Transitioning From Additive to Multiplicative Thinking: A Design and Teaching Experiment With Third Through Fifth Graders

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Dedication

This research is dedicated to all of the students and teachers who opened their minds and gave of their time for me to inquire and learn their ways of thinking while conducting this research. They are all a continuous source of inspiration and the motivation for my own learning and development as a teacher and researcher.

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Abstract

The maturation of multiplicative thinking is key to student progress in middle school as rational number, ratio, and proportion concepts are encountered. But many students arrive from the intermediate grades and falter in developing this essential disposition. Elementary students have historically learned multiplication and division as operation procedures, but this does not mean the students are thinking multiplicatively. Understanding how intermediate elementary students begin to shift from additive approaches to multiplicative thinking in scale and proportion is not fully understood.

This study asks interrelated questions of how the development of additive to multiplicative thinking evolves over time and what students' thinking looks like at various points of transition as they move through third, fourth, and fifth grades. Secondly, what integrated and coordinated series of tasks can teachers use to reveal the emergence of multiplicative thinking? The study took place over the first six months of the school year. Using the structures of a design study and teaching experiment, the principal investigator and classroom teachers collaborated using instructional tasks conjectured to reveal the development of multiplicative thinking among students. Reported here are results from the pre and post interviews of a sample group at each grade level (*n =11, 12, 11,* respectively). Student transitions over "unit confusion," the justified emergence of a place value cover pattern, the role of number size in influencing multiplicative thinking, and role of language in monitoring students' level of thinking is discussed. The development and effectiveness of the interview protocol and scoring rubric used to analyze student level of multiplicative thinking is included.

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CHAPTER ONE. INTRODUCTION

Introduction

Multiplicative thinking and reasoning takes many years to develop. It involves the making of units of units to form new composite units (Steffe, 2002), to think simultaneously across the units (Kamii & Housman, 2000; Lamon, 1994), to work "flexibly and efficiently with an extended range of numbers" and contexts (Siemon, Breed & Virgona, 2005, p. 2), and to think proportionally (Vergnaud, 1988; Lesh, Post, & Behr, 1988). The roots of multiplicative thinking can be seen in children's development of skip counting. The act of counting by fives and placing a finger of ones hand up with each count is a simple form of multiplicative thinking (Steffe, 1994). As each "one" finger extends, it represents an aggregate of five ones. Thus, "one" is five, "two" is ten (a second five), and "three" is fifteen (a third five).

A robust understanding of place value involves multiplicative thinking. For example, for every one ten there are ten ones thus forming a proportional relationship of 1:10. Its converse is also true in that for every ten ones there is one ten, a 10:1 relationship. Place value units need to be flexibly conceptualized not only within a particular place, a location, but also across places as place value is conceptualized as a rate of ten. The number 254 needs to be understood not only as 254 ones (254x1), but also as 25.4 tens (25.4 x 10), and 2.54 hundreds (2.54x10x10) all at the same time. In other circumstances this same number needs be understood as a quantity of tenths or hundredths. This ability to unitize a number (Lamon, 1994) becomes an important basis in understanding decimals and fractions. It's origin, however, begins with whole number and the development of a robust understanding of place value.

The maturation of multiplicative thinking is key to student progress at the middle school level as rational number, ratio, and proportion concepts are encountered. Probability and statistics require even more sophisticated levels of multiplicative thinking. Elementary teachers must be acutely aware of the vertical knowledge students transverse, both before and after their grade level, in developing key conceptual underpinnings of mathematics. Teachers need to be able to "trace the flirtation [of] children's most primitive conceptions of proportion, fraction, and ratio" (Vergnaud, 1988, p 86).

Curriculum resources provide lessons to teach the operations of multiplication and division. Those lessons teach steps and procedures. Lessons are sequenced, presumably, to develop conceptual understanding. But what students and teachers are focusing on may not be what is intended (Lobato, Ellis, & Munoz, 2003). Teaching someone how to multiply and divide does not mean that the student is thinking multiplicatively. The student could be working additively and still arrive at a correct solution and the teacher may not at all be aware of the student's thinking. Yet the teacher may have high fidelity to following the book. Multiplicative thinking, however, is a disposition that emerges over time. It is nurtured. It cannot be taught in a lesson or unit.

While research has identified various aspects of the foundational underpinnings of the transition from additive to multiplicative thinking, little has been done to support teachers in gathering these threads together to provide a comprehensive picture of how to support students through this process. The various learning trajectories remain unclear.

What is it then during the instruction of whole and rational number that needs to be adjusted to develop multiplicative thinking? If the end prize is middle school proportional and relational thinking, what do elementary teachers need to refocus in terms of instructional tasks and conversational patterns that, over time, nurture thinking multiplicatively? How do students at various stages of additive to multiplicative thinking sound like, think like? What mathematical strand topics need to be rethought, e.g. place value, as entry points for students to begin to think about the multiplicative relations inherent in how whole numbers and decimals are constructed? What instructional tasks scaffold students to think in more consolidated units of units en route to thinking proportionally? How do teachers engender these classroom conditions?

Student achievement is impacted by the comprehensive nature of a teacher's classroom discourse practices (Franke, Webb, Chan, Ing, Freund & Battey, 2009). The teacher, therefore, needs to be aware of the stages of, and how to support students' transition from additive to multiplicative thinking. Without a clearer understanding of the possible student learning trajectories, teachers are limited in how to shape such discourse among students. This research seeks to add further insight in these possible trajectories within the context of the classroom environment. Employing the research techniques of a design study and teaching experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, Steffe & Thompson, 2000), elaborating on these student transition points as they move from third to fifth grade provides teachers with a clearer road map of the mathematics classroom.

Research Questions

To answer the larger question of how teachers can influence the emergence of multiplicative thinking among students in third through fifth grade, two specific questions are being investigated in this research project.

1. *How does the development of additive to multiplicative thinking evolve over time and what does students' thinking look like as they transition from additive to multiplicative thinking?*
2. *What integrated and coordinated series of mathematical tasks with place value, rational number, measure, and algebra can teachers use to help reveal the emergence and development of multiplicative thinking?*

Significance

Research over the past three decades has shed light on how children develop their mathematical content knowledge within specific number content domains at the elementary age level (Verschaffel, Greer, & DeCorte, 2007; Lamon, 2007; Carraher & Schliemann, 2007). More is understood about the relationship between teachers' knowledge of student thinking (Carpenter, Fennema, Peterson, & Carey, 1988) and how that knowledge is used to shape classroom discourse (Franke, et. al, 2009), and for instructional decision-making (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). What is less understood, particularly *at the practitioner level,* is how to develop multiplicative thinking capabilities in students in grades three through five as these students engage across myriad strand areas encountered over the course of any one particular year of study. Which conceptual threads, woven together by teachers, nurture this disposition towards multiplicative thinking forming a functional way to comprehend and operate on number relations? How teachers enact these understandings and what the subsequent effects on student achievement is also unknown.

Theoretical Position

A multi-perspective theoretical lens is taken in this study to incorporate the complex dynamics of student and teacher actions that take place within the multivariate conditions of classroom learning. Understanding individual student reasoning can be viewed through a cognitive psychology lens (Siegler & Crowley, 1991, Siegler, 2000). As students and teachers make their thinking public and meaning is negotiated, a social cultural lens can be utilized in understanding how the construction of community norms and practices that encourages the public thinking and shaping of identities evolve (Lave & Wenger, 1991; Wenger, 1998). The evolution of student and teacher reasoning as each singly and in coordination interact with instructional tasks, tools, and symbols allow a distributed cognition lens to be taken (Pea, 1993). Sensitivity to issues of equity and student voice (Confrey & Lachance, 2000) brings additional consideration to how community norms and practices are used to interpret the evolution of mathematical ideas within each classroom. The attention to these issues also serves as a means of monitoring shifts in identity as individuals negotiate their legitimate peripheral participation (Lave & Wenger, 1991).

Research Structure

The work of this study is exploratory. Probing student thinking as they engage with particular instructional tasks is vital to comprehending how progressions in thinking occur. Conducting research in the context of the actual classroom environment allows investigation of the myriad interactions that potentially influence student thinking. It also draws in the classroom teacher in direct participation in the research. Two research techniques, the teaching experiment (Steffe & Thompson, 2000) and the design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) provide the structure through which data on student thinking and instructional tasks can be collected and analyzed.

Conclusion

This study seeks to investigate how various mathematical concepts combine to enhance the development of multiplicative thinking among third, fourth, and fifth grade students. The proposed teaching and design experiment will explore various pathways students take building off of understandings in one conceptual node to other related conceptual nodes. The study will investigate how various instructional supports and discourse practices allow multiplicative thinking to emerge more deliberately. Adding to the literature in this area will provide teacher practitioners an insight into how students develop multiplicative thinking over time and what instructional supports and discourse practices engender that growth.

This first chapter presents the research questions and rationale for investigating them in this study. Chapter two provides a review of the literature being drawn upon to guide the formation of the study has well as to extend the working knowledge of the field. Chapter three describes the methods and methodologies to be employed in the data collection phase and in how the analysis takes places. Chapter four describes the findings describing various transition points the students across the three grade levels traverse as well as describes the interview protocols and scoring rubric designed and tested to capture students’ range of abilities to think multiplicatively. Chapter five concludes the description of the study with a summary review of the findings, a discussion of the study's limitations, and recommendation for classroom practitioners and for future research.

CHAPTER 2. LITERATURE REVIEW

Introduction

… the way students understand an idea can have strong implications for how, or whether, they understand other ideas. (Thompson & Saldanha, 2003).

Understanding how the development of additive to multiplicative thinking evolves over time and what those transitions look like as well as understanding what integrated and coordinated series of mathematical tasks can teachers use to help reveal the emergence and development of multiplicative thinking is the focus of this study. Much has already been mapped in understanding the role of multiplicative thinking plays in a student's mathematical development. This chapter reviews the literature on the unique properties of multiplication and the construction of knowledge and the nature of learning in a multiplicative conceptual field (Vergnaud, 1988, 1994, Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). The review continues with an analysis of the literature on particular nodes, elements, of that conceptual field that is conjectured to inform this research study. The chapter concludes with a summary of how this study seeks to extend the literature in the field of student learning particularly in the intermediate grades as well as in instructional practice.

A Multiplicative Conceptual Field

Multiplication as an Operation

Multiplication as an operation has unique properties that are distinctive from addition and subtraction. Multiplication is about the making of units of units to form composite units that can then be iterated, repeated (Steffe, 2002). Multiplication requires an ability to think simultaneously across units (Kamii & Housman, 2000, Lamon, 1994) and to work "flexibly and efficiently with an extended range of numbers" and contexts (Siemon, Breed, & Virgona, 2005, p. 2). The act of multiplying is unit transforming in that the product results in a new unit or relationship not evident in its component parts (e.g., 3 bags, 4 candies per bag, 12 candies; 5 hours, 15 miles per hour, 60 miles). To have a robust understanding of multiplication is to think proportionally (Vergnaud, 1988, Lesh, Post, & Behr, 1988).

Consider the scenario of three bags of candy with four candies per bag: as the single composite unit (a bag) of a collection of individual items (4 candies in one bag) is increased by the multiplier (3 times the unit-bags), the collection of items in the single unit increases by the same multiple (3 x 1 bag: 3 x 4 candies in one bag). To reason multiplicatively is to coordinate these attributes in such a manner that the ideas are generalized across problem contexts. The individual components of unit composition, unit transformation, and unit coordination connect to form a larger multiplicative conceptual field (Vergnaud, 1988, 1994, Confrey, et al., 2009), providing the basis of working with higher levels of mathematics. What distinguishes emergent additive structures from those that are multiplicative is evidence of anticipatory thinking (Piaget, Inhelder & Szeminska, 1960), a reference to the advance, planful act of coordinating some aspect of the scaling or sharing between the number of aggregate units with the number of subunits (Empson, Junk, Dominguez, & Turner, 2005).

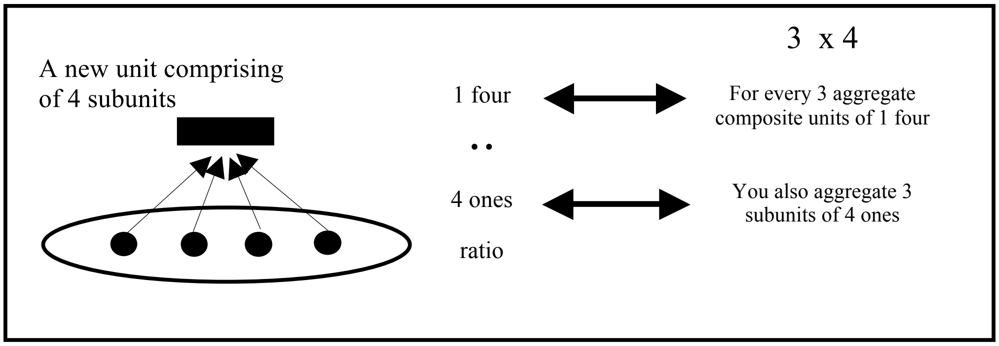


Figure 1: Multiplication as a proportional relationship

The following sections review aspects of the literature regarding core elements of a multiplicative conceptual field. Those sections include learning as conceived through the lens of a conceptual field, place value as a multiplicative relation, research on the emergence of children's strategies in multiplication and division, the concepts of unit rate and scale, multiplication as a ratio, and multiplication as explained through the act of equipartitioning. Additional sections include the role of multiplicative thinking in the conceptualization of fractions and the role of algebraic and relational thinking in the anticipatory thinking required in being able to think multiplicatively. The final section looks at the core concept of decomposition of number as an entry point to the multiplicative conceptual field.

Learning as a Conceptual Field

Case (1992) outlines the idea of a "central conceptual structure," defined as a network of semantic nodes and relationships that becomes the central basis of how a child performs and understands various tasks. Children move through progressions of coordinating information as they engage in learning. Learning is understood to emerge from engaging and generalizing across a broad range of tasks rather than through learning separate tasks in isolation. Children's ability to engage with concepts is influenced by the size of their working memory. Within that memory, understanding the relationships among the nodes is equally as important as the context of the relationships within any particular node. Performance is strongly related to children's capacity in grappling with the representational complexity of instructional tasks relative to their age. As new developmental levels are arrived at, children are able to re-work previous knowledge into a more complex conceptual structure.

Learning is not linear. Siegler (2000) notes how strategies, new and old, cohabitate as they shift over time. Strategies become linked and are drawn upon as a new unit. Multiple strategies can be drawn upon within the solving of a single task. This diversity of thinking ebbs and flows as more advanced ways of thinking are incorporated into the conceptual field. This "overlapping waves theory" observes children's learning over four dimensions: the acquisition of novel ways of thinking, more frequent use of effective ways of thinking among those within the child's current repertoire, the increasing adaptive choices among alternatives, and the increasing execution of alternative approaches.

Knowledge and learning is a generative process (Carpenter & Lehrer, 1999; Franke, Carpenter, Levi, & Fennema, 2001). Learning for understanding allows one to construct meaning for a new idea or process through linkages to existing ideas and processes. Generative knowledge is about the construction of relationships, the extension and application of, in this context, mathematical knowledge, the reflection upon the experiences, the articulation of what one knows, and the making of the knowledge ones own. It is rich in structure and connections. The knowledge is learned in ways that clarify how it can be used across contexts.

Returning to Vergnaud's (1988, 1994) multiplicative conceptual field, building connections among a variety of mathematical concepts and actively reflecting upon the links among individual nodes allows children to build upon their understanding. Analyzing the nature and patterns of individual strategies, and the emerging efficiencies in how those strategies are used, provide insights into the various learning trajectories children follow as their knowledge becomes more generative. Confrey, et al. (2009) outlines a learning trajectory map for rational number reasoning (p. 4). This mapping, in effect, provides one example of the interrelationships among individual nodes of mathematical concepts that rely on elements of multiplicative thinking.

Several researchers have noted attributes of children who come to think multiplicatively and reason proportionally. The following provides an overview of the literature being drawn upon to develop a multiplicative conceptual field for this research.

Place Value and Base Ten Understanding

Place value is a multiplicative relationship. A multidigit number in our primary number system is comprised of a series of composite units based upon an exponential rate of ten. Ten is both a collection of ones and a single composite entity. One hundred must be understood as a set of a hundred single units, ten composite units of ten, and one composite unit of one hundred. Kamii (2000) refers to this comprehensive knowledge as "simultaneous thinking," as a child needs to construct, hold, and flexibly negotiate these relations in a single moment. Place value, as a multiplicative relationship, matures over time. A robust understanding of place value has been found by some not to fully mature until fifth grade (Clark & Kamii, 1996). Children's place value concepts remain fragile up through that time.

Using multiplication and measurement division, where ten is the organizing unit, are powerful tools in nurturing children's base ten understanding (Carpenter, Fennema, Franke, Levi, & Empson, 1999). This connection of using multiplication and division extends to developing place value concepts into decimals (Empson & Levi, 2011). The numbers given in a problem such as, "There are 64 crayons in a tray. If the crayons are placed into boxes where 10 crayons fill up a box, how many full boxes can be made?" allows a child to explore the tens and ones relations within a whole number (64 = 6x10+4). A problem such as, "A painter has been hired to paint the walls inside an apartment building. Each apartment requires 10 gallons of paint. He has 37 gallons of paint. How many apartments can he paint before he runs out and has to go buy more?" engages the child in expressing the remainder as a tenth of a gallon and the place value relationships that it describes in terms of a decimal (37÷10 = 3.7) (Empson & Levi, 2011).

In her early work on children's conceptual development of place value ideas, Fuson (1990) noted how Chinese-based languages develop place value sooner than Western-based languages. The reason is felt largely due to the role of language used to describe number. While English has twelve unique names for its first whole number counts, Chinese-based languages are more descriptive of the place value once past ten. After ten literally comes 'ten-one', 'ten-two…' After what we call 'nineteen' comes 'two-ten' followed by 'two-ten-one.' At this point, the multiplicative relationship of 2x10+1 is verbalized in the very manner in which the number is named. In the English word 'twenty-one,' the multiplicative relationship is more inferential; more effort needs to be made to construct the place value relationship.

In a longitudinal study, researchers showed that first through third grade students allowed to consistently develop invented strategies in multidigit addition and subtraction solidified place value concepts sooner than those students who relied on the traditional algorithms (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). The typical approach to multidigit algorithms is to single-digitize or concatenate the numbers (Fuson, 1990, Fuson, et al., 1997) thus setting aside the multiplicative aspects of place *value* to only focus on "*place*," meaning location, to operate on the numbers. Three implications of these findings can be considered. One is that an earlier solidification of place value relationships indicates the emergence of multiplicative reasoning, at least around ten. The second indicates that the use of single digit language to operate on multidigit numbers suppresses the development of place value as a multiplicative relation. The third is that language significantly shapes the development of place value as a multiplicative relation. These implications have led some in the research field to make the direct recommendation that standard algorithms in their single-digitized form should not be introduced to students until after fourth grade due to the fragility of students' place value understanding up to this point of their development (Kamii & Dominick, 1998).

Multidigit Multiplication and Division Strategies

Studies have shown how students' abstract addition and subtraction give rise to more multiplicative coordination used in developing multidigit multiplication and division strategies. Ambrose, Baek, & Carpenter (2003) analyzed how repeated addition, doubling, and complex doubling give way to more sophisticated decomposition of the multiplicand and multiplier to coordinate actions on multidigit numbers. The decomposition of number evolves into two algebraic forms: decomposition into addends to use the distributive property of multiplication over addition, or decomposition into factors to use the commutative and associative properties.

Improving the ability to think in larger and more efficient units is seen in students' alternative strategies. Ambrose, et al. (2003) noted that in the evolution of students' additive strategies, new composite units of added items were then iterated to form more complex units as a means to find efficiencies. An example would be that of the work of Maddie solving 30 x 300. Maddie added 300 five times to get 1,500. She then iterated that amount three times, stating, "since 3x5 is 15 I added 1500 3 times. Since 15 is half of 30 I added 4,500 2 times. The answer is 9,000" (Brickwedde, 2006, unpublished student work sample). Steffe (1988) proposed that children's ability to construct iterable units was key to moving into multiplicative thinking. While Steffe states that this is hard for younger students to do, Ambrose, et al. found it more common among those students allowed to use invented strategies to solve multiplication and division problems.

Dutch researchers found similar results in analyzing students' long division strategies; a process referred to as "progressive mathematizations" (van Putten, van den Brom-Snijders, & Beishuisen, 2005). These progressive actions involve an increased ability to use iterable amounts as a means to repeatedly subtract from the dividend. As students increased their ability to think in larger and larger composite units, the use of a scale factor emerged. This increase in the use of scale factor reflected the ability to think more multiplicatively.

Unit Rate & Scale Factor, Multiplication as Ratio, & Equipartitioning/Splitting

Vergnaud's (1988, 1994) postulated the idea of a measure space as a means to assist students in developing two mathematical ideas: scale factor and unit rate. In the scenario of, "If 3 tennis balls cost $5.25, how much would 8 tennis balls cost?" the numbers are such that a student might calculate the unit rate of one tennis ball and then multiply it by 8. The same setting but wanting to know what 12 tennis balls would cost may result in a different approach. Seeing a relationship of four threes in twelve, then a student need only multiply $5.25 by the same factor (4) to find the answer. The use of the tabular format of the measure space helps students develop both recursive (scale factor) and explicit (unit rate) capabilities. While each of the above scenarios could be solved additively, the "degree of abstractness" (Kouba, 1989; Mulligan, 1992) increases towards more multiplicative strategies as the ability to work with iterable composite units matures.

"Conceptualized" multiplication - the image one makes while conceiving of multiplication - concerns the creation of units of units (Thompson & Saldanha, 2003). It requires students conceptualizing measured quantities (Vergnaurd, 1988). Measurement entails a ratio comparison (Thompson & Saldanha, 2003). The increase of a measured composite unit results in an equal factor increase of a quantity's measure of subunits.

Thus if a ratio of 10 millimeters to every one centimeter exists (10mm:1cm), increasing the number of centimeters by *a* (1*a*) requires an equivalent factor increase of millimeters (10*a*). That equivalency is in a proportional relationship. Multiplicative thinking, at its core, requires the conceptualization of a ratio and transcends how one operates in multiplication, division, measurement, and fractions.

A related but differing perspective is that multiplicative thinking emerges more out of the act of "splitting," the equal partitioning of objects and relating that division back to the whole (Confrey, 1994, Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). This position argues that splitting, or equipartitioning, is a more primitive action on the part of young children and that multiplication follows as the inverse of the partitioning action.

The two forms of division, measurement and partitive (Carpenter, et al., 1999), lead to two forms of multiplicative thinking. Measurement division – how many twos are in twelve? - is iterative in structure and is most associated with the construct of multiplication as repeated addition and division as repeated subtraction. Partitive division –*n* people sharing a cake equally – generates the relationship of *n* people to 1 cake (*n*/1). The relationship can be inversely understood with the 1 person as 1/*n* of the cake (1:n). A scalar ratio and equivalent inverse relation is established. The 1 cake is *n* times greater than $1/n^{th}$ of a one-person share. Confrey, et al. (2009), along with Thompson and Saldanha (2003), argue that this supports the need for an increase attention to the "*n* times as many" construct of multiplication rather than the over reliance of multiplication as equivalent grouping. It also helps argue the position of some for why multiplication and both forms of division should be taught simultaneously beginning in younger grades (Mulligan & Mitchelmore, 1997, Carpenter, et al., 1999).

Paper folding is an act of partitioning that relies on multiplicative thinking as one reflects upon the series of actions (Emspon & Turner, 2006). While early attempts to predict the number of equal regions that will occur with successive halving are frequently additive in structure, engaging students in reflective discussion as they test and revise their work resulted in shifts from additive to emergent multiplicative thinking. They also noted in their analysis that two forms of doubling strategies were present. One tended to generate multiplicative reasoning while the other did not. The explicit coordination of folds that did emerge reflected the anticipatory thinking (Piaget, et al., 1960) that underpins multiplicative reasoning. This included recursive (how a fold operating on a previous fold is related to the number of partitions) and functional (how a series of folds produces a certain number of partitions) reasoning.

Fractions

Fractions are the first relational numbers that children encounter in the number system (Smith, 2002). Fractions require the context of a unit. The number one-half is only meaningful if it is defined by an explicit unit. One-half of a small cake is not the same physical amount as one-half of a large cake. Fractions also can refer to sets of objects as well as to a partition of a single object. What defines the unit in fractions is broader than what needs to be understood in whole number. A fraction describes an amount relative to the unit (Lamon, 2005).

Establishing the unit, or *unitizing* as described by Lamon (1994, 2005), is a subjective process. It requires the ability to flexibly regard the same commodity from various perspectives depending upon what the “whole” or unit is. Thus a “six-pack” could also be unitized to be three two-packs or two three-packs or one-half of a twelve-pack as well as one fourth of a 24-pack. The reasoning engendered by flexibly re-unitizing a collection of objects is the basis for proportional reasoning. It develops a sense of equivalence and an understanding of rational number as quotients (Lamon, 2005).

There are several constructs that these relationships can be conceived within a person’s mind. These sub-constructs are fractions as part-whole relationships, fractions as quotients, fractions as a operator, fractions as a ratio, and fractions as measure (Kieran, 1988, Behr, Harel, Post, & Lesh, 1993, Behr, Harel, Post, & Lesh, 1992). To have a robust understanding of rational number, in time a child needs to be able to engage in the whole range of constructs and interrelate the thought and action of one structure with those of the others (Behr, et. al. 1992).

The discussion in the previous sections demonstrates how multiplication and division of whole number is directly related to the multiplicative relational thinking necessary to conceptualize fractions. The act of partitioning and ‘splitting’ (Confrey, 1994) units through equal sharing requires the relational coordination among the items being shared and the number of sharers (Empson, Junk, Dominguez, & Turner, 2005). The role of ratio and measure in the formation of composite units are foundational concepts in multiplicative reasoning. Steffe (2002) hypothesizes that the actions of iterating and partitioning, key components of multiplicative reasoning, are parts of the same psychological structure as fractions. The result is that children's ability to construct composite unit fractions is interdependent upon children's multiplying schemes. These acts of unit construction and deconstruction require the coordination of both the action to produce the splits as well as the operation that produces the fractional units.

Algebraic and Relational Thinking

The ability to create and decompose units, to iterate, and scale reflects a student's understanding of equality and other basic algebraic properties (Carpenter, Franke, & Levi, 2003). Nurturing relational thinking allows students to explicitly reflect upon their implicit understandings. This allows students to generalize across contexts and act with deliberateness and understanding. Integrating the learning of arithmetic through the lens of algebra makes both easier and richer. The student is able to comprehend the structure behind the operational choices that are made. The algebraic properties and conjectures provide necessary structure in order for a student's mathematical knowledge to become generative.

"Quantitative reasoning" (Smith & Thompson, 2008) allows one to reason about relationships without the variable assignments that would come with algebraic expressions. It is flexible and general in character. It requires the student to project his or herself into the problem situation to invoke visual imagery in order to reason about the relationships among the quantities involved. Nevertheless, quantitative reasoning is the root of algebraic thinking.

"Relational thinking" is the ability to look "at expressions and equations in their entirety, noticing number relations among and within these expressions and equations"

(Jacobs, Franke, Carpenter, Levi, & Batty, 2007, p. 260). It is a flexible approach to calculations that draw upon the basic underlying algebraic properties of decomposition of number, commutative, associative, and distributive properties. It reflects a shift in focus from seeing arithmetic merely as a means to calculate answers and broadens it to include the examination of relations. The relationships between two expressions are based upon a firm understanding of equivalence. With each legitimate transformation of those expressions in the equation, equivalent ones are created.

Children often operate on an inaccurate definition of the equal sign (Carpenter, Franke, & Levi, 2003). The common misconception is that “equals means the answer comes next.” This is a view of the equal sign being a symbol indicating an act of operation much like the addition and subtraction symbols. Many teachers are unaware of the manner in which their students regard this crucial symbol. This inhibits the transition from arithmetic to algebra in the middle school years. The early development of an accurate and robust understanding that the equal sign is a relational symbol that compares the same relative values is essential. It makes the mathematics easier and the transformation of quantities, and the generalized means that allows for such transformations, better understood.

Decomposition of number

A recurring sub-skill noted in preceding sections of this chapter is the need for a child to decompose a number in order to operate on it. The term ‘decomposition’ refers to the act of breaking a number into sub-units in order to re-aggregate it with other numbers. In multiplication, the algebraic distributive property is based upon the decomposition of a number into sub-units – typically by place value terms – multiply across the addends to create a series of partial sums, then combine all the partials to find the product. A number can also be broken into its factors, reordered (commutative property) and recombined with other factors (associative property) to find a product. Fractions are descriptions of sub-unit decompositions in relation to its original whole.

Steffe (1988) and Fuson (1990) outlined children's conceptual structures for counting sequences and multiplying schemes and for multidigit addition and subtraction strategies, respectively. In each of their analyses of student progressions in thinking, students needed to shift from a unitary concept of number (a number quantity as a collection of ones counted out as such) to various forms of chunking to make more efficient composite units that can then be iterated. Fuson notes the importance of strategies such as making a ten or moving back to a ten in order to add (17 + 5) or subtract (22 – 5) thus moving above or below into the next decade. Decomposing five into 3 + 2 allows for efficiencies of movement rather than relying on the more emergent unitary counting by ones.

Steffe argues that children's developing multiplying schemes are built upon the ability of making units of units. These composite units can be thought of as the decomposed sub-units of the final product. Carpenter and Moser's (1984) framework of children's solution strategies in single digit addition and subtraction notes a developmental progression from direct modeling to counting on strategies (all unitary actions) to deriving and flexible strategies that entail decomposition of number and algebraic relational thinking. An implication of this notion of unit composition/decomposition of number is if a child can move forward into developing multiplication schemes without having the capacity to compose and decompose numbers to work in efficient chunks in addition and subtraction.

Summary

Multiplicative reasoning is a particular reflection of quantitative reasoning and relational thinking. The coordination of composite units, ratios, proportional equivalencies all involve negotiating relations among quantities. Decomposing numbers by addends or factors to use the distributive or the associative properties underlie basic multiplication operations. The properties of zero and one allow for key equivalent transformations. Unitizing up and down (Lamon, 2005) reflect the flexible creation of equivalent relations.

The roots of multiplicative thinking are numerous and complex. It emerges from the generative symbiotic connections among many ideas. There is no linear trajectory through which it emerges. While the multiplicative field can be mapped (Vergnaud, 1988; Confrey, et al., 2009), children's pathways through such a field are web-like and take a long time to develop. As with any system, engaging students only through one of the elements of the system is not enough to effect the development of multiplicative thinking. A child needs consistency in approach across the various nodes so that ideas developed in one supports and instructs development in the other.

The literature of how to support students' transition from additive to multiplicative thinking reflects knowledge about individual nodes. There is only a recent attempt to comprehend the larger conceptual field (Confrey, et al., 2009). Research evidence at times seems in conflict. How do students' additive strategies growing out of repeated addition support (Ambrose, et al. 2003) or hinder (Thompson & Saldanha, 2003) the development of multiplicative thinking? Is partitioning a means to develop multiplicative thinking and fractional understanding (Confrey, 1994; Empson, et al., 2005) rather than the part-whole construct (Lamon, 2007).

