UNIVERSITY OF MINNESOTA
ST. ANTHONY FALLS HYDRAULIC LABORATORY

Technical Paper No. 47, Series B

Two-Dimensional Supercavitating Plate Oscillating Under a Free-Surface

by
C. S. SONG

Prepared for
DAVID TAYLOR MODEL BASIN
Department of the Navy
Washington, D.C.
under
Bureau of Ships Fundamental Hydromechanics Research Program
SR-009-01-01
Office of Naval Research Contract Nonr 710(51)

December 1963
Minneapolis, Minnesota

ERRATA

Eq. (35) on page 17 should read:

$$\tilde{\bar{v}}(0^+, 0) = - jkh_o$$

First unnumbered quantity on page 17 should read:

$$\left[\frac{\sigma}{2(1 + \sigma)} + F_c^{-2}\right] \quad \frac{\Delta \ell}{\tilde{\ell}}$$

First quantity on page 52 should read:

$$Y_1(\lambda, \beta),$$

The quantity after Eq. (A5) on page 52 should read:

$$Y_1(\lambda, \beta) = 0$$

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ABSTRACT

The problem of a supercavitating flat plate at non-zero cavitation number oscillating under a free surface is analyzed by a linearized method using the acceleration potential. The flow is assumed two-dimensional and incompressible. The flow field is made simply connected by using a cut along the wake. The flow field is then mapped on to an upper half plane and the solution is expressed in an integral form by using Cheng and Rott's method.

Equations for the cavity length, total force coefficient, moment coefficient and the frequency response function are expressed in closed form. Numerical results for some special cases are also obtained and presented graphically. When the flow is steady, the present theory agrees with experimental data and other existing theories. For the special case of infinite fluid and infinite cavity the present theory agrees with Parkins' original work. For the special case of zero submergence, the present theory indicates that the total force coefficient is one half that of the value for fully wetted flow in an infinite fluid for both steady and unsteady cases. An alternate analysis is also carried out for the infinite fluid case and the result shows that the effect of the wake assumption is of order of the square of the cavitation number when the cavitation number is small. The effect of the gravity field is also discussed qualitatively.

It is also concluded that the effect of the free-surface is to shorten the cavity and to increase the total force coefficient. The steady part of the force coefficient at an arbitrary submergence is obtained by multiplying the value at infinite submergence by a correction factor, whereas the unsteady part is given by a more complicated function. Even with the presence of a free-surface and oscillation of the foil, the total force coefficient at small cavitation number is approximately equal to the corresponding value at zero cavitation number multiplied by a factor $(1 + \sigma)$.

C O N T E N T S

L I S T   O F   I L L U S T R A T I O N S

LIST OF SYMBOLS

A, A_n - constants

a - a constant related to submergence

B_F - submergence correction factor for steady force coefficient

B_M - submergence correction factor for steady moment coefficient

b - submergence

C - Theodorsen's function

C_F - total force coefficient

C_M - moment coefficient

F - force

$F(z)$, $F(\zeta)$ - complex acceleration potential

F_c - Froude number based on half cavity width

H - homogeneous solution

h_o - amplitude of pitching oscillation

I_o, I_1 etc - integrals

i - unit imaginary number referred to space

J_1, J_2 - Fourior integrals

j - unit imaginary number referred to time

K - complete elliptic integral of first kind

K_1, K_2 - modified Bessel function

k - reduced frequency

k_1, k_2 - modulus of elliptic functions

L_o, L_1 - integrals

ℓ - cavity length

ℓ_1, ℓ_2 - cavity length in transformed plane

P - pressure at infinity

P_o - pressure on free-surface

p - pressure at any point

p_c - pressure in the cavity

q - speed of flow

q_c - speed of flow on cavity surface

r - variable of integration

S - area

t - time

u, v - perturbation velocity

$\bar{v}$ - steady part of v

$\tilde{v}$ - unsteady part of v

W, W', W_1 - frequency response functions

x, y - coordinates

Y_o, Y_1 - integrals

z - complex number

α - angle of attack

α_o - amplitude of pitching oscillation

β - a parameter

γ - specific weight of water

Z - Jacobian zeta function

$\zeta = \xi + i\eta$ - transformed complex plane

θ - argument of the Lambda function

Λ_o - Heuman's Lambda function

λ - a parameter

ρ - density of water

σ - cavitation number

τ - variable of integration

ϕ - acceleration potential

Ψ - imaginary part of the complex acceleration potential

Ψ_o - a frequency response function

$\tilde{\Psi}$ - modulus of the unsteady part of Ψ

ω - angular speed of oscillations

TWO-DIMENSIONAL SUPERCAVITATING PLATE OSCILLATING UNDER A FREE-SURFACE

I. INTRODUCTION

The problem of unsteady flow with a wake or cavity forming behind a solid body has recently stimulated considerable research interest. The problem of accelerated flow of an incompressible infinite fluid with cavity formation in which the flow pattern remained similar with respect to time was solved by Von Karman [1]* in 1949. Recently, Yih [2] extended the theory to the case of an accelerated body without changing the other conditions. General unsteady flow problems with wake or cavity formation are so complicated that their exact solutions seem unlikely to appear in the foreseeable future. At present, only an approximate solution based on linearized theory can be obtained.

