Automorphic Forms on Certain Affine Symmetric Spaces

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Lei Zhang

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
Doctor of Philosophy

Prof. Dr. Dihua Jiang

May, 2011

Acknowledgements

First and foremost, I wish to thank my advisor, Professor Dihua Jiang, for his frequent, illuminating conversations and consistent support. He proposed the thesis project and directed my research on this project. Beyond the education of the knowledge, he also spent much time on training me to work as a professional mathematician. Studying Number Theory under the supervision of Prof. Dihua Jiang is a great treasure for my whole life.

I thank Prof. Jim Cogdell, Dr. A. Aizenbud, Dr. D. Gourevitch and Prof. Binyong Sun. They gave me many helpful advice when I was working on the project, *Glefand pairs*. In addition, I I am grateful to Prof. Fiona Murnaghan. My understanding of the subject of *tame supercuspidal representations* has benefited very much during the communication with her. I would also like to thank Prof. Jeffrey Adler and Prof. Stephen DeBacker to give me an opportunity to join FRG meeting when this project is in progress. In addition, I thank to Prof. Jeffrey Hakim, Prof. Mark Reeder and Prof. Ju-Lee Kim for their helpful advice on this topic.

Finally, I would like to thank my committee, Prof. Steven Sperber and Prof. Alexander A. Voronov, Prof. Richard McGehee for reviewing my thesis and serving on the committee of my thesis defence. There are many people that have earned my gratitude for their contribution to my thesis. Especially, I would like to thank my friends, Mr. Baiying Liu and Mr. Xin Shen etc. for their regular inspired conversation on math and their help.

Abstract

In this thesis, we consider automorphic periods associated to certain affine symmetric spaces such as the symmetric pairs

$$(\mathrm{Sp}_{4n}, \mathrm{Res}_{K/k}\mathrm{Sp}_{2n}) \text{ and } (\mathrm{GSp}_{4n}, \mathrm{Res}_{K/k}\mathrm{GSp}_{2n}),$$

where k is a number field and K is an Étale algebra over k of dimension 2. We consider the period integral of a cusp forms of $\mathrm{Sp}_{4n}(\mathbb{A}_k)$ against with an Eisenstein series of the symmetric subgroup $\mathrm{Res}_{K/k}\mathrm{Sp}_{2n}$. We expect to establish an identity between this period integrals and the special value of the spin L-function of the symplectic group.

In the local theory, using Aizenbud and Gourevitch's generalized Harish-Chandra method and traditional methods, i.e. the Gelfand-Kahzdan theorem, we can prove that these symmetric pairs are Gelfand pairs when K_v is a quadratic extension field over k_v for any n, or K_v is isomorphic to $k_v \times k_v$ for $n \leq 2$. Since $(U(J_{2n}, k_v(\sqrt{\tau})), \mathrm{Sp}_{2n}(k_v))$ is a descendant of $(\mathrm{Sp}_{4n}(k_v), \mathrm{Sp}_{2n}(k_v) \times \mathrm{Sp}_{2n}(k_v))$, we prove that it is a Gelfand pair for both archimedean and non-archimedean fields.

According to the Yu' construction in [76] of irreducible tame supercuspidal representations, we give a parametrization of the distinguished tame supercuspidal representation of symplectic groups in this thesis. Applying the dimension formula of the space $\mathrm{Hom}_H(\pi, \mathbb{1})$ given by Hakim and Murnaghan [28], we prove that if (G, H) is the symmetric pair $(\mathrm{U}(J_{2n}, K_v), \mathrm{Sp}_{2n}(k_v))$ there is no H-distinguished tame supercuspidal representation, where K_v is a quadratic extension over k_v. In addition, for the symmetric pair $(\mathrm{Sp}_{4n}(k_v), \mathrm{Sp}_{2n}(K_v))$, we give the sufficient and necessary conditions of generic cuspidal data such that the corresponding tame supercuspidal representations are H-distinguished. Note that our case is the first case worked out with none of G and H being the general linear groups. Furthermore, motived by a sub-question, we also give an example for the distinguished representations of finite groups of Lie Type in a low rank case. In particular, we show that θ_{10} is the unique $\mathrm{SL}_2(\mathbb{F}_{q^2})$-distinguished cuspidal representation of $\mathrm{Sp}_4(\mathbb{F}_q)$.

Contents

Acknowledgements **i**

Abstract **ii**

1 Introduction **1**

1.1 Global aspect of my work 2

1.2 Local aspect of my work 8

1.2.1 Gelfand Pairs 9

1.2.2 Distinguished Representations 11

1.3 Notation 15

2 Automorphic Periods **16**

2.1 Automorphic periods of (G, H) 17

2.1.1 Stabilizer of the character ψ_β 23

2.2 Global Calculation 24

2.3 Classification of Hermitian Forms 26

2.3.1 Case $n = 1$ 27

2.3.2 Case $n = 2$ 28

2.3.3 Hermitian forms over p-adic fields 28

2.4 Ikeda Lifting 29

2.4.1 Spin L-function of Ikeda's lifting 30

2.4.2 Case $n = 1$ 33

3 Unramified Calculation **34**

3.1 Unramified Split case 37

3.2 An explication formula for Siegel series 40

3.3 Case Sp_4 45

3.4 Case GSp_4 46

4 Uniqueness of the Symmetric Pairs **48**

4.1 Overview 49

4.2 Gelfand Pairs 52

4.3 Regularity 62

5 Distinguished Tame Supercuspidal Representations **71**

5.1 Overview 72

5.2 Involution and Bruhat-Tits buildings 77

5.3 Distinguished tame supercuspidal representations 79

5.3.1 Tame supercuspidal representations 80

5.3.2 Hakim and Murnaghan's formula 83

5.3.3 The symmetric pair $(\mathrm{U}_{2n}(J, E), \mathrm{Sp}_{2n}(k))$ 90

5.3.4 The symmetric pair $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E))$ 92

5.4 Distinguished representations of finite groups of Lie type 96

5.4.1 Cuspidal representations of $\mathrm{Sp}_4(\mathbb{F}_q)$ 99

5.4.2 Lusztig's formula for $\langle \mathrm{tr}(\cdot, R_T^\lambda), \mathrm{Ind}_H^G \mathbb{1} \rangle$ 104

5.4.3 The symmetric pair $(\mathrm{U}_{2n}(\mathbb{F}_{q^2}), \mathrm{Sp}_{2n}(\mathbb{F}_q))$ 105

5.4.4 Unipotent representations of the symplectic group 108

5.4.5 Irreducible representations of the Weyl group of type C_n . . 113

5.4.6 The symmetric pair $(\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{F}_q))$ 114

5.5 Applications of some global results 119

References 122

Chapter 1

Introduction

Let G be a connected reductive group defined over a number field k, and H denote the subgroup of fixed points of an involution θ of G defined over k. In this paper, the involution means an automorphism over k of G of order 2. Such an H is called a *symmetric subgroup* of G and G/H is an *affine symmetric space* via a symmetrization map $g \to g\theta(g)^{-1}$. This symmetrization map induces a homeomorphism from G/H to the image. This pair (G, H) is called a *symmetric pair*.

For example, when θ is a Cartan involution of a semi-simple Lie group $G(\mathbb{R})$, the symmetric subgroup $H(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$ and $G(\mathbb{R})/H(\mathbb{R})$ is a Riemann symmetric space.

The study of algebraic theory of affine symmetric spaces from the invariant-theoretic point of view began with the pioneer work of Kostant and Rallis (1969). Study of the group version is continued by Richardson (1982). Study of the classification theory over rational fields, say, p-adic local fields and number fields, of affine symmetric spaces was carried out by Helminck (2000). More recently, the study of a more general family of algebraic varieties, called *Spherical Varieties*, has been studied by Brion, Knop and others.

As the classical theory of spherical functions has a deep impact on harmonic analysis over Riemannian symmetric spaces and on the general theory of automorphic representations, one expects potential applications of spherical functions or functionals attached to affine symmetric spaces or more generally to spherical varieties in the theory of automorphic forms.

1.1 Global aspect of my work

A period of automorphic forms attached to a symmetric pair (G, H) is given, for example, by

$$\mathcal{P}_H(\varphi) = \int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\, dh, \tag{1.1.1}$$

where φ is an automorphic form on $G(\mathbb{A})$ and $\mathbb{A}$ is the ring of adeles of the number field k. In general, this integral may have a convergence issue, and in that case, suitable regularization is needed, including the Arthur truncation method.

In many examples, these period integrals are related to a special value or a pole of L-functions. One may expect in some cases that there is an identity between an automorphic period and a special value of a certain L-function attached to the automorphic form. Such an identity implies a great deal of arithmetic and analytic information. Examples include the Gross-Prasad conjecture for classical groups and their refinement given recently in Gross, Prasad, and Gan [26] (2010), and the Ichino-Ikeda conjecture on an identity relating the periods studied by Gross and Prasad and the central value of the tensor product L-function of symplectic type for orthogonal groups. More cases have been discussed in Jiang [36] (2007) and the more recent work of Ginzburg, Jiang, and Soudry [24] (2010). Algebraic cycles and periods of automorphic forms are clearly related as discussed in Kudla's paper [43] (2004), for example. In the analytic application, one can refer to Sarnak's conjecture about L^∞-norm in the letter [65] (2004). Lapid and Offen applied the compact unitary period to study this conjecture for automorphic forms on GL_n in their paper [45] (2007). For more applications of this automorphic period, one can refer to Lapid's lecture on ICM 2010.

In this thesis, under the supervision of Dihua Jiang, I work on a particularly interesting affine symmetric pairs

$$(\mathrm{Sp}_{4n}, \mathrm{Res}_{K/k}\mathrm{Sp}_{2n}) \text{ and } (\mathrm{GSp}_{4n}, \mathrm{Res}_{K/k}\mathrm{GSp}_{2n}),$$

where K is an étale algebra over k of degree two, and the related global and local problems in the theory of automorphic forms.

More precisely, given by

$$J_{2n} = \begin{pmatrix} & w_n \\ -w_n & \end{pmatrix} \text{ and } w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

we define a symplectic similitude group

$$\mathrm{GSp}_{2n}(k) = \left\{g \in \mathrm{GL}_{2n}(k) \mid g^t J_{2n} g = \lambda J_{2n}, \text{ for some } \lambda \in k^\times \right\},$$

and a symplectic group

$$\mathrm{Sp}_{2n}(k) = \{g \in \mathrm{GSp}_{2n}(k) \mid \lambda(g) = 1\}.$$

Let G be the group GSp_{4n} or Sp_{4n} over k. Define an invoultion θ of GL_{4n} via $\theta(g) = \varepsilon g \varepsilon^{-1}$, where

$$\varepsilon = \begin{pmatrix} & I_n & & \\ \tau I_n & & & \\ & & & I_n \\ & & \tau I_n & \end{pmatrix}.$$

Denote the embedding $\iota : \mathrm{GL}_{2n}(K) \hookrightarrow \mathrm{GL}_{4n}(k)$ whose images are the fixed points of θ, more explicitly,

$$\iota\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \sqrt{\tau}\begin{pmatrix} X & Y \\ Z & Y \end{pmatrix}\right) = \begin{pmatrix} A & X & B & Y \\ \tau X & A & \tau Y & B \\ C & Z & D & W \\ \tau Z & C & \tau W & D \end{pmatrix}.$$

Since $\varepsilon J_{4n} \varepsilon^t = \tau J_{4n}$, the involution θ is also an involution on G. Let H be G^θ, consisting of the fixed points of θ.

The automorphic period $\mathcal{P}_H(\varphi)$ for this symmetric pair (G, H) has been found to be interesting and important. In [12] (1993), Ash, Ginzburg and Rallis proves the following general result.

Theorem 1.1.1 (Ash, Ginzburg, and Rallis [12] (1993)). *If φ be a cuspidal automorphic form of $\mathrm{Sp}_{2n}(\mathbb{A})$, then $\mathcal{P}_{\mathrm{Sp}_{2n_1} \times \mathrm{Sp}_{2n_2}}(\varphi)$ is always zero, where $n = n_1 + n_2$.*

In particular, when φ is a cuspidal automorphic form on $\mathrm{Sp}_{4n}(\mathbb{A})$ and when $H = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}$, the period $\mathcal{P}_{\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}}(\varphi)$ is zero for all cuspidal automorphic forms φ. Hence it is very interesting to figure out which automorphic forms φ

in the residual spectrum of $\mathrm{Sp}_{4n}(\mathbb{A})$ will support this automorphic period. Even more generally, consider this period for the continuous spectrum of $\mathrm{Sp}_{4n}(\mathbb{A})$. Note that when φ is not cuspidal, it may not be rapidly decreasing in the Siegel set associated with $H(k)\backslash H(\mathbb{A})$. This problem usually is treated by using the Arthur truncation method as first introduced by Jacquet and Rallis in [34] (1992) and by Friedberg and Jacquet in [21] (1993).

One of the special cases was studied by Ginzburg, Rallis, and Soudry in [23] (1999), which was a key ingredient in their theory of automorphic descent that constructs the backward Langlands functorial transfer from the general linear group to a classical group. More precisely, consider the Siegel parabolic subgroup P of Sp_{4n} with the Levi subgroup GL_{2n}. Let τ be a self-dual, irreducible, cuspidal, automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. For the cuspidal datum (GL_{2n}, τ), following Langlands, one defines an Eisenstein series $E(g, \phi_\tau, s)$. Following the Langlands-Shahidi method (Shahidi [68], for instance), the Eisenstein series $E(g, \phi_\tau, s)$ is holomorphic when the real part of s is greater than $\frac{1}{2}$, has meromorphic continuation and a functional equation. Moreover, $E(g, \phi_\tau, s)$ has a pole (which must be simple) at $s = \frac{1}{2}$ if and only if the partial exterior square L-function $L^S(s, \tau, \Lambda^2)$ has a pole at $s = 1$ and the central value of the partial standard L-function $L^S(\frac{1}{2}, \tau)$ is nonzero. The proof uses the work of H. Kim (see [40], for instance) on the normalization of the corresponding local intertwining operators in this case. The residue at $s = \frac{1}{2}$ is denoted by $E_{\frac{1}{2}}(g, \phi)$. In this case, Ginzburg, Rallis and Soudry proved.

Theorem 1.1.2 (Ginzburg, Rallis, and Soudry [23] (1999)). *The following period identity holds*

$$\mathcal{P}_H(E_{\frac{1}{2}}(\cdot, \phi)) = \int_{K_H} \int_{(C_{2n}(\mathbb{A})\mathrm{GL}_n(F)\times\mathrm{GL}_n(F))\backslash(\mathrm{GL}_n(\mathbb{A})\times\mathrm{GL}_n(\mathbb{A}))} \phi(k; \begin{pmatrix} a & \\ & b \end{pmatrix}) \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}k.$$

Here $H = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}$, $K_H = K \cap H$, K *is a maximal compact subgroup of* $\mathrm{Sp}_{4n}(\mathbb{A})$, *and* C_{2n} *is the center of* GL_{2n}.

Following the work of Bump and Friedberg [16] (1990), if the partial exterior square L-function $L^S(s,\tau,\Lambda^2)$ has a pole at $s=1$ and the central value of the partial standard L-function $L^S(\frac{1}{2},\tau)$ is nonzero, then the linear period, i.e. the integration along variables a and b is nonzero. This implies that the right hand side of the identity is nonzero, as proved in Ginzburg, Rallis, and Soudry [23] (1999). Therefore, when the residue $E_{\frac{1}{2}}(g,\phi)$ is nonzero (which is in fact square-integrable following the Langlands criterion), the period $\mathcal{P}_H(E_{\frac{1}{2}}(\cdot,\phi))$ is nonzero. Note that Ginzburg, Rallis, and Soudry used the Arthur truncation method to regularize this period.

In my joint work with Dihua Jiang, we are trying to prove the following result on this period in [37].

Conjecture 1.1.1 (Jiang-Zhang)**.** *Let φ be an automorphic form on $\mathrm{Sp}_{4n}(\mathbb{A})$. If the period $\mathcal{P}_H(\varphi)$ with $H=\mathrm{Sp}_{2n}\times\mathrm{Sp}_{2n}$, is nonzero, assuming the period is defined via suitable regularization, then the cuspidal support of φ is a subgroup of GL_{2n}, where GL_{2n} is the Levi subgroup of the standard Siegel parabolic subgroup of Sp_{4n}.*

It follows that in order to study the period $\mathcal{P}_H(\varphi)$ for general automorphic forms φ on $\mathrm{Sp}_{4n}(\mathbb{A})$, we only need to consider those Eisenstein series or their residues with cuspidal support in GL_{2n}. In this situation, the cuspidal data are generic in the sense that they have a nonzero Whittaker Fourier coefficient. In particular, the residues of Eisenstein series in this family can be completely determined, although it is technically complicated. It is one of my joint research projects with Dihua Jiang to completely understand the period with split $H=\mathrm{Sp}_{2n}\times\mathrm{Sp}_{2n}$ for those Eisenstein series and their residues. The local version of this problem will be discussed in Section 1.2 below.

It is now very natural to ask if there exists a cuspidal automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$, which supports the automorphic period with nonsplit H, i.e. $H=\mathrm{Sp}_{2n}(K)$, with K being a quadratic extension field over k. In the local aspect of my work (Section 1.2), I classify all irreducible supercuspidal representations

It is my long term project to find explicit formula for the automorphic period $\mathcal{P}_H(\varphi)$ in general. When φ is an Eisenstein series or its residue, the explicit formula I am looking for reduces the explicit formula for cuspidal automorphic forms in the cuspidal datum of the Eisenstein series or its residue. The work of Ginzburg, Rallis and Soudry mentioned above could be regarded as one of the examples of this type. The main issue here is to deal with the case when φ is cuspidal. When $n = 1$, a Rankin-Selberg integral is introduced in the work of Piatetski-Shapiro [59] (1997), which expresses the automorphic period in terms of the residue of certain L-function GSp_4 of degree 4. By following a similar consideration, I obtained an infinite sum of eulerian global integrals, which is parametrized by the square classes of the number field k. We discovered that each global integral looks like the integrals considered by Piatetski-Shapiro and Rallis in their paper: A new way to get eulerian products (Piatetski-Shapiro and Rallis [58] 1988). However, in order to figure out the corresponding generating function, which computes the local L-function at the unramified places, we found that the situation is totally different and involves a new type of affine symmetric space. In this thesis, for the symplectic group Sp_4 case, I calculated the local integral at the unramified split places and obtained the following identity in Theorem 2.2.1.

Theorem 1.1.3.

$$\int_{R\backslash H} W(g)|\det g|^{s+\frac{n+1}{2}}\,\mathrm{d}g = L(s+\frac{3}{2}, \Pi_v, \rho),$$

where the L-function is spin L-function of degree 4 and $W(g)$ is a degenerate Whittaker function.

Next, we plan to continue the calculation for the general n in the future. On the other hand, we may also try to consider this problem for the cuspidal automorphic form from the Ikeda lifting. In this thesis, we give the calculation of the lower rank cases such as $n = 1$ and $n = 2$, and establish the identity except over the bad primes.

In general, one may relate the automorphic period for the symmetric pair $(\mathrm{Sp}_{4n}, \mathrm{Res}_{K/k}\mathrm{Sp}_{2n})$ to the geometric and arithmetic issues, since the subgroup H generates Shimura subvarieties in the Shimura variety of Siegel type associated to Sp_{4n}. It is a very interesting problem to figure out the relation of my current work to the Tate conjecture in this case.

1.2 Local aspect of my work

Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. For a symmetric subgroup H of G, the period $\mathcal{P}_H(\varphi_\pi)$, with φ_π in the space V_π of π, defines a linear functional in the following space

$$\mathrm{Hom}_{H(\mathbb{A})}(V_\pi, 1).$$

It is clear that this space is a restricted tensor product

$$\otimes_v \mathrm{Hom}_{H(F_v)}(V_{\pi_v}, 1).$$

Hence the nonzero period $\mathcal{P}_H(\varphi_\pi)$ induces a nonzero functional ℓ_H in

$$\mathrm{Hom}_{H(k_v)}(V_{\pi_v}, 1)$$

for each local place v. This means that if the automorphic representation π is $H(\mathbb{A})$-distinguished, then the local component π_v is $H(k_v)$-distinguished.

Guided by the local-global principle in the theory of automorphic forms or more generally in number theory, it is important to study the local distinguished representations and the relation between local distinction and global automorphic distinction for automorphic representations in general. This leads to the following two basic problems.

Problem A: Let (G, H) be a symmetric pair defined over a p-adic local field F, and let σ be an irreducible admissible representation of $G(F)$. Find the dimension of the space $\mathrm{Hom}_{H(F)}(V_\sigma, 1)$. When F is archimedean, one may consider continuity and Harish-Chandra modules.

Problem B: Classify $H(F)$-distinguished representations of $G(F)$.

These two problems turn out to form the harmonic analysis aspect and representation aspect of the general problem of distinguished representations. The local-global relation is the arithmetic aspect of the problem.

In this thesis, I investigated both **Problem A** and **Problem B** for the particular symmetric pair $(\mathrm{Sp}_{4n}(k_v), \mathrm{Sp}_{2n}(K_v))$ for p-adic local places v. Form now on, we use F to denote a p-adic local field of characteristic zero, and π denote irreducible admissible representation of $G(F)$.

1.2.1 Gelfand Pairs

A symmetric pair (G, H) is called a *Gelfand pair* if the dimension of the vector space $\mathrm{Hom}_H(\pi, 1)$ is at most one for each smooth admissible irreducible representation π of $G(F)$. This is the so called local uniqueness for (G, H). Such a local uniqueness in general is important in representation theory and automorphic forms. In general, a symmetric pair may not be a Gelfand pair. There are some counter examples, given by Hakim and Murnaghan [28] (2008).

In the Chapter 4, we prove the following result.

Theorem 1.2.1 (Zhang [77] (2010)). *All the following pairs are Gelfand pairs for non-archimedean fields of characteristic 0,*

1. $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F))$ *when* $n = 1$ *or 2;*
2. $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(F) \times \mathrm{GSp}_{2n}(F))^{\circ})$ *when* $n = 1$ *or 2;*
3. $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$ *for all* n*;*
4. $(\mathrm{GSp}_{4n}(F), \mathrm{GSp}_{2n}(E)^{\circ})$ *for all* n*;*
5. $(\mathrm{U}_{2n}(J, E), \mathrm{Sp}_{2n}(F))$ *for all* n*.*

Here

$$U(J_{2n}, E) = \{g \in M_{2n\times 2n}(E) \mid \overline{g}^t J_{2n} g = J_{2n}\}.$$

The involution θ is induced from the action of the nontrivial Galois element in the quadratic field on each matrix entry. Then the symmetric subgroup H is $\mathrm{Sp}_{2n}(J_{2n}, F)$. The symmetric pair $(U(J_{2n}, E), \mathrm{Sp}_{2n}(F))$ appears as a descendant of $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$.

To prove this theorem, I applied the criterion which is investigated by the Gelfand and Kazhdan in [22] (1975). Let ρ be an anti-involution of G such that ρ stabilizes H. By the Gelfand-Kazhdan criterion, one needs only to prove that all $H \times H$-invariant distributions on $C_c^\infty(G)$ are preserved by ρ. In order to verify this condition, Bernstein and Zelevinsky [13] (1976) gave a localization principle for p-adic fields. Some important Gelfand pairs are proved based on the Bernstein-Zelevinsky localization principle, such as the Whittaker model (see Gelfand and Kazhdan [22], Shalika [69]), the symmetric pair $(\mathrm{GL}_{2n}(F), \mathrm{Sp}_{2n}(F))$ by Heumos and Rallis [31] (1990), and the Shalika model by Nien [56] (2009), to mention a few.

Following Bernstein's ideas, Aizenbud and Gourevitch [4] (2009) applied the Luna Slice Theorem to generalize the descent technique due to Harish-Chandra to the case of a reductive group acting on a smooth affine variety, which is called *generalized Harish-Chandra method*. Applying this new method, which works for arbitrary local field of characteristic 0, they formulated a conjectural approach to prove that certain symmetric pairs are Gelfand pairs in [4]. This idea was also used by Jacquet and Rallis to prove the uniqueness of linear periods in Jacquet and Rallis [35] (1996).

In the proof of Theorem 3.1, I applied this generalized Harish-Chandra method formulated in [4], and proved that all the symmetric pairs I considered are good and regular in the sense of Aizenbud and Gourevitch in [4]. This finishes the proof.

When n is greater than two, the approach which I used does not work for $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F))$. The issue is to prove that this pair is regular in the sense of [4]. To do this in the $n > 2$ case, one may have to study the invariant distributions on the tangent space of G/H whose supports are nilpotent orbits. The current knowledge has not been well understood about the invariant distributions on the tangent space of G/H, although some rank one cases have been discussed by Rader and Rallis in [62] (1996).

In summary, the theory of the uniqueness of symmetric pairs is eventually reduced to the theory of harmonic analysis on $C_c^\infty(G/H)$. This space $C_c^\infty(G/H)$ is well understood in the archimedean case by P. Delorme, among others. In the non-archimedean case, for instance, Harish-Chandra [29] (1978) thoroughly studied the case $(G\times G, G^{\triangle})$ and discovered lots of properties of the distribution characters of the representations. As a generalization, Rader and Rallis defined the *spherical characters* on the symmetric spaces in [62]. Studying spherical characters with applications to the local uniqueness problem is a long-term goal of mine.

1.2.2 Distinguished Representations

Now I consider **Problem B** for the symmetric pair $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$. This means classifying all H-distinguished representations of $\mathrm{Sp}_{4n}(F)$. From the general classification theory of irreducible admissible representations of a p-adic groups, one may first consider the supercuspidal case and then the case which is parabolically induced from the supercuspidal representations.

In order to classify the H-distinguished supercuspidal representations, I have to use the construction of all supercuspidal representations of p-adic reductive groups by means of Bruhat-Tits buildings. In the tamely ramified case, this construction is carried out by J.-K. Yu in [76] (2001). Let us recall some facts of such constructions.

Morris [53] (1999), and Moy and Prasad [54] (1994) independently gave the level-zero construction for arbitrary reductive group, in terms of Bruhat-Tits buildings, and introduced the depths of quasi-characters and supercuspidal representations of depth zero. Based on these and Adler's work [2] (1998), Yu [76], in the view of the local Langlands conjecture, naturally considered quasi-characters of the centralizer of a certain torus and gave a construction which has a very nice inductive structure: the generic G-datum $(\vec{\mathbf{G}}, y, \rho, \vec{\phi})$. Later, Kim [41] (2007) proved that all tamely ramified supercuspidal representations can be obtained through Yu's construction, subject to some hypotheses on G and F.

The another approach to construct the supercuspidal representations of classical groups began with Bushnell and Kutzko [17] (1993). Following their works, Stevens [72] (2008) constructed all supercuspidal representations of classical groups, which include unitary, symplectic and special orthogonal groups over a non-archimedean local field, except for the residual characteristic 2 case.

All of these constructions have their notable advantages in classifying the H-distinguished supercuspidal representations. For example, applying Bushnell-Kutzko-Stevens' construction, Blondel and Stevens listed all generic supercuspidal representations of $\mathrm{Sp}_4(F)$. Hakim and Murnaghan [28] (2008) applied Yu's construction and gave some necessary conditions for all tame supercuspidal representations of G distinguished by a symmetric subgroup H.

I applied Hakim and Murnaghan's formula in [28] to prove that a tame supercuspidal representation is H-distinguished if and only if each component of the generic G-datum $(\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ satisfies certain conditions. For more details, one can refer to my paper [78] (2010).

Theorem 1.2.2 (Zhang)**.** *Let (G, H) be the symmetric pair $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$. A tame supercuspidal representation $\pi(\Psi)$ is H-distinguished if and only if Ψ satisfies the following conditions, up to a K-conjugation and a θ-symmetric refactorization.*

- *The quadratic extension E is unramified.*

    1. *When d is 0, y is θ-fixed and $\overline{\rho}$ is a distinguished cuspidal representation of the symmetric pair $(G_{y,0}/G_{y,0+}, (G_{y,0}/G_{y,0+})^{\theta})$, that is,*

    $$\prod_i (\mathrm{Sp}_{4r_i}(\mathfrak{f}_i) \times \mathrm{Sp}_{4(n_i-r_i)}(\mathfrak{f}_i), \mathrm{Sp}_{2r}(\mathfrak{f}_{E_i}) \times \mathrm{Sp}_{2(n_i-r_i)}(\mathfrak{f}_{E_i})).$$

    2. *When $d > 0$, the symmetric pair $(G^0, G^{0,\theta})$ is isomorphic to*

    $$(\prod_i \mathrm{U}_2(Q, E_i), \prod_i \mathrm{SU}_1(D_i)).$$

- *The quadratic extension E is ramified.*

1. *The depth d is more than zero.*
2. *The symmetric pair $(G^0, G^{0,\theta})$ is isomorphic to*

$$(\prod_i \mathrm{U}(E_i), \prod_i \mathrm{SU}(D_i)).$$

The point y is θ-fixed point and $\overline{\rho}$ is a distinguished cuspidal representation of the symmetric pair $(G_{y,0}/G_{y,0+}, (G_{y,0}/G_{y,0+})^{\theta})$, that is,

$$\prod_i (\mathrm{Sp}_{4n_i}(\mathfrak{f}_i), \mathrm{Sp}_{2n_i}(\mathfrak{f}_{E_i})).$$

Here, $\mathfrak{f}_i$ is the residue field of a local field and D_i is a quaternion algebra. For $n = 1$, Murnaghan gave a H-distinguished representation [55] (2010), which is consistent with this theorem.

In the case $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$, there is a family of tame supercuspidal representations which are distinguished. For example, the supercuspidal representation $\text{c-Ind}_{\mathrm{Sp}_4(\mathcal{O}_F)}^{\mathrm{Sp}_4(F)} \overline{\theta}_{10}$ is $\mathrm{SL}_2(E)$-distinguished, where θ_{10} is the unipotent cuspidal representation of $\mathrm{Sp}_4(\mathbb{F}_q)$ over the finite field $\mathbb{F}_q$.

As a corollary of the classification of H-distinguished tame supercuspidal representation, I have the following dichotomy theorem.

Theorem 1.2.3 (Zhang [78])**.** *Let π be an irreducible tame supercuspidal representation of $\mathrm{Sp}_{4n}(F)$. If the symmetric pair (G, H) is $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$, then*

$$\dim \mathrm{Hom}_{\mathrm{Sp}_{2n}(E_1)}(\pi, 1) \cdot \dim \mathrm{Hom}_{\mathrm{Sp}_{2n}(E_2)}(\pi, 1) = 0, \tag{1.2.1}$$

where E_1 is the unramified quadratic extension over F and E_2 is a ramified quadratic extension.

For the symmetric pair $(\mathrm{U}_{2n}(E), \mathrm{Sp}_{2n}(F))$, I obtain the following theorem.

Theorem 1.2.4 (Zhang [78])**.** *If a symmetric pair (G, H) is $(\mathrm{U}_{2n}(E), \mathrm{Sp}_{2n}(F))$, then there is no H-distinguished tame supercuspidal representation of G.*

In order to calculate the distinguished representations, more explicitly, one naturally reduces to the consideration of the distinguished representations over finite fields. When n is 1, I have the following theorem.

Theorem 1.2.5 (Zhang [78]). *If a symmetric pair* (G, H) *is* $(\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{F}_{q^2}))$, *then* θ_{10} *is the unique* H*-distinguished irreducible cuspidal representation of* G.

For the H-distinguished representations over finite fields, I apply Deligne-Lusztig's virtual characters and Lusztig's formula on the symmetric spaces to classify all the H-distinguished representations. Since not all irreducible representation of finite groups of Lie type can be expressed as a linear combination of Deligne-Lusztig's virtual representation R_T^θ, there is still a gap between the dimension formula of $\mathrm{Hom}_{H(\mathbb{F}_q)}(R_T^\theta, 1)$ and $H(\mathbb{F}_q)$-distinguished irreducible representations. Theorem 1.2.5 is an example to fill this gap and get a complete list of H-distinguished irreducible representations. For a general n and all other symmetric pairs, I am interested in the classification of H-distinguished representations in terms of Lusztig's parameterization of irreducible representations in [50] (1984) and the construction of linear invariant forms in the finite field case. This is also a long-term project of mine.

Next, when π is not just supercuspidal, there is a framework for distinguished representations given by Kato and Takano in [38] (2008). They generalized the Jacquet's sub-representation theorem to affine symmetric spaces, and introduced the notion of *relative supercuspidal representations*. In addition, there is another general theory about H-distinguished representations by Blanc and Delorme in [14] (2008). For the case under my consideration, based on the setup of Kato and Takano in [38], I can completely list all the relative supercuspidal representations and give a more explicit classification of H-distinguished representations including supercuspidal and non-supercuspidal representations. This is the local version of Theorem 1.1.1 discussed in Section 1.1. In a joint project with Dihua Jiang, we figure out the relation between H-distinguished representations and the structure of their corresponding local Arthur parameters. Some cases for orthogonal groups are discussed in a recent work of Jiang, Nien and Qin.

In conclusion, the classification of distinguished representations over local fields will be used in our explicit calculation of the period formula for $\mathcal{P}_H(\varphi)$ when φ is an automorphic form in the space of an irreducible automorphic representation π occurring in the discrete spectrum of $\mathrm{Sp}_{4n}(\mathbb{A})$. This is one of my future research projects. I omit details here.

1.3 Notation

Let k be a number field, $\mathfrak{o}_k$ be the ring of integers and $\mathfrak{p}$ be the prime ideal of $\mathfrak{o}_k$. Let K be a quadratic extension field of k. Then there is a non-square element τ in $\mathfrak{o}_k$ such that $K = k(\sqrt{\tau})$. We take the non-trivial element in the Galois group $\mathrm{Gal}(K/k)$, which is the non-trivial involution over k, given by

$$x = a + \sqrt{\tau}b \in K \to \bar{x} = a - \sqrt{\tau}b.$$

Denote by $k_\mathfrak{p}$ the completion local field over the place $\mathfrak{p}$.

Let $\mathbb{A}$ be the Adele ring of k. Fix a character ψ of $k\backslash\mathbb{A}$. In this paper, we use F to denote the local field over a place $\mathfrak{p}$ of k and E to denote the completion $K_\mathfrak{p}$, which could be $k_\mathfrak{p} \times k_\mathfrak{p}$ (split case) or $k_\mathfrak{p}(\sqrt{\tau})$ (quadratic extension case). Let $\mathfrak{o}_F$ be the valuation ring of F. Except the Chapter 5, we denote by F the Frobenius map over the finite fields in . In this whole paper, for the nonarchimedean field, we restrict ourselves to the characteristic 0 case, and assume that the characteristics of the residue field and all finite fields are not even.

Usually, we choose the plain upper case letters such as A, B, X, Y, etc. to denote the matrix. Let G and H be the specified reductive groups. According to the subscript, let P be a standard parabolic subgroup with the Levi subgroup M and the unipotent radical N.

Chapter 2

Automorphic Periods

2.1 Automorphic periods of (G, H)

In this section, we introduce our notation and the symplectic pairs. Given a symplectic matrix

$$J_{2n} = \begin{pmatrix} & w_n \\ -w_n & \end{pmatrix} \text{ and } w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}_{n\times n},$$

define a symplectic similitude group

$$\mathrm{GSp}_{2n}(k) = \left\{ g \in \mathrm{GL}_{2n}(k) \mid g^t J_{2n} g = \lambda J_{2n}, \text{ for some } \lambda \in k^\times \right\},$$

and a symplectic group

$$\mathrm{Sp}_{2n}(k) = \{ g \in \mathrm{GSp}_{2n}(k) \mid \lambda(g) = 1 \}.$$

Let G be the group GSp_{4n} or Sp_{4n} defined over k. Define a k-involution θ of GL_{4n} via $\theta(g) = \varepsilon g \varepsilon^{-1}$, where

$$\varepsilon = \begin{pmatrix} & I_n & & \\ \tau I_n & & & \\ & & & I_n \\ & & \tau I_n & \end{pmatrix}.$$

Denote an embedding $\iota : \mathrm{GL}_{2n}(K) \hookrightarrow \mathrm{GL}_{4n}(k)$ whose images are the fixed points of θ, more explicitly,

$$\iota\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \sqrt{\tau} \begin{pmatrix} X & Y \\ Z & Y \end{pmatrix} \right) = \begin{pmatrix} A & X & B & Y \\ \tau X & A & \tau Y & B \\ C & Z & D & W \\ \tau Z & C & \tau W & D \end{pmatrix}.$$

Since $\varepsilon J_{4n} \varepsilon^t = \tau J_{4n}$, the involution θ is also a k-involution on G. Let H be G^θ, consisting of the fixed points of θ.

Lemma 2.1.1. *The symmetric subgroup H is isomorphic to* $\mathrm{Sp}_{2n}(K)$ *if* $G(k)$ *is* $\mathrm{Sp}_{4n}(k)$*. Or, the subgroup H is the group*

$$\left\{g \in \mathrm{GSp}_{2n}(K) \mid \lambda(g) \in k^\times\right\},$$

if $G(k)$ *is* $\mathrm{GSp}_{4n}(k)$*.*

Proof. Let $g \in \mathrm{GL}_{2n}(K)$. Then $\iota(g) \in G(k)$ is equivalent to

$$\iota(g) \cdot J_{4n} \cdot \iota(g)^t \cdot J_{4n}^{-1} = I_{4n} \text{ or } k^\times \cdot I_{4n}.$$

Since $\iota(g)^t = T_1 \iota(g^t) T_1^{-1}$ where

$$T_1 = \begin{pmatrix} I_n & & & \\ & \tau^{-1} I_n & & \\ & & I_n & \\ & & & \tau^{-1} I_n \end{pmatrix},$$

we have $\iota(g)(J_{4n}T_1)\iota(g^t)(J_{4n}T_1)^{-1} = I_{4n}$ or $\in k^\times \cdot I_{4n}$. Therefore,

$$g \cdot \iota^{-1}(J_{4n}T_1) \cdot g^t \cdot \iota^{-1}(J_{4n}T_1) = I_{2n} \text{ or } k^\times \cdot I_{2n}.$$

□

Denote P_G be the standard Siegel parabolic subgroup of G consisting of the matrix of the form $\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$, M_G be the Levi subgroup of P_G, and U_G be the unipotent radical. Then M_G is isomorphic to GL_{2n}. Let $P_H = P_G \cap H$ (*resp.* M_H, U_H).

