal to $\nu = 4$.

- We will now see that the number of real eigenvalues is at least equal to the topological index of the Dirac Operator.



Gattringer-Hip-1998

Spectral Flow at Nonzero Topology

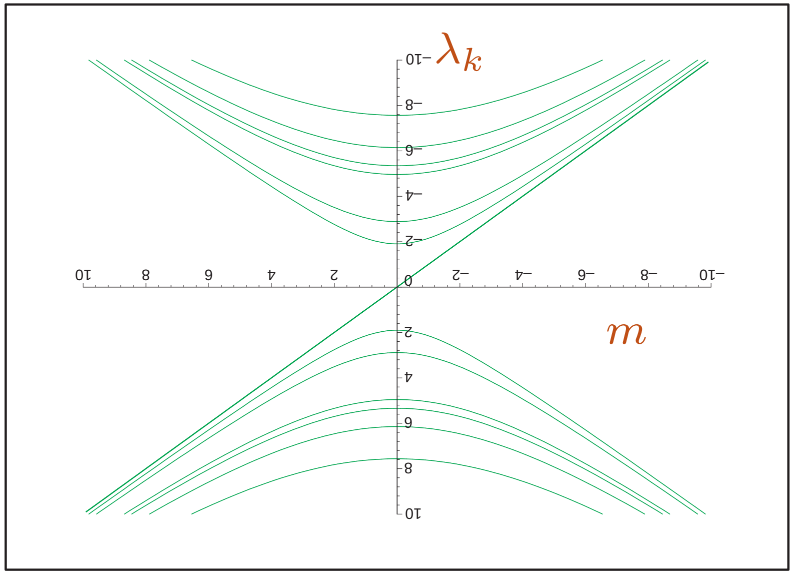
If ϕ is an eigenfunction of a topological zero mode,then

$$D\phi = 0, \qquad \gamma_5\phi = \phi,$$

so that

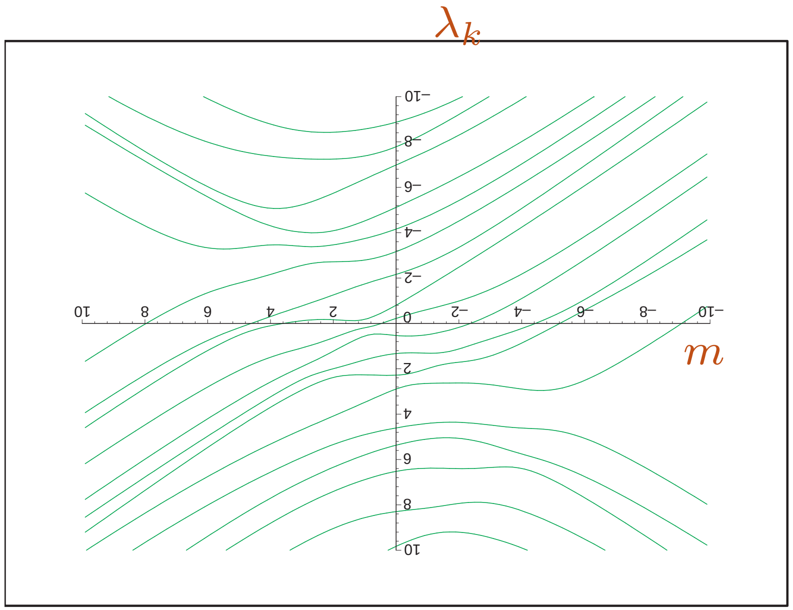
$$\gamma_5(D+m)\phi = m\phi \Rightarrow \lambda_0 = m.$$

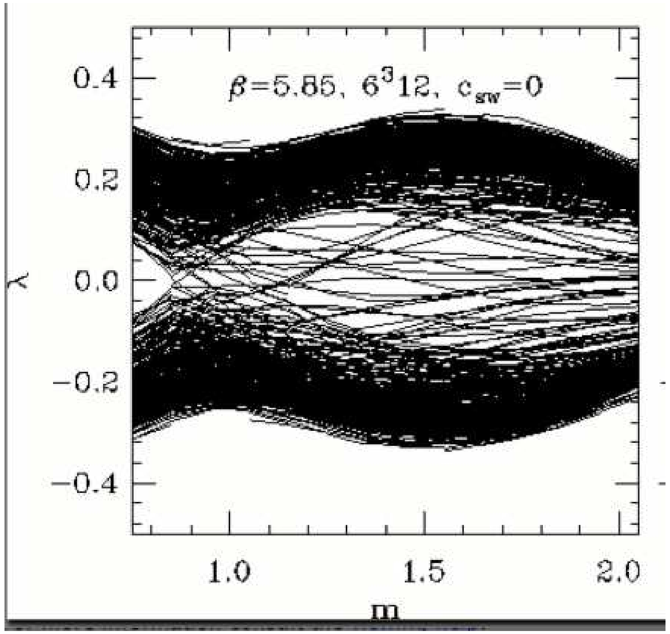
The eigenvalues of the nonzero modes are given by $\lambda_k = \pm\sqrt{c_k^* c_k + m^2}$ while the zero modes depend linearly on m .