The problem of unsteady flow with an infinitely long wake was solved by Woods [3, 4, 5], applying a small perturbation to the basic Helmholtz-Kirchhoff flow. In the United States, the pioneering work in this connection was done by Parkin [6] in 1957. Parkin's theory is an extension of Tulin's [7] linear theory for steady cavity flow and the linear theory of oscillating airfoils [8] using the acceleration potential. Since then a number of papers have appeared in the literature concerning the unsteady two-dimensional flow of an incompressible fluid. Nevertheless, the problem can hardly be considered completely solved. The main difficulty lies in the unique determination of the potential flow problem; thus a satisfactory solution may be obtained only when the real fluid effects (such as viscosity, compressibility, etc.) are taken into account.

The present paper is mainly concerned with the effect of a free surface on an oscillating two-dimensional hydrofoil with a cavity. The effect of the gravity force is also discussed briefly. The cavity is assumed to be long and thin so that linear theory based on the acceleration potential may be applied.

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*Numbers in brackets refer to the List of References on page 35.

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Most of the detailed computations were carried out by Miss Samiha Tadros. The author is grateful to Professor Edward Silberman and Mr. J. M. Wetzel for reviewing the paper. The manuscript was prepared for printing by Mrs. Carol Takyi and Miss Judy Mike.

II. STATEMENT OF THE PROBLEM

Consider a thin hydrofoil with sharp leading edge moving under a free surface with basic speed U parallel to the free surface and undergoing heaving and/or pitching oscillations of small amplitude. A long cavity is assumed to form behind the foil and the separation is assumed to occur at the leading and the trailing edges. The resulting flow pattern is depicted in Fig. 1. Here all linear dimensions are normalized with the chord length.

The amplitude of the oscillation is assumed to be so small that separation points are fixed and there is no reattachment of the flow after separation. Furthermore, the resulting flow is assumed to be only slightly different from the uniform flow so that the linear theory may be applied. The pressure on the free surface, P_o, and the cavity pressure, P_c, are assumed to be constants. Finally, the flow is assumed to be incompressible and irrotational outside the body-cavity-wake region.

According to Bernoulli's equation the speed on the cavity surface for the steady flow conditions, q_c, is given by

$$q_c = U \sqrt{1 + \sigma} \tag{1}$$

where σ is the cavitation number defined as

$$\sigma = \frac{P_o + \gamma b - P_c}{\frac{1}{2} \rho U^2} \tag{2}$$

where γ is the specific weight, ρ is the density, and b is the submergence. If q_c is used as the reference speed and the velocity, $\vec{q}$, at a point is expressed in terms of the perturbation velocity as

$$\vec{q}/q_c = (1 + u, v) \tag{3}$$

the Euler equations of motion may be linearized and written

$$q_c^{-1} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = - \frac{\partial \phi}{\partial x}$$
$$q_c^{-1} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial y} = - \frac{\partial \phi}{\partial y} \tag{4}$$

Here ϕ is an acceleration potential defined as

$$\phi(x, y, t) = \frac{(P_o + \gamma b) - (p + \gamma y)}{\rho q_c^2} \tag{5}$$

Gravity is assumed to act along the negative direction of the y-axis.

It has been shown elsewhere [6, 8] that Eq. (4) and the equation of continuity implied that the acceleration potential is a harmonic function and, hence, the complex acceleration potential.

$$F(z, t) = \phi + i\Psi \tag{6}$$

is an analytic function of

$$z = x + iy$$

Here ϕ and Ψ are related by the Cauchy-Riemann equations,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = - \frac{\partial \Psi}{\partial x} \tag{7}$$

Since the acceleration potential defined by Eq. (5) is continuous everywhere within the flow region, the complex acceleration potential is a regular function everywhere except, possibly, on a finite number of boundary points where weak singularities may have to be admitted.

It is now necessary to define the boundary and determine the boundary condition. Riabouchinsky's model [9], Wagner's reentrant jet model [10] and Roshko's dissipation model [11] are familiar in connection with steady cavity flow problems. Furthermore, in the linear theory, it is customarily assumed that the body-cavity-wake is thin and the boundary condition is applied on a slot. The linearized reentrant jet model was discussed in detail by Geurst [12]. The relationship between various linearized steady cavity flow models was discussed by the author [13]. It was pointed out in Ref. [13] that Tulin's linearized closed cavity model [7] and the linearized energy dissipation model are two special cases of the generalized Riabouchinsky model.

Since the best potential flow model could be obtained only when the role of viscosity is fully taken into consideration it is difficult to compare the accuracies of the existing models. The energy dissipation model is adopted in the present paper mainly because, unlike other models, it reduces the flow region into a simply connected region. The linearized boundary shape is shown in Fig. 2.