What is clear is that all of these elements exist within the central conceptual structure that Case (1992) describes. Engaging in and generalizing across a broad range of tasks rather than through learning separate tasks in isolation reveals the larger structure. To comprehend the structure of a concept is to know the mathematics (Dienes, 1960). Understanding the structure allows the knowledge to become generative.

It is looking at the multiplicative underpinnings of the individual nodes of place value, scale factor, equipartitioning, fractions, and algebraic properties and reasoning while learning the operations of multiplication and division that this study is based. Viewing these mathematical concepts as a connected field of ideas rather than singly is posited to better reveal the various transitions and learning trajectories student transverse as they shift from additive to multiplicative thinking. Understanding these transitions adds to the research field's comprehension for how learning and knowledge is combines to form the larger conceptual field. How these individual conceptual nodes incorporated into instructional tasks combine and interact to move students forward in the increasing levels of abstraction addresses the practical instructional impact for classroom teachers. Therefore, both theory and practice are addressed in this study.

CHAPTER 3. METHODOLOGY

This study asked two questions. The first was how does the development from additive to multiplicative thinking evolve over time and what does student thinking look like as they transition from additive to multiplicative thinking. The second question asked what integrated and coordinated series of tasks can teachers use to help reveal this emergence and development of multiplicative thinking. To investigate these questions, three classrooms of students, one each at third, fourth, and fifth grade, along with their teachers, participated in a design and teaching experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, Steffe and Thompson, 2000) over a six-month period (September through February). The site was a highly diverse school district in a first-ring suburban of a large urban metropolitan area in the Upper Midwest of the United States.

Research Design

A combined design and teaching experiment was conducted in three intermediate grade classrooms over a twenty-two week period (September through February). The nature of a design experiment is such that the researcher is directly involved in a collaborative relationship with the teachers and students in the classroom (Cobb, 2000). This methodology is what Schoenfeld (2007) refers to as a Phase I study; "studies that develop and refine initial ideas, providing evidence that they are worth pursuing" (p. 97).

The principal investigator was embedded as both a co-observer and active co-teacher in the mathematics instruction portion of the day at each grade level. Planning instructional tasks and analyzing student work was done in collaboration with the classroom teachers as they are considered co-participants in the study. The objective is to understand the emergence of the rise of multiplicative thinking, the various learning trajectories (Simon, 1995) different students take in the development of that thinking (research question one), and intentionally injecting and testing the results of learning tasks selected to foster multiplicative thinking (research question two). Working with students at each of the three grade levels allowed for a cross-sectional view of student development that would otherwise require a longitudinal structure to reveal. An iterative process of analysis, revision, redesign, and retesting was used to understand shifts in students' thinking.

Augmenting classroom data, a pre and post one-on-one interview teaching experiment (Steffe and Thompson, 2000) was conducted with a sample of students from within each grade level. These interviews allowed for a closer analysis of various learning trajectories taken by students within a particular classroom. These data provided greater density in understanding the shifts over time made by students as new knowledge was integrated with earlier conceptions of the mathematics. Thus this study drew upon both qualitative and quantitative techniques to understand the shifts in student thinking.

> "A main purpose [of descriptive research is to] explain not only how things took place (descriptive power) but why students learned what they did (explanatory power), thus enabling others to try similar things in the hope of obtaining similar results (prediction)" (Schoenfeld, 2007, p. 86).

The design and teaching experiments were structured to use several analytical means to capture student progressions in thinking. These included observations, direct teaching experiments, interviews, and debriefings with students and teachers. Video and audio recordings, and analysis of student work samples were utilized to provide traces of the public work of the mathematical classroom as well as within the one-on-one or small group settings. The combination of artifacts provided a chain of evidence that contains the density needed to understand student transitions in moving from additive to multiplicative thinking, the various overlapping trajectories that students move along, and the increasing levels of efficiencies and adaptive choices that they made. Incorporating and analyzing teacher perceptions of the core instructional elements that combine to move students forward in that thinking was possible given the density of evidence collected. Use of standardized test scores provided an outside comparison measure to triangulate the analysis from the student work completed within the classroom and interview settings.

Participants and Selection Process

The research was conducted at a K-5 elementary school set within a first ring suburb of a large metropolitan area in the Upper Midwest of the United States. The District included all or part of thee communities. During the 2009-2010 academic year, 2,880 students were enrolled kindergarten through twelfth grade. The district is comprised of three elementary sites, one middle school and one high school site. District demographic data reflected a diverse ethnic and racial student population including 28% of students having Limited English Proficiency (LEP) and 72% on free and reduced lunch. In comparison, the school research site, with a building population of 469 students, had 37% LEP population with 80% of students receiving free and reduced lunch. Across the whole district 555 students open-enrolled into the district, the majority of whom came from the central city under a school integration program. In contrast, 983 district students open-enrolled out of the district reflecting a net loss of the local population to other schools. Table 1 summarizes the comparative demographic information.

Table 1
District & School Demographics

	Am Indian	Asian	Black	Hispanic	White	LEP	Spec. Ed.	Free & Reduced Lunch	Total Enrollment
District	3	6	31	26	34	28	15	72	2880
School	3	5	31	36	26	37	11	80	469

Note: Number of students open enrolling into the district: 555
Number of district students open enrolling outside the district: 983
Site met AYP benchmarks in all categories in Reading and Mathematics for 2009-2010 and was in “safe harbor” status

The principal investigator had a previous one-year relationship with this district providing embedded professional development around elementary mathematics. The nature of this professional development was to provide teachers with a background in the Cognitively Guided Instruction research (Carpenter, et al., 1999) on how children develop their mathematical understanding, assist in infusing this research in how teachers implemented the *Everyday Mathematics* (Wright Group/McGraw Hill, 2007) curriculum adopted by the district, and to provide in-class demonstration teaching support directly with students. This relationship between the district and the principal investigator continued during the 2010-2011 academic year in all three district elementary sites.

The research site was identified by the principal investigator as having a team of third through fifth grade teachers, special education, and English as a Second Language staff who actively used the professional development process in their classrooms in a consistent and reflective manner. This allowed the quick establishment of a team approach to be built and maintained as the design of this research project required.

The teachers at the site controlled the decision as to which classrooms the principal investigator had access. Due to staffing arrangements resulting in clusters of special education and English language learners within specific classrooms, the teachers decided that the research should be conducted in the top level classroom at each grade level. The students in this group was determined at the beginning of each unit within the curriculum with the teachers pre-testing students and then clustering those students accordingly into the three different classrooms at each grade. Sample groups were then drawn from the core of these top tier clusters.

The schedule for when math classes were conducted at each grade level was pre-determined at a building staffing level. An overlap between the third and fifth grade sessions resulted. The implication for the research was that contact with students within the instructional time period varied from one grade to the next. Table 2 outlines the nature of contact with each grade. Adaptations were made so that students experienced, at a minimum, a consistent exposure to the mental math warm-up activities designed during the project. At third grade, the classroom teacher implemented these warm-up activities as the researcher was entering the room. In the other two classrooms, the researcher led these activities on the days when he was present in the building. Audio recordings were made of whole and small group classroom sessions. The majority of the sessions were mental math tasks completed at the start of the class session. The observing teacher of these sessions kept field notes of the students who presented and a short description of the strategy used. Photocopies of pertinent student work samples were collected for those activities completed as part of a full lesson.

Table 2
Nature of Contact With Each Grade

Grade	Portion of Lesson	Nature of Contact with Whole Class
Third	Last 30 minutes	Warm-ups sometimes led by classroom teacher, some held at end of lesson by principle researcher, significant lesson re-writing for some units
Fourth	Full lesson	Warm-up activities and observational support through rest of lesson; Some lesson writing for some units
Fifth	Beginning 20-25 minutes	Warm-up activities with students only

Rationale For Method Selection

A multi-perspective theoretical lens was taken in this study to incorporate the complex dynamics of student and teacher actions that take place within the multivariate conditions of classroom learning. Understanding individual student reasoning can be viewed through a cognitive psychology lens (Steffe, 1988, 1994). As students and teachers make their thinking public and meaning is negotiated, a social cultural lens can be utilized in understanding how the construction of community norms and practices that encourages the public thinking and shaping of identities evolve (Lave & Wenger, 1991; Wenger, 1998). The evolution of student and teacher reasoning, as each singly and in coordination interact with instructional tasks, tools, and symbols, allow a distributed cognition lens to be taken (Pea, 1993). Sensitivity to issues of equity and student voice (Confrey & Lachance, 2000) brings additional consideration to how community norms and practices are used to interpret the evolution of mathematical ideas within each classroom.

Overlapping Waves Theory and its related microgenetic methods of analysis (Siegler, 2000; Siegler & Crowley, 1991) adds further texture to the theoretical lenses being drawn upon. Learning occurs in a series of overlapping sequences of events rather than in a linear fashion. Children use a variety of strategies and ways of thinking. They use diverse strategies and ways of thinking that coexist over a long period of time. New experiences bring on change relative to those original ways of thinking. "Rather than focusing on *the* age at which children develop a given capability, we would trace over time the set of approaches that they use" (Siegler, 2000, p. 30). What then becomes the focus of observations is the manner in which the acquisitions of novel ways of thinking occur, how more frequent use of effective ways of thinking emerge from existing and alternative possibilities, and how increasingly adaptive choices are made.

No one perspective captures the complexities of classroom learning environments. Each theoretical perspective leaves some aspect of learning unexplained. In combination, however, more of the interactions can be understood and described. It is in juxtaposition with each other that a more dynamic interpretation arises from the investigation. In the end, the setting of this research in the context of the classroom allows one "to tease out aspects of the setting that [serve] to support the development of student reasoning" (Cobb, 2007, p. 30).

The questions raised in this research bridge these theoretical views. How does the development of additive to multiplicative thinking evolve over time is one of developmental progression. The social cultural, situated, and distributed cognition lens is particularly useful in understanding the public shifts in student thinking over time. What does student thinking look like as they transition from additive to multiplicative thinking is a more local question. The cognitive psychology and Overlapping Wave theories provide a lens to understand the more individual shifts in thinking. Combining these perspectives adds depth to understanding this transition from one mode of thinking to the other.

Guiding Conjectures

To probe how the development of additive to multiplicative thinking evolves over time, selecting which multiplicative conceptual nodes to juxtapose in this study to foster the emergence of multiplicative thinking was proposed and tested. The initial planning phase of the research was driven by conjectures drawn from the larger research literature (Confrey & Lachance, 2000). Four conjectures were probed as a means of understanding how the inter-nodal relationships influence the emergence of multiplicative thinking among students over time. As the analysis of student thinking in response to the instructional tasks began, the iterative cycle of adapting tasks, plotting the emergence of student thinking, and confronting the veracity of the initial conjectures occurred. In that process, research question two (which tasks) helps to reveal evidence needed to address research question one (understanding transitions). Four initial conjectures were used to plan instructional tasks:

Conjecture 1:

> Place value, deeply understood, requires an understanding of the rate of ten and the ability to simultaneously re-unitize across place the various combinations of units (Kamii & Housman, 2000; Lamon, 1996, 2007). As place value is a multiplicative relationship, such a deep understanding of the rate of ten is foundational to developing multiplicative thinking. Early elements of building an

understanding of the rate of ten include: Consistent usage of the language of ten, meaning talking in terms of a numbers' value rather than in single digits (Fuson, 1990); fluency with organizing around landmarks of ten; fluency in decomposition of number, in particular for developing the use of the distributive property of multiplication over addition and factoring or "splitting" (Confrey, 1994); flexibility in using multiple strategies in solving multidigit addition and subtraction tasks (Carpenter, et. al, 1998).

Subsequent attention to these number reasoning habits and developing the capacity of students to mentally engage in such tasks allow multiplicative thinking to arise earlier and more robustly among learners.

Conjecture 2:

Viewing multiplication as equal groupings (repeated addition) alone, while an important entry point for young learners, is insufficient in developing the capacity to think multiplicatively. The capacity of coming to see multiplication as a ratio, as a proportion, is central to moving on to more advanced mathematical ideas. Elements of building ratio and proportion among early learners include attention to scale factor and unit rate through the use of a measure space (Vergnaud, 1988, 1994), multiplicative comparison situations, equivalency tasks, and spatial structure concepts of measure and similarity (Boester & Lehrer, 2008).

Conjecture 3:

Understanding multiplication of whole number as a ratio allows an earlier robust understanding of fractions as a proportional relationship. Fractions as equipartitioning or equal sharing require increasing coordination among sharers giving rise to multiplicative thinking. Unitizing part-to-whole relationships in geometric structures (Lamon, 2005) requires multiplicative coordination as one works back and forth among the parts to whole and whole to parts.

Conjecture 4:

Understanding the structure of mathematical ideas allows a student to act more explicitly interpreting and solving tasks. To think relationally across the equal sign to compare, contrast, and reflect on the structure of mathematical expressions is foundational to an ability to reason multiplicatively. Thinking algebraically – understanding the general properties of arithmetic – allows a learner to more explicitly solve complex multiplicative relationships (Carpenter, Franke, & Levi, 2003). Explicitly developing the capacity within learners will benefit in the maturation to think multiplicatively.

Design Experiment Elements

The classroom teachers and the principal researcher collaborated on how areas of research interest, as outlined in the four conjectures noted above, would integrate with the flow of mathematical topics outlined both in *Everyday Mathematics* and in the District's pacing calendar. That pacing calendar already reflected local decisions regarding adaptations to the curriculum to best meet state mathematics standards. *Everyday Mathematics* lessons consist of an opening mental math warm-up sequence followed by the main lesson and ending with independent practice. Using this basic structure, significant redesign of the content of the mental math warm-ups around research themes occurred. As indicated in Table 3.2, the mental math warm-ups was consistently used to engage students in verbal exchanges where strategy development, comparing and contrasting of strategies, and the capacity to reflect and verbalize on ones thinking was emphasized.

The evidence from students' thinking in these settings influenced the design and discussion around warm-up sequences on future days. While clear mathematical goals and conjectures focused the planning of tasks, tasks were adapted both in the moment of instruction as well as for the next day's instruction in response to the students' thinking (Hiebert, et al., 1997). These adjustments were to accommodate the various learning trajectories evidenced through analysis of student responses.

As the research study progressed, classroom teachers, particularly at third and fourth grade, initiated conversations with the principal investigator around the lesson portion of various instructional units. Redesign of several units took place to reflect the Cognitively Guided Instruction research base on how students learn the mathematics and the conjectures framing the study. These redesigned units allowed classroom teachers to provide consistency in mathematical content and instructional approach during those times the researcher was absent from the site as well as when co-teaching or in tandem with the researcher while present. Student response and teacher feedback was used to revise and redesign lessons as needed to reflect shifts in student understanding. This design, test, revise, test iterative cycle reflects the core elements of the design experiment model.

Data Collection

Data was collected in two settings: The classroom during regular daily math instruction with the whole class, and in one-on-one interview sessions with a sample of students at each grade level.

Classroom data

Classroom sessions where instructional tasks were implemented by the principal researcher were audio recorded. A few class lessons were also videotaped. Supplementing these recordings were notes from the observing classroom teacher noting the specifics of a student's strategy. Field notes were written immediately following the instructional session wherever possible by the researcher to capture elements of the conversation along with ideas for follow-up on succeeding days. Photocopies of written student work were collected for those instructional tasks designed to foster multiplicative thinking.

One-on-one Interviews

Three interviews were conducted. The pre and post interviews were structured using core identical tasks. The post interview contained additional items used as extensions. An interview was conducted in December that contained items unique to that session. With the exception of number size adaptations for grade level participants, the tasks used within each interview were identical. All sessions were audio recorded. The mid and post interviews were also videotaped.

Schedule of Data Collection

Beginning in mid-September, the researcher was embedded within the three classrooms during the students' math instruction time for approximately two weeks each month. The researcher was present for the full hour of instruction for fourth grade. Scheduling conflicts between third and fifth grade limited the amount of time spent in each classroom. Typically thirty minutes was spent in fifth grade then thirty to thirty-five minutes in the third grade classroom.

Four scheduled half-day debriefing and planning sessions were held across the time frame of the study. During these sessions, the focus of discussion was on the student work, effectiveness of instructional tasks being used, and planning for new instructional tasks. Impromptu meetings with teachers during their preparation periods and before or after school occurred on an as needed basis. Table 3 indicates the general nature of the classroom tasks, the type of data collection that ensued, and the debriefing and planning sessions that occurred with the classrooms teachers.

Table 3
Schedule of Data Collection

September 2010 Through February 2011; 20 Weeks of school year; 12 Weeks Embedded Within Classroom		
September - October Assessment & Establishing Routines	November – December Building Capacity	January - February Increasing efficiency & adaptive expertise
4 weeks embedded in classroom	4 weeks embedded in classroom	2 weeks embedded in classroom 3 weeks of post-interviews
One-on-one Pre Interview Fall MAP Testing	Designing Instructional Tasks Small Group Interview – 3rd grade	Designing Instructional Tasks One-on-one Fraction Interview One-on-one Post Interview Winter MAP Testing
Half-day teacher debriefing & planning meeting	Half-day teacher debriefing & planning meeting	Two Half-day teacher debriefing & planning meeting

Student Sample

A student sample was identified at each grade level for the purpose of conducting one-on-one interviews. A purposeful sample was selected based upon a combination of Fall Measures of Academic Progress (MAP) test scores (Northwest Education

Association, 2008), teacher input, initial classroom observations by the principal investigator, and agreement by parents, guardians, and students to participate. The pre interviews were audio taped. The mid and post interviews were video taped. All field notes and written student work were collected and preserved.

An effort, where possible, was made to balance the sample group by gender and to be representative of the grade level's racial and language composition. Some attrition of sample members occurred at third and fifth grades. Table 4 lists the demographics within each grade level's sample group who participated in the one-on-one interviews.

Table 4
Demographics of Student Sample Groups at Each Grade Level

	Third	Fourth	Fifth
	$n = 11$	$n = 12$	$n = 11$
Male	7	5	3
Female	4	7	8
Asian	1	1	4
Black	0	2	2
Hispanic	4	7	2
White	6	2	3
LEP	4	5	3

Scoring methods

For this analysis, all one-on-one interviews were transcribed and analyzed. A coding rubric was generated and refined from the iterative review of student responses. As the strategies of additional students were subjected to analysis, refinements to the scoring rubric were generated. Definitions of what an additive and a multiplicative strategy were defined and rubrics created to score student work. With the iterative review of the transcripts, the transitional strategy phases from additive to multiplicative were delineated and incorporated into the scoring rubrics.

A microgenetic method of analysis (Siegler, 2000) helped to answer questions about how learning processes unfold. The objective was to map out how consistent over time an individual student used multiplicative thinking as he or she integrates more efficient ways of thinking with the prior additive ones. The trend lines over the twenty-two weeks of data collection were of interest. While it was expected that old and new strategies exist side by side, it was the evidence of how the new emerged and began to transcend the additive structures that was mapped.

Visual profiles of individual students were used to see progressions across period of the study. The rubric scores capturing the spectrum of student solution strategies became the measures with which to plot a student's profile. Overlapping these trends lines was theorized to analyze how adaptive use of new strategies emerges relative to the more established solution strategy patterns.

Description of Instruments

Interview Tasks

The tasks used within the pre and post interviews were organized around four clusters of mathematical ideas. These clusters reflect the working conjectures one, two, and four of the study; ideas of place value and decomposition of number (clusters one and three), multiplicative scale (cluster two), and equality and relational thinking (cluster four). The tasks used for the mid interview reflected the working ideas conjecture three, fractions. Each of the tasks, with the exception of task six, was drawn from either the research literature or from other existing assessment protocols. Table 5 outlines the general nature of the tasks by cluster for the pre and post interviews. Table 6 outlines the tasks selected from the literature to assess students' multiplicative understanding while working with fractions. These tasks formed the basis of the mid interview conducted in December. See Appendix A (pre), B (post), and C (mid) for the complete protocols.

The selection of these tasks was intended to reveal whether or not the students were thinking additively or multiplicatively and if multiplicative thinking was identifiable in some tasks earlier or more easily than in others. Additionally, as an outcome of the design experiment, it was intended to understand if this combination of tasks was effective in revealing student capacity to think multiplicatively and the transitions that occur over the cross-sectional view of the students.

Table 5
Pre and Post Interview Assessment Items

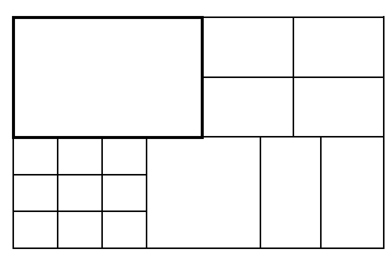
	Pre Interview	Post Interview
Cluster One *Place Value as a multiplicative concept*		
Task 1	Non-context based: How many ones, tens, hundreds in 783?	882 (3^{rd} gr), 832 & 2516 (4^{th} gr), 2516 (5^{th} gr)
Task 4	Context based: How many boxes of ten can be filled with 65/124 (3^{rd} gr.), 465 (4^{th} gr.), 465/1465 (5^{th} gr.) markers?	How many ten-packs can be filled with 356 (3^{rd} gr.), 1462 (4^{th} & 5^{th} gr.) seeds?
Cluster Two *Multiplicative Scale & Comparison*		
Task 2	Non-context - Scaling by fives; Scaling by 100	Scaling by fives; Scaling by 100
Task 5	Context based: (5^{th} grade only) Two bins, 360 in first. That's 10 times larger than the second. How many in the second?	Two cups of seeds. 60 (3^{rd} gr.), 620 (4^{th} & 5^{th} gr.) seeds in first cup. That's 10 times larger than second. How many in second?
Task 8		Missing Value Task – Context based: A 3-pack of tennis balls costs \$5.50. How much would a 9-pack cost?

Table 5 *(continued)*
Pre and Post Interview Assessment Items, continued

	Pre Interview	Post Interview
Cluster Three *Place Value & Decomposition of number*		
Task 3	(3rd gr.) Double 63, Triple 17, Quadruple 13 (4th gr.) Double 163, Triple 27, Quadruple 18 (5th gr.) Double 461, Triple 47, Quadruple 28	(3rd gr.) Double 126, Triple 27, Quadruple 18 (4th & 5th gr.) Double 126, Triple 68, Quadruple 34
Task 6	How much to get from… 57 to 100? 246 to 300? You are at 62, go back 5.	How much to get to… 68 to 100? 141 to 215? You are at 74, go back 7. You are at 82, go back 17.
Cluster Four *Relational Thinking*		
Task 7		$8+4 = \square + 7$ $17 = 12 + 5$ *T/F?* $24 + 73 = 72 + a$

Table 6

Mid Interview Assessment Items

Cluster One *Multiplicative Coordination in use of Fraction Strategies*	
	4 children are sharing 23 chocolate bars so that everyone gets the same amount. The chocolate bars are all the same size. How much will each person get?
	Six children want to share 2 pounds of modeling clay so that everyone gets exactly the same amount. How much clay can each child have?
Cluster Two *Proportionality - Equivalence*	
	The teachers at the grade level plan to have 3 students share a 2-liter bottle of soda for a class celebration. How many bottles of soda will the teachers serve so that 30 students get the same amount as the table of three?
Cluster Three *Part-Whole Unitizing*	
	 Name the fractional part that is indicated in each picture. Explain how you figured this out.

Classroom Tasks

The research occurred within two settings: the classroom during regular math instruction and in the one-on-one interviews. For the classroom, tasks were designed around the ideas framed by the conjectures driving the research. Common to all classrooms at the beginning of the school year was establishing language patterns in order to have students consistently describe numbers in terms of its value rather than the digit's name. A second commonality was to work with students on their capacity to decompose numbers. At third and fourth grade this was initially integrated into the opening units on subtraction. At fifth grade, the decomposition process was integrated into the opening unit on factoring. Across the six months of the research period, multiplicative place value concepts, decomposition of number, scale factor, algebraic relational thinking, along with the doubling, tripling, etc. of multidigit numbers was woven into the opening warm-ups of daily lesson. By November, part of the effort by the researcher and the teachers to provide consistency in ensuring daily practice in these areas, tasks were designed for the teachers to use during those periods when the researcher was not on site. At some grade levels, teachers took on a greater role in designing these tasks themselves.

At third and fourth grade more instances of designing instructional tasks for the main part of the lesson occurred. At third grade, these tasks focused on the two units on multiplication and division that occurred within the six-month timeframe. For fourth grade, these units focused on decimals and multidigit multiplication and division operations. The redesigned lessons were constructed in such a manner that classroom teachers could implement the tasks in the absence of the researcher when off site.

Adaptations and revisions were made to these lessons based upon student performance and teacher feedback.

Data Analysis

The data that is presented in the results chapter of this document, Chapter Four, focuses on the students' performance on the pre and post interviews. The audio and video files of the one-on-one interviews were transcribed. An iterative review process began with the use of an open and axial coding process (Strauss & Corbin, 1990). The first reading of transcripts was for scoring for accuracy of response, followed by classification of the type of thinking that was used. Rubrics emerged as additional transcripts were read. Additional readings of the transcripts allowed for deeper probing of commonalities of strategy, transitions in thinking, and areas of struggle that answering both developmental (evolution over time) and microgenetic (how do the students learn) questions. Categories and themes began to emerge with successive readings. These categories included levels of strategies used to solve tasks, areas of confusion and transition, and language usage. Data from the transcripts were triangulated with data from field notes, student work samples, and video footage. The material provides density to understanding the how and why shifts in student thinking occurred. The data was organized using Excel software to manage frequency counts of the various observed attributes and to allow for comparison of data between and among the various tasks.

This study chose to use both a design and teaching experiment structures in order to parse how students within a classroom setting transition from additive to more multiplicative forms of thinking. The multivariate nature of the classroom setting allowed for these transitions to be witnessed and pondered. The four conjectures formulated from the review of the literature provided an initial focus for the design and implementation of instructional tasks and conversations with students. Data collection and analysis from the classroom setting and from one-on-one interviews provided a lens of how various learning trajectories of students, both collectively and individually, transition over time.

The following chapter is a report of results from the pre and post interview portion of the data to provide an initial understanding of shifts in student thinking over period of the study.

CHAPTER FOUR. DATA ANALYSIS AND RESULTS

Study Context

This research focused on asking two questions:

1. *How does the development of additive to multiplicative thinking evolve over time and what does students' thinking look like as they transition from additive to multiplicative thinking?*
2. *What integrated and coordinated series of mathematical tasks with place value, rational number, measure, and algebra can teachers use to help reveal the emergence and development of multiplicative thinking?*

The full study was set in a mathematics classroom each at third, fourth, and fifth grades at a first-ring suburban community of a large Upper Midwestern urban area with a diverse student population. Data was collected both from within the classroom instructional settings and in one-on-one interviews with students. This chapter reports on the data from the one-on-one interviews only. The analysis of these interviews allows for understanding the evolution of student transitions (question one) and the effectiveness of the interview tasks in capturing and revealing the transition to multiplicative thinking (question two) to be assessed. Analysis of the classroom data will be analyzed at a future time.

To begin understanding the development of the transitions from additive to multiplicative thinking, pre and post one-on-one interviews were conducted with a sample of third (n = 11), fourth (n = 12), and fifth (n = 11) grade students. Interviews were conducted in late September/early October and again in February. The protocol asked five core questions across both interviews. Three extension tasks were asked in the post interview for a total of eight questions. Interviews lasted between a half-hour to an hour depending upon the student and the nature of the conversation.

Analyses of the data from these interviews proceed over the next four sections. Section one introduces the design of the rubric developed to assess an individual's level of additive or multiplicative thinking and uses that rubric to present findings and initial themes that emerged in the pre interview data. Section two describes the instructional tasks used with students within their classrooms between the two interviews. Section three compares pre and post interview data and the shifts in student thinking over the six-month period design experiment. Specific student work samples will be used to illustrate this comparative data. Section four compares an individual student's pre and post scores on the various tasks. This later comparison creates a profile of the individual allowing an understanding of the different trajectories taken in building the capacity to think more multiplicatively.

Various themes emerged from the data and are described. These transitional stages are referred to as "unit confusion", an analysis of students' usage of a "cover pattern", the role of number size in triggering either additive or multiplicative thinking, and the role of language in the development of multiplicative thinking. Each section draws directly from the transcripts, student and interviewer generated artifacts, and field notes created during and after the interview process.

Development of the Scoring Rubric and Student Profiles

The scoring rubric was designed to demarcate the difference between additive and multiplicative thinking when solving a multiplication or division task. Its core structure draws directly from the literature on how multiplication and addition differ. Multiplication is an invariant relation between two known quantities, it is a ratio, a rate (Vergnaud, 1994). Multiplication involves a one-as-many construction of quantities that results in the making of composite units (Steffe, 1994) and the coordination of those units to find a third quantity. While additive strategies have an ascendant link to more advanced multiplicative strategies, the degree of abstractness (Kouba, 1989) distinguish between the two states.

Numerous researchers have outlined children's solution strategies for single digit multiplication and division using primarily an equivalent grouping structure of multiplication (Kouba, 1989, Anghileri, 1989, Carpenter, et al., 1993, Mulligan and Mitchelmore, 1997). The distinction made between additive and multiplicative responses in this area has largely been that a child direct modeled, repeatedly added, or skip counted (additive) or was able to derive or recall a known fact (multiplicative). While this may be satisfactory for single digit combinations, it proves less productive when looking at student thinking around multidigit multiplication and division. Confrey's (1994) position that multiplicative thinking arises more from a *splitting* notion argues for understanding how children come to scale up and down using number and seeing multiplication from an *n* times as many perspective (Confrey, et al., 2009).

Working closely with the taxonomy of invented multidigit multiplication and division algorithms outlined by Ambrose, Baek, and Carpenter (2003) and the work of the Dutch (van Putten, et al., 2005), a collection of attributes emerged as indicators to distinguish the difference of multiplicative from additive thinking. The nature of the decomposition of the multiplier or multiplicand was one determinate. Larger composite decompositions of a number's place value components – example, $4 \times 24 = 4(20 + 4)$ – was multiplicative where the decomposition of the multiplier into individual repeated elements ($4x24 = (1 + 1 + 1 + 1) \times 24 = 24 + 24 + 24 + 24$) was additive. The ability to plan ahead to coordinate the decompositions into known components (also see Empson & Turner, 2006), the creating of new composite units, and the coordination of those units are distinguishing characteristics. The ability to scale, to *split* (Confrey, 1994) and negotiate a measure space (Vergnaud, 1994) are attributes identified by those taking a multiplication as *n* times as many, and of function, $x = f(y)$, approach. The design of the protocol used in the interviews combines this scalar notion of multiplication with the attributes of equivalent grouping perspective of multiplication to form a more comprehensive look at how multiplicative thinking emerges among children. In the blending of these two perspectives, interpreting the "progressive abstraction" (Ambrose, et al., 2003, p. 305) becomes an important discriminator among the various indicators to understand if a child's thinking is more additive or more multiplicative in structure. Table 7 summarizes the attributes used to design the coding system.

