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THE EFFECT OF TUBE VIBRATIONS ON FLOW THROUGH TUBES

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ABSTRACT

This paper presents an approximate analysis and an experimental investigation of the effect of external vibration on flow in tubes. Only flow in straight tubes of uniform cross section was considered. It is shown that external vibration of the tube produces small fluctuations in velocity and pressure in the tube. In spite of these fluctuations, it was found both by analysis and by experiment that the pressure drop in steady flow was not increased and that there was no delay in flow establishment after opening a valve.

An incidental result of this investigation was the experimental finding that laminar flow in tubes at Reynolds numbers up to 15,000 was stable to all small external disturbances. This finding was applicable to both the fully developed region and the region of developing boundary layer beyond at least 60 tube diameters from the entrance.

CONTENTS

L I S T O F F I G U R E S

THE EFFECT OF TUBE VIBRATIONS ON FLOW THROUGH TUBES

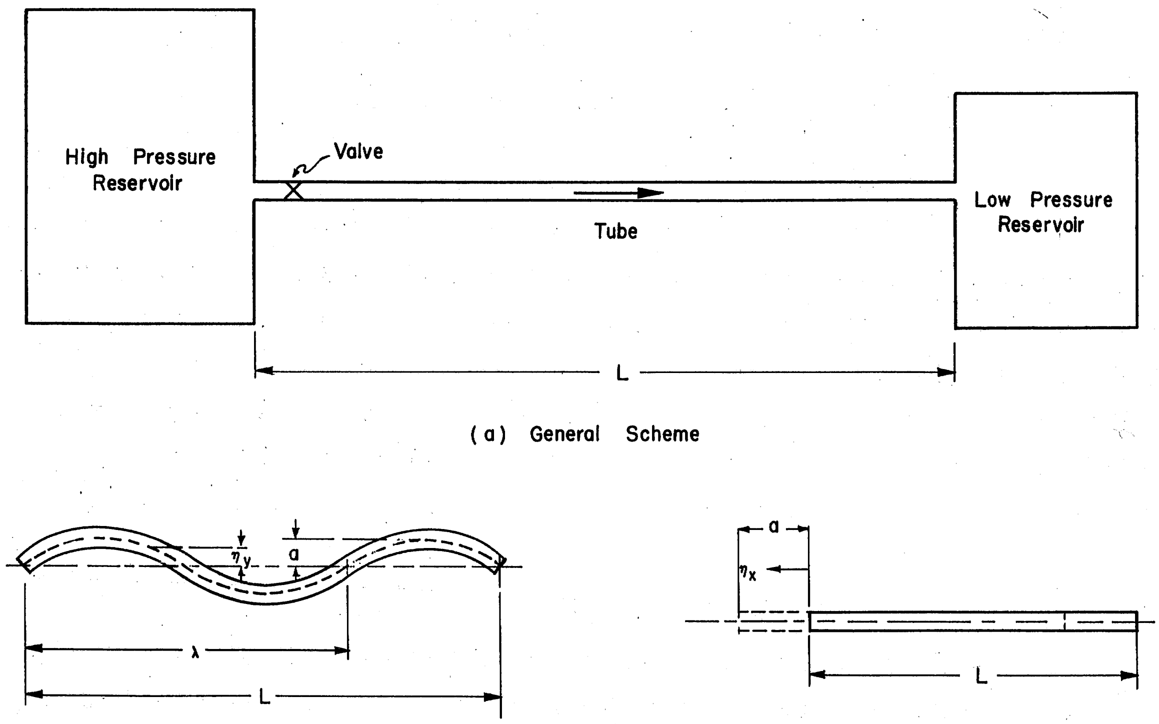
I. INTRODUCTION

This report deals with the effect of external vibrations on flow in tubes. The work described was initiated on behalf of the Bureau of Aeronautics and was sponsored by the Office of Naval Research of the United States Department of the Navy. The Bureau of Aeronautics desired to learn whether tube vibrations could be a possible cause of excessive pressure drop in steady flow or of pressure phase lag in establishing flow in tubing of aircraft hydraulic systems. An affirmative answer to these questions would also have practical applications to flow in many other types of systems and would be of importance in the basic mechanics of pipe flow.

The investigation is limited to the effects of vibration on straight lengths of tubing. No consideration is given to possible effects of vibrations on operation of valves and other devices or on flow through fittings.

The problem investigated is typified by the simplified system sketched in Fig. 1-a. It is assumed that a tube of uniform cross section and some length connects two reservoirs, one of high pressure and one of low pressure. The former is assumed to be so large as to maintain a constant pressure regardless of flow---representative of a pump and accumulator system. The latter is assumed to be of variable volume and pressure---representative of a cylinder and piston. The tube between the reservoirs is vibrated by an outside source (pump, chattering relief valve, or other machinery), the vibrations being conducted through the supports, connecting tube walls, or the surrounding air. The vibrations are assumed to be wave-like transverse, including the case of infinite wave length (Fig. 1-b), longitudinal (Fig. 1-c), or combinations of both.

At some initial time the valve indicated in Fig. 1-a is opened and flow is established from left to right; during establishment of flow, the average pressure in the low-pressure reservoir increases. As the pressure in the low-pressure reservoir increases, its volume also increases; the rate of volume increase varies with time at first, but finally becomes steady as the pressure reaches a steady value, determined by the pressure in the high-pressure reservoir less the steady-flow losses in the tube. Alternatively,



(a) General Scheme

(b) Transverse Vibration of Tube

(c) Longitudinal Vibration of Tube

Fig. 1— System Analysed

the valve in the tube may be replaced by a valve and orifice on the right-hand reservoir; the two reservoirs will then be at equal pressure until the valve is opened. Subsequently, the average pressure in the right-hand reservoir will drop until steady-flow conditions are achieved. (The term "average pressure" has been used to avoid a discussion of pressure surges at this time.)

The problem considered herein may now be stated in terms of this simplified system as follows:

1. During the period of flow establishment, will the rate of change of average pressure in the right-hand reservoir be different in a vibrating system from that in a similar nonvibrating system?

2. In steady flow, will the average pressure in the right-hand reservoir be different in a vibrating system from that in a similar nonvibrating system? In other words, will the pressure drop in the tube be affected by vibrations?

The above problem is analyzed briefly in part II of this report and an experimental investigation is described in part III.

II. ANALYSIS

A. General

External vibration of a tube is simply a means of generating pressure and velocity fluctuations in the fluid in the tube. In fact, if the external vibrations did not produce some form of internal fluctuation, there could be no possible effect on the flow. As a first step, then, it is necessary to determine what form the internal disturbances assume for given external vibrations.

The determination of the form of the internal fluctuations is not an easy matter since, in addition to the inertia of the fluid, three major factors are involved; these factors are the compressibility of the fluid, the viscosity of the fluid, and the elasticity of the tube walls. Since only rough expressions of form for the internal disturbances will be required in the work to follow, the present analysis may be simplified by some rather liberal assumptions. The first of these is that the elasticity of the tube may be included with the compressibility of the fluid in one bulk modulus of elasticity. Another is that the damping due to viscosity may be ignored (this assumption does not eliminate the effect of viscosity). It is assumed, also, that the general, vibratory motion of the tube is small and may be separated into a number of smaller additive components; furthermore, the internal fluctuations produced by these components are also additive. Finally, it is assumed that the fluctuating motion may be added to the mean flow, when necessary, to obtain the resultant motion of the fluid.

The external vibrations are limited to the following typical forms (Fig. 1):

1. Transverse vibrations

$$\eta_y = a \sin \kappa x \cos \sigma t \tag{1}$$

(Walls move as a unit)

2. Longitudinal vibrations

$$\eta_x = a \cos \sigma t \tag{2}$$

(Both ends move together or only one end moves)

where the x-axis is taken along the nonvibrating tube axis,

the y-axis is perpendicular to the x-axis in the plane of transverse vibrations,

η_y is the wall displacement parallel to the y-axis,

η_x is the tube end displacement parallel to the x-axis,

a is the maximum displacement amplitude,

$\kappa = \frac{2\pi}{\lambda}$ and λ is the vibration wave length which is determined principally by the frequency of the forced vibration, the elastic properties, and the method of support of the tube, and

$\sigma = 2\pi f = \frac{2\pi}{\tau}$ and f is the frequency and τ is the period of the vibration.

Also, $a/\lambda \ll 1$, $a/L \ll 1$.

The internal fluctuations for the case of transverse vibration of a circular tube have been analyzed in Appendix II. Since viscosity acts only as a damping force in the case of transverse vibrations, its action has been neglected in the analysis and the fluctuations are found to have the form:

$$dp = \frac{a\rho\sigma^2}{\kappa K} \sin \kappa x \cos \sigma t \frac{\sinh \kappa Ky \cosh \kappa Kz}{\cosh \kappa KR} \tag{3}$$

$$\left.\begin{aligned} u &= \frac{-\sigma a}{K} \cos \kappa x \sin \sigma t \frac{\sinh \kappa Ky \cosh \kappa Kz}{\cosh \kappa KR} + U \\ v &= \sigma a \sin \kappa x \sin \sigma t \left(1 - \frac{\cosh \kappa Ky \cosh \kappa Kz}{\cosh \kappa KR}\right) \\ w &= -\sigma a \sin \kappa x \sin \sigma t \frac{\sinh \kappa Ky \sinh \kappa Kz}{\cosh \kappa KR} \end{aligned}\right\} \tag{4}$$

where dp is the increment in pressure,

u, v, w are the particle velocities relative to the tube wall,

U is a uniform mean flow velocity through the tube,

R is the radius of the circular tube,

ρ is the liquid density before vibration,

$K = \frac{1}{2} \sqrt{2} \sqrt{1 - (\sigma/\kappa c)^2}$,

$c = \sqrt{E/\rho}$ is the sound speed in the liquid, and

E is the bulk modulus of elasticity of the liquid and tube.

The fluctuations are periodic with the same period as the forced vibration of the tube. For infinite wave length, a $\sin \kappa x$ is set equal to the maximum amplitude of vibration in Eq. (1).

In the case of longitudinal fluctuations, the particles of liquid in the interior of a tube are set in motion by both elastic waves and transfer of shear from the adjacent walls through viscosity. The magnitudes of such factors as $x/c\tau$ and $\nu\tau/c^2$, where x is the distance from the closest end of the tube and ν is the kinematic viscosity of the liquid in the tube, will determine the relative importance of compressibility and viscosity. For large $x/c\tau$ and $\nu\tau/c^2$ (the central portions of a long tube containing viscous fluid), elastic waves are almost completely damped, and the resulting motion is almost entirely due to viscosity. For this case, a method of analysis similar to that used for vibrating plane walls may be introduced [1, page 189]*. It is found that for both ends of the tube vibrating together in accordance with Eq. (2), the equation of motion

$$\left[\frac{\delta u'}{\delta t} = -1/\rho \; \frac{\delta p}{\delta x} + \nu\left(\frac{\delta^2 u'}{\delta r^2} + \frac{1}{r}\frac{\delta u'}{\delta r}\right)\right]$$

is satisfied approximately by:

$$u = \sigma a \left\{ \sin\sigma t - \sqrt{\frac{R}{r}} \; e^{-\beta(R-r)} \sin\left[\sigma t - \beta(R-r)\right]\right\} \qquad (5)$$

where, as in the previous definitions,

u is the particle velocity relative to the tube walls (parallel to the tube axis),

u' is the absolute particle velocity,

$r^2 = x^2 + y^2$.

$\beta R = R\sqrt{\frac{\sigma}{2\nu}} \gg 1$, and

e is the natural logarithm base.

The internal motion is also periodic but out of phase with the motion of the tube. The pressure pulses in the core are small of second order and the velocity components v and w are nonexistent.