Lemma 2.1.2. *The subgroup P_H of H is the standard Siegel parabolic subgroup of H.*

Proof. The subgroup P_H consists of the matrices of form

$$\begin{pmatrix} A & X & B & Y \\ \tau X & A & \tau Y & B \\ & & D & \tau W \\ & & \tau W & D \end{pmatrix}.$$

In addition, we have

$$M_H = \left\{ \begin{pmatrix} A & X & & \\ \tau X & A & & \\ & & D & \tau W \\ & & \tau W & D \end{pmatrix} \right\} \text{ and } U_H = \left\{ \begin{pmatrix} I_n & & B & Y \\ & I_n & \tau Y & B \\ & & I_n & \\ & & & I_n \end{pmatrix} \right\}.$$

Then, $M_H \simeq \mathrm{Res}_{K/k}\mathrm{GL}_n$ and $U_H \simeq \mathrm{Res}_{K/k}M_{n\times n}$. Therefore, we have P_H is a standard Siegel parabolic subgroup of H. $\square$

Without any confusion, the intersection M_H and U_H are the Levi subgroup and unipotent radical of P_H. For a n-by-n matrix A, denote $\hat{A} = wA^t w$. Given a matrix $u \in M_{2n\times 2n}$ such that $u = \hat{u}$ and $h \in \mathrm{GL}_{2n}$, define

$$n(u) = \begin{pmatrix} I_{2n} & u \\ & I_{2n} \end{pmatrix} \text{ and } m(h) = \begin{pmatrix} h & \\ & \hat{h}^{-1} \end{pmatrix}.$$

Let $f(g, s)$ be a function in the normalized induced representation

$$\begin{cases} \mathrm{Ind}_{P_H(\mathbb{A})}^{H(\mathbb{A})} |\det|^s & \text{if } G = \mathrm{Sp}_{4n} \\ \mathrm{Ind}_{P_H(\mathbb{A})}^{H(\mathbb{A})} |\det|^{s_1} \rtimes |\lambda|^{s_2} & \text{if } G = \mathrm{GSp}_{4n}. \end{cases}$$

When G is the symplectic similitude group GSp_{4n}, the elements of the Levi subgroup of P_H have the form

$$m(g, \lambda) = \begin{pmatrix} \lambda g & \\ & \hat{g}^{-1}, \end{pmatrix}$$

and the character of $m(g, \lambda)$ in the induced representation is given by $|\det(g)|^{s_1}|\lambda|^{s_2}$. Then we can form an Eisenstein series

$$E(g, s, f) = \sum_{\gamma \in P_H(k)\backslash H(k)} f(\gamma g, s).$$

Let Π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ and φ be a cusps form of Π. In this paper, we consider the following integral,

$$I_G(s, \varphi) = \begin{cases} \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \varphi(g)E(g, s, f)\,\mathrm{d}g & \text{if } G = \mathrm{GSp}_{4n} \\ \int_{H(F)\backslash H(\mathbb{A})} \varphi(g)E(g, s, f)\,\mathrm{d}g & \text{if } G = \mathrm{Sp}_{4n} \end{cases} \tag{2.1.1}$$

First, let us consider the **case** $G = \mathrm{Sp}_{4n}$.

$$\begin{aligned} I_G(s,\varphi) &= \int_{H(k)\backslash H(\mathbb{A})} \varphi(g) \sum_{\gamma\in P_H(k)\backslash H(k)} f(\gamma g, s)\,\mathrm{d}g \\ &= \int_{P_H(k)\backslash H(\mathbb{A})} \varphi(g) f(g,s)\,\mathrm{d}g \end{aligned}$$

Applying the Fourier expansion of φ over the commutative unipotent radical U_G, we obtain

$$\varphi(g) = \sum_{\psi\in(U_G(k)\backslash U_G(\mathbb{A}))^\vee} \varphi_\psi(g),$$

where $(U_G(k)\backslash U_G(\mathbb{A}))^\vee$ is the set of all smooth characters up to isomorphism and the Fourier coefficients φ_ψ (g) are given by

$$\varphi_\psi(g) = \int_{U_G(k)\backslash U_G(\mathbb{A})} \varphi(ug)\psi^{-1}(u)\,\mathrm{d}u.$$

Hence, we identify the space $\mathrm{Sym}^2(k^{2n})$ with the matrices

$$\{A \in M_{2n\times 2n}(k) \mid A = wA^t w\}.$$

Indeed, this space is isomorphic to $U_G(k)$ as abelian groups. Since $U_G(k)\backslash U_G(\mathbb{A})$ is compact,the space $(U_G(k)\backslash U_G(\mathbb{A}))^\vee$ of characters is isomorphic to $U_G(k)$ by Pontryagin Duality Theorem in the work of Ramakrishnan and Valenza [64]. In details, let ψ_0 be a fixed character over $k\backslash\mathbb{A}_k$. We can construct a character ψ_β for all $\beta \in \mathrm{Sym}^2(k^{2n})$ via

$$\psi_\beta(n(u)) = \psi_0(\mathrm{tr}\beta u).$$

For each character ψ on $U_G(k)\backslash U_G(\mathbb{A})$, $\psi(u) = \psi_0(\mathrm{tr}(\beta u))$ for some $\beta \in U_G(k)$. Hence the mapping β to ψ_β gives the isomorphism. Thus we may identity the two spaces by identify ψ and β, and write φ_β for φ_ψ.

Now, let us substitute this Fourier expansion into the integral $I_G(s,\varphi)$, and then obtain

$$\begin{aligned} I_G(s,\varphi) &= \int_{M_H(F)U_H(\mathbb{A})\backslash H(\mathbb{A})} f(g,s) \int_{U_H(F)\backslash U_H(\mathbb{A})} \sum_{\beta\in\mathrm{Sym}^2(k^{2n})} \varphi_\beta(ug)\,\mathrm{d}u\,\mathrm{d}g \\ &= \int_{M_H(F)U_H(\mathbb{A})\backslash H(\mathbb{A})} f(g,s) \sum_{\beta\in\mathrm{Sym}^2(k^{2n})} \varphi_\beta(g) \int_{U_H(F)\backslash U_H(\mathbb{A})} \psi(\mathrm{tr}\beta u)\,\mathrm{d}u\,\mathrm{d}g \end{aligned}$$

Since the Fourier expansion is absolutely converge and the integration domain $U_H(F)\backslash U_H(\mathbb{A})$ is compact, we can exchange the order of the summation and the integral. From now on, we normalize the Haar measure $\mathrm{d}u$ so that $\mathrm{vol}(U_H(k)\backslash U_H(\mathbb{A})) = 1$. The integration of the character ψ_β over $U_H(k)\backslash U_H(\mathbb{A})$ is zero if the characters ψ_β associated with β are not trivial on $U_H(\mathbb{A})$. Define a set

$$\Omega = \{\beta \in \mathrm{Sym}^2(k^{2n}) \setminus \{0\} \mid \psi(\mathrm{tr}\beta u) = 1 \text{ for all } u \in U_H(\mathbb{A})\}.$$

Therefore,

$$I_G(s,\varphi) = \int_{M_H(F)U_H(\mathbb{A})\backslash H(\mathbb{A})} \sum_{\beta\in\Omega} \varphi_\beta(g) f(g,s)\,\mathrm{d}g$$

Lemma 2.1.3. *$\psi(\mathrm{tr}\beta u) = 1$ for all $u \in U_H(\mathbb{A})$ is equivalent to $\mathrm{tr}(\beta u) = 0$ for all $u \in U_H(\mathbb{A})$.*

Proof. If $\mathrm{tr}(\beta u) = 0$ for all $u \in U_H(\mathbb{A})$, then it is easy to conclude $\psi(\mathrm{tr}\beta u) = 1$ for all $u \in U_H(\mathbb{A})$.

Assume $\psi(\mathrm{tr}\beta u) = 1$ for all $u \in U_H(\mathbb{A})$. Applying the fact that xI_{2n} for all $x \in \mathbb{A}_k$ is belong to $U_H(\mathbb{A})$, if $\mathrm{tr}(\beta u_0) \neq 0$ for some $u_0 \in U_H$, we can find an element x such that $\psi(x\mathrm{tr}\beta u) \neq 1$. Therefore, $\mathrm{tr}(\beta u)$ has to be zero for all $u \in U_H(\mathbb{A})$. $\square$

Then, we can easily conclude the following lemma.

Lemma 2.1.4. $\Omega = \{\begin{pmatrix} X & Y \\ -\tau Y & \hat{X} \end{pmatrix} \in \mathrm{Sym}^2(k^{2n}) \setminus \{0\} \mid X = -\hat{X},\ Y = \hat{Y}\}$

Proof. Let

$$\beta = \begin{pmatrix} X & Y \\ Z & \hat{X} \end{pmatrix} \in \Omega,$$

where $Y = \hat{Y}$ and $Z = \hat{Z}$. For all

$$u = \begin{pmatrix} A & B \\ \tau B & A \end{pmatrix} \in U_H \text{ with } A = \hat{A} \text{ and } B = \hat{B},$$

we have $\mathrm{tr}(\beta u) = 0$. Since

$$\mathrm{tr}\begin{pmatrix} X & Y \\ Z & \hat{X} \end{pmatrix}\begin{pmatrix} A & B \\ \tau B & A \end{pmatrix} = \mathrm{tr}((X + \hat{X})A + (\tau Y + Z)B),$$

$\mathrm{tr}(\beta u) = 0$ for all A and B is equivalent to $X + \hat{X} = 0$ and $Z + \tau Y = 0$. Therefore,

$$\beta = \begin{pmatrix} X & Y \\ -\tau Y & \hat{X} \end{pmatrix} \text{ with } X = -\hat{X} \text{ and } Y = \hat{Y}.$$

□

In order to continue unfolding the integral $I_G(s, \varphi)$, let us study the Fourier coefficients under the conjugation of $M_H(k)$. For any element $m(h) \in M_H(k)$, we have

$$\begin{aligned}
\varphi_\beta(m(h)g) =& \int\limits_{\mathrm{Sym}^2(k^{2n})\backslash \mathrm{Sym}^2(\mathbb{A}^{2n})} \varphi(n(u)m(h)g)\psi(\mathrm{tr}(\beta u)))\,\mathrm{d}u \\
=& \int\limits_{\mathrm{Sym}^2(k^{2n})\backslash \mathrm{Sym}^2(\mathbb{A}^{2n})} \varphi(m(h)^{-1}n(u)m(h)g)\psi(tr(\beta u))\,\mathrm{d}u \\
=& \int\limits_{\mathrm{Sym}^2(k^{2n})\backslash \mathrm{Sym}^2(\mathbb{A}^{2n})} \varphi(n(h^{-1}u\hat{h}^{-1})g)\psi(tr(\beta u))\,\mathrm{d}u,
\end{aligned}$$

(changing variables $u' = h^{-1}u\hat{h}^{-1}$,)

$$\begin{aligned}
=& \int\limits_{\mathrm{Sym}^2(k^{2n})\backslash \mathrm{Sym}^2(\mathbb{A}^{2n})} \varphi(n(u')g)\psi(\mathrm{tr}(\beta h u'\hat{h}))\,\mathrm{d}u' \\
=& \int\limits_{\mathrm{Sym}^2(k^{2n})\backslash \mathrm{Sym}^2(\mathbb{A}^{2n})} \varphi(n(u')g)\psi(\mathrm{tr}(\hat{h}\beta h u'))\,\mathrm{d}u' \\
=& \varphi_{\hat{h}\beta h}(g).
\end{aligned}$$

Now, let us classify all the orbits of the adjoint action of $M_H(k)$ on Ω. Assume that

$$h = \begin{pmatrix} X & Y \\ \tau Y & X \end{pmatrix} \in \mathrm{GL}_{2n}(k) \text{ and } \beta = \begin{pmatrix} A & B \\ -\tau B & \hat{A} \end{pmatrix} \in \Omega$$

with $A = -\hat{A}$ and $B = \hat{B}$. Let us introduce a matrix

$$T_2 = \begin{pmatrix} \frac{1}{2} I_n & -\frac{1}{2\sqrt{\tau}} I_n \\ \frac{1}{2} I_n & \frac{1}{2\sqrt{\tau}} I_n \end{pmatrix},$$

satisfying

$$T_2 g T_2^{-1} = \begin{pmatrix} X+\sqrt{\tau}Y & \\ & X-\sqrt{\tau}Y \end{pmatrix} \text{ and } T_2 \beta T_2^{-1} = \begin{pmatrix} & A+\sqrt{\tau}B \\ A-\sqrt{\tau}B & \end{pmatrix}.$$

Under the conjugation of T_2, we have

$$T_2^{-1} \cdot (T_2 g T_2^{-1}) \cdot (T_2 \beta T_2^{-1}) \cdot (T_2 \hat{g} T_2^{-1}) \cdot T_2 = T_2^{-1} \cdot \begin{pmatrix} & Z \cdot C \cdot \hat{\bar{Z}} \\ \bar{Z} \cdot \bar{C} \cdot \hat{Z} & \end{pmatrix} \cdot T_2,$$

where $C = A+\sqrt{\tau}B \in M_{n\times n}(K)$, $\bar{C} = -\hat{C}$ and $Z = X+\sqrt{\tau}Y \in \mathrm{GL}_n(K)$. Then C is a skew-Hermitian matrix. Obviously, the orbits of the adjoint action of $M_H(k)$ on Ω are corresponding to the isometry classes $Herm_n$ of n-dimensional skew-hermitian spaces over K. In Section 2.3, we recall the classification of hermitian forms over a number field k and a local field.

Denote D_β the stabilizer of the action M_H on β. We will calculate the D_β in the next section.

2.1.1 Stabilizer of the character ψ_β

Without confusion, we also write β as an element in $M_{n\times n}(K)$ and then D_β as a subgroup in $\mathrm{GL}_n(K)$. One can choose a representative $\beta = \begin{pmatrix} & \sqrt{\tau}\beta_r \\ 0_{n-r} & \end{pmatrix}$ in Ω. Therefore,

$$D_\beta = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathrm{GL}_n(E) | A\bar{A}^t = \lambda I_r,\ C \in \mathrm{GL}_{n-r}(E),\ B \in M_{r\times(n-r)}(E) \right\} \tag{2.1.2}$$

If $\mathrm{rank}(\beta) = n$, then

$$D_\beta = \{A \in \mathrm{GL}_n(E) | A\beta\bar{A}^t = \lambda\beta\}.$$

Remark that λ is trivial when $G = \mathrm{Sp}_{4n}$.

2.2 Global Calculation

Let us continue the calculation of the integral $I(s,\varphi)$.

$$\begin{aligned}
I_G(s,\varphi) &= \int_{M_H(k)U_H(\mathbb{A})\backslash H(\mathbb{A})} \sum_{\beta\in Herm} \sum_{h\in D_\beta(k)\backslash M_H(k)} \varphi_\beta(hg)f(hg,s)\,\mathrm{d}g \\
&= \sum_{\beta\in Herm} \int_{D_\beta(k)U_H(\mathbb{A})\backslash H(\mathbb{A})} \varphi_\beta(g)f(g,s)\,\mathrm{d}g \\
&= \sum_{\beta\in Herm} \int_{D_\beta(\mathbb{A})U_H(\mathbb{A})\backslash H(\mathbb{A})} \int_{D_\beta(k)\backslash D_\beta(\mathbb{A})} \varphi_\beta(tg)f(tg,s)\,\mathrm{d}t\,\mathrm{d}g \\
&= \sum_{\beta\in Herm} \int_{D_\beta(\mathbb{A})U_H(\mathbb{A})\backslash H(\mathbb{A})} \left(\int_{D_\beta(k)\backslash D_\beta(\mathbb{A})} \varphi_\beta(m(t)g)|\det t|^{s+\frac{n+1}{2}}\,\mathrm{d}t\right) f(g,s)\,\mathrm{d}g \\
&= \sum_{\beta\in Herm} \int_{D_\beta(\mathbb{A})U_H(\mathbb{A})\backslash H(\mathbb{A})} \int_{D_\beta(k)\backslash D_\beta(\mathbb{A})} \int_{U_G(k)\backslash U_G(\mathbb{A})} f(g,s) \\
&\quad \varphi_\beta(m(t)ug)\psi_\beta(u)|\det t|^{s+\frac{n+1}{2}}\,\mathrm{d}u\,\mathrm{d}t\,\mathrm{d}g.
\end{aligned}$$

Denote $R_\beta = D_\beta U_G$ and

$$\varphi_{\beta,s}(g) = \int_{R_\beta(k)\backslash R_\beta(\mathbb{A})} \varphi(rg)\alpha_{\beta,s}(r)\,\mathrm{d}r. \tag{2.2.1}$$

where $\alpha_{\beta,s}$ is a charachter of R_β defined by $\alpha_{\beta,s}(tu) = |\det t|^{s+\frac{n+1}{2}}\psi_\beta(u)$ for $t\in D_\beta$ and $u\in U_G$. Therefore,

$$I_G(s,\varphi) = \sum_{\beta\in Herm} \int_{D_\beta(\mathbb{A})U_H(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{\beta,s}(g)f(g,s)\,\mathrm{d}g. \tag{2.2.2}$$

For **the case** $G = \mathrm{GSp}_{4n}$, we use the similar way and obtain the following identity for GSp_{4n}.

$$I_G(s,\phi) = \sum_{\beta\in Herm/k^\times} \int_{D_\beta(\mathbb{A})U_H(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{\beta,s}(g)f(g,s)\,\mathrm{d}g. \tag{2.2.3}$$

If G is the symplectic similitude group GSp_{4n}, then the similitude $k^\times$ also acts on the set Ω by the dilation and we denote by $Herm/k^\times$ the orbits of this action on the isometry classes of Hermitian forms $Herm$. The stabilizer D_β of the character associated to β can be found at Equation 2.1.2. The integral φ_β for GSp_{4n} is given by

$$\varphi_{\beta,s}(g) = \int_{R_\beta(k)\backslash R_\beta(\mathbb{A})} \varphi(rg)\alpha_{s_1,s_2,\beta}(r)\,\mathrm{d}r. \tag{2.2.4}$$

In this case, the character $\alpha_{s_1,s_2,\beta}$ of R_β is given by

$$\alpha_{\beta,s}(tu) = |\det t|^{s+\frac{n+1}{2}}|\lambda(t)|^{s+\frac{n(n+1)}{4}}\psi_\beta(u).$$

Let all places outside the finite S be finite and unramified with odd residual characteristic. Put

$$I_S(s,\varphi,\beta) = \int_{D_{\beta,S}U_{H,S}\backslash H_S} \varphi_\beta(g)f(g,s)\,\mathrm{d}g$$

where $H_S = \prod_{v\in S} H(k_v)$, $D_{\beta,S} = \prod_{v\in S} D_\beta(k_v)$ and $U_{H,S} = \prod_{v\in S} U_H(k_v)$.

Denote by S' a finite set of places containing S. Let $p \notin S'$. Then

$$I_{S'}(s,\varphi,\beta) = \int_{D_{\beta,S'}U_{H,S'}\backslash H_{S'}} f(g,s)\varphi_\beta(g)\,\mathrm{d}g.$$

$$I_{S'\cup\{v\}}(s,\varphi,\beta) = \int_{D_{\beta,S'}U_{H,S'}\backslash H_{S'}} f(g,s)\int_{D_\beta(k_v)U_H(k_v)\backslash H(k_v)} \varphi_\beta(gg_v)f_p(g_v,s)\,\mathrm{d}g_v\,\mathrm{d}g. \tag{2.2.5}$$

Following the method in the work of Piatetski-Shapiro and Rallis [58], we take the limit for the finite places set S' and expect to evaluate this integral over all the places.

Referring to the following section, for the case $\mathrm{Sp}_4(\mathbb{A})$, we calculate an explicit result of the period integral over a cuspidal form in an irreducible cuspidal automorphic representation from the Ikeda lifting. In addition, we need to normalize the Eisenstein series in a suitable way. For the case $\mathrm{Sp}_4(\mathbb{A})$, we use the zeta function $\zeta_K(s)$ to normalize the Eisenstein series of $\mathrm{SL}_2(\mathbb{A}_K)$. Then, we have the following conjecturally identity.

Theorem 2.2.1.

$$I_G(s,\phi) = \sum_{a\in K^\times/\ker(\mathrm{N}(K))} \prod_{S_a} I_v(s,a) \prod_{K_v\ split} L_v(s+\frac{1}{2},\Pi_v,\rho) \prod_{K_v\ unram} I_v(s,a), \tag{2.2.6}$$

where $S_a = S \cup \{v \mid |a|_v \neq 1\}$.

2.3 Classification of Hermitian Forms

In this section, we will review the classification of the skew-Hermitian forms over K. There are many materials which discussed the classification of the Hermitian forms and quadratic forms over a number field. Here, we refer to O'Meara [57] and Lewis [46]. Indeed, the skew-Hermitian matrices over K can be equivalently treated as the Hermitian matrices via $A \to \sqrt{\tau}^{-1}A$. Let A be a n-by-n Hermitian matrix over K and $A = X + \sqrt{\tau}Y$ for X and Y in $M_{n\times n}(k)$.

First, we assume that A is non-degenerate and rewrite this Hermitian matrix as a 2n-by-2n symmetric matrix via

$$X + \sqrt{\tau}Y \longrightarrow \begin{pmatrix} X & \tau Y \\ -\tau Y & -\tau X \end{pmatrix}.$$

Hence, by Jacobson's theorem, two Hermitian matrices are isometric if and only if the corresponding symmetric matrices are isometric. Referring to O'Meara [57, Remark 66:5], for a non-degenerate quadratic form Q, we have the following complete set of invariants for Q:

1. the dimension of the quadratic space,
2. the discriminant $\mathrm{disc}(Q)$,
3. the Hasse symbols $\mathrm{Hass}_{\mathfrak{p}}(Q)$ at all finite places $\mathfrak{p}$,
4. the number of the positive signatures at all real places $\mathfrak{p}$.

For the quadratic form associated to A, the discriminant is $(-\tau)^n$ and the Hasse symbol $\mathrm{Hass}_{\mathfrak{p}}(Q)$ is equal to the Hilbert symbol $(\mathrm{disc}(A), -\tau)_{\mathfrak{p}}$ at the finite place $\mathfrak{p}$. The isometry class of a non-degenerate Hermitian matrices A is uniquely determined by the discriminant $\mathrm{disc}(A)$ and the positive indices at all real places. In addition, for each isometry class, we can take a diagonal matrix with the entries in k as a representative.

If an Hermitian matrix A is degenerate, let us assume the rank of A is r. Then, the isometry classes of all the Hermitian matrices of rank r are bijective to $Herm_r$. Let us summarize the classification of the Hermitian matrices as the following lemma.

Lemma 2.3.1. *An isometry class of an n-by-n Hermitian matrix A over K is uniquely determined by the following invariants:*

1. *the rank of A,*

2. *the Hilbert symbol* $(\mathrm{disc}(A), -\tau)_{\mathfrak{p}}$ *at all finite places* $\mathfrak{p}$*,*

3. *the number of the positive signatures of A at all real places* $\mathfrak{p}$*.*

Here, if the rank r of A is strictly less than n, then we consider A as an r-by-r non-degenerate Hermitian matrix.

In general, we can choose the representative in Ω,

$$\beta = \begin{pmatrix} 0_n & Y \\ -\tau Y & 0_n \end{pmatrix} \text{ and } Y = \begin{pmatrix} & & & a_1 \\ & & a_2 & \\ & \iddots & & \\ a_n & & & \end{pmatrix}.$$

Next, we show two lower rank examples.

2.3.1 Case $n = 1$

If n=1, the isometry classes $Herm_1$ is isomorphic to $k^\times/\mathrm{N}(K^\times)$. By the Hasse Norm Theorem, a and b in $k^\times$ have the same image in $k^\times/\mathrm{N}(K^\times)$ if and only if a and b have the same image in $k_\mathfrak{p}^\times/\mathrm{N}(K_\mathfrak{p}^\times)$ for all places $\mathfrak{p}$. This condition is equivalent to $(a,\tau)_\mathfrak{p} = (b,\tau)_\mathfrak{p}$ for all places $\mathfrak{p}$. We can choose the representative in Ω,

$$\beta = \begin{pmatrix} 0 & a \\ -\tau a & 0 \end{pmatrix}.$$

2.3.2 Case $n = 2$

We can choose the representative in Ω,

$$\beta = \begin{pmatrix} & & & a \\ & & b & \\ & -\tau a & & \\ \tau b & & & \end{pmatrix}.$$

Here, a is in $k^\times$ and b is in k. According to the Lemma 2.3.1, if b is zero, this representative is for the degenerate Hermitian matrices. The isometric classes are corresponding to $k^\times/\mathrm{N}(K^\times)$. If b is not zero, the isometric classes are uniquely determined by the number of the positive signatures and the determinant ab.

2.3.3 Hermitian forms over p-adic fields

Theorem 2.3.1 (Jacobowitz [33, Theorem 3.1])**.** *Let V and W be a hermitian spaces over E (E could be any quadratic extension over F). Then $V \simeq W$ if and only if* $\dim V = \dim W$ *and $dV \simeq dW$, where dV and dW are discriminant.*

Let A be a nondegenerate skew-hermitian matrix with entries in E. Skew-hermitian matrices could be equivalently treated as hermitian matrices by $A \to \sqrt{\tau}A$. Applying Theorem 2.3.1,

$$A \simeq \sqrt{\tau}1_n \text{ or } \sqrt{\tau}\mathrm{diag}\{\underbrace{1,\cdots,1}_{n-1},u\},$$

where $u \in \{\varpi, \epsilon\}$, accordingly as E is unramified extension or ramified extension.

2.4 Ikeda Lifting

In this section, we recall a family of irreducible cuspidal representation of $\mathrm{Sp}_{4n}(\mathbb{A})$ by Ikeda in [32], which is usually called the Ikeda lifting. let k be a totally real number field over $\mathbb{Q}$ of degree d. Denote $\mathfrak{S}_\infty$ the set of all archimedean places of k. Let π be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$. Assume that $\pi = \otimes_v \pi_v$ satisfies the following conditions:

(A1) For $v \notin \mathfrak{S}_\infty$, π_v is a degenerate principal series $I_{\mathrm{GL}_2}(\mu_v, \mu_v^{-1})$.

(A2) For $v \in \mathfrak{S}_\infty$, π_v is a discrete series representation with lowest weight $\pm 2\kappa_v$.

(A3) The root number $\varepsilon(1/2, \pi)$ is equal to 1.

Here, the global root number $\epsilon(1/2, \pi)$ is defined by

$$\epsilon(1/2, \pi) = \prod_v \epsilon(1/2, \pi_v),$$

where the local root number $\epsilon(1/2, \pi_v)$ is given by

$$\epsilon(1/2, \pi_v) = \begin{cases} \mu_v(-1) & \text{if } v \notin \mathfrak{S}_\infty, \\ (-1)^{\kappa_v} & \text{if } v \in \mathfrak{S}_\infty. \end{cases}$$

Let $\langle , \rangle_v$ be the Hilbert symbol for k_v. Define a character χ_{-1} of $k_v^\times$ via $\chi_{-1}(x) = \langle -1, x \rangle_v$.

For $v \notin \mathfrak{S}_\infty$, define

$$\Pi_{2n,v} = \Pi(2n, \pi_v) = \mathrm{Ind}_{P_{2n}(k_v)}^{\mathrm{Sp}_{4n}(k_v)}(\chi_{-1}(\det A)^n)\mu(\det(A)).$$

If $v \in \mathfrak{S}_\infty$, let $\Pi_{2n,v}$ be the lowest weight representation of Sp_{4n} with lowest $\mathrm{U}(2n)$-type $(\det)^{\kappa_v + n}$. The restricted tensor product

$$\Pi_{2n} = \otimes_v' \Pi(n, \pi_v),$$

Ikeda proved that Π_{2n} is an irreducible cuspidal automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$ in [32] and is called the *Ikeda lifting* of π. The partial of standard L-function of Π_{2n} is given by,

$$L^S(s,\Pi_{2n}) = \zeta_k(s)\prod_{i=1}^{2n} L(s+n-i+\frac{1}{2},\tau\otimes\chi_{(-1)^n}), \tag{2.4.1}$$

where $\zeta_k(s)$ is the Dedekind zeta function of k.

2.4.1 Spin L-function of Ikeda's lifting

In [67], Schmidt computed the spin L-function of the Ikeda lifting Π of Sp_{4n}. The formula is given by

$$L(s,\Pi,\rho) = \delta(s)\prod_{j=0}^{n}\prod_{\substack{r=j(j-2n)\\ r\equiv j \bmod 2}}^{j(2n-j)} L(s+\frac{r}{2},\pi,\mathrm{Sym}^{n-j})^{\beta(r,j,n)}, \tag{2.4.2}$$

which expresses the spin L-function of the Ikeda lifting in terms of symmetric power L-function of the cuspidal representation π of $\mathrm{PGL}_2(\mathbb{A})$. The function $\delta(s)$ is a polynomial of s, only dependent on the weight of the local representation over the archimedean place. In particular, the L-function $L(s,\pi,\mathrm{Sym}^0)$ is the completed Riemann zeta function.

In this thesis, we only consider the nonarchimedean place, and then have not recalled the definition of $\delta(s)$ here. Let us recall the combinatoric datum in Equation 2.4.2.

$$\alpha(r,j,n) = \#\{(x_1,x_2,\dots,x_j) \mid x_i\in\{1-2n,3-2n,\dots,2n-1\}, \sum_{i=1}^{j} x_i = r,$$
$$\text{and } x_{i_1}\neq x_{i_2} \text{for all } 1\le i,i_1,i_2\le j\}.$$

Let $\beta(r,j,n)=\alpha(r,j,n)-\alpha(r,j-2,n)$. If $j<0$, default $\alpha(r,j,n)=0$.

Example 2.4.1. *We list three examples of $\alpha(r,j,n)$ and $\beta(r,j,n)$ for $n=1,2,3$, which can be found at the appendix of [67]. The entires in the first column are the* *value of* r *and the entries in the first row are the value of* j. *Based on the definition, one can easily find that* $\alpha(r,j,n) = \alpha(r, 2n-j, n)$ *and* $\alpha(r,j,n) = \alpha(-r,j,n)$. *It is enough to consider* $r \geq 0$ *and* $j \leq n$.

Table 2.1: $\alpha(r,j,1)$

(r,j)	0	1
0	1	
1		1

Table 2.2: $\beta(r,j,1)$

(r,j)	0	1
0	1	
1		1

Table 2.3: $\alpha(r,j,2)$

(r,j)	0	1	2
0	1		2
1		1	
2			1
3		1	
4			1

Table 2.4: $\beta(r,j,2)$

(r,j)	0	1	2
0	1		1
1		1	
2			1
3		1	
4			1

Table 2.5: $\alpha(r, j, 3)$

(r,j)	0	1	2	3
0	1		3	
1		1		3
2			2	
3		1		3
4			2	
5		1		2
6			1	
7				1
8			1	
9				1

Table 2.6: $\beta(r, j, 3)$

(r,j)	0	1	2	3
0	1		2	
1		1		2
2			2	
3		1		2
4			2	
5		1		1
6			1	
7				1
8			1	
9				1

Example 2.4.2. *From Equation 2.4.2, we calculate the spin L-function of* Sp_4. *Assume that* π *is an irreducible cuspidal representation of* $\mathrm{PGL}_2(\mathbb{A})$. *Let* π_v *be a principal series* $I_{\mathrm{PGL}_2}(||^{\nu}, ||^{-\nu})$ *of* $\mathrm{PGL}_2(k_v)$ *and* S *contains all the places except the unramified places of the Ikeda lifting* Π. *Therefore, the partial of the spin L-function of* Sp_4 *is*

$$L^S(s, \Pi, \rho) = \prod_{v \notin S} \frac{1}{(1 - q^{-(s-1/2)})(1 - q^{-(s+1/2)})(1 - p^{-(s-\nu)})(1 - p^{-(s+\nu)})}.$$

Example 2.4.3. *Following the notation as above, we state the local factor of the spin L-function of* Sp_8:

$$L_v(s, \Pi_v, \rho) = L(s, \pi_v, \mathrm{Sym}^2) \prod_{i=0}^{3} L(s + \frac{3}{2} - i, \pi_v) \prod_{i=0}^{4} \zeta(s + 2 - i).$$

In details,

$$L(s, \pi_v) = \frac{1}{(1 - p^{-(s-\nu)})(1 - p^{-(s+\nu)})}, \qquad \zeta(s) = \frac{1}{1 - q^{-s}},$$

and

$$L(s, \pi_v, \mathrm{Sym}^2) = \frac{1}{(1 - q^{-(s-2\nu)})(1 - q^{-s})(1 - q^{-(s+2\nu)})}.$$

2.4.2 Case $n = 1$

Now, let us calculate the period integral $I(s, \phi)$ for a cuspidal form ϕ in an irreducible cuspidal automorphic representation Π of Ikeda lifting. By the unfolding in the last section, we conclude that the period integral is summation over the rational isometry classes of the Hermitian forms over k.

In this example, we assume that G is Sp_4. Thus, we can choose $a \in k$ as a representative for each Hermitian form. Further, we normalize the Eisenstein series in the following way

$$E^*(g, s, f) = \zeta_K(s+1)E(g, s, f).$$

Then the normalized Eisenstein series have meromorphic continuation and functional equation. More generally,

$$E^*(g, s, f) = \prod_{i=1}^{[\frac{n}{2}]} \zeta_{K_v}(2s + n + 1 - 2i) \cdot \zeta_{K_v}(s + \frac{n+1}{2})E(g, s, f).$$

In next chapter, we calculate the period integral over the unramified places

$$I_G(s, \phi) = \sum_{a \in K^\times / \ker(\mathrm{N}(K))} \prod_{S_a} I_v(s, a) \prod_{K_v \text{ split}} L_v(s + \frac{1}{2}, \Pi_v, \rho) \prod_{K_v \text{ unram}} I_v(s, a),$$

where $S_a = S \cup \{v \mid |a|_v \neq 1\}$.

Chapter 3

Unramified Calculation

In this chapter, we consider the following local integral

$$\int_{D_\beta(k_v)U_H(k_v)\backslash H(k_v)} \varphi_\beta(g_v) f_p(g_v,s)\,\mathrm{d}g_v$$

from the global integral (2.2.2) and (2.2.3), for an unramified representation

$$I_{\mathrm{Sp}_{4n}}(\chi_{-1}\eta) := \mathrm{ind}_{P_{2n}(F)}^{\mathrm{Sp}_{4n}(F)} \chi_{-1}(\det)^n \eta(\det) |\det|^{\frac{2n+1}{2}},$$

which is the degenerate principal series. In order to simplify the notation, we can assume that $\chi_{-1}\eta = |\ |^\nu$ and then $I_{\mathrm{Sp}_{4n}}(\chi_{-1}\eta) = I_{\mathrm{Sp}_{4n}}(\nu)$. In the last chapter, we know this degenerate principal series are the unramified representation of the cuspidal automorphic representation Π of the Ikeda lifting.

Similarly, If G is the symplectic similitude group GSp_{4n}, we consider the degenerate principal series

$$I_{\mathrm{GSp}_{4n}}(\nu_1,\nu_2) := \mathrm{ind}_{P_{2n}(F)}^{\mathrm{GSp}_{4n}(F)} |\det|^{\nu_1+\frac{2n+1}{2}} \rtimes |\lambda|^{\nu_2-\frac{n(2n+1)}{2}}.$$

Here the character $|\det g|^{\nu_1+\frac{2n+1}{2}} \rtimes |\lambda|^{\nu_2-n(2n+1)}$ is defined over the element

$$\begin{pmatrix} g & \\ & \lambda\hat{g}^{-1}\end{pmatrix},$$

which is slightly different from the induced representation associated to the Eisenstein series.

For a prime ideal $\mathfrak{p}$ in $\mathfrak{o}_k$ corresponding to the place v, the completion local field K_v has the following three cases:

1. K_v is the unramified quadratic extension over k_v;

2. K_v is a ramified quadratic extension over k_v;

3. K_v is split, which is isomorphic to $k_v \times k_v$.

Remark that there are only finitely many prime ideals of k which are ramified in K. The Dirichlet densities of both split prime ideals and unramified prime ideals are $\frac{1}{2}$. Therefore, for the local calculation, we consider the unramified extension K_v and the split case $k_v \times k_v$ first.

In oder to simplify our notation, we use F instead of k_v to denote the local field over a place and E to denote the completion K_v. In this chapter, let G, H, P_G, H_G and R_β, etc, represent the same groups but F-points. Denote by $K_G = \mathrm{Sp}_{4n}(\mathfrak{o}_F)$ (or $\mathrm{GSp}_{4n}(\mathfrak{o}_F)$) a standard maximal open compact subgroup of G. Let (Π, V) be an irreducible unramified representation of G and v_0 be a K fixed vector in V. Let ℓ_β be a linear functional on V such that

$$\ell_\beta(\Pi(r)v) = \alpha_{s,\beta}^{-1}(r)\ell_\beta, \text{ for all } r \in R_\beta(v).$$

Remark that this linear functional arise from Integral (2.2.1) and (2.2.4).

For the **symplectic group** Sp_{4n}, by Equation (2.2.5), we need to calculate the following local integral:

$$I_v(s, \beta) = \int_{D_\beta(F)U_H(F)\backslash H(F)} \varphi_\beta(gk) f_v(k, s)\,\mathrm{d}k. \tag{3.0.1}$$

Here we choose a right H-invariant Haar measure $\mathrm{d}g$, so normalized that $\mathrm{vol}(K) = 1$. Let $f_v(g, s)$ be the unramified vector of

$$I_{H(E)}(s) = \mathrm{ind}_{P_H}^{H(E)} |\det|_E^{s+\frac{n+1}{2}}.$$

By the Iwasawa decomposition,

$$\begin{aligned} I_v(s, \beta) &= \int_{D_\beta \backslash M_H} \ell_\beta(\Pi(g)v_0) |\det g|^{s+\frac{n+1}{2}} \delta_{P_H}(g)^{-1} \mathrm{d}g & (3.0.2) \\ &= \int_{D_\beta \backslash M_H} \ell_\beta(\Pi(g)v_0) |\det g|^{s-\frac{(n+1)}{2}} \mathrm{d}g. & (3.0.3) \end{aligned}$$

In order to calculate the linear functional ℓ_β, we show a straightforward lemma in a general setting.

Lemma 3.0.1. *Let G be a reductive group defined over a nonarchimedean field of F, N be a closed subgroup of G, and M be the normalizer of N in G. Let ψ be a smooth character of N, and N_1 be the kernel of ψ. Then M naturally* *acts on ψ. Let (Π, V) be a smooth representation of G, and $v_0 \in V$ be a K-fixed vector where K is an open compact subgroup of G. Assume that there is a linear functional ℓ in the space* $\mathrm{Hom}_N(\Pi, \psi)$*. If $h \in M$ satisfies that the restriction of ψ on $h(K \cap N)h^{-1}$ is nontrivial, then $\ell(\Pi(h)v_0) = 0$.*

Proof. Let n be in $K \cap N$ such that $\psi(hnh^{-1}) \neq 1$. We have

$$\ell(\Pi(h)v_0) = \ell(\Pi(h)\Pi(n)v_0) = \psi(hnh^{-1})\ell(\Pi(h)v_0).$$

Therefore, $\ell(\Pi(h)v_0) = 0$. $\square$

Define

$$W(g, \beta) = \ell_\beta(\Pi(g)(v_0)),$$

for the unramified vector v_0 of Π. This function $W(g, \beta)$ is called *degenerate Whittaker functions* associated to ℓ_β. For simplicity, we call it the degenerate Whittaker functions in this chaper.

3.1 Unramified Split case

In this section, we consider the split case $k_v(\sqrt{\tau}) \cong k_v \times k_v$. In order to have an easisily readable matrix form, we use another $\mathrm{Sp}_{4n}(F)$-conjugate involution

$$\varepsilon = \mathrm{Ad}\begin{pmatrix} I_n & & \\ & -I_{2n} & \\ & & I_n \end{pmatrix} \text{ instead of } \mathrm{Ad}\begin{pmatrix} & I_n & & \\ \tau I_n & & & \\ & & & I_n \\ & & \tau I_n & \end{pmatrix}.$$

The embedding of $H = \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F)$ into $\mathrm{Sp}_{4n}(F)$ is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} A & & & B \\ & X & Y & \\ & Z & W & \\ C & & & D \end{pmatrix}.$$

Therefore, we can choose the following matrix for the symmetric matrices β associated to the character ψ_β of U_G,

$$\beta_r = \begin{pmatrix} I_r & & \\ & 0_{2n-2r} & \\ & & I_r \end{pmatrix}.$$

Then the stabilizer D_{β_r} is

$$\left\{ m(\begin{pmatrix} A & B & & \\ 0_{(n-r)\times r} & C & & \\ & & X & 0_{(n-r)\times r} \\ & & Y & \hat{A}^{-1} \end{pmatrix}) \in M_G(F) \mid X, C \in \mathrm{GL}_{n-r}(F), A \in \mathrm{GL}_r(F) \right\}.$$

Lemma 3.1.1. *The double coset* $\mathrm{GL}_n^{\triangle}(F)\backslash \mathrm{GL}_n(F) \times \mathrm{GL}_n(F)/\mathrm{GL}_n(\mathfrak{o}) \times \mathrm{GL}_n(\mathfrak{o})$ *is corresponding*

$$\{(\varpi^{a_1}, \varpi^{a_2}, \dots, \varpi^{a_n}) \mid a_1 \geq a_2 \geq \cdots \geq a_n\}.$$

Denote the set

$$\Lambda^+ = \{\vec{a} = (a_1, a_2, \cdots, a_n) \mid a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}.$$

Lemma 3.1.2. *Given a character* η *of* $F^\times$*, we have the following uniqueness,*

$$\dim \mathrm{Hom}_{R_{\beta_r}(F)}(I(\eta), \psi \otimes |\det|^{-s-\frac{n+1}{2}}) = \begin{cases} 1 & \text{if } r = 2n, \\ 1 & \text{if } \eta = |\ |^s \text{ and } r < 2n, \\ 0 & \text{if } \eta \neq |\ |^s \text{ and } r < 2n. \end{cases}$$

Indeed, we define the following linear functional on the degenerate principal series $I(\chi_{-1}\eta)$ as following,

$$\ell_{I_{2n}}(f) = \int_{U_G(F)} f(Jn)\psi_\beta(n)\, \mathrm{d}n.$$

For the case $\beta = I_{2n}$, the degenerate Whittaker function $W(g, I_{2n})$ associated to $\ell_{I_{2n}}$ is well studied by Kudla, Piateiski-Shapiro and Rallis in [63, 44]. Moreover, let

$$b_{2n}(s) = \prod_{k=1}^{n} L(2s + 2n + 1 - 2k)L(s + \frac{2n+1}{2}).$$

Then

$$W(e, \beta) = \frac{L(s + \frac{1}{2}, \chi_\beta)}{b_{2n}(s)},$$

where $\beta \in M_{2n\times 2n}(\mathfrak{o}) \cap \mathrm{GL}_{2n}(\mathfrak{o})$, $|\det(\beta)| = 1$ and

$$\chi_\beta(x) = (x, (-1)^n \det(\beta)).$$

In order to evaluate the integral $I_v(s, I_{2n})$, we need to calculate the value of the degenerate Whittaker function $W(g, \beta)$ on $D_\beta\backslash M_H$. For all $h \in D_\beta(F)$, $g \in M_H(F)$ and $k \in K_G \cap M_H(F)$, we have

$$\begin{aligned}
W(m(hgk), \beta) =& \ell_\beta(r(m(g)) \cdot f) \\
=& \int_{U_G(F)} f(Jn(u)m(g)\psi(\mathrm{tr}\beta u)\,\mathrm{d}u \\
=& \int_{U_G(F)} f(Jm(g)n(g^{-1}u\hat{g}^{-1})\psi(\mathrm{tr}\beta u)\,\mathrm{d}u \\
=& \int_{U_G(F)} f(m(\hat{g}^{-1})Jn(g^{-1}u\hat{g}^{-1})\psi(\mathrm{tr}\beta u)\,\mathrm{d}u \\
=& |\det g|^{-\nu-\frac{(2n+1)}{2}} \int_{U_G(F)} f(Jn(g^{-1}u\hat{g}^{-1})\psi(\mathrm{tr}\beta u)\,\mathrm{d}u \\
=& |\det g|^{-\nu-\frac{(2n+1)}{2}} \delta_{P_G}(m(g)) \int_{U_G(F)} f(Ju)\psi(\mathrm{tr}\beta g u\hat{g})\,\mathrm{d}u \\
=& |\det g|^{-\nu+\frac{(2n+1)}{2}} W(e, \hat{g}\beta g).
\end{aligned}$$

Recall that the character η in the degenerate principal series $I_{\mathrm{Sp}_{4n}}(\chi_{-1}\eta)$ are defined by the character in the local component $I_{\mathrm{GL}_2}(\eta, \eta^{-1})$ of the cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$ and this local component is a unitary representation of $\mathrm{GL}_2(F)$. If we rewrite η as $|\ |^\nu$, then $\mathrm{Re}\mu = 0$ or $\mu \in [-\frac{1}{2}, \frac{1}{2}]$. By Lemma 3.1.2, we have $\ell_{\beta_r}(v) = 0$ for all $v \in I(\chi_{-1}\eta)$ if $s > \frac{1}{2}$.