Table 7
Attributes of Additive and Multiplicative Thinking

additive thinking *multiplicative thinking*

increase in progressive abstraction →

Attributes of measure (Evidence of…)	• Accuracy • Automaticity of execution • Language of values • Nature & Efficiency of the decomposition of number • Place value as a rate of ten, the capacity to re-unitize a number • Formation of composite units • Coordination and transformation of units • Strategy type • Algebraic/Relational/Simultaneous thinking ° Use of scale factor or unit rate ° Derived & flexible strategies ° Equal sign as comparative symbol ° Properties of operation (distributive, associative, commutative, identity)
Qualifiers	• Accuracy ° Both additive and multiplicative strategies can result in high levels of accuracy ° Both could have inaccuracies in computation while using appropriate strategies • Automaticity ° Automaticity with algorithmic procedures may not be an indicator of multiplicative thinking, merely fluency with procedural steps while working very additively • Language of values ° Someone persistently using single digit language may have the capacity to think multiplicative but one cannot tell without probing for understanding • Place value, interpreted only by a digit's location, and subsequent inability to re-unitize to other units of ten, may be the result of lack of exposure rather than an inability to think multiplicatively

Table 7 *(continued)*
Attributes of Additive and Multiplicative Thinking

additive thinking *multiplicative thinking*

increase in progressive abstraction →

Multiplicative Thinking (descriptors	• Automaticity in execution • Efficiency in decomposition of numbers (Addends and or factors) • Place value as a rate of ten, the capacity to re-unitize a number • Making and use of composite units • Coordination and transformation of units • Planning ahead • Strategy type: Building Up by Other Factors, Distributive, associative (with factors), and/or commutative properties • Relational/Simultaneous thinking ° Use of scale factor or unit rate ° Derived fact strategies • Mental abstractness
Additive (descriptors)	• Smaller, incremental steps • Decompositions in smaller aggregations • Place value: ten as a unitary or counted unit • Minimal if no use of composite units • Confusion over/ignoring of units • Additive strategies (Skip Counting, Repeated Addition, Doubling, Complex Doubling) • Calculation as oppose to relational thinking • More grounded visually than mentally

Mindful of Siegler's overlapping wave theory (2000) that both old and new strategies co-exist and intermingle as new ideas slowly comes to supplant earlier strategies, a dual coding system was devised to parse the various elements of the transition stages from additive to multiplicative thinking. The goal was to create a workable continuum that reflects shifts in student thinking yet captures seemingly overlapping strategy usage.

The decision-making process to determine if a solution to a task was additive or multiplicative at its core structure had complications, even after the attributes listed above were considered. For example, Task 8 in the post interview (missing value problem: If a three-pack of tennis balls costs $5.50, how much will a nine-pack cost?), a correct response of $16.50 as well as the incorrect response of $11.00 could both be assessed as being additive in the solution strategy used. Child A, while recognizing correctly that three 3-packs are needed, uses repeated addition to add $5.50 three times. Child B sees the difference between a 3-pack and a 9-pack is six so adds $5.50 twice. Child A understands the multiplicative structure of the problem but uses an additive calculation processes. Child B does not understand the problem's multiplicative structure, instead solves the task operating on its additive surface features, not its multiplicative structural features. The fact that Child A's solution was correct means that at some level the problem's multiplicative structure was understood.

As a result of conundrums such as this, the child's solution was first assessed to determine if the strategy falls within an additive or multiplicative *structure*. A second code analyzed the particular elements of the solution to determine if the *strategy* was emergent (more additive in its calculation) or more abstract (more scalar in its calculation). The composite sum of those two codes allowed a 1 through 4 rubric score for each task in the clinical interview to be formed. The effect of the two summed scores was that there were limited means by which a score of 1 or 4 could be earned – clearly additive, clearly multiplicative, respectively. There was more than one avenue by which a 2 or a 3 could be created. These two middle scores indicated the overlapping areas where multiplicative thinking may be entwined with the more emergent additive calculations.

Figure 2 represents the portion of the rubric designed to analyze Task Eight.

<table>
<tr><td colspan="2">Pre interview:
Not Given
Post interview:
[Presented in tabular form] If a 3-pack of tennis balls costs $5.50, how much will a 9-pack cost?</td></tr>
<tr><td>Understanding of the Multiplicative Structure of the Task</td><td>Classification of the Solution Strategy</td></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon reasoning around the scale factor of three</td><td>2
• Accuracy
• Automaticity
• Justification is based reasoning with a multiple of three</td></tr>
<tr><td>1
• Recognizes the multiple of three but adds to solve
or
• Some computational inaccuracies
• Self-corrects due to prompts and light scaffolds</td></tr>
<tr><td>1 - Additive
• Sees an additive difference between the quantities and calculates accordingly</td><td>0
• Interprets problem as a additive structure not a multiplicative
or
• Only self-corrects due to extensive scaffolding
or
• Or even with extensive scaffolding, does not arrive at an accurate solution</td></tr>
</table>

Figure 2. Task 8 Rubric

Thus for the purposes of this study, the first code determined whether or not the multiplicative *structure* of the task was understood. The second code analyzed the *solution strategy* utilized by the child. Again using the example of Task 8 in the post interview, Child A would receive a score of 2 (understanding the multiplicative structure) but receive a 1 or a 0 in terms of the solution strategy. This would result in a composite score of a 3 or 2. Child B would receive an initial score of 1 (not understanding the multiplicative structure) and a subsequent solution strategy score of either a 1 or a 0. The resulting composite score would be at most a 2 or at minimum 1. See Appendix D for the full rubric and examples of scored passages.

Line plots combining a student's pre and post interview were created to note shifts in student thinking over time. The line plot of the scores from each interview protocol tasks were placed side by side with each other to created a profile of the range of strategies a child used at any one slice in time. Plotting the series provided a density of information mapping how additive, how multiplicative, or how transitional a child might appear. Plotting both the pre and post interview scores provided a means to analyze shifts in ways of thinking students make over time. Figure 3 provides an example of one student's profile. The dashed line represent scores received for the twelve sub-tasks for the core five questions used in the pre interview. The solid line represents the scores for the twelve core sub-tasks along with the six extension sub-tasks given during the post interview. Each individual line plot demonstrates the range of strategies that exist within a student's repertoire at any one time as well as some of the common tasks that appear to cause a student to think more additively (falls in the line) or multiplicatively (rises in the line). The two lines together on the same plot visually capture how shifts in the capacity to think multiplicatively (scores of 3 and 4) are more evident in the post interview (solid line) than additively (scores of 1 or 2) in the pre interview (dashed line).

The next section used evidence from student transcripts to demonstrate the design of the scoring rubric (Figure 2). The fourth section used the line plot profiles to illustrate various shifts in students' capacity and possible learning trajectories taken in becoming more multiplicative in their thinking.

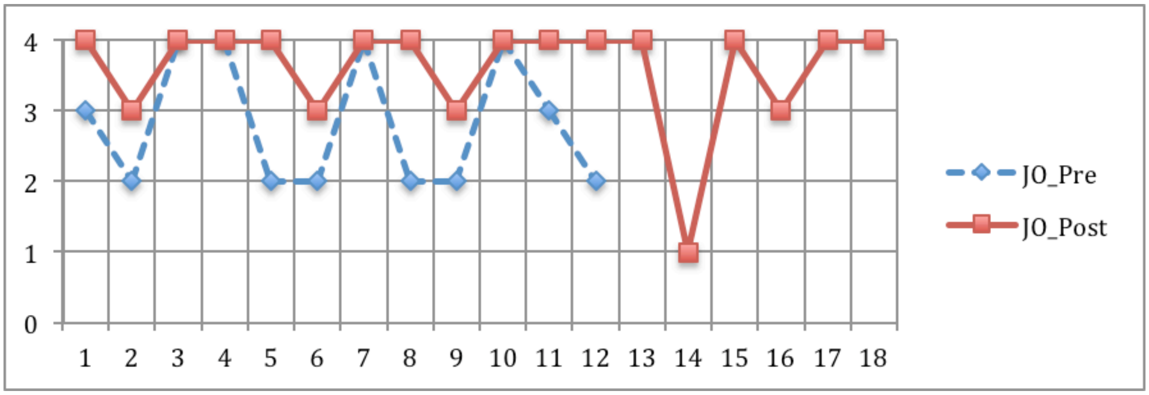


Figure 3. Third Grader's (JO) pre and post interview results on protocol sub-task items. A score of 1 or 2 (*y-axis*) indicates an additive composite score. A 3 or 4 indicate a multiplicative composite score. A 3 indicates more emergent additive calculations were used compared to more abstract and efficient multiplicative calculations. The dashed line are scores on the pre interview (September), the solid line the post interview. The *x-axis* numbers represent the sub-tasks in the interview protocol. Twelve sub-tasks (13 for fifth graders) were presented in the pre interview, 18 during the post interview.

Analysis of the Data

Pre Interview

A core of five tasks was presented in both the pre and post interviews. Numbers and context changed between the two interviews, yet the tasks remained essentially identical. A sixth task (Task 5 on the protocol) was used in the pre interview with all fifth graders, with fourth graders if fluent on certain previous tasks, and not at all with third graders. This was due to the level of difficulty of the task. The post interview contained two additional tasks (three for third and the balance of fourth graders – task five on the protocol) serving as extensions. See Appendix A and B for the full interview protocol.

Tasks One, Two, Three, Six, and Seven each contained subtasks. Task Four contained yet another additional subtask for fourth graders only on the post interview. These variations among the grade level protocols reflect potential areas of overlap in learning development with the grades below and above. The pre interview resulted in twelve points of data for each child in the third and fourth grade sample groups, with thirteen for fifth grade. The post interviews result in eighteen data points for each child. Grade four had two additional subtasks in task one bringing the data elements to total of twenty.

The protocol was designed to have four clusters of tasks that probed different aspects of the multiplicative conceptual field as outlined in the conjectures governing this study. Cluster one (Tasks One and Four) focused on direct place value concepts, the rate of ten and the capacity to re-unitize a number. Cluster two (Tasks Two, Five, and Eight) focused on issues of multiplicative scale. Cluster three (Tasks Three and Six) probed ideas of decomposition of number, the distributive property, and structuring around landmarks of ten. Cluster four (Task Seven) explored equality and relational thinking.

The following four subsections, describes results of each of the clusters mentioned above as it emerged from both the pre and post interviews.

Cluster One: Task One (How many ones, tens, hundreds are in the number 783?) and Task Four (Rate of ten in a context)

<table>
<tr><td colspan="2">Pre Interview: How many ones, tens, hundreds, thousands in 783?
Post interview: How many ones, tens, hundreds, thousands in 832 (3rd Gr.), 832 & 2516 (4th Gr.) or 2516 (5th Gr.)?</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon …
…reasoning around the factors of 10
…the making of composite units
…coordination of units

• May recognize the surface pattern</td><td>2
• Accurate
• Automaticity
• Justification is based upon factors of ten; could include an explanation of the place value pattern but reasoning as to why it works is demonstrated
or
• First response may involve a misinterpretation of the problem but quickly, and with explanation, adjusts to a correct response. The ability to justify is key.</td></tr>
<tr><td>1
• Some calculation involved with one or more of the units
or
• Some initial inaccuracies
• Self-corrects due to prompts and minor scaffolds
or
• Justification is based solely on a covering pattern without any explanation as to why it can be trusted</td></tr>
<tr><td>1 - Additive
• Skip counts to solve
• Direct Models to solve
or
• Struggles to see beyond the value of the digit's location
or
• Sees the numbers as a string of single digits</td><td>0
• Several inaccuracies
• Only self-corrects due to extensive scaffolding</td></tr>
</table>

Figure 4. Task One Rubric: How many ones, tens, hundreds, are in a given number?

Eighty percent of the third graders responding to Task One in the pre interviews gave an initial response of “3… Because it shows the number three” in the ones place. Once prompted that, “You would be correct if I was asking you how many ones are in the ones place, but I am asking you a slightly different question. How many ones are in the whole number 783?” Depending upon the amount of scaffolding required to understand the difference between these two questions, a child was scored a two (a quick response of 783) or a one (required extended conversation, needed blocks, did some counting…) to solve the task. While 100% and 70% of the third graders could be scored as having a multiplicative understanding of the number of ones and hundreds in 783, only 30% could do so when asked about the number of tens in that number. Fourth grade followed a similar pattern. Two-thirds required some scaffolding to understand the nature of the question. Once understood, responding to the tasks about the ones and hundreds resulted in 92% and 58% using an initial multiplicative structure. 100% of the fifth graders responded multiplicatively with the ones and hundreds tasks. Only 64% initially misunderstood the initial question about the ones place.

What is consistent between third and fourth grades, and to a lesser extent fifth grade, are the number of students who shifted to more additive approaches to understand how many tens are in the number. Among the third graders, 70% struggled with this task with 30% requiring base ten blocks to calculate it. 50% required a significant level of scaffolding support to solve it, the end result of which was a combined score of one. Fourth graders, too, struggled with how many tens were in the number. Two-thirds used an additive approach with 55% requiring such significant scaffolding resulting in a combined score of one.

What appeared among the fourth graders, which was different than those in third, was what is best described as “unit confusion.” Third graders, if confused, used direct modeling through the use of the base ten blocks to reason their way through the task. They did not, with the exception of one case, demonstrate this behavior. Fourth graders, however, worked more abstractly trying to reason through the task mentally. What emerged from the fourth grade data was a number of incidences (seven out of eleven) where a child kept losing track of if he or she was working in ones or tens, thus confusing his or herself upon which unit to attend. This included four more extreme incidences where students ended up confusing the number of tens in the hundreds place as a single digit collection of ten and then added that quantity to the tens place or confusing the number of tens with being a quantity of ones and added it to the hundreds place. For example, NI, when asked how many tens are in 783, responded, “Fifteen… because ten times ten is a hundred so it’s then seventy plus eighty equals fifteen.” The conversation continues through a series of queries that indicates that NI clearly knows that “eighty is eight tens” and that there are “seventy” tens in seven hundred. Her difficulty was how to keep the quantities and their corresponding units separate. Her initial response was to combine the “8” (tens) and the “7” (hundreds) and to get “15.” EZ knew he had “70 tens” in seven hundred and that there were “eight” tens in eighty but his initial total of tens was “150.” Each lost track of the units to which they needed to attend and combined the wrong quantities.

This unit confusion is different than that of JQ who, when asked how many ones were in the number 783, said, “Eighteen… because seven plus eight, I got fifteen and I added three more and I got eighteen.” This response is based upon seeing multidigit numbers as a concatenated string of single digits (Fuson, 1990). It is also different from the occasional misspoken statement that the child quickly self-corrects his or herself. NI and EZ both understood the multiplicative relations of how many tens are in eighty and seven hundred, respectively. It is the need to keep track of these two quantities and resulting two units (a combination of four entities) simultaneously that results in an increased cognitive load, e. g., 8 tens in 80 (ones) and 70 tens in 7 hundreds or 700 (ones) that NI came up with 15 (combining 8 + 7) and EZ came up with 150 (combining 80 + 70).

Only three incidences of unit confusion occurred in fifth grade pre interview sample group. The three were so significant, however, that either a correct answer was never arrived at (two cases) or so much scaffolding occurred that only a combined score of two resulted (one case).

It was in the fifth grade pre interviews that the first expressions of the place value/"cover" pattern were verbalized. The following exchange occurred when KI was asked how many tens are in 783:

KI: 780?

JB: And why 780?

KI: Because three isn't a whole ten and seven, ah… eighty, no it's 78.

JB: And why are you changing your mind?

KI: Because seven or 70 tens is 700 because you add the zero times ten. And 80, it'd be eight times ten.

In that exchange, three things are notable. First she articulates that the "three isn't a whole ten" and thus can be ignored. She also notes the effect of the zero pattern of when something is either multiplied or divided by ten. The exchange also notes how confusion in coordinating the units results in the initial answer of 780 which she self-corrects once she begins articulating her thinking out loud. Only three incidents of this cover pattern occurred in task one. The noticing of that pattern, however, increased while solving Task 4.

Task four was designed to essentially ask the same question as task one, part two. The difference was that task four was set in the context of colored markers being placed into boxes with ten markers in each box. Third graders were asked how many boxes of ten could 65 markers fill, with fourth graders starting with 465 markers, and fifth graders with 1,465 markers. See figure 5 for the scoring rubric for task four.

<table>
<tr><td colspan="2">Pre interview:
The factory that makes colored markers has a bin filed with 65/ 124 (3^{rd} Gr.), 465 (4^{th} Gr.), 465/1465 (5^{th} Gr.) markers. If the sorting machine places 10 markers in every box, how many full boxes can the machine fill?
Post interview:
The greenhouse growers are planting flower seeds now so plants will be ready to sell in the spring. The grower sows seeds in 10-packs. How many full 10-packs can be planted with 356 (3^{rd} Gr.), 1462 (4^{th} & 5^{th} grades) seeds?</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon reasoning around the factors of 10
• May recognizes the surface pattern</td><td>2
• Accuracy
• Automaticity
• Justification is based upon factors of ten; could include an explanation of the covering pattern but reasoning as to why it works is demonstrated</td></tr>
<tr><td>1
• Accurate but additive in strategy solution
or
• Some initial inaccuracies
• Confusion in the coordination of units
• Self-corrects due to prompts and light scaffolds
or
• Justification is based solely on a covering pattern without any explanation as to why it can be trusted</td></tr>
<tr><td>1 - Additive
• Skip counts to solve
• Direct Models to solve</td><td>0
• Several inaccuracies
• Only self-corrects due to extensive scaffolding
or
• Even with extensive scaffolding, does not arrive at an accurate solution</td></tr>
</table>

Figure 5. Task Four Rubric: How many boxes of ten can be filled given a starting number?

This task was structured to see if the type of thinking demonstrated in task one, a pure number task, was the same or different than when the same task was given within a tangible context. Among the third graders a more additive structure governed the solution strategies. More direct modeling through picture drawing, particularly tallies occurred even though the number quantity was smaller (65) than in task one (763). However, one student, JA, quickly answered, "six." When asked why so fast, he stated, "Because it is too easy. Then the other five [markers], they would just add five more to get to seven [boxes]." He was then posed an extension of what it there had been 124 markers, how many of boxes could be filled then? His response was, "I could make twelve… because it has the number twelve right here," pointing to the tens and hundreds places, thus indicating the cover pattern. He answered similarly to the number 491 markers. An interesting change happened when the number was extended to 1,646. The following exchanged then occurred.

JA: That would be a hundred and ten.

JB: A hundred what?

JA: A hundred ten… Because it was easy. If you cover, added these two together it would equal ten [pointing to the digits 6 and 4 in the hundreds and tens place].

JB: Oh! Add the four and the six together?

JA: Oh! I, uh, nope. They would be a hundred and six… no, a hundred… a hundred…

This exchange is of note since with two and three digits numbers he has able to answer knowledgably. The four-digit number has more multiplicative combinations to track and he was not able to sustain his reasoning. The cover pattern was not just merely a procedural matter to him but rather was a result of a series of multiplicative relationships. Had he just relied upon the procedural knowledge of the cover pattern, he likely might have said 164 without any trouble. Kamii (1994) talks about the increasing multiplicative relationships students have to comprehend with each increase in a magnitude of ten. It also indicates how number size can trigger different levels of response among children.

Fourth grade, too, was mostly additive in responding to task four (465 markers, how many boxes of 10). The conversation with ER extends the analysis of how number size and unit coordination interact with different number combinations.

ER: Uh, let's see. Four hundred, like make four hundred and sixty but the five is not in there but…

JB: Okay. The five's not in there. Makes sense. And why is the five not in there?

ER: Because it [the machine] only can make ten so the five is left out.

JB: All right.

ER: So it can only make, uh um, four hundred sixty.

JB: So it's only going to use four hundred sixty markers? But how many boxes of ten would that make?

ER: Four hundred sixty?

JB: Well, but that would be one marker in every box. I am going to put ten markers in every box. So how many boxes would that be?

ER: Sixty.

JB: Why?

ER: Because ten, the six is in the tens, so like 10, 20, 30, 40, 50, 60. So, six boxes of ten, so that's sixty.

The episode continues with trying to skip count to determine then how many tens are in four hundred. This is one instance where the student does not determine a complete answer. ER recognizes that the five can be ignored because it is not a complete ten. He also determines through skip counting that the sixty is six tens. But the passage also indicates that he is working hard at determining if he is to focus on the four hundred sixty, the sixty, or the six tens. After some additional scaffolding to determine how many tens are in 400, the conversation continues.

JB: …so how many total boxes is that?

ER: Forty.

JB: Forty. Okay. And so, the four hundred makes…

ER: Forty.

JB: Forty boxes. You said the sixty made…

ER: Six boxes.

JB: …six boxes. So, how many boxes is that all together?

ER: Four hundred sixty?

ER loses track of what unit, which piece of information to which he needs to attend. This type of struggle occurred with several students who were more emergent. He more easily understands the multiplicative relations with the sixty-five, but struggles with the added four hundred. The shift from additive to multiplicative thinking requires students to grapple with coordinating these elements. The interview is capturing ER in a moment in time where this ability to coordinate is difficult.

Three of the twelve fourth graders were able to respond using clear multiplicative reasoning. KA knew the answer was forty-six.

> KA: So, I already know that the tens is sixty so there will be six right there, and then… [pause] and then I would take the four hundred and then do that again…
>
> JB: And by that again you are covering over… tell me what you are doing.
>
> KA: Covering over the ones and then you get forty. So then forty-six.
>
> JB: So forty-six boxes.
>
> KA: And then you have five left over, but you can't add anything.

This exchange exemplifies how comprehending that sixty is six tens and that four hundred is forty tens evolves and how the cover pattern comes to be used to express that rate of ten. This capacity to reason with the values across place unit continued with the fifth graders. The fifth graders needed to reason with either 1,465 or 1,521 markers needing to be placed into boxes of ten. Ten of the eleven fifth graders in the sample group used a multiplicative structure with five of those providing a multiplicative rationale supporting their reasoning. EL worked systematically place by place to solve the problem.

> EL: I am not going to be able to put the extra little one in there [referring to the one in the ones place]… The two…
>
> JB: The what?
>
> EL: The twenty.
>
> JB: Thank you. Thank you.
>
> EL: That would be two boxes… and the five hundred would be fifty boxes. And the thousand would be a hundred boxes.

JB: And so how many boxes is that all together?

EL: One hundred fifty-two.

Others more specifically used the cover pattern to quickly determine the answer to how many boxes of ten could be generated. MA answers “one hundred forty-six” boxes of ten quickly. Asked how she knew that so fast, she responded,

MA: Because I know five, you know, it’s not quite a ten yet. So I just take the other numbers and minus a place value and just get the number.

Cluster Two: Task Two (If there are five fives in twenty-five, how many fives in seventy-five? If there are three tens in thirty, how many tens are there in three hundred?), Task Five (There are two bins. The first has 360 markers in it. That’s ten times larger than the second bin. How many are in the second bin?), and Task Eight (A three-pack of tennis balls costs $5.50 cents, how much will a nine pack cost?)

The numbers were differentiated among the grade levels. Third graders were asked if they agreed that that there were two fives in ten. If so, then how many fives were in thirty? Fourth and fifth graders were asked if there were five fives in twenty-five, how many fives were in seventy-five. The commonality among the grades was that each set of numbers was built upon a scale of three times more.

<table>
<tr><td colspan="2">Pre interview:
3rd Gr.: If 2 fives in 10, how many fives in 30? If in 10 tens 100, how many tens in 400?
4th & 5th Gr.: If 5 fives in 25, how many in 75? If 3 tens in 30, how many tens in 300?
Post interview:
3rd Gr.: If 2 fives in 10, how many fives in 40? If in 10 tens 100, how many tens in 600?
4th & 5th Gr.: If 5 fives in 25, how many in 150? If 4 tens in 40, how many tens in 120?</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon reasoning around multiples of the factors either using scale factor or unit rate</td><td>• Accurate
• Automaticity
• Justification is based upon factors of ten; could include an explanation of the covering/zero pattern but reasoning as to why it works is demonstrated
or
• Derives using the distributive property</td></tr>
<tr><td>1
• Has a multiplicative construct but needs to additively calculate to resolve (skip count, repeated addition, etc.)
or
• Some initial inaccuracies
• Self-corrects due to prompts and light scaffolds
or
• Justification is based upon additive structures</td></tr>
<tr><td>1 - Additive
• Skip counts to solve
• Direct Models to solve</td><td>0
• Several inaccuracies
• Only self-corrects due to extensive scaffolding
or
• Even with extensive scaffolding, does not arrive at an accurate solution</td></tr>
</table>

Figure 6. Task Two Rubric: Scaling up – If five fives are in twenty-five, how many fives are in seventy-five?

Eighty percent of third graders, 91.7% of fourth graders, and 36% of fifth graders approached task two from an additive structural perspective. Third grade had four individual cases where direct modeling via individual tallies was made to determine the number of fives in thirty. Most of the additive approaches were skip counting strategies. BT, a third grader, used a strategy, while essentially skip counting, demonstrates the creation of new composite units by which she then skip counted. When asked how many fives in thirty, she said:

BT: I would need to use… uh um… [pause], uh, to use thirty, no, twenty of these, the fives.

JB: Okay. And why twenty of those?

BT: Because I counted by fives.

JB: And can you do that out loud so I can see how you kept track?

BT: Ten and twenty and thirty.

JB: And so, that is three fingers. So when you did ten, twenty, thirty, how many fives are in each one?

BT: Uh um, thirty?

JB: Well, how many…. You've already… Humm… How were you able to, how were you keeping track of the number of fives as you go ten, twenty, thirty?

BT: Because I, I count them.

JB: So when you do ten, how many fives is that?

BT: Two.

JB: Two. And when you do twenty?

BT: That's, uh um, that's four.

JB: That's four fives, right? And when you do thirty, how many fives is that?

BT: Six.

JB: And how do you know there are six?

BT: Because I counted them and see how I can get how many fives I need to make a total of ten.

What is notable in this passage is that it captures early elements of multiplicative thinking. BT used two fives for every new ten. Ten then became the new countable unit. Each finger that she raised was a one-ten to two-fives unit. What was difficult for her was that once she made ten her countable unit, she struggled to maintain simultaneously her ability to track how many fives that was. She could think in fives or she could think in tens, but she struggled to think in both tens and fives at the same time. The scaffolding provided her allowed her to reflect back and forth between the two units, thus being able to finalize her answer. Without that support, it is unclear that she would have arrived at a correct answer.

In contrast, CH, a fourth grader, initially seems to have used a skip counting strategy but when probed based upon the speed at which he arrived at his answer described a more multiplicative understanding.

CH: Because it's like quarters. Seventy-five, fifty, twenty-five, fifty, seventy-five. So there are five, fifteen [sic], fifteen fives

.

.

JB: Like three quarters you said. And how did you know so quickly that would also be fifteen total?

CH: Because there are five, there are five fives, so five times three is [inaudible], I mean fifteen.

The difference in the two levels of explanation follows a pattern found in another study where a student' s first verbal explanation may be, in fact, a less efficient strategy since the actual strategy used is harder to verbalize (Brickwedde, 1993, unpublished Master Thesis).

KI, a fifth grader, derives her answer, "Five times ten is fifty and then five times two is ten so that is sixty, plus another ten is seventy, and then plus one more." In doing this, she is simultaneously accumulating fifteen fives while tracking the number of fives that also needed to be tracked. GV, also a fifth grader, summarized his thinking,

GV: Because I had it three times, 'cause twenty-five times three equals seventy-five so I did five, uh, five times three, err, well five times three equals fifteen, so…

Each of these three demonstrate a scaling up processing. CH links his scalar thinking with a skip counting strategy. KI scales simultaneously using a derived strategy. GV recognizes and uses the specific scale factor of three in his work. Each is multiplicative in structure. Each demonstrates various trajectories students process that type of thinking.

Task five in the pre interviews was only administered to four fourth graders. This task was only given if a level of multiplicative thinking was demonstrated in tasks one and four (cluster one tasks). All but one fifth grader was presented this task in the pre interviews. Figure 7 exhibits the context of the task, built off of the imagery of task four, and the attributes used to score a student's response.

<table>
<tr><td colspan="2">Pre interview:
There are two bins of markers ready to go to the sorting machine on the factory floor. One bin has 90 (3rd Gr.), 360 (4th & 5th Gr.) markers in it. That's ten times as many as in the second bin, How many markers are in the second bin?
Post interview:
There are two cups of seeds ready for the grower to use. The first cup has 60 (3rd Gr.), 620 (4th & 5th Gr.) seeds in it. That is ten times more than the seeds in the second cup. How many seeds are in the second cup?</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon reasoning around the scale factor of ten</td><td>2
• Accuracy
• Automaticity
• Justification is based reasoning with a multiple of ten</td></tr>
<tr><td>1
• Some initial inaccuracies, example: reasoning the second bin/cup was 10 times more – 3600 markers or 6200 seeds
• Self-corrects due to prompts and light scaffolds
or
• Justification is based upon additive structures</td></tr>
<tr><td>1 - Additive
• Sees an additive difference between the quantities and calculates accordingly
• Skip counts to solve
• Direct Models to solve</td><td>0
• Several inaccuracies
• Only self-corrects due to extensive scaffolding
or
• Does not arrive at an accurate solution</td></tr>
</table>

Figure 7. Task Five Rubric – Two bins. First has 360. That's ten times more than the second bin. How many in the second bin?

Of the four fourth graders presented the task, one didn't know how to start. Another, JE, used repeated subtraction removing one ten at a time until she got to 260, i.e., subtracted ten tens. KA paused to contemplate the task, finally responding,

KA: I am either thinking that, uh, thirty-six or sixty.

JB: Tell me why… Why those two estimates?

KA: Because ten times is like… like it has part of the same number but just like adds another number.

JB: Okay. Okay. So which one do you think it might be? Which one might make more sense than the other do you think?

KA: The thirty-six.

JB: And why the thirty-six?

KA: Because there is a, they are all [inaudible] and usually when you times ten you add a zero.

KA is articulating the place value pattern that occurs as a result of multiplying a number by ten. Yet she is insecure in trusting that this will work for her. Six of the eleven fifth graders referenced this place value cover pattern in their approach to solving the task. EL answers "thirty-six" quickly. When asked to explain her response was "I kind of cheated and took away the zero." When pressed as to why this pattern worked and why she could trust it, she responded:

EL: Because it is like adding a ten, so then if you add or minus a ten you just have to like, timesing, or… dividing. You could just take, put, add or take away the…

Compare KI's explanation with EL's. "…three hundred divided by ten, it's the same as just saying cause times ten is adding a zero then dividing by zero would be minus a zero." MA stated that, "…when you divide by ten you minus one place value." These three statements are various levels of understanding of the linkage of the pattern resulting from the effect of dividing any number by ten. In two of the cases, EL and KI, the student answered the question rapidly then used the place value cover pattern to justify their thinking. MA did what some other students did and confused which vessel was ten times more than the other. But MA's explanation for 3600 and 36 clearly articulated the pattern effect of the rate of ten when either multiplying or dividing.

Those that could not complete this task struggled to conceptualize the directionality of the scale effect as well as reasoning through the task itself. SH struggled initially in this manner but after a prompt referencing Task Four, she systematically decomposed the 360 into 300 and 60.

SH: Use the sixty first.

JB: Okay. Go ahead.

SH: So then… so then that's six?

JB: Okay.

SH: Like six…

JB: So there are six tens in the sixty?

SH: Uh huh... The there's thirty in the three hundreds. So…

JB: So that would be a total of what?

SH: Three hundred six.