At the other extreme, consider a tube in which the high-pressure end (the upstream end) is fixed in position and in which the other end (the downstream end) vibrates in accordance with Eq. (2) as though a solid wall were placed across the tube. Pressure fluctuations at the downstream end of the tube will be determined almost exclusively by the compressibility of the fluid. Without considering viscosity, the fluctuating quantities may be found as shown in Appendix II. The quantities are:

$$u = -\sigma a\left(\frac{\sin\sigma\frac{x}{c'}}{\sin\sigma\frac{L}{c'}} - \frac{\sin\sigma\frac{x}{c}}{\sin\sigma\frac{L}{c}}\right)\sin\sigma t \qquad (6)$$

*Numbers in brackets refer to the bibliography on page 30.

$$\left.\begin{aligned} p &= p_o + \rho c \sigma a \left(\frac{1 - \cos\sigma\frac{x}{c}}{\sin\sigma\frac{L}{c}}\right)\sin\sigma t \\ &= p_o - \rho c\, u_o \left(\frac{1 - \cos\sigma\frac{x}{c}}{\sin\sigma\frac{L}{c}}\right) \end{aligned}\right\} \quad (7)$$

where, as in the previous definitions,

u is the particle velocity relative to the adjacent wall,

$p - p_o = dp$ is the pressure increment previously defined,

x is measured from the high-pressure end of the tube ($x = L$ is the closed end),

c' is the sound speed in the tube wall, and

$u_o = \frac{\delta\eta_c}{\delta t}$ is the velocity of the tube at $x = L$.

The resultant fluctuations are again periodic of the same period as the tube vibrations.

The actual internal fluctuations due to vibration of a tube wall are, of course, considerably more complex than indicated by Eqs. (3) to (7), especially when a mean flow is superimposed on the fluctuating motions. These equations will serve, however, to give a qualitative picture of the form of the motion.

B. Previous Work

As far as could be determined, except for one rather special case, no analytical or experimental studies relating to the direct effect of external vibrations on flow in tubes have been reported in the literature. The special case mentioned [2] is an experiment wherein a section of pipe was free to vibrate longitudinally with respect to the fluid which flowed within it. Fixed sections of pipe supported each end of the vibrated section, the latter slipping over the former. The longitudinal vibrations of this free section are reported to have produced a characteristic turbulent velocity profile downstream from the vibrated section where the profile was of characteristic laminar shape before vibration.

Another paper with a promising title was found [3], but this concerned the opposite problem—the effect of flow on pipe vibrations—and had no bearing on the present problem*. It should also be noted that several individuals connected with aircraft and industrial hydraulic applications have stated categorically, from their experience, that external vibrations of a tube do not produce excess pressure drop or pressure phase lag when flow is established.

It also appears that there is no prior work available dealing with the effect of internal vibrations superimposed on a mean flow which can be adapted to the present investigation. Other related work will be drawn upon as required.

C. Steady Flow

The effect of the internal fluctuations, Eqs. (3) to (7), on steady flow will be examined first. It would appear that the velocity expressions could be used to compute energy loss due to viscous dissipation, and the pressure fluctuations could be used to compute losses in the elastic pipe walls. Such computations are useless, however, when applied to determining the additional pressure drop in the tube. By the conditions of the problem, the energy which generates the fluctuations comes from an external source and is not extracted from the liquid in the tube; it is brought to the portion of the tube under consideration through the tube walls or the tube supports. If the external source of energy were removed, vibrations and fluctuations would cease through viscous damping in the liquid and elastic damping in the tube walls. The damping of the internal fluctuations may actually add a small quantity of heat energy to the main flow.

Another possibility, however, is that the induced internal fluctuations alter the profile of the mean flow and thus change the wall shear and the pressure drop in the tube. For a circular tube of radius R, the mean pressure drop along the tube is given by

$$\frac{dp}{dx} = 2\,\frac{\tau_o}{R} \tag{8}$$

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*The paper by Holt and Haviland demonstrates that a tube vibrating as the one shown in Fig. 1-b does work on the fluid over a part of its length and that the fluid does work on the tube over the remainder, and as a result there is no net work on the fluid. The longitudinal motion of the fluid simply serves to transfer the transverse momentum of the tube from one part of its length to another and to damp its motion thereby.

where $\tau_o = \mu \frac{dU}{dy}\Big|_{y=o}$ is the wall shear,

μ is the coefficient of viscosity,

U is the mean flow velocity measured parallel to the tube axis, and

y is the distance from the tube wall toward the center.

If the fluctuations in velocity or pressure can materially alter the mean value of $dU/dy|_{y=o}$, a pressure drop different from the pressure drop without vibrations would be experienced.

The most obvious application of this change in profile appears to be the possible excitation of transition from laminar to turbulent flow in the tube, due to vibration. It should be noted that a necessary prerequisite for the production of transition is the existence of a laminar flow at a Reynolds number in excess of the minimum critical Reynolds number for tubes. The existence of such a flow will be assumed for the argument to follow, but it is unlikely to be obtained in ordinary tubing or piping applications; furthermore, the nonvibrating, steady-flow pressure drop above the minimum critical Reynolds number will ordinarily be computed on the basis of a turbulent flow. As a result, regardless of the outcome of the argument to follow, the excitation of transition by external vibrations could be of no importance in producing excess pressure drop except in extremely special cases.

The possibility of transition due to self-excitation of small disturbances applied in the fluid near the boundary stems from Tollmein's original analysis [4] as verified experimentally by Schubauer and Skramstad [5] for two-dimensional boundary layer flows without pressure gradient. In this work, it was determined that such boundary layers (which occur near the leading edge of flat plates) are unstable to disturbances of certain frequencies in flows where the Reynolds number exceeds a critical value. A similar analysis for three-dimensional, fully developed, laminar flow in tubes is not available. Sexl [6] analyzed this problem, but neglected the effect of viscosity; he found the flow stable for all disturbances. Lin [7] investigated theoretically the related two-dimensional problem of fully developed, laminar flow between parallel walls; he found this flow unstable for disturbances of certain frequencies. However, it cannot be inferred from the two-dimensional analysis that similar pipe flows should be unstable. It may be concluded that the question of possible transition due to small external disturbances of fully developed, laminar flow in tubes is still unanswered theoretically.

The experiment cited earlier from Richardson [2] might be considered as experimental verification of the instability of laminar flow profiles in tubes. However, it appears that the vibrated section of tube in these experiments was so near the entrance that the tube boundary layer may have been approximately two-dimensional in the sense studied by Tollmein [4], Schubauer and Skramstad [5], and Lin [7]. Experimental data from the present investigation, to be presented later, indicates that the fully developed, turbulent flow in tubes is stable to external disturbances.

The possibility still exists of a change in the mean velocity profile near the wall within the purely laminar or purely turbulent regimens. The change in profile would be brought about by a higher rate of interchange of momentum between parallel streamlines produced by the internal fluctuations resulting from vibration. Thus, τ_o in Eq. (8), and dp/dx would increase.

If the mean flow in the tube is assumed to be parallel to the axis and steady with time, and to have superimposed upon it the fluctuations in velocity given by Eqs. (4), for example, Reynolds stresses [1, page 192] will arise because of the correlation in velocity fluctuations with time. In the x-y plane there will be additional shear given by $\tau_R = \rho\overline{uv}$, where τ_R is the Reynolds stress, ρ is the liquid density, u and v are the fluctuating velocity components from Eqs. (4), and the bar indicates that the mean value (with time) is to be taken. If the indicated product is formed, there results:

$$\tau_R = \rho\overline{uv} = -\frac{\sigma^2 a^2}{2K} \sin 2\kappa x \, \overline{\sin^2 \sigma t} \left(\frac{\sinh \kappa Ky \cosh \kappa Kz}{\cosh \kappa KR} - \frac{\sinh 2\kappa Ky \cosh^2 \kappa Kz}{\cosh^2 \kappa KR}\right)$$

for which the symbols have been defined previously. The mean value of the sine-squared factor over one period is 1/2. Thus,

$$\rho\overline{uv} = \frac{\sigma^2 a^2}{4K} \sin 2\kappa x \left(\frac{\sinh \kappa Ky \cosh \kappa Kz}{\cosh \kappa KR} - \frac{\sinh 2\kappa Ky \cosh^2 \kappa Kz}{\cosh^2 \kappa KR}\right) \tag{9}$$

It is seen that, in general, $\rho\overline{uv}$ is not zero even at the wall; however, it does alternate in sign depending on x. This alternating characteristic will tend to cancel the effect of periodic changes in profile and there will be little net effect on the pressure drop from this cause.

The above analysis is only rough, in that the fluctuations for a uniform velocity profile have been superimposed on a nonuniform profile

($dU/dy \neq 0$), and it is limited to the transverse vibrations only. However, it is probably correct in indicating that the effect, if any, is certainly of second order and probably is not measurable by ordinary instruments.

The possibility has also been suggested that a transversely vibrating tube may act (instantaneously) as a curved tube or as a rough or corrugated tube. Exact similarity is lacking in both analogies. In the case of a curved tube, the excess pressure drop is caused by separation effects primarily due to accumulation of boundary layer fluid by secondary currents on the inside of the bend. In the vibration problem, the secondary currents of a normal bend hardly have time to develop, let alone accumulate low-energy fluid, and the primary cause of separation is removed. In the case of a rough or corrugated tube, separation behind the individual roughness forms is responsible for additional pressure drop. In the vibration problem, both the time factor and the small relative magnitude of the disturbance ($a/\lambda << 1$ in Fig. 1-b) tend to prevent separation.

So far as can be determined from all reasonable arguments, it would appear that no measurable, additional pressure drop should occur as a result of vibration of a tube containing liquid in steady, mean flow. The above arguments are independent of the actual pressure of the liquid as long as the pressure is not so low that negative fluctuation peaks could cause cavitation or air separation.