The topological charge is given by the difference of the number of positive eigenvalues and negative eigenvalues for large positive mass.

Spectral Flow for $a \neq 0$





Edwards-Heller-Narayanan-1998

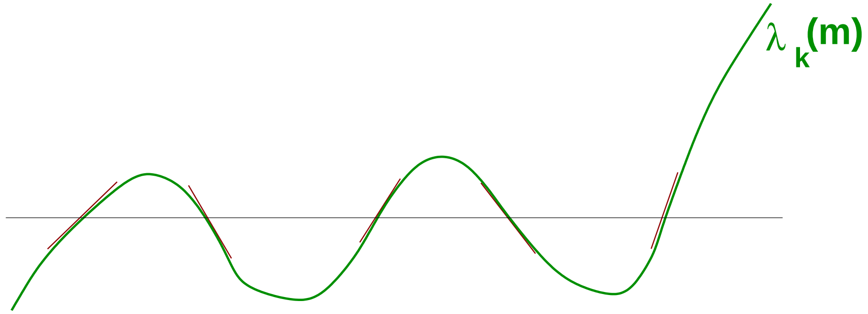
At the crossing point Narayanan-Neuberger-1995, Edwards-Heller-Narayanan-1998

$$\gamma_5(D_W + m_c)\phi = 0 \quad \Longrightarrow \quad D_W\phi = -m_c\phi.$$

Therefore the number or real eigenvalues of D_w is at least equal to the topological charge. Note that multiple crossings may occur.

Index of D_W

The total number of flow lines with a net flow across the real axis is a topological invariant of the Dirac operator.



This number can be obtained from the number of crossings in the following way

$$\nu = \sum_{m_k, \lambda(m_k)=0} \mathrm{sign}\left(\frac{d\lambda(m)}{dm}\right)_{m=m_k} .$$

For the specialists: this can be rewritten in terms of the expectation values of γ_5

$$\nu = \sum_k \mathrm{sign}(\langle \phi_k | \gamma_5 | \phi_k \rangle),$$

where the sum is over all eigenfunctions of D_W . Itoh-Iwasaki-Yoshie-1987

Index of D_W and Chiral Condensate

The index of D_W can be obtained from the chiral condensate

$$\Sigma(m) = \left\langle \mathrm{Tr} \frac{1}{D_W + m - i\epsilon\gamma_5} \right\rangle$$

and the spectral density

$$\rho_\chi(m) = \frac{1}{2\pi} \mathrm{Im}\, \Sigma(m).$$

Using that the real eigenvalues of D_W are given by the crossing points of the spectral flow lines of D_5 one can show

$$\rho_\chi(\lambda^W) = \left\langle \sum_{\lambda_k^W \in R} \delta(\lambda_k^W + \lambda^W) \mathrm{sign}(\langle k|\gamma_5|k\rangle) \right\rangle$$

This provides us with a lower limit of the density of real eigenvalues of D_W .

$$\int d\lambda^W \rho_\chi(\lambda^w) = \nu.$$

$i\epsilon$ Prescription for the Wilson Dirac Operator

- Since the spectrum of D_W is complex, the addition or $i\epsilon$ does not take us away from the singularities of D_W^{-1} .
- However, $D_5^{-1} = [\gamma_5(D_w + m)]^{-1}$ is regularized by the addition of $i\epsilon$ to D_5 . This makes it possible to define a generating function for the spectrum of D_5 .
- In order to access the spectrum of $\gamma_5(D_W + m)$ we add a source $z\gamma_5$ so that the total fermion determinant is given by

$$\det[D_W + m + \gamma_5 z] = \det[\gamma_5(D_W + m) + z].$$

- Since $\det(\gamma_5(D_W + m) - i\epsilon) = \det(D_W + m - i\epsilon\gamma_5)$, we can extract information on the spectrum of $D_W - i\epsilon\gamma_5$ from the partition function for D_5 .
- The low energy limit of the corresponding partition function is given by a chiral Lagrangian that up to low energy constants is uniquely determined by symmetries.