This passage once again demonstrates how coordinating the scaling process and the coordination of units requires a more complex cognitive load than addition or subtraction. SH's response of "three hundred six" is an indication of that complexity. She solved the problem by reasoning how many tens were in sixty, how many tens were in three hundred, yet struggled to coordinate those tens into the answer of thirty-six. Instead, she responds with a hybrid answer of 300 and 6. Eventually through scaffolding she resolves her answer as thirty-six but the exchanges reflect the cognitive toggling back and forth upon which attribute she needs to focus.

Cluster Three: Task Three (Double, Triple, and Quadruple Numbers) and Task Six (How much to get from one number to another either forwards or backwards)

Figure 8 outlines the attributes by which Task 3 was scored. Again, numbers were differentiated among the grade levels. Numbers that students were asked to double were higher than the subsequent numbers of for tripling and quadrupling. An emphasis was placed in the interviews for the student to do as much of the work mentally as they could with any writing being done by the interviewer. Not having immediate access to a pen was an attempt to scaffold the child to think more abstractly and flexibly where otherwise the student might revert to more procedural habits.

Doubling was clearly more accessible for third graders in the pre interviews. While it is often difficult to discern whether or not a student is thinking multiplicatively or additively when doubling a number, fifty percent gave a multiplicative response. JO was asked to double twenty-three. When asked to confirm that he knew what double meant, he responded, “Yeah. It would be forty-six.” His answer was immediate and internal. RE, in contrast, had to work out the answer on paper.

<table>
<tr><td colspan="2">Pre interview:
3^{rd} Gr.: Double 63, Triple 17, Quadruple 13
4^{th} Gr.: Double 163, Triple 27, Quadruple 18
5^{th} Gr.: Double 461 Triple 47, Quadruple 28
Post interview:
3^{rd} Gr.: Double 126, Triple 27, Quadruple 18
4^{th} & 5^{th} Gr.: Double 264, Triple 68, Quadruple 34</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon the distributive property of multiplication over addition or associative property through factoring

• Building Up by Other Factors (multiplicative in structure, additive in solution)</td><td>2
• Accuracy
• Automaticity
• Consistent language of value or capacity to scale single digit combinations to its extension
• A stronger degree of mental abstractness</td></tr>
<tr><td>1
• Accurate but needs to do some extended adding to find answers
or
• Some initial inaccuracies
• Self-corrects due to prompts and light scaffolds
or
• Is accurate but fluctuates among language of value and single-digits but without connections between the two forms, e.g., is insecure in following the values in a consistent manner
or
• Accurate but bound by the standard algorithm and the use of single digit language</td></tr>
<tr><td>1 - Additive
• Uses Tens & Ones/Partial Sums/Standard Algorithm, Repeated Addition, Doubling, or Complex Doubling</td><td>0
• Several inaccuracies
• Consistent single-digit language even in the face of scaffolding
• Only self-corrects due to extensive scaffolding</td></tr>
</table>

Figure 8. Task Three Rubric: Double, Triple, and Quadruple a Number

Both tripling and quadrupling resulted in far more additive approaches, including needing to work out the answers themselves on paper. Eighty percent used additive approaches to calculate triple 17 and quadruple 13. Where JO was able to immediately answer forty-six for doubling twenty-three, after writing three seventeens on paper, he initially resorted to base ten blocks to solve the task.

JO: Hmmm… I'm going to start with seven. Okay. Oh, dang, this is hard!

JB: Okay. So, if you need to do something on paper you can, but tell me out loud what you were just starting.

JO: Uh um, triple seventeen equals… [sound of blocks]…

JB: What is the easiest part to triple, do you know?

JO: Uh um, ten.

JB: So what are three tens?

JO: Thirty.

JB: So you know that one, right?

JO: Yeah, but then the seventeen [sic] makes another ten which is forty-four [But he writes 14 on his paper, so, he means 30 + 14 = 44.]

JB: So, forty-four.

JO: …that's so… and then I am going to take some cubes off, [whispering] one, two, three, four… [sound of cubes]…

JO does solve the task accurately with the answer of fifty-one with some effort. He begins to get lost in the values that he is working with but with some strategic scaffolding is able to right himself and finish the task. He goes on to quadrupling thirteen also working on paper but this time no blocks are used.

JO: Oh dang! Four times thirteen equals something.

JB: Yeah. So what is that going to be?

JO: Ah… I do not know.

JB: Okay. Then figure it out on paper then… [JO writes four thirteens vertically on the paper.] So there are four thirteens.

JO: [Whispering] And all of the tens makes forty… And then three plus three equals six which makes forty-six which crosses out these two and three plus six, I mean forty-six plus three equals forty-nine, which crosses out that, and then plus nine, forty-nine equals, uh um, fifty-two.

Working with sevens, which triggered a direct modeling response, compared to working with threes, which triggered a repeated addition response, seems to be the stimulus for the difference in the two approaches. He clearly knew what three tens and four tens each combined to make instantaneously but the other numbers required calculation.

Fourth graders were only marginally better in their solution strategies for the three subtasks. Sixty-three and six-tenths percent and 72.7% were additive (score 2) when asked to triple 27 and quadruple 18, respectively. Substantial shifts occurred at fifth grade where 81.8% used a multiplicative structure to both triple 47 and quadruple 28. As an example of a fluent fifth grader, MA who states, “I would triple the seven first. And that’ll equal twenty-one. And I will triple the forty and that’ll equal a hundred twenty. So I will add a hundred twenty and twenty-one which will equal a hundred forty-one.”

While the comparisons among the grades may not be surprising with the third graders early in the year, the results of the fourth graders raise interesting questions. Is this a developmental issue or environmental, meaning a lack of experience with using something other than an additive approach to multiplication prior to entering the grade level? On the other hand, are fourth graders in September still very emergent in their capacity to think multiplicatively?

Task Six (see figure 9) was designed to see how fluid a student was in using landmarks of ten to span the difference between two numbers. The efficiency of the quantities spanned and the fluidness in response resulted in a score of four. Moving in small increments would indicate a more emergent, almost skip counting level of processing. While not directly measuring or requiring the use of multiplicative thinking, the task was important in capturing a student's ability to decompose numbers. The ability to fluidly decompose numbers is a key element of using the distributive property and in factoring when multiplying and dividing quantities, thus the inclusion of the task in the assessment.

In looking at the pre interview data among the three grade levels at who scored a four, there was a decreasing level of fluency among the third graders as the numbers got larger and when they were asked to move backwards under a decade; 60, 40, 11 percent, respectively. Results were stronger at fourth grade with 83, 83, and 50 percent, respectively scoring a four. For the fifth graders, the task was more novel as it had not been used within the general classroom at that point in the school year. That might reflect the lower levels of fluency (72.7, 72.7, and 55.6 percent, respectively.) However, when you look at combining both scores three and four, 90.9, 90.9, and 100 percent of the fifth graders fell into this range.

<table>
<tr><td colspan="2">Pre interview:
3rd Gr.: How much to get from 7 to 20, 46 to 100? You are at 62, go back 5. What number are you at?
4th Gr.: How much to get from 57 to 100? 246 to 300? You are at 62, go back 5. What number are you at?
5th Gr.: How much to get from 246 to 300? 457 to 1000? 872 to 2,138?
Post interview:
3rd, 4th & 5th Gr.: How much to get from 68 to 100? 141 to 215? You're at 74, go back 7. You're at 82, go back 17.</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon fluid decomposition of numbers to span distances both forwards and backwards</td><td>2
• Accuracy
• A fluid response
• Consistent language of value
• Easily organizes around landmarks of ten</td></tr>
<tr><td>1
• May be fluid in formation of jumps but required some calculation time to determine the combined span.
or
• Works in smaller incremental chunks
or
• Some initial inaccuracies
• Self-corrects due to prompts and light scaffolds
or
• Fluctuates among language of value and single-digits but without connections between the two forms</td></tr>
<tr><td>1 - Additive
• Uses ones or skip counting to span distances
or
• Decomposes number but does not use them to organize around efficient landmarks</td><td>0
• Several inaccuracies
• Consistent single-digit language even in the face of scaffolding
• Only self-corrects due to extensive scaffolding</td></tr>
</table>

Figure 9. Task Six Rubric: Get to a Number

Cluster Four: Task Seven - Relational Thinking

Task Seven (see figure 10) was not included in the pre interview. It served as an extension task in the post interview. It is presented here at this time to complete the full range of the rubric. The three subtasks were drawn from two studies (Carpenter, Franke, & Levi, 2003, Knuth, Stephens, McNeil, & Alibali, 2006) that looked at a student's algebraic reasoning and that student's ability to think relationally across the equal sign. Its inclusion in the interview protocol was to test if there was a pattern of behavior with the ability to scale up and the ability to think relationally. Data from this task will be presented in a later section.

Many students hold a misconception of what the equal sign means (Carpenter, Franke, & Levi, 2003). The typical misconception is that equals means "the answer comes next." While the specific data from the post interviews are presented in a later section of this chapter, it is worth while contemplating some of the varied responses students gave for this task and to understand the nuances between scores of 3 and 4 compared to scores of 1 and 2.

JA, a third grader whose work on Task One was so automatic and extensive, exemplifies how having strong mathematical capacity in one area does not mean that the equal sign is accurately understood. For the first sub-task, 8 + 4 = ___ + 7, JA immediately said that twelve needed to go in the box. He states first, "It goes with these ones [indicating the 8 + 4] but not with the seven." He continues with the statement, "This is the only thing that it equals." With a challenge to the veracity of his statement JA does switch to "Five. Five goes in the box." But immediately goes states, "No… not the whole thing because there has to be a twelve right here." While some cognitive dissonance made him reconsider his initial answer, it was momentary and his original definition that twelve "has to be right here" indicates that his "rule" for what is mathematically necessary is firmly believed.

<table>
<tr><td colspan="2">Pre interview:
Not given
Post interview:
3rd, 4th & 5th Gr.: 8+4 = $\square$ + 7; 17 = 12 + 5 T/F?; 24 + 73 = 72 + a</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Relational
• Compares values across the equal sign</td><td>2
• Accuracy
• Automaticity
• Provided a relational justification for first and third sub-items
• Provides a definition of the equal sign as a relational/comparative symbol</td></tr>
<tr><td>1
• Accuracy
• Calculated to find answers for sub-items one and three
• Provided a definition of the equal sign as a relational/comparative symbol</td></tr>
<tr><td>1 – Non-Relational
• Equal sign as an operational symbol</td><td>0
• Inaccurate response
or
• Uses a definition of equals as "the answer comes next" or something similar
or
• Orientation of sub-item two (17=12+5) is considered "backwards"</td></tr>
</table>

Figure 10. Task Seven Rubric: Algebraic Reasoning and Relational Thinking

Carpenter, Franke, and Levi (2003) note that two benchmarks exist capturing student thinking once the child recognizes the equal sign as a relational, comparative symbol. The first benchmark describes students who, while he or she sees the equal as a relation symbol, had to calculate each side in order to determine the truth of the equivalence. GA, another third grade solves the 8 + 4 equals box plus 7 by counting, "Eight plus…9, 10, 11, 12. So it would be… 8, 9, 10, 11, 12… Five." Each mathematical expression had to be determined. He calculated 8 + 4 to determine it was 12, then solved $7 + x = 12$ to solve for the unknown.

The second benchmark describe by Carpenter, et al., describes students who begin to look at the relationships among the quantities across the equal sign in order to determine whether or not there is an equivalence in values. JO, the third grader immediately answers "five" to the same task presented to the other two students above. When asked why five, he stated, "Because I put… eight minus one equals seven and I put, and then I put that one into the four." JO did not have to consider whether or not 8 + 4 or 7 + 5 was equal to twelve. He recognized that the values needed to be in an equivalent relationship, so by decomposing the eight into seven plus one and re-associating the one with the four he would end up with an equivalent mathematical expression of seven plus five.

The difference between sub-task Part 1 and Part 3 provide insight into how number size can effect a student's response. ER, a fourth grader used a derived strategy to determine the unknown quantity for the first task saying, "because seven plus three is ten and then two more would be twelve and three plus two is five." However, the larger number in the third sub-item, $24 + 73 = 72 + a$, answered "25" "because they had seventy-three but they minused one so that means that the other number, the twenty-four, would have to go up." No student at any of the grade levels in the study responded relationally to the first item then calculated the third item. Thinking relationally on the first transferred to thinking relationally on the other. However, there were six cases, one at third, three at fourth, and two at fifth, where the larger numbers elevated the student's thinking from one of needing to calculate and being able to think relationally across the equal sign.

After many iterations, the design of the rubric, with its dual code structure, captured the variations of student thinking as new student interviews were analyzed and scored. With reach transcript coded and analyzed the pattern of unit confusion, the emergence of the cover pattern, and the effect of number sizes on the types of strategies students' utilized became consistent themes. The next section, briefly describes how student thinking influenced the types of instructional tasks used within the classroom setting. These classroom tasks are framed by the pre and post interviews and serve as the incubator for the emergence and maturation of more multiplicative thinking.

Between the Interviews: Classroom Instructional Tasks

This second section describes in general the work conducted with students in the three different classrooms. Instructional tasks were given at the different grade levels to see if the capacity to think multiplicatively could be nurtured. The sample students were joined with the remainder of their mathematics class peers. Depending upon the unit being taught at the grade level there was some variation on the total number of students in each mathematics class. Third grade typically contained 20 students, fourth grade 30, and fifth grade 25.

In negotiating access to the site, the teachers decided that students in the top tier mathematics classrooms would be involved in the research. Teachers conducted pre and post tests with students before each unit in the curriculum. Students were then clustered into different classrooms depending upon their scores. Due to the scheduling of support staff into the various mathematics classrooms, there was overlap in the schedule between third and fifth grade. The result was that the principal researcher's ability to work with students directly within the classroom varied from grade level to grade level. In all, ten weeks spread across twenty-two weeks was spent in direct contact with students in their math classrooms. The typical pattern was to be in the classroom for two weeks and then out for two weeks. Additional time was spent conducting one-on-one interviews with members of each grade level's sample group members.

Individual time with the classroom teachers was negotiated to allow for debriefing around student thinking witnessed during lessons and to also plan for future warm-ups and lessons. In addition, three half-day debriefing and planning sessions were conducted across the twenty-two weeks with the whole grade level staff. These conversations focused on how students in their classrooms were engaging with the tasks being used in the research classroom and what differentiation adjustments could be designed. This collaboration allowed a level of consistency to permeate across each grade level and allow for a free flow of information among the teachers as they thought about the needs of particular types of students.

The following describes the type of work completed with students for their classroom mathematics instruction. The grades will be discussed in the order of the day in which the math classes occurred across a typical morning. Table 8 outlines mathematical topics that were the foci at each grade level across the twenty-two week period.

Table 8

Summary of "Mental Math" Warm-Up Activities and Instructional Units: September - November

	September	October	November
Third Grade	• Subtraction Strategies - Decomposition of number - Strategy development & flexibility • Get to a Number (Example: How much to get to ten, twenty) • Make a Ten Strategy Decomposition of number	• Make a Ten/Next Ten Strategy (working in larger chunks of numbers) • Combinations of 100 and 1000 • Relational Thinking • Place value – Reunitizing a number (Colored Marker Activity) • Place Value Scaling up (Example: If 10 tens in 100, how many tens in 400)	• Get to a number • Doubling a Number • Language development • Equivalency Tasks • Splitting & Sharing (Example: Share 18 with 6)
Fourth Grade	• Doubling, Tripling Numbers - Language development - Decomposition of number - Strategy development • Get to a Number (Example: 48 → 100) - Fluency forwards and backwards around landmarks of ten • Scaling up lesson: If five triangles in one pattern stamp, how many in…	• Getting over/going back under a decade (working in larger chunks of numbers) • Place value – - Reunitizing a number (Colored Marker Activity) - Factors of ten (Example: 6 x 50 = 6 x 5 x 10) - Scaling up (Example: If 10 tens in 100, how many tens in 400) • Scaling Up by multiplies other than 10 (Example: 2 x 8, 6 x 8, 18 x 8) • Doubling, Tripling Numbers • Decomposition of Number Strategies - Distributive property - Factoring • Algebraic Thinking	• Rewritten decimal unit – Place value o Reunitizing across places o Decomposition of Number • Equivalency tasks
Fifth Grade	• Doubling, Tripling, and Quadrupling Numbers - Language development - Strategy development • Decomposition of Number - Distributive property - Factoring • Algebraic Thinking	• Place value - Reunitizing a number (Colored Marker Activity) - Factors of ten (Examples: 50 x 60 = 5 x 10 x 6 x 10 40 x 300 = 12 x _____ - Multiplying largest "partial" in multidigit multiplication • Doubling, Tripling, and Quadrupling Numbers	• Splitting and Sharing (Example: Share 64 with 3) • Scale Factor (If 6 sixes in 36, how many sixes in 72?) including need to scale up remainders (If 16 r 4 sixes in 100, how many sixes in 400?)

Table 8 *(continued)*
Summary of "Mental Math" Warm-Up Activities and Instructional Units: December - January

	December	**January**
Third Grade	• Rewritten multiplication unit - Deriving strategies development - Decomposition of Number o Distributive o Factoring - Scaling up o Basic Facts: 3x4, 6x4, 12x2) o Place Value: If 3 tens in 30, how many tens in 60? - Place Value using Multiplication and Measurement Division Tasks - *n* times as many tasks	• Scaling up around 10 • Reunitizing number across place values • Double, triple, quadruple, & quintuple numbers • Basic facts derived strategies - Distributive Property - Scaling up through factors of 2 and 3
Fourth Grade	• Get to a Number – Decimals (Example: 0.38 → 1.99) • Rewritten multiplication unit • Relational thinking • Doubling, Tripling & Quadrupling numbers • Multiplication Basic Fact deriving strategies - Distributive Property - Factoring - Relational thinking	• Rewritten Division unit • Scaling up with factors of ten and non-ten multiples • Splitting & Sharing • Missing Factor scaling tasks • Tripling, Quadrupling, Quintupling, & Sextupling
Fifth Grade	• Splitting & Sharing – Fractions (Example: 4 sharing 10, 4 sharing 3, 4 sharing 2 ½) • Relational thinking with fractions • Equivalency • Order & Comparing Fractions	• Reunitizing numbers across places (Example: 1.5 thousands → ___ ones • Ordering & comparing fractions • Place value as a rate of ten • Relational thinking with negative numbers

Grade Four

Fourth grade had the most contact with the principal investigator (PI) in their classroom. Math began first thing in the morning. When present, the PI led each morning's warm-up activities. These activities had the dual intention of exploring new concepts and skills as well as to build capacity to think more fluidly and efficiently. Particular emphasis was placed on using the language of value rather than single digit language, decomposition of number which led to the use of the distributive property, factoring, and compensation strategies, scaling up by factors of 10, 2 and 3, and re-unitizing a number across places.

After the warm-up session, typically fifteen to twenty minutes in length, the classroom teacher would lead the main body of the lesson as presented in the published curriculum resource used by the District. During this portion of the lesson, the PI would move about the room having students process aloud their thinking while working on assigned tasks. Particular emphasis was placed upon communicating with members of the sample group although not exclusively.

As the fall semester progressed, the teacher and PI began collaborating more regarding assessing students, shaping of existing lessons, and wholesale rewriting of units based upon evidence from student work. Particular focus was placed on rewriting a unit on decimals and the units on multidigit multiplication and division. These decisions were based upon the need to strengthen students' multiplicative place value understanding and to increase the presence of scaling up activities not otherwise emphasized in the published resource materials. The rewritten units also provided the classroom teacher materials to use when the PI cycled out of the classroom thus providing students more consistency in types of mathematical tasks and conversations to which they were being exposed.

Grade Five

Work within the fifth grade classroom was limited to just warm-up activities. This was due to overlapping math instruction schedules between fifth and third grades. Fifth grade began at 10:00 each morning. Third grade began at 10:15. The PI worked with the fifth grade doing warm-up activities until approximately 10:25 to 10:30 and then transitioned into the middle of the third grade lesson.

The focus of fifth grade began more immediately with multiplicative decomposition of number strategies particularly around factoring. This matched the opening unit that students were encountering within the published curriculum resource used by the school district. This allowed conversations around the rate of ten when multiplying multidigit number combinations, example: 50 x 60 = 5 x 10 x 6 x 10, as well as explicit conversations around the distributive property and concepts of equality.

As the twenty-two weeks progressed, thinking in scale, exploring multiplicative relations while working with fractions, and re-unitizing numbers across place values were explored. Several students at this grade level were initially reluctant to publicly engage in the mathematical conversations resulting in more passive engagement. Effort was then placed in having partners verbalize with each other as they solved the task and then presenting together as a means to increase verbalizing ones thinking to others.

No rewriting of main units occurred at the grade level. However, since the teachers rotated with each unit who taught the top tier students, the PI collaborated with all the teachers directly across the data collection period. In addition, the planning and designing of the warm-ups was completed with the involvement of the entire fifth grade team so that they could conduct the same warm-ups with the other sections of students. Some of the warm-ups were exclusively designed by the teachers to maintain continuity in the type of tasks students were exposed to during times when I was not at the school site.

Grade Three

Grade three held its math class at the end of the morning with students leaving for recess and lunch immediately following. The math class was already underway when I entered the classroom from being with the fifth graders. In the early autumn months, this led to the warm-ups being conducted typically at the end of the lesson. The emphasis was placed on developing language patterns with having the students talk in value and to gain fluency in the area of subtraction strategies. Thinking of place value as a multiplicative relation was infused into the conversation through the conversations around subtraction.

A high level of collaboration among two of the third grade teachers and the PI took place in order to coordinate what they would start the lesson with before the PI entered the classroom and also what tasks that they would engage students in while the PI was out of the district. As a result, students came to see the PI as a co-teacher in the classroom who would ask questions during the middle of public sharing episodes as well as when circulating around the room. Two units were redesigned and largely implemented by the classroom teachers when multiplication and division were formerly introduced to the students. The new units were in line with similar units being used at fourth grade where an emphasis on decomposition of number, place value as a rate of ten, and scaling up relational thinking was being nurtured.

The analysis of the data from the classroom has yet to occur. That effort will follow in the near future. Nevertheless, the collaborative role of the PI and classroom teachers while working with the students during math instruction added depth and texture to how students' multiplicative thinking emerged over the twenty-two weeks. The next section returns to the one-on-one interview process. The focus is to introduce the post interview data and compare student shifts in thinking over time.

Comparing Pre and Post Interview Data

Post interviews with all sample group students were conducted over a three-week interval in February. The following section compares shifts in student performance when particular pre and post tasks are compared. The comparisons were conducted to understand how students transition from additive to multiplicative thinking over time (research question one), and to look at the potential effects of certain types of tasks in revealing student's multiplicative thinking (research question two). To understand the transitions students move through over time, the comparisons have the potential of clarifying what types of thinking can be considered developmental, a natural progression in terms of cognitive maturation, as oppose to environmental, meaning as a result of a lack of exposure to being asked to think multiplicatively prior to that point in time.

To assist in monitoring the specific tasks referenced in the comparisons, Table 9 notes the cluster of tasks used to test the initial conjectures of the study, the specific tasks outlined in the scoring rubrics, and the basic content of the sub-items within each task. For further details, see Appendix A, B (pre and post interview instruments), and D (scoring rubrics).

Table 9
Summary of Item Numbers and Tasks

	Cluster One				Cluster Two	
	Task 1			Task 4	Task 2	
Sub-Item	P1	P2	P3	P1	P1	P2
	How many ones in 783?	How many tens?	How many hundreds?	How many boxes of ten?	Scale by five	Scale by ten
Item	1	2	3	4	5	6

Table 9 *(continued)*
Summary of Item Numbers and Tasks

	Cluster Three						
	Task 3			Task 6			
Sub-Item	P1	P2	P3	P1	P2	P3	P4
	Double a Number	Triple a Number	Quadruple a Number	Get to a Number	Get to a Number	Get to a Number	Get to a Number
Item	7	8	9	10	11	12	13

Table 9 *(continued)*
Summary of Item Numbers and Tasks

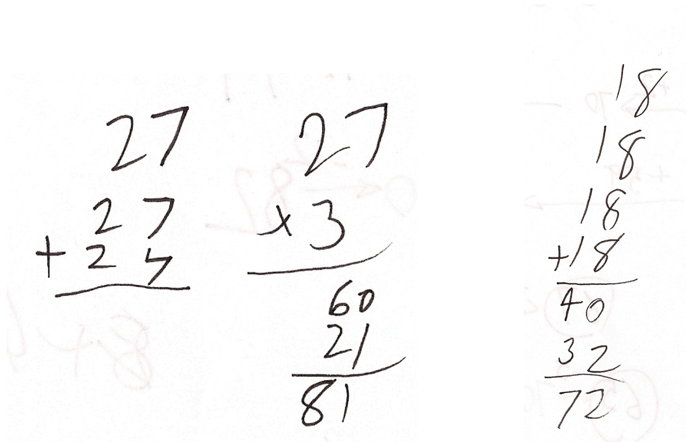
	Cluster Two	Cluster Four			Cluster Two
	Task5	Task 7			Task 8
Sub-Item	P1	P1	P2	P3	P1
	Scale by 10	Relational Thinking	Relational Thinking	Relational Thinking	Missing Value
Item	14	15	16	17	18

Tripling and Quadrupling: Task 3_Part 2 and Part 3

In September, nearly all the sample third graders used additive structures when asked to triple17 and quadruple 13. During the February interview, 90.9% of those same students used a multiplicative structure when asked to triple 27 and 72.7% when asked to quadruple 18. However, with the third graders, when you look closely within the strategies used by those following a multiplicative structure when tripling 27, 45.5% used an efficient multiplicative strategy (score of 4) with 45.5% needed to do more calculating (score of 3). When quadrupling 18, the level of efficiency dropped with 9.1% scoring a four and 63.6% scoring a three. Looking at a variety of strategies used by the third graders who scored a three, it was the amount of calculating to determine what four eights were that distinguished the two scores.

BT typified a more additive solution to tripling 27 in the post interview. "I count the tens by two [sic]… It is sixty… and seven plus seven is fourteen plus, uh um, more, plus seven more is… twenty-one and sixty plus twenty-one is… eighty-one." A similar mix of multiplication and repeated addition was used when she quadrupled eighteen. "Count four times for the tens. It is forty… And then if eight plus eight is sixteen and then I add eight more is twenty-four… and I had to add it one more time, it's… thirty-two… seventy-two."

DN when asked to complete the same tasks begins more multiplicatively. When asked to triple 27, he initially writes out three twenty-sevens but then stops, rewrites the task as a multiplication problem and quickly says, "So then that is sixty and that is twenty-one so then it is eighty-one." (See figure 11 (a).) When asked to quadruple 18, DN, while he wrote out the tasks as a repeated addition problem, determined the answer so rapidly that it raises the issue of how multiplicatively he was processed the answer. "Eighteen… eighteen, eighteen, eighteen, so then forty… add thirty-two, put that together and that's seventy-two." See figure 11 (b) to see what he wrote compared to what he said.



(a) (b)

Figure 11 – DN work samples: Triple 27 (a) and Quadruple 18 (b). Comparison of what was written and what was spoken

Fourth grade followed a similar pattern between the September and the February interviews. 72.7% of the sample students used an additive structure when tripling 27 and quadrupling 18. During the February interview, 100% and 91.7% used a multiplicative structure to triple 68 and quadruple 34 respectively. When that data for the post interviews is dissected, 75% and 66.7% scored a four for each of the two items indicating that multiplicative efficiency was more consistent among the fourth graders compared to the third graders. (See Table 10.)

Table 10
Task 3, Part 2 and Part 3 Comparisons

		Tripling		Quadrupling	
		Pre	Post	Pre	Post
Grade 3					
	Mx	20.0	90.9	10.0	72.7
	Add	80.0	9.1	90	27.3
Grade 4					
	Mx	27.3	100.0	27.3	91.7
	Add	72.7	0.0	72.7	8.3
Grade 5					
	Mx	81.8	81.8	81.8	81.8
	Add	18.2	18.2	18.2	18.2

JQ, who was not able to complete the tripling and quadrupling tasks in the September pre interview, sounded very different by the post interview in February. While she makes an error, she is able to self-correct and reason appropriately to find the correct answer.

> JQ: I'd do sixty times three… And I got eighteen, I mean a hundred eighty… I did eight times three and I got twenty-four. Add it together and I got a hundred ninety-four. A hundred ninety-four.
>
> JB: And why a hundred ninety-four?
>
> JQ: 'Cause I added one hundred eighty and I added ten. That would get me to ninety. Then I added… Oh, wait! Two hundred four, actually.
>
> JB: You're correcting it to two hundred four?
>
> JQ: Yeah… 'Cause it would be fourteen to a hundred ninety-four so then I added another fourteen and it was going to be two hundred four, and then I double-

> checked so I did it a hundred eighty plus ten is ninety plus another ten is two hundred plus four is two hundred four.

Quadrupling thirty-four was much easier. “I did thirty times four, got a hundred twenty. Then I did four times three, I mean four times four, and I got sixteen. I add it together and I got a hundred thirty-six.” During the post interviews, unless the student asked to write, the PI wrote down the partials on paper that the student verbalized. The student’s actions were largely mental. This was intentionally done to elevate the cognitive difficulty of the task to better understand what the student was constructing internally rather than what procedure might be enacted on paper. JQ’s error, while she visually could see the numbers 180 and 24 side by side on the paper in front of her require her to compose a new hundred, rather than a new ten when adding 120 and 16 when she quadrupled 34. This is another indication how certain number sizes or number combinations can result in more cognitive processing than others resulting in more additive or more multiplicative responses. Having effective strategies in both operations overlapping each other allows the student to successfully navigate the more complex combinations when they are encountered.

The fifth graders were consistent across both interviews and with both items with 81.8% using a multiplicative structure. In the pre interview, students were asked to triple 47 and quadruple 28. In the post interview, students were asked to triple 68 and quadruple 34. When analyzing the levels of efficiency used by the fifth graders in the post interview, of the 81.8% cluster, all scored a four. With quadrupling, 54.5% scored a four and 27.3% scored a three.

GV represents a student who was fluent with tripling 68 but reverted to skip counting when asked to quadruple 34. When asked to triple 68 GV quickly stated, "I would start with the sixty. It would be one eighty. And that would be twenty-four. Wait! Yeah, that would be twenty-four… and I would add that together and equals two hundred four." However, when asked to quadruple 34, where for JQ in fourth grade this was the easiest, GV did the following after first groaning upon hearing the task.

GV: Uh, I would start with the three – 3, 6, 9, 12 – a hundred and twelve, or a hundred and twenty.

JB: Which is it?

GV: A hundred twenty. Uh, - 4, 8, 12, 16 for the four. Add that together equals a hundred thirty-six.

The strategies verbalized by EL and SH represent transitions in thinking that deal with the language of value that is different than with the unit confusion demonstrated by others. EL begins tripling 68 by saying,

EL: Eight times three is twenty-four. Six times three… eighteen. Eighteen plus twenty four… ten plus twenty is thirty, eight plus four is twelve… forty-two. No! Because you're tripling…

JB: There's something that I noticed you did in this first one [doubling 264] which was good, that got you into trouble because you didn't do it here. You didn't watch your language.

EL: Ahhh! One eighty!

JB: …because it is not a six that's a…?

EL: Sixty.

JB: Sixty. Okay, so how does that change your answer?