D. Flow Establishment

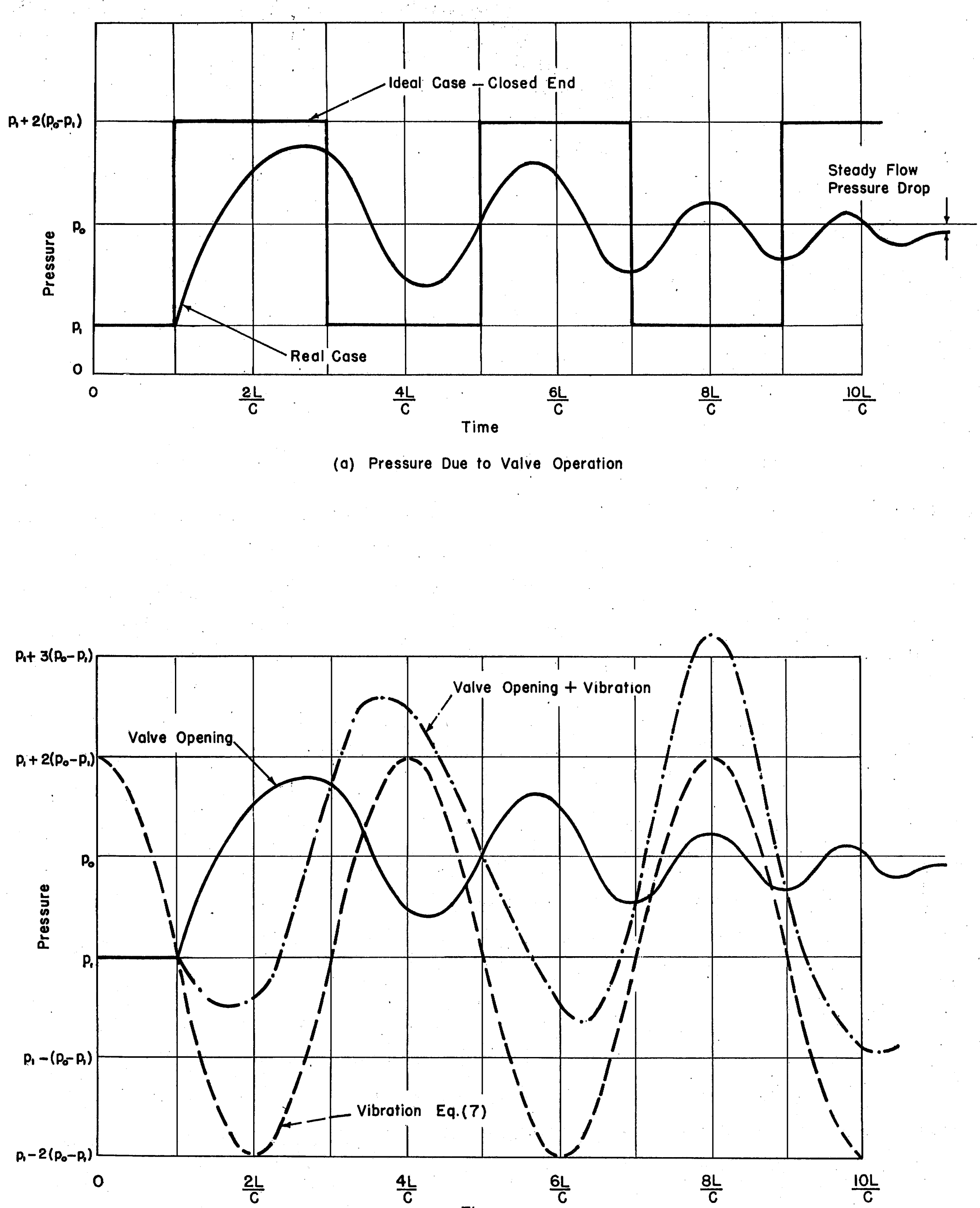
The effect of pressure fluctuations caused by tube vibration on flow establishment will now be considered. Referring to Fig. 1-a and the description which accompanied it, it is seen that opening the valve will result in a pressure pulse traveling with the speed of sound in the liquid through the length of the tube. The magnitude of the pressure pulse will depend on the rate of valve opening and will equal the pressure in the high-pressure reservoir, p_o, for instantaneous opening; it will be less for finite rates of opening. At the entrance to the low-pressure reservoir, the pulse will be partially reflected and partially transmitted, and there will be further reflection within the reservoir. The first reflected pulse will be again reflected (negatively) at the high-pressure reservoir; the reflection process will continue until the effect of the pulse is completely damped. Detailed methods of analysis of this phenomenon may be found in reference [8].

The changing volume of the reservoir in the general case just discussed makes the exact pressure determination in the reservoir quite complex. In a special case, obtained by substituting a fixed, closed end for the low-pressure reservoir, the determination of the pressure is much simpler; the pressure for instantaneous valve opening is shown by the solid curve in Fig. 2-a. Damping has been neglected for this case. The maximum pressure is $2\ (p_0 - p_1)$ in excess of the original pressure in the tube, where p_1 is the pressure in the tube before the valve is opened; this peak repeats in periods of $4L/c$, where L is the length of the tube and c is the sound speed in the liquid. In an actual installation, the valve is opened rapidly but not instantaneously, there is the equivalent of a low-pressure reservoir at the downstream end of the tube, and pressure pulses are damped in the liquid and in the tube walls. The resulting pressure fluctuations are different from the case represented by the solid curve in Fig. 2-a, but the solid curve defines the limits of magnitude of the pulses. The broken curve in Fig. 2-a represents an estimate of how the pressure curve at the downstream end of a tube might look in an actual installation. The exact shape is not important to the present argument.

If tube vibration is to delay the attainment of maximum pressure at the low-pressure reservoir, it will be necessary for the pressure fluctuations caused by vibration to be of such magnitude and phase relationship that, when added to the fluctuations caused by valve opening, the result will be a curve of lower average pressure than exists as a result of valve opening alone. Eq. (7), page 6, gives the pressure curve produced by an extreme case of longitudinal vibration, as already noted. (Pressure pulses produced by transverse vibrations have an average value of zero over a cross section of the tube at any instant of time for any value of x; this may be seen by integrating Eq. (3) on page 4 over the cross section of the circular tube. Any other form of longitudinal vibration will give smaller pressure pulses than will Eq. (7).) If L is substituted for x, and dp, the pressure increment due to vibration, is written for $p - p_0$, Eq. (7) becomes:

$$dp = \rho c \sigma a \left(\frac{1 - \cos \sigma \frac{L}{c}}{\sin \sigma \frac{L}{c}} \right) \sin \sigma t$$

The absolute value of the factor in parentheses varies from 0 to 1 as $\sigma L/c$ varies from 0 to $(2n - 1)\ \pi/2$ where n = 1, 2, 3 .... The maximum value of



(a) Pressure Due to Valve Operation

(b) Pressure Due to Valve Opening and Vibration

Fig. 2— Pressure at Low Pressure End of Tube

dp is $\rho c \sigma a$. This maximum must be of the same order of magnitude as the pressure rise due to valve opening (and of opposite sign) if the pressure rise due to valve opening is to be delayed; that is,

$$\rho c \sigma a \approx 2\ (p_o - p_1)$$

Hence, for longitudinal vibrations of maximum effectiveness,

$$\sigma \left(= \frac{2\pi}{\tau}\right) \approx \frac{(2n - 1)\pi c}{2L}$$

and

$$a \approx \frac{4\ (p_o - p_1)\ L}{\rho c^2\ (2n - 1)\pi}$$

For n = 1, $\tau = 4L/c$ and the vibrational period of maximum effectiveness is closely equal to the period of the first positive pressure pulse due to valve opening. In Fig. 2-b, the curve given by Eq. (7) with $\tau = 4L/c$ and

$$a = 4\ (p_o - p_1)\ L/\pi\rho\ c^2$$

has been plotted as a solid line with the origin of the time axis so chosen that the negative peak due to vibration approximately cancels the positive peak due to valve opening taken from the broken line of Fig. 2-a. The dotted curve in Fig. 2-b is obtained by summing the ordinates of the two pressure curves, one due to vibration and the other due to valve opening, and gives the resultant pressure. It is seen that the average pressure is hardly changed by the vibration, and if there is any delay in the high pressure reaching the low-pressure end of the tube, the delay is certainly less than one period of the vibratory motion. If n had been taken as 2 or more in plotting Eq. (7) on Fig. 2-b, then two or more periods of the vibratory motion would have occurred during the first period of the pressure pulse due to valve opening and the maximum possible delay would have been less than with n = 1.

As a numerical illustration of the factors involved in the preceding estimate, consider a tube of 10-ft length containing hydraulic oil with $p_o - p_1 = 1000$ psi. Take the density ρ as 1.65 slugs per cu ft and the bulk modulus of elasticity E as 2×10^7 psf [9]. Then $c \approx 3500$ fps, and for longitudinal vibrations of the tube with n = 1 and $dp = 2\ (p_o - p_1)$, $\tau = 0.0114$ sec and $a \approx 0.09$ ft > 1 inch. The likelihood of obtaining a maximum displacement of more than 1 in. at one end of a 10-ft tube, the other end being fixed, is very small indeed. For lesser amplitudes and the same frequency, the maximum pressure pulse due to vibration would be smaller in direct proportion to the amplitude and would cancel less of the positive pulse due to valve opening. For longitudinal vibrations with $n = 2$, $\tau = 0.0038$ sec and $a \approx 0.03$ ft, a more reasonable value.

In any event, the possible delay in transmission of a pressure pulse due to valve opening appears so small in the preceding estimate as to be of no consequence in practice. Since the case analyzed was an exaggerated one to begin with, it is not anticipated that measurable pressure phase lags due to valve manipulation will be encountered in systems of the type sketched in Fig. 1-a. Systems in which the valve is located at the low-pressure end of the tube or on an orifice in the low-pressure reservoir would be similar to that analyzed.

III. EXPERIMENTAL WORK

The arguments presented in the preceding section require some experimental verification. This section of the report describes experiments designed to check the results of the preceding section.

A. Experimental Apparatus

The majority of the experiments were made on a recirculating hydraulic system consisting of a pump, tank, and accumulator unit; a length of straight tubing, hereinafter called the test tube; an energy dissipating device composed of valves and additional lengths of tubing; and, a return line to the tank. For experiments on flow establishment, a quick-acting valve was added preceding or following the test tube. The test tube was mounted to permit either transverse or longitudinal vibration. For measuring purposes, piezometer taps were installed at both ends of the test tube, a rotameter (which is a discharge-measuring device) was placed in the return line, and a thermometer was installed at the downstream end of the test tube.

Fig. 3 is a photograph of the experimental apparatus. The essential parts are labeled on the figure. The housing on the right includes the tank, pump, accumulator, and relief valve. The photograph shows the quick-acting valve for flow establishment at its downstream position. The valve was also