III. Chiral Lagrangian

Microscopic Limit

Chiral Lagrangian

Generating Function for the Wilson Dirac Spectrum

Results in terms of an integral over a diffusion kernel.

Microscopic Limit

In the microscopic domain, where the combinations

$$mV, \qquad zV, \qquad a^2V$$

are kept fixed in the thermodynamic limit, the m , z and a dependence of the chiral Lagrangian resides in the zero momentum part of the partition function that factorizes form the nonzero momentum part.

Chiral Lagrangian

In the microscopic domain the QCD partition function

$$\langle \det^{N_f}(D_W + m + \gamma_5 z)\rangle$$

in the sector of topological charge ν the partition function that describes the zero momentum part of the Goldstone modes is given by

$$Z_\nu^\chi(m,z,a) = \int_{U\in U(N_f)} dU \det{}^\nu U e^{\frac{1}{2}mV\Sigma \mathrm{Tr}(U+U^\dagger)+\frac{1}{2}zV\Sigma\mathrm{Tr}(U-U^\dagger)-a^2VW_8\mathrm{Tr}(U^2+U^{-2})}.$$

Sharpe-Singleton-1998, Rupak-Shoresh-2002, Bär-Rupak-Shoresh-2004, Damgaard-Splittorff-JV-2010

Up to the constants Σ and W_8 this partition function is determined uniquely by the symmetries and transformation properties of the QCD partition function.

To this order the chiral Lagrangian also contains

$$a^2W_6[\mathrm{Tr}(U+U^\dagger)]^2 + a^2W_7[\mathrm{Tr}(U+U^\dagger)]^2$$

but they can be added to the mass term after linearizing the square at the expense of a Gaussian integral.

Generating Function for Wilson Dirac spectrum

To find the Dirac spectrum we have to use the supersymmetric method. The generating function for the Wilson Dirac spectrum is given by

$$Z(m,z,z',a) = \left\langle \det(D_W+m)\frac{\det(D_W+m+\gamma_5 z)}{\det(D_W+m+\gamma_5 z' - i\epsilon\gamma_5)} \right\rangle .$$

Damgaard-Splittorff-JV-2010

The resolvent is equal to

$$G(z,m,a) = \lim_{z'\to z} \frac{d}{dz} Z(m,z,z',a)\Big|_{z'=z},$$

and the spectral density is given by

$$\rho(z) = \frac{1}{\pi}\mathrm{Im}\, G(z), \qquad \text{for} \quad z \quad \text{real}.$$

For $z \ll \Lambda_{\rm QCD}$ the z -dependence of the generating function is given by a chiral Lagrangian that is uniquely determined by symmetries compatible with the convergence of the bosonic integrals. In the microscopic domain the partition function reduces to a super-unitary matrix integral.

Generating Function for the Wilson Dirac Spectrum

The low-energy limit of the generating function for the Wilson Dirac spectrum is given by

$$Z(m,z,a) = \int_{U\in Gl(N_f+1|1)} dU \mathrm{Sdet}^{\nu} U e^{i\frac{1}{2}mV\Sigma \mathrm{Str}(U-U^\dagger)+i\frac{1}{2}V\Sigma\mathrm{Str}(\zeta U+U^\dagger\zeta)-i^2a^2VW_8\mathrm{Str}(U^2+U^{-2})}$$

Here, $\zeta = \mathrm{diag}(0,0,z,z')$.

Damgaard-Splittorff-JV-2010

- With respect to the usual chiral Lagrangian we have $U \to \pm iU$. This made it possible to analytically continue the partition function to real a .
- We require that the integral is regular for $a \to 0$. So, the imaginary part of z determines the sign of the mass term. Notice that because m is multiplied by γ_5 in $\det[\gamma_5(D_W+m)+z]$ it is not possible to obtain convergent integrals by adding $i\epsilon$ to m .
- For $N_f=0$ the integral reduces to products of one dimensional integrals. For $N_f>0$ a direct evaluation of the integrals is cumbersome.

Evaluation of the Integrals

The integrals can be evaluated conveniently by linearizing the quadratic terms in U

$$\begin{aligned} e^{a^2 \mathrm{Str}(U^2+U^{\dagger 2})} &= e^{-2N_f a^2 + a^2 \mathrm{Str}(U+U^\dagger)^2}, \\ &= e^{-2N_f a^2} \int d\sigma e^{\mathrm{Str}\,\sigma^2/16a^2 + \frac{i}{2}\mathrm{Str}(U+U^\dagger)\sigma}, \end{aligned}$$

where σ is an $(N_f+1|1)$ graded “Hermitian” matrix.