The boundary conditions are listed below:

(1) Since the pressure, P_o, on the free surface is a constant and equal to the atmospheric pressure, it follows that

$$\phi = 0 \quad \text{on} \quad y = b$$

(2) If the cavity is defined to be a space wherein the pressure, P_c, is essentially a constant then the boundary condition on the cavity surface is

$$\phi = \frac{\sigma}{2(1+\sigma)} - F_c^{-2} \qquad \text{on } y = 0^+, \quad 0 < x \leq \ell$$

$$\phi = \frac{\sigma}{2(1+\sigma)} + F_c^{-2} \qquad \text{on } y = 0^-, \quad 1 \leq x \leq \ell$$

where

$$F_c = \frac{q_c}{\sqrt{g\delta}} = \text{Froude number}$$

δ = half cavity thickness

ℓ = cavity length

(3) Adopting the energy dissipation model or a special case of the generalized Riabouchinsky model wherein the tail-body is a pair of parallel flat plates, the vertical velocity component, v, on the wake boundary is identical to zero. Then, according to Eqs. (4) and (7), Ψ can be a function of time only. Since the acceleration potential can be determined uniquely only up to an arbitrary function of time, we may set

$$\Psi = 0 \qquad \text{on} \qquad y = 0^{\pm}, \qquad \ell \leq x \leq +\infty$$

It may also be argued that the pressure distribution far downstream is essentially hydrostatic and, hence, $\phi = 0$ on the wake boundary. Therefore, an alternate boundary condition on the wake boundary is

$$\phi = 0 \qquad \text{on} \qquad y = 0^{\pm}, \qquad \ell \leq x \leq +\infty$$

As a partial justification of the assumption, the pressure distribution in the wake behind a steady supercavitating flat plate in the free-jet tunnel at the St. Anthony Falls Hydraulic Laboratory was measured and the result is plotted in Fig. 3. The data show that the pressure recovery at the tail of the cavity is about 30 per cent of the stagnation head and the pressure in the wake is, indeed, almost constant and equal to P_o.

(4) The normal component of the relative velocity on the wetted part of the foil surface is equal to zero. If the equation of the foil surface is given by

$$Y = Y(x, t)$$

then the condition is

$$-\Psi = q_c^{-2} \int Y_{tt} dx + 2q_c^{-1} Y_t + Y_x + \Psi' (t) = - \Psi_1$$

$$\text{on} \quad y = 0^-, \qquad 0 < x \leq 1$$

(5) To relate the acceleration field and the velocity field, the function $\Psi'(t)$ should be determined by the general integral of Eq. (4). That is

$$v(x, y, t) = - \int_{-\infty}^{x} \frac{\partial}{\partial x} \Psi(\tau, y, t - x/q_c + \tau/q_c) \; d\tau$$

where τ is a variable of integration. The detailed derivation of this condition may be found in Ref. [6].

(6) Finally, to make the solution unique, we require that

$$\Psi = 0 \quad \text{at} \quad x = -\infty\,, \quad y = b$$

It will be seen later that this condition is also equivalent to

$$\phi = 0 \quad \text{at} \quad x = +\infty\,, \quad y = 0$$

which implies that the pressure distribution near the wake at $x = +\infty$ is hydrostatic.

III. SOLUTION OF THE PROBLEM

A. The General Solution

In seeking the general solution it is more convenient to transform the flow region onto the upper half plane. This is accomplished by using the following mapping function:

$$z = \frac{b}{\pi a}\left[-\zeta + a\, \ell n \left(\frac{a}{a - \zeta}\right)\right] \tag{8}$$

where $\zeta = \xi + i\eta$ and a is the solution of

$$b = \frac{\pi a}{1 - a\, \ell n \left(\frac{a + 1}{a}\right)} \tag{9}$$

The ζ-plane is depicted in Fig. 2b.

It is readily seen that the points b and b' corresponding to the end of the cavity are mapped on $(-\ell_1, 0)$ and $(\ell_2, 0)$ respectively. Here $-\ell_1$ and ℓ_2 are the pair of solutions of the following equation:

$$\ell = \frac{b}{\pi a}\left[-\xi + a\, \ell n \left(\frac{a}{a - \xi}\right)\right] \tag{10}$$

To solve the mixed boundary value problem the method of Cheng and Rott [14] will be used. It was shown by the author [15] that the general solution satisfying the Hölder condition and the first four boundary conditions may be written as

$$F(\zeta) = \frac{H(\zeta)}{\pi}\left\{\int_{-1}^{0} \frac{\Psi_1(\tau)\, d\tau}{(\tau - \zeta)H(\tau)} + \int_{-\ell_1}^{-1} \frac{\left[\frac{\sigma}{2(1+\sigma)} + F_c^{-2}\right]d\tau}{i(\tau - \zeta)H(\tau)}\right.$$

$$\left. + \int_{0}^{\ell_2} \frac{\left[\frac{\sigma}{2(1+\sigma)} - F_c^{-2}\right]d\tau}{i(\tau - \zeta)H(\tau)} + \sum_{n=0}^{\infty} A_n \zeta^n\right\} \tag{11}$$

where

$$H(\zeta) = \sqrt{(\ell_1 + \zeta)(1 + \zeta)(\zeta - \ell_2)(\zeta - a)/(-\zeta)} \tag{12}$$

and A_n are real constants.

Note that the first alternative of the third boundary condition, $\Psi = 0$ on the wake, is chosen.

It should be noted that $H(\zeta)$ is one of the many possible homogeneous solutions satisfying the Hölder condition. The Hölder condition is required to limit the order of singularities so that the function is integrable and the resulting force will be finite. The leading edge singularity is explicitly indicated in Eq. (12) because, as usual, the stagnation point is assumed to be at the leading edge.