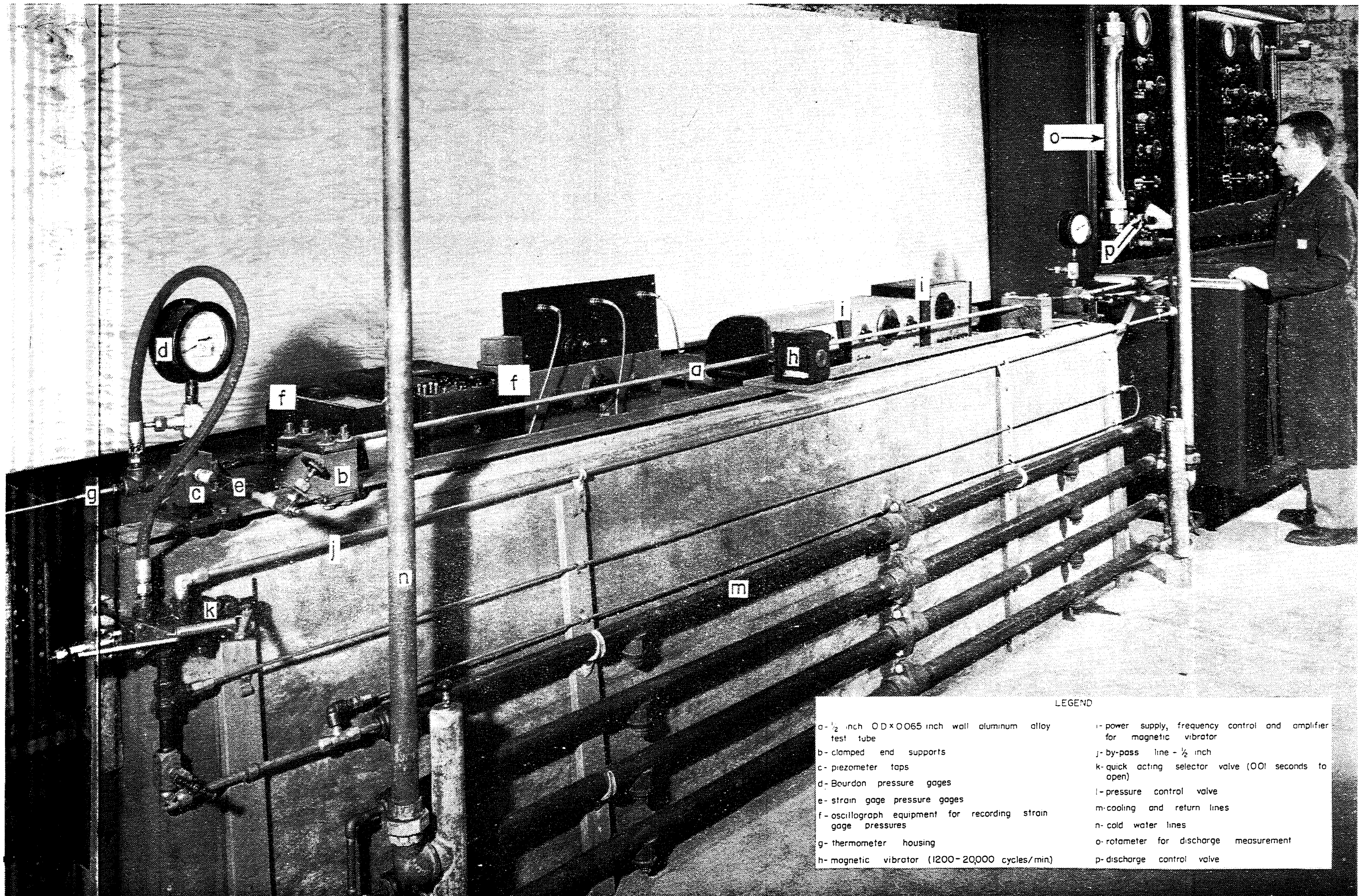


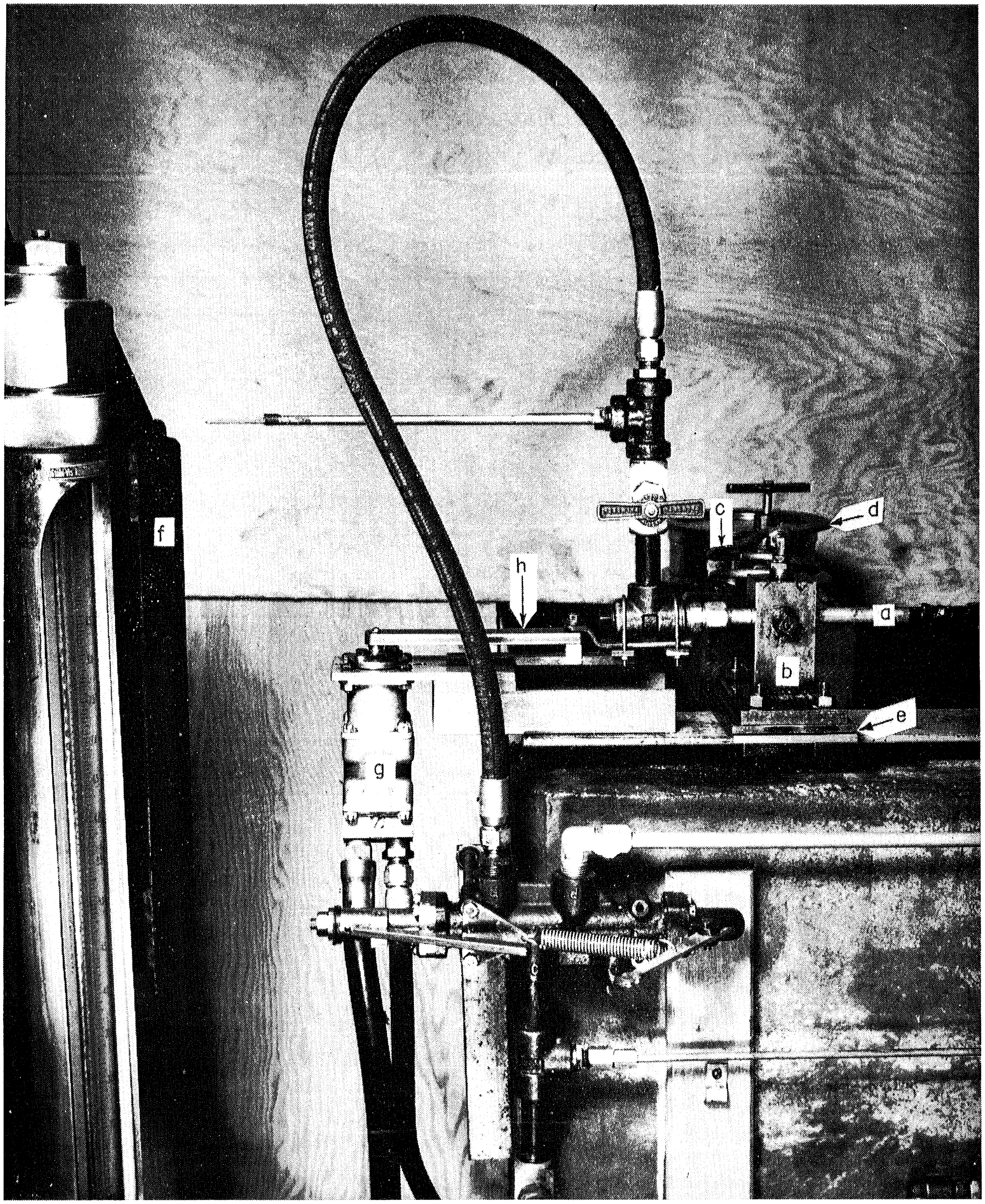
Fig. 3 - View of Experimental Apparatus

used at the upstream end of the test tube between the tube and the flexible hose. In either location, the valve was arranged so that in the uncocked position the by-pass line was closed and the test tube was open. This was the normal position for all steady-flow experiments. For the flow-establishment experiments the valve was cocked, thereby forcing flow through the by-pass line instead of through the test tube. Release of the cocking lever caused the valve to return to the uncocked position under spring action, establishing flow in the test tube. From fully closed to fully open, the valve operated in about 0.01 sec.

Also to be seen in the photograph is the magnetic vibrator set up for transverse vibration of the tube. A hydraulically operated vibrator was also available. This is shown set for longitudinal vibration of the test tube in the photograph in Fig. 4. Figure 4 is a view of the downstream end of the apparatus shown in Fig. 3. The hydraulic vibrator consisted of a rod connected eccentrically to the shaft of a hydraulic motor driven by an independent pump and electric motor unit. The hydraulic vibrator was capable of producing frequencies up to 60 cps with a nonresonant amplitude of about 0.03 inch. The magnetic vibrator was capable of frequencies from 20 to 20,000 cps at a nonresonant amplitude of about 0.005 inch. Under resonant conditions, these amplitudes increased to as much as 0.25 in. for the hydraulic vibrator and 0.09 in. for the magnetic vibrator. The magnetic vibrator was driven by suitable power-amplifying and frequency-modulating equipment, but the unit was capable of accelerating small masses only. Frequency was measured stroboscopically with both vibrators. Vibrational patterns similar to those shown schematically in Fig. 1 were obtained.

For transverse vibration, the tube was clamped in fixed-end supports which may be seen in Fig. 3. The spacing of the supports could be varied to change the wave length for a given frequency. After a few such changes, however, it was decided that wave length was not an important factor and most of the experiments were performed with a spacing of just over 9 ft between clamped ends. Both vibrators were used for transverse vibration, but only the hydraulic vibrator could be used for longitudinal vibration because of the large mass which had to be accelerated (the piezometer-tap blocks were vibrated with the tube).

The entire test installation was mounted on a steel rail embedded in concrete resting on a solid rock foundation. This was done to eliminate



a-test tube
b-piezometer tap block and support
c-copper tube coil between piezometer and gage
d-fixed Bourdon pressure gage
e-roller assembly Rollers 1/8 in. O.D are located between plates.
f-hydraulic pump unit
g-hydraulic motor
h-drive rod

Fig. 4- Detailed View of Apparatus for Longitudinal Vibration

extraneous vibrations and to confine to the test stand the disturbance due to vibration of the tube. Even so, some vibration from outside was transmitted through the flexible hose connection and a 60 cps vibration was picked up through the foundation. In comparison to the amplitudes used in the experiments, both of these vibrations were very small.

The hydraulic system was capable of producing a flow of 10 gpm and 2600 psi pressure at the pump when aircraft hydraulic oil (specification AN-O-366) was used. However, at the maximum pressure, the flow dropped to about 6 gpm. The rotameter for measuring flow rates was calibrated with a weighing tank and a stop watch and was accurate within 1 per cent of the indicated flow rate over the range that was used. The apparatus was designed to permit operation at any selected average working pressure in the range from just over atmospheric to over 2000 psi. This was done to determine whether the pressure level might be a factor in the problem under study. Working pressure was controlled by manipulation of the needle valve downstream from the test tube, shown in Fig. 3.

Two sizes of test tubes were used, both of aluminum alloy and both 12 ft long. One was 1/2-in. OD by 0.065-in. wall thickness and the other was 1/4-in. OD by 0.035-in. wall thickness. The two sizes of tubes permitted covering a range of Reynolds numbers with the fixed maximum discharge of the pump unit. The test tubes were drilled for piezometer taps near each end. Four holes, 90° apart, in a ring 3 in. from each end were used. Burrs were carefully removed from the inside surface of the tube where the holes were drilled. The tubes were then inserted in heavy piezometer-tap blocks through holes drilled slightly larger than the tubes and fitted with O-rings. The blocks were drilled and tapped to receive various types of pressure gages and to connect these with the piezometer holes in the tubes. The blocks may be seen at either end of the tube in Fig. 3. They are 11.5 ft apart. When experiments were run using longitudinal vibrations, a copper coil was installed between the piezometer blocks and the gages to permit the gages to remain stationary; this arrangement can be seen in Fig. 4.

Several types of gages were available for pressure measurement. For low working pressures, a U-tube manometer and a pair of 60 psi Bourdon gages (graduated in 1 psi divisions) were used. For higher pressures, a pair of 500 psi and a pair of 3000 psi Bourdon gages (graduated in 10 psi and in 50 psi divisions, respectively) were used. The Bourdon gages were calibrated with a dead-weight tester prior to use; they could be read accurately to one-quarter of the smallest division. In use, wavering of the pointer on the Bourdon gages, due to small fluctuations in pump and accumulator pressure, was damped to plus or minus one-quarter of the smallest division by a needle valve between each gage and its block. At some frequencies of longitudinal vibration, the pointer vibrated in resonance with the external vibration and satisfactory damping was impossible. Unfortunately, the pressure drop over the 11.5-ft length of test section between piezometer taps at low flow rates was no larger than the smallest graduation of the high-pressure gages; this made accurate pressure-drop readings at high working pressures and low Reynolds numbers impossible to obtain. A differential-type Bourdon gage with a small scale reading and a high case working pressure was required, but such a gage could not be obtained in time to be of use in these experiments.

For the flow-establishment experiments, electronic pressure gages made up of crossed pairs of strain gages were used. The strain gages were wrapped on a short length of aluminum tubing which could be fitted into the piezometer-tap blocks. This arrangement is shown in Fig. 3. The aluminum tubes were provided with bleed valves at their free ends. The leads from the strain gages were connected to a Brush BL-310 universal analyzer and a Brush BL-202 magnetic oscillograph. Information furnished by the manufacturer indicated that this equipment should respond satisfactorily to frequencies of 100 cps or less. The electronic gages were not calibrated because they were used only on a comparative basis. During the experiments, no provision was made to control the oil temperature except to limit its maximum value. The cooling coils shown in Fig. 3 served this purpose and the maximum temperature was limited to about 145° F at maximum working pressure and flow rate. The thermometer well provided at the end of the test tube and shown in Fig. 3 was adequate for obtaining oil temperatures since the equipment was always permitted to run long enough under test conditions before an experiment to establish constant temperature. The well was originally designed to receive a thermocouple unit, but the thermometer was found to give the same results with much less trouble.

The photograph in Fig. 5 shows a rearrangement of the apparatus described previously and used for experiments on transition from laminar to turbulent flow at high Reynolds numbers. The tank-pump-accumulator unit has been replaced by a constant-level tank and the return line is disconnected. Water was used in this system in place of hydraulic oil. The entrance to the tube was fitted with a bell mouth which joined the constant-level tank smoothly.