EL: One eighty plus twenty-four… one hundred plus… okay keep the one hundred. Eighty plus twenty is one hundred so one hundred plus one hundred is two hundred…four.

SH had similar issues that reflected thinking about numbers as single digits or seeing ones actions as a series of procedural steps. The statement, "Because like in my head I brought this down here and I did this four with the zero equals four…" indicates how SH is thinking procedurally rather than in value. In contrast, KI, a fifth grade classmate, links what she knows about single digit combinations to understand the extended combination. In tripling sixty-eight, she states:

KI: I know that eight times three is twenty-four, and six times three would be eighteen, but it's sixty times three so it's a hundred and eighty. And a hundred eighty plus twenty-four is two hundred four.

While an indication that the capacity for multiplicative thinking improves across the grade levels, number size and number combination may also be a factor in determining the consistency of efficiency that a student is able to bring to bear on a mathematical task. Language of value also seems to be a factor in how students trace the effects on the decomposition of the numbers that they are operating upon and their ability to track the units that they need to coordinate. The use of single digit language may be confounding a student's capacity to self-monitor the transformations of unit elements that is a structural aspect of multiplicative thinking. KI's linkage between the smaller number combinations with larger ones is a matter of scaling up. EL and SH's language descriptions indicate a loss of value of the quantity independent of scale; thus the source of their computational errors.

Thinking in Scale: Task 2_Part 1 and Part 2

A similar paired analysis was done with Task Two, parts one and two. These items asked students to use a known fact to then scale up by fives and then tens. In scaling with fives, (Pre: If there are two fives in ten, then how many fives in thirty? If there are ten tens in one hundred, how many tens are in four hundred? Post: If there were two fives in tens, how many fives in forty? If there are ten tens in one hundred, how many tens are in six hundred?) third graders were consistently additive in the September interviews whether scaling with fives (80%) or tens (90%). (See Table 11.)

Table 11
Task 2, Part 1 and Part 2 Comparisons

		Scale with 5		Scale with 10	
		Pre_P1	Post_P1	Pre_P2	Post_P2
Grade 3					
	Mx	20	63.6	10	100.0
	Add	80	36.4	90	0.0
Grade 4					
	Mx	8.3	75	58.3	83.3
	Add	91.7	25	41.7	16.7
Grade 5					
	Mx	63.6	81.8	81.8	90
	Add	36.4	18.2	18.2	10

JS was typical in his approach as he skip counted by fives to thirty to determine he would need six fives. He began determining how many tens were in four hundred by skip counting as well. However, when he got up to 280 keeping track on his fingers, he stopped and said “forty.” When asked what made him stop in the middle of that count to realize it would be forty, he responded, “When I was counting I was thinking like I was counting in my head too and thinking and it was forty.” While his explanation is not explicit, he clearly made a realization that the work he had already done could be projected forward to either determine or predict the answer was forty, or that his counting confirmed an earlier prediction he might have had.

AN used a derived strategy to scale up by fives, “because I know that five times five is twenty-five and five times six equals thirty.” His attempt to scale by tens hints at the use of the cover pattern but his understanding reflects a more superficial one. Asked how many tens would be in 400 if ten tens were in one hundred, AN responded, “uh um, fourteen… because… if ten times ten equals one hundred, ten times fourteen is four hundred.” As AN was an ELL student, and wondering if language was being misunderstood by the interview, a follow-up probe was asked.

JB: So if there are ten tens in one hundred, how many tens would there be in two hundred?

AN: Twelve.

JB: And why twelve?

AN: Because… [pause]… ten tens… [pause]…

A series of prompts and scaffolds continued with the student to see if his thinking became clearer but AN was not able to come to a final conclusion to the question. Other third grade students reverted to direct modeling to solve the task. Four such cases occur when trying to scale with fives and three when scaling with ten.

This shifted by the post interviews in February where 63.6% were able use a multiplicative structure to scale by fives and 100% to scale by tens. Table 4.5 outlines the shifts student level of thinking at each grade level between the pre and post interviews.

Such a shift was also seen among the fourth graders where 91.7% used an additive structure towards scaling by fives where 75% were able to use a multiplicative structure by the post interview. What is notable is that the capacity to scale by tens was already predominant in the pre interviews (58.3%) and strengthened to 83.3% by the post interviews.

Two strategies emerged among the fourth graders during the post interview. VA used scale factor to reason both the fives and the tens tasks. When asked how many fives were in one hundred fifty given five fives in twenty-five, she first responded twenty-five. When asked why, her response was…

> VA: Mm… because if you… if you add twenty-five five times, I mean six times, it would get, um, there's twenty-five cents in a dollar and then you add fifty more and that would be six.
>
> JB: So then how many fives would that be?
>
> VA: Thirty.
>
> JB: So is it twenty-five or is it thirty?
>
> VA: Thirty.

VA saw the relationship of and increase of six twenty-fives would equal a hundred fifty, therefore six fives would equal thirty fives. She worked "within measures" as Vergnaud (1994) describes it. EZ, however, worked "between measures" or with the unit rate. When asked how many fives were in one hundred fifty, he responded, "thirty." Asked to tell what he had just done to figure that out, he said,

> EZ: Because three plus… Like if you take away that zero it's fifteen, and then if there's a three, thirty, then you take away the zero on the thirty, then it would be three times five equals fifteen. So if you add zeros in there.

Where VA's focus is on 25 x 6 = 150 therefore 5 x 6 = 30, EZ's focus is on noticing the relationships between 3 x 5 =15, therefore 30 x 5 = 150. EZ ignores 5 x 5 = 25 in interpreting the task. EZ reasons using the cover pattern to justify the relationships that he comprehends.

Among the fifth graders, the use of multiplicative structures was already pre-dominant in scale both by fives (63.6%) and tens (81.8%) in the pre interviews and continued to strengthen in the post interviews (81.8% and 90%, respectively). The strategies used by the fifth graders were typically more efficient variations of the strategies found among the third and fourth graders: derived, scaling within measures and scaling between measures.

Two observations are worth noting. Scaling by fives consistently was harder than by tens across all three grades supporting the notion that the capacity to think multiplicatively is influenced by number combinations. Secondly, while the combinations asked the third graders and the fourth graders were different in their respective interviews, the fact that the third graders post interview results were similar to the fourth grader results suggests that attention to multiplicative thinking in third grade has potential longitudinal consequences. Fifth graders began to more consistently think in multiplicative terms than the two prior grades. The shifts in the fifth graders' thinking was more towards automaticity and fluency particularly when asked how many fives in 150 knowing there are five fives in twenty-five. This augurs for a developmental capacity among the different ages. That said, the evidence is also suggesting that there is a capacity to develop multiplicative thinking starting as early as third grade with certain number combinations. If this conjecture holds up to further scrutiny, than lack of exposure to tasks fostering multiplicative thinking may be depressing student results in the upper grades.

Re-unitizing Place Value with and without a context: Task 1_Part 2 and Task 4_Part 1

These two items were paired and analyzed to understand the effect of context on solving a similar task. (See Table 12.) Task one in the pre interview involved presenting the student with the number 783 and asking how many ones, tens, and hundreds were in the whole number. The post interview asked third and fourth graders the same set of questions of the number 832. Fourth graders were also asked in the post interview a second number that matched what the fifth graders (2516) were presented. Task Four involved presenting students with a scenario of a large quantity of colored markers being placed into boxes of ten (pre interview) or at a greenhouse where seeds were being planted into ten packs (post interview). (See Appendix A and B for full text of Task 4 used in the pre and post interviews.)

The fourth grade post interview data indicates a drop in the number of efficient multiplicative strategies scoring a four when working with 356 seeds in the post interview. However, as a group 91.7% used a multiplicative structure in the context of Task Four compared to only 66.6% on task one. Fourth graders were also asked an extension of how many ten-packs if there were 1462 seeds. This is the same quantity as fifth grade students were asked to consider. Just over sixty-three percent used an efficient multiplicative strategy. This considerable improvement is likely ascribed to the carryover effect of conversations around the 356 seed task. Fifth grade post interview shows a drop in multiplicative thinking in Task Four compared to their performance on task one.

Table 12
Task 1, Part 2 and Task 4, Part 2 Comparisons

	Pre Interview		Post Interview		
	Task One	Task Four	Task One	Task Four	
	P2	P1	P2	P1	P2
Grade 3					
Mx	30.0	20.0	81.8	72.7	
Add	70.0	80.0	18.2	27.3	
Grade 4					
Mx	33.3	25.0	66.7	91.7	83.3
Add	66.7	75.0	33.3	8.3	0.0
Grade 5					
Mx	72.7	81.8	90.9		81.8
Add	27.3	18.2	9.1		18.2

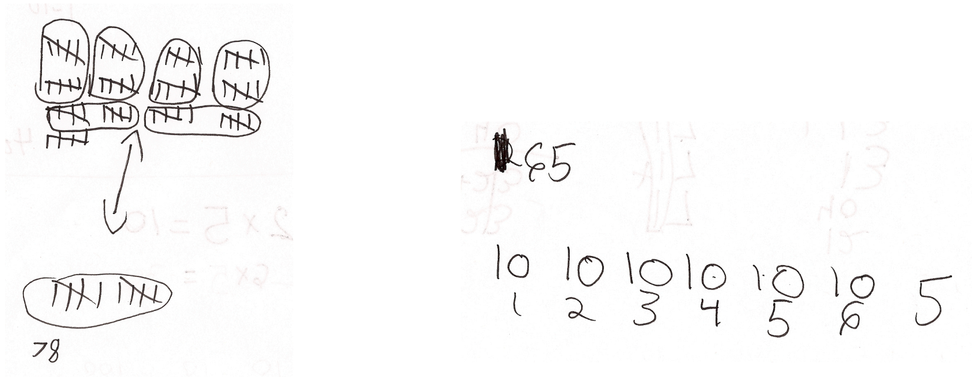
When looking at patterns across the collapsed scores of who used multiplicative or additive structures, there seems to be no improvement. In fact, a decline in multiplicative thinking with the context-based task at third and fourth grade appears to have occurred. However, when the percentages of who was scored a four, three, two, or one (see Table 13), context does seem to improve a student's capacity to think multiplicatively.

Table 13
Task One, Part 2 and Task 4, Part 1: Pre and Post Interview Comparisons

		Pre Interview		Post Interview		
		Task One	Task Four	Task One	Task Four	
	Score	P2	P1	P2	P1	
Grade 3	4	10.0	20.0	27.3	45.5	
	3	20.0	0.0	54.5	27.3	
	2	20.0	80.0	0.0	9.1	
	1	50.0	0.0	18.2	18.2	
Grade 4	4	16.7	25.0	58.3	41.7	63.6
	3	16.7	0.0	8.3	50.0	36.4
	2	16.7	50.0	8.3	8.3	0.0
	1	50.0	25.0	25.0	0.0	0.0
Grade 5	4	45.5	54.5	81.8		45.5
	3	27.3	27.3	9.1		36.4
	2	9.1	9.1	9.1		9.1
	1	18.2	9.1	0.0		9.1

The difference in strategy usage among the third graders between the pre and post interviews is interesting. While highly additive in the pre interviews what is of note is that direct modeling as a solution strategy was used in three cases on Task one (non-context) but increased to seven cases on Task four (context). Figure 12 highlights two variations of direct modeling strategies used.

In the pre-interview, DN made sixty-five tallies in groups of five counting to himself, "20… I have 30, 40, 50, 60, 65." He continues by saying, "So if they are making groups of ten, I add, I keep adding five plus five… [circling two groups of five as he goes]… there's four… five, I can only make six boxes and they will be left with five crayons." In contrast, for Task One - How many tens are in 783? – DN reasons,



(a) (b)

Figure 12 – DN direct modeling (a) and BR direct modeling transitioning to a counting strategy (b) work samples from the September pre interview

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> "…if ten of these are in a hundred [the sound of base ten blocks being moved is heard] there's ten, so there would be seventy of these in seven hundred… plus eight more because of the eighty… to [inaudible], to figure it out because seventy plus eighty…. I know how I would do it. I would do it like this, seven plus eight equals, what's that, it will be one hundred fifty. "

DN, as with other students presented several times elsewhere in this chapter, loses track of his units when trying to determine the final quantity. With scaffolding in reviewing his thinking, DN states, "Oh! Seventy-eight!" What is of note, however, is that when he arrives at Task Four with 65 crayons, a much smaller quantity than 783, he reasons through direct modeling.

The bottom part of his work displayed in figure 12 (a) indicates extensions that were presented to him in reaction to the level of thinking displayed in Task One. DN was asked, "What happens if it had been 78 markers?" he quickly responded, "They would make seven boxes." Asked why so quickly, his explanation included drawing the tallies for the seventh box. Asked about if there had been one hundred twenty-four markers, DN stated, "Uh um, so if there's seven plus {sic] seventy-eight, then for a hundred it would be ten boxes plus twenty it would be two more boxes so it is twelve boxes."

BR had to do extensive direct modeling with the base ten blocks to determine the number of tens in 783. This conversation included repeated issues of keeping track of the units being followed. In Task Four, with how many full boxes of ten can be made with 65 markers, her initial response was "sixty boxes." The immediate exchanges indicate that coordinating the units was confusing for her. Figure 12 (b) captures what she writes down as she solves the task.

> BR: So there can't be sixty boxes or sixty-five. I would have to, uh, I would have to put them into groups, put them into groups of ten… just going to do that… make groups of pictures. … 10, 20, 30, 40, 50, 60… now there's five left. How do I do that?
>
> JB: Well, so the question is how many full boxes can you make.
>
> BR: You can make sixty full…
>
> JB: How many boxes did you make?
>
> BR: [writing her count under each ten] 1, 2, 3, 4, 5, I made six whole boxes.

As the third grades progressed toward February and the post interview, all evidence for direct modeling disappeared. What became more evident as the students become more abstract in their thinking was a significant increase in unit confusion. Five cases of unit confusion occurred for both Tasks One and Four involving eight members of the sample group. In four of these cases, losing track of the units led to combining quantities across units as documented earlier in this chapter. In the post interview, JS first indicated that there would be 110 tens in 832. When asked why 110 tens, JS said he knew, “ten tens is a 100 and so 80 tens are 800 and 80 and 30 equals 110.”

The data shows this pattern of unit confusion prevalent among the fourth grades in both September and February on both tasks. Fifth grade results reflect only a slight lower incident rate of unit confusion in September on both tasks. In the post interview, unit confusion remained an issue with Task One (How many tens in 2516?), however, completely disappeared on Task Four (How many ten-packs can be filled with 1462 seeds?).

Evident at both fourth and fifth grade data is how the emergence of the cover pattern as a reasoned explanation for a solution to the task evolved. The cover pattern was non-existent among fourth graders in September but was utilized by 5 out of the twelve cases for Task One and in two cases with Task Four. Among the fifth graders, the cover pattern strategy was already present in September (3 cases for Task One, four for Task Four) and relatively equally present in the post interviews.

The following three fifth graders’ statements made during the post interview reflect a progression in conceptualization, if not in articulation of why the cover pattern can be trusted. Each statement is in response to the question, “How many tens are in the number 2516?” GV states, “Two hundred fifty-one… Because you don’t count the ones out because you have to go right here [pointing to the tens place] and chose… just put zero there.” PR, in response to the same question answers,

PR: Two hundred and fifty-one.

JB: Now I noticed you did the cover up as you explained what you do. How can you trust that that will always work for you?

PR: Because if you look at it in the tens, because the tens of course will start at the tens right here. So you can just, but the sixteen, or the six isn't really like a ten as a whole. It's just six. Like you could put it in like a decimal form or a fraction but it's not going to be a full ten. So you can just cover that up and the two [hundred] fifty-one.

KI's response was "Two hundred fifty-one and six tenths." Asked why she knew that with such confidence she answered, "Because it's like saying the tens place and up. That's how many tens there are. And then the ones place would be how many of it, uh um, half of it or fourth of it or how many."[1]

Those students who answered about the value of tens correctly, but did not rely upon the cover pattern, used a reasoning process around factors of ten. OR, a fourth grader in response to how many tens in 832, said, "Eighty-three… because there would be eighty tens in eight hundred because there's ten in a hundred… and then from the thirties I count it because there would be three tens." ZA, a fifth grader in response to the number 2516, justified his answer of 251 with, "Because six isn't all the way up to ten so it's like six would be a remainder for like division. If you're to divide it by ten, it's like dividing it by ten."

---

[1] There is a mathematical point that can be argued that the only correct answer to, "How many tens are in the number 2516?" would be 251.6 tens and that 251 is not a mathematically correct answer. However, for most primary and intermediate grades, the *inferred* question is how many *whole tens* are in 2516, to which 251 is the correct answer. For the purposes of this study, both answers were deemed correct responses with the inclusion of the decimal amount considered more abstract than the how many whole tens there were.

Taking a cross-sectional look across the three grades, the transition from direct modeling to the awkwardness of unit confusion, and to the emergence of the cover pattern and reasoning around factors of ten seems evident. What is not clear is how much of the unit confusion, particularly at fifth grade, is a result of students never having been asked to reason about a number's values across places which the capacity to think multiplicatively demands. There appears to be a developmental trajectory present but this trajectory is difficult to benchmark to a grade level without further investigation.

Multiplicative Comparison/Scalar: Task 5 and Task 8

Task Five was asked only in the post interview with grades three and four; not in the pre interview. The context was a multiplicative compare task that built off of the imagery of the greenhouse where one was working with seeds. Fourth and fifth graders were presented the same number combinations: two cups, the first cup has 620 seeds in it, that's ten times more than the second cup, how many in the second cup? Third graders began with 60 seeds in the first cup. Task Eight was a missing value problem: If a three-pack of tennis balls cost $5.50, how much money does a nine-pack cost? All grade levels were presented the same numbers. Both tasks aimed at understanding the students' ability to think in scale, particularly in the *n-times as many* construct of multiplication. Table 14 outlines the percentages for each grade based upon the composite score given to a student's response.

Task Five was difficult for the majority of third graders with 63.6% not able to determine a correct answer if they determined an answer at all. 58.4% of fourth graders used a multiplicative structure to solve the task where 81.8% of the fifth graders were successful. Task Eight, in comparison, was much easier. 72.8%, 83.3%, and 90.9% of third, fourth, and fifth graders, respectively, were able to solve the task using a multiplicative structure. When the level of efficiency (score of 4) and amount of calculation (score of 3) is taken into consideration, 45.5% of third graders needed to do more calculating to find the answer. 33.3% of fourth graders did as well. The results between fourth and fifth grade are interesting. More fourth graders (50%) solved the task using efficient multiplicative strategies than fifth graders (36.4%). It appears that the ability to comprehend the structure comes easily to both, however, the ability to work multiplicatively may be an issue of exposure as the rewritten four grade units nurtured more explicitly the *n-times as many* construct of multiplication where fifth grade students only had more limited exposure during warm-up tasks and not in the additional exploration during main lessons.

Among the third graders, the answer of 50 or 30 was the most common response. BR's response was, "So you just need to minus ten so it will be fifty. Fifty seeds." AN first answered 120, "because I know sixty plus sixty is a hundred twenty." With a scaffold to clarify that the first cup was ten times larger than the second, AN countered with, "Thirty… because I have been thinking that if thirty plus thirty equals sixty." These students were clearly responding to the additive structure of *minus 10 less* meaning ten ones less (answer 50) or to *half as much* (answer 30) being the only multiplicative relationship they could draw upon.

Table 14
Tasks 5 and 8 Post Interview Extension Comparisons

	Post Interview Extensions		
	Score	Task Five	Task Eight
Grade 3	4	9.1	27.3
	3	18.2	45.5
	2	9.1	9.1
	1	63.6	18.2
Grade 4	4	41.7	50.0
	3	16.7	33.3
	2	16.7	0.0
	1	25.0	16.7
Grade 5	4	54.5	36.4
	3	27.3	54.5
	2	9.1	9.1
	1	9.1	0.0

GA, a third grader, represents how the multiplicative processing can begin to emerge. GA initially answered 50, thus conceptualizing the problem through an additive lens. When, however, his response was challenged he states, "Wait! Hold on! No, wait… see ten. I don't get it, kind of. Because I am thinking of, like, ten more and I am thinking of, like, six, like five times ten or six times ten." After a little more conversation to have GA clarify his ideas, he states, "Six?... Yeah, because six times ten equals sixty."

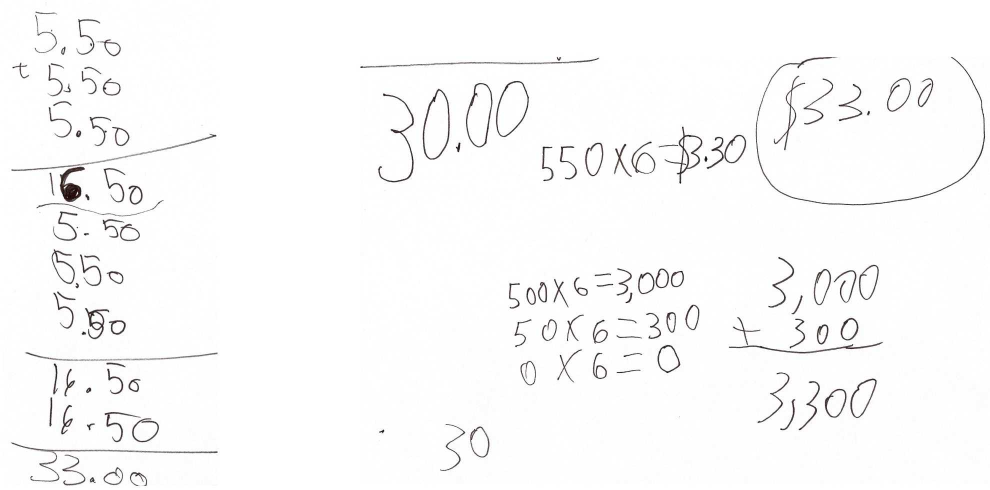
OR, a fourth grader, and MA, a fifth grader had similar approaches. OR said, "I think sixty-two… because sixty-two times ten is six hundred twenty." MA's statement was from the perspective of division, "Because ten can go into six hundred twenty sixty-two times." Both statements are based on recognizing the relationship using a factor of ten. The most pervasive explanation among the fourth and fifth graders was based upon variations of the cover pattern. EL, a fifth grader, stated, "I know that ten times more would be like adding another zero," or ZA's statement, I was thinking at first by division. It's like dividing ten by six hundred and twenty and that would get you into sixty-two. It would just be mainly taking out the zero, taking out the factor of ten.

Task Eight showed a similar array of strategies. Those students who could not solve the task, focused on the additive structure of the situation rather than on the multiplicative scale of three that formed the task. Task Eight was shown in a visual form to each student to show the units and quantities involved (See Figure 13).

Tennis Balls	Cost
3-pack	$5.50
9-pack	?

Figure 13 – Task 8: A missing value task set in a "measure space" (Vergnaud, 1994)

BR, a third grader, and SA, a fourth grader, each interpreted the task using an additive structure. BR noticed that it would be "$5.50 times six, so six times." Asked why she responded, "Because three plus six equals nine. I would have to be five, fifty [meaning $5.50] times six." She then proceeded to add $5.50 in two sets of three, each adding to $16.50 then adding those to get a total of $33.00. (See figure 14 (a).) SA also focuses on six because "six plus three equals nine." He uses multiplication to determine the thirty-three dollar total. When asked if that answer sounded right, his response was, "No… No one would really buy nine packs of tennis balls for thirty-three dollars. So it really doesn't make sense." (See figure 14 (b).) After a little more conversation, however, SA rationalizes the answer by saying, "It actually does sound right because if a famous tennis player walks into a store and they have a tennis court in their back yard and they want nine packs of tennis balls…"



(a) (b)

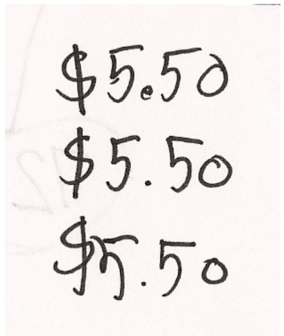
Figure 14 – BR's (a) and SA's (b) work samples to the question, "If a 3-pack of tennis balls costs $5.50, how much will a nine-pack cost?"

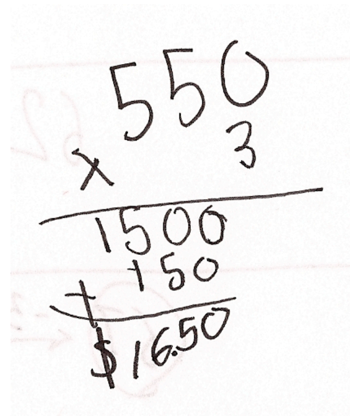
JQ demonstrate a different strategy while misconstruing the problem. The fourth grader "noticed that three and nine, if you do three times three is nine, so I knew that you need to add six." Her response was $11.56 because, "I did five times six, I mean, five plus six and that equals eleven. And then I did fifty plus six, and that is fifty-six." At some level she recognized she needed three times as many balls, but she saw the *difference* between the two numbers and added accordingly, albeit in a unique manner.

GA initially began using an additive structure in solving the task saying, "5, 10, 15, 20, 25, 30 plus [inaudible]… hold on… thirty… hold on… thirty dollars and… thirty…wait!" When asked to tell what he had noticed, GA responded,

> I know the three, that the three-pack is six, that the nine-pack is six more than the three-pack and six more needs to be, needs to be… five hundred fifty tens, five dollars and fifty cents times six. So…

In the middle of his reflecting on his thinking, GA switches to a multiplicative understanding of the problem. He remains confused, however, about whether or not he needed to do more. With the continued conversation, he eventually turns to the paper and write out each $5.50. (See figure 15 (a).) and arrives as $16.50. While he added to calculate his answer, he could only do so once he had correctly understood the multiplicative structure of the task. KA , figure 15 (b), states immediately that, "You would have to times that by three because three times three is nine. KA asked for the paper to work through her thinking consciously removing "the dollar sign and the decimal" then arrived at her answer.





(a) (b)

Figure 15 – GA third grade (a), and KA's fourth grade (b) work sample for Task 8: If a 3-pack of tennis balls costs $5.50, how much would a 9-pack cost?

Many other students, including three third graders, completed the task in their heads. OR, a fourth grader, after a pause said, "I think sixteen dollars and fifty cents." Ask what she had done in her head she responded,

> OR: In my head I added fifty cents and I got a dollar fifty. And then I added the five three times because three times three equals nine. And then so I did five dollars, ten dollars, fifteen dollars, and I… then I added them and got sixteen dollars and fifty cents.

OR's mental work indicates that she understands the multiplicative structure – "three times three equals nine" – but then added the $5.50 through skip counting. DN, a third grader, quickly stated, "You just have to triple five would be fifteen, and then I triple [$.50], plus a one hundred fifty cents, so then, so then that will be sixteen dollars and fifty cents." Nearly all of the fifth grade sample students solved the task mentally using one or the other strategies used by either OR or DN.

Taking a cross-sectional view of the three grades, one can see an emerging capacity to think in scale. The increasing ability to comprehend the multiplicative structure improves with the students' ages as well as a diminishing need to rely on more additive calculations. The *n-times as many* construct, however, appears to be extremely difficult for third graders at least at that late winter timeframe of the year.

Relational & Scalar Thinking: Task7_Part 3 and Task 8 and Task7_Part 3 and Task 5

Task Seven contained three sub-items that assessed students' interpretation of the equal sign. A score of three or four indicates that the individual sees the equal sign as a relation symbol rather than the misconception of it being an operational symbol. A score of four indicates that the individual student can think relationally across the equal sign. An example presented in a previous section, JO answered 5 stating, "Because… eight minus one equals seven… then I put that one into the four." JO did not have to consider whether or not 8 + 4 or 7 + 5 was equal to twelve. He recognized that the values needed to be in an equivalent relationship, so by decomposing the eight into seven plus one and re-associating the one with the four he would end up with an equivalent mathematical expression of seven plus five. A score of three indicates seeing the equal sign as a comparative relation but needs to calculate each side to test the veracity of equivalence (Carpenter, Franke, & Levi, 2003). GA counted, "Eight plus…9, 10, 11, 12. So it would be… 8, 9, 10, 11, 12… Five." Each mathematical expression had to be determined. He calculated 8 + 4 to determine it was 12, then solved $7 + x = 12$ to solve for the unknown.

Pairing these two tasks together was done to see if the capacity to think relationally in task seven matches with those more complex scaling tasks. The ability to scale up requires the coordination of two elements not dissimilar to thinking relationally with elements across the equal sign. In comparing Task Seven with Eight there is a reasonable alignment between the two. (See Table 15.) Recognizing the equal sign as a relational symbol was the norm among nearly all of the students at each grade level (a score of 4 or 3). Being able to recognize the multiplicative structure of Task Eight was recognized by a large percentage of students at each grade level. How much calculation was needed to determine the missing value discriminated who scored a three or a four.

Table 15

Task 7, Part 3 and Task 8 Comparisons

	Relational Thinking & Scale		
	Score	Task 7_P3	Task 8
Grade 3	4	27.3	27.3
	3	63.6	45.5
	2	0.0	9.1
	1	9.1	18.2
Grade 4	4	41.7	50.0
	3	50.0	33.3
	2	8.3	0.0
	1	0.0	16.7
Grade 5	4	81.8	36.4
	3	18.2	54.5
	2	0.0	9.1
	1	0.0	0.0

The level of difficulty of Task Five, in general, makes using Task Seven as a predictor unclear especially at third grade. (See Table 16.) Of the five students at fourth grade who scored a four on task five, three also scored a four on Task Seven with the other two scoring a three. The alignment improves in fifth grade. Of the six sample students who scored a four on Task Five, they also scored a four on Task Seven. Three students scored a three on Task Five. Two of those scored a four and one scored a three on Task Seven. While not consistent across the students at the two grades levels, one thing that is evident is that a student who succeeds on Task Five at least sees the equal sign as a relation symbol rather than as an operational symbol. It may be that the items presented to students in Task Seven were not capturing enough of student's capacity to think relationally in a context other than across the equal sign. The conjecture about the ability to think relationally as a predictor of the ability to think in scale requires further exploration.

Table 16
Task 7 – Part 3 and Task 5 Comparisons

	Relational Thinking & Scale		
	Score	Task 7_P3	Task Five
Grade 3	4	27.3	9.1
	3	63.6	18.2
	2	0.0	9.1
	1	9.1	63.6
Grade 4	4	41.7	41.7
	3	50.0	16.7
	2	8.3	16.7
	1	0.0	25.0
Grade 5	4	81.8	54.5
	3	18.2	27.3
	2	0.0	9.1
	1	0.0	9.1

In looking across all three tasks – seven-part 3, five, and eight – there was one third grade who scored a four on all three tasks. Two scored all fours at fourth grade and four scored all fours at fifth grade. There were two additional fifth grade students who scored fours for Tasks Five and Seven-part three, but only a three for Task Eight. When the solution strategies are looked at for these two students for that particular item, the one, LA, first added only two to get \$11.50 – the difference between the two packs – but readily corrected herself in her verbal reflection on her strategy. MA answered \$16.50 quickly but gave addition based explanation of how she determined the answer. Fourth grade, in comparison, had only one additional student who fell into this profile. One other scored a four one both Tasks Five and Eight but gave a calculated response for Task Seven-part three.