Now, let us consider that $\beta = I_{2n}$, and then

$$D_\beta(F) \simeq \mathrm{GL}_n(F)^{\triangle} \hookrightarrow M_H(F) \simeq \mathrm{GL}_n(F) \times \mathrm{GL}_n(F).$$

Applying Lemma 3.1.1, for all $m(g) \in M_H(F)$, we have $m(g) = m(htk)$ for some $h \in \mathrm{GL}_n(F)^{\triangle}$, $k \in M_H(F) \cap K_G$ and

$$t = \mathrm{diag}\{\varpi^{m_1}, \varpi^{m_2}, \cdots, \varpi^{m_n}, I_n\} \text{ where } m_1 \geq m_2 \cdots, \geq m_n.$$

By Lemma 3.0.1, we conclude that

$$W(m(t), I_{2n}) = 0 \text{ if } m_n < 0.$$

Referring to Equation (3.0.3), we have

$$\begin{aligned} I_v(s, I_{2n}) &= \int_{\mathrm{GL}_n(F)^{\triangle}\backslash \mathrm{GL}_{2n}(F)\times \mathrm{GL}_{2n}(F)} W(g, I_{2n})|\det g|^{s-\frac{n+1}{2}}\mathrm{d}g \\ &= \sum_{\vec{m}\in\Lambda^+} W(m(t(\vec{m})), I_{2n})q^{-(\sum_{i=1}^n m_i)(s-\frac{n+1}{2})}\mathrm{vol}(\mathrm{GL}_n(\mathfrak{o})t(\vec{m})\mathrm{GL}_n(\mathfrak{o})) \\ &= \sum_{\vec{m}\in\Lambda^+} W(e, B(\vec{m}))q^{-(\sum_{i=1}^n m_i)(s-\nu+\frac{n}{2})}\mathrm{vol}(\mathrm{GL}_n(\mathfrak{o})t(\vec{m})\mathrm{GL}_n(\mathfrak{o})), \quad (3.1.1) \end{aligned}$$

where

$$B(\vec{m}) = \mathrm{diag}\{\varpi^{m_1}, \varpi^{m_2}, \cdots, \varpi^{m_n}, \varpi^{m_n}, \cdots, \varpi^{m_1}\}.$$

Next, we will apply the explicit formula for $W(e, B)$ to evaluate the above integral.

3.2 An explication formula for Siegel series

In this section, we introduce the explicit formula for the Fourier coefficients of the local Siegel series in the work of Katsurada [39]. First, let recall some notation in [39].

For a prime number $p \neq 2$, let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\mathbb{Z}_p$ be the valuation ring. Let $S_n(\mathbb{Z}_p)$ be the set of $n \times n$ symmetric matrices with entries in $\mathbb{Z}_p$. Denote by ord the normalized additive valuation on $\mathbb{Q}_p$. Let ψ_0 be a fixed unramified additive character of $\mathbb{Q}_p$. Remark that in our notation the matrix β is symmetric with respect to the anti-diagonal.

For $B \in S_n(\mathbb{Z}_p)$ we define the local Siegel series $b(B,s)$ by

$$b_p(B,s) = \sum_{R \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \psi_0(\mathrm{tr}(B \cdot R)) \cdot p^{-\mathrm{ord}(\mu(R))s}, \tag{3.2.1}$$

where $\mu(R)$ is the product of denominators of elementary divisors of R.

Define a function χ_p on ${\mathbb{Q}_p}^{\times}$ via

$$\chi_p(x) = \begin{cases} 1 & \text{if } \mathbb{Q}_p(\sqrt{x}) = \mathbb{Q}_p, \\ -1 & \text{if } \mathbb{Q}_p(\sqrt{x}) \text{ is the unramified field extension,} \\ 0 & \text{if } \mathbb{Q}_p(\sqrt{x}) \text{ is a ramified field extension.} \end{cases}$$

Let B be in $S_n(\mathbb{Z}_p)$ and $\det(B) \neq 0$. Define a polynomial $\gamma_p(B;X)$ in X by

$$\gamma_p(B,X) = \begin{cases} (1-X)\Pi_{i=1}^{n/2}(1-p^{2i}X^2)(1-p^{n/2}\xi_p(B)X)^{-1} & \text{if } n \text{ is even} \\ (1-X)\Pi_{i=1}^{(n-1)/2}(1-p^{2i}X^2) & \text{if } n \text{ is odd} \end{cases}.$$

where $\xi_p(B) = \chi_p((-1)^{n/2}\det B)$.

Let $\langle , \rangle_p$ be the Hilbert symbol over $\mathbb{Q}_p$ and h_p the Hasse invariant of the symmetric matrices. Now, we introduce the following notation,

$$\eta(B) = h_p(B) \cdot (\det B, (-1)^{(n-1)/2}\det B)_p, \text{ if } n \text{ is odd,} \tag{3.2.2}$$

$$\xi(B) = \xi_p(B), \text{ if } n \text{ is even,} \tag{3.2.3}$$

$$d(B) = \mathrm{ord}(\det(B)), \tag{3.2.4}$$

$$\delta(B) = \begin{cases} 2[(d(B)+1)/2] & \text{if } n \text{ is even} \\ d(B) & \text{if } n \text{ is odd,} \end{cases} \tag{3.2.5}$$

$$e(B) = \begin{cases} \delta(B) - 2 + 2\xi(B)^2 & \text{if } n \text{ is even} \\ \delta(B) & \text{if } n \text{ is odd.} \end{cases} \tag{3.2.6}$$

For a matrix B of even degree, let $\xi'(B) = 1 + \xi(B) - \xi(B)^2$. Here we make the convention that $\xi(B) = \xi'(B) = 1$ if B is the empty matrix.

As a quadratic form over $\mathbb{Z}_p$-module, B is $\mathbb{Z}_p$-equivalent to

$$J(B) := p^{e_1}U_1 \perp \cdots \perp p^{e_s}U_s,$$

where $e_1 > e_2 > \cdots > e_s \geq 0$ and

$$U_i = \underbrace{1 \perp 1 \cdots \perp 1}_{n_i - 1} \perp u_i$$

with $u_i \in \{1, \epsilon\}$. Here τ is a non-square element in $\mathbb{Z}_p$. Let b_i be the i-th diagonal component of $J(B)$ and $J_i = b_i \perp \cdots \perp b_n$. For each $1 \leq i \leq n$, define

$$d_i(B) = d(J_i) \text{ and } \delta_i(B) = \delta(J_i).$$

We note that $d_i(B) = \mathrm{ord}(a_1 \cdots a_{n-i+1})$, where $a_1, \cdots, a_n$ are elementary divisors of B such that $a_1|a_2 \cdots |a_n$. Now we define invariants $\xi_i(B)$ for $1 \leq i \leq [n/2]$ and $\eta_i(B)$ for $1 \leq i \leq [(n-1)/2]$ of B by

$$\xi_i(B) = \begin{cases} \xi(J_{2i-1}) & \text{if } n \text{ is even} \\ \xi(J_{2i}) & \text{if } n \text{ is odd} \end{cases} \tag{3.2.7}$$

$$\eta_i(B) = \begin{cases} \eta(J_{2i}) & \text{if } n \text{ is even} \\ \eta(J_{2i-1}) & \text{if } n \text{ is odd} \end{cases} \tag{3.2.8}$$

Example 3.2.1. *Let* $B = \mathrm{diag}\{\varpi^{m_1}, \tau\varpi^{m_1}, \ldots, \varpi^{m_n}, \tau\varpi^{m_n}\}$ *and* $|\tau| = 1|$. *Then if* n *is even,* $\xi_{2i} = \chi(-\tau)$ *and* $\xi_{2i-1} = 1$ *for* $1 \leq i \leq [\frac{n+1}{2}]$.

If n *is odd,* $\xi_{2i} = 1$ *and* $\xi_{2i-1} = \chi(-\tau)$ *for* $1 \leq i \leq [\frac{n+1}{2}]$. *For all* n, $\xi'_i = \xi_i$.

If n *is even, then*

$$\delta_{2i-1} = 2\sum_{j=i}^{n} m_j$$

$$\delta_{2i} = m_i + 2\sum_{j=i+1}^{n} m_j$$

Katsurada gave the following explicit formula for

$$F_p(B;p^{-s}) = \frac{b_p(B,s)}{\gamma_p(B;p^{-s})}. \tag{3.2.9}$$

Equation (3.2.9), we have an explicit formula for the Fourier coefficients of the local Siegel series.

Theorem 3.2.1 (Theorem 4.3 [39])**.** *Let B be a non-degenerate symmetric integral matrix of degree n over $\mathbb{Z}_p$, and write ξ_i, δ_i etc. instead of $\xi_i(B)$, $\delta_i(B)$ etc.*

If n is even,

$$F(B;p^{-s}) = \sum_{(j_1,\cdots,j_n)} \prod_{i=1}^{n/2} [\frac{1-\xi_i p^{n/2+1-i+j_1+\cdots+j_{2i-2}+j_{2i-1}-s}}{1-p^{n+3-2i+2j_1+\cdots+2j_{2i-2}-2s}}$$
$$\times \{(-1)^{\xi_{i+1}}\xi_i'\eta_i(p^{n/2+1-i+j_1+\cdots+j_{2i-2}-s})^{\delta_{2i-1}-\delta_{2i}+\xi_i^2}p^{\delta_{2i-1}/2}\}^{1-j_{2i-1}}]$$
$$\times \prod_{i=1}^{n/2} \frac{\{(-1)^{\xi_i+1}\xi_{i+1}'\eta_i(p^{n/2-i+j_1+\cdots+j_{2i-1}-s})^{\delta_{2i}-\delta_{2i+1}+2-\xi_{i+1}^2}p^{(2\delta_{2i}-\delta_{2i+1}+2)/2}\}^{1-j_{2i}}}{1-\xi_{i+1}p^{n/2+1-i+j_1+\cdots+j_{2i-1}-s}}.$$

If n is odd,

$$F(B;p^{-s}) = \sum_{(j_1,\cdots,j_n)} \prod_{i=1}^{(n-1)/2} [\frac{1-\xi_i p^{(n+3)/2-i+j_1+\cdots+j_{2i-1}-j_{2i}-s}}{1-p^{n+2-2i+2j_1+\cdots+2j_{2i-1}-2s}}$$
$$\times\{(-1)^{\xi_i+1}\xi_i'\eta_{i+1}(p^{(n+1)/2-i+j_1+\cdots+j_{2i-1}-s})^{\delta_{2i}-\delta_{2i+1}+\xi_i^2}p^{\delta_{2i}/2}\}^{1-j_{2i}}]$$
$$\times \prod_{i=1}^{(n+1)/2} \frac{\{(-1)^{\xi_i}\xi_i'\eta_i(p^{(n+1)/2-i+j_1+\cdots+j_{2i-2}-s})^{\delta_{2i-1}-\delta_{2i}+2-\xi_i^2}p^{(2\delta_{2i-1}-\delta_{2i}+2)/2}\}^{1-j_{2i-1}}}{1-\xi_i p^{(n+3)/2-i+j_1+\cdots+j_{2i-2}-s}}.$$

Here $(j_1,\cdots,j_n)$ runs over all elements of $\{0,1\}^n$. We make the convention that $\xi_{[(n+2)/2]} = \xi'_{[(n+2)/2]} = \eta_{[(n+1)/2]} = 1$ and $\delta_{n+1} = 0$, and that $j_1 + \cdots j_{2i-2} = 2j_1 + \cdots + 2j_{2i-2} = 0$ if i=1 and $\prod_{i=1}^{(n-1)/2}(*) = 1$ if $n = 1$.

Example 3.2.2. *If $n = 1$ and $B = p^m$, then*

$$F(B;X) = \frac{1-(pX)^{m+1}}{1-pX}.$$

Example 3.2.3. *In the case $n = 2$ and $B = B(m) = \mathrm{diag}\{\varpi^m, \tau\varpi^m\}$, we have the following explicit formula.*

- *When* $|\tau| = 1$,

$$F(B,X) = \frac{(1-\chi_p(-\tau)pX)(1-(p^2X)^{m+1})}{(1-p^3X^2)(1-p^2X)} + \frac{\chi_p(-\tau)p^{2m+1}X^{m+1}(1-\chi_p(-1)p^2X)(1-(pX)^{m+1})}{(1-p^3X^2)(1-pX)}.$$

- *When* $|\tau| = p$,

$$F(B,X) = \frac{1-(p^2X)^{m+1}}{(1-p^3X^2)(1-p^2X)} - \frac{p^{2m+3}X^{m+2}(1-(pX)^{m+1})}{(1-p^3X^2)(1-pX)}$$

Example 3.2.4. *Let* $B = \mathrm{diag}\{\varpi^{m_1}, -\varpi^{m_1}, \varpi^{m_2}, -\varpi^{m_2}\}$*, where* $m_1 \geq m_2$*. Then*

$$\xi_1 = \xi_2 = 1,$$
$$\delta_1 = 2m_1 + 2m_2,\ \ \delta_2 = m_1 + 2m_2,\ \ \delta_3 = 2m_2,\ \ \delta_4 = m_2,$$
$$\eta_1 = \langle -\varpi^{m_1}, \varpi^{m_2}\rangle_p \langle -\varpi^{m_1}, -\varpi^{m_2}\rangle_p,\ \ \eta_2 = 1.$$

Furthermore, referring to Table 3.1 and Table 3.2, we have

$$\eta_1 = \begin{cases} 1 & \text{if } -1 \in F^2, \\ (-1)^{m_1} & \text{if } -1 \notin F^2. \end{cases}$$

Therefore,

$$\begin{aligned} F(B,p^{-s}) = & \frac{(1-p^{2-s})}{(1-p^{5-2s})(1-p^{3-s})} \times \\ & \left[\frac{(1-p^{3-s})(1-p^{(4-s)(m_2+1)})}{(1-p^{7-2s})(1-p^{4-s})} + \frac{p^{m_2}p^{(3-s)(m_2+1)}(1-p^{4-s})(1-p^{(3-s)(m_2+1)})}{(1-p^{7-2s})(1-p^{3-s})}\right] \\ & + \left[\frac{\eta_1 p^{m_1+m_2}p^{(2-s)(m_1+1)}(1-p^{3-s})}{(1-p^{5-2s})(1-p^{2-s})} - \frac{p^{m_1+m_2+1}p^{(2-s)(m_1+1)}(1-p^{2-s})}{(1-p^{5-2s})(1-p^{3-s})}\right] \times \\ & \left[\frac{(1-p^{2-s})(1-p^{(3-s)(m_2+1)})}{(1-p^{5-2s})(1-p^{3-s})} + \frac{p^{m_2}p^{(2-s)(m_2+1)}(1-p^{3-s})(1-p^{(2-s)(m_2+1)})}{(1-p^{5-2s})(1-p^{2-s})}\right] \\ & - \frac{\eta_1 p^{2m_1+2m_2+1}p^{2(2-s)(m_1+1)}(1-p^{3-s})}{(1-p^{5-2s})(1-p^{2-s})} \times \\ & \left[\frac{(1-p^{1-s})(1-p^{(2-s)(m_2+1)})}{(1-p^{3-2s})(1-p^{2-s})} + \frac{p^{m_2}p^{(1-s)(m_2+1)}(1-p^{2-s})(1-p^{(1-s)(m_2+1)})}{(1-p^{3-2s})(1-p^{1-s})}\right]. \end{aligned}$$

The following tables are the Hilbert symbols for a p-adic field F of characteristic 0.

Table 3.1: $-1 \in F^2$

	1	ϵ	ϖ	$\epsilon\varpi$
1	1	1	1	1
ϵ	1	1	-1	-1
ϖ	1	-1	1	-1
$\epsilon\varpi$	1	-1	-1	1

Table 3.2: $-1 \notin F^2$

	1	ϵ	ϖ	$\epsilon\varpi$
1	1	1	1	1
ϵ	1	1	-1	-1
ϖ	1	-1	-1	1
$\epsilon\varpi$	1	-1	1	-1

3.3 Case Sp_4

In this section, we only consider that τ is split over p and $|\tau|_v = |a|_v = 1$. Then, we continue to calculate Equation (3.1.1) and obtain,

$$
\begin{aligned}
I_v(s, I_{2n}) &= \sum_{\vec{m}\in\Lambda^+} W(e, B(\vec{m}))q^{-(\sum_{i=1}^{n} m_i)(s-\nu+\frac{n}{2})}\mathrm{vol}(\mathrm{GL}_n(\mathfrak{o})t(\vec{m})\mathrm{GL}_n(\mathfrak{o})) \\
&= \sum_{m=0}^{\infty} F(\mathrm{diag}\{p^m, -p^m\}, p^{-(\nu+\frac{3}{2})})p^{-m(s-\nu+\frac{1}{2})} \\
&= \sum_{m=0}^{\infty} \Bigg(\frac{(1-pX)(1-(p^2X)^{m+1})}{(1-p^3X^2)(1-p^2X)} \\
&+ \frac{p^{2m+1}X^{m+1}(1-p^2X)(1-(pX)^{m+1})}{(1-p^3X^2)(1-pX)} \Bigg) p^{-m(s-\nu+\frac{1}{2})} \\
&= \frac{(1-pX)}{(1-p^3X^2)(1-p^2X)(1-p^{-(s-\nu+\frac{1}{2})})} - \frac{(1-pX)p^2X}{(1-p^3X^2)(1-p^2X)(1-p^{-s})} \\
&+ \frac{(1-p^2X)pX}{(1-p^3X^2)(1-pX)(1-p^{-s})} - \frac{(1-p^2X)(pX)^2}{(1-p^3X^2)(1-pX)(1-p^{-(s+\nu+\frac{1}{2})})} \\
&= \frac{1-p^{-s-1}}{(1-p^{-s})(1-p^{-s-v-\frac{1}{2}})(1-p^{-s+v-\frac{1}{2}})}
\end{aligned}
$$

where $X = p^{-(\nu+\frac{3}{2})}$.

3.4 Case GSp_4

In order to verify our calculation, we apply our formulation into the GSp_4 case and recalculate it. The result also can be found in the work of Piatetski-Shapiro [59].

If G is GSp_4, then the set $Herm/k^\times$ contains only one element and we take $\beta = 1$ as a representative in the orbit $Herm/k^\times$. Since β is trivial, we omit β as a subscript or a variable in the integral. Therefore, the stabilizer D as a subgroup of G is

$$\left\{ \begin{pmatrix} a & b & & \\ \tau b & a & & \\ & & a & -b \\ & & -\tau b & a \end{pmatrix} \right\}.$$

In this case, the involution ε is

$$\begin{pmatrix} & 1 & & \\ \tau & & & \\ & & & 1 \\ & & \tau & \end{pmatrix},$$

where τ is a non-square unit element or the identity element. Similar to Lemma 3.1.2, we also can construct the degenerate Whittaker functions and have the following lemma for the symplectic similitude group.

Lemma 3.4.1. $\dim \mathrm{Hom}_R(I_{\mathrm{GSp}_4}(\nu_1, \nu_2), \alpha_{s_1,s_2}^{-1}) = 1$ *if and only if* $\nu_1 + \nu_2 = s_2 - s_1$.

In order to obtain a nonzero linear integral, let $s = s_2$ and $s_1 = s - \nu_1 - \nu_2$. Since for both cases of τ, we have

$$D\backslash M_H \simeq \left\{ \mathbf{t} = \begin{pmatrix} t & & & \\ & t & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

Therefore, applying the calculation similar to the Sp_4 case, we have

$$\begin{aligned}I_v(s) &= \int_{D\backslash M_H} W(\mathbf{t})|t|^{s-1}\,\mathrm{d}t \\ &= \int_{F^\times} F(\begin{pmatrix} t & \\ & -\tau t \end{pmatrix}, X)|t|^{\nu_2+\frac{3}{2}}|t|^{s-1}\,\mathrm{d}t \\ &= \frac{1-\chi(\tau)p^{-(s+\nu_1+\nu_2+1)}}{(1-p^{-(s+\nu_1+\nu_2)})(1-p^{-(s+\nu_1+\frac{1}{2})})(1-p^{-(s+2\nu_1+\nu_2+\frac{1}{2})})}.\end{aligned}$$

In this calculation, we apply the fact that the local Siegel series $F(B,X)$ for GSp_4 are same as these for Sp_4. In addition, for the Eisenstein series of H, we use $\zeta_K(2s-s_1+1)$ to normalize the Eisenstein series. Thus,

$$I_v(s)\zeta_{K_v}(2s-s_1+1) = L(s+\frac{1}{2},\Pi_v,r),$$

where the L-function is the standard L-function of GSp_4 of degree 4. Remark that this result is compatible with the result of Piatetski-Shapiro [59].

Chapter 4

Uniqueness of the Symmetric Pairs

In this chapter, we attempt to prove that the symmetric pairs

$$(\mathrm{Sp}_{4n}(k_v), \mathrm{Sp}_{2n}(K_v)) \text{ and } (\mathrm{GSp}_{4n}(k_v), \mathrm{GSp}_{2n}(K_v)^\circ)$$

are Gelfand pairs. Considering a descendant of $(\mathrm{Sp}_{4n}(k_v), \mathrm{Sp}_{2n}(k_v) \times \mathrm{Sp}_{2n}(k_v))$, we also prove that $(U(J_{2n}, k_v(\sqrt{\tau}), \mathrm{Sp}_{2n}(k_v))$ is a Gelfand pair for both archimedean and non-archimedean fields. The most content of this chapter are discussed by the author in Zhang [77].

4.1 Overview

The theory of Gelfand pairs has a wide range of applications in harmonic analysis on symmetric spaces, automorphic forms and L-functions. For instance, the uniqueness of Whittaker models for quasi-split reductive groups over local fields plays an important role in the Langlands-Shahidi method and the Rankin-Selberg method to study automorphic L-functions.

The main tool to verify whether a pair (G, H) is a Gelfand pair is the Gelfand-Kazhdan criterion for groups defined over complex, real, and p-adic fields. Bernstein and Zelevinsky in [13] give a localization principle for p-adic fields to verify the assumption in the Gelfand-Kazhdan criterion. Some important Gelfand pairs are found based on the Bernstein-Zelevinsky localization principle, such as $(\mathrm{GL}_{2n}(F), \mathrm{Sp}_{2n}(F))$ in [31], and Shalika model in [56], to mention a few.

Following Bernstein's ideas, Aizenbud and Gourevitch applied the Luna Slice Theorem to generalize the descent technique due to Harish-Chandra to the case of a reductive group acting on a smooth affine variety. By applying this new method which works for arbitrary local field F of characteristic 0, they formulated an approach to prove that certain symmetric pairs were Gelfand pairs in [4]. This idea was also used by Jacquet and Rallis to prove the uniqueness of linear periods in [35].

Following their theorem in [4], if a symmetric pair is both a good pair and all descendants are regular, then it is a Gelfand pair. Here a symmetric pair (G, H, θ)

means that a symmetric subgroup H consists of the fixed elements of an involution θ. By this method, they proved an archimedean analogue of uniqueness of linear periods and the multiplicity one theorem for $(\mathrm{GL}(n+1,F),\mathrm{GL}(n,F))$ in [5].

However, not all symmetric pairs are Gelfand pairs. For instance, $(\mathrm{SL}_{2n}(E),\mathrm{SL}_{2n}(F))$ and $(U(1,1),T(F))$ are not Gelfand pairs, as proved in [9] and [28] respectively. Even if π is a supercuspidal representation, there still exists such a π that $\dim\ \mathrm{Hom}_H(\pi,\mathbb{C})>1$.

In this chapter, we apply the generalized Harish-Chandra descent method of Aizenbud and Gourevitch to prove that certain families of symmetric pairs are Gelfand pairs. When no confusion is possible, we use (G,H) instead of (G,H,θ).

Let

$$J'_{4n}=\begin{pmatrix} J_{2n} & \\ & J_{2n}\end{pmatrix}$$

For the non-split case, in order to easily calculate the H-conjugation on the tangent space of $H\backslash G$, we realize $\mathrm{Sp}_{4n}(F)$ as a subgroup of $\mathrm{GL}_{4n}(E)$. That is,

$$\mathrm{Sp}_{4n}(F)=\{g=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix}\in \mathrm{GL}_{4n}(E)\mid gJ'_{4n}g^t=J'_{4n}\}.$$

We denote the embedding $\iota:\mathrm{Sp}_{4n}(J'_{4n},F)\hookrightarrow \mathrm{GL}_{4n}(E)$. The involution is $\mathrm{Ad}(\varepsilon)$, where $\varepsilon=\mathrm{diag}\{\sqrt{\tau}1_{2n},-\sqrt{\tau}1_{2n}\}$, and the symmetric subgroup H is $\iota(\mathrm{Sp}_{2n}(J_{2n},E))$.

Using the similar description, one have the symmetric pairs

$$(\mathrm{GSp}_{4n}(F),(\mathrm{GSp}_{2n}(F)\times\mathrm{GSp}_{2n}(F))^\circ)\text{ and }(\mathrm{GSp}_{4n}(F),\mathrm{GSp}_{2n}(E)^\circ)$$

for the split case and the non-split case respectively. Here $(\mathrm{GSp}_{2n}(F)\times\mathrm{GSp}_{2n}(F))^\circ$ consists of pairs of elements in $\mathrm{GSp}_{2n}(F)$ with same similitude and $\mathrm{GSp}_{2n}(E)^\circ$ consists of elements in $\mathrm{GSp}_{2n}(E)$ with similitude in $F^\times$. The involutions for both of them are same as the previous cases.

Define

$$U(J_{2n},E)=\{g\in M_{2n\times 2n}(E)\mid \overline{g}^tJ_{2n}g=J_{2n}\}.$$

The involution θ is induced from the action of the nontrivial Galois element in the quadratic field on each matrix entry. Then the symmetric subgroup H is

$\mathrm{Sp}_{2n}(J_{2n}, F)$. The symmetric pair $(U(J_{2n}, E), \mathrm{Sp}_{2n}(F))$ appears as a descendant of $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$.

To prove that a symmetric pair (G, H, θ) is a Gelfand pair, it is sufficient to show that all H-invariant distribution on G is invariant under some anti-involution of G. This main approach is due to the Gelfand-Kazhdan criterion [22] for non-archimedean F and a version of the Gelfand-Kazhdan criterion for archimedean F which is given by Aizenbud, Gourevitch and Sayag in [8]. Sun and Zhu also proved a generalized version of the Gelfand-Kazhdan criterion for the archimedean case in [73].

Applying Aizenbud and Gourevitch's generalized Harish-Chandra decent in [4], one reduces to proving that a pair is a good pair and all descendants are regular instead of proving that all H-invariant distributions on G are invariant under some anti-involution of G. A good pair means that every closed orbit with respect to the left and right translation of H is preserved by an anti-involution of G. In order to show a symmetric pair is a good pair, one can calculate the Galois cohomology of the centralizer groups of semi-simple elements in the symmetric subgroup. If the cohomology is trivial, then the pair is good. We show in this manner that $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$, $(\mathrm{GSp}_{4n}(F), \mathrm{GSp}_{2n}(E)^{\circ})$ and $(U(J_{2n}, E), \mathrm{Sp}_{2n}(F))$ are good, for any local field F of characteristic 0.

A descendant consists of the centralizer of certain semi-simple elements in G and the restriction of the involution in the centralizer, and it is also a symmetric pair. A symmetric pair (G, H, θ) is regular if all H conjugation invariant distributions on the tangent space of the symmetric space G/H are preserved by the adjoint actions of all admissible elements. A usual way to prove the regularity of a symmetric pair is based on considering the support and homogenity of H-invariant distributions. One can see Theorem 4.3.4 for any local field of characteristic 0 and Theorem 4.3.1 for non-archimedean fields.

In the sequel, we need to calculate all descendants of the symmetric pairs and prove them to be regular. Explicitly, the descendants of the symmetric pair

$(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$ are

$$(U(J_{2m}, E), \mathrm{Sp}_{2m}(F)), (GL_{2m}(F), \mathrm{Sp}_{2m}(F)) \text{ and } (\mathrm{Sp}_{4m}(F), \mathrm{Sp}_{2m}(E))$$

and the descendants of GSp case are similar to those of Sp case. The symmetric pair $(\mathrm{GL}_{2m}(F), \mathrm{Sp}_{2m}(F))$ is a Gelfand pair, as proved in [31] for the non-archimedean case and in [7] for the archimedean case. These proofs also imply the regularity of this pair. Therefore we show

Theorem 4.1.1. *The symmetric pair $(U(J_{2n}, E), \mathrm{Sp}_{2n}(F))$ is a Gelfand pair for both archimedean and non-archimedean fields.*

Applying Theorem 4.3.1, we show the following theorem.

Theorem 4.1.2. *All of these following pairs are Gelfand pairs for non-archimedean fields,*

1. $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F))$ *when* $n \leq 2$*;*
2. $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(F) \times \mathrm{GSp}_{2n}(F))^{\circ})$ *when* $n \leq 2$*;*
3. $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$ *for all* n*;*
4. $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(E))^{\circ})$ *for all* n*.*

If $E = F \times F$ for $n > 2$, this method does not work for $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F))$. For instance, when $n = 3$, we can find a distinguished orbit such that the homogenity of this orbit is equal to $\frac{1}{2} \dim \mathfrak{g}^{\sigma}$, which is the same as the homogenity of the invariant distributions of the nilpotent cone. The regularity of this symmetric pair is still in progress. Assume this symmetric pair is regular, then it is a Gelfand pair.

4.2 Gelfand Pairs

In this section, we introduce some notation, definitions and a usual process to prove that a pair is a Gelfand pair. First, let us recall some techniques and terminologies due to Aizenbud and Gourevitch. For more details, see [4].

About basic definitions and facts on representations of p-adic groups, I will refer to the reference [13]. Let X be an l-space. Denote by $\mathfrak{S}(X)$ the space of Schwartz functions on X and $\mathfrak{S}^*(X) := \mathfrak{S}(X)^*$ to be the dual space to $\mathfrak{S}(X)$. For a group G acting on a set X and an element $x \in X$ we denote by G_x the stabilizer of x and X^G the fixed points of G.

We recall the definition of Gelfand pairs in [4]. Let $H \subset G$ be a closed subgroup of G. (G, H) is a **Gelfand pair** if for any irreducible admissible representation (π, V) of G, we have

$$\dim \ \mathrm{Hom}_H(\pi, \mathbb{C}) \leq 1.$$

By an **admissible representation** of G, we mean a smooth admissible representation of $G(F)$ over a non-archimedean F.

To prove that a pair is a Gelfand pair, one can apply the following generalized Gelfand-Kazhdan criterion which is generalized for both archimedean and non-archimedean fields, and has an inequality for the dimension of H-invariant linear functionals over any irreducible admissible representation.

Theorem 4.2.1 (Theorem 8.1.4 [4]). *Let H be a closed subgroup of a reductive group G and let τ be an anti-involution of G and such that $\tau(H) = H$. Suppose $\tau(T) = T$ for all bi H-invariant Schwartz distributions T on $G(F)$. Then for any irreducible admissible representation (π, V) of G, we have*

$$\dim \ \mathrm{Hom}_H(\pi, \mathbb{C}) \cdot \dim \ \mathrm{Hom}_H(\tilde{\pi}, \mathbb{C}) \leq 1,$$

where $\tilde{\pi}$ is the smooth contragredient representation of π.

Theorem 4.2.2 (Theorem 8.2.1 [4]). *Let G be a reductive group and let σ be an $Ad(G)$-admissible anti-involution of G. Let θ be the involution of G defined by $\theta(g) := \sigma(g^{-1})$. Let (π, V) be an irreducible admissible representation of G. Then $\tilde{\pi} \cong \pi^\theta$, where π^θ is defined as twisting π by θ.*

Here for a smooth affine variety X which is associated an action π of a reductive group G, an algebraic automorphism τ on X is called G-**admissible** if

1. $\pi(G(F))$ is of index at most 2 in the group of automorphisms of X generated by $\pi(G(F))$ and τ.

2. For any closed $G(F)$ orbit $O \subset X(F)$, we have $\tau(O) = O$.

For our symmetric pairs, it is a well-known result, as Theorem 4.2.2, that there is an involution θ' such that $\pi^{\theta'} = \tilde{\pi}$. We can choose a suitable θ' such that θ' stabilizes our symmetric subgroups respectively. Hence we have dim $\mathrm{Hom}_H(\pi, \mathbb{C}) =$ dim $\mathrm{Hom}_H(\tilde{\pi}, \mathbb{C})$. Therefore, it is enough to show that all H-invariant distributions over G are invariant under the anti-involution σ. To prove this, let us introduce generalized Harish-Chandra descent method in [4].

Definition 4.2.1. *A symmetric pair (G, H, θ) is called* ***good pair*** *if for any closed $H \times H$ orbit $O \in G$, where $H \times H$ acts on G by left and right translation, we have $\sigma(O) = O$, where σ is the anti-involution on G defined by $\sigma(g) := \theta(g^{-1})$.*

Due to Rader and Rallis [62], if a reductive group acts on an affine variety, an orbit is closed in the analytic topology if and only if this orbit is closed in the Zariski topology. Therefore, there is no confusion when we say a closed orbit.

For a symmetric pair, denote $\mathfrak{g} = Lie(G), \mathfrak{h} = Lie(H)$. Let θ and σ act on $\mathfrak{g}$ by their differentials and denote $\mathfrak{g}^\sigma = \{a \in \mathfrak{g} \mid \theta(a) = -a\}$. Note that H acts on $\mathfrak{g}^\sigma$ by the adjoint action. Denote $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ and define a symmetrization map $\rho : G \to G$ by $\rho(g) = g\sigma(g)$. The symmetrization map induces a homeomorphism from G/H to $\mathrm{Im}\rho$ and translates the left multiplication of H on G/H to H conjugation on $\mathrm{Im}\rho$. Therefore the $H \backslash G/H$ are bijective to H-conjugate classes on $\mathrm{Im}\rho$. Let us describe $\mathfrak{g}^\sigma$ for $\mathrm{Sp}_{4n}(F)$ in the split case and the non-split case,

1. $\mathfrak{g}^\sigma = \left\{ \begin{pmatrix} & \beta \\ -\tilde{\beta} & \end{pmatrix} \mid \beta \in M_{2n \times 2n}(F) \right\}$, if $K_v \simeq F \times F$;

2. $\mathfrak{g}^\sigma = \left\{ \begin{pmatrix} & \beta \\ -\tilde{\beta} & \end{pmatrix} \mid \beta \in M_{2n \times 2n}(E), \bar{\beta} + \tilde{\beta} = 0 \right\}$, if $K_v \simeq E$.

Denote $\tilde{\beta} = J_{2n}\beta^t J_{2n}^{-1}$ throughout this chapter. Note that $\mathfrak{g}^\sigma$ are same for $\mathrm{Sp}_{4n}(F)$ and $\mathrm{GSp}_{4n}(F)$.

Definition 4.2.2. *Let x be a semi-simple element and $x = \rho(g)$ for some g. We will say that the pair $(G_x, H_x, \theta|_{G_x})$ is a **descendant** of (G, H, θ), where G_x (resp. H_x) is the centralizer of x in G (resp. H).*

Denote $Q(\mathfrak{g}^\sigma) := \mathfrak{g}^\sigma/(\mathfrak{g}^\sigma)^H$. Since H is reductive, there is a canonical embedding $Q(\mathfrak{g}^\sigma) \hookrightarrow \mathfrak{g}^\sigma$. Let $\phi : \mathfrak{g}^\sigma \to \mathfrak{g}^\sigma/H$ be the standard projection. Denote $\Gamma(\mathfrak{g}^\sigma) := \phi^{-1}(\phi(0)) \cap Q(\mathfrak{g}^\sigma)$. Denote also $R(\mathfrak{g}^\sigma) := Q(\mathfrak{g}^\sigma) - \Gamma(\mathfrak{g}^\sigma)$. Applying to Gelfand pairs, we need to consider $Ad(H)$-invariant distributions over $Q(\mathfrak{g}^\sigma)$. By Lemma 5.2.2 (ii) in [3], we have $Q(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$ as $G = \mathrm{Sp}_{4n}(F)$. Since $\mathfrak{g}^\sigma$ are same for $\mathrm{Sp}_{4n}(F)$ and $\mathrm{GSp}_{4n}(F)$, $Q(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$ is also true when $G = \mathrm{GSp}_{4n}(F)$. Further, applying the Lemma 7.3.8 [4], $\Gamma(\mathfrak{g}^\sigma)$ consists of all nilpotent elements in $\mathfrak{g}^\sigma$.

Definition 4.2.3. *A symmetric pair (G, H, θ) is called **regular** if for any admissible $g \in G$, which means $\rho(g) \in Z(G)$ (the center of G) and $Ad(g)|_{\mathfrak{g}^\sigma}$ is H-admissible, such that $\mathfrak{S}^*(R(\mathfrak{g}^\sigma))^H \subset \mathfrak{S}^*(R(\mathfrak{g}^\sigma))^{Ad(g)}$, we have*

$$\mathfrak{S}^*(Q(\mathfrak{g}^\sigma))^H \subset \mathfrak{S}^*(Q(\mathfrak{g}^\sigma))^{Ad(g)}.$$

Applying generalized Harish-Chandra method, Aizenbud and Gourevitch [4] have the following theorem.

Theorem 4.2.3 (Theorem 7.4.5[4])**.** *Let (G, H, θ) be a good symmetric pair such that all its descendants are regular. Then*

$$\mathfrak{S}^*(G(F))^{H(F)\times H(F)} \subset \mathfrak{S}^*(G(F))^\sigma.$$

By this theorem, if a symmetric pair is good and all its descendants are regular, one can directly get that all H bi-invariant Schwartz distributions T on $G(F)$ are invariant under the anti-involution σ. Hence, applying generalized Gelfand-Kazhdan criterion (Theorem 4.2.1) and Theorem 4.2.2, one can show that this symmetric pair is a Gelfand pair. In the rest of this chapter, we will show that our pairs which we consider are good and study their regularity.