The partition function at $a \neq 0$ can be interpreted as diffusion in a^2 of a partition function at $a = 0$ with shifted quark masses.

Result for Spectral Resolvent

The final result for the spectral resolvent is remarkably simple

$$
\begin{aligned}
G^{\nu}_{N_f+1|1}(m,z;a) &= \int dS J(S) e^{(1/16a^2)\mathrm{Trg}(S-z)^2} \mathrm{Sdet}^{\nu}(S-m) \\
&\times Z^{\nu}_{N_f+1|1}\left(\{(m^2-S_k^2)^{1/2}\}; a=0\right),
\end{aligned}
$$

where $J(S)$ is the Jacobian of transforming σ to the diagonal representation S .

It can be shown that this integral can be rewritten into a sum of products of two-dimensional integrals and can easily be evaluated numerically.

Akemann-Kieburg-Splittorf-JV-2011.

The microscopic quenched spectral density is equal to

$$
\rho_5^{\nu}(x,m;a) = \frac{1}{\pi}\mathrm{Im}[G^{\nu}_{N_f+1|1}(x,m;a))].
$$

Damgaard-Splittorff-JV-2010

IV. The Wilson Dirac Spectrum

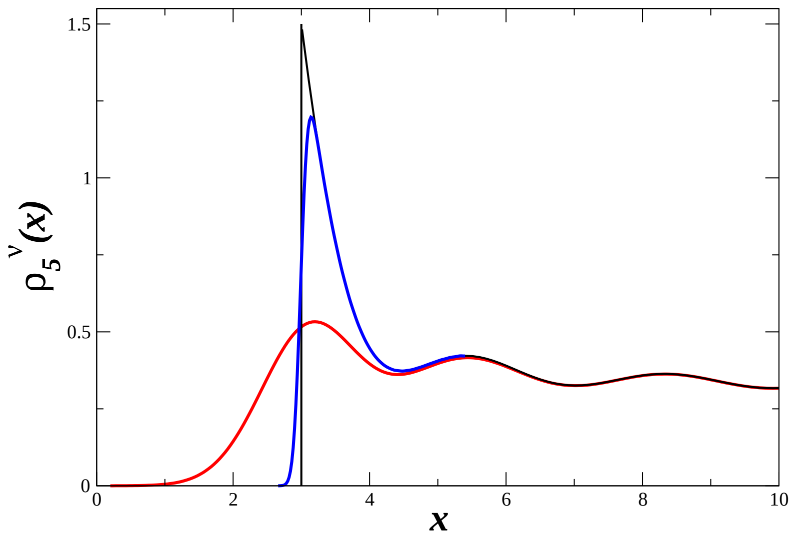
The Quenched Spectral Density

Distribution of Topological Eigenvalues

Dirac Spectra for Dynamical Wilson Fermions

Density of the real eigenvalues of D_W

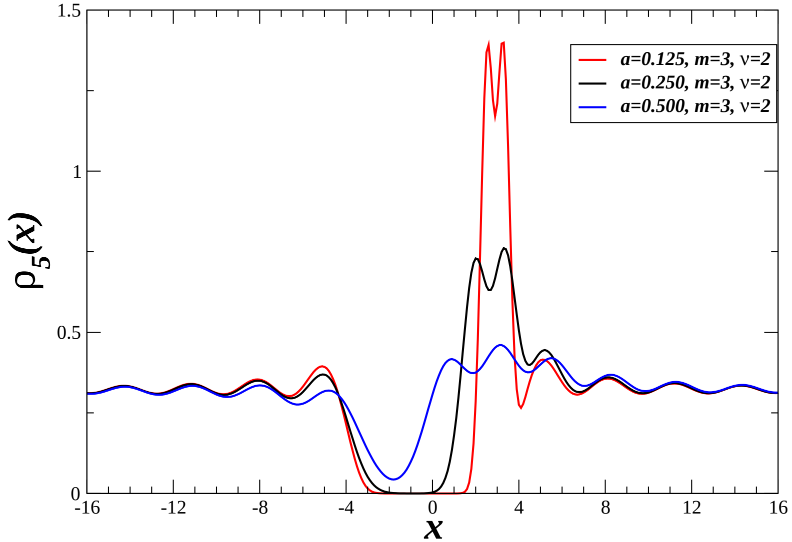
The Quenched Spectral Density



The microscopic spectrum of $\gamma_5(D_W + m)$ for $mV\Sigma = 3$, $\nu = 0$ and $a\sqrt{W_8 V} = 0$, 0.03 , and 0.250 . The $\nu = 0$ spectrum is reflection symmetric about $x = 0$.