Temperature was controlled and constant temperature maintained by immersing the constant-level tank in a larger tank containing water maintained at the desired temperature. Water was supplied to the constant-level tank through a number of orifices in a ring-shaped supply pipe above the tank; the level was maintained by admitting a surplus and wasting the excess over the sides of the constant-level tank into the outside tank. The U-tube manometer for pressure measurement may be seen in Fig. 5. With this apparatus, rate of discharge was measured using a calibrated volumetric tank and stop watch, and the temperature was obtained by holding a thermometer in the discharge stream. Also to be seen in the photograph is a special antivibration mounting which was used to eliminate the 60 cps vibration previously picked up through the foundation.

For the hydraulic oil, a kinematic viscosity versus temperature chart was prepared, using an Engler viscosimeter at atmospheric pressure; the chart was checked with the viscosimeter periodically during the course of the experiments. Corrections to viscosity for pressures above atmospheric, and data pertaining to density and moduli of elasticity of the oil were obtained from reference [9]. Standard tables were used to obtain the physical properties of water.

B. Steady-Flow Experiments

These experiments consisted of a comparison of the pressure drop between steady flow without and with vibration. The comparison was made over the 11.5-ft length of test section between piezometer taps. The apparatus shown in Fig. 3 was used. The procedure consisted essentially of the following steps:

1. A steady rate of flow was established at a predetermined working pressure by manipulation of the discharge-control valve on the pump unit and the energy-dissipating valves in the return line. The rate of flow, pressures at both ends of the test length, and temperature were measured after steady conditions were obtained.

2. Forced vibration over the range of frequencies from about 8 to 350 cps was then applied to the test length under the conditions already described. Both transverse and longitudinal vibration was applied, but the latter was limited to a maximum of 60 cps by

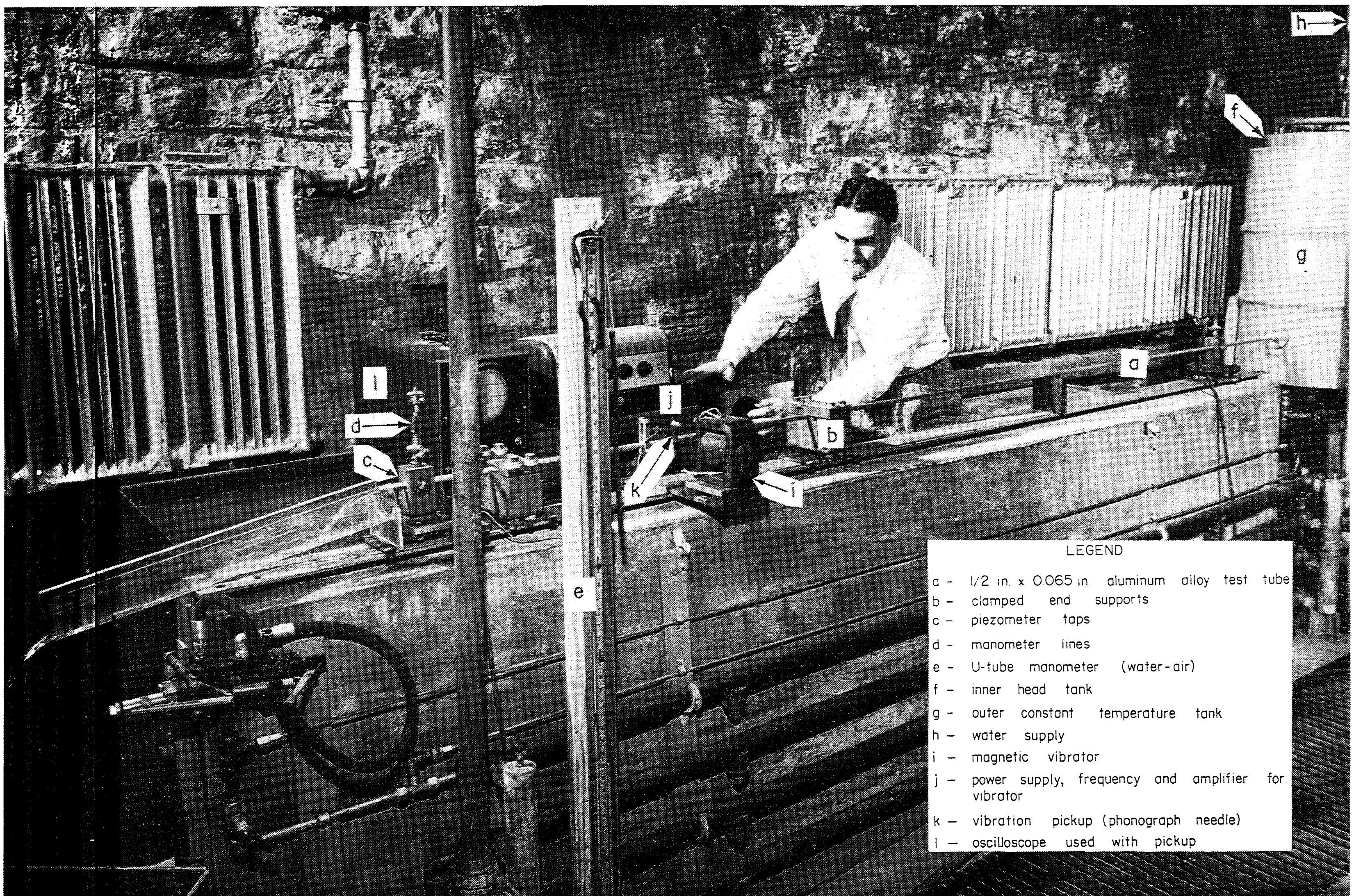


Fig. 5 - View of Modified Experimental Apparatus Used for Laminar Flow at High Reynolds Number

use of the hydraulic vibrator only, as previously noted. As the frequency was gradually changed under any one set of conditions, pressures at both ends of the test tube were constantly observed. However, the rate of flow, pressures, and temperature were recorded only at intervals during the changes of frequency. Under any one set of conditions, the frequency was changed both upward and downward over the entire range to ensure that nothing was missed. (Since considerable time was involved in the process of recording observations during frequency variation, the temperature, pressures, and flow rate were sometimes found to vary slightly during the course of a run.)

3. After readings were completed at one rate of flow and pressure, the rate of flow and/or pressure was changed and the entire process was repeated. As already noted, both 1/2-in. and 1/4-in. OD tubes were used and eventually a range of Reynolds numbers from 750 to slightly over 10,000, at a range of average working pressures from just above atmospheric to over 2000 psi, was covered. (The Reynolds number is defined as $2\bar{\bar{U}}R/\nu$ where $\bar{U}$ is the mean velocity of flow, R is the tube radius, and ν is the kinematic viscosity.)

It was observed that there was no measurable change in pressure drop due to vibration over the entire range of Reynolds numbers, working pressures, and vibrational frequencies covered.

At the higher working pressures, as has been observed, the smallest graduation of the high-pressure Bourdon gages was large in comparison to the pressure drop, so that some change in pressure drop could occur without being recorded on the gage. Likewise, under identical flow conditions two different observers might report readings differing by as much as one-quarter of the smallest graduation. Therefore, pressure variations of this magnitude could not be attributed to vibration, but were considered as experimental error. With these points in mind, some representative data are plotted in Fig. 6 on a standard friction factor versus Reynolds number diagram (f versus Re). For comparison, the theoretical laminar flow curve ($f = 64/Re$) and the Blasius curve for turbulent flow in smooth pipes are also shown.

The points plotted in Fig. 6 include some data from all working pressure ranges. With two or three exceptions, all of the points above a

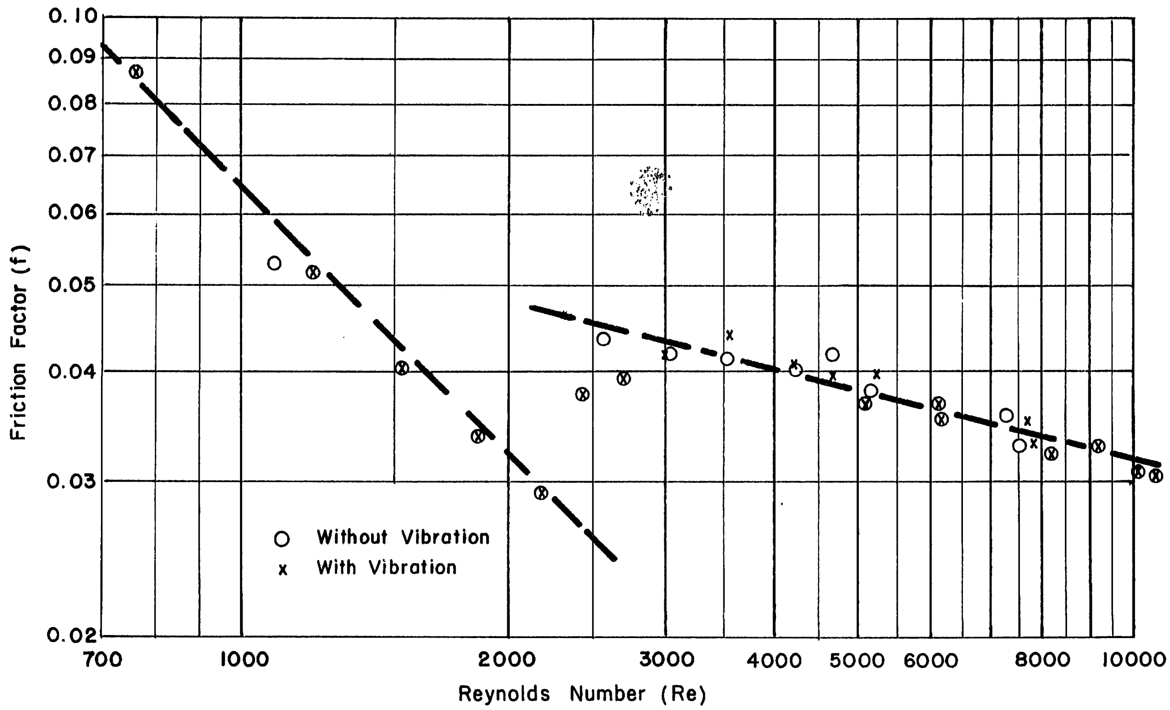


Fig. 6—Friction Factor Versus Reynolds Number With and Without Vibration

Reynolds number of 3500 are taken with the 3000 psi Bourdon gages with smallest graduations of 50 psi and where the probable error in reading was of the order of 10 psi, at the least, out of a total pressure drop of 30 psi at this Reynolds number. Furthermore, two of these points with the largest spread are taken from records for longitudinal vibration where additional difficulty in reading was caused by pointer vibration. The deviation from the plotted curves and the occasional spread between the without-vibration and the with-vibration points must be placed within the experimental error of the gages. This error is reduced as the Reynolds number increases and the total pressure drop in the test tube increases. No attempt has been made to select the points in Fig. 6, other than to cover the Reynolds number range rather uniformly, to select data from both modes of vibrations and from a range of frequencies, and to cover the working pressure range. Only a small percentage of the recorded test data are plotted in Fig. 6; in addition, there was a great quantity of similar unrecorded data obtained while the frequency was gradually varied. If the plotted points had been confined to records taken with the 60 psi and 500 psi gages, the deviation from the plotted curves in Fig. 6 would have been negligible.

One other experiment in connection with steady flow in tubes should also be mentioned. After all of the other experiments had been completed, a 1/4-in. OD tube which was in place in the apparatus shown in Fig. 3 was cut to remove an 8.5-ft long section between piezometer taps. This section was replaced with a 1/4-in. ID tube suitably lapped and fitted with O-rings at its ends so that the new section could be vibrated over the remaining ends of the 1/4-in. OD tube. Longitudinal vibrations at frequencies of 5 to 20 cps were applied at amplitudes from about 1/32 in. to about 1/4 in. The Reynolds number was varied from 1200 to 8900, but working pressures were held below 500 psi. (The conditions of this experiment corresponded closely to the conditions leading to Eq. (5) on page 5.)