To prove that a symmetric pair is a good pair, one can apply the following proposition.

Proposition 4.2.1 (Corollary 7.1.5 [4]). *Let (G, H, θ) be a symmetric pair and G/H be connected. Let $g \in G(F)$ such that $H(F)gH(F)$ is a closed orbit in $G(F)$. Suppose that $H^1(F, H_{\rho(g)}))$ is trivial,then $\sigma(g) \in H(F)gH(F)$.*

In the original corollary, the assumption is that $H^1(F, (H \times H)_g)$ is trivial. By $H_{\rho(g)} \simeq (H \times H)_g$ in Proposition 7.2.1 (ii) [4], the assumption can be replaced by the triviality of $H^1(F, H_{\rho(g)}))$. Hence we have this proposition. Further, by Proposition 7.2.1 (i) [4], if $H(F)gH(F)$ is closed in $G(F)$, then $\rho(g)$ is semi-simple. To prove that the symmetric pairs are good, we need to consider the centralizer $H_{\rho(g)}$ for each semi-simple $\rho(g)$ and show that $H^1(F, H_{\rho(g)})) = 1$. Since $H_{\rho(g)}$ is just a part of the descendants, let us compute all the descendants of these symmetric pairs $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$ and $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(E))^{\circ})$.

To calculate the descendants of symmetric pairs, we refer calculation for some symmetric pairs in [6], in which they apply the method of computing centralizers of semi-simple elements of classical groups in [70]. In the paper [78] and Chapter 5, I wrote down explicit representatives of all the semi-simple H-conjugate classes in G^{σ} and calculate their centralizers in G and H respectively. After Aizenbud and Gourevitch found an easier calculation in [6], I decide to adopt their argument to calculate the descendants of our symmetric pairs. Let (G, H, θ) be one of following symmetric pairs,

1. $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F))$;
2. $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$;
3. $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(F) \times \mathrm{GSp}_{2n}(F))^{\circ})$;
4. $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(E))^{\circ})$.

We recall some notation in Notation 6.1.1 [8]. Let V be the symplectic space and $G = \mathrm{Sp}(V)$ or $\mathrm{GSp}(V)$. Let $x \in G^{\sigma}$ be a semi-simple element and P be its minimal polynomial. For groups of symplectic similitudes, since $\lambda(\rho(g)) = \lambda(g)\lambda(\theta(g))^{-1} = 1$ for all $g \in \mathrm{GSp}_{4n}(F)$, we have $\mathrm{Im}\rho(\mathrm{GSp}_{4n}(F)) \subset \mathrm{Sp}_{4n}(F)^{\sigma}$. Hence it is enough to calculate the centralizers of semisimple element x in $\mathrm{Sp}_{4n}(F)^{\sigma}$.

Let $Q = \sum_{i=0}^{n} a_i t^i \in F[t]$ (where $a_n \neq 0$) be an irreducible polynomial. Denote $F_Q := F[t]/Q$, $\mathrm{inv}(Q) := \sum_{i=0}^{n} a_{n-i} t^i$, $V^0_{Q,x} := Ker(Q(x))$ and $V^1_{Q,x} := Ker(\mathrm{inv}(Q)(x))$. We define an F_Q-linear space structure on $V^i_{Q,x}$ by letting t act on $V^0_{Q,x}$ by x and on $V^1_{Q,x}$ by x^{-1}. We will consider $V^i_{Q,x}$ as a linear space over F_Q. Let P be the minimal polynomial of the semisimple element x. Note that the minimal polynomial of x^{-1} is $\mathrm{inv}(P)$ and hence P is proportional to $\mathrm{inv}(P)$. Let $P = \Pi_{i \in I} P_i$ be the decomposition of P into irreducible factors over F. Since P is proportional to $\mathrm{inv}(P)$, every P_i is proportional to $P_{s(i)}$ where s is some permutation of I of order ≤ 2. Let $I = \bigsqcup I_\alpha$ be the decomposition of I to orbits of s. Denote $V_\alpha := Ker(\Pi_{i \in \alpha} P_i(x))$. Clearly $V = \oplus V_\alpha$ and V_α are orthogonal to each other and each V_α is invariant under ε for $\theta = \mathrm{Ad}\varepsilon$. It is enough to consider cases $P = \Pi_{i \in \alpha} P_i$ over a single orbit α. Hence the pair (G_x, H_x) is a product of pairs for irreducible multiples. Let us calculate the descendants of (G, H, θ) case by case.

Theorem 4.2.4. *All the descendants of the pair* $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$ *are products of the pairs of the following types, where E is a commutative semi-simple algebra over F of dimension 2.*

1. $(\mathrm{GL}(W), \mathrm{Sp}(W))$ *for a space W over some extension field F' over F.*

2. $(U(W_{E'}), \mathrm{Sp}(W))$ *for a symplectic space W over some extension field F' of F and a skew-hermitian space $W_{E'} := W \otimes E'$ over a quadratice extension field E' of F', i.e.* $(U(J_{2m}, E'), \mathrm{Sp}_{2m}(F'))$.

3. $(\mathrm{Sp}(W), \mathrm{Sp}(W)_{\varepsilon})$ *for a symplectic space W over F.*

   If E is isomorphic to $F \times F$, $\mathrm{Sp}(W)_{\varepsilon}$ *is isomorphic to* $\mathrm{Sp}_{2m_1}(F) \times \mathrm{Sp}_{2m_2}(F))$ *for some positive integers $m = m_1 + m_2$.*

   If E is a quadratic extension over F, $\mathrm{Sp}(W)_{\varepsilon}$ *is isomorphic to* $\mathrm{Sp}_{2m}(E)$ *for some positive integer m.*

Proof. We will discuss 3 special cases and then deduce the general case from them.

Case 1. $P = Q \cdot \mathrm{inv}(Q)$, where Q is an irreducible polynomial. Note that $\mathrm{GL}(V)_x \cong \Pi_i \mathrm{GL}(V^i_{Q,x})$. Since the symplectic form defines a non-degenerate pairing $V^0_{Q,x} \cong (V^1_{Q,x})^*$, and $V^i_{Q,x}$ are isotropic, we have

$$\mathrm{Sp}(V)_x \cong \Delta \mathrm{GL}(V^0_{Q,x}) < \Pi_i \mathrm{GL}(V^i_{Q,x}).$$

Since $\varepsilon x = x^{-1}\varepsilon$, ε gives an isomorphism $V^i_{Q,x} \cong V^{1-i}_{Q,x}$. Compose this isomorphism $V^i_{Q,x} \cong (V^{1-i}_{Q,x})$ given by ε with the isomorphism $V^i_{Q,x} \cong (V^{1-1}_{Q,x})^*$ given by the non-degenerate pairing. This gives a symplectic structure on $V^0_{Q,x}$. Clearly, ε gives an siomorphism $V^i_{Q,x} \cong V^{1-i}_{Q,x}$ as symplectic spaces and hence

$$(\mathrm{Sp}(V)_\varepsilon)_x \cong \Delta \mathrm{Sp}(V^0_{Q,x}) < \Delta \mathrm{GL}(V^0_{Q,x}).$$

Case 2. P is irreducible and $x \neq x^{-1}$. In this case $\mathrm{GL}(V)_x \cong \mathrm{GL}(V^0_{P,x})$. Also, $V^0_{P,x}$ and $V^1_{P,x}$ are identical as F-vector spaces but the action of F_P on them differs by a twist by η. Therefore the isomorphism $V^0_{P,x} \cong (V^1_{P,x})^*$ gives a skew-hermitian structure on $V^0_{P,x}$ over (F_P, η) and ε gives an (F_P, η)-antilinear automorphism of $V^0_{P,x}$. Then

$$\mathrm{Sp}(V)_x \cong U(V^0_{P,x}).$$

Denote $W := (V^0_{P,x})^\varepsilon$. It is a linear space over $(F_P)^\eta$. It has a symplectic structure. Hence

$$(\mathrm{Sp}(V)_\varepsilon)_x \cong \mathrm{Sp}(W) < U(V^0_{P,x}).$$

Case 3. P is irreducible and $x^2 = 1$. Again, $\mathrm{GL}(V)_x \cong \mathrm{GL}(V^0_{P,x})$. However, in this case $F_P = F$ and $V^0_{P,x} = V$. Also $\mathrm{Sp}(V)_x \cong \mathrm{Sp}(V^0_{P,x})$. Since ε commutes with x and hence $\mathrm{Sp}(V)_x \cong \mathrm{Sp}(V^0_{P,x})$, then

$$(\mathrm{Sp}(V)_\varepsilon)_x \cong (\mathrm{Sp}(V^0_{P,x}))_\varepsilon < \mathrm{Sp}(V^0_{P,x}).$$

□

Theorem 4.2.5. *All the descendants of the pair* $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(E))^\circ)$ *are products of the pairs of the following types, where E is a commutative semi-simple algebra over F of dimension 2.*

1. $(\mathrm{GL}(W) \times \mathrm{GL}_1(F'), \mathrm{GSp}(W))$ *for aspace W over some extension field F' over F.*

2. $(\mathrm{GU}(W_{E'}), \mathrm{GSp}(W))$ *for a symplectic space W over some extension field F' of F and a skew-hermitian space $W_{E'} := W \otimes E'$ over some quadratic extension field E' of F', i.e.* $(\mathrm{GU}(J_{2m}, E'), \mathrm{GSp}_{2m}(F'))$.

3. $(\mathrm{GSp}(W), (\mathrm{GSp}(W)_\varepsilon)^\circ)$ *for a symplectic space W over some extension field F' over F.*

   If E is isomorphic to $F \times F$, $(\mathrm{GSp}(W)_\varepsilon)^\circ$ is isomorphic to $(\mathrm{GSp}_{2m_1}(F) \times \mathrm{GSp}_{2m_2}(F))^\circ$ for some positive integers $m = m_1 + m_2$.

   If E is a quadratic extension over F, $(\mathrm{GSp}(W)_\varepsilon)^\circ$ is isomorphic to $(\mathrm{GSp}_{2m}(E))^\circ$ for some positive integer m.

Proof. We will discuss 3 special cases and then deduce the general case from them.

Case 1. $P = Q \cdot \mathrm{inv}(Q)$, where Q is an irreducible polynomial. Note that $\mathrm{GL}(V)_x \cong \Pi_i \mathrm{GL}(V^i_{Q,x})$. Since the symplectic form defines a non-degenerate pairing $V^0_{Q,x} \cong (V^1_{Q,x})^*$, and $V^i_{Q,x}$ are isotropic, we have

$$\mathrm{GSp}(V)_x \cong \Delta \mathrm{GL}(V^0_{Q,x}) \times F_Q^\times < \Pi_i \mathrm{GL}(V^i_{Q,x}).$$

Since $\varepsilon x = x^{-1}\varepsilon$, ε gives an isomorphism $V^i_{Q,x} \cong (V^{1-i}_{Q,x})$. Compose this isomorphism $V^i_{Q,x} \cong V^{1-i}_{Q,x}$ given by ε with the isomorphism $V^i_{Q,x} \cong (V^{1-1}_{Q,x})^*$ given by the non-degenerate pairing. This gives a symplectic structure on $V^0_{Q,x}$. Clearly, ε gives an siomorphism $V^i_{Q,x} \cong (V^{1-i}_{Q,x})$ as symplectic spaces and hence

$$(\mathrm{GSp}(V)_\varepsilon)_x \cong \Delta \mathrm{GSp}(V^0_{Q,x}) < \Delta \mathrm{GL}(V^0_{Q,x}) \times F^\times.$$

Case 2. P is irreducible and $x \neq x^{-1}$. In this case $\mathrm{GL}(V)_x \cong \mathrm{GL}(V^0_{P,x})$. Also, $V^0_{P,x}$ and $V^1_{P,x}$ are identical as F-vector spaces but the action of F_P on them differs by a twist by η. Therefore the isomorphism $V^0_{P,x} \cong (V^1_{P,x})^*$ gives a skew-hermitian structure on $V^0_{P,x}$ over (F_P, η) and ε gives an (F_P, η)-antilinear automorphism of $V^0_{P,x}$. Then

$$\mathrm{GSp}(V)_x \cong \mathrm{GU}(V^0_{P,x}).$$

Denote $W := (V^0_{P,x})^{\varepsilon}$. It is a linear space over $(F_P)^{\eta}$. It has a symplectic structure. Hence

$$(\mathrm{GSp}(V)_{\varepsilon})_x \cong \mathrm{GSp}(W) < \mathrm{GU}(V^0_{P,x}).$$

Case 3. P is irreducible and $x^2 = 1$. Again, $\mathrm{GL}(V)_x \cong \mathrm{GL}(V^0_{P,x})$. However, in this case $F_P = F$ and $V^0_{P,x} = V$. Also $\mathrm{GSp}(V)_x \cong \mathrm{GSp}(V^0_{P,x})$. Since ε commutes with x and hence $\mathrm{GSp}(V)_x \cong \mathrm{GSp}(V^0_{P,x})$, then

$$(\mathrm{GSp}(V)_{\varepsilon})_x \cong (\mathrm{GSp}(V^0_{P,x}))_{\varepsilon} < \mathrm{GSp}(V^0_{P,x}).$$

$\square$

Remark 4.2.1. *The previous two proofs are analogous to the proof of Theorem 6.5.1 in [8].*

There is another descendent $(\mathrm{U}(Q_{2n}, E), \mathrm{SU}_n(D))$ *which is explained at Chapter 5. Since the regularity of this symmetric pair is same to the symmetric pair* $(\mathrm{U}(J_{2n}, E), \mathrm{Sp}_{2n}(J_{2n}))$*. Therefore, we omit the details of this pair in this chapter.*

Let us calculate the descendants of the symmetric pair $(U(V_E, J), \mathrm{Sp}(V, J))$, where (V, J) is a symplectic space over F and $E = E$. Denote η the non-trivial Galois element. The involution θ in this symmetric pair is extended from η. We also identify θ as an involution on V_E, $V_E = V \otimes E$ with a skew-symmetric hermitian form. We recall some notation in Notation 6.1.4 [8]. Let $x \in U(V_E, J)^{\sigma}$ be a semi-simple element and P be the minimal polynomial of x. Let $Q = \sum_{i=0}^{n} a_i t^i \in E[t]$ (where $a_n \neq 0$) be an irreducible polynomial. Denote $E_Q := E[t]/Q$, $Q^* := \eta(\mathrm{inv}(Q))$, $V^0_{Q,x} := Ker(Q(x))$, and $V^1_{Q,x} := Ker(Q^*(x))$. We twist the action of E on $V^1_{Q,x}$ by η. We define E_Q-linear space structure on $V^i_{Q,x}$ by letting t act on $V^i_{Q,x}$ by x. We will consider $V^i_{Q,x}$ as linear spaces over E_Q. If Q is proportional to Q^* we define an involution μ on E_Q by $\mu(f(t)) := \eta(f)(t^{-1})$ for all $f(t) \in E[t]/Q$. Note that $\theta(x) = x^{-1}$ and hence P is proportional to P^*. Let $P = \Pi_{i \in I} P_i$ be the decomposition of I to irreducible multiples. Since P is proportional to $\mathrm{inv}(P)$, every P_i is proportional to $P_{s(i)}$ where s is some permutation of I of order ≤ 2. Let $I = \bigsqcup I_{\alpha}$ be the decomposition of I to orbits of s.

Denote $V_\alpha := Ker(\Pi_{i\in\alpha}P_i(x))$. Clearly $V_E = \oplus V_\alpha$ and V_α are orthogonal to each other and each V_α is invariant under θ. It is enough to consider cases $P = \Pi_{i\in\alpha}P_i$ over a single orbit α. Hence the pair $(U(V_E, J)_x, \mathrm{Sp}(V, J)_x)$ is a product of pairs for irreducible multiples.

Theorem 4.2.6. *Let (V, J) be a symplectic space over F and $E = E$. Let $V_E = V \otimes E$ be a skew-symmetric hermitian space. Then all the descendants of the pair $(U_\epsilon(V_E), \mathrm{Sp}(V))$ are products of pairs of the following types.*

1. *$(\mathrm{GL}(W), \mathrm{Sp}(W))$ for some space W over some extension field F' over F.*
2. *$(U(W'_E, J), \mathrm{Sp}(W, J))$ for some space W over some extension field F' over F and some quadratic extension E' of F'.*

Proof. We will discuss 2 special cases and then deduce the general case from them.

Case 1. $P = QQ^*$ where Q is irreducible over E. Clearly $\mathrm{GL}(V_E)_x \cong \mathrm{GL}(V^0_{Q,x}) \times \mathrm{GL}(V^1_{Q,x})$. Recall that the skew hermitian form gives a non-degerate pairing between $V^0_{Q,x}$ and $V^1_{Q,x}$, and the space $V^i_{Q,x}$ are isotropic.

Therefore $\mathrm{GL}(V^0_{Q,x}) \cong \mathrm{GL}(V^1_{Q,x})$,

$$\mathrm{GL}(V_E)_x \cong \Pi_i \mathrm{GL}(V^i_{Q,x}) \text{ and } U(V_E)_x \cong \Delta\mathrm{GL}(V^0_{Q,x}) < \Pi_i \mathrm{GL}(V^i_{Q,x}).$$

Since $\theta(x) = x^{-1}$, θ gives an isomorphism $V^i_{Q,x} \cong (V^{1-i}_{Q,x})$. Compose this isomorphism $V^i_{Q,x} \cong (V^{1-i}_{Q,x})$ given by θ with the isomorphism $V^i_{Q,x} \cong (V^{1-1}_{Q,x})^*$ given by the non-degenerate pairing. This gives a symplectic structure on $V^0_{Q,x}$. Then

$$\mathrm{Sp}(V)_x \cong \Delta\mathrm{Sp}(V^0_{Q,x}) < \Delta\mathrm{GL}(V^0_{P,x}).$$

Case 2. P is irreducible over E. Clearly $\mathrm{GL}(V_E)_x \cong \mathrm{GL}(V^0_{P,x})$ and $V^0_{P,x}$ as F-linear spaces but the actions of E_P on them differ by a twist by μ. Hence the isomorphism $V^0_{P,x} \cong (V^0_{P,x})^*$ given by the skew-hermitian form gives a skew-hermitian structure on $V^0_{P,x}$ over (E_P, μ) and the isomorphism $V^0_{P,x} \cong V^1_{P,x}$ given by θ gives an antilinear involution of $V^0_{P,x}$. Therefore,

$$U(V_E)_x \cong U(V^0_{P,x}) < \mathrm{GL}(V^0_{P,x}) \text{ and } \mathrm{Sp}(V)_x \cong \mathrm{Sp}(V^0_{P,x}) < U(V^0_{P,x}).$$

□

Remark 4.2.2. *The above proof is analogous to the proof of the Theorem 6.4.1 [8].*

Theorem 4.2.7. *Symmetric pairs* $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$, $(\mathrm{GSp}_{4n}(F), \mathrm{GSp}_{2n}(E)^\circ)$, *and* $(U(J_{2n}, E), \mathrm{Sp}_{2n})$ *are good pairs.*

Proof. By Proposition 4.2.1, it is sufficient to show that $H^1(F, R_{F'/F}(\mathrm{Sp}_{2n})) = 1$. According to Shapiro's Lemma, we have

$$H^1(F, R_{F'/F}(\mathrm{Sp}_{2n})) = H^1(F, \mathrm{Sp}_{2n}(F)).$$

Since $H^1(F, \mathrm{Sp}_{2n}(F)) = 1$ in (29.25) [42], $H^1(F, R_{F'/F}(\mathrm{Sp}_{2n})) = 1$.

Hence $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$ is a good pair. For GSp case, since $\rho(\mathrm{GSp}_{4n}) = \rho(\mathrm{Sp}_{4n})$, the paur $(\mathrm{GSp}_{4n}(F), \mathrm{GSp}_{2n}(E)^\circ)$ is also a good pair.

□

4.3 Regularity

In previous section, we have already computed all descendants of these symmetric pairs. By generalized Harish-Chandra method, we have to show that all of these descendants are regular, and hence conclude that these symmetric pairs are Gelfand pairs. By Proposition 7.4.4 [4], a product of regular symmetric pairs is regular. It is enough to prove that descendants in Theorem 4.2.4, Theorem 4.2.5, and Theorem 4.2.6, are regular. In this section, we introduce some definitions related to the nilpotent orbits in $\Gamma(\mathfrak{g}^\sigma)$ and summarize several methods to prove the regularity of symmetric pairs.

By Lemma 7.1.11 [4], for each nilpotent element $x \in \mathfrak{g}^\sigma$, there exists a $\mathfrak{sl}_2$-triple (x, h, f) such that the semisimple element $h \in \mathfrak{h}$ and $f \in \mathfrak{g}^\sigma$. Denote by $d(x)$ the semisimple element in $\mathfrak{sl}_2$.

Definition 4.3.1. *An orbit* $\mathrm{Ad}H \cdot x$ *in* $\Gamma(\mathfrak{g}^\sigma)$ *is called* ***negative defect*** *if*

$$\mathrm{tr}(\mathrm{ad}(d(x))|_{\mathfrak{h}_x}) < \dim Q(\mathfrak{g}^\sigma).$$

Definition 4.3.2. *A nilpotent element $x \in \mathfrak{g}^\sigma$ is* ***distinguished*** *if*

$$\mathfrak{g}_x \cap Q(\mathfrak{g}^\sigma) \subset \Gamma(Q(\mathfrak{g}^\sigma)),$$

where $\mathfrak{g}_x$ is the centralizer of x in $\mathfrak{g}$.

Definition 4.3.3. *A symmetric pair (G, H, θ) is a pair of* ***negative distinguished defect*** *if all the distinguished elements in $\Gamma(\mathfrak{g}^\sigma)$ have negative defect.*

Considering the Fourier Transform of H-invariant distributions which is also H-invariant, one have the following theorems to prove a symmetric pair is regular.

Theorem 4.3.1. *Let (G, H, θ) be a symmetric pair over a non-archimedean field. For each element $x \in \Gamma(\mathfrak{g}^\sigma)$, it satisfies one of following condition:*

1. $\mathrm{tr}(\mathrm{ad}(d(x))|_{\mathfrak{h}_x}) \neq \dim \mathfrak{g}^\sigma$ *(Proposition 7.3.7 [4]);*

2. $\mathrm{Ad}(g) \cdot x$ *is H-conjugate to x for each admissible element g.*

Then the pair (G, H, θ) is regular.

Proof. This proposition is a straightforward of localization principle, Proposition 7.3.7, Proposition 7.3.5 and Remark 7.4.3 in [4]. □

We use this theorem to prove the main theorems in this chapter.

Theorem 4.3.2. *For any local field of characteristic 0,*
the symmetric pair $(U(J_{2n}, E), \mathrm{Sp}_{2n}(F))$ is a Gelfand pair. Furthermore, $(\mathrm{GL}_{2n}(F) \times \mathrm{GL}_1(F), \mathrm{GSp}_{2n}(F)$ and $(\mathrm{GU}(J_{2n}, E), \mathrm{GSp}(J_{2n}, F))$ are regular.

Proof. We need to prove that all descendants of this symmetric pair are regular. Since $(\mathrm{GL}_{2n}(F), \mathrm{Sp}_{2n}(F))$ is regular for any local field of characteristic 0, proved in [7], [66] and [31], it is enough to prove the regularity of $(U(J_{2n}, F(\sqrt{\tau})), \mathrm{Sp}_2(J_{2n}, F))$. Now let us analysis H adjoint action on $\mathfrak{g}^\sigma$. In this case,

$$\mathfrak{g}^\sigma = \{\sqrt{\tau}A \mid A = \tilde{A}, A \in M_{2n\times 2n}(F)\}.$$

Then $\mathfrak{g}^\sigma$ are same for $(\mathrm{GL}_{2n}(F), \mathrm{Sp}_{2n}(F))$ and $(U(J_{2n}, F(\sqrt{\tau})), \mathrm{Sp}_2(J_{2n}, F))$ up to a scalar. Even though all admissible elements for these two symmetric pairs could be different, the argument in [7] and [66] still work for unitary symmetric pairs over archimedean field. For non-archimedean field, according the classification of Sp_{2n} conjugate nilpotent orbits in $Q(\mathfrak{g}^\sigma)$ in [7] and [31], one can prove that each nilpotent orbit is invariant under the adjoint action of all admissible elements. Therefore we show the regularity of $(U(J_{2n}, F(\sqrt{\tau})), \mathrm{Sp}_2(J_{2n}, F))$ for both archimedean and non-archimedean. $(\mathrm{GL}_{2n}(F) \times \mathrm{GL}_1(F), \mathrm{GSp}_{2n}(F)$ and $(\mathrm{GU}(J_{2n}, E), \mathrm{GSp}(J_{2n}, F))$ which are the descendants of $(\mathrm{GSp}_{4n}(F), \mathrm{GSp}_{2n}(E)^\circ)$ are regular by the same arguments. □

In symplectic cases, if g is admissible in $\mathrm{Sp}_{4n}(F)$, then $\rho(g) \in Z(G)$ and $\rho(g) \in \{\pm I_{4n}\}$. We can easily verify $\rho(I_{4n}) = I_{4n}$ and $\rho(\omega) = -I_{4n}$, where $\omega = \begin{pmatrix} & I_{2n} \\ I_{2n} & \end{pmatrix}$. Therefore $g \in H \cup \omega H$. If g is admissible in $\mathrm{GSp}_{4n}(F)$, then $\rho(g) \in \{\pm I_{4n}\}$ and $g \in H \cup \omega H$, by $\lambda(\rho(g)) = 1$. To prove that these symmetric pairs are regular, it is enough to prove $\mathfrak{S}^*(Q(\mathfrak{g}^\sigma))^H \subset \mathfrak{S}^*(Q(\mathfrak{g}^\sigma))^{\mathrm{Ad}(\omega)}$.

Now let us prove the case $E \simeq E$ for any n and non-archimedean field F.

Theorem 4.3.3. *If E is a quadratic extension field over F, symmetric pairs $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$ and $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(E))^\circ)$ are Gelfand pairs for any non-archimedean field F.*

Proof. We have already proved that these pairs are good pairs. It is enough to prove that these pairs are regular. Applying the following lemma 4.3.1 and $\mathrm{Ad}(\omega) \cdot \iota(\beta) = \iota(\bar{\beta})$, we show $\mathrm{Ad}(\omega)$ preserve each H-adjoint orbit on $\mathfrak{g}^\sigma$. By Theorem 4.3.1, we prove that these pairs are regular and hence Gelfand pairs. □

Lemma 4.3.1. *For any $\iota(\beta) = \begin{pmatrix} & \beta \\ \bar{\beta} & \end{pmatrix} \in \Gamma(\mathfrak{g}^\sigma)$, there exists an element $h \in H$ such that $h\beta\bar{h}^{-1} \in M_{2n\times 2n}(F)$.*

Proof. Since $\iota(\beta) \in \Gamma(\mathfrak{g}^\sigma)$ is equivalent to $(\beta\bar{\beta})^{2n} = 0$, $\mathrm{rank}(\beta) < 2n$. Denote $c_i(\beta)$ the i-th column of β and $r_i(\beta)$ the i-th row of β. Then there exists some $a_i \neq 0$ such that $\sum_{i=1}^{2n} a_i c_i(\beta) = 0$. By the adjoint action of some Weyl elements in H, we can switch columns such that $a_1 \neq 0$. Then $c_1(\bar{h}^{-1}\beta h) = 0_{2n\times 1}$ and $r_{2n}(\bar{h}^{-1}\beta h) = 0_{1\times 2n}$ if we take

$$h = \begin{pmatrix} 1 & v^* & a_{2n}a_1^{-1} \\ & 1_{2n-2} & v \\ & & 1 \end{pmatrix}$$

where $v = (a_2a_1^{-1}, \cdots, a_{2n-1}a_1^{-1})^t$ and v^* is uniquely determined by v. Hence we can assume $\beta = \left(\begin{smallmatrix} 0 & v & a \\ 0 & A & v^* \\ 0 & 0 & 0 \end{smallmatrix}\right)$. By $\tilde{\beta} + \bar{\beta} = 0$, we have conditions, $v^* = J_{2n-2}\bar{v}^t$, $a \in F$ and $\tilde{A} + \bar{A} = 0$, $(A\bar{A})^{2n-2} = 0$. If n=1, β is obviously in $M_{2\times 2}(F)$. By induction, there exists $h \in H$ such that $hA\bar{h}^{-1} \in M_{2(n-1)\times 2(n-1)}(F)$. Thus we can assume that A is a nilpotent element in the Lie algebra of $\mathrm{Sp}_{2n}(F)$ and conjugate to the standard form in [56]. We treat A as a skew adjoint endmorphsim over F-vector space V which has a symplectic form J_{2n-2}, then it can be decomposed to $\oplus_{i\in I}V_i(j)$ according to a $\mathfrak{sl}_2$ triple including A, where j is the highest weight. We can restrict A to $V_i(j)$, if j is even, then

$$A|_{V_i(j)} = \begin{pmatrix} D_j & 0 \\ 0 & -D_j \end{pmatrix}.$$

If j is odd, we have

$$A|_{V_i(j)} = D_j \cdot \mathrm{diag}(1, \cdots, c_i, -1, \cdots, -1)$$

for some $c_i \in F^\times$ which is the $(j/2+1)$-th entry of the diagonal matrix. Here D_j is a Jordan block of $j \times j$ matrix, i.e.

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

The subscript i of $V_i(j)$ means if j is even, $c_i(A) = 0$, and if j is odd, $c_i(A) = c_{2n+2-i-j}(A) = 0$.

Taking $h = \begin{pmatrix} 1 & w & 0 \\ & 1 & J_{2n-2}w^t \\ & & 1 \end{pmatrix}$ and choosing a suitable vector w such that $c_i(v - \bar{w}A) \cdot c_i(A) = 0$ for each $1 \leq i \leq 2n-2$ and under the action of this h, we reduce to the case $c_i(v) \cdot c_i(A) = 0$. Next we choose $h_1|_{V_i(j)} = \mathrm{diag}(c, \bar{c}, \cdots, c, \bar{c}, \bar{c}, c, \cdots, \bar{c}, c)$, if $c^{-1} = c_i(v) \neq 0$, otherwise $h_1|_{V_i(j)} = 1$. Remark that the subscript i of $c_i(v)$ is same as that of $V_i(j)$, i.e. in the index set I. Then we get $c_i(v) = 0$ or 1 for $i \in I$ and all entries are in F except $c_{2n+2-i-j}(v)$ for all odd j. If $c_{2n+2-i-j}(v) \notin F$, we define h_2 restriction to this $V_i(j)$ by

$$h_2|_i(j) = \begin{pmatrix} 1 & T \\ & 1 \end{pmatrix}$$

where $T = \mathrm{diag}(c, \bar{c}, \cdots, c, \bar{c}, c, \cdots, \bar{c}, c)$ and $c = -c_{2n+2-i-j}(v)$. In the rest $V_i(j)$, h_2 is the identity matrix. Under action of h_2, we get that all entries of β are belong to F. $\square$

Next, we consider the split case $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F))$. First we introduce another theorem in [3] to prove the regularity of symmetric pairs. Then applying Theorem 4.3.1 for $n = 1, 2$, we prove that these symmetric pairs are Gelfand pairs. However, both of these two theorems fail to prove the regularity for $n \geq 3$. For $n \geq 3$, the regularity is still in progress.

Theorem 4.3.4 (Theorem 5.2.5 [3])**.** *Let (G, H, θ) be a symmetric pair of negative distinguished defect. Then it is regular.*

For each nilpotent element $e \in \mathfrak{g}^\sigma$, we take the $\mathfrak{sl}_2$-triple (e, h, f) such that the semi-simple element $h \in \mathfrak{h}$ and $f \in \mathfrak{g}^\sigma$. The centralizer $\mathfrak{g}_e$ of a nilpotent element $e \in \mathfrak{g}^\sigma$ is invariant under $\mathrm{ad}(h), h \in H$ and under θ. This implies that $\mathfrak{g}_e = \mathfrak{h}_e + \mathfrak{g}_e^\sigma$ is direct sum and $\mathrm{ad}(h)$ also preserves $\mathfrak{h}_e$ and $\mathfrak{g}_e^\sigma$. Then we can take a basis $w_1, \cdots, w_r$ of $\mathfrak{g}_e^\sigma$ which are eigenvectors on $\mathrm{ad}(h)$. Denote n_i the corresponding eigenvalues for each w_i. Indeed, $n_i (1 \leq i \leq r)$ are the highest weights of (e, h, f) of $\mathfrak{g}^\sigma$. We have following criterion for $\mathfrak{g}^\sigma$-distinguished nilpotent element.

Lemma 4.3.2 ([75])**.** *e is $\mathfrak{g}^\sigma$-distinguished nilpotent element if and only if $n_i > 0$ for all $1 \leq i \leq r$.*

In the following definition, we consider that the $\mathfrak{sl}_2$ triple (e, h, f) is a graded Lie algebra, where $e, f \in \mathfrak{sl}_2(F)_1$ are of grade 1 and $h \in \mathfrak{sl}_2(F)_0$ is of grade 0.

Definition 4.3.4. *A graded representation of* $\mathfrak{sl}_2$ *triple* (e, h, f) *is a representation of* $\mathfrak{sl}_2$ *on a graded vector space* $V = V_0 \oplus V_1$ *such that* $\mathfrak{sl}_2(F)_i(V_j) \subset V_{i+j}$*, where* $i, j \in \mathbb{Z}/2\mathbb{Z}$*.*

In our case, we have the involution $\theta = \mathrm{Ad}(\varepsilon)$, and take $(e,\ h,\ f)$ as above and V_i invariant by ε. Therefore, we have a graded representation of $\mathfrak{sl}_2$.

Decompose V to the direct sum of irreducible representations, i.e. $\oplus V(d)$, where d is the highest weight of $\mathfrak{sl}_2$. Let $L(d)$ be the subspace of the lowest weight vectors of $V(d)$. Since $X \in \mathfrak{g}^\sigma$, $L(d)$ can be decomposed to the direct sum of $L(d)_0$ and $L(d)_1$, where $L(d)_i \in V_i$. Denote $x_{d+1} = \dim(L(d)_0)$ and $y_{d+1} = \dim(L(d)_1)$.

Proposition 4.3.1. *For symmetric pairs*

$$(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F)) \text{ and } (\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(F) \times \mathrm{GSp}_{2n}(F))^\circ),$$

we have

1. *If* d *is odd,* $x_{d+1} = y_{d+1}$*;*
2. *If* d *is even,* x_{d+1} *and* y_{d+1} *are even;*
3. $\sum_{d\ even} x_{d+1} = \sum_{d\ even} y_{d+1}$*.*

Proof. The proof for GSp case is similar to the proof of Sp case. Here we give the proof for Sp case.

1. Let $v \neq 0 \in L(d)_0$, then $e^d v \in L(d)_1$, since d is odd. Since the restriction of $<,>$ to $V(d)$ naturally induces a nondegenerate form, and $< v, e^d v >= 0$, there exists a vector $w \in V(d)_1$ such that $< w, e^d v >\neq 0$. Hence $e^d w$ is a non-zero vector of the highest weight and $e^d w \in L(d)_1$. Let V' be a $\mathfrak{sl}_2$-module which is generated by v and w. It is clear that V' is a nondegenerate symplectic vector space. We have $V(d) = V' \oplus W$. By induction, we conclude (1).

2. For $\mathrm{Sp}_{2n}(F)$, $V(d)$ with the odd highest weight has even dimension, it is enough to show that e_{d+1} is even. Similarly, if $e_{d+1} = 0$, there is nothing to prove. If not, taking a non-zero vector $v \in L(d)_0$, then $< v, e^d v >=< e^{d/2}v, e^{d/2}v >= 0$. By the similar argument, we can find a vector w such that $e^d w \in L(d)_0$ and a nondegenerate symplectic vector space V' such that $\dim L(V')_0 = 2$ and $V = V' \oplus W$. By induction, we have x_{d+1} is even.

3. It is straight forward by the skew-symmetry of e.

$\square$

Proposition 4.3.2. *For symmetric pairs*

$$(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F)) \text{ and} (\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(F) \times \mathrm{GSp}_{2n}(F))^\circ),$$

$e \in \Gamma(\mathfrak{g}^\sigma)$ is a distinguished nilpotent element , then the corresponding $x_{d+1} = 1$ if d is odd and $x_{d+1}y_{d+1} = 0$ if d is even.

Proof. By Lemma 4.3.2, we need to prove $\dim((\mathfrak{g}_e^\sigma)_h) = 0$. Referring to [19], any $z \in \mathfrak{g}_e$ is completely determined by its restriction to $L(d)$. If $Z \in (\mathfrak{g}_e)_h$, Z sends $L(d)$ to $L(d)$, then it defines a bilinear form $(\cdot, \cdot)$ on $L(d)$ via $(v, w) =< v, e^d \cdot Zw >$. This form is symmetric or sympletic according as d is even or odd. The set of possible forms $(\cdot, \cdot)$ and $Z \in \mathfrak{g}^\sigma$ is thus a vector space of dimension $x_{d+1}y_{d+1}$ if d is even and $x_{d+1}(x_{d+1} - 1)$ if d is odd. Since $\dim((\mathfrak{g}_e^\sigma)_h) = 0$, $x_{d+1} = y_{d+1} = 1$ if d is odd and $x_{d+1}y_{d+1} = 0$ if d is even. $\square$

Theorem 4.3.5. *If $n = 1$, symmetric pairs $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F))$ and $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(F) \times \mathrm{GSp}_{2n}(F))^\circ)$ are Gelfand pairs for non-archimedean fields.*

Proof. It is enough to show that these pair are regular. We have to prove $\mathfrak{S}^*(\Gamma(\mathfrak{g}^\sigma))^{\mathrm{Ad}H} \subset \mathfrak{S}^*(\Gamma(\mathfrak{g}^\sigma))^{\mathrm{Ad}(\omega)}$. By Theorem 4.3.1, it is sufficient to show that each orbit of $\mathrm{Ad}H$ over $N(\mathfrak{g}^\sigma)$ is invariant under $\mathrm{Ad}(\omega)$. We show this in the following.

Let $\begin{pmatrix} 0 & A \\ -\tilde{A} & 0 \end{pmatrix} \in \mathfrak{g}^\sigma$, where $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

If $\begin{pmatrix} 0 & A \\ -\tilde{A} & 0 \end{pmatrix} \in \Gamma(\mathfrak{g}^\sigma)$, then $A\tilde{A} = 0$ which is equivalent to $\det(A) = 0$.

There exist h_1 and h_2 in $\mathrm{Sp}_2(F)$ such that $h_1 A h_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if $A \neq 0$ and $h_1 A h_2^{-1} = 0$ if $A = 0$.

Therefore, $N(\mathfrak{g}^\sigma) = \mathrm{Ad} H \cdot e \cup \{0\}$, where $e = \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & & \\ 0 & -1 & & \end{array}\right)$.

□

Theorem 4.3.6. *If* $n = 2$, *symmetric pairs* $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F))$ *and* $(\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(F) \times \mathrm{GSp}_{2n}(F))^\circ)$ *are Gelfand pairs for non-archimedean field* F*.*

Proof. By Proposition 4.3.1, nilpotent orbits in $\mathfrak{g}^\sigma$ have partition $[4^2]$, $[3^2 1^2]$, $[2^4]$ and $[2^2 1^4]$. The corresponding $\mathrm{tr}(\mathrm{ad}(d(x))|_{\mathfrak{h}_x}) \in \{8, 12, 20\}$. Therefore,

$$\mathrm{tr}(\mathrm{ad}(d(x)) \neq 16 \text{ by } \dim \mathfrak{g}^\sigma = 4n^2.$$

Applying Proposition 4.3.1, we have $(\mathrm{Sp}_8(F), \mathrm{Sp}_4(F) \times \mathrm{Sp}_4(F))$ is regular.