Damgaard-Splittorff-JV-2010

The peak at $x = m$ in the left figure is due to the measure. The peak height for $a = 0$ is equal to $mV\Sigma$.



The microscopic spectrum of $\gamma_5(D_W + m)$ for $mV\Sigma = 3$, $\nu = 2$ and $a\sqrt{W_8 V} = 0.125,\ 0.250$ and 0.500 , respectively.

Damgaard-Splittorff-JV-2010

Distribution of "Topological" Eigenvalues

For $a = 0$ the eigenvalue density of D_5 can be decomposed as

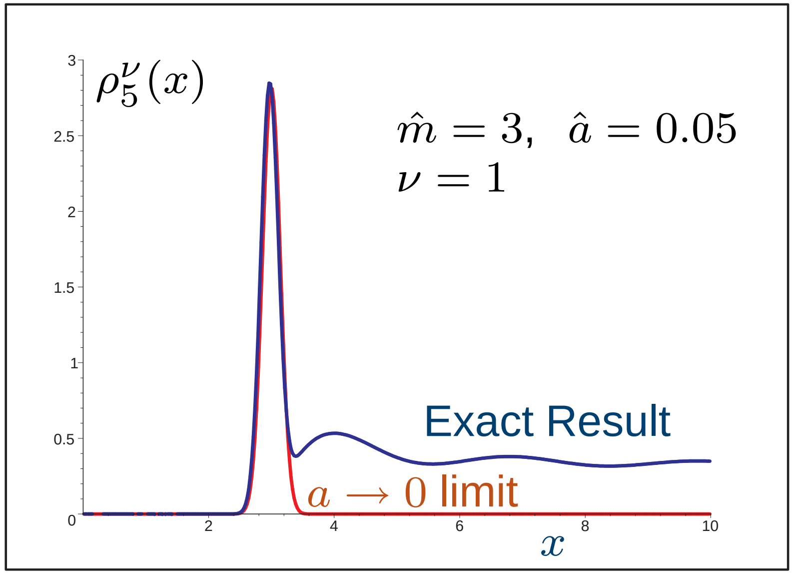
$$\rho_5^\nu(\lambda) = \nu\delta(\lambda - m) + \rho_{\lambda>m}(\lambda).$$

For $a \neq 0$ the width of the peak at $\lambda = m$ becomes finite.

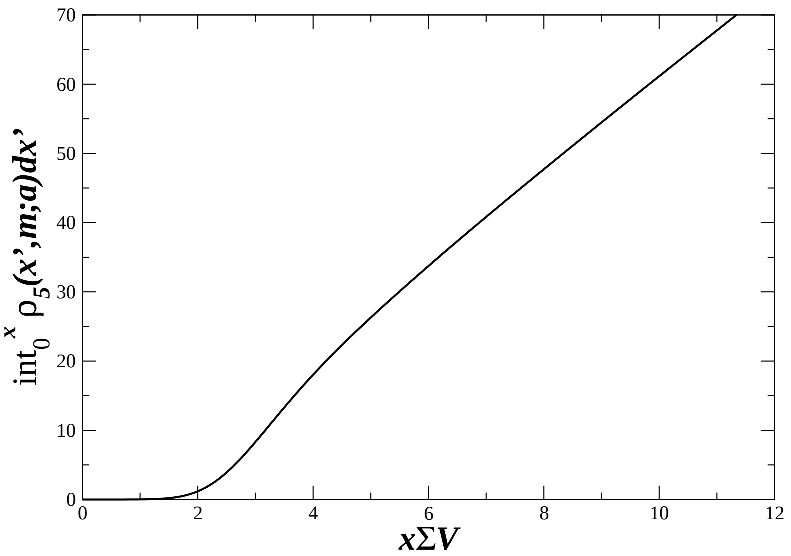
- For $\nu = 1$ the small- a result is given by (see red curve in figure)

$$\rho^{\nu=1}_{5,\mathrm{topo}}(x) = \frac{1}{4a\sqrt{\pi V W_8}} e^{-\frac{V\Sigma^2(x-m)^2}{16a^2W_8}}.$$