Again, it was observed that there was no measurable change in pressure drop between the tube without vibration and the tube with vibration.

C. Steady Laminar Flow at High Reynolds Numbers

In the experiments just described, transition from laminar to turbulent flow took place between Reynolds numbers of 2100 and 2200 with or without vibration. Since this is very near the minimum value of Reynolds number for transition, it was not possible to determine from the experiments whether vibration could be a cause of transition. To satisfy this point the experimental apparatus was modified, as shown in Fig. 5 and described on page 19. In this apparatus, water was used at constant head and the rate of flow was controlled by controlling the temperature and, thus, the viscosity. Only the 1/2-in. OD tube was used and this was fitted with a bell-mouth entrance, as already noted. The piezometer taps in the 1/2-in. tube were exactly 10 ft apart with the upstream tap 1.67 ft from the point where the bell mouth joined the tube. Reynolds numbers of 15,000 in laminar flow were obtained by permitting the water in the head tank to remain at rest at constant temperature for several hours before experiments were started.

The existence of laminar flow was established by two criteria, namely, the appearance of the emerging stream and the pressure drop between piezometer taps. With respect to the first criterion, the emerging stream was transparent in laminar flow and cloudy in turbulent flow. The second was not so simple; most of the 12-ft length of test section was within the inlet length for laminar flow at high Reynolds numbers (the inlet length is given roughly by $x = 0.0575\ R \cdot Re$ [1, page 301] where x is the distance from the bell-mouth entrance to the fully developed flow region). To compute the pressure drop

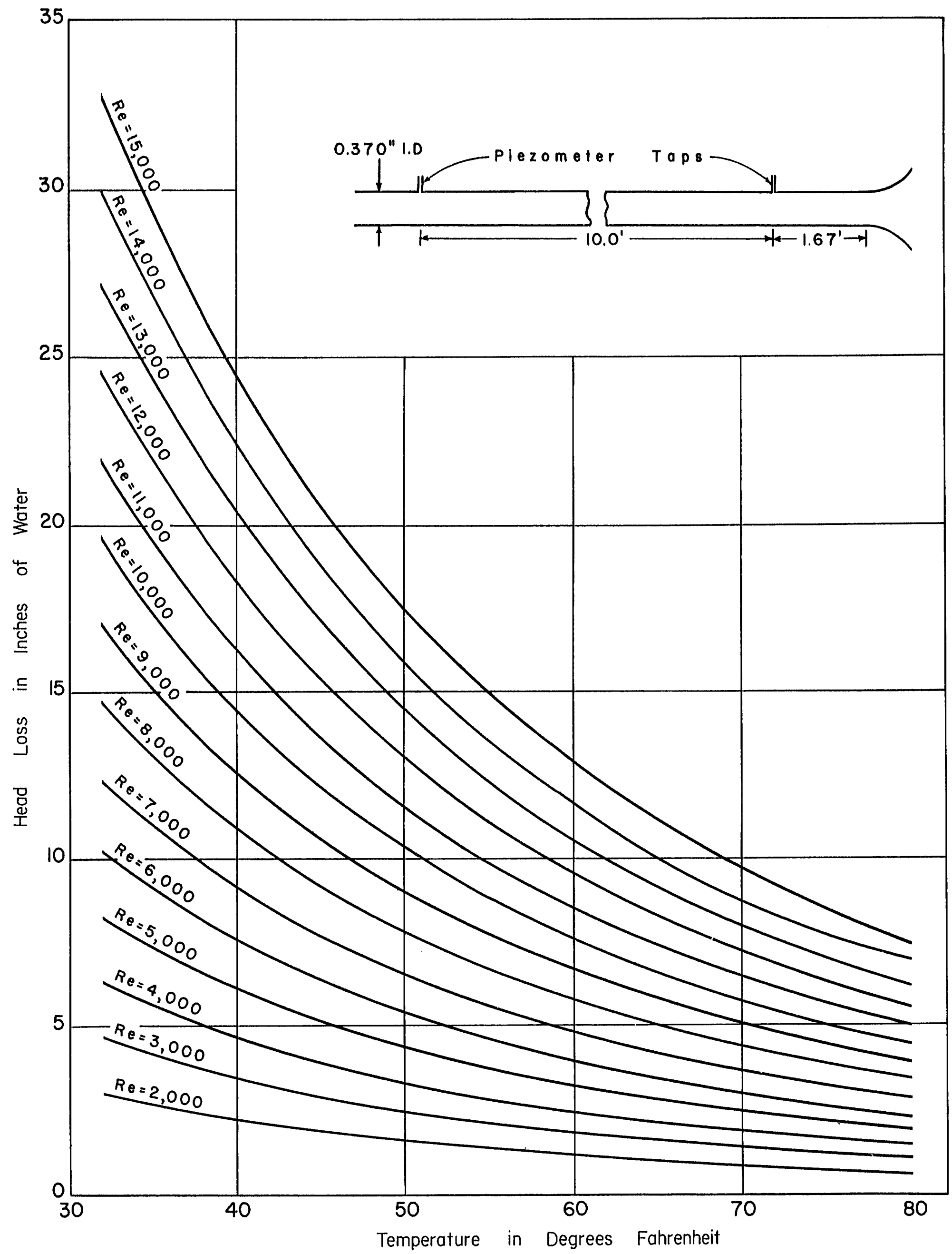


Fig. 7 Head Loss Versus Temperature for Laminar Flow in Entrance Region of Smooth Tube

between piezometer taps, use was made of Table XII in Goldstein [1, page 307]. The pressure drop was computed for several Reynolds numbers at several temperatures and curves were plotted to facilitate interpolation. These curves are shown in Fig. 7 as curves of head loss versus temperature for several Reynolds numbers. (In making the computations, the entrance of the tube was assumed to be located at the point of tangency of the bell mouth with the tube as indicated in the sketch in Fig. 7.) Pressure drops between piezometer taps were measured with the U-tube manometer, using an air-water interface, and were compared with the curves of Fig. 7 to determine whether a given flow was laminar. Surprisingly good agreement was obtained between the computed and measured pressure drops in laminar flow; discrepancies were less than 1 per cent of the pressure drop

Vibration in these experiments was applied only transversely over the entire range of frequencies from about 5 to 20,000 cps. The amplitude and frequency of the vibrations were measured with a crystal pickup (natural frequency 10,000 cps) and an oscilloscope; these may be seen in Fig. 5. The length of the tube vibrated was limited to a maximum of about 45 in. and was varied from that length down to about 20 in. to change the wave length of the vibration at fixed frequencies. Most of the runs were made using the latter length. The maximum vibrated length was limited to allow vibration of the upstream and downstream ends of the tube separately. The downstream position was adjacent to the downstream piezometer tap in a region approaching completely developed laminar flow; the upstream position, which began about 24 in. from the bell mouth, was in a region in which the boundary layer was still developing, but was certainly beyond the two-dimensional region in the sense of the discussion on page 8.

Over the entire range of frequencies tested in the range of Reynolds numbers from 3000 to 15,000, it was found that the flow remained laminar both with and without vibration. This was true for both regions of vibration. The only exception to the above statement occurred at frequencies of 220 cps, and multiples thereof, for all Reynolds numbers. At these frequencies some sort of resonance occurred in the entire structure and it is believed that large magnitude disturbances reached the bell-mouth entrance. Similar transition could be produced by striking a sharp blow to the outside of the tank or to the tube within a few inches of the tank.

From these results it appears that laminar flow downstream from the actual inlet of a tube is stable to all small external disturbances.

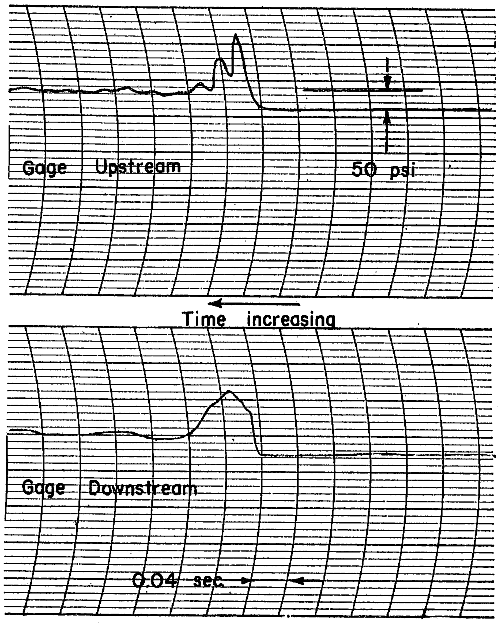
D. Flow-Establishment Experiments

These experiments were made to check the results of the analysis, which indicated that vibration during flow establishment should be ineffective in delaying the transmission of pressure pulses. Equipment was not available to reproduce the idealized plan of Fig. 1-a. The low-pressure end of the test tube in the experiments was connected to more tubing instead of to a low-pressure reservoir. There was, therefore, no reflection of pressure pulses from this end, nor was it possible to generate pressure pulses at this end by means of the longitudinal vibrations visualized in Eq. (7). Nevertheless, the apparatus is believed to be representative of a typical hydraulic system.

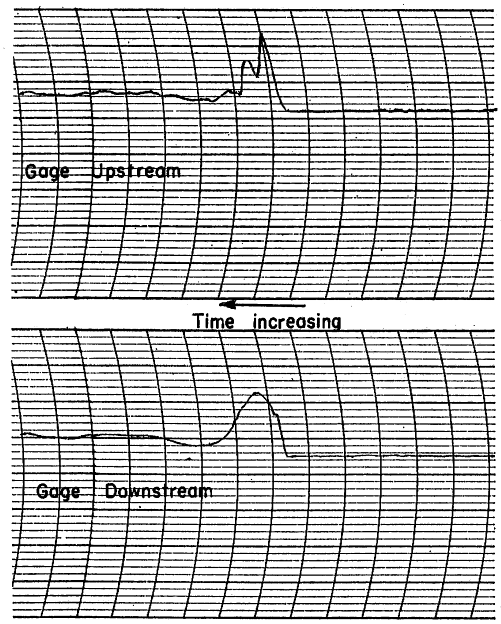
The experiments were organized in such a way that steady flow was first eatablished in the test tube at a given rate and working pressure, as had been done for the steady-flow experiments. The quick-acting valve was then cocked, causing the flow to be diverted to the by-pass line (Fig. 3). The by-pass line had the same total resistance to flow as the test tube so that there were no changes in flow characteristics while flow occurred in the by-pass line. While the flow was diverted, the test tube was set in vibration at a given frequency; then the cocking lever was released, reestablishing flow in the test tube.

Just before, during, and shortly after reestablishment of the flow, the oscillograph to which the electronic strain gages were attached was switched on. The piezometer taps were 11.5 ft apart. Data recorded, in addition to the oscillograph trace, were rate of flow, temperature, working pressure, frequency of vibration, and location of the valve. (As already noted, the quick-acting valve was used both upstream and downstream of the test tube.) The vibrational frequency was varied in steps during these experiments; the following frequencies were used: 500, 1000, 1300, 1500, 2250, 2500, 3000, 3500, 4500, 6000, 9000, 12,000, 16,000, and 20,000 cycles per minute. The experiments were repeated for two sets of working pressures—about 150 psi and about 1600 psi. Only one Reynolds number, about 6000 for steady flow, and only the 1/2-in. OD test tube were used in these experiments.