While the sample size for each of the grade levels is small, there is some indication that the fifth graders collectively were more capable of thinking in scale than fourth graders. This is not surprising given the decades of accumulated knowledge of developmental psychology. What is worth noting is that fourth graders, by February of the school year, had nearly half of the sample group capable of thinking relationally and in scale. Thinking in scale was difficult for the third graders especially with Task Five. However, Task Eight showed that 72.8% of the third graders understood the multiplicative structure of situation at hand. The third graders just remained more additive in their solution strategies to the task. Taking a cross-section look across the three grades one can see the emergence of thinking in scale. While further investigation needs to occur to understand the capacity to think relationally in simpler tasks aids the growth to think relationally in more scalar contexts.

Individual Profiles Across Tasks in Pre and Post Interview Results

This section looks at presenting individual student profiles comparing his or her score on each task in the pre interview with those in the post interview. A composite score of one or two falls within an additive approach to a task where a composite score of three or four would indicate a multiplicative approach. Using the format of a line plot, the scores create a profile of how additive or how multiplicative the conceptual understanding of an individual would be. Plotting just the pre or just the post interview scores provides a glimpse of how additive or how multiplicative an individual is relative to the other tasks in the protocol. By plotting the pre and post interview scores on the same graph, the shifts in the student's thinking over time can be assessed. Comparing plots with each other provides a level of understanding of the different learning trajectories students take within the same classroom learning environment.

The following represent a selection of individual students at each grade level to provide further evidence to the progressions in individual student learning trajectories. Table 4.3, presented earlier, serves as a reminder for what task each item assessed and to which the numbered sub-task on the *x*-axis refers. The *y*-axis scores – the composite scores of four, three, two, one – refer to the level of abstractness students used in solving the task. Four represents a multiplicative or relational level of understanding, one the most additive or emergent level of thinking. Dashed lines indicated the scores on the pre interview tasks; the solid line indicates the post interview.

Grade Three

Four student profiles are represented here. JR represents the most emergent within the grade three sample group. While he progressed in his strategy development on some tasks, he did worse in the post interview than previously. Re-unitizing a number across places (Items 2 & 4) remained very additive as did how he doubled, tripled, and quadrupled numbers. Scaling by ten (Item 6) was automatic where scaling by five (Item 5) and scaling ten in the multiplicative comparison setting (Item 14) was extremely difficult. (See figure 16.)

Yet, in spite of those difficulties, JR solved the missing value task where he knew the cost of a three-pack of tennis balls was $5.50, so what was the cost of a nine-pack (Item 18). He readily understood the multiplicative structure of task. He needed to add his way through the calculation but he understood the core concept of the task.

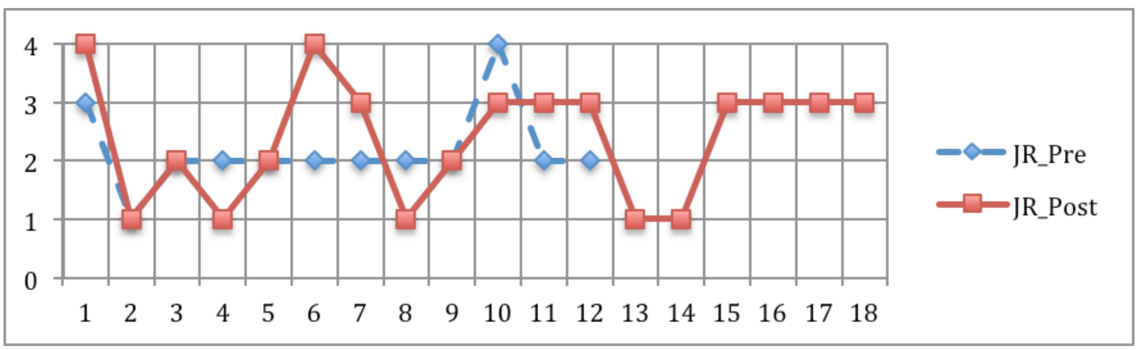


Figure 16. JR: Third grader pre and post interview profile

BR was also one of the more emergent third graders in the sample group. Her profile, while stronger over all than JR, also indicated a spiked set of strengths on the one hand yet weaknesses in others particularly on tasks involving scale (Item 5, 14, and 18). Her strategies to double, triple, and quadruple numbers became more multiplicative in structure while still needing to use smaller additive increments to find the solution. Scaling by ten became easier than scaling by five. More telling was what was difficult for her. She was very operational in her understanding of the equal sign (Items 15-17) and she could not complete the missing value task at all. (See figure 17.)

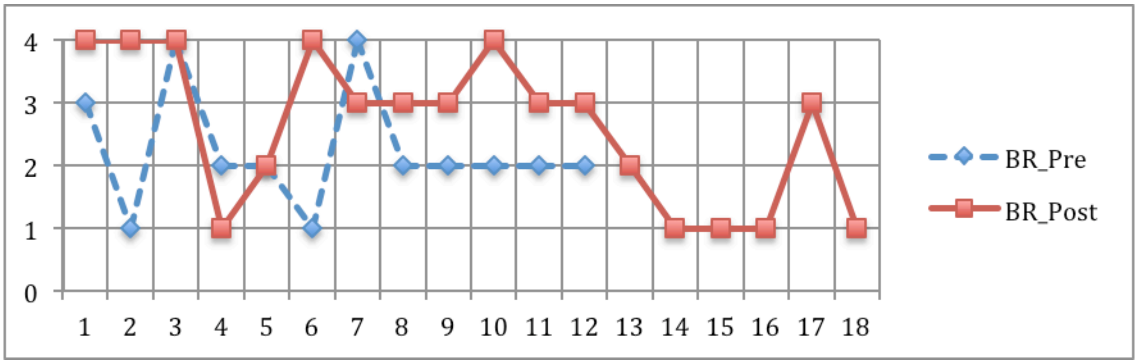


Figure 17. BR: Third grader pre and post interview profile

In contrast, JS showed consistent growth from the September to February interviews. While re-unitizing across a place value (Item 4) and comprehending the multiplicative comparison scale (Item 14) remained additive in structure and difficult to comprehend, on all other items his level of understanding shifted to recognize the inherent multiplicative structure. He still needed to calculate in smaller increments in order to complete several tasks (scores of three) but they were improved shifts in understanding from where he started. (See figure 18.)

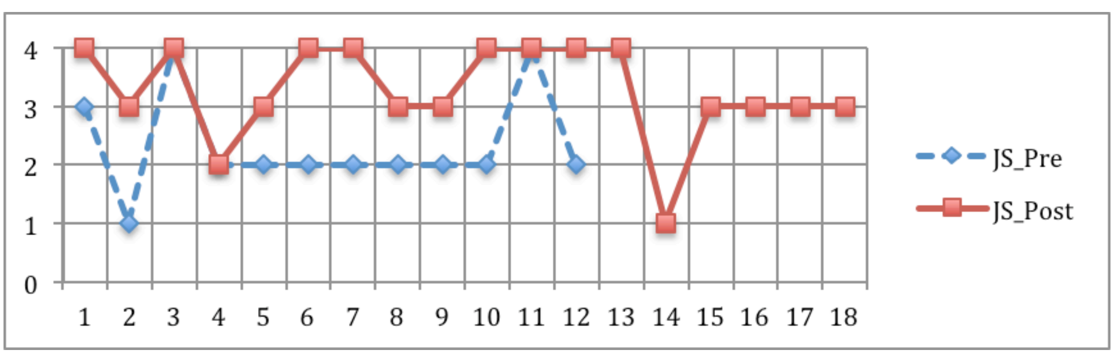


Figure 18. JS: Third grader pre and post interview profile

DA represents the most multiplicative in thinking amongst the third graders in the sample group. In September he was already scoring a three or four on all items except to Items 4 (How many boxes of 10?), 5 (Scaling by 5), and 9 (Quadruple a number). By the February interview, he had become multiplicative in his understanding of all tasks including the difficult items with which other third graders struggled. For Items 14 (multiplicative comparison scale) and 18 (Missing value task) DA gave fluid and efficient explanations when solving the task. While DA does not represent the typical third grade profile the way that JS does, he does represent the capacity of a certain percentage of third graders who consistently think in multiplicative terms. (See figure 19.)

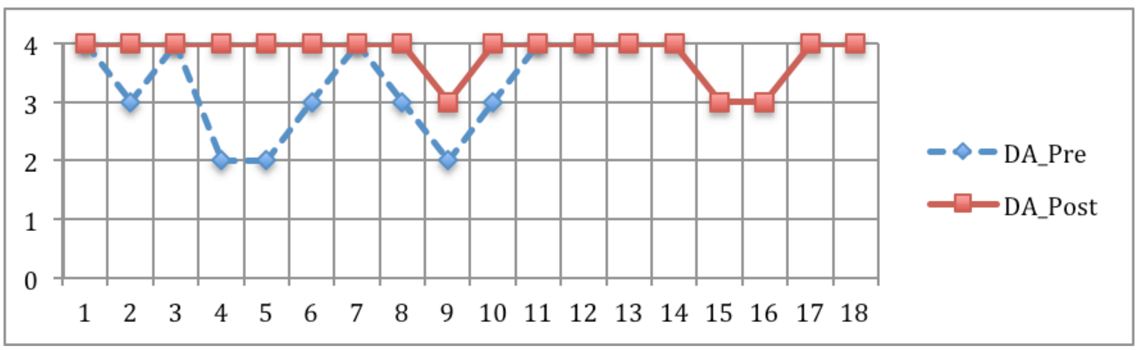


Figure 19. DA: Third grader pre and post interview profile

Grade Four

The two students presented here, JQ and OR, reflect how growth shifted from mostly additive to near uniformly multiplicative. JQ in September had difficulty even completing the interview protocol (thus the absence of scores for Items 8 and 9). The February interview also began additively with the first two items but then shifted to more multiplicative thinking. JQ is bilingual and this raises the question of how much code switching (Moschkovich, 2010) may be a factor. A similar pattern can be discerned with items 15 through 17 where the first item in the series scored a one but subsequent tasks that build off of the discussion of the first potentially influenced the following items where she scored a three. (See figure 20.)

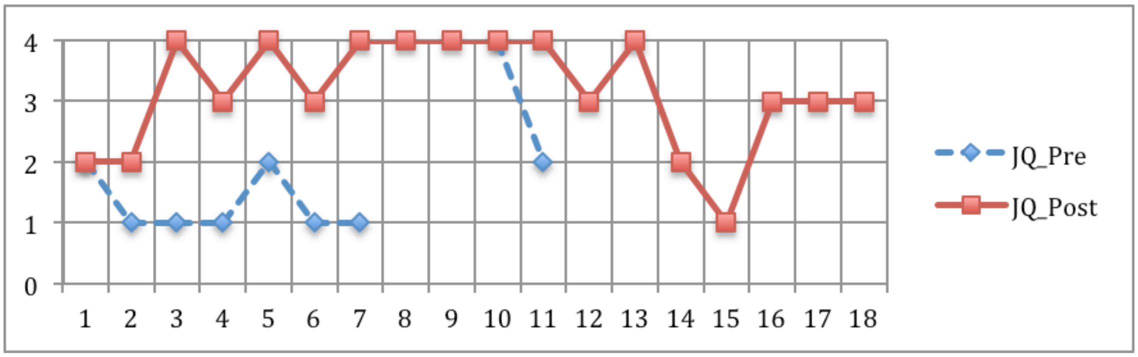


Figure 20. JQ: Fourth grader pre and post interview profile

OR shows a similar but slightly stronger additive profile with the September interview. By the February interview she is consistently solving tasks using a multiplicative structure. Items 5 (scaling by 5), 9 (quadrupling a number), and 18 (missing value tasks) are key items that resulted in less efficient, smaller calculated solutions. Items 11, 16, and 17 were similar. Overall, these two individuals reflect a pattern of similar growth among the fourth graders over the course of the design experiment. (See figure 21.)

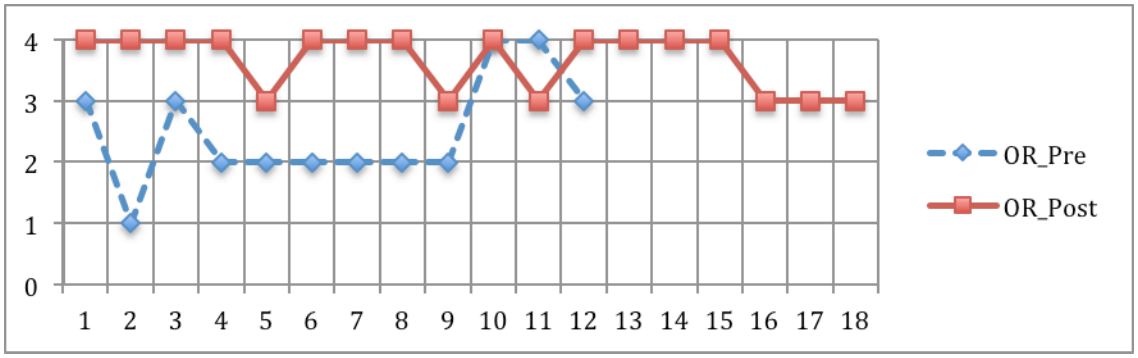


Figure 21. OR: Fourth grader pre and post interview profile

NI and CH both reflect fourth graders in the study who began the year demonstrating evidence of comprehending the multiplicative structure of the tasks and continued to solidify that knowledge as the year continued. It is not clear why CH would require the need to calculate more on Item 4 (How many boxes of 10) and on Item 18 (missing value task) when he was so automatic on Item 14 (Multiplicative Comparison). This does speak to Seigler's overlapping wave theory (2000) that new and old strategies exist side by side and that the context of a task triggers one response over the other. (See figures 22 and 23)

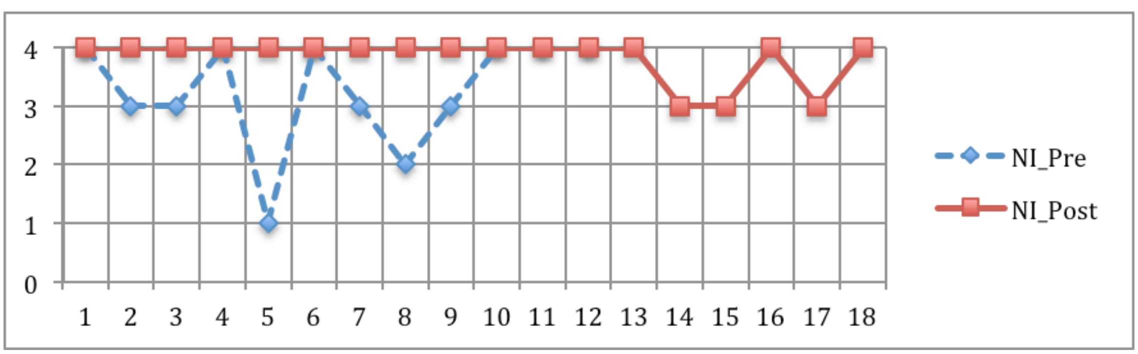


Figure 22. NI: Fourth grader pre and post interview profile

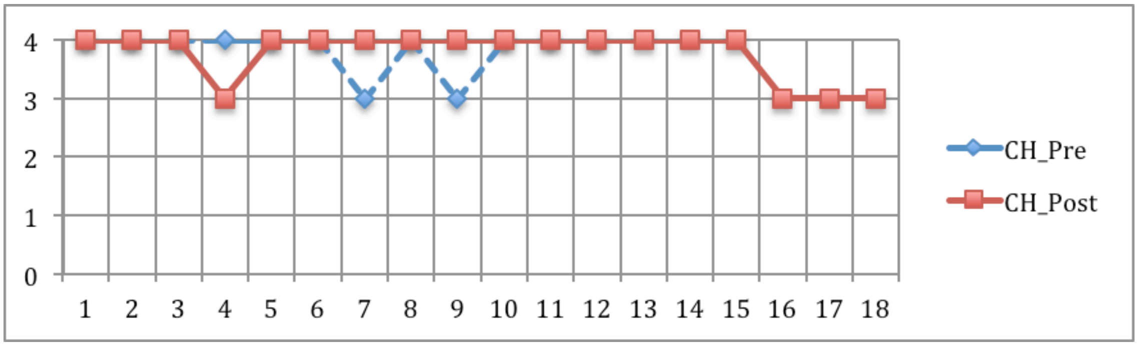


Figure 23. CH: Fourth grader pre and post interview profile

Grade Five

LA represents a core of the sample fifth graders who demonstrated Siegler's theory of comingled strategies, where in one interview a level of automaticity was used and at another more calculation was required. In each case, however, a consistent profile of understanding the multiplicative structure of the task and an increase of that capacity over time is evident. (See figure 24.)

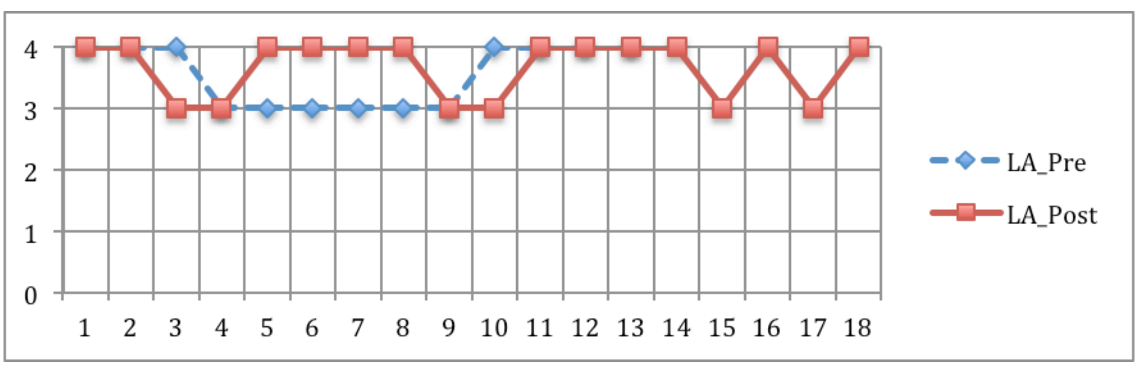


Figure 24. LA: Fifth grader pre and post interview profile

VI represents the profile of a student who shows strong growth over time yet also shows some variation in working more multiplicative in the earlier interview than on the latter interview. Task 9 may be a good case in point where being asked to quadruple 28 (September) versus quadruple 34 (February) triggered an automatic response (September) compared to an additive response (February). Was it the number size or the moment is unclear. She does, nevertheless, exhibit a profile across the tasks of gaining a capacity to think multiplicatively. (See figure 25.)

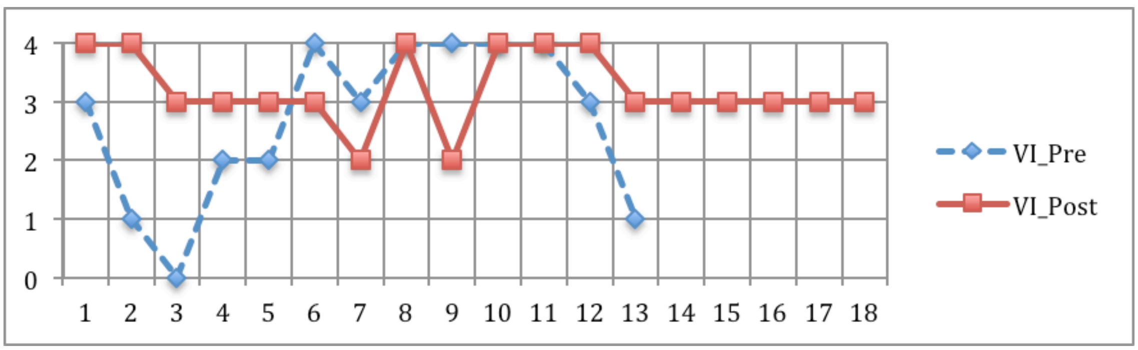


Figure 25. VI: Fifth grader pre and post interview profile

SE is a student that reflects a less consistent profile. While she exhibits growth on some tasks she is inconsistent with many. SE's work reflects a student who struggles with core ideas of place value, re-unitizing a number across places, and in decomposing numbers. Items 10 through 13 (Get to Number) were consistently the most automatic for students even in third grade. But here, SE's increments to span the distance between two numbers were in smaller, less efficient quantities or was the result of confusion requiring a level of scaffolding to complete the task. Yet her response to Item 17 (relational thinking with higher numbers) and Item 18, (missing value) were multiplicative in structure and, in the case of Item 17, automatic. From the perspective of the classroom teacher, her profile does indicate the need for close monitoring and additional probing to parse out what mathematical concepts remain unformed or unconnected. Even more curious, is when this profile is matched with her winter Measure of Academic Progress (MAP) (NWEA, 2005) test score. A nationally norm-referenced computer-based test, the winter benchmark for on-progress growth for fifth grade is a score of 215. Her score was 217 with the spring benchmark expectation of 219. This is a student, based upon the testing criteria, who is doing well. However, the level of reasoning exposed by the interview protocol used in this study does not match the level of thinking exhibited by others in the sample. This issue will be explored more in the next chapter. (See figure 26.)

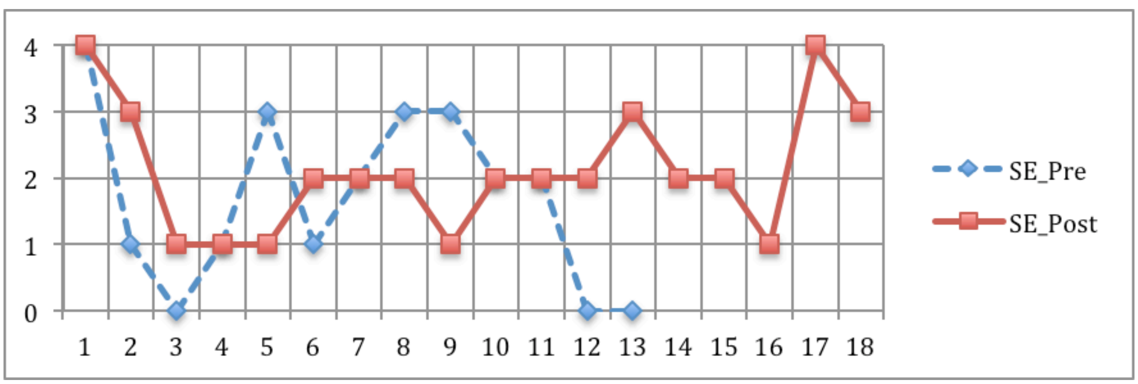


Figure 26. SE: Fifth grader pre and post interview profile

Summary

These four sections have outlined the design of the scoring rubric developed to assess the structure and automaticity of a student's solution to selected tasks, outlined themes of transition as students move from an additive to a multiplicative level, compared the differences between various paired tasks to identify potential relationships among the tasks used in the interview protocol, and outlined the use of a representation to visually capture the growth and consistency of the level of thinking over time capturing different learning trajectories students pass through. These finding will now be discussed for their implications in the next chapter.

CHAPTER FIVE. DISCUSSION, CONCLUSIONS, AND RECOMMENDATIONS

Discussion

Two questions were explored in this study. The first was how does the development of additive to multiplicative thinking evolve over time and what does students' thinking look like as they transition from one to the other? A second question asked what integrated and coordinated series of mathematical tasks can teachers use to help reveal the emergence and development of multiplicative thinking. The previous chapter presented the results of student responses to selected tasks in pre and post interviews of a design and teaching experiment with a classroom each of third, fourth and fifth grade students from the beginning of the school year in September through the end of February. In between the interviews, the principal investigator collaborated and co-taught with the classroom teachers at each grade level presenting instructional tasks and engaging students in conversations around various elements of a multiplicative conceptual field (Vergnaud, 1988, 1994; Confrey, et. al, 2009). These tasks were conjectured to stimulate and nurture the emergence of multiplicative thinking.

The pre and post interview results are used here to analyze the shifts in student thinking over the length of the design and teaching experiment. Using the conceptual framework of Overlapping Waves Theory (Siegler, 2000), the analysis identifies how newer strategies gradually replace earlier additive structures through a transitional process of co-existence with each other. This theoretical lens takes into account that the multiplicative conceptual field is broad and interconnected. Shifts in the structure of ones capacity to think multiplicatively may develop in one area of the conceptual field but needs time to mature and interconnect with other areas of the conceptual field. A natural ebb and flow exists in how that new knowledge is integrated to create a more comprehensive capacity to consistently think in a multiplicative manner.

To address the question of what the differences between additive and multiplicative thinking look like, analysis began with the design and refinement of a rubric to be used to assess a student's thinking strategy on the various tasks used in the interview protocol. Drawing upon prior research of others in the field as well as using evidence from the emerging data in this study, a rubric was needed by which student responses could be scored. What began to emerge from the data was a need to differentiate the multiplicative and additive *structure* of a task from the multiplicative or additive *solution strategy* invoked by the student. An additive interpretation of the structure, as in the case of the missing value, 3-pack of tennis ball task (task eight), led to an incorrect answer. A student in this instance focused on the difference between a 3-pack and a 9-pack being six instead of the multiplicative relationship of three times as many. The student's *strategy* in some instances was more additive than multiplicative in that to determine the cost of the 9-pack, if a 3-pack cost $5.50, a repeated addition approach was used rather than multiplying $5.50 by three. On the other hand, using an additive *structure* and *strategy* on some tasks allowed students to arrive at a correct response. A means to separate these two approaches and the "correctness" of answer needed to be found.

The analysis of the individual transcripts revealed initial themes indicating potential student transitions as they began to think more multiplicatively. Four themes being explored here are "unit confusion," the development and usage of a "cover pattern," the role of number size in triggering either an additive or multiplicative solution strategy, and the effect of language in shaping multiplicative thinking. A fifth theme, more of a question requiring further investigation, is how much of what is being identified in this small-scale study is developmental or environmental. How much is due to the natural age and maturation of a child as they progress through these intermediate elementary grades? How much is environmental, meaning the students have never been asked to think about multiplicative relationships or in scale so do not develop the skill when in fact they are capable of doing so?

Multiplicative thinking is foundational to the mathematics that students engage in at the middle school level with its focus on rational number, ratio, and proportional reasoning. The questions investigated here in this study addresses how does multiplicative thinking arise out of the arithmetic underpinnings of learning place value, the operations of multiplication and division, and fractions in the intermediate grades. The following section discusses the five themes identified in this study and how these findings fit within the literature on the development and role of multiplicative thinking in the later elementary years. The themes begin to uncover for further consideration the various learning trajectories student evolve through as they navigate the broader multiplicative conceptual field. These sections are followed by a discussion of the rubric design and the interview protocol itself.

Unit Confusion

Multiplication is about the making of new composite units of smaller units that can then be iterated (Steffe, 2002). This requires an ability to think simultaneously across units (Kamii, 2000) and to unitize and re-unitize a number flexibly and efficiently (Lamon, 1994, Siemon, Breed, & Virgona, 2005). The act of multiplying is unit transforming in that the product results in a new unit or relationship not evident in its component parts. To have a robust understanding of multiplication, one ultimately needs to think proportionally (Vergnaud, 1988, Lesh, Post, & Behr, 1988).

Lamon describes unitizing as the cognitive assignment of a unit of measure to a given quantity (1996). To re-unitize is to reassign a different unit of measure to a quantity that continues to maintain an equivalent relation with the original unit. The ability to think multiplicatively requires the coordination of units as those units transform as a consequence of operating on the quantities. "Unit confusion" is being defined here as a student's solving a task using an appropriate strategy but then becoming confused when *coordinating* the units within the task, thus coming up with a wrong answer. It is posited that this confusion is reflective of a natural developmental cognitive struggle students need to engage in in order to build the capacity to coordinate and re-unitize number and to think multiplicatively.

Returning to the example of NI, when she was asked how many tens are in 783, she responded, "Fifteen… because ten times ten is a hundred so it's then seventy plus eighty equals fifteen." The interview continues through a series of queries that indicates that NI clearly knows that "eighty is eight tens" and that there are "seventy" tens in seven hundred. Her difficulty was how to keep the pieces and their corresponding units separate. Her initial response was to combine the "8" (tens) and the "7" (hundreds) and to get "15." EZ knew he had "70 tens" in seven hundred and that there were "eight" tens in eighty but his initial total of tens was "150." Each student lost track of the units to which they needed to attend. Their errors are a map of their cognitive struggle to develop the capacity to effectively coordinate the units.

The third graders typically did not demonstrate this unit confusion behavior at the beginning of the school year because many of them directly modeled the tasks through physical representations such as tallies or cubes. This more emergent, structurally additive approach, allows them to potentially work around the coordination process. However, as some of the third graders in February and most fourth graders across both interviews began to become both more abstract and multiplicative in their work, an increasing percentage of this confusion emerged from the evidence.[2]

Rather than seeing this confusion as merely errors being made by students, it is suggested that this is an important instructional moment to publically engage students in finding ways to explicitly resolve the coordination. This mathematical idea lies at the heart of developing the mental capacity to follow the transformation of the units and how to coordinate among the different units with which the students are working. The confusion is an indicator that they are transitioning to more sophisticated levels of understanding and finding ways to coordinate the units becomes part of the learning process.

Among the fifth graders in the study, the frequency of unit confusion diminished. The most common occurrence, both in the pre and post interviews, was when students were asked to re-unitize a multidigit number into complete units of ten (Task 1, Part 2).

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[2] A possible alternative argument could be made that some of these students may be misapplying rules each has constructed about mathematics similar to those articulated by Benny in Erlwanger's classic description (1973). The analysis in this study, however, is that these students accurately understand the parts but are having difficulty coordinating the end result. This represents a significant difference with Benny's attempt to create order out of mathematical procedures he did not conceptually understand.

Across all the tasks presented, however, most occurrences of unit confusion were with two individuals in the fifth grade sample group, one with a total of five instances, the other six. What is notable about these two individuals is when their Winter MAP (NWEA, 2008) scores are taken into account the first student scored 217 (215 is the benchmark) and the second scored 225, substantially above grade level. This raises the issue, to be addressed further in this discussion, that one can succeed on standardized tests using less mature thinking strategies.

Taking a cross-sectional look over the three grade levels, there appears to be a progression from direct modeling where unit coordination is minimal, to a period of unit confusion where abstract multiplicative thinking is developing, to an ability to efficiently coordinate the units while solving tasks. The concepts of “progressive abstraction” (Ambrose, Baek, & Carpenter, 2003) and “progressive mathematization” (van Putten, van den Brom-Snijgers, & Beishuizen, 2005) speak to the increasing coordination of the decomposition of number and use of scale, respectively, to efficiently multiply and divide numbers. The results of this study posit that the increasing capacity to coordinate the units assigned to the numbers being operated upon is part of this progressive maturation process. Posing tasks that heighten the need to coordinate the units exposes a student’s capacity to coordinate units. It is in the explicit grappling with the coordination of the units that a student matures in his or her multiplicative thinking.

The Emergence of the Cover Pattern

As students became more abstract in their thinking, a “cover pattern” began to emerge. The students were asked to visualize the factory where colored markers were being placed into boxes. Given a particular starting quantity, they were then asked to determine how many full boxes of ten the machine could complete. Recounting JA, a third grader when he was presented with the quantity of 65, JA quickly answered, “six.” When asked why so fast, he stated, “Because it is too easy. Then the other five, they would just add five more to get to seven” [boxes of ten]. He was then posed an extension of what if there had been 124 markers, how many of boxes of ten could then be filled? His response was, “I could make twelve… because it has the number twelve right here,” pointing to the tens and hundreds places. He answered similarly to the number 491 markers. His response, however, changes when the number is extended to 1,646. He answered, “That would be a hundred and ten… Because is easy if you cover, added these two together it would equal ten” [pointing to the digits 4 and 6 in the tens and hundreds place].