B. Cavity Length

To complete the formal solution it is necessary to determine the remaining unknowns such as $\Psi'(t)$, $\ell(t)$, and A_n. To satisfy the requirements imposed by the last boundary condition ($\Psi = 0$ at $\zeta = +\infty$) it is obvious that all coefficients, A_n, must vanish. Furthermore, it may be observed that, for a very large ζ, $H(\zeta)/(\tau - \zeta)$ is of order of $\sqrt{\zeta}$. Consequently, the last boundary condition may be satisfied only if

$$\int_{-1}^{0} \frac{\Psi_1(\tau)d\tau}{H(\tau)} + \int_{-\ell_1}^{-1} \frac{\left[\frac{\sigma}{2(1+\sigma)} + F_c^{-2}\right]d\tau}{iH(\tau)} + \int_{0}^{\ell_2} \frac{\left[\frac{\sigma}{2(1+\sigma)} - F_c^{-2}\right]d\tau}{iH(\tau)} = 0 \tag{13}$$

Equation (13) and the equation which may be derived by using the fifth boundary condition constitute a pair of simultaneous equations by which two unknown functions, $\Psi'(t)$ and $\ell(t)$, may be determined.

In principle the solution is now complete.

C. The Alternate Solution

The solution given by Eqs. (11), (12) and (13) involves hyperelliptic integrals whose values are not yet tabulated. Therefore, to obtain numerical results it is desirable to simplify the solution somewhat. This can be done if the alternate condition is used for the third boundary condition. There are only two branch points in this case, and the homogeneous solution will be

$$H_1(\zeta) = \sqrt{\frac{1+\zeta}{-\zeta}} \tag{12a}$$

instead of $H(\zeta)$ given by Eq. (12).

It will now be demonstrated that the two solutions are almost identical when the cavity is very long ($\ell >> 1$); hence, the simpler solution may be usable in the evaluation of the force and moment.

By using the identity

$$\frac{a - \tau}{(a - \zeta)(\tau - \zeta)} + \frac{1}{a - \zeta} = \frac{1}{\tau - \zeta}$$

Eq. (11) may be written as

$$F(\zeta) = \frac{H(\zeta)}{\pi(a - \zeta)} \Bigg\{ \int_{-1}^{0} \frac{(a - \tau)\Psi_1(\tau)d\tau}{(\tau - \zeta)H(\tau)}$$

$$+ \int_{-\ell_1}^{-1} \frac{(a - \tau)\left[\frac{\sigma}{2(1 + \sigma)} + F_c^{-2}\right]d\tau}{i(\tau - \zeta)H(\tau)}$$

$$+ \int_{0}^{\ell_2} \frac{(a - \tau)\left[\frac{\sigma}{2(1 + \sigma)} - F_c^{-2}\right]d\tau}{i(\tau - \zeta)H(\tau)} + \int_{-1}^{0} \frac{\Psi_1(\tau)d\tau}{H(\tau)}$$

$$+ \int_{-\ell_1}^{-1} \frac{\left[\frac{\sigma}{2(1 + \sigma)} + F_c^{-2}\right]d\tau}{iH(\tau)} + \int_{0}^{\ell_2} \frac{\left[\frac{\sigma}{2(1 + \sigma)} - F_c^{-2}\right]d\tau}{iH(\tau)} \Bigg\}$$

The sum of the last three terms in this equation is, according to Eq. (13), equal to zero. Furthermore, when $\ell >> 1$, it can be shown that $\ell_1 >> 1$ and $\ell_2 \approx a$. Consequently, when the cavity is very long, we have, approximately

$$F(\zeta) = F_1(\zeta) \tag{14}$$

where $F_1(\zeta)$ is the alternate solution.

IV. FLAT PLATE WITH HEAVE OSCILLATION

When a flat plate is performing a simple harmonic heave motion, the equation of the wetted surface may be written as

$$Y = -\alpha x - h_o e^{j\omega t} \tag{15}$$

where j is the unit imaginary number and only the real part of Eq. (15) expresses the physical motion. It is now possible to write

$$\Psi'(t) = \Psi_o e^{j\omega t} \tag{16}$$

and the boundary condition on the wetted surface is

$$\Psi_1 = \alpha + (\Psi_o - k^2 h_o x) e^{j\omega t} \tag{17}$$

where

α = angle of attack

h_o = amplitude of heaving oscillation

$k = \omega/q_c$ = reduced frequency

Ψ_o = a frequency response function to be determined later

The rigorous evaluation of the gravity effect is extremely complicated because F_c is an unknown function of x. In the following only a qualitative estimate of the gravity effect will be made by replacing the actual value of F_c by its average value (a constant).