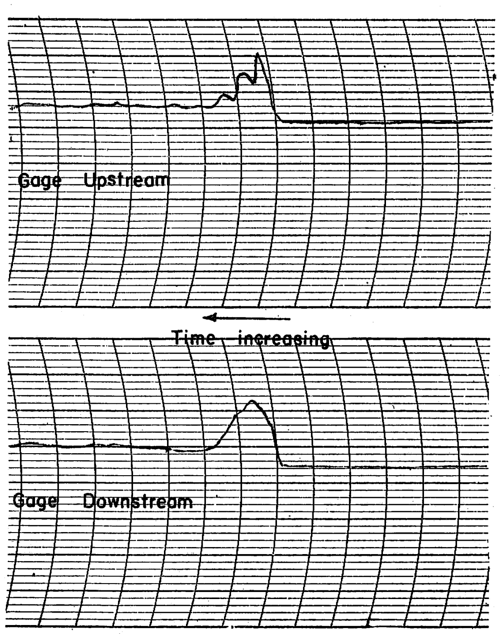
Typical oscillograph traces are shown in Figs. 8 and 9. In Fig. 8, separate traces made by the upstream and downstream gages are shown, while in Fig. 9 single traces obtained by connecting the leads from the two gages to give differential readings are recorded. The wave-like form of the trace



(a) No Vibration



(b) Transverse Vibration at 3500 CPM and 0.031 Inches Maximum Amplitude



(c) Transverse Vibration at 20,000 CPM and 0.005 Inches Maximum Amplitude

Valve upstream
Working pressure = 143 psi
Reynolds number = 6000

Fig. 8 - Flow Establishment. Pressure - Gage Oscillograph Traces

from the upstream gage in Fig. 8 probably resulted from action of the accumulator. The magnitudes of the initial peaks in Figs. 8 and 9 resulted from a construction detail of the four-way valve used as a quick-acting valve. During release of the valve to establish flow, all flow was momentarily stopped, producing the high peak which is roughly four times the working pressure in the low-pressure experiments.

In all of the experiments, the traces obtained with vibration have been compared with those obtained with no vibration, all flow conditions being the same. All of the comparisons are the same as those shown in Figs. 8 and 9. There is no apparent difference in the traces with and without vibration. As already noted, the response of the amplifier and oscillograph unit was limited to frequencies less than 100 cps; it can, therefore, be stated that the time delay in transmission of pressure pulses and in flow establishment, if any, was less than 0.01 sec over the 11.5-ft length between piezometer taps.

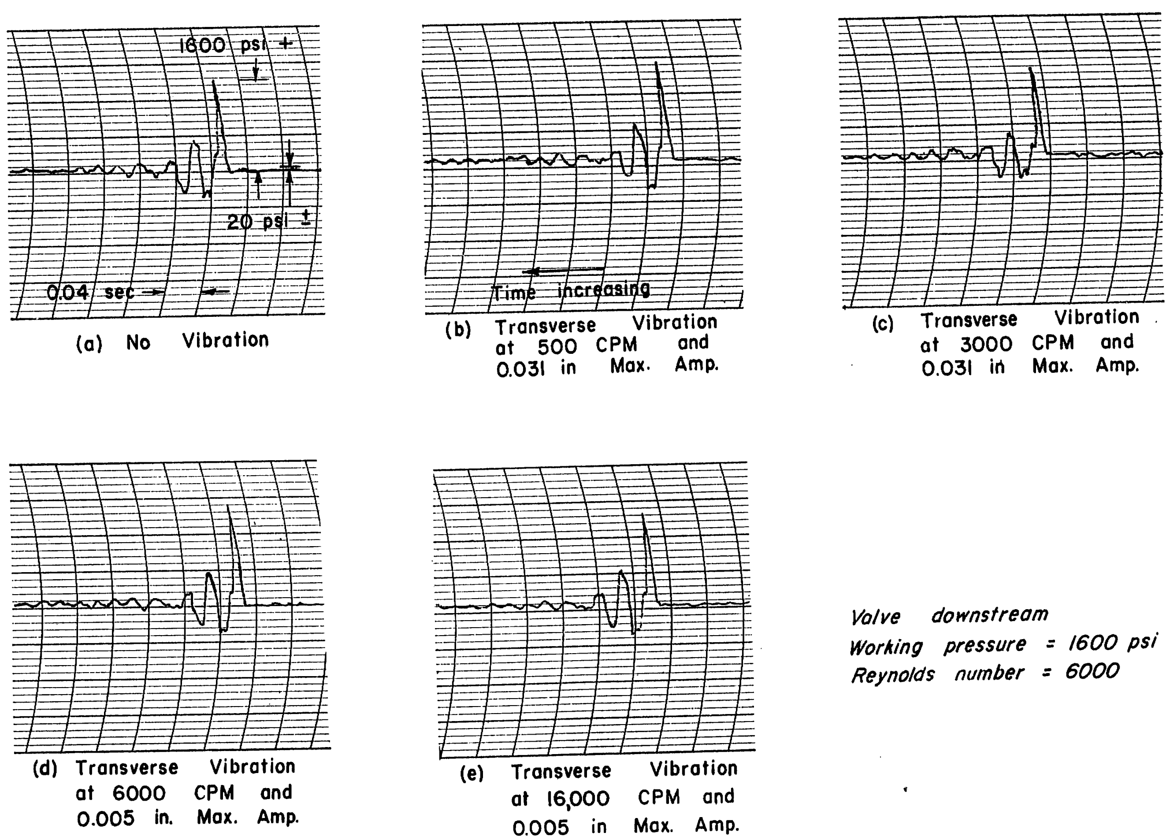


Fig 9 — Flow Establishment. Pressure-Gage Oscillograph Traces

IV. CONCLUSIONS

It was found in the preceding analysis and experimental work that:

1. There was no additional pressure drop produced by vibration of a straight, smooth tube above the normal pressure drop for that tube in a nonvibrating condition.

2. In a smooth, straight tube there was no measurable time delay due to vibration in transmission of pressure pulses or in establishment of flow after opening a valve.

Strictly speaking, the above results were obtained for circular tubes only; it is believed, however, that the results are equally applicable to tubes of other shapes. The experimental work in connection with the second point was not as complete as that for the first point, but it is believed that the conclusion is warranted from the analysis and is substantiated by the data.

An incidental result of the work was the experimental finding that laminar flow in tubes at Reynolds numbers up to 15,000 (based on tube diameter and mean velocity) was stable to all small external disturbances. This finding included both the fully developed flow and the region of the developing boundary layer beyond at least the first 60 diameters of tube length from the entrance.

BIBLIOGRAPHY

[1] Goldstein, S., (ed.). Modern Developments in Fluid Dynamics. Oxford: Clarendon Press, 1938.

[2] Richardson, E. G. Dynamics of Real Fluids. London: Edward Arnold and Company, 1950.

[3] Holt, A., and Haviland, G. "Bending Vibrations of a Pipe Line Containing Flowing Fluid." Journal of Applied Mechanics, Vol. 17, No. 3, pp. 229-32. September, 1950.

[4] Tollmein, W. "Ueber die Entstebung der Turbulenz." Nachrichten der Gesellschaft der Wessenschaften zu Göttingen (Report I of the Göttingen Scientific Society), Mathematisch-Physikalisihe Klasse. (Translated from the German in National Advisory Committee for Aeronautics Technical Memorandum No. 609, 1931.)

[5] Schubauer, G. B., and Skramstad, H. K. Laminar-Boundary-Layer Oscillations and Transition on a Flat Plate. National Advisory Committee for Aeronautics Technical Report No. 909, 1948.

[6] Sexl, T. "Uberdreidimensionale Storungen der Poiseuilleschen Stromung." Annales der Physik Bd. 84, Heft 22, Folge 4, 807-22. 1927.

[7] Lin, C. C. "On the Stability of Two-Dimensional Parallel Flows." Quarterly of Applied Mathematics, Vol. 3, pp. 117-42. July, 1945.

[8] McNown, J. S. "Surges and Water Hammer," from Engineering Hydraulics. H. Rouse (ed.). New York: John Wiley and Sons, Inc., 1950.

[9] Campbell, S. E. Investigation of the Fundamental Characteristics of High Performancy Hydraulic Systems. United States Air Force, Technical Report No. 5997, 1950.

[10] Lamb, H. Hydrodynamics. New York: Dover Publishing Company, 1945.

[11] Coulson, C. A. Waves (Third Edition). New York: Interscience Publishers, 1944.

APPENDIX

APPENDIX I

LIST OF SYMBOLS

A, B, C, D, F, G, H - constants.

a - maximum amplitude of vibration.

c - sound speed in liquid.

c' - sound speed in tube wall material.

E - bulk modulus of elasticity of liquid and tube.

f - frequency (cycles per second).

g - acceleration of gravity.

K - parameter = $\sqrt{2}/2 \sqrt{1 - (\sigma/\kappa c)^2}$.

L - length of tube.

m_1, m_2, etc. - constant parameters.

p - unit pressure.

p_o - nominal working pressure in tube.

p_1 - pressure in tube downstream of valve prior to valve opening.

R - radius of tube.

Re - Reynolds number = $2\overline{U}R/\nu$.

r - distance from tube axis along a diameter ($r^2 = x^2 + y^2$).

t - time.

U - velocity of mean flow.

$\overline{U}$ - mean velocity.

u - relative particle velocity parallel to tube axis with respect to tube wall.

u' - absolute particle velocity parallel to tube axis.

v - relative particle velocity parallel to y-axis.

w - relative particle velocity parallel to z-axis.

x - coordinate axis coincident with tube axis; also, distance from end of tube along x-axis.

y - coordinate axis normal to x in plane of transverse vibration.

z - coordinate axis normal to x and y.

x_o, y_o, z_o - coordinate axes parallel to x, y, and z in Lagrangian notation.

β - parameter = $\sqrt{\sigma/2\nu}$.

η_x - tube wall displacement parallel to x-axis.

η_y - tube wall displacement parallel to y-axis.

κ - parameter = $2\pi/\lambda$.

λ - wave length of transverse vibration.

μ - coefficient of viscosity.

ν - kinematic viscosity = μ/ρ.

ξ - displacement of a fluid particle due to vibration, components ξ_x, ξ_y, and ξ_z, along the coordinate axis.

ρ - liquid density.

σ - parameter = $2\pi/\tau$.

τ - period of vibration.

τ_o - wall shear.

τ_R - Reynolds stress = $\rho\,\overline{uv}$.

Subscript o - denotes initial conditions unless otherwise noted.

A P P E N D I X   II

PRESSURE AND VELOCITY FLUCTUATIONS DUE TO COMPRESSIBILITY IN A VIBRATING TUBE

A. Transverse Vibrations

Referring to Fig. 1-b and Eq. (1) on pages 2 and 3, it is given that the tube walls have the displacement

$$\eta_y = a \sin \kappa\, x \cos \sigma\, t \tag{A-1}$$

Because of the wall displacement, every particle in the interior of the tube has a displacement ξ. In general, ξ is different from η because of the finite time of travel of pressure pulses through the slightly compressible liquid filling the tube. The problem is three-dimensional, notwithstanding the two-dimensional vibratory motion, and ξ has the components ξ_x, ξ_y, and ξ_z, where the z-axis is normal to the plane of vibration. Viscosity enters this problem only through its damping effect on the internal motion; this is assumed to be negligible.