Referring Proposition 5.2 in [15], Bosman and van Dijk classify all H-conjugate orbits in $N(\mathfrak{g}^\sigma)$ when $G = \mathrm{Sp}_{2n}(F)$ and $H = \mathrm{Sp}_{2n-2}(F) \times \mathrm{Sp}_2(F)$ for $n > 2$ and real field F. However the proof still works for arbitrary non-archimedean field F. We can easily verify that all orbits are invariant under $\mathrm{Ad}(g)$ for all admissible element g. Hence by Proposition 4.3.1, we can show that $(\mathrm{Sp}_{2n}(F), \mathrm{Sp}_{2n-2}(F) \times \mathrm{Sp}_2(F))$ are regular for $n > 2$. According to Theorem 4.3.5, $(\mathrm{Sp}_{2n}(F), \mathrm{Sp}_{2n-2}(F) \times \mathrm{Sp}_2(F))$ are regular for $n > 1$. Therefore, all the descendants of symmetric pairs $(\mathrm{Sp}_8(F), \mathrm{Sp}_4(F) \times \mathrm{Sp}_4(F))$ and $(\mathrm{GSp}_8(F), (\mathrm{GSp}_4(F) \times \mathrm{GSp}_4(F))^\circ)$ are regular and then these symmetric pairs are Gelfand pairs. □

Remark 4.3.1. *If $n = 3$, denote $e = \begin{pmatrix} & A \\ -\tilde{A} & \end{pmatrix}$, where*

$$A = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}.$$

We can easily find the corresponding semi-simple element $d(e)$, calculate

$$\mathrm{tr}(\mathrm{ad}(d(e))|_{\mathfrak{h}_e}) = \dim \mathfrak{g}^\theta = 36.$$

Additionally, this orbit is not invariant under $\mathrm{Ad}(w)$ and a distinguished orbit. Therefore, for general n, we can not use Proposition 4.3.1 or Theorem 4.3.5 to prove that this symmetric pair is regular.

Under the assumption of the regularity of the symmetric pairs $(\mathrm{Sp}_{2m}(F), \mathrm{Sp}_{2m_1}(F) \times \mathrm{Sp}_{2m_2}(F))$, we have the following theorem.

Theorem 4.3.7. *If $(\mathrm{Sp}_{2(m_1+m_2)}(F), \mathrm{Sp}_{2m_1}(F) \times \mathrm{Sp}_{2m_2}(F))$ are regular for any $m_1 + m_2 \leq 2n$,*

$$(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F)) \text{ and } (\mathrm{GSp}_{4n}(F), (\mathrm{GSp}_{2n}(F) \times \mathrm{GSp}_{2n}(F)))^\circ$$

are Gelfand pairs for any non-archimedean field of characteristic 0.

Chapter 5

Distinguished Tame Supercuspidal Representations

In this chapter, we classify all H-distinguished tame supercuspidal representations of G. The most constant of this chapter is considered in [78]. Since we use the symbol F as a Frobenious map on reductive groups defined over a finite field $\mathbb{F}_q$, we replace the k as the local field only in this chapter.

5.1 Overview

Define a unitary group

$$\mathrm{U}(J_{2n}, E) = \{g \in M_{2n\times 2n}(E) \mid \overline{g}^t J_{2n} g = J_{2n}\}.$$

An involution θ of the unitary group is induced from the action of the nontrivial Galois element in $\mathrm{Gal}(E/k)$ on each matrix entry. Then, the symmetric pair is $(U(J_{2n}, E), \mathrm{Sp}_{2n}(k))$.

Let

$$Q_{2n} = \mathrm{diag}\{1, \tau, 1, \tau, \cdots, 1, \tau\},$$

and

$$\varepsilon'_{2n} = \mathrm{diag}\{\begin{pmatrix} & 1 \\ \tau & \end{pmatrix}, \begin{pmatrix} & 1 \\ \tau & \end{pmatrix}, \cdots, \begin{pmatrix} & 1 \\ \tau & \end{pmatrix}\}.$$

Let $k(\sqrt{\omega})$ be a quadratic extension over k. Here $\tau \in k(\sqrt{\omega})^\times$ is not in the image of norm map on $k(\sqrt{\omega})^\times$. Define a unitary group

$$\mathrm{U}(Q_{2n}, k(\sqrt{\omega})) = \{g \in M_{2n\times 2n}(k(\sqrt{\omega})) \mid \overline{g}^t Q_{2n} g = Q_{2n}\}.$$

An involution θ is given by $\theta(g) := \varepsilon'\overline{g}\varepsilon'^{-1}$. Then, the symmetric subgroup is isomorphic to $\mathrm{SU}_n(D)$, where D is the quaternion algebra over k and $\mathrm{SU}_n(D)$ preserves a nondegenerate hermitian form relative to an involution of the first kind.

Let π be an irreducible smooth admissible representation of G. For a symmetric par (G, H), the representation π is called *H-distinguished* if the H-invariant linearly functional space $\mathrm{Hom}_H(\pi, \mathbb{1})$ is nonzero. In this chapter, we study H-distinguished supercuspidal representations of the symmetric pairs

$$(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E)) \text{ and } (U(J_{2n}, E), \mathrm{Sp}_{2n}(k)).$$

We have proved that these two pairs are Gelfand pairs in [77]. Here a pair (G, H) is called a *Gelfand pair* if the dimension of the space $\mathrm{Hom}_H(\pi, \mathbb{1})$ is at most one for all irreducible smooth representations π of G, where H is a closed subgroup of G. Hence, for these two pairs, if π is H-distinguished, then all linear forms in $\mathrm{Hom}_H(\pi, \mathbb{1})$ are proportional to each other. In general, the dimension of $\mathrm{Hom}_H(\pi, \mathbb{1})$ is not always 1. One can find some symmetric pairs in [28, Section 5.9] such that the dimension of the space $\mathrm{Hom}_H(\pi, \mathbb{1})$ for some supercuspidal π is 2.

First, let us recall Yu's construction of tame supercuspidal representations. Assume that $\mathbf{G}$ splits over a tamely ramified extension of k. Yu defined generic cuspidal G-data in [76]. A cuspidal G-datum Ψ is a 5-tuple $(\vec{\mathbf{G}}, y, \vec{r}, \rho, \vec{\phi})$ that satisfies certain conditions. Its first component $\vec{\mathbf{G}}$ is a tamely ramified twisted Levi sequence $(\mathbf{G}^0, \mathbf{G}^1, \ldots, \mathbf{G}^d = \mathbf{G})$. The second component y is a vertex of the Bruhat-Tits building $\mathscr{B}(G^0)$. Under a natural embedding in the Bruhat-Tits building $\mathscr{B}(G)$, the point y is realized as a point in $\mathscr{B}(G)$. The third sequence is a series of increasing non-negative real numbers which are the depths of the quasi-characters in the fifth component $\vec{\phi} = (\phi_0, \ldots, \phi_d)$. Each ϕ_i is the quasi-character of G^i. The forth component ρ is the depth-zero supercuspidal representation of G^0.

For each generic cuspidal G-datum $\Psi = (\vec{\mathbf{G}}, y, \vec{r}, \rho, \vec{\phi})$, Yu associated a compact-mod-center subgroup $K(\Psi)$ of G and an irreducible smooth representation $\kappa(\Psi)$ of $K(\Psi)$ such that the smooth and compactly supported induced representation $\pi(\Psi) = \mathrm{ind}_{K(\Psi)}^G \kappa(\Psi)$ is irreducible and supercuspidal. These representations are referred as *tame supercuspidal representations*. Indeed, the terminology of Hakim and Murnaghan in [28] is slightly different from Yu's in [76]. In this chapter, we use the terminology of Hakim and Murnaghan in [28].

Next, applying Mackey's theorem, the space $\mathrm{Hom}_H(\pi(\Psi), \mathbb{1})$ decomposes canonically as

$$\mathrm{Hom}_H(\pi, \mathbb{1}) = \bigoplus_{K(\Psi)gH \in K(\Psi)\backslash G/H} \mathrm{Hom}_{K(\Psi)\cap gHg^{-1}}(\kappa(\Psi), \mathbb{1}).$$

Hakim and Murnaghan proved that there are only finite double cosets $K(\Psi)gH$ such that $\mathrm{Hom}_{K\cap gHg^{-1}}(\kappa(\Psi), \mathbb{1})$ is nonzero. If the space $\mathrm{Hom}_{K\cap gHg^{-1}}(\kappa(\Psi), \mathbb{1})$ is nonzero for some g, one can choose a suitable Ψ such that Ψ is θ-symmetric. Referring to [28], a cuspidal G-datum Ψ is *θ-symmetric* if $\theta(G^i) = G^i$ and $\phi_i \circ \theta = \phi_i^{-1}$ for $0 \leq i \leq d$, and $\theta([y]) = [y]$, where $[y]$ is the image of y in the reduced building of G.

For this nonzero homomorphism space and θ-symmetric datum Ψ, Hakim and Murnaghan gave the following isomorphism,

$$\mathrm{Hom}_{K\cap gHg^{-1}}(\kappa(\Psi), \mathbb{1}) \cong \mathrm{Hom}_{K^0\cap gHg^{-1}}(\rho \otimes \phi, \eta),$$

where ϕ is $\prod_{i=0}^{d}(\phi_i|G^0)$ and η is a quadratic character.

The recent work [55] of Murnaghan links the θ-symmetric generic cuspidal data and certain elements in G that are θ-split tamely ramified elliptic regular elements. We classify all H-conjugate θ-split element and apply Murnaghan's result to identity all the θ-symmetric generic cuspidal data.

For the quadratic character η, we refer to another recent work [27] of Hakim and Lansky. They classified all the distinguished tame supercuspidal representations of the symmetric pair $(\mathrm{GL}_{2n+1}(k), \mathrm{O}_{2n+1}(k))$, and showed the triviality of the quadratic characters in their cases. We will apply their techniques to work on our cases and show the triviality. Finally, we have the following theorem for the symmetric pair $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E))$.

Theorem 5.1.1. *Let (G, H) be the symmetric pair $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E))$. A tame supercuspidal representation $\pi(\Psi)$ is H-distinguished if and only if Ψ satisfies the following conditions, up to a K-conjugation and a θ-symmetric refactorization.*

- *The quadratic extension E is unramified.*

1. *When d is 0, y is θ-fixed and $\overline{\rho}$ is a distinguished cuspidal representation of the symmetric pair $(G_{y,0}/G_{y,0+}, (G_{y,0}/G_{y,0+})^\theta)$, that is,*

$$\prod_i (\mathrm{Sp}_{4r_i}(\mathfrak{f}_i) \times \mathrm{Sp}_{4(n_i-r_i)}(\mathfrak{f}_i), \mathrm{Sp}_{2r}(\mathfrak{f}_{E_i}) \times \mathrm{Sp}_{2(n_i-r_i)}(\mathfrak{f}_{E_i})).$$

2. *When $d > 0$, the symmetric pair $(G^0, G^{0,\theta})$ is isomorphic to*

$$(\prod_i \mathrm{U}_2(Q, E_i), \prod_i \mathrm{SU}_1(D_i).$$

- *The quadratic extension E is ramified.*

  1. *The depth d is more than zero.*
  2. *The symmetric pair $(G^0, G^{0,\theta})$ is isomorphic to*

  $$(\prod_i \mathrm{U}(E_i), \prod_i \mathrm{SU}(D_i)).$$

  The point y is θ-fixed point and $\overline{\rho}$ is a distinguished cuspidal representation of the symmetric pair $(G_{y,0}/G_{y,0+}, (G_{y,0}/G_{y,0+})^\theta)$, that is,

  $$\prod_i (\mathrm{Sp}_{4n_i}(\mathfrak{f}_i), \mathrm{Sp}_{2n_i}(\mathfrak{f}_{E_i})).$$

For $n = 1$ case, Murnghan gave a H-dsitinguished representation in [55], which is consistent with this theorem.

By this theorem, we can easily conclude that $\mathrm{Sp}_{2n}(k(\sqrt{\tau}))$-distinguished tame supercuspidal representations over unramified and ramified quadratic extensions are mutually exclusive.

Corollary 5.1.1. *If π is a tame supercuspidal representation of $\mathrm{Sp}_{4n}(k)$, then*

$$\dim \mathrm{Hom}_{\mathrm{Sp}_{2n}(E_1)}(\pi, 1) \cdot \dim \mathrm{Hom}_{\mathrm{Sp}_{2n}(E_2)}(\pi, \mathbb{1}) = 0, \tag{5.1.1}$$

where E_1 is the unramified quadratic extension over k and E_2 is a ramified quadratic extension.

In general, this corollary is not true for all irreducible representations. For instance, if π is the trivial representation, then the multiplication of the dimensions is 1.

By Theorem 5.1.1, we reduce the question, of whether a tame supercuspidal representation is H-distinguished, to the question, of whether a cuspidal representation of a finite group of Lie type is H-distinguished. For finite groups of Lie type, Deligne and Lusztig proved that each irreducible representation is a constitute of a virtual characters R_T^λ in [20]. Then, Lusztig [50] parameterized all irreducible representations by unipotent representations of subgroups of the dual group. In order to get a more explicit result of Theorem 5.1.1, we plan to identify all H-distinguished cuspidal representations in terms of the Lusztig's parameterization in [50]. Lusztig already gave a dimension formula for the character R_T^λ distinguished by H in [51]. However, this is still not enough to complete the classification in terms of Lusztig's parameterization. Indeed, not all irreducible representations are linear combinations of R_T^λ. For instance, one can refer to Example 5.4.2. Even if the H-distinguished dimension formula of all R_T^λ is known, there is still a gap between$H$-distinguished dimension formula of all R_T^λ and the dimension formula of all irreducible representations. Essentially, the distinguished formula of the unipotent representations plays an important role on the distinguished formula of all irreducible representations.

For the symmetric pair $(\mathrm{U}(\mathbb{F}_{q^2}), \mathrm{Sp}_{2n}(\mathbb{F}_q))$, Henderson [30] applied Lusztig's formula to obtain the dimension formula of $\mathrm{Hom}_{\mathrm{Sp}_{2n}(\mathbb{F}_q)}(\pi, \mathbb{1})$ for all irreducible representations π. For this case, every irreducible representation of the unitary group over finite fields is a linear combination of R_T^λ. Based on Henderson's result, we conclude that there is no $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-distinguished cuspidal representation of $\mathrm{U}(\mathbb{F}_{q^2})$. Then, we have the following theorem.

Theorem 5.1.2. *If (G, H) is the symmetric pair $(\mathrm{U}_{2n}(J, E), \mathrm{Sp}_{2n}(k))$, then there is no H-distinguished tame supercuspidal representations of G.*

For the case $(\mathrm{Sp}_{4n}(\mathbb{F}_q), \mathrm{Sp}_{4n}(\mathbb{F}_q)^\theta)$, we calculate the dimension formula for a lower rank case $n = 1$. In this lower rank case, we apply the fact that the dimension formula is less than one for all irreducible representations, and complete the classification of distinguished cuspidal representations.

Theorem 5.1.3. *If a symmetric pair (G, H) is $(\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{F}_{q^2}))$, then θ_{10} is the unique H-distinguished irreducible cuspidal representation of G.*

Besides the approach of Hakim and Murnaghan in [28], the result of the automorphic period integral could also imply some results of the H-distinguished representations over local fields. At the end of this chapter, we recall that the automorphic period integral of cuspidal automorphic forms in the case $(\mathrm{Sp}_{2n}(\mathbb{A}), \mathrm{Sp}_{2m}(\mathbb{A}) \times \mathrm{Sp}_{2(n-m)}(\mathbb{A}))$ is vanishing which is proved by Ash, Ginzburg and Rallis [11]. Moreover, Prasad and Schulze-Pillot [60] proved that each H-distinguished supercuspidal representation is a local component of a cuspial automorphic representation whose period integral is non zero. Therefore, we obtain the following corollary.

Corollary 5.1.2. *There is no $\mathrm{Sp}_{2m}(k) \times \mathrm{Sp}_{2(n-m)}(k)$-distinguished supercuspidal representation of $\mathrm{Sp}_{2n}(k)$.*

5.2 Involution and Bruhat-Tits buildings

In this section, we identify the θ-fixed vertices of the Bruhat-Tits building of G. Denote by $\mathscr{B}(G)$ the (extended) Bruhat-Tits building (see [74, §2.1]) of G. Let $\mathfrak{f}$ (resp. $\mathfrak{f}_E$) be the residue field of k (resp. E).

Let $\widetilde{\mathbf{G}} = \mathbf{G} \rtimes \langle\theta\rangle$ be the semi-product of the order 2 group $\langle\theta\rangle$ with $\mathbf{G}$. The group $\widetilde{\mathbf{G}}$ is a non-connected reductive group over k, whose Bruhat-Tits building $\mathscr{B}(\widetilde{G})$ is same as the building $\mathscr{B}(G)$. Therefore, the action θ on $\mathscr{B}(\widetilde{G})$ induces an involution of $\mathscr{B}(G)$, also denoted by θ. The relation between the θ-fix points $\mathscr{B}(G)^\theta$ and $\mathscr{B}(H)$ are given by the following theorem in Prasad and Yu's paper [61].

Proposition 5.2.1 ([61, (1.9) The main theorem])**.** *Let the residue characteristic of k be odd and H be the connected component of the symmetric subgroup*

G^θ. *There is an* $H(k)$*-equivariant toral map* $\iota\colon \mathscr{B}(H) \to \mathscr{B}(G)$ *whose image is* $\mathscr{B}(G)^\theta$*, uniquely defined up to translation by an element of* $X_*(Z(H)^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$*, and compatible with unramified change of base field.*

Here, a toral map ι maps $A(H,T)$ to $A(G,S)$ by affine transformation, for every maximal k-split torus T of H and a maximal k-split torus S of G such that $T \subset S$. Applying this theorem, we can find which vertices of $\mathscr{B}(G)$ are stable under θ.

Let x be a point in the building $\mathscr{B}(G)$ and G_x be the stabilizer of x. Referring to §3.4 and §3.5 in [74], let $\overline{G}_x$ be the group defined over $\mathfrak{f}$ of the reduction mod $\mathfrak{p}$ of the $\mathfrak{o}$-group scheme associated to x. If G is simply connected, then $\overline{G}_x$ is connected. By the section §4 in [74], we can identify the finite groups $\overline{\mathrm{SU}(J)}_x$, $\overline{\mathrm{SU}(Q)}_x$, and $\overline{\mathrm{Sp}}_x$.

By Proposition 5.2.1, we can easily obtain the following lemma.

Lemma 5.2.1. *Let* G *be a connected simple and simply-connected group over* k*. If* x *is a vertex in* $\mathscr{B}(G)$ *and* $\theta(x) = x$*, then* $\overline{G}_x^\theta$ *is isomorphic to* $\overline{H}_{\iota^{-1}(x)}$*.*

Using the definition of the action of an involution on a building and Lemma 5.2.1, we conclude the following lemmas.

Lemma 5.2.2. *If* (G,H) *is the symmetric pair* $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(k) \times \mathrm{Sp}_{2n}(k))$*, then, up to* H*-conjugate, there are* $n+1$ *vertices that are* θ*-fixed. Moreover,* $(\overline{G}_x, \overline{G}_x^\theta)$ *is isomorphic to*

$$(\mathrm{Sp}_{4r}(\mathfrak{f}) \times \mathrm{Sp}_{4(n-r)}(\mathfrak{f}), \mathrm{Sp}_{2r}(\mathfrak{f}) \times \mathrm{Sp}_{2r}(\mathfrak{f}) \times \mathrm{Sp}_{2(n-r)}(\mathfrak{f}) \times \mathrm{Sp}_{2(n-r)}(\mathfrak{f})).$$

Lemma 5.2.3. *If* (G,H) *is the symmetric pair* $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E)$ *where* E *is the unramified quadratic extension of* k*, then, up to* H*-conjugate, there are* $n+1$ *vertices that are* θ*-fixed. Moreover,* $(\overline{G}_x, \overline{G}_x^\theta)$ *is isomorphic to*

$$(\mathrm{Sp}_{4r}(\mathfrak{f}) \times \mathrm{Sp}_{4(n-r)}(\mathfrak{f}), \mathrm{Sp}_{2r}(\mathfrak{f}_E) \times \mathrm{Sp}_{2(n-r)}(\mathfrak{f}_E)).$$

Here $\mathfrak{f}_E$ is the quadratic extension of the finite field $\mathfrak{f}$.

Lemma 5.2.4. *If* (G, H) *is the symmetric pair* $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E)$ *where* E *is a ramified quadratic extension of* k*, then there is no* θ*-fixed vertex.*

Lemma 5.2.5. *If* (G, H) *is the symmetric pair* $(\mathrm{SU}_{2n}(E), \mathrm{Sp}_{2n}(k))$ *when* E *is unramified, then, up to* H*-conjugate, there are* $n+1$ *vertices that are* θ*-stable. Moreover,* $(\overline{G}_x, \overline{G}_x^{\theta})$ *is isomorphic to*

$$(\mathrm{SU}_{2r}(\mathfrak{f}) \times \mathrm{SU}_{2(n-r)}(J, \mathfrak{f}), \mathrm{Sp}_{2r}(\mathfrak{f}) \times \mathrm{Sp}_{2(n-r)}(\mathfrak{f})).$$

Lemma 5.2.6. *If* (G, H) *is the symmetric pair* $(\mathrm{SU}_{2n}(J, E), \mathrm{Sp}_{2n}(k))$ *when* E *is ramified, then, up to* H*-conjugate, there is only one* θ*-fixed vertex such that* $(\overline{G}_x, \overline{G}_x^{\theta})$ *is isomorphic to* $(\mathrm{Sp}_{2n}(\mathfrak{f}), \mathrm{Sp}_{2n}(\mathfrak{f}))$.

Lemma 5.2.7. *Let* (G, H) *be the symmetric pair* $(\mathrm{SU}_{4n}(Q, E), \mathrm{SU}_{2n}(D))$*. If* E *is the unramified quadratic extension, then, up to* H*-conjugate, there is no* θ*-fixed vertex. If* E *is a ramified quadratic extension, then, up to* H*-conjugate, there is only one* θ*-fixed vertex such that* $(\overline{G}_x, \overline{G}_x^{\theta})$ *is isomorphic to* $(\mathrm{Sp}_{4n}(\mathfrak{f}), \mathrm{Sp}_{2n}(\mathfrak{f}_E))$.

Lemma 5.2.8. *Let* (G, H) *be the symmetric pair* $(\mathrm{SU}_{4n+2}(Q, E), \mathrm{SU}_{2n+1}(D))$ *for* $n \geq 2$*. If* E *is the unramified quadratic extension, then, up to* H*-conjugate, there is no* θ*-fixed vertex. If* E *is a ramified quadratic extension, then, up to* H*-conjugate, there is only one* θ*-fixed vertex such that* $(\overline{G}_x, \overline{G}_x^{\theta})$ *is isomorphic to* $(\mathrm{Sp}_{4n}(\mathfrak{f}), \mathrm{Sp}_{2n}(\mathfrak{f}_E))$.

5.3 Distinguished tame supercuspidal representations

In this section, we will recall Yu's construction of tame supercuspidal representations, and Hakim and Murnaghan's formula for distinguished tame supercuspidal representations.

5.3.1 Tame supercuspidal representations

Let G be split over a tamely ramified extension of k. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let $\mathscr{B}_{\mathrm{red}}(G)$ be the reduced building of G. That is the building $\mathscr{B}_{red}(G)$ of the derived group of G^{der} of G. Denote by $\mathbf{Z}$ the center of $\mathbf{G}$ and $X_*(\mathbf{Z})$ the group of k-rational cocharacters of $\mathbf{Z}$. Then, $\mathscr{B}(G) = \mathscr{B}_{\mathrm{red}}(G) \times (X_*(\mathbf{Z}) \otimes \mathbb{R})$.

Definition 5.3.1. *A sequence $\vec{\mathbf{G}} = (\mathbf{G}^0, \ldots, \mathbf{G}^d)$ of connected reductive k-groups is a* **twisted Levi sequence** *in $\mathbf{G}$ if*

$$\mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \cdots \mathbf{G}^d = \mathbf{G}$$

and there exists a finite extension E of k such that $\mathbf{G}^0 \otimes E$ splits over E and $\mathbf{G}^i \otimes E$ is a Levi subgroup (that is, an E-Levi E-subgroup) of $\mathbf{G}^d \otimes E$, for all $i \in \{0, \ldots, d\}$. If E can be chosen to be tamely ramified over k, we say that $\vec{\mathbf{G}}$ is tamely ramified.

For a point $x \in \mathscr{B}(G)$, let $\{G_{x,r}\}_{r \geq 0}$ and $\{\mathfrak{g}_{x,r}\}_{r \geq 0}$ be the Moy-Prasad filtrations (see [54]) for the parahoric subgroup $G_{x,0}$ and the Lie algebra $\mathfrak{g}$.

Definition 5.3.2 ([28, Page 50])**.** *A 5-tuple $(\vec{\mathbf{G}}, y, \vec{r}, \rho, \vec{\phi})$ satisfying the following conditions will be called a* **cuspidal G-datum***:*

D1 *$\vec{\mathbf{G}}$ is a tamely ramified twisted Levi sequence $\vec{\mathbf{G}} = (\mathbf{G}^0, \ldots, \mathbf{G}^d)$ in $\mathbf{G}$ and $\mathbf{Z}^0/\mathbf{Z}$ is k-anisotropic, where $\mathbf{Z}^0$ and $\mathbf{Z}$ are the centers of $\mathbf{G}^0$ and $\mathbf{G}^d$, respectively.*

D2 *y is a point in the apartment $\mathscr{A}(\mathbf{G}, \mathbf{T}, k)$, where $\mathbf{T}$ is a tame maximal k-torus of $\mathbf{G}^0$ and E is a Galois tamely ramified extension of k over which $\mathbf{T}$ (hence $\vec{\mathbf{G}}$) splits. (Recall that $\mathscr{A}(\mathbf{G}, \mathbf{T}, k)$ denote the apartment in $\mathscr{B}(\mathbf{G}(E))$ corresponding to $\mathbf{T}$ and $\mathscr{A}(\mathbf{G}, \mathbf{T}, k) = \mathscr{A}(\mathbf{G}, \mathbf{T}, k) \cap \mathscr{B}(G)$.)*

D3 *$\vec{r} = (r_0, \ldots, r_d)$ is a sequence of real numbers satisfying $0 < r_0 < r_1 < \cdots < r_{d-1} \leq r_d$, if $d > 0$, and $r_0 \geq 0$ if $d = 0$.*

D4 *ρ is an irreducible representation of the stabilizer $K^0 = G^0_{[y]}$ of $[y]$ in G^0 such that $\rho|G^0_{y,0+}$ is 1-isotypic and the compactly induced representation $\pi_{-1} = \mathrm{ind}_{K^0}^{G^0}\rho$ is irreducible (hence supercuspidal). Here $[y]$ denotes the image of y in the reduced building of G.*

D5 *$\vec{\phi} = (\phi_0, \dots, \phi_d)$ is a sequence of quasicharacters, where ϕ_i is a quasicharacter of G^i. We assume that $\phi_d = 1$ if $r_d = r_{d-1}$ (with r_{-1} defined to be 0), and in all other cases if $i \in \{0, \dots, d\}$ then ϕ_i is trivial on G^i_{y,r_i^+} but nontrivial on G^i_{y,r_i}.*

Given a cuspidal G-datum $\Psi = (\vec{\mathbf{G}}, y, \vec{r}, \rho, \vec{\phi})$. Let $s_i = r_i/2$, for $i \in \{0, \dots, d-1\}$. Define a series of compact subgroups of G,

$$\begin{aligned} K^0 &= G^0_{[y]}, K^0_+ = G^0_{y,0+}, \\ K^{i+1} &= K^0 G^1_{y,s_0} \cdots G^{i+1}_{y,s_i}, \quad i \in \{0, \dots, d-1\}, \\ K^{i+1}_+ &= K^0_+ G^1_{y,s_0^+} \cdots G^{i+1}_{y,s_i^+}, \quad i \in \{0, \dots, d-1\}. \end{aligned}$$

Let $K = K^d$ and $K_+ = K^d_+$.

Definition 5.3.3 ([28, Definition 2.46])**.** *If ϕ is a quasicharacter of G then the **depth** of ϕ, denoted $r(\phi)$, is the smallest nonnegative real number r that satisfies $\phi|G_{x,r+} = 1$, for some $x \in \mathscr{B}(G)$,*

Definition 5.3.4 ([28, Definition 2.48])**.** *If $r > 0$, $s = r/2$ and $x \in \mathscr{B}(G)$ and S is a subgroup of $G_{x,s+}$ that contains $G_{x,r}$, let $\mathfrak{s}$ be the lattice in $\mathfrak{g}_{x,s+}$ such that $s \supset \mathfrak{g}_{x,r}$ and $e(\mathfrak{s}/\mathfrak{g}_{x,r+}) = S/G_{x,r+}$. An element $X^* \in \mathfrak{g}^*_{x,-r}$ defines a character of S that is trivial on $G_{x,r+}$ as follows:*

$$e(Y + \mathfrak{g}_{x,r^+}) \to \psi(X^*(Y)), \quad Y \subset \mathfrak{s}.$$

*This character of S is said to be **realized** by the element X^* of $g^*_{x,-r}$, or by the coset $X^* + \mathfrak{s}^\bullet$, where*

$$\mathfrak{s}^\bullet = \{Y^* \in \mathfrak{g}^*_{x,-r} \mid Y^*(\mathfrak{s}) \subset \mathfrak{p}\}.$$

As above s and r, $e = e_{x,r} \colon \mathfrak{g}_{x,s^+:r^+} \to G_{x,s^+:r^+}$ is an isomorphism, which is stated in [28].

Definition 5.3.5 (Definition 3.7 [28]). *An element $X^* \in \mathfrak{z}'^{,*}_{-r}$ is G-generic of depth $-r$ if it satisfies the following conditions:*

GE1 *$v(X^*(H_a)) = -r$, for all $a \in \Phi(\mathbf{G}, \mathbf{T}) \backslash \Phi(\mathbf{G}', \mathbf{T})$, where $H_a = da^\vee(1)$, $a^\vee$ is the coroot associated to a.*

GE2 *Suppose ϖ_r is an element of the algebraic closure $\overline{k}$ of k of valuation r and $\tilde{X}^*$ is the residue class of $\varpi_r X^*$ in the residue field of $\overline{k}$. Then the isotropy subgroup of$\tilde{X}^*$ in the Weyl group of $\Phi(\mathbf{G}, \mathbf{T})$ coincides with the Weyl group of $\Phi(\mathbf{G}', \mathbf{T})$.*

Definition 5.3.6 (Definition 3.9 [28]). *Let $r \in \mathbb{R}$, $r > 0$. A quasicharacter ϕ of G' is said to be G-generic (relative to y) of depth r if ϕ is trivial on G'_{y,r^+}, and nontrivial on $G'_{y,r}$, and there exists a G-generic element $X^* \in \mathfrak{z}'^{,*}_{-r}$ of depth r that realizes the restriction of ϕ to $G'_{y,r}$.*

Lemma 5.3.1 (Lemma 8.1 [76]). *If the residual characteristic of k is not a torsion prime for $\psi(G)^\vee$, then **GE1** implies **GE2**.*

Corollary 5.3.1. *If $\mathbf{G}$ is $\mathbf{SU}(Q)$ or $\mathbf{Sp}$ and the character of the residue field $\mathfrak{f}$ of k is odd, then the condition **GE1** implies **GE2**.*

Proof. This is a straightforward to prove this corollary by Lemma 5.3.1. □

Definition 5.3.7 (Definition 3.11 [28]). *If $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ satisfies Conditions **D1-D5** (that is, Ψ is a cuspidal G-datum), and if ϕ_i is G^{i+1}-generic of depth r_i relative to y for all $i \in \{0, \ldots, d-1\}$, then the G-datum Ψ is said to be* **generic**. *In this case, the reduced G-datum $(\vec{\mathbf{G}}, \pi_{-1}, \phi)$ is called generic.*

In order to construct tame supercuspidal representations, we also need that $\mathbf{G}^i$ in $\vec{\mathbf{G}}$ for all $1 \leq i \leq d$ satisfies the following Hypothesis $C(\mathbf{G})$.

Hypothesis $C(\mathbf{G})$. Let ϕ be a quasicharacter of G of positive depth r. If $x \in \mathscr{B}(G)$, then $\phi|G_{x,(r/2)^+}$ is realized by an element of $\mathfrak{z}^*_{-r}$, where $\mathfrak{z}^*$ is the dual of the center of g.

Lemma 5.3.2. *Let n be an integer and more than two. Then Hypothesis $C(\mathbf{U}(J))$, $C(\mathbf{U}(Q))$ and $C(\mathbf{Sp})$ are satisfied, where the unitary group $\mathbf{U}$ is defined over k.*

Proof. The proofs for Hypothesis $C(\mathbf{U}(J))$ and $C(\mathbf{U}(Q))$ are similar to the proof for Hypothesis $C(\mathbf{GL_n})$. One can refer to [28, Lemma 2.50]. For Hypothesis $C(\mathbf{Sp})$, since the derived subgroup $\mathrm{Sp}(k)$ is itself, the quasicharacter is trivial. It is a straightforward to prove Hypothesis $C(\mathbf{Sp})$. □

By Lemma 5.3.2, we drop this hypothesis in our cases.

Using this generic data, Yu constructed tame supercuspidal representations One can refer to [76] and [28, Section 3].

5.3.2 Hakim and Murnaghan's formula

In this section, we will recall Hakim and Murnaghan's formula in [28] for possible distinguished tame supercuspidal representations.

We follow the notations of [28]. Let G act on θ via $g\cdot\theta = \mathrm{Ad}(g)\circ\theta\circ\mathrm{Ad}(g)^{-1}$, and Θ be the G-orbit of θ. Fix $\theta \in \Theta$. Let Θ' be the K-orbit of θ. Define $m_K(\Theta') = \#\{KgG^\theta \in K\backslash G/G^\theta \mid g\cdot\theta \in \Theta'\}$. Indeed, $m_K(\Theta')$ is independent with the choice of $\theta \in \Theta'$. Given a G-orbit Θ and $\theta \in \Theta$, define abelian groups

$$\begin{aligned} G_\theta &= \{g \in G \mid g\theta(g)^{-1} \in Z\}, \\ Z^1_\Theta &= \{z \in Z \mid \theta(z) = z^{-1}\}, \\ B^1_\Theta &= \{z\theta(z)^{-1} \mid z \in Z\}, \\ H^1_\Theta &= Z^1_\Theta / B^1_\Theta. \end{aligned}$$

Let $\mathcal{G}_\theta = G_\theta/(ZG^\theta)$ and $\mathcal{K}_\theta$ be the image of $K \cap G_\theta$ in $\mathcal{G}_\theta$. Then, Hakim and Murnaghan have the following two lemma to evaluate the m_K

Lemma 5.3.3 ([28, Lemma 2.8])**.** *If Θ' is a K-orbit in Θ and $Z \subset K$ then $m_K(\Theta') \leq |H^1_\Theta| < \infty$*

Lemma 5.3.4 ([27, Lemma 3.1])**.** *The constant $m_K(\Theta')$ is identical to the order of the elementary finite abelian 2-group $\mathcal{G}_\theta/\mathcal{K}_\theta$ and thus is is a power of 2.*

Definition 5.3.8 (Definition 3.13 [28])**.** *If θ is an involution of G and $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ is a cuspidal G-datum then Ψ is* **weakly θ-symmetric** *if $\theta(\vec{\mathbf{G}}) = \vec{\mathbf{G}}$, and $\phi_i \circ \theta = \phi_i^{-1}$, for all $i \in \{0, \ldots, d\}$, that is, each ϕ_i is θ-symmetric. If Ψ is weakly θ-symmetric and $\theta([y]) = [y]$ then we say* **θ-symmetric**.

Recall that $K^0 = G^0_{[y]}$. Define a representation of $K^0(\Psi)$ by

$$\rho'(\Psi) = \rho(\Psi) \otimes \prod_{i=0}^{d} (\phi_i | K^0(\Psi)).$$

Let $\chi^{\mathcal{M}_i}(k) = \det(\mathrm{Int}(k))|W_i^+)^{\frac{q-1}{2}}$, referring to [28, Section 5.4]. Define a quadratic character of $K^0(\Psi)^\theta$, where Ψ is θ-symmetric, by

$$\eta'_\theta(k) = \prod_{i=0}^{d-1} \chi^{\mathcal{M}_i}(f'_i(k)).$$

Theorem 5.3.1 (Theorem 5.26 [28])**.** *Let Ψ, ξ, Θ, K, etc. be as above.*

(1) $\langle \Theta, \xi \rangle_G = \sum_{\Theta' \in \Theta^K} m_K(\Theta') \langle \Theta', \xi \rangle_K$.

(2) If $\theta \in \Theta' \in \Theta^K$ and if there exists a θ-symmetric refactorization of Ψ then

$$\langle \Theta', \xi \rangle_K = \dim \mathrm{Hom}_{K^0(\Psi)^\theta}(\rho'(\Psi), \eta'_\theta(\Psi)).$$

(3) If $\theta \in \Theta' \in \Theta^K$ and $\langle \Theta', \xi \rangle_K \neq 0$, then there exists $k \in K$ and a refactorization $\dot{\Psi}$ of Ψ such that ${}^k\dot{\Psi}$ is θ-symmetric element of ξ. In this case, $\dot{\Psi}$ is θ'-symmetric where $\theta' = k^{-1} \cdot \theta \in \Theta'$.

(4) Suppose that $\langle \Theta', \xi \rangle_G \neq 0$, Choose $\theta \in \Theta$ such that some refactorization of Ψ is θ-symmetric. Let $g_1, \cdots, g_m$ be a maximal (finite) sequence in G such

that $g_j\theta(g_j^{-1}) \in K^0 = K^0(\Psi)$ *and the* K*-orbits of the elements* $\theta_j = g_j \cdot \theta$ *are distinct. Then*

$$\mathrm{Hom}_{G^\theta}(\pi(\Psi), 1) \cong \bigoplus_{j=1}^{m} m_K(\Theta')\mathrm{Hom}_{K^{0,\theta_j}}(\rho'(\Psi), \eta'_{\theta_j}(\Psi)). \tag{5.3.1}$$

Hence, letting Θ'_j *be the* K*-orbit of* θ_j, $j \in \{1, \cdots, m\}$,

$$\langle \Theta, \xi \rangle_G = \sum_{j=1}^{m} m_K(\Theta'_j)\langle \Theta'_j, \xi \rangle_K.$$

For $0 \le i \le d$, let Z^i be the center of G^i, let $\mathfrak{z}$ be the center of the Lie algebra $\mathfrak{g}^i$ of G^i, and let $\mathfrak{z}^{i*}$ be the dual of $\mathfrak{z}^i$. Let $Z^i_{\mathrm{reg}} = \{g \in Z^i \mid Z_G(g)^\circ = G^i\}$, $\mathfrak{z}^i_{\mathrm{reg}} = \{X \in \mathfrak{z}^i \mid Z_G(X)^\circ = G^i\}$, and $\mathfrak{z}^{i*}_{\mathrm{reg}} = \{X^* \in \mathfrak{z}^{i*} \mid Z_G(X^*)^\circ = G^i\}$.

Proposition 5.3.1 ([55, Lemma 7.6])**.** *Let* $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ *be a* θ*-symmetric cuspidal* G*-datum such that* $d \ge 1$*. Fix* i *such that* $0 \le i \le d-1$*. Then* Z^i_{reg}, $\mathfrak{z}^i_{\mathrm{reg}}$, *and* $\mathfrak{z}^{i*}_{\mathrm{reg}}$ *contain* θ*-split elements.*

In order to calculate the H-conjugacy class of θ-split semi-simple elements, we choose the following different symplectic form for the symplectic groups.

If τ is in ${k^\times}^2$, we can choose another symplectic form J'_{4n}

$$\mathrm{Sp}_{4n}(J') = \{g \in M_{4n\times 4n}(k) \mid gJ'_{4n}g^t = J'_{4n}\},$$

where

$$J'_{4n} = \begin{pmatrix} J_{2n} & \\ & J_{2n} \end{pmatrix}.$$

The involution is $\mathrm{Ad}(\varepsilon)$, where $\varepsilon = \mathrm{diag}\{I_{2n}, -I_{2n}\}$.

If τ is in $k^\times \smallsetminus {k^\times}^2$ and E is the quadratic extension $k(\sqrt{\tau})$, we realize $\mathrm{Sp}_{4n}(k)$ as a subgroup of $\mathrm{GL}_{4n}(E)$, that is,

$$\mathrm{Sp}_{4n}(k) \cong \{g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{GL}_{4n}(E) \mid gJ'_{4n}g^t = J'_{4n}\}.$$

The involution is $\mathrm{Ad}(\varepsilon)$, where $\varepsilon = \mathrm{diag}\{\sqrt{\tau}1_{2n}, -\sqrt{\tau}1_{2n}\}$, and the symmetric subgroup is isomorphic to $\mathrm{Sp}_{2n}(E)$.

Recall that $\mathfrak{g} = Lie(G)$ and $\mathfrak{h} = Lie(H)$. Let θ act on $\mathfrak{g}$ by their differentials and denote $\mathfrak{g}^- = \{a \in \mathfrak{g} \mid \theta(a) = -a\}$. Note that H acts on $\mathfrak{g}^-$ by the adjoint action.