The notable aspect of that exchange is that JA was using the cover pattern of ignoring the ones place while looking at the remaining portion of the number. He could, however, only coordinate those elements up to a three-digit number. Once he regarded the same question, but with a four-digit number, his ability to coordinate the units of ten disintegrated and he exhibited the unit confusion behavior described previously. The cover pattern was not just merely a procedural matter to him but rather was a result of a series of multiplicative relationships.

By fifth grade, others more specifically used the cover pattern to quickly determine the answer to how many boxes of ten could be generated. MA answers “one hundred forty-six” boxes of ten quickly. Asked how she knew that so fast, she responded, “Because I know five, you know, it’s not quite a ten yet. So I just take the other numbers and minus a place value and just get the number.”

The ability to re-unitize a number requires simultaneous thinking (Kamii, 2000) and the ability to reason up and down (Lamon , 2005, Confrey, 1994). A conjecture that formed this study was that place value, understood as a multiplicative relation, is an early site for students to develop the capacity for multiplicative thinking. The elegance of base ten system in terms of the social convention of how to write and read the units by which one is describing is the basis for the cover pattern.

However, this pattern can quickly become procedural and shallow as students learn it without comprehending its multiplicative underpinnings. As AX, a third grader explained, “Because I know these tricks of the other ones, like a hundred, how many tens there are. Always cover up the last number and you get your answer.” Appeals to authority (Carpenter, Franke, & Levi, 2003) also appeared with PR, a fifth grader’s explanation, “My third grade teacher taught me if you were to multiply anything by ten, then you would like put, like pretty much place a zero at the end multiplying it by ten.” The commercial reform-based curriculum formally used by the host district, in fact, encourages students through repeated fact extension exercises for students to recognize the “zero pattern” and “cover pattern.” As the teacher’s manual spoke nothing about justifying the rate of ten, factors of ten underlying the pattern, the pattern is merely a rule, a procedure which a student may or may not have the capacity to justify mathematically.

The cover pattern by itself is not evidence of multiplicative thinking. It is a student’s *level of justification* providing evidence supporting the veracity of the pattern that must be analyzed. A fifth grade student, after calculating her solution for the third time in the interview involving a rate of ten claimed, “I did it again!” This statement of exacerbation indicated her recognition of the cover pattern only after doing all that calculation effort yet again. Having her reflect upon her observations each time, however, provides foundational support that in the future will allow her to mathematically trust the pattern. Where many students stopped with the recognition that the quantity in the ones place was “not quite a ten yet,” a more robust understanding was demonstrated by a fifth grader, KI, in her justification. “Because it’s like saying the tens place and up, that’s how many tens there are. And then the ones place would be how many of it, um, half of it or fourth of it, or how many,” meaning a fractional quantity of the next ten. So with the next question asking how many hundreds in 2,516, KI easily responded, “twenty-five hundreds and sixteen hundredths” hundreds.

The justification provided by OR that there are eighty-three tens in 832 “because there would be eighty tens in eight hundred because there’s ten in a hundred… and then from the thirties I count it because there would be three tens” as well as ZA’s explanation to justify that there are 251 tens in the number 2516, “Because six isn’t all the way up to ten so it’s like six would be a remainder for like division. If you’re to divide it by ten, it’s like dividing it by ten” are mathematical at the core. These are students who are not acting merely on procedure. They are capable of producing a mathematical basis for what underlies the pattern.

This capacity to look at a number (example: 783) and re-unitize it as a simultaneous set of relations of all ones, all tens, all hundreds, or a combination of hundreds, tens, and ones is a core function of multiplicative thinking. It augurs for why place value/base ten understanding is a multiplicative relationship that can be an early source for developing a student's capacity to coordinate and transform units along with developing scalar thinking and a *n-times* as many rate of change. The caution for teachers and curriculum developers is how does the emergence of this useful pattern develop through a robust level of mathematical justification rather than a quick surface feature that is learned only at the procedural level.

Effect of Number Size on the Capacity to Think Multiplicatively

The passage with JA noted above, where he stumbles answering the question of how many tens are in 1646, raises the third theme identified among the student transcripts: that number size or number combination can determine whether or not an individual thinks multiplicatively or additively. JO, one of the third graders highlighted in this analysis, when asked to triple 17 knew that three tens were 30 but had to direct model with cubes to determine what three sevens were. The very next task, quadruple 13, triggered an automatic response of knowing four tens was forty and an additive response to calculating the four threes. This varying among multiplicative and levels of additive responses, even within the same task, was evidenced across the varied number combinations with which a particular student worked. This occurred at all three grade levels. However, substantial shifts occurred at fifth grade where 81.8% used a multiplicative structure to both triple 47 and quadruple 28. An example of a fluent fifth grader, MA stated, "I would triple the seven first. And that'll equal twenty-one. And I will triple the forty and that'll equal a hundred twenty. So I will add a hundred twenty and twenty-one which will equal a hundred forty-one."

Task Two asked students to first scale up by fives followed by scaling up by tens. The shifts in student capacity over the three grade levels indicated that scaling in tens was easier for third and fourth graders compared to scaling by fives. By fifth grade, the difference largely disappeared. The idea that students can think multiplicatively in some instances but not in others can be explained using Siegler's Overlapping Waves Theory (2000). New ideas as they become understood sit side by side with older ideas. It is only with support over time that the new ideas become integrated into an even newer, larger conceptual framework. In this instance, scaling in tens may allow students to reflect upon strategies to then extend those ideas to scaling by fives. This transition of being able to think multiplicatively with some number combinations more easily than others is similar to findings described with early number strategies where number size influences the level of abstract strategies upon which a student draws (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). With lower numbers a student may be able to act as a flexible thinker or work at the number recall level, where with higher numbers direct modeling or counting strategies are required (Carpenter, Fennema, Franke, Levi, & Empson, 1999).

The data from Task 8 (scaling by three) indicates that the transition to multiplicative thinking certainly improves as the students increase by grade level, but it also shows that the capacity for multiplicative thinking is present as early as third grade. Recognizing the multiplicative structure of Task 8 was understood by 72.8% of third grades. While only 27.3% had gained automaticity and operated at a higher multiplicative level (score of 4) and the others required more calculation (score of 3), the emergence of a multiplicative understanding a scale of three was discernibly present. The instructional implication of this is that *structure* can be understood prior to automaticity and efficiency of *strategy*. Out of understanding structure comes automaticity and efficiency.

The Role of the Language of Value

Based upon the research of Fuson (1990), Fuson, et. al. (1997) and Carpenter, et. al. (1998), an emphasis was placed in this study on developing student consistency in describing numbers by value rather than by the number's digit. The later is what Fuson refers to as children "concatenating" a multidigit number into a string of single digits and operating on the digits accordingly. It was also found that students who used invented strategies in addition and subtraction where the value of numbers is maintained that students solidified their place value knowledge sooner and were more flexible in solving mental tasks than those students who used the single digit language. These students who used single digit language were more procedurally bound with the algorithms they used. In this study, when students decomposed numbers in order to operate with them, there was an insistence that the language of value be used. This explains the exchange between EL, a fifth grade student, and the interviewer when he noted the source of her error because "you didn't watch your language."

Those students who were most emergent among the sample struggled with understanding the values of the numbers with which they were working. This was evident in the language that they used to describe their thinking. EL's error when tripling 68 because "eight times three is twenty-four. Six times three [is] eighteen…" is the result of treating the 60 in 68 as a literal 6. Her number sense was strong enough that her initial answer of 42 was nonsensical – "No! Because you're tripling!" Once reminded to "watch your language" she quickly realized the error and self-corrected her response.

EL was careless, not an emergent student. JQ, who was noted in the analysis of how many ones are in 783, answered, "Eighteen… because seven plus eight, I got fifteen and I added three more and I got eighteen." To her at the beginning of fourth grade, multidigit numbers were more a string of single digits rather than an aggregate of various values that totaled an equivalence of 783 ones. By February she was able to catch herself as she worked as in the case of, "I'd do sixty times three… And I got eighteen, I mean a hundred eighty…"

SE is another student who worked typically from a single digit perspective. Her solution to double 264 in February, while accurate, was absent of value: "And then I add the six plus six, which is twelve. I put the two down there, carry over the one. And then I do two plus two, which is four, and it's going to be a five because I carried the one over." Two other students were similar in that their thinking was more procedural and their performance was lower than their peers, particularly on scaling tasks.

The language of value can reveal students' algebraic thinking in how they are decomposing numbers and then operating on the partials. JQ's February solution to quadrupling thirty-four captures that algebraic usage of the distributive property of multiplication over addition: "I did thirty times four, got a hundred twenty. Then I did four times three, I mean four times four, and I got sixteen. I add it together and I got a hundred thirty-six."

The issue of language with unit confusion is different than the issue of language with which JQ initially started and SE continues. NI and EZ's struggle with determining how many tens were in 783 (NI originally said "15", EZ "150") is an issue with coordination of values *with their corresponding units* rather than with the concept of the value of the original quantities with which they are working. These two students were eventually able to navigate the struggle with coordination of units *because they understood the values of the original quantities.* JQ in September had no such ability to navigate the relationships.

Place value is a multiplicative relation organized around the rate of ten. It is being argued here that having students consistently use the language of value allows students to explicitly grapple with the underlying mathematics involved when decomposing numbers and operating upon them according to algebraic properties of the various operations. Some of the early third graders saw 65 as 10 + 10 + 10 + 10 + 10 + 10 + 5, an accurate albeit additive decomposition of the number. Those thinking multiplicatively know that "six tens is sixty," therefore, 65 = 6 x 10 + 5. Language can capture and make those mathematical ideas explicit. The students in this study, who were multiplicative in their thinking, demonstrate the capacity to communicate in the language of value. For those who started out or remained emergent at the end of the data period, one element that distinguished them from the others was their struggle with tracking the values of the basic quantities with which they were working.

Developmental or Environmental?

While the comparisons of the extent of multiplicative thinking among the grades may not be surprising, especially among the third graders early in the year, the results of the fourth graders raise interesting questions. Is this a developmental issue (a question of maturation) or environmental (a lack of exposure) with using something other than an additive approach to multiplication prior to entering the grade level? An example of this question is when students were presented two similar tasks as noted previously where students were asked to scale up by fives and then by tens. As noted, scaling by fives consistently was harder than by tens across all three grades supporting the notion that the capacity to think multiplicatively is influenced by number combinations. Secondly, while the combinations asked the third graders and the fourth graders were different in their respective interviews, the fact that the third graders post interview results were similar to the fourth grade results suggests that attention to multiplicative thinking in third grade has potential longitudinal consequences.

When analyzing the results of task eight, 50% of fourth graders and only 36.4% of fifth graders scored a four in the level of automaticity and efficiency. Is the data on fourth graders reflective of a stronger developmental capacity than previously understood? Are the lower fifth grade results a reflection of a lack of environmental exposure in previous grades? With such a small scale study, quirks in sampling has to be considered. Fifth graders certainly began to more consistently think in multiplicative terms than the two prior grades especially as measured by tasks one, four, and five. This augurs for a developmental capacity among the different ages. That said, the evidence is also suggesting that there is a capacity to develop multiplicative thinking starting as early as third grade with certain number combinations and certain mathematical contexts. If this conjecture holds up to further scrutiny, than lack of exposure to tasks fostering multiplicative thinking may be depressing student results in upper grades.

Interview Protocol and Design of the Rubric

This study used the structures of both a design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) and teaching experiment (Steffe & Thompson, 2000) as a means to capture the dynamics of how multiplicative thinking arises in a classroom learning environment as well as within individuals. The design experiment entails the engineering of particular forms of learning as well as being concerned with the development of theories of learning within the direct context of the learning environment. The process is one of a design, test, refine, iterative process. The interview protocol and subsequent scoring rubric is one engineered outcome of this design experiment.

Addressing the second question of this study – what integrated and coordinated series of mathematical tasks can teachers use to help reveal the emergence and development of multiplicative thinking – the interview protocol and subsequent scoring rubric captures multiplicative thinking as it emerges over time. The tasks on the interview protocol were selected to analyze an individual student's thinking on different aspects of multiplicative thinking. Four clusters of tasks were selected to assess place value as a rate of ten, the capacity to think in scale, computational strategy development, and algebraic and relational thinking. With the exception of a few, most tasks were drawn from the literature. The following discusses the interview protocol as a measure of student's capacity to think multiplicatively. The discussion focuses on strengths and potential adjustments to the protocol for future use in other settings.

Task one asked students how many ones, tens, and hundreds were in the whole number 783. Three sub-scores resulted for each part. The number was changed in the post interview and four digit numbers were available as extensions. After analyzing many student transcripts, particularly of those who struggled with the number of tens (78) in 783, it was difficult to score whether or not the student's response to how many hundreds (7) was one of relief supporting their single digit construction of place value or if they were truly thinking flexibly in their comprehension of the units. In the analysis of the data, Task 1 – Part 3, the hundreds place, was not used as it was felt that the data was not reliable. The recommendation would be to ask for the number of ones and tens but drop asking how many hundreds in the future use of the protocol. Subsequently, with a four digit number (2516) asking about the hundreds place would be informative but the thousands place would be eliminated. For those demonstrating automaticity across all sub-parts of this task, probing how much of the *next ten* is, the quantity in the ones place, would be worth probing. In the current iteration of the protocol this follow-up probe was asked spontaneously as a result of student initiated conversation.

Task two provided useful data in answering questions about the capacity to think in scale. Conversations around these two sub-parts were quite extensive. One future question is if asking the student to scale first by ten and then by five would have any effect on student responses. Since scaling by tens has proven easier for students than by fives, particularly among third and fourth graders, would starting with the easier scalar task stimulate a higher level of thinking when solving the harder task? This would be worth doing investigating in further exploratory studies.

Task three which asked students to double, triple, and quadruple numbers also revealed fruitful insights into the automaticity and efficiency of student computational strategies. Doubling is difficult to sometimes discern if an additive or multiplicative structure was being used. A careful parsing of language often determined a composite score of two or three. The automaticity of response, even in the face of additive language was also used to discern the difference. The data on doubling, therefore, was not used in any comparative analysis with other tasks. While it could be argued that this task be dropped, it does serve as an entry task to the successively harder sub-tasks that follow. It is recommended that the sub-task of doubling be reviewed further for future inclusion.

Task six contained at least three sub-tasks that probed student fluency in organizing around landmarks of ten and decomposing numbers to efficiently and fluidly move forward and backwards on a number line. The inclusion of these tasks was based on an organizing conjecture that lack of fluency on the tasks would be an indicator of weak base ten understanding. Those in this study who struggled with these tasks did show weakness with base ten tasks. However, most students were highly fluent on the task, particularly in the post interview that the data was not of high need. A possible suggestion is that task six be used as an extension task for those who are less fluent with previous tasks to assess if issues with decomposition of number and organizing around landmarks of ten exists. Its inclusion in the protocol with all students may not be as informative as initially conjectured to assess multiplicative thinking.

Task seven was crafted to assess students understanding of equality and the capacity to think relationally across the equal sign. Thinking in scale requires an individual to consider relations among and between quantities. Rates and proportions are quantities that co-vary with each other. This task included three sub-parts drawn from specific studies. Part one (8+4= ___+7) and part three (24+73 = 72+*a*) were from work reported by Carpenter, Franke & Levi (2003). Part two (17 = 12 +5) was used in assessing students' understanding of equality as reported by Knuth, et al. (2005). Only

Part three (24+73 = 72+*a*) proved reliably productive in comparing a student's capacity to think relationally and to think in scale. The number quantities in part one (8+4= ___+7) were low enough that students gave explanations involving calculation where the numbers were high enough on part three that the task forced verbal explanations that were more consistently relational in thinking. Like in task three where doubling is worth keeping as a warm-up to the more complex succeeding tasks, so too does the initial task in seven serve as a warm-up to part three. However, while there was some useful discussions around orientation and equality with part two (17 = 12 +5), the data produced little useful data in how the capacity to think relationally supports being able to think in scale. It is recommended that part two be dropped from the design of the protocol.

In general, the protocol used tasks that elicited differences in a student's conceptualization of the *structure* of the task as well as captured a student's additive or multiplicative *strategy* used to solve the task. The repetition of tasks used in both the pre and post interview protocols captured discernable shifts in students' conceptual understanding and range of solution strategies. With the revisions proposed above, the protocol is a useful tool to form the basis of a larger study.

The rubric, used to score student responses, went through a series of design, test, and revise iterations resulting in its current state. The dual phase scoring process began to emerge as pre interview transcripts and work from within the classrooms were analyzed. The current rubric no longer was being revised after most to the third and fourth grade sample students were scored. The rubric is descriptive enough and flexibly accommodates a variety of student responses that it is in condition for use in a larger study where multiple scorers could assess its reliability.

Conclusions

Ambrose, Baek, and Carpenter (2003) discuss how students' multidigit multiplication and division strategies emerge out of the invented strategies students use for multidigit addition and subtraction tasks. It is expected then that early conceptualization of multiplication will reflect these additive roots. As the capacity to unitize quantities and coordinate the transformation of those units, a student's capacity to think in scale emerges. The nature of a student's decomposition of the number, the algebraic properties incorporated to form efficiencies reflect the increasing progressive abstraction that thinking multiplicatively requires. The students' capacity to express themselves in language helps to facilitate the maturation of such thinking.

This 'progressive abstraction' (Ambrose, et al., 2003) or 'progressive mathematization' (van Putten, et al., 2005) inherently captures the need of students' work across units and coordinate the transformation. This study seeks to expand those authors' use of those terms to explicitly include students' capacity to re-unitize and coordinate the transformations of units as they operate on quantities. It is argued that this is an important link towards thinking in scale. It is also being argued in this study that place value as a multiplicative relations is a productive source of developing this level of coordination, therefore extending and integrating into the multiplicative field this aspect of the literature.

As students gain a greater capacity to think in scale, their capacity to extend that conceptualization to their acts of splitting and partitioning (Confrey, 1994) and to rational number (Steffe, 2008; Lamon, 2005) they form connections with other nodes in the wider multiplicative conceptual field (Vergnaud, 1994, Confrey, et al., 2009). This foundation is what students need in middle school to effectively reason proportionally. What this study adds to this literature is additional clarity into the possible learning trajectories, as outlined by Confrey, et al. (2009), students take as they develop the capacity to think multiplicatively.

This study sought to better understand what the transitions from additive to multiplicative thinking looks like and how it evolves over time. Are there ways to nurture student capacity to think more multiplicatively so that as they arrive in middle school the foundation for solidifying their ability to think proportionally is stronger? This study adds details to the transitions, the learning trajectories, intermediate grade students pass through in this process. The process of coordinating units as one operates on numbers is an area that this study has identified as essential with which students need to grapple. As a student begins to think more multiplicatively, tracking the transformation of the units becomes more necessary. A certain messiness in developing this capacity to coordinate the units becomes apparent. Rather than dismissing this messiness by ignoring attention to units at all, as is typical of most current curricular materials, it is suggested by this study that explicitly focusing students' attention on unit coordination has a positive influence on students shifting from additive to multiplicative thinking.

Third graders demonstrated capacity to respond to the multiplicative structure of the task. Tapping into the multiplicative structure of the base ten system and place value concepts as well as working in easier number combinations allowed them to begin to think beyond the additive structures with which they are developmentally familiar. Further exploration needs to occur to find which number combinations allow students to move beyond additive structures. Nevertheless, the results of this study indicate that certain combinations allow them to use more abstract and efficient strategies than others, particularly ten. Developing place value as a multiplicative relations rather than merely an issue of location seems to have a positive influence in moving students to think multiplicatively. Rethinking how place value is developed in the elementary grades may be necessary to take advantage of this fruitful source of learning.

Surface patterns such as covering over the ones place to determine the quantities of ten in a number as well as such examples of "counting the zeros" when multiplying two multidigit quantities (example: 30 x 40) may allow students to quickly find correct answers. What students demonstrated in this study is that explicit expectation to justify why those patterns worked mathematically was within the realm of students at each grade level. Establishing justification and proof of classroom conjectures helped to elevate student capacity to support mathematically why such surface patterns could be trusted. A few students who struggled with justifying the mathematics scored at or above grade level on an outside norm-referenced standardized assessment. This indicates that students can determine correct answers using less efficient thinking structures. The question that arises is at what point does the inability to perceive the multiplicative structures hinder his or her progress to fully comprehend the more abstract mathematics they will encounter in upper grades, particularly once they arrive in middle school.

The language a student uses reveals how he or she conceptualizes the mathematics. Understanding if the error is one of unit confusion or a lack of value assignment is evident by how the student verbally processes his or her thinking. Listening to a student speak in terms of value also reveals the decompositions of number and algebraic properties that is being activated. Language also reveals the level of abstraction the student engages in, is he or she thinking additively or multiplicatively. The research on effect of classroom discourse is widely understood. This study adds strength to a particular area of that discourse literature by arguing for the role of having students consistently speak in terms of value.

The capacity of third and fourth graders to comprehend the multiplicative structure of the extension tasks used in the interview protocol raises the question of what is developmental and what is environmental. It is clear that there is a developmental progression in the number size and quantities students can work with at an abstract level. The fifth graders as a whole did better than the students in the earlier grades. Nevertheless, the evidence indicates that it is possible to stimulate multiplicative thinking among third and fourth graders at a level that fifth graders in September demonstrated. This data may be indicating that typical instructional practices in teaching the procedures to multiply and divide numbers may be suppressing student capacity to reason multiplicatively. Students may be more capable of higher levels of progressive abstraction than has been historically expected.

A secondary question raised in this design experiment was what combination of integrated and coordinated series of mathematical tasks could teachers use to help reveal the emergence and development of multiplicative thinking. Four conjectures were proposed in the design of this study to address this question. Those propositions focused on the role of place value and the base ten system as a multiplicative structure as means to stimulate multiplicative thinking, the role of scale factor and unit rate in developing early multiplicative thinking, the interconnection with multiplicative thinking with emerging conceptions of fractions as relational numbers, and the role of algebraic relational thinking in building capacity to think in scale.

This data presents an analysis that focuses on the results of such instructional actions as measured by the pre and post interviews. Direct analyses of the classroom instructional sessions are not reported here. However, the data from the two interview sessions does allow for interpretation of the effect such instructional tasks had on students. This is particularly so with the data of the third and fourth grade students where a more significant presence within the classroom by the PI had on shaping the full daily lesson. As indicated previously, explicit attention to developing the concepts related to multiplicative scale, particularly starting with the base ten system, and to algebraic thinking results in data indicating a greater capacity for multiplicative thinking than previously reported. In fact, a lack of attention to these multiplicative concepts may be potentially depressing student capacity to transition to the more sophisticated multiplicative reasoning needed by these students in middle school.

Multiplicative thinking is not a lesson or a unit that is taught. It is a disposition that matures over time. Correct answers to multiplicative structures can be arrived at through additive strategies. A teacher cannot know at which level students are operating on unless classroom discourse norms allow that thinking to be transparent. How instruction is shaped to attend to students' growth to think more abstractly and in more multiplicative terms requires attention of the research and teaching communities. Student capacity to reason multiplicatively may be inadvertently suppressed by current instructional practices. Explicit attention to unit coordination and transformation, finding familiar number ranges that scaffolds students to think more abstractly – particularly the explicit use of the base ten system – and building the capacity to justify mathematically the patterns that emerge from the effects of multiplication moves students to think in scale and to view multiplication as a rate of change.

Limitations

This design experiment was exploratory in nature. The research structure is what Schoenfeld (2007) describes as a Phase One study. The intent was to "develop and refine initial ideas, providing evidence that they are worth pursuing" (p. 97). Such studies are intended to address both descriptive and explanatory power, meaning explaining how things took place and why the students learned what they did. While such research is completed with the intent that others may obtain similar results in other instructional settings, the capacity to extend the findings in this study to predict results in other settings is limited.

This study was designed to be cross-sectional in looking at a similar sample of students across three grade levels. Nevertheless, the sample size ($n = 11$, $n = 12$, $n=11$, respectively) is small. Thus the predictive power of the study is limited. The sample group was from the same top tier of each grade level making cross-sectional observations possible. However, each grade level sample group is not representative of a cross-section of each grade level. This indicates a level of caution in the veracity of the descriptive and explanatory power of what, for example, all third graders progress through in their development of multiplicative thinking.

The access to each grade level's mathematics instructional time was not identical. While there was a common approach in terms of the types of instructional tasks students were asked to engage in – and reflected in the common tasks in the pre and post interviews – fourth grade had a higher presence of the principal investigator than fifth grade. Third and fourth grades experienced more re-written curriculum units shaped around student needs than fifth grade. While the study has a cross-sectional structure to it, the issue of access again tempers the predictive power of this study.

Attentive to those limitations, it is being argued here that the instructional tasks and concomitant discourse practices students engaged with around the four conjectures stimulated student capacity to think multiplicatively. While this design experiment has limitations, it provides evidence that is worth pursuing in further, larger scale studies.

Recommendations

Classroom Instructional Implications

The tasks used in the interview protocols reflect similar instructional tasks utilized in the classroom environment. Within the classrooms, these tasks were typically delivered as part of a daily "mental math" exercise. The phrase mental math is being used here to describe how typically students did not have paper and pencil with which to work. Frequent partner sharing was used to allow individuals to think out loud with another in order to work through a task before more public sharing occurred. Beginning number combinations typically began with lower number combinations making the activity accessible to the large majority of participants. The numbers or combinations of numbers were increased in terms of complexity and quantity. As students publicly shared, the principal researcher or classroom teacher transcribed the strategy on the board for visual consideration by both the sharer as well as the whole group. These tasks can be looked upon as capacity building activities that over time scaffold students' ability to decompose number, use the language of value instead of single digits, think in scale, follow the explicit transformation of the units in the task, to think relationally, and consider various forms of representation in which to record their thinking processes. Adding such activities to the daily routine of the mathematics classroom is one recommendation of this study.

In addition to the time and space in daily math instruction to build the above capacity is that place value and the larger base ten system needs to be rethought as an explicit multiplicative relation. The cursory attention in most curricula to the surface features of place value leads to a limited understanding of the base ten system. Explicit attention, even as early as third grade, to the factors of ten that underlie and the capacity to re-unitize a number flexibly and simultaneously is both possible and essential.

Future Study

The multiplicative conceptual field has been mapped by others (Vergnaud, 1994, Confrey, et al., 2009). This study sought to better comprehend some of the learning trajectories students take in developing the capacity to think in more abstract multiplicative ways across some of the individual nodes within that larger field. To determine the veracity of findings in this study, the next step is a larger study that would entail the entire grade at each of the three grade levels. Comparing these students to sites that do not use an enhanced approach to multiplicative thinking would allow for a statistical analysis of differences among student achievement. In order to have middle school mathematics, with its focus on rational number, ratio and proportional reasoning shifts in content and practice need to be enhanced in the intermediate grades. This study points a direction to how and where some of those enhancements might lie. The research questions remain for the next larger scale study to more clearly discern if these enhancements prove effective in building the capacity of intermediate grade students to think more multiplicatively. This initial study does provide enough evidence that further research is worth pursuing.

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APPENDIX A1: PRE INTERVIEW PROTOCOLS - GRADE THREE

Interview Items to Assess the Presence of Multiplicative Thinking – Grade Three	
Assessment Item	
Task 1: Direct Place Value concepts *On a sheet of paper, write the number 783 and show it to the student. Ask the following questions:* • How many ones are in 783? • How many tens? • How many hundreds?	
Task 2: Multiplicative scaling questions *(Scale with 5)* If there are 2 fives in 10, how many fives are in 30? *(Scale with 10)* If there are 10 tens in 100, how many tens are there in 400?	
Task 3: Use of place value and/or other strategies involving decomposition of number (Ask student to first solve mentally, if necessary present paper) • Double 63 • Triple 17 • Quadruple 13	
Task 4: Measurement Division • The factory that makes colored markers has a bin filled with 65/124 markers. If the sorting machine places 10 markers in every box, how many full boxes can it fill?	
Task 5:Multiplicative Compare Problem (Not presented at third grade)	
Task 6: Get to a Number • How much to get from 7 to 20? • How much to get from 46 to 100? • You are at 62. Go back 5.	

APPENDIX A2: PRE INTERVIEW PROTOCOLS – GRADE FOUR

Interview Items to Assess the Presence of Multiplicative Thinking – Grade Four	
Assessment Item	
Task 1: Direct Place Value concepts *On a sheet of paper, write the number 783 and show it to the student. Ask the following questions:* • How many ones are in 783? • How many tens? • How many hundreds?	
Task 2: Multiplicative scaling questions *(Scale with 5)* If there are 5 fives in 25, how many fives are in 75? *(Scale with 10)* If there are 3 tens in 30, how many tens are there in 300?	
Task 3: Use of place value and/or other strategies involving decomposition of number (Ask student to first solve mentally, if necessary present paper) • Double 163 • Triple 27 • Quadruple 18	
Task 4: Measurement Division • The factory that makes colored markers has a bin filled with 465 markers. If the sorting machine places 10 markers in every box, how many full boxes can it fill?	
Task 5: Multiplicative Compare Problem • There are two bins of markers ready to go to the sorting machine on the factory floor. One bin has 360 markers in it. That's ten times as many as in the second bin. How many markers are in the second bin?	
Task 6: Get to a Number • How much to get from 57 to 100? • How much to get from 246 to 300? • You are at 62. Go back 5.	

APPENDIX A3: PRE INTERVIEW PROTOCOLS – GRADE FIVE

Interview Items to Assess the Presence of Multiplicative Thinking – Grade Five	
Assessment Item	
Task 1: Direct Place Value concepts *On a sheet of paper, write the number 783 and show it to the student. Ask the following questions:* • How many ones are in 783? • How many tens? • How many hundreds?	
Task 2: Multiplicative scaling questions *(Scale with 5)* If there are 5 fives in 25, how many fives are in 75? *(Scale with 10)* If there are 3 tens in 30, how many tens are there in 300?	
Task 3: Use of place value and/or other strategies involving decomposition of number (Ask student to first solve mentally, if necessary present paper) • Double 461 • Triple 47 • Quadruple 28	
Task 4: Measurement Division • The factory that makes colored markers has a bin filled with 1465 markers. If the sorting machine places 10 markers in every box, how many full boxes can it fill?	
Task 5: Multiplicative Compare Problem • There are two bins of markers ready to go to the sorting machine on the factory floor. One bin has 360 markers in it. That's ten times as many as in the second bin. How many markers are in the second bin?	
Task 6: Get to a Number • How much to get from 57 to 100? • How much to get from 246 to 300? • You are at 62. Go back 5.	

APPENDIX B1: POST INTERVIEW PROTOCOLS - GRADE THREE

<table>
<tr><th colspan="2">Interview Items to Assess the Presence of Multiplicative Thinking – Grade Three</th></tr>
<tr><td colspan="2"></td></tr>
<tr><td>Assessment Item</td><td></td></tr>
<tr><td>Task 1: Direct Place Value concepts

On a sheet of paper, write the number 832 and show it to the student. Ask the following questions:
• How many ones are in 832?
• How many tens?
• How many hundreds?</td><td></td></tr>
<tr><td>Task 2: Multiplicative scaling questions

(Scale with 5)
If there are 2 fives in 10, how many fives are in 40?