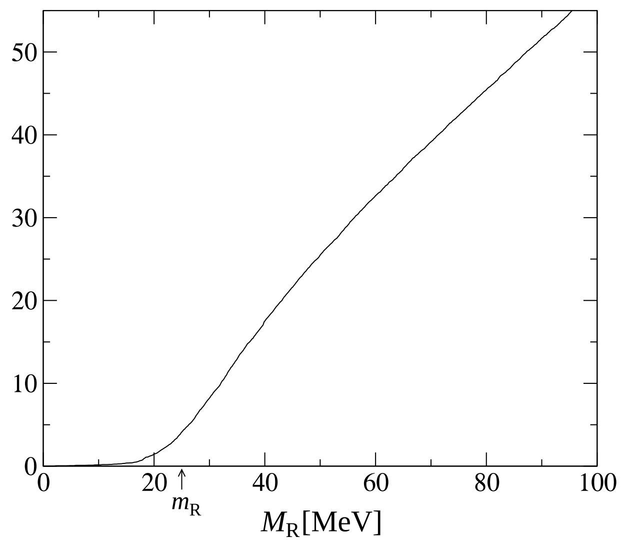
- For small a , this distribution coincides with the spectral density of the real eigenvalues of D_W (up to a shift by m).
- The low-energy constant W_8 can be obtained from the width of the distribution of the smallest eigenvalue.



Integrated Spectral Density at fixed θ



Akemann-Damgaard-Splittorff-JV-2010



Lüscher-Palombo-2010

$$\rho(\lambda, \theta = 0) = \sum_\nu \frac{Z_\nu}{Z} \rho^\nu(\lambda).$$

In the quenched case Z_ν/Z is a Gaussian with width given by the topological susceptibility.

Tail States

For $|z-m|/a$ fixed for $a \to 0$ the spectral density inside the gap can also be obtained from a saddle point analysis:

$$\rho(z) \sim e^{-\Sigma^2 V (z-m)^2/16a^2 W_8} \qquad \text{for} \quad 0 < z \ll m.$$

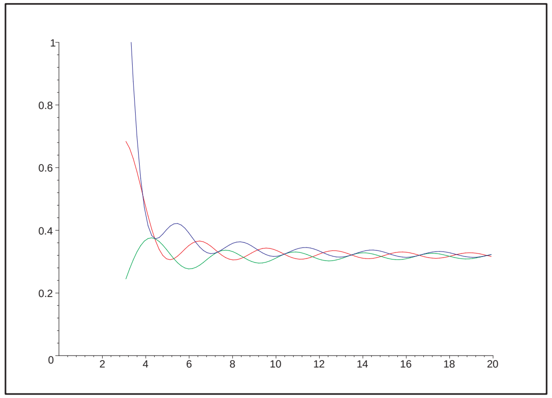
The width parameter is given by Damgaard-Splittorff-JV-2010

$$\sigma^2 = \frac{8a^2 W_8}{V\Sigma^2}, \qquad \frac{\sigma}{\Delta\lambda} = \frac{\sqrt{8}}{\pi} a\sqrt{W_8 V}.$$

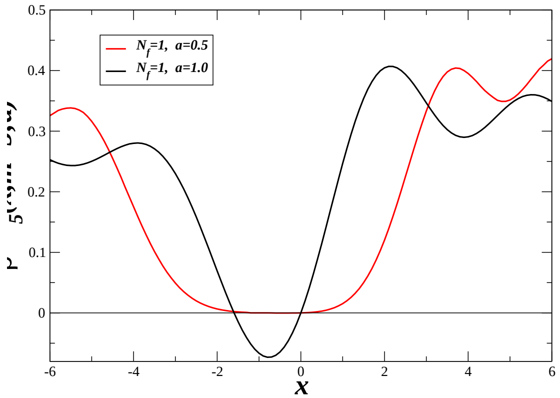
This is exactly the scaling behavior found by Del Debbio-et al-2006.

√ Typical lattice parameters are $mV\Sigma = 6$ and $a\sqrt{W_8 V} = 0.2 - 0.5$.

Effect of Light Flavors



The microscopic spectral density of $\gamma_5(D_W + m)$ for $m = 3$, $\nu = 0$ and $a = 0$ for different number of flavors. Results are shown for $N_f = 0$ (blue), $N_f = 1$ (red) and $N_f = 2$ (green).