A. Cavity Length

We shall consider first the infinite fluid case by letting b approach infinity. Thus, the transformation formula Eq. (8) is reduced to:

$$z = \zeta^2 \tag{18}$$

and hence:

$$\ell_1 = \ell_2 = \sqrt{\ell}$$

Equation (13) is now reduced to

$$\int_{-1}^{0} \Psi_1(\tau)\sqrt{\frac{-\tau}{1+\tau}}\,d\tau - \left[\frac{\sigma}{2(1+\sigma)} + F_c^{-2}\right]\sqrt{\ell}\int_{-\sqrt{\ell}}^{1}\sqrt{\frac{\tau}{(\ell-\tau^2)(1+\tau)}}\,d\tau$$

$$- \left[\frac{\sigma}{2(1+\sigma)} - F_c^{-2}\right]\sqrt{\ell}\int_{0}^{\sqrt{\ell}}\sqrt{\frac{\tau}{(\ell-\tau^2)(1+\tau)}}\,d\tau = 0 \tag{19}$$

The integrals appearing in Eq. (19) are given in a table of elliptic integrals [16]. After performing the integrations, Eq. (19) is reduced to:

$$\alpha + (\Psi_o - \tfrac{5}{2}k^2 h_o)\, e^{j\omega t}$$

$$= \left(\frac{\sigma}{1+\sigma} + 2F_c^{-2}\right)\left\{\frac{2}{\pi}\sqrt{\frac{\sqrt{\ell}}{\sqrt{\ell}+1}}\; K(k_1) + \sqrt{\ell}\,[1 - \Lambda_o(\Theta, k_1)]\right\} \tag{20}$$

$$+ \left(\frac{\sigma}{1+\sigma} - 2F_c^{-2}\right)\sqrt{\ell}\;[1 - \Lambda_o(\tfrac{\pi}{4}, k_1)]$$

where:

$K(k_1)$ = complete elliptic integral of first kind

$\Lambda_o(\Theta, h_1), \Lambda_o(\frac{\pi}{4}\, k_1)$ = Heuman's Lambda function

$$k_1 = \sqrt{\frac{\sqrt{\ell}-1}{\sqrt{\ell}+1}},\quad \Theta = \sin^{-1}\sqrt{\frac{\sqrt{\ell}+1}{2\sqrt{\ell}}}$$

In particular, when gravity is neglected $(F_c \rightarrow \infty)$, then Eq. (20) is reduced to:

$$[\alpha + (\Psi_o - \frac{5}{8} k^2 h_o) e^{j\omega t}] (\frac{1 + \sigma}{\sigma}) = L_1 (\ell) \tag{21}$$

where

$$L_1(\ell) = \frac{2}{\pi} \sqrt{\frac{\sqrt{\ell}}{\sqrt{\ell} + 1}} K(k_1) + \sqrt{\ell} [2 - \Lambda_o(\Theta,k_1) - \Lambda_o(\frac{\pi}{4}, k_1)] \tag{22}$$

The fact that the variable, ℓ, is separated from the other variables makes Eq. (21) convenient for practical use. It should be noted that when the flow is steady

$$\alpha(\frac{1 + \sigma}{\sigma}) = L_1 (\ell) \tag{21a}$$

The function L_1 is plotted on a log-log graph in Fig. 4. The resulting line is almost a straight line with a slope of 0.5 indicating that the cavity length is almost proportional to α^2/σ^2. A similar function obtained by using a slightly different method (without the wake assumption) which will be discussed in Section VI is also plotted in the same figure for comparison. A good agreement between the two methods is indicated. The steady flow cavity length given by Eq. (21a) is compared with that calculated by the quasi-linear theory [15] in Fig. 5. Here a good agreement is also noted.

An indication as to the effect of gravity on the cavity length may be obtained if the cavitation number in Eq. (20) is set equal to zero. The result is

$$[\alpha + (\Psi_o - \frac{5}{8} k^2 h_o) e^{j\omega t}] \frac{F_c^2}{2} = L_2(\ell) \tag{23}$$

where

$$L_2(\ell) = \frac{2}{\pi} \sqrt{\frac{\sqrt{\ell}}{\sqrt{\ell}+1}}\; K(k_1) - \sqrt{\ell}\; [\Lambda_o(\Theta,k_1) - \Lambda_o(\tfrac{\pi}{4}, k_1)] \tag{24}$$

The function $L_2(\ell)$ is plotted in Fig. 6. It may be observed that $\sigma = 0$ implies an infinite cavity only when the Froude number is infinite. Moreover the effect of gravity is to shorten the cavity.

To study the finite submergence case it should be noted that when ℓ is large ℓ_1 is also large and ℓ_2 is approximately equal to a. In fact, when the submergence is less than 10 chords, a is less than one and

$$\ell_2/a \approx 1 - \exp(-\ell/a) \tag{25}$$

Under this condition $(b<10,\ \ell>1)$, Eq. (13) may be approximated by

$$\int_{-1}^{0} \Psi_1(\tau) \sqrt{\frac{-\tau}{1+\tau}}\ \frac{d\tau}{a-\tau}$$

$$-\left[\frac{\sigma}{2(1+\sigma)} + F_c^{-2}\right] \sqrt{\ell_1} \int_{-\ell_1}^{-1} \sqrt{\frac{\tau}{(\ell_1+\tau)(1+\tau)}}\ \frac{d\tau}{a-\tau} \tag{26}$$

$$-\left[\frac{\sigma}{2(1+\sigma)} - F_c^{-2}\right] \int_{0}^{\ell_2} \sqrt{\frac{\tau}{(1+\tau)(\tau-\ell_2)(\tau-a)}}\ d\tau = 0$$