(Scale with 10)
If there are 10 tens in 100, how many tens are there in 600?</td><td></td></tr>
<tr><td>Task 3: Use of place value and/or other strategies involving decomposition of number (Ask student to first solve mentally, if necessary present paper)

• Double 126
• Triple 27
• Quadruple 34</td><td></td></tr>
<tr><td>Task 4: Measurement Division

• The greenhouse growers are planting flower seeds now so plants will be ready to sell in the spring. The grower sows seeds in 10-packs. How many full 10-packs can be planted with 356?</td><td></td></tr>
<tr><td>Task 5: Multiplicative Compare Problem

There are two cups of seeds ready for the grower to use. The first cup has 60 seeds in it. That is ten times more than the seeds in the second cup. How many seeds are in the second cup?</td><td></td></tr>
<tr><td>Task 6: Get to a Number

• How much to get from 68 to 100?
• How much to get from 141 to 215?
• You are at 74. Go back 7.
• You are at 82. Go back 17.</td><td></td></tr>
</table>

APPENDIX B1: POST INTERVIEW PROTOCOLS - GRADE THREE *(continued)*

<table>
<tr><th colspan="2">Interview Items to Assess the Presence of Multiplicative Thinking – Grade Three (continued)</th></tr>
<tr><td>Task 7: Algebraic & Relational Thinking

• 8+4 = □ + 7
• 17 = 12 + 5 T/F?
• 24 + 73 = 72 + a</td><td></td></tr>
<tr><td>Task 8: Missing Value Task

<table><tr><th>Tennis Balls</th><th>Cost</th></tr><tr><td>3- pack</td><td>$5.50</td></tr><tr><td>9-pack</td><td></td></tr></table></td><td></td></tr>
</table>

APPENDIX B2: POST INTERVIEW PROTOCOLS – GRADE FOUR

<table>
<tr><th colspan="2">Interview Items to Assess the Presence of Multiplicative Thinking – Grade Four</th></tr>
<tr><td colspan="2"></td></tr>
<tr><th>Assessment Item</th><th></th></tr>
<tr><td>Task 1: Direct Place Value concepts

On a sheet of paper, write the number 832/2516 and show it to the student. Ask the following questions:
• How many ones are in 832/ 2516?
• How many tens?
• How many hundreds?</td><td></td></tr>
<tr><td>Task 2: Multiplicative scaling questions

(Scale with 5)
If there are 5 fives in 25, how many fives are in 150?

(Scale with 10)
If there are 4 tens in 40, how many tens are there in 120?</td><td></td></tr>
<tr><td>Task 3: Use of place value and/or other strategies involving decomposition of number (Ask student to first solve mentally, if necessary present paper)

• Double 264
• Triple 68
• Quadruple 34</td><td></td></tr>
<tr><td>Task 4: Measurement Division

• The greenhouse growers are planting flower seeds now so plants will be ready to sell in the spring. The grower sows the seeds in 10-packs. How many full 10-packs can be planted with 356/1462 seeds?</td><td></td></tr>
<tr><td>Task 5: Multiplicative Compare Problem

• There are two cups of seed ready for the grower to use. One cup has 620 seeds in it. That first cup has 10 times as many seeds as the second cup. How many seeds are in the second cup?</td><td></td></tr>
<tr><td>Task 6: Get to a Number

• How much to get from 68 to 100?
• How much to get from 141 to 215?
• You are at 74. Go back 7.
• You are at 82. Go back 17.</td><td></td></tr>
</table>

APPENDIX B2: POST INTERVIEW PROTOCOLS – GRADE FOUR *(continued)*

<table>
<tr><td>Task 7: Algebraic & Relational Thinking

• 8+4 = □ + 7
• 17 = 12 + 5 T/F?
• 24 + 73 = 72 + a</td><td></td></tr>
<tr><td>Task 8: Missing Value Task

<table><tr><th>Tennis Balls</th><th>Cost</th></tr><tr><td>3- pack</td><td>$5.50</td></tr><tr><td>9-pack</td><td></td></tr></table></td><td></td></tr>
</table>

APPENDIX B3: POST INTERVIEW PROTOCOLS – GRADE FIVE

Interview Items to Assess the Presence of Multiplicative Thinking – Grade Four	
Assessment Item	
Task 1: Direct Place Value concepts *On a sheet of paper, write the number 2516 and show it to the student. Ask the following questions:* • How many ones are in 2516? • How many tens? • How many hundreds?	
Task 2: Multiplicative scaling questions *(Scale with 5)* If there are 5 fives in 25, how many fives are in 150? *(Scale with 10)* If there are 4 tens in 40, how many tens are there in 120?	
Task 3: Use of place value and/or other strategies involving decomposition of number (Ask student to first solve mentally, if necessary present paper) • Double 264 • Triple 68 • Quadruple 34	
Task 4: Measurement Division • The greenhouse growers are planting flower seeds now so plants will be ready to sell in the spring. The grower sows the seeds in 10-packs. How many full 10-packs can be planted with 1462 seeds?	
Task 5: Multiplicative Compare Problem • There are two cups of seed ready for the grower to use. One cup has 620 seeds in it. That first cup has 10 times as many seeds as the second cup. How many seeds are in the second cup?	
Task 6: Get to a Number • How much to get from 68 to 100? • How much to get from 141 to 215? • You are at 74. Go back 7. • You are at 82. Go back 17.	

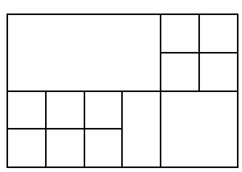
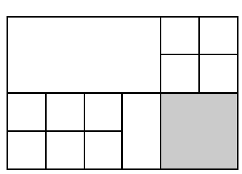
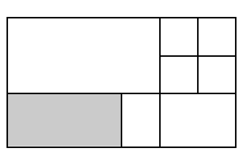
APPENDIX B3: POST INTERVIEW PROTOCOLS – GRADE FIVE *(continued)*

<table>
<tr><td>Task 7: Algebraic & Relational Thinking

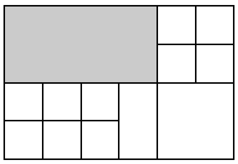
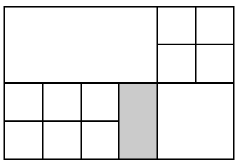
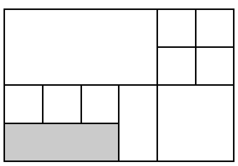
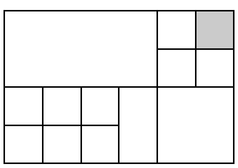
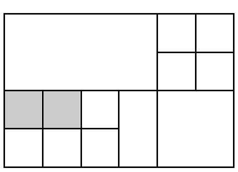
• $8+4 = \square + 7$
• 17 = 12 + 5 T/F?
• 24 + 73 = 72 + a</td><td></td></tr>
<tr><td>Task 8: Missing Value Task

<table><tr><th>Tennis Balls</th><th>Cost</th></tr><tr><td>3- pack</td><td>$5.50</td></tr><tr><td>9-pack</td><td></td></tr></table></td><td></td></tr>
</table>

APPENDIX C: MID INTERVIEW PROTOCOL – GRADES THREE THROUGH FIVE

Mid Interview: Assessment of Multiplicative Thinking – Fractions	
Four children are sharing 23 chocolate bars so that everyone gets the same amount. The chocolate bars are all the same size. How much will each person get?	
Six children want to share 2 pounds of modeling clay so that everyone gets exactly the same amount. How much clay can each child have?	
The teachers at the grade level plan to have 3 children share a two-liter bottle of soda for a class celebration. How many bottles of soda will the teachers serve so that 30 students get the same amount as the table of three?	
	*Tell me how to see: a. 1/2 b. 1/4 c. 1/6 d. 1/3 e. 1/12 f. 1/8*
	*Tell me what part of the whole gird is shaded?*
	*Tell me what part of the whole gird is shaded?*

APPENDIX C: MID INTERVIEW PROTOCOL – GRADES THREE THROUGH FIVE *(continued)*

Mid Interview: Assessment of Multiplicative Thinking – Fractions *(continued)*	
	*Tell me what part of the whole gird is shaded?*
	*Tell me what part of the whole gird is shaded?*
	*Tell me what part of the whole gird is shaded?*
	*Tell me what part of the whole gird is shaded?*
	*Tell me what part of the whole gird is shaded?*

APPENDIX D: SCORING RUBRIC WITH ANNOTATIONS

Task 1: *Direct Place Value Concepts – Rate of Ten*

<table>
<tr><td colspan="2">Pre-Interview: How many ones, tens, hundreds in 783?
Post-Interview: How many ones, tens, hundreds, thousands in 832 (3rd Gr.), 832 & 2516 (4th Gr.) or 2516 (5th Gr.)?</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon …
…reasoning around the factors of 10
…the making of composite units
…coordination of units

• May recognize the surface pattern</td><td>2
• Accurate
• Automaticity
• Justification is based upon factors of ten; could include an explanation of the place value pattern but reasoning as to why it works is demonstrated
or
• First response may involve a misinterpretation of the problem but quickly, and with explanation, adjusts to a correct response. The ability to justify is key.</td></tr>
<tr><td>1
• Some calculation involved with one or more of the units
or
• Some initial inaccuracies
• Self-corrects due to prompts and minor scaffolds
or
• Justification is based solely on a covering pattern without any explanation as to why it can be trusted</td></tr>
<tr><td>1 - Additive
• Skip counts to solve
• Direct Models to solve
or
• Struggles to see beyond the value of the digit's location
or
• Sees the numbers as a string of single digits</td><td>0
• Several inaccuracies
• Only self-corrects due to extensive scaffolding</td></tr>
</table>

Score 2-1 – AX – Grade 3 – "Oh! Seven hundred eighty-three… So it would be like this. You take seven of these, eight of these, and three cubes [pointing to base ten blocks] and count them all together… [How many tens] … So far I've got seventy… I know that there are ten in each hundred… I am counting by tens 'til I get to seventy… seventy-eight…because you take, for example, all of these and add the eight [sound of base ten blocks being handled] it would be seventy-eight… [How many hundreds]… Super easy. Seven."

Score 2-2 – Factors of ten – EL – Grade 5 – [How many tens in 783?] "Because 783, you can't, the three isn't a ten so you know that there's 70 [tens], that's 700. And then there's eight [tens], and break the 700 into tens is 70 and then the eight for the 80, seventy-eight."

Score 2-2 – Place value pattern – MA – Grade 5 – [How many tens in 783?] "Because I know tens always have a zero in the ones, or have to have at least a zero in the ones place and so I don't count in the one in the zeros place, I mean the number in the zeros place."

APPENDIX D: SCORING RUBRIC WITH ANNOTATIONS *(continued)*

Task 2: *Multiplicative Scale*

<table>
<tr><td colspan="2">Pre-Interview:
3rd Gr.: If 2 fives in 10, how many fives in 30? If in 10 tens 100, how many tens in 400?
4th & 5th Gr.: If 5 fives in 25, how many in 75? If 3 tens in 30, how many tens in 300?
Post-Interview:
3rd Gr.: If 2 fives in 10, how many fives in 40? If in 10 tens 100, how many tens in 600?
4th & 5th Gr.: If 5 fives in 25, how many in 150? If 4 tens in 40, how many tens in 120?</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon reasoning around multiples of the factors either using scale factor or unit rate</td><td>• Accurate
• Automaticity
• Justification is based upon factors of ten; could include an explanation of the covering/zero pattern but reasoning as to why it works is demonstrated
or
• Derives using the distributive property</td></tr>
<tr><td>1
• Has a multiplicative construct but needs to additively calculate to resolve (skip count, repeated addition, etc.)
or
• Some initial inaccuracies
• Self-corrects due to prompts and light scaffolds
or
• Justification is based upon additive structures</td></tr>
<tr><td>1 - Additive
• Skip counts to solve
• Direct Models to solve</td><td>0
• Several inaccuracies
• Only self-corrects due to extensive scaffolding
or
• Even with extensive scaffolding, does not arrive at an accurate solution</td></tr>
</table>

Score 2-2 – a multiplicative derived strategy – AN – Grade 3 - [If 2 fives in 10, how many fives in thirty?] “Six… Because I know that five times five is twenty-five and five times six equals thirty.”
– Kira – Grade 5 – (If 5 fives in 25, how many in 75?] “Five times ten is 50 and then five times two is 10 so that is 60, plus another 10 is 70, and then plus one more.”

Score 2-2 – multiplicative relationship – CH – Grade 4 – [If 5 fives in twenty-five, how many fives in seventy-five?] “So the are… fifteen fives…Because there are five, there are five fives, so five times three is [inaudible], I mean fifteen.”

Score 2-1 – a multiplicative organization require scaffolding to complete – BT – Grade 3 - [If 2 fives in 10, how many fives in thirty?] “I would need to use, um, [pause] uh, I would need to use thirty, no, twenty of these, the fives… because I counted by fives…ten, twenty, thirty [JB: And so that is three fingers. So, when you did 10, 20, 30, how many fives are in each one?]… um, thirty?... because I counted them. [JB: So when you do ten, how many fives is that?] …two [20]… that’s four [30]… six… because I counted them and see how I can get how many fives I need to make a total of ten.” – What makes this multiplicative is that she combined the two fives into one unit of ten and counted the tens. While still skip counting, her skip count is a new unit of two fives equaling a one ten.

Score 1-1 – additive strategy – BR – Grade 3 - [If 2 fives in 10, how many fives in thirty?] Counting on her fingers, “Five, ten, fifteen, twenty, twenty-five, thirty. So there would be six fives.”

APPENDIX D: SCORING RUBRIC WITH ANNOTATIONS *(continued)*

Task 3: *Use of Place Value in Decomposition of Number*

Score Where student starts – in the tens or ones place – separately; 1 – tens place; 0 – ones place

<table>
<tr><td colspan="2">Pre-Interview:
3rd Gr.: Double 63, Triple 17, Quadruple 13
4th Gr.: Double 163, Triple 27, Quadruple 18
5th Gr.: Double 461 Triple 47, Quadruple 28
Post-Interview:
3rd Gr.: Double 126, Triple 27, Quadruple 18
4th & 5th Gr.: Double 264, Triple 68, Quadruple 34</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon the distributive property of multiplication over addition or associative property through factoring

• Building Up by Other Factors (multiplicative in structure, additive in solution)</td><td>2
• Accuracy
• Automaticity
• Consistent language of value or capacity to scale single digit combinations to its extension
• A stronger degree of mental abstractness</td></tr>
<tr><td>1
• Accurate but needs to do some extended adding to find answers
or
• Some initial inaccuracies
• Self-corrects due to prompts and light scaffolds
or
• Is accurate but fluctuates among language of value and single-digits but without connections between the two forms, e.g., is insecure in following the values in a consistent manner
or
• Accurate but bound by the standard algorithm and the use of single digit language</td></tr>
<tr><td>1 - Additive
• Uses Tens & Ones/Partial Sums/Standard Algorithm, Repeated Addition, Doubling, or Complex Doubling</td><td>0
• Several inaccuracies
• Consistent single-digit language even in the face of scaffolding
• Only self-corrects due to extensive scaffolding</td></tr>
</table>

Score 2-2 – Distributive property – CH – Grade 4 – [Triple 27] "Well twenty times three is sixty… and seven times three is twenty-one, so eighty-one."

Score 2-2 – Distributive property – KI – Grade 5 – [Double 461] "I would do 61 times two and that would be 122… and then I would do 400 times two, 800 and then I would add them together, 922."

Score 2-1 – Mix of multiplicative thinking and deriving [completed mentally with minor visual scaffold] – CH – Grade 4 – [Quadruple 18] "um, forty, thinking of the tens, and eight plus eight is sixteen and so fifty-six, and another ten for the other [sixteen] so you add six… Wait! What was I…? Sixty, sixty-two, I think… [verbal review scaffold, followed by visual scaffold – *JB: So I wrote those two numbers down so what would you do there?*] Add ten, sixty-six, I guess that, um, take four from the six and add it to that, that's seventy, seventy-two."

Score 1-1 – Tens & Ones/Partial Sums strategy – DN – Grade 3 – [Quadruple 13] "Add up all four tens, forty… then add up all the threes… three, six, nine, twelve… and forty plus twelve equals fifty-two."

Score 1-1 – Standard addition algorithm with understanding – AL – Grade 4 [Double 163] "…I plus three plus three equals six, sixty plus sixty equals one hundred twenty so I put the twenty here and I put the one hundred right here… and then one hundred plus one hundred equals two hundred and plus the one hundred is three hundred. So three hundred twenty-six. " [completed on paper]

Score 1–1 – Doubling Strategy – AL – Grade 4 [Quadruple 18] "[pause]… Forty… ten plus ten plus ten plus ten equals forty… And then eight plus eight equals, equals sixteen and the other ones is sixteen… sixteen plus sixteen equals, um, um… thirty-two… and then thirty-two plus forty equals , equals to [some whispering] is seventy-two. [jb: …I noticed… you paused for a moment and did something with your fingers…] I just did it for one, two, three equals seven so I put, I added the zero."

APPENDIX D: SCORING RUBRIC WITH ANNOTATIONS *(continued)*

Task 4: *Place Value – Rate of Ten via Measurement Division in Context*

Pre-Interview:
The factory that makes colored markers has a bin filed with 65/ 124 (3rd Gr.), 465 (4th Gr.), 1465 (5th Gr.) markers. If the sorting machine places 10 markers in every box, how many full boxes can the machine fill?
Post-Interview:
The greenhouse growers are planting flower seeds now so plants will be ready to sell in the spring. The grower sows seeds in 10-packs. How many full 10-packs can be planted with 356 (3rd Gr.), 1462 seeds?

Understanding of the Multiplicative *Structure* of the Task	Classification of the *Solution Strategy*
2 - Multiplicative • Responses based upon reasoning around the factors of 10 • May recognizes the surface pattern	**2** • Accuracy • Automaticity • Justification is based upon factors of ten; could include an explanation of the covering pattern but reasoning as to why it works is demonstrated
	1 • Accurate but additive or had to methodically calculate in strategy solution or • Some initial inaccuracies • Confusion in the coordination of units • Self-corrects due to prompts and light scaffolds or • Justification is based solely on a covering pattern without any explanation as to why it can be trusted
1 - Additive • Skip counts to solve • Direct Models to solve or • Uses the standard algorithm with single digit processing	**0** • Several inaccuracies • Only self-corrects due to extensive scaffolding or • Even with extensive scaffolding, does not arrive at an accurate solution

Score 1-1 – Skip Counting, additive strategy – BR – Grade 3 – Pre-interview [65 markers, how many full boxes of ten?] - “Sixty boxes…Oh! Sixty-five. There’s sixty-five. I am getting all confused here… So, there can’t be sixty boxes or sixty-five. I would have to, um, I would have to put them into groups, put them into groups of ten… um, just going to do that…make groups of pictures… [writes on paper one ten at a time while saying] 10, 20, 30, 40, 50, 60, now there’s five left. How do I do that? [*JB: Well the question is how many full boxes can you make?*] … you can make sixty full… [*JB: How many boxes did you make?*] … 1, 2, 3, 4, 5, 6… whole boxes…”

Score 2-2 – multiplicative thinking – JO – Grade 3 – [65 markers, how many full boxes of ten?] “Okay. So… That’s easy! Sixty… I mean six… because um, you can only put ten in a box so, six, there’s um, there’s six tens, which makes six boxes.”

Score 2-2 – multiplicative thinking, answer within the number – MO – Grade 5 – [1,465 markers, how many full boxes of ten?] “Ah, because I know five, it’s not quite a ten, so I just take the other numbers and minus the place value and just get the number.”

APPENDIX D: SCORING RUBRIC WITH ANNOTATIONS *(continued)*

Task 5: *Multiplicative Compare Problem – In Context*

<table>
<tr><td colspan="2">Pre-Interview:
There are two bins of markers ready to go to the sorting machine on the factory floor. One bin has 90 (3^{rd} Gr.), 360 (4^{th} & 5^{th} Gr.) markers in it. That's ten times as many as in the second bin, How many markers are in the second bin?
Post-Interview:
There are two cups of seeds ready for the grower to use. The first cup has 60 (3^{rd} Gr.), 620 (4^{th} & 5^{th} Gr.) seeds in it. That is ten times more than the seeds in the second cup. How many seeds are in the second cup?</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon reasoning around the scale factor of ten</td><td>2
• Accuracy
• Automaticity
• Justification is based reasoning with a multiple of ten</td></tr>
<tr><td>1
• Some initial inaccuracies, example: reasoning the second bin/cup was 10 times more – 3600 markers or 6200 seeds
• Self-corrects due to prompts and light scaffolds
or
• Justification is based upon additive structures</td></tr>
<tr><td>1 - Additive
• Sees an additive difference between the quantities and calculates accordingly
• Skip counts to solve
• Direct Models to solve</td><td>0
• Several inaccuracies
• Only self-corrects due to extensive scaffolding
or
• Does not arrive at an accurate solution</td></tr>
</table>

Score 2-2 – Multiplicative scale – CH – Grade 4 – "Okay. Thirty-six times ten is three hundred sixty so I am going to have to go with thirty-six."

Score 2-2 – Multiplicative scale – place value pattern – KI – Grade 5 – "I did three hundred sixty divided by 10. It's the same as just saying…because times ten is adding a zero the dividing by zero would be minus a zero.

Score 2-0 – Initial additive, switches to multiplicative thinking in response to scaffolding – AX – Grade 3 – [Two cups, 60 in first, that's ten times more than the second] – condensed passage – "So far I have been thinking about fifty. I totally, I'm not sure." [*And why does fifty make sense?*] "I don't know… That's why I am double checking… [*Is sixty ten times bigger than fifty?*] "It's ten more than fifty… Ten! … [*Is sixty ten times more than ten?*] "[pause] I am not sure… Okay. Now I was just thinking about ten. If there were a hundred in here then the answer would be ten. [*Could you use that to help figure out what might be here?*] "That's what I am trying to do… Oh! Six!... Six times ten equals sixty."

APPENDIX D: SCORING RUBRIC WITH ANNOTATIONS *(continued)*

Task 6: *Place Value – Fluency Around Landmarks of Ten & Decomposition of Number*

<table>
<tr><td colspan="2">Pre-Interview:
3rd Gr.: How much to get from 7 to 20, 46 to 100? You are at 62, go back 5. What number are you at?
4th Gr.: How much to get from 57 to 100? 246 to 300? You are at 62, go back 5. What number are you at?
5th Gr.: How much to get from 246 to 300? 457 to 1000? 872 to 2,138?
Post-Interview:
3rd, 4th & 5th Gr.: How much to get from 68 to 100? 141 to 215? You're at 74, go back 7. You're at 82, go back 17.</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon fluid decomposition of numbers to span distances both forwards and backwards</td><td>2
• Accuracy
• A fluid response
• Consistent language of value
• Easily organizes around landmarks of ten</td></tr>
<tr><td>1
• May be fluid in formation of jumps but required some calculation time to determine the combined span.
or
• Works in smaller incremental chunks
or
• Some initial inaccuracies
• Self-corrects due to prompts and light scaffolds
or
• Fluctuates among language of value and single-digits but without connections between the two forms</td></tr>
<tr><td>1 - Additive
• Uses ones or skip counting to span distances
or
• Decomposes number but does not use them to organize around efficient landmarks</td><td>0
• Several inaccuracies
• Consistent single-digit language even in the face of scaffolding
• Only self-corrects due to extensive scaffolding</td></tr>
</table>

Score 2-2 – Instant response – CH – Grade 4 – [You are at 57, how much to get to 100?] "Forty-three… because if you ad three to here that would be sixty and sixty plus forty is, you know, is a hundred."

Score 1-1 – additive strategy – BR – Grade 3 – [How much to get from 46 to 100?] "Forty-seven, 48, 49, 50. Four!... Plus four equals fifty… Plus fifty, plus fifty… fifty-four."

BR – Grade 3 – [How much to get from 46 to 100?] "I would go plus four because then I get to fifty…And I would go ten again, and that's sixty. And so I have, I am sure I could put plus twenty… to eighty [*jb: Oh another ten. Oh, I'm sorry. You did another ten and that got you to where?*] …To seventy. And now I can do plus thirty… a hundred… uh, fifty-four."

BR – Grade 3 – [You are at 62. Go back 5.] "I would just count five ones back. So… Two, sixty-two, 61, 60, [pause] 59, 58, 57. You would end up at fifty-seven."

APPENDIX D: SCORING RUBRIC WITH ANNOTATIONS *(continued)*

Task 7: *Algebraic/Relational Thinking*

Pre-Interview:
Not given
Post-Interview:
3rd, 4th & 5th Gr.: *8+4 =* $\square$ *+ 7; 17 = 12 + 5 T/F?; 24 + 73 = 72 + a*

Understanding of the Multiplicative *Structure* of the Task	**Classification of the *Solution Strategy***
2 - Relational • Compares values across the equal sign	**2** • Accuracy • Automaticity • Provided a relational justification for first and third sub-items • Provides a definition of the equal sign as a relational/comparative symbol
	1 • Accuracy • Calculated to find answers for sub-items one and three • Provided a definition of the equal sign as a relational/comparative symbol
1 – Non-Relational • Equal sign as an operational symbol	**0** • Inaccurate response or • Uses a definition of equals as "the answer comes next" or something similar or • Orientation of sub-item two (17=12+5) is considered "backwards"

Score 2-2 – Relational thinking – RN – Grade 3 – [17=5+12 Is that true or false?] "True… Because 17, they kind of tell you the number there and then 12+5, I just had right now and then 12+5 is 17 also. And they are on a seesaw and they are kind, they are balanced."

Score 2-1 – Relational with calculation – AN – Grade 3 – [8+4 = 7 + $\square$] "Because I know seven plus four equals eleven then plus one more equals five."

Score 1-0 – Non-relational, inaccurate response –BR – Grade 3 – [8+4 = 7 + $\square$] "Twelve… because eight plus four equals twelve. [*So eight plus four equals twelve plus seven?*] – [pause] Nineteen… because if you, I just of thought of it. [*So does eight plus four equal nineteen plus seven?*] Eight plus four equals twelve plus seven. That's how I did it."

Score 2-2 – Relational Thinking – EZ – Grade 4 – [24 + 73 = 72 + *a*] "Twenty-five… because they had seventy-three but they minused one so that means that the other number, the twenty-four, would have to go up."

Score 2-2 – Relational Thinking – NI – Grade 4 – [8 + 4 = ___ + 7] "Because if you add one to the seven it would be eight, and then it would be four more."

Score 2-1 – Relational understanding of the equal sign but needing to calculate - JQ – Grade 4 [24 + 73 = 72 + *a*] "So seventy two plus something has to be ninety-seven… So, I know that five plus two is seven sp the five would be a …and, and…it would be zero and two so it would be twenty-five, seventy-two plus twenty-five."

APPENDIX D: SCORING RUBRIC WITH ANNOTATIONS *(continued)*

Task 8: *Scale Factor – Missing Value Task*

<table>
<tr><td colspan="2">Pre-Interview:
Not Given
Post-Interview:
[Presented in tabular form] If a 3-pack of tennis balls costs $5.50, how much will a 9-pack cost?</td></tr>
<tr><th>Understanding of the Multiplicative Structure of the Task</th><th>Classification of the Solution Strategy</th></tr>
<tr><td rowspan="2">2 - Multiplicative
• Responses based upon reasoning around the scale factor of three</td><td>2
• Accuracy
• Automaticity
• Justification is based reasoning with a multiple of three</td></tr>
<tr><td>1
• Recognizes the multiple of three but adds to solve
or
• Some computational inaccuracies
• Self-corrects due to prompts and light scaffolds</td></tr>
<tr><td>1 - Additive
• Sees an additive difference between the quantities and calculates accordingly</td><td>0
• Interprets problem as a additive structure not a multiplicative
or
• Only self-corrects due to extensive scaffolding
or
• Or even with extensive scaffolding, does not arrive at an accurate solution</td></tr>
</table>

Score 2-2 – Multiplicative response – DN – Grade 3 [If a three-pack costs $5.50, how much would a nine-pack cost?] “So then you just have to triple five would be fifteen, and then I triple, plus a one hundred fifty cents, so then, so then that will be sixteen dollars and… fifty cents.”

Score 2-1 – Multiplicative in response to a scaffold to an inaccurate multiplicative response – GA – Grade 3 [If a three-pack costs $5.50, how much would a nine-pack cost?] “5, 10, 15, 20, 25, 30 plus [inaudible], hold on, thirty, hold on thirty dollars and, thirty, wait! … I know the three, that the three-pack is six, that the nine-pack is six more than the three-pack and six more needs to be, needs to be [pause] five hundred fifty tens, five dollars and fifty cents times six. So… [scaffold *Oh, you are trying to do that six times. Now you would get, you are saying you would get six more balls, when you go from the three-pack to the nine-pack. Is that what you are saying?*] Uh huh. [*But my question is are you buying six more packs when you get those six more? I thought there were three to a pack.*] Yeah. Three’s three to a pack… Well, nine tennis balls would cost three times so, can I use this [paper]? …Okay. [inaudible self-talk] fifteen… Okay, so, fifteen dollars and… fifty times ten… 10, 20, 30, 40 50, 60, 70, 80, 90, 100… so that will be sixteen and sixteen dollars and …fifty cents.”

Score 2-0 – Multiplicative in response to a scaffold to an inaccurate additive response – AX – Grade 3 [If a three-pack costs $5.50, how much would a nine-pack cost?] “So just so that I know is three plus six equals nine. So I am thinking it is somewhere around six dollars, I’m thinking… I know what it is not going to be, I just think… [pause] Now I think it is somewhere around eleven dollars… because, like, I sat down six plus three equals nine so I put the six dollars under the five and I was thinking now that’s going to be around eleven dollars… [scaffold: *I can’t tell if you are adding something. Are you still adding something?... Or are you done?*] No… I am just thinking about what [inaudible] … [Scaffold: *So the question would be, if you got, let’s just think, how many three-packs do you have to have nine?*] Two more. [*Two more. But it’s a total of how many?*] Six more. [*Well, you said you had…two more packs but you have to buy the first one, too, right?… So how many three-packs are you buying?*] So, I would need three and [inaudible]… Oh! Now I figured that it will be fifteen… because, I was thinking of, I was pretending there was three more of these and so far I added up all of the fives… fifteen dollars, fifteen dollars and fifteen cents is what I'm thinking. [*Fifteen cents. So why…?*] Because pretty much these two equal fifteen because there is nothing in the place of the zeros… [scaffold] Oh! A dollar and fifty… sixteen dollars and… Oh! Sixteen dollars and fifty cents."

Score 1-0 – Sees an additive difference between the quantities - JQ – Grade 4 - [If a three-pack costs $5.50, how much would a nine-pack cost?] "Eleven dollars and fifty-six cents… Because I knew… I noticed that three and nine, if you do three time three is nine, so I knew to add a six [*So you knew that three times three is nine. Tell me about the six.*] Because six plus three is nine… I did five time six, I mean five plus six, and that equals eleven, and then I did fifty plus six, and that is fifty-six."