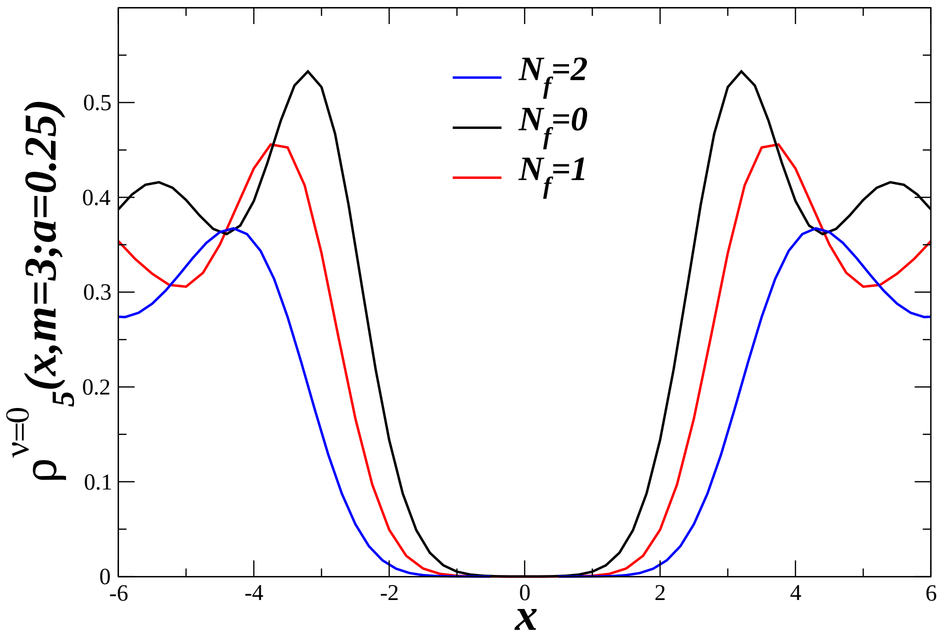
The microscopic spectral density for $N_f = 1$ and $\nu = 1$. The mass $m = 3$ and a is given in the label of the figure.

For $a = 0$ the spectral density of $\gamma_5(D_W + m)$ is obtained by a simple variable transformation

$$\rho_5^\nu(x) = \frac{x}{\sqrt{x^2 - m^2}} \rho^\nu(\sqrt{x^2 - m^2})\theta(|x| - |m|).$$

Akemann-Damgaard-Splittorff-JV-2010

Result for $N_f = 0$, 1 and 2 Flavors

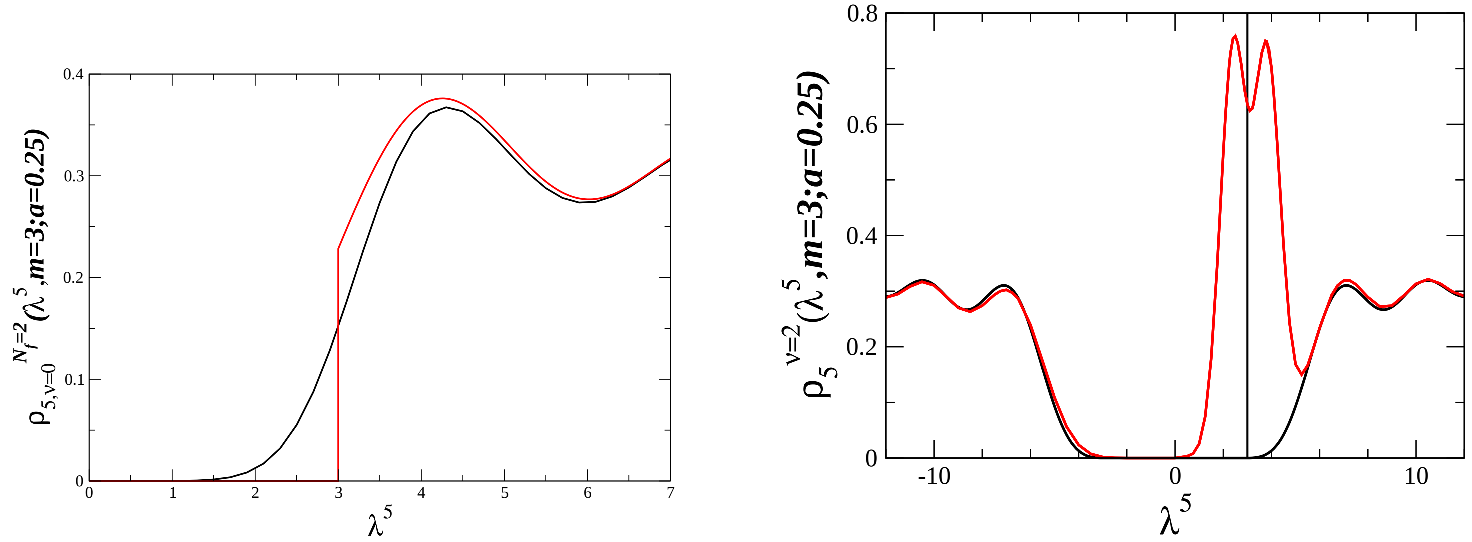


The spectral density of the Wilson Dirac operator for $N_f = 0$, 1 and 2 .