After integration, this is reduced to

$$(\alpha + \Psi_o e^{j\omega t})\, I_o - k^2 h_o I_1 e^{j\omega t}$$

$$= \left(\frac{\sigma}{1 + \sigma} + 2F_c^{-2}\right)\left[\frac{1}{1 + a} + \sqrt{\frac{a\ell_1}{(1 + a)(\ell_1 + a)}}\, Z\,(\beta, k_2)\right] K(k_2) \tag{27}$$

$$+ \left(\frac{\sigma}{1 + \sigma} - 2F_c^{-2}\right) K(k_3)\; Z\,(\beta,\, k_3)$$

where

$$I_o = \pi\left(1 - \sqrt{\frac{a}{1 + a}}\right) \tag{28}$$

$$I_1 = \frac{b}{\pi a} \int_0^1 \left[\frac{\tau}{\tau + a} + \frac{a}{\tau + a}\, \ell_n \frac{a}{a + \tau}\right] \sqrt{\frac{\tau}{1 - \tau}}\, d\tau \tag{29}$$

$Z(\beta,\, k_2)$, $Z(\beta,\, k_3)$ = Jacobian zeta function

$$k_2 = \sqrt{\frac{\ell_1 - 1}{\ell_1}},\quad k_3 = \sqrt{\frac{\ell_2(a + 1)}{a(\ell_2 + 1)}},\quad \beta = \sin^{-1}\sqrt{\frac{a}{1 + a}}$$

The integral I_1 is evaluated in the appendix.

In particular, when the flow is steady and when gravity is neglected, Eq. (27) is simplified to

$$\pi(1 + \sigma)\, \frac{\alpha}{\sigma}\, I_o = K(k_2)/(1 + a) \tag{30}$$

$$+ \sqrt{\frac{a\ell_1}{(1 + a)(\ell_1 + a)}}\; K(k_2)\; Z\,(\beta, k_2) + K(k_3) Z(\beta, k_3)$$

The steady flow cavity length given by this equation is computed for several angles of attack at $b = 1$ and $b = 2.52$. The result is shown in Fig. 7. Some unpublished experimental data obtained at the St. Anthony Falls Hydraulic Laboratory for an aspect ratio of 4 are compared with the present two-dimensional theory, in Fig. 8. A slight difference between the theory and data may be due to the effect of finite span. It should be noted that the measured cavity length was divided by a factor 2 on the assumption that the cavity length defined in the present theory is about $\frac{1}{2}$ of the cavity length by the usual definition. This is done because the tail of the cavity by the present definition corresponds to the point of maximum thickness whereas, by the usual definition, it corresponds to the point of minimum thickness. Eq. (30) also indicates that when $b = 0$ the cavitation number must be also zero. This result is to be expected since when submergence is zero the cavity is open to the atmosphere and $P_c = P_o$.

B. The Frequency Response Function, Ψ_o

For harmonic oscillations, it is possible to write

$$v(x,y,t) = \bar{v}(x,y) + \tilde{v}(x,y)\, e^{j\omega t} \tag{31}$$

$$\Psi(x,y,t) = \bar{\Psi}(x,y) + \tilde{\Psi}(x,y)\, e^{j\omega t} \tag{32}$$

and the 4th boundary condition is reduced to [6]

$$\tilde{v}(x,y) = -e^{-jkx} \int_{-\infty}^{x} e^{jk\tau} \frac{\partial \tilde{\Psi}}{\partial \tau} d\tau \tag{33}$$

Here the condition $v = \Psi = 0$ at $x = -\infty$ is incorporated in the above equation. Equation (33) is applicable to any point on the foil and independent of the path of integration.

Integrating Eq. (33) by parts and setting the end point at the origin, it follows that

$$\tilde{v}(0^+, 0) = - \tilde{\Psi}(0^+,0) + jk \int_{-\infty}^{0^+} e^{jkx} \; \tilde{\Psi} \, dx \tag{34}$$

where

$$\bar{v}(0^+, 0) = - jkh_o \tag{35}$$

$$\tilde{\Psi}(0^+, 0) = \Psi_o \tag{36}$$

To obtain the function $\tilde{\Psi}$, it is necessary to separate the steady and the unsteady parts of the complex acceleration potential. This is a difficult task because the cavity length appears in a non-linear way in Eq. (11). However, for a long cavity and a small amplitude oscillation, the contribution of the last two integrals, in Eq. (11) to $\tilde{\Psi}$ is very small. In fact, their order of magnitude may be estimated. If $\bar{\ell}$ is the average cavity length and $\Delta\ell$ is the amplitude of the cavity length change, then the contribution of the last two terms to $\tilde{\Psi}$ is of order of

$$\left[\frac{\sigma}{2(1 + \sigma)} + F_c^{-2}\right] \frac{\Delta\ell}{\ell}$$

Since $\bar{\ell}$ is of order of $1/\sigma^2$, if $\Delta\ell$ is of order of one, then the contribution is of order of

$$\left[\frac{\sigma}{2(1 + \sigma)} + F_c^{-2}\right] \sigma^2$$

which is very small for long cavities. When there is no gravity this is only of order of σ^3 and, certainly, the last two terms are negligibly small. Taking the first term only, it follows that

$$\tilde{\Psi} = \frac{1}{\pi} \, \mathrm{Im} \left\{ \sqrt{\frac{1 + \zeta}{-\zeta}} \int_{-1}^{0} [\Psi_o - k^2 h_o x(\tau)] \sqrt{\frac{-\tau}{1 + \tau}} \, \frac{d\tau}{\tau - \zeta} \right\} \tag{37}$$