Let us describe $\mathfrak{g}^-$ for $\mathrm{Sp}_{4n}(J')$ in the split case and the non-split case,

1. $\mathfrak{g}^- = \{\begin{pmatrix} & \beta \\ -\tilde{\beta} & \end{pmatrix} \mid \beta \in M_{2n\times 2n}(F)\}$, if τ is in $k^{\times 2}$ };

2. $\mathfrak{g}^- = \{\begin{pmatrix} & \beta \\ -\tilde{\beta} & \end{pmatrix} \mid \beta \in M_{2n\times 2n}(E), \overline{\beta} + \tilde{\beta} = 0\}$, if τ is in $k^\times \smallsetminus k^{\times 2}$.

Let $X \in M_{2n\times 2n}(k)$ and $Y \in M_{n\times n}(k)$. Denote $\widetilde{X} := J_{2n}^{-1} X^t J_{2n}$ and $\widehat{Y} := w_n^{-1} Y^t w_n$.

Proposition 5.3.2 ([25, Corollary 2.10])**.** *If $A \in M_{2n\times 2n}(k)$ and $\widetilde{A} = A$, then there exists $g \in \mathrm{Sp}(J_{2n})$ such that gAg^{-1} has the form $\begin{pmatrix} C & \\ & \widehat{C} \end{pmatrix}$.*

Lemma 5.3.5. *Let $B \in M_{2n\times 2n}(k)$. If $B\widetilde{B}$ and $\widetilde{B}B$ are semi-simple and* $\mathrm{rank}(B) = \mathrm{rank}(B\widetilde{B})$*, then there exist g and h in $\mathrm{Sp}_{2n}(k)$ such that*

$$B = h \begin{pmatrix} I_m & & & \\ & 0_{n-m} & & \\ & & 0_{n-m} & \\ & & & \widehat{B_1} \end{pmatrix} g$$

for some $B_1 \in \mathrm{GL}_m(k)$.

Proof. Since $B\widetilde{B}$ is semi-simple and $B\widetilde{B} = \widetilde{B\widetilde{B}}$, by Proposition 5.3.2, there exists an element h in $\mathrm{Sp}_{2n}(J)$ such that

$$hB\widetilde{B}h^{-1} = \begin{pmatrix} B_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \widehat{B_1} \end{pmatrix} \tag{5.3.2}$$

.

Without loss of generality, assume that $B\widetilde{B}$ is equal to the right side of Equation (5.3.2). Then, BJB^t has the form $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$. Let v_i be the i-th row of B. Then, the vector space spanned by $\{v_1, v_2, \ldots, v_n\}$ is an isotropic space under the symplectic form J. There exists $g \in \mathrm{Sp}_{2n}(k)$ such that $Bg_1 = \left(\begin{smallmatrix} \eta & 0 \\ \alpha & \beta \end{smallmatrix}\right)$, where $\eta = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)_{n\times n}$. Therefore, we have

$$\eta\widehat{\beta} = \begin{pmatrix} B_1 & \\ & 0 \end{pmatrix}, \tag{5.3.3}$$

$$\alpha\widehat{\beta} = \beta\widehat{\alpha}. \tag{5.3.4}$$

By Equation (5.3.3) and $\mathrm{rank}(B) = \mathrm{rank}(B\widetilde{B})$, we obtain

$$\begin{aligned} &\mathrm{rank}(B_1) = \mathrm{rank}(\eta\widehat{\beta}) \le \min\{\mathrm{rank}(\eta), \mathrm{rank}(\beta)\}, \\ &\mathrm{rank}(B) \ge \mathrm{rank}(\eta) + \mathrm{rank}(\beta) \ge 2\mathrm{rank}(B_1). \end{aligned}$$

Then, $\mathrm{rank}(\eta) = \mathrm{rank}(\beta) = \mathrm{rank}(B_1)$.

Assume that $\alpha = \left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)$ and $\beta = \left(\begin{smallmatrix} U & V \\ S & T \end{smallmatrix}\right)$. According to Equation (5.3.3), we have $T = \widehat{B_1}$ and $V = 0$. Since $\mathrm{rank}(\beta) = \mathrm{rank}(B_1)$, the matrix U is also 0. Since $U = V = 0$ and $\mathrm{rank}(B) = \mathrm{rank}(\eta) + \mathrm{rank}(\beta)$, it follows that $Y = 0$. According to Equation (5.3.4), we conclude that $X = 0$ and

$$ZB_1 + W\widehat{S} = S\widehat{W} + \widehat{B_1}\widehat{Z}. \tag{5.3.5}$$

Let

$$g_2 = \begin{pmatrix} I & & & \\ \widehat{S}B_1^{-1} & I & & \\ -\widehat{w}B_1^{-1} & 0 & I & \\ -\widehat{Z}B_1^{-1} & -\widehat{B}_1^{-1}W & -\widehat{B}_1^{-1}S & I \end{pmatrix}.$$

By Equation (5.3.5), g_2 is in $\mathrm{Sp}_{2n}(J)$. Then, we have

$$B = \begin{pmatrix} I_m & & & \\ & 0 & & \\ & & 0 & \\ & & & \widehat{B_1} \end{pmatrix} g_2^{-1}g_1^{-1}.$$

□

If τ is in $k^\times \setminus k^{\times 2}$, we identify $\mathfrak{g}^-$ as

$$\{A \in M_{2n\times 2n}(E) \mid \overline{A} + \widetilde{A} = 0\}.$$

The symmetric subgroup H is $\mathrm{Sp}_{2n}(J, E)$. The H-adjoint action on $\mathfrak{g}^-$ is same as the action of H given by $g \cdot A \to gA\overline{g}^{-1}$.

Lemma 5.3.6. *If A is in $\mathfrak{g}^-$, then there exists an element g in $\mathrm{Sp}_{2n}(E)$ such that*

$$gA\overline{g}^{-1} = \begin{pmatrix} 0_{n-m} & & & \\ & & B & \\ & I_m & & \\ & & & 0_{n-m} \end{pmatrix}.$$

Proof. Let $V \simeq E^{2n}$ be the vector space of $\mathrm{Sp}_{2n}(E)$ with the symplectic form J. We consider A as a linear transformation over V, satisfying $\langle Ax, y\rangle = -\langle x, \overline{A}y\rangle$. Let V_1 be the kernel of of A and V_2 be the image of A. For any x in V_1, since $\langle Ax, y\rangle = 0$ for all y in V and $\langle Ax, y\rangle = -\langle x, \overline{A}y\rangle$, we have $\langle x, \overline{B}y\rangle = 0$ for all y. Hence, V_1 is orthogonal to $\overline{V_2}$. By [35, Lemma 4.2], $A\overline{A}$ has the same rank with A. Then, $\overline{V_1}$ is the kernel of $A\overline{A}$.

Next, we prove that $V_1 \cap \overline{V_2} = 0$. If v be in the intersection $V_1 \cap \overline{V_2}$, then there exists a vector u such that $\overline{A}u = v$ and $Av = 0$. We have $A\overline{A}u = 0$ and $u \in \overline{V_1}$. Hence, $\overline{B}u = 0$ and $v = 0$.

Now, we have $V = V_1 \oplus \overline{V_2}$. In addition, V_1 and $\overline{V_2}$ are symplectic subspace of V. Therefore, there exists a element g in $\mathrm{Sp}_{2n}(E)$ such that $gA\overline{g}^{-1}$ has the form $\begin{pmatrix} 0 & \\ & B_1 \\ & & 0 \end{pmatrix}$, where $\overline{B_1} + \widetilde{B_1} = 0$ and $\det(B_1) \neq 0$.

Without loss of generality, assume that $\det(B) \neq 0$. We consider the group $R_{E/k}(\mathbf{Sp}_{2n}(J))$ acts on $\mathfrak{g}^-$. According to Lemma 5.3.5, A and B are in the same orbit under the action of $R_{E/k}(\mathbf{Sp}_{2n}(J))$ if and only if $A\overline{A}$ and $B\overline{B}$ are $\mathrm{Sp}_{2n}(E)$-conjugate. Since $A\overline{A}$ is conjugate to $\overline{A}A$ and $A\overline{A} = \widetilde{A\overline{A}}$, by Proposition 5.3.2, $A\overline{A}$

is $\mathrm{Sp}_{2n}(E)$-conjugate to the matrix of the form

$$\begin{pmatrix} B & & & & & \\ & C & & & & \\ & & \overline{B} & & & \\ & & & \widehat{\overline{B}} & & \\ & & & & \widehat{C} & \\ & & & & & \widehat{B} \end{pmatrix},$$

where C is a semi-simple element in $\mathrm{GL}_n(k)$, and $B \in \mathrm{GL}_n(E)$ is semi-simple and not conjugate to $\overline{B}$.

Applying the argument similar to [27, Section 4.2], for any semi-simple element $C \in M_{n\times n}(k)$, we can find an element D such that D is $\mathrm{GL}_n(k)$-conjugate to C and $D = \widehat{D}$. Let

$$A' = \begin{pmatrix} & & B & & & \\ & 0 & & & D & \\ I & & & & & \\ & & & & & \widehat{\overline{B}} \\ & I & & & 0 & \\ & & & I & & \end{pmatrix}.$$

Then, $A'\overline{A'}$ is $\mathrm{Sp}_{2n}(E)$-conjugate to $A\overline{A}$. Hence, A and A' are in the same orbit under the action of $R_{E/k}(\mathbf{Sp}_{2n}(J))$. Next, we shall prove that A and A' are also in the same orbit under the action of $\mathrm{Sp}_{2n}(J, E)$.

Let $\mathcal{O}$ be the orbit of A' under the action of $R_{E/k}(\mathbf{Sp}_{2n}(J))$, and $\mathcal{O}(k)$ be the k-rational points of $\mathcal{O}$. Let $\mathbf{H}_{A'}$ be the stabilizer of A' in $R_{E/k}(\mathrm{Sp}_{2n}(J))$. Then, $\mathbf{H}_{A'}$ is defined over k. Since the $\mathrm{Sp}_{2n}(E)$-orbits in $\mathcal{O}(k)$ correspond to the kernel from the Galois cohomology $H^1(k, \mathbf{H}_{A'})$ to $H^1(k, R_{E/k}(\mathbf{Sp}_{2n}(J)))$. We will show that $H^1(k, \mathbf{H}_{A'})$ is trivial. Therefore, two elements in $\mathfrak{h}^-$ are in the same orbit under the action of $R_{E/k}(\mathbf{Sp}_{2n}(J))$ if and only if these two elements are in the same orbit under the action of $\mathrm{Sp}_{2n}(J, E)$. Indeed, applying the standard form

A', we can conclude that $\mathbf{H}_{A'}$ is isomorphic to

$$\prod_{E'} R_{E'/k}(\mathbf{Sp}) \times \prod_{E''} R_{E''/k}(\mathbf{SU}(D_{E''})).$$

Here E' and E'' are finite field extensions of k and $D_{E''}$ is the quaternion algebra of E''. Further, $\mathrm{SU}(D_{E''})$ is E''-form of the symplectic group. According to Shapiro's Lemma, we have

$$H^1(k, R_{E'/k}(\mathbf{Sp})) = H^1(k, \mathbf{Sp}(J)) \text{ and } H^1(k, R_{E''/k}(\mathbf{SU}(D_{E''})) = H^1(k, \mathbf{SU}(D_k)).$$

In addition, $H^1(k, \mathbf{Sp})$ and $H^1(k, \mathbf{SU}(D_k))$ are trivial. Therefore, $H^1(k, \mathbf{H}_{A'})$ is trivial. A and A' are in the same orbit under the action of $\mathrm{Sp}_{2n}(E)$. We prove this lemma. $\square$

5.3.3 The symmetric pair $(\mathrm{U}_{2n}(J, E), \mathrm{Sp}_{2n}(k))$

In this section, let (G, H) be the symmetric pair $(\mathrm{U}_{2n}(J, E), \mathrm{Sp}_{2n}(k))$. We will prove that there is no H-distinguished tame supercuspidal representations of G. In order to prove this, it is sufficient to show that for any θ-symmetric data, $\langle \Theta', \xi \rangle_K$ is zero.

First, we use an argument, similar to [27, Section 5.3], to prove the triviality of η'_θ.

Lemma 5.3.7. *The quadratic character η'_θ is trivial.*

Proof. Recall that the character $\chi^{\mathcal{M}_i}$ is given by $\det(\mathrm{Int}(k)|W_i^+)^{(p-1)/2}$, where $W_i^+ = J^{i+1,\theta}/J_+^{i+1,\theta}$. We consider the problem on the Lie algebra and obtain that the character is $\det(\mathrm{Ad}(k)|\mathfrak{M}_i^+)^{(p-1)/2}$, where $\mathfrak{M}_i^+ = \mathfrak{J}^{i+1,\theta}/\mathfrak{J}_+^{i+1,\theta}$ and is an $\mathbb{F}_p$-vector space.

In fact, the $\mathbb{F}_p$-linear structure on $\mathfrak{M}_i^+$ extends naturally to an $\mathfrak{f}$-linear structure. By a classical "transitivity of norms" formula, we have

$$\det_{\mathfrak{f}}(\mathrm{Ad}(k)|\mathfrak{M}_i^+) = N_{\mathfrak{f}/\mathbb{F}_p}(\det_{\mathbb{F}_q}(\mathrm{Ad}(k)|\mathfrak{M}_i^+)).$$

Since η'_θ is a character of $K^{0,\theta}$ and factors through $K_+^{0,\theta}$, the quadratic character η'_θ is a character of $K^{0,\theta}/K_+^{0,\theta}$. Using the fact that $K^{0,\theta}/K_+^{0,\theta}$ is isomorphic to $\prod_i \mathrm{Sp}_{2i}(\mathbb{F}_q)$, the character η'_θ is a character of $\prod_i \mathrm{Sp}_{2i}(\mathbb{F}_{q_i})$ by Lemma 5.2.5 and 5.2.6. However, the derived group of $\prod_i \mathrm{Sp}_{2i}(\mathbb{F}_q)$ is itself. Hence, η'_θ is trivial. □

Theorem 5.3.2. *If (G, H) is the symmetric pair $(\mathrm{U}_{2n}(J, E), \mathrm{Sp}_{2n}(k))$, then there is no H-distinguished tame supercuspidal representation of G.*

Proof. To prove that there is no H-distinguished tame supercuspidal representation of G, by Theorem 5.3.1 we need only to show that $\mathrm{Hom}_{K^{0,\theta}}(\rho'(\Psi), \eta'_\theta(\Psi))$ is zero for all θ-symmetric data Ψ. Since η'_θ is trivial by Lemma 5.3.7,

$$\mathrm{Hom}_{K^{0,\theta}}(\rho'(\Psi), \mathbb{1}) = \mathrm{Hom}_{K^{0,\theta}}(\rho(\Psi), \prod_{i=0}^{d}(\phi_i | K^0(\Psi))).$$

Since $\mathfrak{g}^- = \{A \in \sqrt{\tau} M_{2n\times 2n}(F) \mid \widetilde{A} = A\}$, by Proposition 5.3.2, every θ-split semi-simple element in the Lie algebra is H-conjugate to $\begin{pmatrix} C & \\ & \widehat{C} \end{pmatrix}$. By Proposition 5.3.1, $\mathfrak{z}^i_{\mathrm{reg}}$ contains θ-split elements. Since Ψ is θ-symmetric, $(G^0, G^{0,\theta})$ is isomorphic to $(\prod_i \mathrm{U}(J, E'_i), \prod_i \mathrm{Sp}(E_i))$ where E_i is an extension field of k and E'_i is a quadratic extension of E_i. Since θ fixes the point $[y]$, the finite group $K^{0,\theta}/K_+^{0,\theta}$ is isomorphic to a product of some symplectic groups by Lemma 5.2.5 and 5.2.6. In order to

$$\mathrm{Hom}_{K^{0,\theta}}(\rho(\Psi), \prod_{i=0}^{d}(\phi_i | K^0(\Psi))) \neq 0,$$

we need that $\prod_{i=0}^{d}(\phi_i | K^0(\Psi))$ is trivial on $K_+^{0,\theta}$. Since the derived group of $K^{0,\theta}/K_+^{0,\theta}$ is itself, $\prod_{i=0}^{d}(\phi_i | K^0(\Psi))$ is trivial on $K^{0,\theta}$. Therefore,

$$\mathrm{Hom}_{K^{0,\theta}}(\rho(\Psi), \prod_{i=0}^{d}(\phi_i | K^0(\Psi))) = \mathrm{Hom}_{K^{0,\theta}/K_+^{0,\theta}}(\overline{\rho}, \mathbb{1}). \tag{5.3.6}$$

Referring to Lemma 5.2.5 and Lemma 5.2.6, we can reduce to the following cases.

If E_i' is the unramified quadratic extension, then

$$\mathrm{Hom}_{K^{0,\theta}/K_+^{0,\theta}}(\overline{\rho}, \mathbb{1}) = \mathrm{Hom}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q_i})}(\overline{\rho}, \mathbb{1}),$$

where $\overline{\rho}$ is a cuspidal representation of $\mathrm{U}_{2n}(J, \mathbb{F}_{q_i^2})$. By Corollary 5.4.1,

$$\mathrm{Hom}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q_i})}(\overline{\rho}, \mathbb{1}) = 0.$$

If E_i' is a ramified quadratic extension,

$$\mathrm{Hom}_{K^{0,\theta}/K_+^{0,\theta}}(\overline{\rho}, \mathbb{1}) = \mathrm{Hom}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q_i})}(\overline{\rho}, \mathbb{1})$$

where $\overline{\rho}$ is a cuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{F}_{q_i})$. By the cuspidality of $\overline{\rho}$,

$$\mathrm{Hom}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q_i})}(\overline{\rho}, \mathbb{1}) = 0.$$

By Equation (5.3.6), $\mathrm{Hom}_{K^{0,\theta}}(\rho(\Psi), \prod_{i=0}^{d}(\phi_i|K^0(\Psi)))$ is zero. Then,

$$\mathrm{Hom}_{K^{0,\theta}}(\rho'(\Psi), \eta'_{\theta_j}) = 0$$

for all θ-symmetric data Ψ. □

5.3.4 The symmetric pair $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E))$

In this section, let (G, H) be the symmetric pair $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E))$.

First, we calculate the double cosets $K\backslash G/H$, where K is the maximal compact subgroup $\mathrm{Sp}_{4n}(\mathfrak{o})$. Let E be the unramified field extension over k and X be the space of skew-symmetric matrices $g \in G$. Then $g \to g^{-1}\varepsilon g J$ gives a bijection to from $H\backslash G$ to X. The group G acts on X by $g \cdot x = gxg^t$.

Let $\Lambda_n^+ = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$. Denote the matrix $\left(\begin{smallmatrix} & \varpi^a \\ -\varpi^a & \end{smallmatrix}\right)$ by Π_a. For $\lambda \in \Lambda_n^+$, let Π^λ be the matrix

$$\mathrm{diag}\{\Pi_{\lambda_1}, \Pi_{\lambda_2}, \ldots, \Pi_{\lambda_n}, -\Pi_{-\lambda_n}, \ldots, -\Pi_{-\lambda_1}\},$$

and zeros elsewhere.

Lemma 5.3.8. *As a disjoint union,*

$$X = \cup_{\lambda \in \Lambda_n^+} K \cdot \Pi^{\Lambda}.$$

Proof. The proof is similar to [52, Lemma 3.1]. $\square$

Lemma 5.3.9. *The quadratic character η'_θ is trivial.*

Proof. An argument similar to the one used in Lemma 5.3.7 shows that η'_θ is trivial. $\square$

Theorem 5.3.3. *Let (G, H) be the symmetric pair $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E))$. A tame supercuspidal representation $\pi(\Psi)$ is H-distinguished if and only if Ψ satisfies the following conditions, up to a K-conjugation and a θ-symmetric refactorization.*

- *The quadratic extension E is unramified.*

  1. *When d is 0, y is θ-fixed and $\overline{\rho}$ is a distinguished cuspidal representation of the symmetric pair $(G_{y,0}/G_{y,0+}, (G_{y,0}/G_{y,0+})^\theta)$, that is,*

  $$\prod_i (\mathrm{Sp}_{4r_i}(\mathfrak{f}_i) \times \mathrm{Sp}_{4(n_i - r_i)}(\mathfrak{f}_i), \mathrm{Sp}_{2r}(\mathfrak{f}_{E_i}) \times \mathrm{Sp}_{2(n_i - r_i)}(\mathfrak{f}_{E_i})).$$

  2. *When $d > 0$, the symmetric pair $(G^0, G^{0,\theta})$ is isomorphic to*

  $$(\prod_i \mathrm{U}_2(Q, E_i), \prod_i \mathrm{SU}_1(D_i).$$

- *The quadratic extension E is ramified.*

  1. *The depth d is more than zero.*
  2. *The symmetric pair $(G^0, G^{0,\theta})$ is isomorphic to*

  $$(\prod_i \mathrm{U}(E_i), \prod_i \mathrm{SU}(D_i)).$$

  The point y is θ-fixed point and $\overline{\rho}$ is a distinguished cuspidal representation of the symmetric pair $(G_{y,0}/G_{y,0+}, (G_{y,0}/G_{y,0+})^\theta)$, that is,

  $$\prod_i (\mathrm{Sp}_{4n_i}(\mathfrak{f}_i), \mathrm{Sp}_{2n_i}(\mathfrak{f}_{E_i})).$$

Proof. First, assume that $\pi(\Psi)$ is H-distinguished tame supercuspidal representation. Then, there exist Θ' and ξ such that $\langle \Theta', \xi \rangle_K$ is nonzero. After a K-conjugation and a refactorization, we can assume that Ψ is θ-symmetric.

In order to prove this theorem, we consider two cases, $d = 0$ and $d > 0$.

Let $d = 0$. The supercuspidal representation $\pi(\Psi)$ is isomorphic to c-$\mathrm{Ind}_{G_y}^{G}\rho$, where $\overline{\rho}$ is a cuspidal representation of $\overline{G}_y$. If E is a ramified quadratic extension of k, by Lemma 5.2.4, there does not exist a vertex y of $\mathscr{B}(G)$ such that y is θ-fixed. By Theorem 5.3.1, there is no θ-symmetric data Ψ with $d = 0$. Therefore, there is no H-distinguished depth-zero distinguished representation. If E is the unramified quadratic extension of k, then for a θ-fixed y, $(\overline{G}_y, \overline{G}_y^{\theta})$ is isomorphic to

$$(\mathrm{Sp}_{4r}(\mathfrak{f}) \times \mathrm{Sp}_{4(n-r)}(\mathfrak{f}), \mathrm{Sp}_{2r}(\mathfrak{f}_E) \times \mathrm{Sp}_{2(n-r)}(\mathfrak{f}_E)),$$

by Lemma 5.2.3. According to Lemma 5.3.8, if $r = 0$, then there is only one element g such that $g\theta(g^{-1}) \in K$. For $r > 0$, as a conclusion of Lemma 5.3.8, there is also only one element g such that $g\theta(g^{-1}) \in K$. Then, by Equation 5.3.1,

$$\mathrm{Hom}_{G^{\theta}}(\pi(\Psi), 1) \cong m_K(\Theta')\mathrm{Hom}_{G_{y,0}/G_{y,0+}}(\overline{\rho}, \mathbb{1}).$$

According to $\dim \mathrm{Hom}_H(\pi, \mathbb{1}) \leq 1$, $m_K(\Theta') = 1$. Therefore, $\pi(\Psi)$ is H-distinguished if and only if Ψ satisfies the condition (1).

Let $d > 0$. According to Proposition 5.3.1 and Lemma 5.3.6, we conclude that $(G^0, G^{0,\theta})$ is isomorphic to

$$(\prod_i \mathrm{U}_{2i}(E_i) \times \prod_j \mathrm{U}_{2j}(Q, E_j), \prod \mathrm{Sp}_{2i}(k_i) \times \prod_j \mathrm{SU}_j(D_j)).$$

Then considering that $\mathrm{Hom}_{K^0(\Psi)^{\theta}}(\rho'(\Psi), \eta'_{\theta}(\Psi))$ restricts to the components of $(\prod_i \mathrm{U}_{2i}(E_j), \prod_j \mathrm{Sp}_{2i}(k_i))$, we conclude that $\mathrm{Hom}_{K^0(\Psi)^{\theta}}(\rho'(\Psi), \eta'_{\theta}(\Psi))$ is zero by Theorem 5.3.2. Therefore, $(G^0, G^{0,\theta})$ is isomorphic to

$$(\prod \mathrm{U}_{2j}(Q, E_j), \mathrm{SU}_j(D_j)).$$

Since $\dim \mathrm{Hom}_H(\pi(\Psi), \mathbb{1}) = 1$, we conclude that $\langle \Theta', \xi \rangle_K$ is the only nonzero term in Equation (5.3.1). By Theorem 5.3.1, $\langle \Theta', \xi \rangle_K = \dim \mathrm{Hom}_{K^0(\Psi)^{\theta}}(\rho'(\Psi), \eta'_{\theta}(\Psi))$.

Since η'_θ is trivial by Lemma 5.3.9,

$$\mathrm{Hom}_{K^{0,\theta}}(\rho'(\Psi), \mathbb{1}) = \mathrm{Hom}_{K^{0,\theta}}(\rho(\Psi), \prod_{i=0}^{d}(\phi_i|K^0(\Psi))).$$

Since the derived group of $K^{0,\theta}/K_+^{0,\theta}$ is itself, $\prod_{i=0}^{d}(\phi_i|K^0(\Psi))$ is trivial on $K^{0,\theta}$. Therefore,

$$\mathrm{Hom}_{K^{0,\theta}}(\rho(\Psi), \prod_{i=0}^{d}(\phi_i|K^0(\Psi))) = \mathrm{Hom}_{K^{0,\theta}/K_+^{0,\theta}}(\overline{\rho}, \mathbb{1}).$$

According to Lemma 5.2.7, if E is unramfied and y is θ-fixed, then $(G^0, G^{0,\theta})$ is isomorphic to $(\prod_i \mathrm{U}_2(E_i), \prod_i \mathrm{SU}_1(D_i)$. Then, ρ is a character and trivial on $K^{0,\theta}$.

According to Lemma 5.2.8, if E is ramified and y is θ-fixed, then the space $\mathrm{Hom}_{K^{0,\theta}/K_+^{0,\theta}}(\overline{\rho}, \mathbb{1})$ is nonzero if and only if Ψ satisfies the condition (2). In summary, if $\pi(\Psi)$ is H-distinguished, then Ψ satisfies the conditions in this theorem, up to a K-conjugation and a θ-symmetric refactorization.

Next, if Ψ satisfies the conditions in this theorem, we can easily construct a non-zero H-invariant linear form on $\pi(\Psi)$. Therefore, $\pi(\Psi)$ is a tame supercuspidal H-distinguished representation if and only if Ψ satisfies the conditions in this theorem. □

Corollary 5.3.2. *If the symmetric pair (G, H) is $(\mathrm{Sp}_{4n}(k), \mathrm{Sp}_{2n}(E))$ and $\pi(\Psi)$ is a tame supercuspidal representation, then*

$$\dim \mathrm{Hom}_{\mathrm{Sp}_{2n}(E_1)}(\pi, 1) \cdot \dim \mathrm{Hom}_{\mathrm{Sp}_{2n}(E_2)}(\pi, \mathbb{1}) = 0, \tag{5.3.7}$$

where E_1 is the unramified quadratic extension over F and E_2 is a ramified quadratic extension.

Proof. By Theorem 5.3.3, if $\pi(\Psi)$ is G^{θ_1}-distinguished tame supercuspidal representation, then G^0 is the centralizer of a θ_1-split element z and isomorphic to $\prod \mathrm{U}_i(Q_i, E_i)$. Since E_1 is the unramified quadratic extension and E_2 are not, z is not G-conjugate to a θ_2-split element by Lemma 5.3.6. Up to G-conjugate, Ψ is not θ_2-symmetric. Therefore, $\pi(\Psi)$ is not a G^{θ_2}-distinguished tame supercuspidal representation. $\square$

5.4 Distinguished representations of finite groups of Lie type

Referring to Theorem 5.3.1, we can reduce the problem of distinguished tame supercuspidal representations to the problem of distinguished depth zero supercuspidal representations relative to characters, that is, the calculation of the dimension of $\mathrm{Hom}_{K^{0,\theta}}(\rho'(\Psi), \eta'_\theta(\Psi))$. Recall that $\eta'(\Psi)$ is an irreducible depth zero supercuspidal representation of G^0 tensored with a character of K^0. According to section 5.3.2, this depth zero supercuspidal representations are the inflation of cuspidal representations of finite groups of Lie type.

In this section, we shall consider the dimension of symmetric subgroup invariant linear functionals in irreducible cuspidal representations of finite groups of Lie type.

In this section, we shall introduce the Deligne-Lusztig's representations in [20], and Lusztig's classification of the irreducible representations of the symplectic groups over a finite field. Now let us recall some notation for reductive groups over finite fields.

Let G be a connected reductive group defined over a finite field $\mathbb{F}_q$ of odd characteristic p, a connected reductive group G^* defined over $\mathbb{F}_q$ be its dual group. Let F be a Frobenius morphism on G, and the fixed subgroup be G^F. Let T be an F-stable maximal torus of G and $\widehat{T}^F$ be the set of characters of T^F over $\overline{\mathbb{Q}}_l^\times$, which is the algebraic closed field of $\mathbb{Q}_l$ and l is prime to q. Let λ be an element in $\widehat{T}^F$. For each pair (T, λ), Deligne and Lusztig [20] attach a virtual representation R_T^λ of G^F.

Define a *norm map* $N\colon T^{F^n} \to T^F$, given by

$$t \to t \cdot F(t) \cdot F^2(t) \cdots F^{n-1}(t).$$

For each λ, the composite $\lambda \circ N$ is a character of T^{F^n}.

Definition 5.4.1 ([20, Definition 5.5])**.** *The pairs (T, λ), (T', λ') are geometrically conjugate if the pairs $(T, \lambda \circ N)$ and $(T', \lambda' \circ N)$ are G^{F^n}-conjugate, for some n.*

There is a bijection between geometric conjugacy classes of the pairs (T, λ) and G^{*F}-conjugacy semi-simple elements in G^F. Referring to [47, Section 7.5], we have a bijection between G^F-conjugacy classes (T, λ) and G^{*F}-conjugacy classes (T', s), with T' an k-stable maximal torus in G^* and $s \in T'^F$. Sometimes, to emphasize the semi-simple element s, we also use $R^s_{T'}$ to denote R^λ_T.

Let $\mathscr{E}(G)$ be the set of isomorphism classes of irreducible representations of G^F over $\overline{Q}_l$. Let $\mathscr{E}(G, s)$ be a subset of $\mathscr{E}(G)$, consisting of all irreducible representations ρ such that $\langle \rho, R^s_T \rangle \neq 0$ for some T. Since, for every irreducible representation ρ, there exists a representation R^s_T such that $\langle \rho, R^s_T \rangle \neq 0$, it follows that $\mathscr{E}(G)$ is the union of $\mathscr{E}(G, s)$ over the set of semi-simple G^{*F}-conjugacy classes in G^{*F}. In addition, if the semi-simple elements s and s' are not G^{*F}-conjugate, then R^s_T and $R^{s'}_{T'}$ have no common constitute for any F-stable maximal tori T and T', containing s and s' respectively. Therefore, we have a partition $\mathscr{E}(G) = \coprod_s \mathscr{E}(G, s)$, where s runs through the set of semi-simple G^{*F}-conjugacy classes in G^{*F}.

An irreducible representation ρ of G^F is *unipotent* if $\langle \rho, R^{\mathbb{1}}_T \rangle_{G^F} \neq 0$ for some k-stable maximal torus T. We have that $\mathscr{E}(G, 1)$ is the set of unipotent representations. By [20, Proposition 7.10], we have that the restriction to G^F of a unipotent representations of the adjoint group $(G^{ad})^F$ is irreducible, and every unipotent representation of G^F is such a restriction. There is a bijection between the sets of isomorphism classes of unipotent representations of $(G^{ad})^F$ and G^F.

Example 5.4.1. *Let G^F be the group* $\mathrm{PGL}_2(\mathbb{F}_q)$ *or* $\mathrm{PU}_2(\mathbb{F}_{q^2})$*. Then, G has two F-stable maximal tori T_1 and T_2. Assume that T_1 is split over $\mathbb{F}_q$. Therefore,* *the unipotent representations of* G^F *are* $(R^{\mathbb{1}}_{T_1} + R^{\mathbb{1}}_{T_2})/2$ *and* $(R^{\mathbb{1}}_{T_1} - R^{\mathbb{1}}_{T_2})/2$. *Moreover, the representation* $(R^{\mathbb{1}}_{T_1} + R^{\mathbb{1}}_{T_2})/2$ *is the trivial representation of* G^F, *and the representation* $(R^{\mathbb{1}}_{T_1} - R^{\mathbb{1}}_{T_2})/2$ *is not cuspidal.*

Example 5.4.2. *According to [71], there is a well-known result that the representation* θ_{10} *of* $\mathrm{Sp}_4(\mathbb{F}_q)$ *is the unipotent cuspidal representation. In addition,* θ_{10} *is not a linear combination of* R^{λ}_{T}. *We will continue to discuss this in section 5.4.4*

Let $\varepsilon(G)$ be $(-1)^{\mathbb{F}_q\text{-rank of } G}$. Let G^*_s be the centralizer of s in G^*. We have the following theorem to classify the irreducible representations of G^F.

Proposition 5.4.1 ([48, Corollary 6.1], [50])**.** *If the center of* G *is connected, then there exists a bijection* $\psi : \mathscr{E}(G, s) \to \mathscr{E}(G^{*\circ}_s, 1)$ *such that*

$$\langle \rho, R^s_T \rangle_{G^F} = \varepsilon(G)\varepsilon(G^{*\circ}_s)\langle \psi(\rho), R^{\mathbb{1}}_T \rangle_{G^{*\circ}_s},$$

for all $\rho \in \mathscr{E}(G, s)$ *and* F*-stable maximal torus* $T \subset G^*$ *containing* s.

In [50], Lusztig parametrized all the unipotent representations of reductive groups. We refer to the parametrization of classical groups in Lusztig [47] and recall the unipotent representations of symplectic groups in section 5.4.4.

Next, we recall from [48] the definitions and theorems for representations of reductive groups with disconnected center.

Let $\pi\colon G \to G^{ad}$ be the adjoint quotient of G. There is a natural isomorphism $(G^{ad})^F/\pi(G^F) \cong (Z_G/Z^{\circ}_G)_F$. Here the subscript F means the largest quotient on which F acts trivially, and Z_G is the center of G. The isomorphism is defined by the correspondence $g \in (G^{ad})^F \to \tilde{g}^{-1}F(\tilde{g}) \in Z_G$, where $g \in G$ and $\pi(\tilde{g}) = g$. The adjoint group $(G^{ad})^F$ acts on $\mathscr{E}(G)$ via the adjoint action on G^F. Since the action is trivial on the subgroup $\pi(G^F)$ and leaves stable each subset $\mathscr{E}(G, s)$, we have that an induced action of $(G^{ad})^F/\pi(G^F)$ on $\mathscr{E}(G, s)$. Hence, we get an action of $(Z_G/Z^{\circ}_G)_F$ on $\mathscr{E}(G, s)$. The action is defined by $\rho \in \mathscr{E}(G, s) \to \rho^{\tilde{g}_0}$, where $\rho^{\tilde{g}_0}(g) = \rho(\mathrm{Ad}(\tilde{g}_0)g)$.

Let s be a semi-simple element of G^{*F}. Since the adjoint action of G^*_s on $\mathscr{E}(G^*_s)$ leaves stable the set of unipotent representations $\mathscr{E}(G^{*\circ}_s, 1)$, we have a natural action of $G^{*F}_s/(G^{*\circ}_s)^F$ on $\mathscr{E}(G^{*\circ}_s, 1)$.

For a reductive group with disconnected center, Lusztig [48] generalized Proposition 5.4.1 and gave the following proposition.

Proposition 5.4.2 ([48, Proposition 5.1])**.** *There exists a surjective map* $\psi : \mathscr{E}(G, s) \to \mathscr{E}(G^*_s, 1)$ *mod action of* $G^{*F}_s/(G^{*\circ}_s)^F$ *with the following properties.*

The fibers of ψ *are precisely the orbits of the action of* $(Z_G/Z^\circ_G)_F$ *on* $\mathscr{E}(G, s)$. *If* Ψ *is a* $G^{*F}_s/(G^{*\circ}_s)^F$*-orbit on* $\mathscr{E}(G^*_s, 1)$ *and* Γ *is the stabilizer in* $G^{*F}_s/(G^{*\circ}_s)^F$ *of an element in* Ψ, *then the fiber* $\psi^{-1}(\Psi)$ *has precisely* $\#\Gamma$ *elements. If* $\rho \in \psi^{-1}(\Psi)$ *and* T *is an* k*-stable maximal torus of* G^* *containing* s, *then*

$$\langle \rho, R^s_T \rangle_{G^F} = \varepsilon_G \varepsilon_{G^*_s} \sum_{\overline{\rho} \in \Psi} \langle \overline{\rho}, R^{\mathbb{1}}_T \rangle_{G^{*F}_s} \tag{5.4.1}$$

To characterize the cuspidal representations in $\mathscr{E}(G, s)$, the statement [47, (7.5.4)] can be generalized for the reductive groups with disconnected centers, by Proposition 8.2 in [20] and Proposition 5.4.2, and can be stated as follows.

Proposition 5.4.3 ([47, (7.5.4)])**.** *An irreducible representation* ρ *of* G^F *is cuspidal if and only if* $\langle \rho, R^s_T \rangle_{G^F} = 0$ *for any pair* (T, s) *with* T *an* F*-stable maximal torus contained in a proper* F*-stable parabolic subgroup of* G^*.

We shall apply Proposition 5.4.2 and Proposition 5.4.3 to identify the cuspidal representations in $\mathscr{E}(\mathrm{Sp}_4(\mathbb{F}_q))$.

5.4.1 Cuspidal representations of $\mathrm{Sp}_4(\mathbb{F}_q)$

In this section, let G be the symplectic group Sp_4. Hence, the dual group G^* is $\mathrm{SO}_5(V)$, whose quadratic form is given by ω_4. By Proposition 5.4.2, we need to find the centralizers of semi-simple elements of the dual group G^{*F}. Let s be a semi-simple element of G^{*F} and $P_s(x)$ be the minimal polynomial of s. Let A_s be the $\mathbb{F}_q$-algebra of linear map of V generated by s. Moreover, the algebra A_s

is isomorphic to $\mathbb{F}_q[x]/P_s\mathbb{F}_q[x]$. We consider V as an A_s-module, denoted by V_s. The inverse map $s \to s^{-1}$ extends an involution over the algebra A_s, denoted by inv.

By Springer's statement 2.4 in [1, Page 254], there exists a $\mathbb{F}_q$-linear function l on A_s such that the symmetric bilinear form $(a, b) \to l(ab)$ is nondegenerate and $l(\mathrm{inv}(x)) = l(x)$. We can define a bilinear form $\langle\ ,\ \rangle_s$ on V_s such that for all $a \in A_s$,

$$l(a\langle x, y\rangle_s) = \langle ax, y\rangle. \tag{5.4.2}$$

Recall that $\langle x, y\rangle$ is the quadratic form over V. We then have $\langle x, y\rangle_s = \mathrm{inv}(\langle x, y\rangle_s)$.

For a polynomial $P(x) = \sum_{i=1}^n a_i x^i$ in $\mathbb{F}_q[x]$, we also define the operator inv for $\mathrm{inv}(P)(x) = \sum_{i=1}^n a_{n-i}x^i$. For a semi-simple element s of an orthogonal group, we have that $P_s(x)$ is proportional to $\mathrm{inv}(P_s)(x)$. Let $P_s(x) = \prod_{\alpha\in\Omega} P_{s,\alpha}(x)$ be the decomposition of P into irreducible factors over $\mathbb{F}_q$. Then, the action of inv on the factors of $P_s(x)$ is a permutation of the index set Ω of order less than 2. Therefore, we have

$$P_s(x) = \prod_{i\in\Omega\backslash\Omega^{\mathrm{inv}}} P_{s,i}(x)\mathrm{inv}(P_{s,i})(x) \cdot \prod_{j\in\Omega^{\mathrm{inv}}} P_{s,j}(x).$$

Here Ω^{inv} is the set of the inv-fixed points. Applying this factorization, we can calculate the centralizer of s.