Splittorff-JV-2011

Small eigenvalues are less likely for dynamical quarks.

Result for $N_f = 2$ Flavors

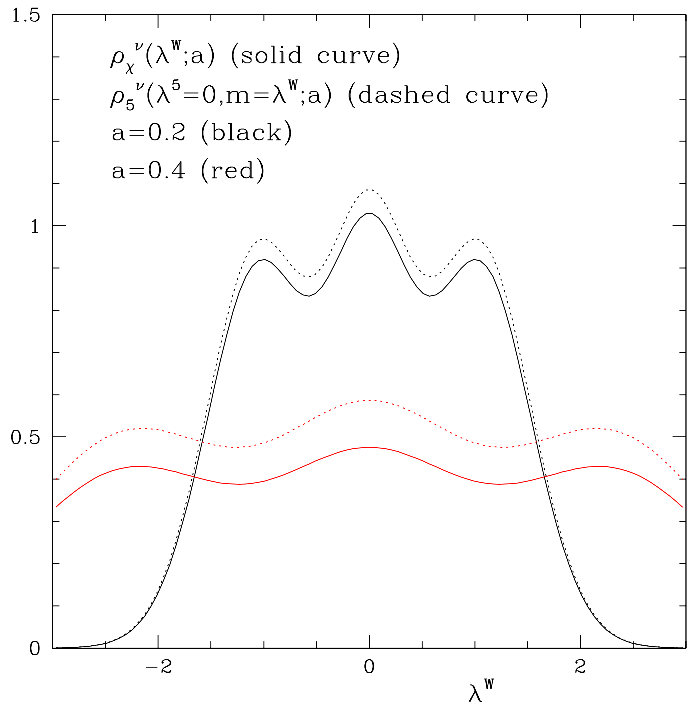


Spectral density for two flavors with quark mass $mV\Sigma = 3$ for $\nu = 0$ (left) and $\nu = 2$ (right). We show the result for $a = 0$ (red) and $aW_8\sqrt{V} = 0.25$ (black).

Splittorff-JV-2011

Nonzero topology makes the inversion of the Dirac operator much harder.

Density of Real Eigenvalues



$$\rho_\chi(\lambda^W) \leq \rho_{\rm real}(\lambda^W) \leq \rho_5(\lambda_5 = 0, m = \lambda^W).$$

The quenched spectral density of the real eigenvalues of D_W for $\nu = 3$ and $aW_8\sqrt{V} = 0.2$ and 0.4 .

Akemann Damgaard-Splittorff-JV-2010

For large ν and small $a\sqrt{V}$ the distribution of real eigenvalues approaches a semi-circle. Note that since this result is derived from a chiral Lagrangian, it is universal.

Random Matrix Theory for the Wilson Dirac Operator

A Random Matrix Theory for the Wilson Dirac operator is given by an ensemble of random matrices with the global symmetries of the of the Wilson Dirac operator. In the sector of topological charge ν, the random matrix partition function is given by

$$Z_{N_f}^{\nu} = \int dAdBdW \, \det{}^{N_f}(D_W + m + z\gamma_5) \, P(D_W),$$

with

$$D_W = \begin{pmatrix} aA & C \\ -C^\dagger & aB \end{pmatrix}. \qquad \text{and} \quad A^\dagger = A, \qquad B^\dagger = B.$$

A is a square matrix of size $n \times n$, and B is a square matrix of size $(n+\nu) \times (n+\nu)$. The matrix C is a complex $n \times (n+\nu)$ matrix. Damgaard-Splittorff-JV-2010

For $a = 0$ this is the chiral Random Matrix ensemble.

This random matrix ensemble can be defined for each of the three possible anti-unitary symmetries, but we have only studied $\beta = 2$.

In the microscopic domain, the Random Matrix Theory partition function reduces to the chiral Lagrangian introduced before with $W_8 > 0$.

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- The width of the distribution of the smallest eigenvalue scales as $a/\sqrt{V}$ suggested by lattice simulations.
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- The low-energy constant W_8 can be extracted from the distribution of the smallest eigenvalue.