Substituting Eqs. (35), (36) and (37) into Eq. (34) and solving for Ψ_o, the following is obtained:

$$\Psi_o = jkh_o\, W(k, b) \tag{38}$$

where

$$W(k, b) = \frac{1 + jkJ_2}{1 + jkJ_1} \tag{39}$$

$$J_1 = -\frac{1}{\pi} \int_0^{\infty} e^{-jkx}\, \mathrm{Im}\left\{ \sqrt{\frac{1+\zeta}{-\zeta}} \int_{-1}^{0} \sqrt{\frac{-\tau}{1+\tau}}\, \frac{d\tau}{\tau - \zeta} \right\} dx \tag{40}$$

$$J_2 = \frac{jk}{\pi} \int_0^{\infty} e^{-jkx}\, \mathrm{Im}\left\{ \sqrt{\frac{1+\zeta}{-\zeta}} \int_{-1}^{0} x(\tau) \sqrt{\frac{-\tau}{1+\tau}}\, \frac{d\tau}{\tau - \zeta} \right\} dx \tag{41}$$

To compute J_1 and J_2, it is convenient to consider the cases for $b = 0$ and $b > 0$ separately.

1. Zero Submergence

When $b = 0$ the transformation formula degenerates to

$$z = -\zeta \tag{42}$$

If the negative x-axis (positive ξ-axis), is chosen as the path of the integration, Eqs. (40) and (41) may readily be integrated to yield

$$J_1 = -\int_0^{\infty} e^{-jkx}\left(\sqrt{\frac{1+x}{x}} - 1\right) dx \tag{43}$$

$$= \exp(\tfrac{1}{2}jk)[K_1(\tfrac{1}{2}jk) + K_o(\tfrac{1}{2}jk)] - \frac{1}{jk}$$

and

$$J_2 = -jk \int_0^{\infty} e^{-jkx}\left[\frac{1}{2}\sqrt{\frac{1+x}{x}} - \sqrt{x(1+x)} + x\right] dx \tag{44}$$

$$= -\frac{jk}{4}\exp(\tfrac{1}{2}jk)[K_1(\tfrac{1}{2}jk) + K_o(\tfrac{1}{2}jk)] + \frac{1}{2}\exp(\tfrac{1}{2}jk)K_1(\tfrac{1}{2}jk) - \frac{1}{jk}$$

where $K_1(\frac{1}{2}jk)$ and $K_o(\frac{1}{2}jk)$ are modified Bessel functions [17]. Finally, Eq. (39) and Eq. (38) are reduced to

$$W(k,0) = -\tfrac{1}{2}jk + C(\tfrac{1}{2}k) \tag{45}$$

and

$$\Psi_o = \tfrac{1}{2}k^2 h_o + jkh_o C(\tfrac{1}{2}k) \tag{46}$$

where $C(\frac{1}{2}k)$ is the well-known Theodorsen's function [8].

2. Non-Zero Submergence

In this case, it seems most convenient to choose a curved path on the z-plane so that it corresponds to the positive η-axis on the ζ-plane. When ζ is replaced by $i\eta$, Eq. (40) may be integrated once and yields

$$J_1 = \int_0^{\infty} e^{-jkx}\left(\sqrt{\frac{r+1}{2}} - 1\right) dx \tag{40a}$$

where

$$r = \sqrt{1+\eta^2}/\eta$$

Equation (41) may also be integrated once and yields

$$J_2 = \frac{jkb}{\pi a} \int_0^{\infty} e^{-jkx} \left\{ \eta \sqrt{\frac{r-1}{2}} \left(1 + \sqrt{\frac{r+1}{2}} - \frac{1}{r} \sqrt{\frac{r+1}{2}} \right) \right.$$

$$- \frac{1}{2}\sqrt{\frac{r+1}{2}} + 2a \sqrt{\frac{r+1}{2}}\, \ell n \frac{\sqrt{a} + \sqrt{a+1}}{2\sqrt{a}} \tag{41a}$$

$$\left. + a \ell n \frac{[1 + r + \sqrt{2(1+r)}]\sqrt{1 + a^2(r^2 - 1)}}{[a + 1 + \sqrt{2a(a+1)(r+1)} + ar]\sqrt{r^2 - 1}} - \frac{a}{2}\, \ell n \left(1 + \frac{\eta^2}{a^2}\right) \right\} dx$$

For the infinite submergence case we have $a \longrightarrow \infty$ and Eq. (39) is reduced to

$$W(k, \infty) = W_1(k) - \frac{3}{8} jk \tag{47}$$

where $W_1(k)$ is the frequency response function obtained by Parkin [6]. The numerical value of $W_1(k)$ was computed by Parkin by means of a series expansion. The frequency response function, $W(k, b)$, is plotted in Fig. 9 for special cases of $b = 0$ and $b = \infty$.