Lemma 5.4.1. *Let G^* be an odd special orthogonal group and s be a semi-simple element in G^{*F}. If s has the eigenvalue -1, then*

$$G_s^{*F} \cong \prod_i \mathrm{GL}_i(\mathbb{F}_{q^{n_i}}) \times \prod_j \mathrm{U}_j(\mathbb{F}_{q^{2m_j}}) \times \mathrm{SO}_{2m+1}(\mathbb{F}_q) \times \mathrm{O}^{\pm}_{2m'}(\mathbb{F}_q).$$

Here $\mathrm{O}^{\pm}_{2m'}(\mathbb{F}_q)$ is either an $\mathbb{F}_q$-split even orthogonal group or a quasi-split even orthogonal group.

If s does not have the eigenvalue -1, then

$$G_s^{*F} \cong \prod_i \mathrm{GL}_i(\mathbb{F}_{q^{n_i}}) \times \prod_j \mathrm{U}_j(\mathbb{F}_{q^{2m_j}}) \times \mathrm{SO}_{2m+1}(\mathbb{F}_q).$$

Proof. To prove this lemma, we need only to prove the lemma in three special cases and then deduce the general case from them.

Assume that $P_s(x) = Q(x) \cdot \mathrm{inv}(Q)(x)$, and $Q(x)$ is irreducible. Then, A_s is isomorphic to $\mathbb{F}_q[x]/(Q \cdot \mathbb{F}_q[x]) \oplus \mathbb{F}_q[x]/\mathrm{inv}(Q)\mathbb{F}_q[x]$. Therefore, the centralizer G_s^{*F} is a general linear group over a finite extension field of $\mathbb{F}_q$. Hence we have $C_{G^*}(s)^F$ is connected and isomorphic to $\mathrm{GL}_{n/d}(\mathbb{F}_{q^d})$.

Assume that $P_s(x)$ is irreducible. Then, A_s is isomorphic to a field $\mathbb{F}_{q^{2m}}$, where the integer $2m$ is the degree of $P_s(x)$. The automorphism inv on A_s is the element of oder 2 in the Galois group $\mathrm{Gal}(\mathbb{F}_{q^{2m}}/\mathbb{F}_q)$. Hence, the inner form (5.4.2) on V_s is a symmetric hermitian form. The centralizer G_s^{*F} is a unitary group. Therefore, we have G_s^{*F} is connected and isomorphic to $\mathrm{U}_{n/m}(\mathbb{F}_{q^{2m}})$.

Assume that $P_s = x^2 - 1$. The centralizer G_s^* in the group is the subgroup of the product of an even orthogonal group and an odd orthogonal group, consisting of the elements with the same determinant. Then, G_s^* is not connect and the Galois cohomology $H^1(\mathbb{F}_q, G_s^*)$ has two elements that are corresponding to the two isometry classes of quadratic forms. Therefore, there are two G^{*F}-conjugate classes of s. The centralizer G_s^{*F} is isomorphic to an $\mathbb{F}_q$-split group $\mathrm{SO}_{2m+1}(\mathbb{F}_q) \times \mathrm{O}_{2m'}(\mathbb{F}_q)$ or $\mathrm{SO}_{2m+1}(\mathbb{F}_q) \times \mathrm{O}^-_{2m'}(\mathbb{F}_q)$.

In general, for any semi-simple element s, if the centralizer G_s^* is connected, then the Galois cohomology $H^1(\mathbb{F}_q, G_s^*)$ is trivial and there is only one G^{*F}-conjugacy class of s. If the centralizer G_s^* is disconnected, then there are two G^{*F}-conjugate classes of s. Therefore, we have that the centralizer $C_{G^*}(s)$ is the product of these three cases. $\square$

In the rest of this section, we will identify the cuspidal representations of $\mathrm{Sp}_4(\mathbb{F}_q)$. Since $(Z_G/Z_G^\circ)_F$ is $\{\pm \mathrm{I}_4\}$, we can find an element

$$\tilde{g} = \mathrm{diag}\{\sqrt{\tau}, \sqrt{\tau}, \sqrt{\tau}^{-1}, \sqrt{\tau}^{-1}\}$$

such that $\tilde{g}^{-1}F(\tilde{g}) = -\mathrm{I}_4$. Therefore, the action of $-\mathrm{I}_4 \in (Z_G/Z_G^\circ)_F$ on $\mathscr{E}(G,s)$ is given by $\rho \in \mathscr{E}(G,s) \to \rho^{\tilde{g}}$.

Let us show an example of the action of $(Z_G/Z_G^\circ)_F$ on $\mathscr{E}(G,s)$. Let s_0 be a semi-simple element in G^{*F} with the minimal polynomial $P_{s_0}(x) = (x^2-1)Q(x)$, where $Q(x)$ is an irreducible quadratic polynomial over $\mathbb{F}_q$ and proportional to $\mathrm{inv}(Q)$. In addition, assume that $(G_s^*)^F$ is isomorphic to $\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{O}_2^-(\mathbb{F}_q)$. Indeed, $\mathrm{SO}_2^-(\mathbb{F}_q)$ is isomorphic to $\mathrm{U}_1(\mathbb{F}_{q^2})$. The unipotent representation of $(G_s^{*\circ})^F$ is only the trivial representation. Recall that Ψ in Proposition 5.4.2 is a ${G_{s_0}^*}^F/({G_{s_0}^*}^\circ)^F$-orbit on $\mathscr{E}(G_{s_0}^*,1)$. In this case, since Ψ contains only the trivial representation, Γ contains two elements. We denote one of them by ρ_0, and the other is $\rho_0^{\tilde{g}}$. By Equation (5.4.1), we have that $\mathscr{E}(G,s_0)=\{\rho_0,\rho_0^{\tilde{g}}\}$ and $R_T^{s_0}=\rho_0\oplus\rho_0^{\tilde{g}}$, where T is $({G_{s_0}^*}^\circ)^F$ and isomorphic to $\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{U}_1(\mathbb{F}_{q^2})$.

Theorem 5.4.1. *If ρ is an irreducible cuspidal representation of $\mathrm{Sp}_4(\mathbb{F}_q)$, then ρ is in the following set*

$$\{R^{s_1}_{\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{U}_1(\mathbb{F}_{q^2})}, R^{s_2}_{\mathrm{U}_1(\mathbb{F}_{q^4})}, \rho_0, \rho_0^{\tilde{g}}, \theta_{10}\},$$

where ${G_{s_1}^*}^F$ and ${G_{s_2}^*}^F$ are isomorphic to $\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{U}_1(\mathbb{F}_{q^2})$ and $\mathrm{U}_1(\mathbb{F}_{q^4})$.

Proof. Let s be a semi-simple element in G^{*F}. If the minimal polynomial $P_s(x)$ of s has an irreducible factor Q such that $\mathrm{inv}(Q)$ is not proportional to Q, then each k-stable torus T containing s is contained in a proper k-stable parabolic subgroup of G^*. By Proposition 5.4.3, there is no cuspidal representation in $\mathscr{E}(G,s)$. Now, we need only to consider the semi-simple elements whose minimal polynomials only contain irreducible factor Q such that $\mathrm{inv}(Q)$ is proportional to Q.

If the centralizer is connected, then the map ψ is a bijection from $\mathscr{E}(G,s)$ to $\mathscr{E}(G_s^*,1)$, and maps cuspidal representations to unipotent cuspidal representations of G_s^{*F}. To identify the cuspidal representations in $\mathscr{E}(G,s)$, we need only to identify the unipotent cuspidal representations of G_s^{*F}. By Lemma 5.4.1, if the centralizer is connected, then it is isomorphic to one of the groups in the following set,

$$\{\mathrm{U}_1(\mathbb{F}_{q^4}),\mathrm{U}_1(\mathbb{F}_q)\times\mathrm{U}_1(\mathbb{F}_q),\mathrm{U}_2(\mathbb{F}_{q^2}),\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{SO}_3(\mathbb{F}_q),\mathrm{SO}_5(\mathbb{F}_q)\}.$$

If G_s^{*F} is one of groups $\mathrm{U}_1(\mathbb{F}_{q^4})$ and $\mathrm{U}_1(\mathbb{F}_q)\times\mathrm{U}_1(\mathbb{F}_q)$, then the trivial representation is the unipotent cuspidal representation of G_s^{*F}. Therefore, $\mathscr{E}(G,s)$ has only one representation which is a cuspidal representation. The cuspidal representations are $R^{s_1}_{\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{U}_1(\mathbb{F}_{q^2})}$ and $R^{s_2}_{\mathrm{U}_1(\mathbb{F}_{q^4})}$, where $G^*_{s_1}{}^F$ and $G^*_{s_2}{}^F$ are isomorphic to $\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{U}_1(\mathbb{F}_{q^2})$ and $\mathrm{U}_1(\mathbb{F}_{q^4})$.

If G_s^{*F} is one of groups $\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{SO}_3(\mathbb{F}_q)$ and $\mathrm{U}_2(\mathbb{F}_{q^2})$, then the adjoint group is isomorphic to $\{\mathrm{PGL}_2(\mathbb{F}_q),\mathrm{PU}_2(q^2)\}$. Since there is no unipotent cuspidal representation of $\mathrm{PGL}_2(\mathbb{F}_q)$ or $\mathrm{PU}_2(q^2)$, the set $\mathscr{E}(G,s)$ does not contain any cuspidal representation.

For the case $\mathrm{SO}_5(\mathbb{F}_q)$, the representations in $\mathscr{E}(G,1)$ are unipotent representations of $\mathrm{Sp}_4(\mathbb{F}_q)$. It is a well-known result that θ_{10} is the cuspidal unipotent representation of $\mathrm{Sp}_4(\mathbb{F}_q)$ in Srinivasan [71].

If the centralizer is not connected, then $(G_s^{*\circ})^F$ is isomorphic to one of the groups in,

$$\{\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{SO}_2^{\pm}(\mathbb{F}_q),\mathrm{SO}_4^{\pm}(\mathbb{F}_q),\mathrm{SO}_2^{\pm}(\mathbb{F}_q)\times\mathrm{SO}_3(\mathbb{F}_q)\}.$$

For the cases $\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{SO}_2(\mathbb{F}_q)$ and $\mathrm{SO}_2(\mathbb{F}_q)\times\mathrm{SO}_3(\mathbb{F}_q)$, since $\mathrm{SO}_2(\mathbb{F}_q)$ is isomorphic to $\mathbb{F}_q^\times$, each F-stable maximal torus T containing s is contained in a proper F-stable parabolic subgroup of G^*. Therefore, there is no cuspidal representation in $\mathscr{E}(G,s)$.

For the case $\mathrm{U}_1(\mathbb{F}_{q^2})\times\mathrm{SO}_2^-(\mathbb{F}_q)$, the group $\mathrm{SO}_2^-(\mathbb{F}_q)$ is isomorphic to $\mathrm{U}_1(\mathbb{F}_{q^2})$. Since all F-stable maximal tori containing s are not contained in a proper F-stable parabolic subgroup of G^*, the representations in $\mathscr{E}(G,s)$ are cuspidal. By the discussion in the example, ρ_0 and $\rho^{\tilde{g}}$ are two elements in $\mathscr{E}(G,s)$, and they are all cuspidal.

If $(G_s^{*\circ})^F$ is isomorphic to one of the groups $\mathrm{SO}_4^-(\mathbb{F}_q)$ and $\mathrm{SO}_2^-(\mathbb{F}_q)\times\mathrm{SO}_3(\mathbb{F}_q)$, then the adjoint group is in $\{\mathrm{PGL}_2(\mathbb{F}_{q^2}),\mathrm{PGL}_2(\mathbb{F}_q)\}$. The unipotent representations of these adjoint groups are only the trivial representation and the non-trivial representation. Both of them are not cuspidal. Thus, $G_s^{*F}/(G_s^{*\circ})^F$ acts trivially on the unipotent representations. By Equation (5.4.1), the pre-image of a cuspidal representation of ψ is cuspidal. Therefore, there is no cuspidal representation in $\mathscr{E}(G,s)$.

If $(G_s^{*\circ})^F$ is isomorphic to $\mathrm{SO}_4(\mathbb{F}_q)$, then the adjoint group is $\mathrm{PGL}_2(\mathbb{F}_q)\times\mathrm{PGL}_2(\mathbb{F}_q)$. The set of unipotent representations of $\mathrm{PGL}_2(\mathbb{F}_q)\times\mathrm{PGL}_2(\mathbb{F}_q)$ is $\{\mathbb{1},\chi\otimes\mathbb{1},\mathbb{1}\otimes\chi,\chi\otimes\chi\}$, where χ is the nontrivial unipotent representation of $\mathrm{PGL}_2(\mathbb{F}_q)$. Then, the group $G_s^{*F}/(G_s^{*\circ})^F$ stabilizes on $\{\mathbb{1},\chi\otimes\chi\}$, and switches $\chi\otimes\mathbb{1}$ and $\mathbb{1}\otimes\chi$. Since $\mathbb{1}$ and $\chi\otimes\chi$ are not cuspidal, the pre-image of $\mathbb{1}$ and $\chi\otimes\chi$ are not cuspidal representations. For the representations $\chi\otimes\mathbb{1}$ and $\mathbb{1}\otimes\chi$, the pre-image are same. By Example 5.4.1, we have that both $\langle\chi\otimes\mathbb{1},R^{\mathbb{1}}_{T_1\times T_1}\rangle$ and $\langle\mathbb{1}\otimes\chi,R^{\mathbb{1}}_{T_1\times T_1}\rangle$ are positive, where T_1 is an $\mathbb{F}_q$ split torus in $\mathrm{PGL}_2(\mathbb{F}_q)$. Therefore,

$$\langle\psi^{-1}(\chi\otimes\mathbb{1}),R^s_{T'}\rangle=\langle\chi\otimes\mathbb{1}\oplus\mathbb{1}\otimes\chi,R^{\mathbb{1}}_{T_1\times T_1}\rangle$$

is not zero. It follows that $\psi^{-1}(\chi\otimes\mathbb{1})$ is not cuspidal. In this case, there is no cuspidal representation in $\mathscr{E}(G,s)$.

Combining all cases together, we obtain all irreducible cuspidal representations of $\mathrm{Sp}_4(\mathbb{F}_q)$ and prove this theorem. $\square$

5.4.2 Lusztig's formula for $\langle\mathrm{tr}(\cdot,R^\lambda_T),\mathrm{Ind}_H^G\mathbb{1}\rangle$

Let θ be an involution of G and commute with F. Let H be the subgroup of G, consisting of the θ-fixed elements. Hence, H is also F-stable. Our purpose is to calculate the multiplicity $\langle\rho,\mathrm{Ind}_{H^F}^{G^F}\mathbb{1}\rangle$ for every irreducible cuspidal representation ρ of G^F.

For the Deligne-Lusztig virtual representations R^s_T, Lusztig gave a formula in [51, Theorem 3.3] to computer $\langle\mathrm{tr}(\cdot,R^\lambda_T),\mathrm{Ind}_{H^F}^{G^F}\mathbb{1}\rangle$. Applying this formula, Henderson in [30] gave explicit formulas for the dimensions of all symmetric subgroups invariant linear functionals of all irreducible representations of general linear groups or unitary groups.

Now, we introduce the Lusztig's formula. Define

$$\Theta_T=\{f\in G\mid\theta(f^{-1}Tf)=f^{-1}Tf\}.$$

Then T (resp. H) acts on Θ_T by left (resp. right) multiplication. Indeed, the double cosets $T\backslash\Theta_T/H$ is one-one correspondence to the double cosets $B\backslash G/H$, where B is a Borel subgroup containing T. Let Θ_T^F be the set of F fixed elements. The double cosets $T^F\backslash\Theta_T^F/H^F$ are also bijective to the G^F conjugate classes of (θ, F)-stable maximal tori of G^F.

For any $f \in \Theta_T^F$, define a morphism $\varepsilon_{T,\lambda}$ of $(T \cap fHf^{-1})^F$ by

$$\varepsilon_{T,\lambda}(t) = (-1)^{\mathbb{F}_q\text{-rank}(Z_G((T\cap fHf^{-1})^\circ))+\mathbb{F}_q\text{-rank}(Z_G^\circ(t)\cap Z_G((T\cap fHf^{-1})^\circ))}. \tag{5.4.3}$$

Indeed, $\varepsilon_{T,\lambda}$ is a character and trivial on $((T \cap fHf^{-1})^\circ)^F$. Finally define

$$\Theta_{T,\lambda}^F = \{f \in \Theta_T^F \mid \lambda_{(T\cap fHf^{-1})^F} = \epsilon_{T,\lambda}\}.$$

The groups T^F and H^F still act by left and right multiplication.

Theorem 5.4.2 ([51, Theorem 3.3])**.**

$$\langle \mathrm{tr}(\cdot, R_T^\lambda), \mathrm{Ind}_H^G \mathbb{1}\rangle = \sum_{f\in T^F\backslash\Theta_{T,\lambda}^F/H^F} (-1)^{\mathbb{F}_q\text{-}rank(T)+\mathbb{F}_q\text{-}rank(Z_G((T\cap fHf^{-1})^\circ))}$$

5.4.3 The symmetric pair $(\mathrm{U}_{2n}(\mathbb{F}_{q^2}), \mathrm{Sp}_{2n}(\mathbb{F}_q))$

In this section, we will prove that there is no $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-distinguished cuspidal representation of $\mathrm{U}_{2n}(\mathbb{F}_{q^2})$. For this symmetric pair, Henderson [30] applied the Theorem 5.4.2, and gave an explicit formula for the dimensions $\langle \rho, \mathrm{Ind}_{\mathrm{Sp}_{2n}(\mathbb{F}_q)}^{\mathrm{U}_{2n}(\mathbb{F}_{q^2})} \mathbb{1}\rangle_{\mathrm{U}_{2n}(\mathbb{F}_{q^2})}$, for any irreducible representation ρ of $\mathrm{U}_{2n}(\mathbb{F}_{q^2})$. This formula is given in terms of the partition, and does not obviously imply the dimension formula of cuspidal representations. In order to obtain the dimension formula of cuspidal representations, we just identify the parametrization of cuspidal representations, and then apply Henderson's formula for cuspidal representations.

Let us recall the classification of irreducible representations of $\mathrm{U}_{2n}(\mathbb{F}_{q^2})$, and some related definitions.

Let μ be a partition of n. Its size, length, and transpose are denoted by $|\mu|$, $l(\mu)$, and μ'. A partition $\mu = (\mu_1, \mu_2, \cdots, \mu_{l(\mu)})$ is called *even* if μ_i for all $1 \le i \le l(\mu)$ are even. The multiplicity of i as a part of μ is denoted by $\mu(i)$.

It is well-known that all irreducible representations of the symmetric group S_n are corresponding to the partitions of n. Let ρ^μ be the irreducible representation of S_n, corresponding to the partition μ. In addition, a partition v corresponds to a cycle-type in S_n which is a conjugate class in S_n. Thus, define that ρ^μ_v is the character value of ρ^μ at a partition v.

Next, let us recall some notation to parametrize the G^F-conjugate classes of (T, λ). Let $\widehat{\mathbb{F}^\times_{q^e}}$ be the characters of one dimension representations of $\mathbb{F}^\times_{q^e}$. We consider the system maps of $\widehat{\mathbb{F}^\times_{q^e}} \to \widehat{\mathbb{F}^\times_{q^{e'}}}$ for $e|e'$. Let L be its direct limit. Let σ be the q-th power on both $\overline{\mathbb{F}}^\times_q$ and L. Let $\overline{\mathbb{F}}^{\times\sigma^e}_q$ and L^{σ^e} be the fixed points under σ^e for all $e \in \mathbb{N}$.

Let $\tilde{\sigma}$ be the composition map of the inverse map and the map σ. We have the canonical pairing $\langle \cdot, \cdot \rangle^{F^e} : \overline{\mathbb{F}}^{\times\sigma^e}_q \times L^{\sigma^e} \to \overline{\mathbb{Q}}^\times_l$. In addition, we can replace the superscript σ^e by $(\tilde{\sigma})^e$. Note that $(\tilde{\sigma})^2 = \sigma^2$.

Let $\langle \tilde{\sigma} \rangle \backslash L$ be the set of orbits of L under the action $\langle \tilde{\sigma} \rangle$, where $\langle \tilde{\sigma} \rangle$ is the group generated by $\tilde{\sigma}$ on L. Let m_ξ be the cardinality of the orbit ξ in $\langle \tilde{\sigma} \rangle \backslash L$.

Let $\widehat{\mathcal{P}}_n$ be the set of collections of partitions $\underline{\nu} = (\nu_\alpha)_{\alpha \in L}$, almost all zero, and $\sum_{\alpha \in L} |\nu_\alpha| = n$. Let $\widehat{\mathcal{P}}^{\tilde{\sigma}}_n$ be the subset of $\widehat{\mathcal{P}}_n$ consisting of all $\underline{\nu}$ such that $\nu_{\tilde{\sigma}(\alpha)} = \nu_\alpha$ for all α. Recall that $(\mu_\alpha)_1, (\mu_\alpha)_2, \cdots, (\mu_\alpha)_{l(\nu_\alpha)}$ are the parts of the partition of μ_α. For $\underline{\nu} \in \widehat{\mathcal{P}}^{\tilde{\sigma}}_n$,

$$\sum_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} \sum_{j=1}^{l(\nu_\xi)} \tilde{m}_\xi (\nu_\xi)_j = \sum_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} \tilde{m}_\xi |\nu_\xi| = n$$

For $\underline{\nu}, \underline{\rho} \in \widehat{\mathcal{P}}^{\tilde{\sigma}}_n$, we write $|\underline{\nu}| = |\underline{\rho}|$ to mean that $|\nu_\xi| = |\rho_\xi|$ for all ξ.

There is a bijection between $\widehat{\mathcal{P}}^{\tilde{\sigma}}_n$ and the set of G^F-orbits of pairs (T, λ). Furthermore, (T, λ) is corresponding to $\underline{\nu}$ as following:

1. $T^F_{\underline{\nu}}$ is isomorphic to

$$\prod_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} \prod_{j=1}^{l(\nu_\xi)} (\overline{\mathbb{F}}^\times_q)^{\tilde{\sigma}^{\tilde{m}_\xi (\nu_\xi)_j}};$$

2. $\lambda_{\underline{\nu}}$ corresponds to

$$\prod_{\xi\in\langle\tilde{\sigma}\rangle\backslash L}\prod_{j=1}^{l(\nu_\xi)}\langle\cdot,\xi\rangle^{\tilde{\sigma}^{\tilde{m}_\xi(\nu_\xi)_j}}.$$

For $\underline{\nu}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$, let $B_{\underline{\nu}}=\mathrm{tr}(\cdot,R_T^\lambda)$ for (T,λ) in the corresponding G^F-orbit. The Deligne-Lusztig characters of unitary group are a basis of the space of conjugate classes functions. That is, each irreducible character χ of G^F is linear combination of $B_{\underline{\nu}}$. Hence, in this notation, Henderson [30, Section 1.3] introduced the following irreducible character of G^F, corresponding to the partition $\underline{\rho}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

$$\chi^{\underline{\rho}}=(-1)^{\lceil\frac{n}{2}\rceil+\sum_{\xi\in\langle\tilde{\sigma}\rangle\backslash L}\tilde{m}_\xi n(\rho'_\xi)+|\rho_\xi|}\sum_{\substack{\underline{\nu}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}\\ |\underline{\nu}|=|\underline{\rho}|}}\left(\prod_{\xi\in\langle\tilde{\sigma}\rangle\backslash L}(z_{\nu_\xi})^{-1}\chi_{\nu_\xi}^{\rho_\xi}\right)B_{\underline{\nu}},$$

where z_{ν_ξ} is the cardinality of the centralizer of cycle-type μ_ξ in $\mathrm{S}_{|\mu_\xi|}$. All irreducible characters arise in this way for unique $\underline{\rho}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$. Alternatively, the transformed formula of $B_{\underline{\nu}}$ for all $\underline{\nu}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ is,

$$B_{\underline{\nu}}=\sum_{\substack{\underline{\rho}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}\\ |\underline{\rho}|=|\underline{\nu}|}}(-1)^{\lceil\frac{n}{2}\rceil+\sum_{\xi\in\langle\tilde{\sigma}\rangle\backslash L}\tilde{m}_\xi n(\rho'_\xi)+|\rho_\xi|}\left(\prod_{\xi\in\langle\tilde{\sigma}\rangle\backslash L}\chi_{\nu_\xi}^{\rho_\xi}\right)\chi^{\underline{\rho}}.$$

Using the parametrization $\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ of irreducible representations of $\mathrm{U}_n(\mathbb{F}_{q^2})$, Henderson have the following dimension formula.

Proposition 5.4.4 ([30, Theorem 2.2.1])**.** *For any $\underline{\rho}\in\widehat{\mathcal{P}}_{2n}$,*

$$\langle\chi^{\underline{\rho}},\mathrm{Ind}_{\mathrm{Sp}_{2n}(\mathbb{F}_q)}^{\mathrm{U}_{2n}(\mathbb{F}_{q^2})}\mathbb{1}\rangle=\begin{cases}1 & \textit{if all } \rho_\xi \textit{ are even;}\\ 0 & \textit{otherwise.}\end{cases}$$

To calculate the dimension formula of the irreducible cuspidal representations of $\mathrm{U}_{2n}(\mathbb{F}_{q^2})$, we have to identify all irreducible cuspidal representations in terms of the elements in $\underline{\rho}\in\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$. Hence, we will prove the following lemma.

Lemma 5.4.2. *An irreducible representation $\chi^{\underline{\rho}}$ is an irreducible cuspidal representation if and only if for all ρ_ξ of $\underline{\rho}$, ρ_ξ is the partition $(s_\xi, s_\xi - 1, \cdots, 2, 1)$, where s_ξ is some positive integer, depending on ξ.*

Proof. By Proposition 5.4.3, we have that $\chi^{\underline{\rho}}$ is cuspidal if and only if $\langle \chi^{\underline{\rho}}, B_{\underline{\nu}} \rangle = 0$ for all $T^F_{\underline{\nu}}$ contained in a proper F-stable parabolic subgroup of $\mathrm{U}_n(\mathbb{F}_{q^2})$. Since $\tilde{\sigma}^2 = \sigma^2$ for an even part $(\nu_\xi)_j$, it follows that $T^F_{\underline{\nu}} \subset \mathrm{GL}_{\xi(\nu_\xi)_j}(\mathbb{F}_q) \times \mathrm{U}(\mathbb{F}_{q^2})$ is the Levi subgroup of a proper F-stable parabolic subgroup of $\mathrm{U}_n(\mathbb{F}_{q^2})$. Hence, a F-stable maximal torus $T^F_{\underline{\nu}}$ contained in a proper F-stable parabolic subgroup of $\mathrm{U}(\mathbb{F}_{q^2})$ is equivalent that a part $(\nu_\xi)_j$ is even.

Assume that $|\underline{\nu}| = |\underline{\rho}|$, otherwise $\langle \chi_{\underline{\rho}}, B_{\underline{\nu}} \rangle = 0$. Since $\langle \chi_{\underline{\rho}}, B_{\underline{\nu}} \rangle$ is determined by $(\prod_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} (z_{\nu_\xi})^{-1} \chi^{\rho_\xi}_{\nu_\xi})$, the inner product $\langle \chi^{\underline{\rho}}, B_{\underline{\nu}} \rangle = 0$ is equivalent to $\chi^{\rho_\xi}_{\nu_\xi} = 0$ if ν_ξ has an even part for some ξ. For the cuspidal representation $\chi^{\underline{\rho}}$, since $\langle \chi^{\underline{\rho}}, B_{\underline{\nu}} \rangle = 0$ for all $B_{\underline{\nu}}$ with even part, we have that $\chi^{\rho_\xi}_{\nu_\xi}$ is 0 for all cycle-type ν_ξ with an even part. By Lusztig [47, Proposition 9.4], if $\chi^{\rho_\xi}_{\nu_\xi}$ is 0 for all cycle-type ν_ξ with an even part, then ρ_ξ is the partition $(s, s-1, \cdots, 1)$ for some positive integer s. Therefore, we prove this lemma. □

Applying Proposition 5.4.4 and Lemma 5.4.2, we conclude that if $\chi^{\underline{\rho}}$ is an irreducible cuspidal representation, then all ρ_ξ are not even. Therefore, we have the following dimension formula for irreducible cuspidal representations.

Corollary 5.4.1. *If χ is an irreducible cuspidal representation of $\mathrm{U}_{2n}(\mathbb{F}_{q^2})$, then $\langle \chi^{\underline{\rho}}, \mathrm{Ind}^{\mathrm{U}_{2n}(\mathbb{F}_{q^2})}_{\mathrm{Sp}_{2n}(\mathbb{F}_q)} \mathbb{1} \rangle = 0$.*

5.4.4 Unipotent representations of the symplectic group

In this section, we recall the unipotent representations of the symplectic group. These representations will play an important role in identifying the distinguished representations.

Lusztig [49] parametrized the unipotent representations in terms of symbol classes. Let us recall some definitions about symbol classes.

A *symbol* is an unordered pair $\Lambda = \binom{S}{T}$, where S and T are finite sets consisting of non-negative integers. The *rank* $\mathrm{rk}(\Lambda)$of Λ is defined by

$$\mathrm{rk}(\Lambda) = \sum_{\lambda \in S} \lambda + \sum_{\mu \in T} \mu - \left[\left(\frac{\#S + \#T - 1}{2} \right)^2 \right],$$

where the square bracket is Gaussian function. The *defect* $\mathrm{def}(\Lambda)$ is $\mathrm{def}(\Lambda) = |\#S - \#T|$. Obviously, the rank of a symbol is a non-negative integer and $rk(\Lambda) \geq \left[\left(\frac{\mathrm{def}(\Lambda)}{2} \right)^2 \right]$.

An equivalence relation on the symbols is given as follows. A *shift* operation $\binom{S}{T} \to \binom{S'}{T'}$ is given by $S' = \{0\} \cup (S+1)$, $T' = \{0\} \cup (T+1)$. This shift operation generates an equivalence relation on the set of symbols. The equivalence classes are called *symbol classes*. The rank and defect of a symbol are invariant under the shift. In each symbol class, we can find a unique representative $\binom{S}{T}$ such that $0 \notin S \cap T$ such a symbol is said to be *reduced*.

A symbol

$$Z = \begin{pmatrix} z_0, z_2, \cdots, z_{2m} \\ z_1, z_3, \cdots, z_{2m-1} \end{pmatrix}$$

of rank n and defect 1 is called *special symbol* if $z_0 \leq z_1 \leq z_2 \leq \cdots \leq z_{2m-1} \leq z_{2m}$ holds.

Let $Z = \binom{S}{T}$ be a special symbol of rank n and defect 1, and let Z_1 be the set of singles of Z, that is $S \cup T \setminus S \cap T$. Denote Z_1^* the set of entries of Z_1 in the first row of Z and $(Z_1)_*$ the set of entries of Z_1 in the second row of Z. Then we have $Z_1 = Z_1^* \coprod (Z_1)_*$. Let d be $(\#Z_1 - 1)/2$. We have $\#Z_1^* = d + 1$, and $\#(Z_1)_* = d$. Let Z_2 be the set of elements which appear in both rows of Z. Thus, $Z = \binom{Z_2 \coprod (Z_1)^*}{Z_2 \coprod (Z_1)_*}$

Let $\mathscr{S}_Z$ be the set of all symbols of rank n and odd defect which contain the same entries as Z. There is a one-one correspondence between $\mathscr{S}_Z$ and the set V_{Z_1} of subsets of Z_1 of even cardinality. Indeed, each symbol Λ is corresponding to a subset M of Z_1 such that $\#M \equiv d \pmod 2$, by $\Lambda_M = \binom{Z_2 \coprod (Z_1 - M)}{Z_2 \coprod M}$. In further, we associate to M the set $M^\# \subset Z_1$ defined by $M^\# = M \cup (Z_1)_* \setminus (M \cap (Z_1)_*)$.

The set $M^{\#}$ is a subset of Z_1 of even cardinality. Therefore, Λ_M is one-one correspondence to $M^{\#}$.

First, the set V_{Z_1} has a natural structure of F_2-vector space of dimension $2d$. Indeed, let the sum of $M_1^{\#}$ and $M_2^{\#}$ be $(M_1^{\#} \cup M_2^{\#}) - (M_1^{\#} \cap M_2^{\#})$. Then, the non-singular symplectic form on V_{Z_1} is given by

$$(M_1^{\#}, M_2^{\#}) = \# M_1^{\#} \cap M_2^{\#} \pmod 2.$$

By the bijection $\mathscr{S}_Z \leftrightarrow V_{Z_1}$, we also have a non-singualar symplectic form on $\mathscr{S}_z$. For instance, the zero vector is the special symbol Z.

Recall that a unipotent representation is a constitute representation of $R_T^{\mathbb{1}}$ for some F-stable maximal torus T of G. To discuss the virtual representation $R_T^{\mathbb{1}}$, let us classify the G^F-conjugacy classes of F-stable maximal tori of G.

Two elements $w, w' \in W$ are called *F-conjugate* if there exists $x \in W$ such that $w' = x^{-1} w F(x)$. By [18, Proposition 3.3.3], there is a bijection between the G^F-conjugacy classes of F-stable maximal tori of G and the F-conjugacy classes of W. Without confusion, we use R_w to denote the $R_T^{\mathbb{1}}$ which T is corresponding to $w \in W$.

Recall that the Frobenius map F acts trivially on the Weyl group W_n of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. For each virtual representation M of W_n, Lusztig defined the following virtual $\overline{\mathbb{Q}}_l$-representation in [49, Section 4.6],

$$R(M) = |W|^{-1} \sum_{w \in W} \mathrm{tr}(w, M) R_w.$$

In section 5.4.5, we know that there is a one-one correspondence between the symbol classes of rank n and defect one, and the irreducible representations of W_n. Therefore, we also use the symbols to denote the corresponding irreducible representations of the Weyl group.

Now, we recall Lusztig's theorem on the unipotent representations. Let $\Phi_{n,d}$ be the set of symbol classes of rank n and defect d, and $\Phi_n = \coprod_{odd\ d} \Phi_{n,d}$.

Proposition 5.4.5 ([47, Theorem 8.2])**.** *For the symplectic group* Sp_{2n}*, there exists a one-one correspondence between the unipotent representations and the the* *set of symbol classes of rank* n *and odd defect, that is,* Φ_n*. Moreover, the symbol* Λ *with* $\mathrm{rk}\Lambda = [(\mathrm{def}(\Lambda)/2)^2]$ *is corresponding to the cuspidal unipotent representation.*

We use $\rho(\Lambda)$ to denote the unipotent representation corresponding to Λ.

Under this parametrization of the unipotent representations, Lusztig in [49, Theorem 5.8] gave the formula between the Deligne-Lusztig representations and the unipotent representations, for q sufficient large. Asai proved that the formula was still true for the small q in [10]. The complete theorem can be found in [50].

Proposition 5.4.6 ([49, Theorem 5.8])**.** *For the symplectic group* Sp_{2n}*, and any* $\Lambda \in \mathscr{S}_Z$ *of defect one, we have*

$$R[\Lambda] = 2^{-d} \sum_{\Lambda' \in \mathscr{S}_Z} (-1)^{(\Lambda,\Lambda')} \rho(\Lambda')$$

where $(,)$ *is the symplectic form on* $\mathscr{S}_Z$ *described as above.*

Unipotent representations of $\mathrm{Sp}_4(\mathbb{F}_q)$

In this part, we apply the previous theorems on the unipotent representations of the symplectic groups, and list all the unipotent representations. The following results also can be found in Carter's book [18, Chapter 13].

First, let us see the symbol classes Φ_2 of rank 2 and odd defect.

$$\begin{aligned}
\Phi_{2,1} &= \left\{ \begin{pmatrix} 2 \\ - \end{pmatrix}, \begin{pmatrix} 1,2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0,2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0,1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0,1,2 \\ 1,2 \end{pmatrix} \right\}; \\
\Phi_{2,3} &= \left\{ \begin{pmatrix} 0,1,2 \\ - \end{pmatrix} \right\}
\end{aligned}$$

For the symbol $\Lambda = \binom{0,1,2}{-}$, since the rank of Λ is same as $[(\mathrm{def}(\Lambda)/2)^2]$, by Proposition 5.4.5, we have that the unipotent representation $\rho\binom{0,1,2}{-}$ is the unique cuspidal unipotent representation of $\mathrm{Sp}_4(\mathbb{F}_q)$, that is, θ_{10}.

Next, we list the special symbols and the dimension of corresponding F_2-vector spaces V_{Z_1}.

Special symbol Z	$\binom{2}{-}$	$\binom{1,2}{0}$	$\binom{0,1,2}{-}$
Dimension $2d$ of V_{Z_1}	0	2	0

For the special symbol $\Lambda = \binom{1,2}{0}$, we have $\mathscr{S}_Z$ and the vector space V_{Z_1} of dimension 2 in the following.

$\mathscr{S}_Z$	$\binom{1,2}{0}$	$\binom{0,2}{1}$	$\binom{0,1}{2}$	$\binom{0,1,2}{-}$
$M^\#$ in V_{Z_1}	$\emptyset$	$\{0,1\}$	$\{0,2\}$	$\{1,2\}$

Applying Proposition 5.4.6, we have following equations:

$$R\left[\begin{pmatrix}2\\-\end{pmatrix}\right] = \mathbb{1} \tag{5.4.4}$$

$$2R\left[\begin{pmatrix}1,2\\0\end{pmatrix}\right] = \rho\begin{pmatrix}1,2\\0\end{pmatrix} + \rho\begin{pmatrix}0,2\\1\end{pmatrix} + \rho\begin{pmatrix}0,1\\2\end{pmatrix} + \rho\begin{pmatrix}0,1,2\\-\end{pmatrix} \tag{5.4.5}$$

$$2R\left[\begin{pmatrix}0,2\\1\end{pmatrix}\right] = \rho\begin{pmatrix}1,2\\0\end{pmatrix} + \rho\begin{pmatrix}0,2\\1\end{pmatrix} - \rho\begin{pmatrix}0,1\\2\end{pmatrix} - \rho\begin{pmatrix}0,1,2\\-\end{pmatrix} \tag{5.4.6}$$

$$2R\left[\begin{pmatrix}0,1\\2\end{pmatrix}\right] = \rho\begin{pmatrix}1,2\\0\end{pmatrix} - \rho\begin{pmatrix}0,2\\1\end{pmatrix} + \rho\begin{pmatrix}0,1\\2\end{pmatrix} - \rho\begin{pmatrix}0,1,2\\-\end{pmatrix} \tag{5.4.7}$$

$$R\left[\begin{pmatrix}0,1,2\\1,2\end{pmatrix}\right] = \rho\begin{pmatrix}0,1,2\\1,2\end{pmatrix} \tag{5.4.8}$$

To simplify the notation, let the left sides of Equation (5.4.4)–(5.4.8) be R_1, R_2, R_3, R_4, and R_5. Let the summands of the right side of Equation (5.4.5) be ρ_1, ρ_2, ρ_3, and θ_{10}. Let the right side of Equation (5.4.8) be ρ_5.

5.4.5 Irreducible representations of the Weyl group of type C_n

In this section, we recall the construction of the irreducible representation of a Weyl group of type C_n in [49, §2]. Let W_n be the Weyl group of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. W_n acts on the set

$$\mathscr{S}_n = \{1, 2, \cdots, n, n', \cdots, 2', 1'\},$$

and is generalized by the permutations $\langle(1,2), \cdots, (n-1,n), (n,n')\rangle$. Then we can define a quadratic character χ of W_n via

$$\chi_n(w) = (-1)^{\#\{w(1),w(2),\cdots,w(n)\}\cap\{1',2',\cdots,n'\}}.$$

Let π_1 and π_2 be irreducible representations of S_k and S_{n-k}. Since W_k (resp. W_{n-k}) is the semi-product of S_k (resp. S_{n-k}) and $(\mathbb{Z}/2\mathbb{Z})^k$ (resp. $(\mathbb{Z}/2\mathbb{Z})^{n-k}$), there is a natural lifting $\overline{\pi}_1$ (resp. $\overline{\pi}_2$) as a representation of W_k (resp. W_{n-k}). Then the induced representation $\mathrm{Ind}_{W_k \times W_{n-k}}^{W_n} \overline{\pi}_1 \otimes (\overline{\pi}_2 \otimes \chi)$ is an irreducible representation of W_n, corresponding to the ordered pair (π_1, π_2).