Let the individual particles in the liquid be identified by their coordinates x_o, y_o, and z_o at the time, t_o, prior to vibration. Then, at any time, $x = x_o + \xi_x$, $y = y_o + \xi_y$, and $z = z_o + \xi_z$. The equations of motion and continuity are written in Lagrangian form [10] as follows (there being no external forces, and viscous forces being neglected):

$$\left.\begin{aligned}
&\frac{\delta^2\xi_x}{\delta t^2}\left(1 + \frac{\delta\xi_x}{\delta x_o}\right) + \frac{\delta^2\xi_y}{\delta t^2} \cdot \frac{\delta\xi_y}{\delta x_o} + \frac{\delta^2\xi_z}{\delta t^2} \cdot \frac{\delta\xi_z}{\delta x_o} = -\frac{1}{\rho}\frac{\delta p}{\delta x_o}\\
&\frac{\delta^2\xi_x}{\delta t^2} \cdot \frac{\delta\xi_x}{\delta y_o} + \frac{\delta^2\xi_y}{\delta t^2}\left(1 + \frac{\delta\xi_y}{\delta y_o}\right) + \frac{\delta^2\xi_z}{\delta t^2} \cdot \frac{\delta\xi_z}{\delta y_o} = -\frac{1}{\rho}\frac{\delta p}{\delta y_o}\\
&\frac{\delta^2\xi_x}{\delta t^2}\frac{\delta\xi_x}{\delta z_o} + \frac{\delta^2\xi_y}{\delta t^2}\frac{\delta\xi_y}{\delta z_o} + \frac{\delta^2\xi_z}{\delta t^2}\left(1 + \frac{\delta\xi_z}{\delta z_o}\right) = -\frac{1}{\rho}\frac{\delta p}{\delta z_o}
\end{aligned}\right] \tag{A-2}$$

$$\rho_o = (\rho_o + d\rho)\,\frac{\delta(x_o + \xi_x,\ y_o + \xi_y,\ z_o + \xi_z)}{\delta(x_o,\ y_o,\ z_o)} \tag{A-3}$$

where p is the pressure,

ρ_o is the density at the time, t_o, and

$d\rho$ is the (small) increment in particle density between the time, t_o, and some later time, t.

These equations are too complicated to solve with the given boundary conditions as they stand. However, if the following assumptions are made

$$\frac{\delta \xi_x}{\delta x_o}, \frac{\delta \xi_x}{\delta y_o}, \ldots\ldots\ldots\ldots\ldots \frac{\delta \xi_z}{\delta z_o} \ll 1 \tag{A-4}$$

the equations become soluble. The limitations imposed by these assumptions will be indicated presently.

With the assumptions of Eqs. (A-4), Eqs. (A-2) and (A-3) become, in first approximation,

$$\frac{\delta^2 \xi_x}{\delta t^2} = -\frac{1}{\rho}\frac{\delta p}{\delta x_o}, \quad \frac{\delta^2 \xi_y}{\delta t^2} = -\frac{1}{\rho}\frac{\delta p}{\delta y_o}, \quad \frac{\delta^2 \xi_z}{\delta t^2} = -\frac{1}{\rho}\frac{\delta p}{\delta z_o} \tag{A-2a}$$

$$\rho_o = (\rho_o + d\rho)\left(1 + \frac{\delta \xi_x}{\delta x_o} + \frac{\delta \xi_y}{\delta y_o} + \frac{\delta \xi_z}{\delta z_o}\right) \tag{A-3a}$$

or

$$-\frac{d\rho}{\rho_o} \approx \frac{\delta \xi_x}{\delta x_o} + \frac{\delta \xi_y}{\delta y_o} + \frac{\delta \xi_z}{\delta z_o} \tag{A-3b}$$

where ρ_o has been written for $\rho_o + d\rho$ in the denominator of the left side of the equation.

Introducing the bulk modulus of elasticity of the liquid E, assumed to include the elasticity of the tube,

$$E = \frac{dp}{d\rho/\rho_o} \tag{A-5}$$

where dp is the increment in pressure (not necessarily small) accompanying the increment in density $d\rho$.

Substituting Eq. (A-5) in Eq. (A-3b) yields

$$-\frac{dp}{E} = \frac{\delta \xi_x}{\delta x_o} + \frac{\delta \xi_y}{\delta y_o} + \frac{\delta \xi_z}{\delta z_o} \tag{A-6}$$

Equation (A-6) gives the increment in pressure of a particle in first approximation caused by relative displacement of the sides of the particle. The total pressure on the particle is $p = p_o + dp$ where p_o is the pressure at the time t_o and it must be assumed that p_o is independent of position. Then,

$$\frac{\partial p}{\partial x_o} = \frac{\partial (dp)}{\partial x_o}, \quad \frac{\partial p}{\partial y_o} = \frac{\partial (dp)}{\partial y_o}, \quad \frac{\partial p}{\partial z_o} = \frac{\partial (dp)}{\partial z_o} \tag{A-7}$$

Substituting Eqs. (A-7) in Eqs. (A-2a), taking $\frac{\partial}{\partial x_o}$ of the first, $\frac{\partial}{\partial y_o}$ of the second, and $\frac{\partial}{\partial z_o}$ of the third, and adding, yields:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial \xi_x}{\partial x_o} + \frac{\partial \xi_y}{\partial y_o} + \frac{\partial \xi_z}{\partial z_o}\right) = -\frac{1}{\rho}\left(\frac{\partial^2}{\partial x_o^2} + \frac{\partial^2}{\partial y_o^2} + \frac{\partial^2}{\partial z_o^2}\right)(dp)$$

Substitution of Eq. (A-6) in the above then leads to the familiar wave equation:

$$\left(\frac{\partial^2}{\partial x_o^2} + \frac{\partial^2}{\partial y_o^2} + \frac{\partial^2}{\partial z_o^2}\right)(dp) = \frac{1}{c^2}\frac{\partial^2 (dp)}{\partial t^2} \tag{A-8}$$

where $c = \sqrt{E/\rho}$ is the speed of transmission of pressure pulses (sound speed) in the liquid. The solution of Eq. (A-8) for standing waves [11, page 14] may be written in the form

$$dp = (A_1 \cos m_1 x_o + A_2 \sin m_1 x_o)(B_1 e^{m_2 y_o} + B_2 e^{-m_2 y_o})$$

$$(C_1 e^{m_3 z_o} + C_2 e^{-m_3 z_o})(D_1 \cos m_4 ct + D_2 \sin m_4 ct) \tag{A-9}$$

where $m_4^2 = m_1^2 - m_2^2 - m_3^2$, the m's, A's, B's, C'c, and D's are constants and either A_1 or A_2 and D_1 or D_2 are zero.

The constants are evaluated by taking the partial derivative with respect to y of Eq. (A-9) and substituting in the second of Eqs. (A-2a) to obtain $\partial^2\xi_y/\partial t^2$. The value of $\partial^2\xi_y/\partial t^2$ at the boundary is then found from Eq. (A-1) (replacing x by x_o). Assuming a tube of circular cross section of ~~radius R, at the boundary $y_o^2 + z_o^2 = R^2$. The boundary condition cannot be~~ satisfied exactly at all points on the boundary; only four points may be chosen. By choosing these as $z_o = 0$, $y_o = \pm R$ and $y_o = 0$, $z_o = \pm R$, the constants become

$$A_1 = D_2 = 0$$

$$A_1B_1C_1D_1 = A_1B_1C_2D_1 = - A_1B_2C_1D_1 = - A_1B_2C_2D_1 = \frac{\sigma^2 \; a\rho}{4m_2 \cosh m_2 R}$$

$$m_1 = \kappa = \frac{2\pi}{\lambda}, \; m_4 = \frac{\sigma}{c} = \frac{2\pi}{\tau c}$$

$$m_2 = \pm m_3 = \pm \frac{\sqrt{2}}{2}\kappa \sqrt{1 - \left(\frac{\sigma}{\kappa c}\right)^2} = \pm \kappa K \text{ where } K = \frac{\sqrt{2}}{2}\sqrt{1 - \left(\frac{\sigma}{\kappa c}\right)^2}$$

Using the above constants yields for the fluctuations

$$dp = a\rho \frac{\sigma^2}{\kappa K} \sin\kappa x_o \cos\sigma t \frac{\sinh \kappa K y_o \cosh \kappa K z_o}{\cosh \kappa K R} \tag{A-9a}$$

$$u = \frac{\delta\xi_x}{\delta t} = -\frac{\sigma a}{K} \cos \kappa x_o \sigma t \frac{\sinh \kappa K y_o \cosh \kappa K z_o}{\cosh R} + F \tag{A-10}$$

$$v = \frac{\delta\xi_y}{\delta t} = -\sigma a \sin\kappa x_o \sigma t \frac{\cosh \kappa K y_o \cosh \kappa K z_o}{\cosh \kappa K R} + G \tag{A-11}$$

$$w = \frac{\delta\xi_z}{\delta t} = -\sigma a \sin\kappa x_o \sin\sigma t \frac{\sinh \kappa K y_o \sinh \kappa K z_o}{\cosh \kappa K R} + H \tag{A-12}$$

The additive constants result from the integration of $\delta^2\xi/\delta t^2$ and are made absolute rather than functions of the space variables to satisfy the assumptions of Eqs. (A-4). If F is set equal to U, where U is the velocity of a uniform mean flow through the tube and G = H = 0, the equations are satisfied as long as U is independent of x. (This addition of fluctuating velocity to mean velocity is possible because the fluctuating motion studied here is a potential motion.)

The assumptions of Eqs. (A-4) are now tested by integrating Eqs. (A-10), (A-11), and (A-12) with respect to time to obtain ξ_x, ξ_y, and ξ_z. It is found that

$$\frac{a}{\lambda} \ll 0.11$$

must hold for all values of λ. It may also be noted that for

$$y_o^2 + z_o^2 = R^2$$

by Eq. (A-1), Eq. (A-11) must yield

$$v = \frac{\delta \eta y}{\delta t} = -\sigma a \sin \kappa x \cos \sigma t$$

This is satisfied exactly where the boundary conditions were originally satisfied ($z_o = 0$, $y_o = \pm R$ and $y_o = 0$, $z_o = \pm R$) and very closely for all other points on the boundary.

B. Longitudinal Vibrations

Referring to Fig. 1-c and to Eq. (2) on pages 2 and 3, it is given that the high-pressure end of the tube ($x = 0$) is fixed in position and maintains the constant pressure p_o while the other end ($x = L$) is closed and has the displacement

$$\eta_x = a \cos \sigma t \tag{A-13}$$

The motion may be considered one-dimensional. Using methods similar to those for transverse vibrations, the one-dimensional equation of wave motion is obtained in the form

$$\frac{\delta^2 \xi_x}{\delta x_o^{\,2}} = \frac{1}{c^2} \frac{\delta^2 \xi_x}{\delta t^2} \tag{A-14}$$

The general solution is

$$\xi_x = (A_1 \cos m_1 x + A_2 \sin m_1 x)(D_1 \cos m_4 ct + D_2 \sin m_4 ct) \tag{A-15}$$

where $m_1 = m_4$.

Evaluating the constants from the boundary conditions yields

$$A_2 = D_2 = 0$$

$$m_1 = m_4 = \frac{\sigma}{c} = \frac{2\pi}{\tau c}$$

$$A_1 D_1 = \frac{a}{\sin m_1 L}$$

Finally,

$$\xi_x = a \frac{\sin m_1 x_o}{\sin m_1 L} \cos \sigma t \tag{A-16}$$

$$u = \frac{\delta \xi_x}{\delta t} = -\sigma a \frac{\sin m_1 x_o}{\sin m_1 L} \sin \sigma t \tag{A-17}$$

$$p = p_o - \rho c\, u_o \left(\frac{1 - \cos m_1 x_o}{\sin m_1 L}\right) \tag{A-18}$$

where $u_o = \frac{\delta \eta_x}{\delta t}$ is the velocity of the closed end of the tube.

Equation (A-17) is to be compared with the equation of motion of the pipe wall, which is of similar form but with the parameter m_1 containing the speed of sound in the tube wall material rather than in the liquid. Thus, the relative velocity between fluid and wall is

$$-\sigma a \left(\frac{\sin \sigma \frac{x_o}{c'}}{\sin \sigma \frac{L}{c'}} - \frac{\sin \sigma \frac{x_o}{c}}{\sin \sigma \frac{L}{c}} \right) \sin \sigma t$$

where c' is the speed of sound in the tube wall.

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