Now σ_1 corresponds to a partition $0 \leqq \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_{m'}$ of k ($\sum \alpha_i = k$), in the following way: it is the unique irreducible representation of S_k whose restriction to $S_{\alpha_1} \times S_{\alpha_2} \times \cdots S_{\alpha_{m'}} \subset S_k$ contains the unit representation and its restriction to $S_{\alpha_1^*} \times S_{\alpha_2^*} \times \cdots \subset S_k$ (where $\alpha_1^* \leqq \alpha_2^* \leqq \cdots$ is the dual partition) contains the sign representation. Similarly, σ_2 corresponds to a partition $0 \leqq \beta_1 \leqq \cdots \leqq \beta_{m''}$ of I. Since m' and m'' can be increased at our will (by adding zeroes) we may assume that $m' = m+1$ and $m'' = m$. We now set $\lambda_i = \alpha_i + i - 1, (1 \leqq i \leqq m+1), \mu_i = \beta_i + i - 1 (1 \leqq i \leqq m)$. Let Λ denote the tableau

$$\begin{pmatrix} \lambda_1, \lambda_2, \cdots, \lambda_{m+1} \\ \mu_1, \mu_2, \cdots, \mu_m \end{pmatrix}$$

Then Λ is a symbol of *rank* n and *defect 1*. As in the section 5.4.4, we denote a symbol with its equivalence class under shift by $[\Lambda]$. Thus, we have a 1-1 correspondence between irreducible representations of W_n and symbols $[\Lambda]$ of rank n

and defect one, modulo shift. We also denote by $[\Lambda]$ the irreducible representation of W_n.

Example of representations of the Weyl group of $\mathrm{Sp}_4(\mathbb{F}_q)$

Referring to section 5.4.4, we use the symbols of rank n and odd defects to parametrize the unipotent representations of the symplectic groups. We have the list of all symbols of rank 4 and odd defects. The symbols of rank 4 and defect 1 are the following:

Symbol	(π_1, π_2)	Representation of W_2
$\begin{pmatrix} 2 \\ - \end{pmatrix}$	$2, -$	$\mathbb{1}$
$\begin{pmatrix} 1, 2 \\ 0 \end{pmatrix}$	$(1,1), -$	$\overline{\mathrm{sgn}}_2$
$\begin{pmatrix} 0, 2 \\ 1 \end{pmatrix}$	$1, 1$	$\mathrm{Ind}_{W_1 \times W_1}^{W_2} \mathbb{1} \otimes \chi_1$
$\begin{pmatrix} 0, 1 \\ 2 \end{pmatrix}$	$-, 2$	χ_2
$\begin{pmatrix} 0, 1, 2 \\ 1, 2 \end{pmatrix}$	$-, (1,1)$	$\overline{\mathrm{sgn}}_2 \otimes \chi_2$

In this table, the first column is the symbol of rank 2 and defect 1. The second column is the corresponding ordered partition pairs. The last column is the corresponding irreducible representation of W_2. Here $W_1 \times W_1$ is the normal subgroup $\langle(1,1')\rangle \times \langle(2,2')\rangle$ of W_2 , and χ is the non-trivial character of $\langle(2,2')\rangle$.

5.4.6 The symmetric pair $(\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{F}_q))$

In this section we will identify all $\mathrm{SL}_2(\mathbb{F}_{q^2})$ or $\mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{SL}_2(\mathbb{F}_q)$ distinguished cuspidal representations of $\mathrm{Sp}_4(\mathbb{F}_q)$.

To find the distinguished representations of $\mathrm{Sp}_4(\mathbb{F}_q)$, we give an estimate of the dimension formula.

Theorem 5.4.3. *For any irreducible representation ρ of $\mathrm{Sp}_{4n}(\mathbb{F}_q)$, we have*

$$\langle \rho, \mathrm{Ind}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})}^{\mathrm{Sp}_{4n}(\mathbb{F}_q)} \mathbb{1} \rangle \leq 1.$$

Proof. By the calculation of orbits in [77], there exists an anti-involution which preserves the double cosets

$$\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}) \backslash \mathrm{Sp}_{4n}(\mathbb{F}_q) / \mathrm{Sp}_{2n}(\mathbb{F}_{q^2}).$$

Then applying Mackey theorem, we conclude that $\langle \rho, \mathrm{Ind}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})}^{\mathrm{Sp}_{4n}(\mathbb{F}_q)} \mathbb{1} \rangle \leq 1$. □

Theorem 5.4.4. *For any irreducible representation ρ of $\mathrm{Sp}_4(J, \mathbb{F}_q)$, we have*

$$\langle \rho, \mathrm{Ind}_{\mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{SL}_2(\mathbb{F}_q)}^{\mathrm{Sp}_4(\mathbb{F}_q)} \mathbb{1} \rangle \leq 1.$$

Proof. This proof is similar with Theorem 5.4.3 □

According to [18, Theorem 3.5.6], for a connected reductive group G, if its derived group is simply connected, then $C_G(s)$ is connected for any semi-simple element s. Since G is the symplectic group $\mathrm{Sp}_{4n}(\mathbb{F}_q)$, $Z_G(t)$ in the definition 5.4.3 is connected. Therefore $\varepsilon_{T,f}$ is always trivial.

If the character λ is trivial, then $T^F \backslash \Theta_{T,\lambda}^F / H^F$ is same as the double cosets $T^F \backslash \Theta_T^F / H^F$.

Fix T to be the set of diagonal matrices of G, and B to be the set of upper triangular matrices of G. Then B is an F-stable Borel subgroup of G, containing an F-stable maximal torus T. Let g be an element of G such that $g^{-1}F(g) \in N_G(T)$. Hence gTg^{-1} is also an F-stable maximal torus of G. The map from gTg^{-1} to the image of $g^{-1}F(g)$ in W gives a bijection between the G^F-conjugacy classes of F-stable maximal tori of G and the conjugacy classes of W.

Let $\{f_i\}$ be a set of representatives of $T \backslash \Theta_T / H$. Then, $\{gf_i\}$ is a set of representatives of $gTg^{-1} \backslash \Theta_{gTg^{-1}} / H$. Let ϑ be a map on G, defined by $\vartheta(g) = g\theta(g^{-1})$. Since gTg^{-1} and H are F-stable, and θ and F are commutative, we conclude that F stabilizes double cosets $gTg^{-1} \backslash \Theta_{gTg^{-1}} / H$. Then, F permutes the representatives $\{gf_i\}$. Moreover, we obtain the following lemma.

Lemma 5.4.3. *An element $F(gf_i)$ in the double coset $gTg^{-1}\cdot gf_j\cdot H$ is equivalent to $g^{-1}F(g)\vartheta(F(f_i))\theta(g^{-1}F(g))^{-1} = t\vartheta(f_j)\theta(t)^{-1}$ for some t in T.*

Let

$$w_1 = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} \text{ and } w_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}.$$

The images of these two elements in $N_G(T)/T$ generate W. Let I_4 be the 4-by-4 identity matrix. Then, the set $\{I_4, w_1, w_2, w_1w_2, (w_1w_2)^2\}$ consists of representatives of all conjugacy classes of W. We can choose g such that $g^{-1}F(g) \in \{I_4, w_1, w_2, w_1w_2, (w_1w_2)^2\}$. Then, we use the elements in this set to parametrize the k-stable maximal tori, up to G^F-conjugate.

If the symmetric pair (G, H) is $(\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{F}_q)\times \mathrm{SL}_2(\mathbb{F}_q))$, then it is easily to check that the number of double cosets $T\backslash\Theta_T/H$ is 4, and there exists a representative set $\{f_i\}_{1\leq i\leq 4}$ such that f_i for all $1\leq i\leq 4$ satisfies the following equations:

$$\vartheta(f_1) = I_4, \vartheta(f_2) = -I_4, \vartheta(f_3) = w_1, \vartheta(f_4) = w_2^{-1}w_1w_2.$$

By Lemma 5.4.3, we obtain the F-fixed double cosets for each F-stable maximal torus gTg^{-1}.

gTg^{-1}	1	w_1	w_2	w_1w_2	$(w_1w_2)^2$
$\{gf_i\}^F_{1\leq i\leq 4}$	$\{gf_i\}_{1\leq i\leq 4}$	$\{gf_3, gf_4\}$	$\{gf_1, gf_2\}$	$\emptyset$	$\{gf_i\}_{1\leq i\leq 4}$

The first row is the G^F-conjugate class of F-stable maximal torus gTg^{-1}. The second row consists of the representatives of F-fixed double cosets $gTg^{-1}\backslash\Theta_T/H$.

To apply Lusztig's formula 5.4.2, we calculate the $\mathbb{F}_q$-ranks of gTg^{-1} for all g, and $Z_G((T\cap fHf^{-1})^\circ))$. Let G_i be $Z_G((T\cap fHf^{-1})^\circ))$.

gTg^{-1}	1	w_1	w_2	w_1w_2	$(w_1w_2)^2$
$(gTg^{-1})^F$	$\mathbb{F}_q^\times \times \mathbb{F}_q^\times$	$\mathbb{F}_{q^2}^\times$	$\mathbb{F}_q^\times \times \mathbb{F}_{q^2}^1$	$\mathbb{F}_{q^4}^1$	$\mathbb{F}_{q^2}^1 \times \mathbb{F}_{q^2}^1$
G_1^F	$\mathbb{F}_q^\times \times \mathbb{F}_q^\times$	$-$	$\mathbb{F}_q^\times \times \mathbb{F}_{q^2}^1$	$-$	$\mathbb{F}_{q^2}^1 \times \mathbb{F}_{q^2}^1$
G_2^F	$\mathbb{F}_q^\times \times \mathbb{F}_q^\times$	$-$	$\mathbb{F}_q^\times \times \mathbb{F}_{q^2}^1$	$-$	$\mathbb{F}_{q^2}^1 \times \mathbb{F}_{q^2}^1$
G_3^F	$\mathrm{GL}_2(\mathbb{F}_q)$	$\mathrm{GL}_2(\mathbb{F}_q)$	$-$	$-$	$\mathrm{U}_2(J, \mathbb{F}_{q^2})$
G_4^F	$\mathrm{GL}_2(\mathbb{F}_q)$	$\mathrm{SL}_2(\mathbb{F}_{q^2})$	$-$	$-$	$\mathrm{U}_2(J, \mathbb{F}_{q^2})$
$\boxed{R_w^{\mathbb{1}}}$	4	0	2	0	0

Therefore, we have the dimensions $\boxed{R_i}$ for $1 \leq i \leq 5$,

$$\boxed{R_1} = 1, \boxed{R_2} = 2, \boxed{R_3} = 2, \boxed{R_4} = 0, \boxed{R_5} = 0.$$

Since $\boxed{\theta_{10}}$ and $\boxed{\rho_i} \in \{0, 1\}$ for all i, we conclude the following lemma.

Lemma 5.4.4. *A unipotent irreducible representation ρ is H-distinguished if and only if $\rho \in \{\mathbb{1}, \rho_1, \rho_2\}$.*

If the symmetric pair (G, H) is $(\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{F}_{q^2}))$, then it is easily to check that the number of double cosets $T\backslash\Theta_T/H$ is 4, and there exists a representative set $\{f_i'\}_{1\leq i\leq 4}$ such that f_i' for all $1 \leq i \leq 4$ satisfies the following equations: $\vartheta(f_1') = I_4$, $\vartheta(f_3') = -\vartheta(f_2')$

$$\vartheta(f_2') = \begin{pmatrix} & \sqrt{\tau}^{-1} & & \\ -\sqrt{\tau} & & & \\ & & & -\sqrt{\tau}^{-1} \\ & & \sqrt{\tau} & \end{pmatrix}, \vartheta(f_4') = \begin{pmatrix} & & & 1 \\ & & -\tau & \\ & \tau^{-1} & & \\ -1 & & & \end{pmatrix}.$$

By Lemma 5.4.3, we obtain the F-fixed double cosets for each F-stable maximal torus gTg^{-1}.

gTg^{-1}	1	w_1	w_2	w_1w_2	$(w_1w_2)^2$
$\{gf_i'\}_{1\leq i\leq 4}^F$	$\{gf_1', gf_4'\}$	$\{gf_i'\}_{1\leq i\leq 4}$	$\emptyset$	$\{gf_2', gf_3'\}$	$\{gf_1', gf_4'\}$

The first row is the G^F-conjugate class of F-stable maximal torus gTg^{-1}. The second row consists of the representatives of F-fixed double cosets $gTg^{-1}\backslash\Theta_T/H$.

To apply Lusztig's formula 5.4.2, we calculate the $\mathbb{F}_q$-ranks of gTg^{-1} for all g, and $Z_G((T\cap f'Hf'^{-1})^\circ))$. Let G'_i be $Z_G((T\cap f'Hf'^{-1})^\circ))$.

gTg^{-1}	1	w_1	w_2	w_1w_2	$(w_1w_2)^2$
$(gTg^{-1})^F$	$\mathbb{F}_q^\times\times\mathbb{F}_q^\times$	$\mathbb{F}_{q^2}^\times$	$\mathbb{F}_q^\times\times\mathbb{F}_{q^2}^1$	$\mathbb{F}_{q^4}^1$	$\mathbb{F}_{q^2}^1\times\mathbb{F}_{q^2}^1$
$G_{1'}^F$	$\mathrm{GL}_2(\mathbb{F}_q)$	$\mathrm{GL}_2(\mathbb{F}_q)$	$-$	$-$	$\mathrm{U}_2(J,\mathbb{F}_{q^2})$
$G_{2'}^F$	$-$	$\mathbb{F}_{q^2}^\times$	$-$	$\mathbb{F}_{q^4}^1$	$-$
$G_{3'}^F$	$-$	$\mathbb{F}_{q^2}^\times$	$-$	$\mathbb{F}_{q^4}^1$	$-$
$G_{4'}^F$	$\mathrm{GL}_2(\mathbb{F}_q)$	$\mathrm{SL}_2(\mathbb{F}_{q^2})$	$-$	$-$	$\mathrm{U}_2(J,\mathbb{F}_{q^2})$
$\boxed{R_w^{\mathbb{1}}}$	2	2	0	2	-2

Therefore, we have the dimensions of R_i for $1\le i\le 5$.

$$\boxed{R_1}=1,\boxed{R_2}=-1,\boxed{R_3}=1,\boxed{R_4}=0,\boxed{R_5}=0.$$

Since $\boxed{\theta_{10}}$ and $\boxed{\rho_i}\in\{0,1\}$ for all i, we conclude the following lemma.

Lemma 5.4.5. *A unipotent irreducible representation ρ is H-distinguished if and only if $\rho\in\{\mathbb{1},\rho_3,\theta_{10}\}$.*

Now, we can calculate the dimension $\boxed{\rho}$, for any irreducible cuspidal representation ρ of $\mathrm{Sp}_4(\mathbb{F}_q)$.

Theorem 5.4.5. *If a symmetric pair (G,H) is $(\mathrm{Sp}_4(\mathbb{F}_q),\mathrm{SL}_2(\mathbb{F}_q)\times\mathrm{SL}_2(\mathbb{F}_q))$, then there does not exists H-distinguished cuspidal representation of G.*

Proof. In Lemma 5.4.4, we show that θ_{10} is not H-distinguished. According to Theorem 5.4.1, we need only to prove that $R_{w_1w_2}^{\lambda_1}$, $R_{w_1w_2}^{\lambda_2}$, and $R_{(w_1w_2)^2}^{\lambda_0}$ are not H-distinguished. Here, $(\lambda_1,T_{(w_1w_2)^2})$, $(\lambda_2,T_{w_1w_2})$, and $(\lambda_0,T_{(w_1w_2)^2})$ are corresponding to $(s_1,\mathrm{U}_1(q^2)\times\mathrm{U}_1(q^2))$, $(s_2,\mathrm{U}_1(\mathbb{F}_{q^4}))$, and $(s_0,\mathrm{U}_1(q^2)\times\mathrm{U}_1(q^2))$.

By the table in the split case, we have that $\Theta_{T_{w_1w_2}}^F$ is empty. Therefore, $\boxed{R_{\mathrm{U}_1(\mathbb{F}_{q^4})}^{s_2}}$ is zero. For the case $R_{(w_1w_2)^2}^{\lambda_i}$ for $0\le i\le 1$, we have that $\Theta_{T_{(w_1w_2)^2},\lambda_i}^F$ for $0\le i\le 1$ are empty. Then, $R_{\mathrm{U}_1(q^2)\times\mathrm{U}_1(q^2))}^{s_i}$ are not distinguished by H.

Therefore, there is no H-distinguished cuspidal representation of G. ∎

Theorem 5.4.6. *If a symmetric pair (G, H) is $(\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{F}_{q^2}))$, then θ_{10} is the unique H-distinguished irreducible cuspidal representation of G.*

Proof. The proof is similar to Theorem 5.4.5. In Lemma 5.4.5, we have that θ_{10} is a H-distinguished irreducible cuspidal representation of G.

By the table in the non-split case, we have that $\Theta^F_{T_{w_1 w_2}, \lambda_2}$ and $\Theta^F_{T_{(w_1 w_2)^2}, \lambda_i}$ for $0 \leq i \leq 1$ are empty. Therefore, $\boxed{R^{s_2}_{\mathrm{U}_1(\mathbb{F}_{q^4})}}$ and $\boxed{R^{s_i}_{\mathrm{U}_1(q^2)\times \mathrm{U}_1(q^2)}}$ for $0 \leq i \leq 1$ are zero. Then, we prove this theorem. $\square$

5.5 Applications of some global results

In this section, we recall some results on automorphic forms. Applying these results, we study the distinguished representations of the symmetric pair

$$(\mathrm{Sp}_{4n}(\mathbb{F}_q), \mathrm{Sp}_{2m}(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-m)}(\mathbb{F}_q)).$$

Proposition 5.5.1 ([11, Proposition 2])**.** *For all cuspidal automorphic forms ϕ of $\mathrm{Sp}_{2n}(\mathbb{A}_k)$ and for all $0 < m < n$,*

$$\int_{H_m(k)\backslash H_m(\mathbb{A}_k)} \phi(h)\, \mathrm{d}h = 0.$$

Prasad and Schulze-Pillot proved that each H-distinguished supercuspidal representation can be embedded as a component of a cuspidal automorphic representation whose period integral over H is nonvanishing.

Proposition 5.5.2 ([60, Theorem 4.1])**.** *Let H be a closed subgroup of a reductive group G with finite center, both defined over a number field k. Suppose that S is a finite set of non-archimedean places of k, and π_v a supercuspidal representation of $G(k_v)$ for all $v \in S$, which is distinguished by $H(k_v)$. Let T be a finite set of places containing S and all the infinite places, such that G is quasi-split at places outside T. Then there exists a global automorphic form $\Pi = \bigotimes \Pi_v$ of $G(\mathbb{A}_k)$, necessarily* *cuspidal, such that* $\Pi_v = \pi_v$ *for* $v \in S$*, and* Π_v *is unramified at all finite places of* k *outside* T*, and an* $\phi \in \Pi$ *such that*

$$\int_{H(k)\backslash H(\mathbb{A}_k)} \phi(h)\,\mathrm{d}h \neq 0.$$

Applying Proposition 5.5.2 and Proposition 5.5.1, we obtain the following corollary.

Corollary 5.5.1. *If the symmetric pair* (G, H) *is* $(\mathrm{Sp}_{2n}(k), \mathrm{Sp}_{2m}(k)\times\mathrm{Sp}_{2(n-m)}(k))$ *and* k *is a non-archimedean field, then there is no* H*-distinguished supercuspidal representation of* G*.*

For the symmetric pair $(\mathrm{Sp}_{2n}(\mathbb{F}_q), \mathrm{Sp}_{2m}(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-m)}(\mathbb{F}_q)$ over a finite field, we lift the cuspidal representation $\overline{\rho}$ of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ to a representation of $\mathrm{Sp}_{2n}(\mathfrak{o})$, where $\mathfrak{o}$ is the integer ring of k and k has the residue field that is isomorphic to $\mathbb{F}_q$. Then, we get a depth-zero supercuspidal representation π of $\mathrm{Sp}_{2n}(k)$. If $\overline{\rho}$ is distinguished by $\mathrm{Sp}_{2m}(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-m)}(\mathbb{F}_q)$, then π is a $\mathrm{Sp}_{2m}(k) \times \mathrm{Sp}_{2(n-m)}(k)$-distinguished.

Corollary 5.5.2. *There is no* $\mathrm{Sp}_{2m}(\mathbb{F}_q)\times\mathrm{Sp}_{2(n-m)}(\mathbb{F}_q)$*-distinguished cuspidal representation of* $\mathrm{Sp}_{2n}(\mathbb{F}_q)$*.*

Proof. Let (G, H) be the symmetric pair $(\mathrm{Sp}_{2n}(k), \mathrm{Sp}_{2m}(k) \times \mathrm{Sp}_{2(n-m)}(k))$. We choose the involution $\theta = \mathrm{Ad}(\varepsilon)$, where $\varepsilon = \mathrm{diag}\{I_m, -I_{2(n-m),I_m}\}$. Then, it is clear that θ is stable $G_{0,0} = \mathrm{Sp}_{2n}(\mathfrak{o})$. The finite field $G_{0,0}/G_{0,0^+}$ is isomorphic to $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ and $(G_{0,0^+}/G_{0,0})^\theta$ is isomorphic to $\mathrm{Sp}_{2m}(\mathbb{F}_q) \times \mathrm{Sp}_{2(n-m)}(\mathbb{F}_q)$.

Let $\overline{\rho}$ is an irreducible cuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. There is a representation ρ of $G(\mathfrak{o}_k)$ such that is trivial on $G_{0,0^+}$ and factors though the representation $\overline{\rho}$ of $G_{0,0}/G_{0,0^+}$. Let π be the induced representation $\mathrm{ind}_{G_{0,0}}^G\rho$. Then π is a depth-zero supercuspidal representation of G. By Mackey's theorem, we have that $\mathrm{Hom}_{G_{0,0}\cap H}(\pi, \mathbb{1})$ is a summand of $\mathrm{Hom}_H(\pi, \mathbb{1})$. Since

$$\mathrm{Hom}_{G_{0,0}\cap H}(\pi, \mathbb{1}) \cong \mathrm{Hom}_{(G_{0,0^+}/G_{0,0})^\theta}(\overline{\rho}, \mathbb{1}) \text{ and } \mathrm{Hom}_H(\pi, \mathbb{1}) = 0,$$

we have

$$\mathrm{Hom}_{\mathrm{Sp}_{2m}(\mathbb{F}_q)\times \mathrm{Sp}_{2(n-m)}(\mathbb{F}_q)}(\overline{\rho}, \mathbb{1}) = 0.$$

$\square$

References

[1] *Seminar on Algebraic Groups and Related Finite Groups. (Held at The Institute for Advanced Study, Princeton, N. J., 1968/69).* Lecture Notes in Mathematics, Vol. 131. Springer-Verlag, Berlin, 1970.

[2] Jeffrey D. Adler. Refined anisotropic K-types and supercuspidal representations. *Pacific J. Math.*, 185(1):1–32, 1998. ISSN 0030-8730.

[3] Avraham Aizenbud. A partial analog of integrability theorem for distributions on p-adic spaces and applications. 11 2008. URL http://arxiv.org/abs/0811.2768.

[4] Avraham Aizenbud and Dmitry Gourevitch. Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis's theorem. *Duke Math. J.*, 149(3):509–567, 2009. ISSN 0012-7094. doi: 10.1215/00127094-2009-044. URL http://dx.doi.org/10.1215/00127094-2009-044. With an appendix by the authors and Eitan Sayag.

[5] Avraham Aizenbud and Dmitry Gourevitch. Multiplicity one theorem for $(\mathrm{GL}_{n+1}(\mathbb{R}), \mathrm{GL}_n(\mathbb{R}))$. *Selecta Math. (N.S.)*, 15(2):271–294, 2009. ISSN 1022-1824. doi: 10.1007/s00029-009-0544-7. URL http://dx.doi.org/10.1007/s00029-009-0544-7.

[6] Avraham Aizenbud and Dmitry Gourevitch. Some regular symmetric pairs. *Trans. Amer. Math. Soc.*, 362(7):3757–3777, 2010. ISSN 0002-9947.

doi: 10.1090/S0002-9947-10-05008-7. URL http://dx.doi.org/10.1090/S0002-9947-10-05008-7.

[7] Avraham Aizenbud and Eitan Sayag. Invariant distributions on non-distinguished nilpotent orbits with application to the gelfand property of (gl(2n,r),sp(2n,r)). 10 2008. URL http://arxiv.org/abs/0810.1853.

[8] Avraham Aizenbud, Dmitry Gourevitch, and Eitan Sayag. $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a Gelfand pair for any local field F. *Compos. Math.*, 144(6):1504–1524, 2008. ISSN 0010-437X. doi: 10.1112/S0010437X08003746. URL http://dx.doi.org/10.1112/S0010437X08003746.

[9] U. K. Anandavardhanan. Distinguished non-Archimedean representations. In *Algebra and number theory*, pages 183–192. Hindustan Book Agency, Delhi, 2005.

[10] Teruaki Asai. Unipotent characters of $\mathrm{SO}^{\pm}_{2n}$, Sp_{2n} and SO_{2n+1} over F_q with small q. *Osaka J. Math.*, 20(3):631–643, 1983. ISSN 0030-6126. URL http://projecteuclid.org/getRecord?id=euclid.ojm/1200776326.

[11] Avner Ash, David Ginzburg, and Steven Rallis. Vanishing periods of cusp forms over modular symbols. *Math. Ann.*, 296(4):709–723, 1993. ISSN 0025-5831. doi: 10.1007/BF01445131. URL http://dx.doi.org/10.1007/BF01445131.

[12] Avner Ash, David Ginzburg, and Steven Rallis. Vanishing periods of cusp forms over modular symbols. *Math. Ann.*, 296(4):709–723, 1993. ISSN 0025-5831. doi: 10.1007/BF01445131. URL http://dx.doi.org/10.1007/BF01445131.

[13] I. N. Bernšteĭn and A. V. Zelevinskiĭ. Representations of the group $GL(n, F)$, where F is a local non-Archimedean field. *Uspehi Mat. Nauk*, 31(3(189)):5–70, 1976. ISSN 0042-1316.

[14] Philippe Blanc and Patrick Delorme. Vecteurs distributions H-invariants de représentations induites, pour un espace symétrique réductif p-adique G/H. *Ann. Inst. Fourier (Grenoble)*, 58(1):213–261, 2008. ISSN 0373-0956. URL http://aif.cedram.org/item?id=AIF_2008__58_1_213_0.

[15] E. P. H. Bosman and G. van Dijk. A new class of Gel′fand pairs. *Geom. Dedicata*, 50(3):261–282, 1994. ISSN 0046-5755. doi: 10.1007/BF01267869. URL http://dx.doi.org/10.1007/BF01267869.

[16] Daniel Bump and Solomon Friedberg. The exterior square automorphic L-functions on $\mathrm{GL}(n)$. In *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, volume 3 of *Israel Math. Conf. Proc.*, pages 47–65. Weizmann, Jerusalem, 1990.

[17] Colin J. Bushnell and Philip C. Kutzko. *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, volume 129 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. ISBN 0-691-03256-4; 0-691-02114-7.

[18] Roger W. Carter. *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. ISBN 0-471-94109-3. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[19] David H. Collingwood and William M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993. ISBN 0-534-18834-6.

[20] P. Deligne and G. Lusztig. Representations of reductive groups over finite fields. *Ann. of Math. (2)*, 103(1):103–161, 1976. ISSN 0003-486X.

[21] Solomon Friedberg and Hervé Jacquet. Linear periods. *J. Reine Angew. Math.*, 443:91–139, 1993. ISSN 0075-4102. doi: 10.1515/crll.1993.443.91. URL http://dx.doi.org/10.1515/crll.1993.443.91.

[22] I. M. Gel′fand and D. A. Kajdan. Representations of the group $\mathrm{GL}(n, K)$ where K is a local field. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 95–118. Halsted, New York, 1975.

[23] David Ginzburg, Stephen Rallis, and David Soudry. On explicit lifts of cusp forms from GL_m to classical groups. *Ann. of Math. (2)*, 150(3):807–866, 1999. ISSN 0003-486X. doi: 10.2307/121057. URL `http://dx.doi.org/10.2307/121057`.

[24] David Ginzburg, Dihua Jiang, and David Soudry. Periods of automorphic forms, poles of l-functions and functorial lifting. *SCIENCE CHINA MATHEMATICS*, 53(9):2215–2238, 2010.

[25] Daniel Goldstein and Robert M. Guralnick. Alternating forms and self-adjoint operators. *J. Algebra*, 308(1):330–349, 2007. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2006.06.009. URL `http://dx.doi.org/10.1016/j.jalgebra.2006.06.009`.

[26] Benedict H. Gross, Dipendra Prasad, and Wee Teck Gan. Symplectic local root numbers, central critical l-values and restriction problems in the representation theory of classical groups. *to appear in Asterisque. Accessed online at http://www. math. ucsd. edu/˜ wgan*, 2010.

[27] Jeffrey Hakim and Joshua Lansky. Orthogonal periods of tame supercuspidal representations of gl(n) with n odd. *preprint*, 2010.

[28] Jeffrey Hakim and Fiona Murnaghan. Distinguished tame supercuspidal representations. *Int. Math. Res. Pap. IMRP*, (2):Art. ID rpn005, 166, 2008. ISSN 1687-3017.

[29] Harish-Chandra. Admissible invariant distributions on reductive p-adic groups. In *Lie theories and their applications (Proc. Ann. Sem. Canad. Math.*

Congr., Queen's Univ., Kingston, Ont., 1977), pages 281–347. Queen's Papers in Pure Appl. Math., No. 48. Queen's Univ., Kingston, Ont., 1978.

[30] Anthony Henderson. Symmetric subgroup invariants in irreducible representations of G^F, when $G = \mathrm{GL}_n$. *J. Algebra*, 261(1):102–144, 2003. ISSN 0021-8693. doi: 10.1016/S0021-8693(02)00559-8. URL http://dx.doi.org/10.1016/S0021-8693(02)00559-8.

[31] Michael J. Heumos and Stephen Rallis. Symplectic-Whittaker models for Gl_n. *Pacific J. Math.*, 146(2):247–279, 1990. ISSN 0030-8730. URL http://projecteuclid.org/getRecord?id=euclid.pjm/1102645157.

[32] Tamutsu Ikeda. On the lifting of automorphic representations of $\mathrm{PGL}_2(\mathbb{A})$ to $\mathrm{S}p_{2n}(\mathbb{A})$ or $\widetilde{\mathrm{Sp}_{2n}(\mathbb{A})}$ over a toally real field. URL http://www.math.kyoto-u.ac.jp/~ikeda/preprints.html.

[33] Ronald Jacobowitz. Hermitian forms over local fields. *Amer. J. Math.*, 84: 441–465, 1962. ISSN 0002-9327.

[34] Hervé Jacquet and Stephen Rallis. Symplectic periods. *J. Reine Angew. Math.*, 423:175–197, 1992. ISSN 0075-4102.

[35] Hervé Jacquet and Stephen Rallis. Uniqueness of linear periods. *Compositio Math.*, 102(1):65–123, 1996. ISSN 0010-437X. URL http://www.numdam.org/item?id=CM_1996__102_1_65_0.

[36] Dihua Jiang. Periods of automorphic forms. In *Proceedings of the International Conference on Complex Geometry and Related Fields*, volume 39 of *AMS/IP Stud. Adv. Math.*, pages 125–148, Providence, RI, 2007. Amer. Math. Soc.

[37] Dihua Jiang and Lei Zhang. Distinguished representations of $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$. *In preparation*, 2011.

[38] Shin-ichi Kato and Keiji Takano. Subrepresentation theorem for p-adic symmetric spaces. *Int. Math. Res. Not. IMRN*, (11):Art. ID rnn028, 40, 2008. ISSN 1073-7928.

[39] Hidenori Katsurada. An explicit formula for Siegel series. *Amer. J. Math.*, 121 (2):415–452, 1999. ISSN 0002-9327. URL http://muse.jhu.edu/journals/american_journal_of_mathematics/v121/121.2katsurada.pdf.

[40] Henry H. Kim. On local L-functions and normalized intertwining operators. *Canad. J. Math.*, 57(3):535–597, 2005. ISSN 0008-414X. URL http://journals.cms.math.ca/ams/ams-redirect.php?Journal=CJM&Volume=57&FirstPage=535.

[41] Ju-Lee Kim. Supercuspidal representations: an exhaustion theorem. *J. Amer. Math. Soc.*, 20(2):273–320 (electronic), 2007. ISSN 0894-0347. doi: 10.1090/S0894-0347-06-00544-3. URL http://dx.doi.org/10.1090/S0894-0347-06-00544-3.

[42] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. ISBN 0-8218-0904-0. With a preface in French by J. Tits.

[43] Stephen S. Kudla. Special cycles and derivatives of Eisenstein series. In *Heegner points and Rankin L-series*, volume 49 of *Math. Sci. Res. Inst. Publ.*, pages 243–270. Cambridge Univ. Press, Cambridge, 2004. doi: 10.1017/CBO9780511756375.009. URL http://dx.doi.org/10.1017/CBO9780511756375.009.

[44] Stephen S. Kudla and Stephen Rallis. A regularized Siegel-Weil formula: the first term identity. *Ann. of Math. (2)*, 140(1):1–80, 1994. ISSN 0003-486X. doi: 10.2307/2118540. URL http://dx.doi.org/10.2307/2118540.

[45] Erez Lapid and Omer Offen. Compact unitary periods. *Compos. Math.*, 143 (2):323–338, 2007. ISSN 0010-437X.

[46] D. W. Lewis. The isometry classification of Hermitian forms over division algebras. *Linear Algebra Appl.*, 43:245–272, 1982. ISSN 0024-3795. doi: 10.1016/0024-3795(82)90258-0. URL http://dx.doi.org/10.1016/0024-3795(82)90258-0.

[47] G. Lusztig. Irreducible representations of finite classical groups. *Invent. Math.*, 43(2):125–175, 1977. ISSN 0020-9910.

[48] G. Lusztig. On the representations of reductive groups with disconnected centre. *Astérisque*, (168):10, 157–166, 1988. ISSN 0303-1179. Orbites unipotentes et représentations, I.

[49] George Lusztig. Unipotent characters of the symplectic and odd orthogonal groups over a finite field. *Invent. Math.*, 64(2):263–296, 1981. ISSN 0020-9910. doi: 10.1007/BF01389170. URL http://dx.doi.org/10.1007/BF01389170.

[50] George Lusztig. *Characters of reductive groups over a finite field*, volume 107 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1984. ISBN 0-691-08350-9; 0-691-08351-7.

[51] George Lusztig. Symmetric spaces over a finite field. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 57–81. Birkhäuser Boston, Boston, MA, 1990. doi: 10.1007/978-0-8176-4576-2_3. URL http://dx.doi.org/10.1007/978-0-8176-4576-2_3.

[52] Zhengyu Mao and Stephen Rallis. A Plancherel formula for $\mathrm{Sp}_{2n}/\mathrm{Sp}_n \times \mathrm{Sp}_n$ and its application. *Compos. Math.*, 145(2):501–527, 2009. ISSN 0010-437X. doi: 10.1112/S0010437X08003904. URL http://dx.doi.org/10.1112/S0010437X08003904.

[53] Lawrence Morris. Level zero **G**-types. *Compositio Math.*, 118(2):135–157, 1999. ISSN 0010-437X. doi: 10.1023/A:1001019027614. URL http://dx.doi.org/10.1023/A:1001019027614.

[54] Allen Moy and Gopal Prasad. Unrefined minimal K-types for p-adic groups. *Invent. Math.*, 116(1-3):393–408, 1994. ISSN 0020-9910. doi: 10.1007/BF01231566. URL http://dx.doi.org/10.1007/BF01231566.

[55] Fiona Murnaghan. Regularity and distinction of supercuspidal representations. *preprint*, 2010.

[56] Chufeng Nien. Uniqueness of Shalika models. *Canad. J. Math.*, 61(6):1325–1340, 2009. ISSN 0008-414X. URL http://journals.cms.math.ca/ams/ams-redirect.php?Journal=CJM&Volume=61&FirstPage=1325.

[57] O. Timothy O'Meara. *Introduction to quadratic forms*. Classics in Mathematics. Springer-Verlag, Berlin, 2000. ISBN 3-540-66564-1. Reprint of the 1973 edition.

[58] I. Piatetski-Shapiro and S. Rallis. A new way to get Euler products. *J. Reine Angew. Math.*, 392:110–124, 1988. ISSN 0075-4102. doi: 10.1515/crll.1988.392.110. URL http://dx.doi.org/10.1515/crll.1988.392.110.

[59] I. I. Piatetski-Shapiro. L-functions for GSp_4. *Pacific J. Math.*, (Special Issue):259–275, 1997. ISSN 0030-8730. doi: 10.2140/pjm.1997.181.259. URL http://dx.doi.org/10.2140/pjm.1997.181.259. Olga Taussky-Todd: in memoriam.

[60] Dipendra Prasad and Rainer Schulze-Pillot. Generalised form of a conjecture of Jacquet and a local consequence. *J. Reine Angew. Math.*, 616:219–236, 2008. ISSN 0075-4102. doi: 10.1515/CRELLE.2008.023. URL http://dx.doi.org/10.1515/CRELLE.2008.023.

[61] Gopal Prasad and Jiu-Kang Yu. On finite group actions on reductive groups and buildings. *Invent. Math.*, 147(3):545–560, 2002. ISSN 0020-9910. doi: 10.1007/s002220100182. URL http://dx.doi.org/10.1007/s002220100182.

[62] Cary Rader and Steve Rallis. Spherical characters on $\mathfrak{p}$-adic symmetric spaces. *Amer. J. Math.*, 118(1):91–178, 1996. ISSN 0002-9327. URL http://muse.jhu.edu/journals/american_journal_of_mathematics/v118/118.1rader.pdf.

[63] Stephen Rallis. *L-functions and the oscillator representation*, volume 1245 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987. ISBN 3-540-17694-2.

[64] Dinakar Ramakrishnan and Robert J. Valenza. *Fourier analysis on number fields*, volume 186 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999. ISBN 0-387-98436-4.

[65] Peter Sarnak. A letter to Cathleen Morawetz. 2004.

[66] Eitan Sayag. (gl(2n,c),sp(2n,c)) is a gelfand pair. 05 2008. URL http://arxiv.org/abs/0805.2625.

[67] Ralf Schmidt. On the spin L-function of Ikeda's lifts. *Comment. Math. Univ. St. Pauli*, 52(1):1–46, 2003. ISSN 0010-258X.

[68] F. Shahidi. *Eisenstein Series and Automorphic l-functions*. Amer Mathematical Society, 2010.

[69] J. A. Shalika. The multiplicity one theorem for GL_n. *Ann. of Math. (2)*, 100: 171–193, 1974. ISSN 0003-486X.

[70] T. A. Springer and R. Steinberg. Conjugacy classes. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, Lecture Notes in Mathematics, Vol. 131, pages 167–266. Springer, Berlin, 1970.

[71] Bhama Srinivasan. The characters of the finite symplectic group Sp(4, q). *Trans. Amer. Math. Soc.*, 131:488–525, 1968. ISSN 0002-9947.

[72] Shaun Stevens. The supercuspidal representations of p-adic classical groups. *Invent. Math.*, 172(2):289–352, 2008. ISSN 0020-9910. doi: 10.1007/s00222-007-0099-1. URL http://dx.doi.org/10.1007/s00222-007-0099-1.

[73] Binyong Sun and Chen-Bo Zhu. A general form of gelfand-kazhdan criterion. 03 2009. URL http://arxiv.org/abs/0903.1409.

[74] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.

[75] G. van Dijk. Invariant eigendistributions on the tangent space of a rank one semisimple symmetric space. *Math. Ann.*, 268(3):405–416, 1984. ISSN 0025-5831. doi: 10.1007/BF01457067. URL http://dx.doi.org/10.1007/BF01457067.

[76] Jiu-Kang Yu. Construction of tame supercuspidal representations. *J. Amer. Math. Soc.*, 14(3):579–622 (electronic), 2001. ISSN 0894-0347. doi: 10.1090/S0894-0347-01-00363-0. URL http://dx.doi.org/10.1090/S0894-0347-01-00363-0.

[77] Lei Zhang. Gelfand pairs $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$. *Journal of Number Theory*, 130(11):2428–2441, November 2010.

[78] Lei Zhang. Distinguished tame supercuspidal representations of symmetric pairs $(\mathrm{Sp}_{4n}(F)), \mathrm{Sp}_{2n}(E)$. *Ready to submit*, 2010.