C. Force and Moment

The force on the plate per unit span is

$$F = \int_0^1 (p - P_c)\, dx \tag{48}$$

The total force coefficient defined as

$$C_F = \frac{F}{\frac{1}{2}\rho\, U^2} = 2(1+\sigma) \int_0^1 \frac{p - P_c}{\rho q_c^2}\, dx$$

may be written in terms of the acceleration potential in the following form:

$$C_F = \sigma + \frac{2gS}{U^2} - 2(1+\sigma) \int_0^1 \phi\, dx \tag{49}$$

where

$$S = \int_0^1 y dx$$

is the area between the plate and the x-axis. For a small angle of attack S is very small and the second term in Eq. (49) may be neglected.

In a similar manner, the moment coefficient with respect to the leading edge (nose down moment as positive) may be written as

$$C_M = \frac{\sigma}{2} + \frac{2gS\bar{x}}{U^2} - 2(1 + \sigma) \int_0^1 x\phi dx \tag{50}$$

where $\bar{x}$ is the moment arm of S with respect to the leading edge.

After performing the necessary substitutions and integrations, the following formula for the force coefficient is obtained:

$$C_F = C_F(0)(1 + \sigma) + (1 - L_3 - L_4)\sigma + 2(L_4 - L_3)(1 + \sigma)\ F_c^{-2} + \frac{2gS}{U} \tag{51}$$

where

$$C_F\ (0) = B_F C_F(S) + C_F(k) \tag{52}$$

$$B_F = \frac{4b}{\pi a} \left[\frac{1}{2} + a - \sqrt{a(a + 1)} \right] \tag{53}$$

$$C_F(S) = \frac{\pi}{2} \alpha \tag{54}$$

$$C_F(k) = (\tfrac{\pi}{2} B_F \Psi_o + h_o I_2 k^2)\ e^{j\omega t} \tag{55}$$

$$I_2 = \frac{2b^2}{\pi a^2} \left\{ \frac{1}{2}\left[\sqrt{a(a+1)} - a\right] - \frac{3}{8} + a\left[\sqrt{a(a+1)} + \frac{1}{2}\right] \right.$$

$$\left. \ell n \frac{4a}{2a + 1 + 2\sqrt{a(a+1)}} + \frac{a^2}{2}\, \ell n \frac{4(a+1)}{2a + 1 + 2\sqrt{a(a+1)}} \right\} \tag{56}$$

$$L_3 = \frac{b}{\pi a} \left[\sqrt{\ell_1(\ell_1 - 1)} + \ln\left(\sqrt{\ell_1} + \sqrt{\ell_1 - 1}\right) \right.$$

$$-2\sqrt{a(a+1)}\, \ln\left(\sqrt{\ell_1} + \sqrt{\ell_1 - 1}\right) \tag{57}$$

$$\left. + 2a\ell n \frac{\sqrt{\ell_1(a+1)} + \sqrt{a(\ell_1 - 1)}}{\sqrt{\ell_1 + a}} - \ell_1 + 1 + a\ell n \frac{\ell_1 + a}{1 + a} \right]$$

$$L_4 = \frac{b}{\pi a} \left[\sqrt{\ell_2(\ell_2 + 1)} - \ell n\left(\sqrt{\ell_2} + \sqrt{\ell_2 + 1}\right) \right.$$

$$+ 2\sqrt{a(a+1)}\, \ell n\left(\sqrt{\ell_2} + \sqrt{\ell_2 + 1}\right) \tag{58}$$

$$\left. - 2a\ell n \frac{\sqrt{\ell_2\,(a+1)} + \sqrt{a(\ell_2 + 1)}}{\sqrt{a - \ell_2}} - \ell_2 + a\ell n \frac{a}{a - \ell_2} \right]$$

It should be noted that the last two terms in Eq. (51) are the gravity force correction terms which are negligible at normal operating conditions of a supercavitating hydrofoil. The second term is a correction term due to finite cavity length. When $b \rightarrow \infty$, it may be shown easily that the second term is of order of σ^2. When $b = 0$, then, $\sigma = 0$ and the second term is identical to zero. The following additional comments with regard to Eq. (51) may facilitate further understanding:

(1) The force coefficient at zero cavitation number is the sum of the steady term, $B_F C_F(s)$ and the unsteady term. The steady term is obtained by multiplying the infinite fluid term by the submergence correction factor B_F. The submergence correction factor is plotted in Fig. 10. It is seen that the limiting values of B_F for the infinite fluid case and the zero submergence case are 1 and 2, respectively.

(2) The force coefficient at small cavitation numbers are approximately equal to the force coefficient at zero cavitation number multiplied by the factor $(1 + \sigma)$. This is a familiar steady flow result from linearized theories.

(3) For the infinite fluid, gravity-free and zero cavitation number case, it may be readily calculated that

$$C_F = \frac{\pi}{2}\alpha + \frac{\pi}{2}\left[jkW_1(k, \infty) - \frac{9}{16}k^2\right] h_o e^{j\omega t} \tag{59}$$

This formula agrees with Parkin's results [6].

(4) For the zero submergence, gravity-free case, we have

$$C_F = \pi\alpha + \pi\left[jkC\left(\frac{1}{2}k\right) - \frac{1}{4}k^2\right] h_o e^{j\omega t} \tag{60}$$

It is noted that Eq. (60) is exactly one half of the lift coefficient for the fully wetted plate in an infinite fluid.

The moment coefficient is

$$C_M = C_M(0)(1 + \sigma) + (\tfrac{1}{2} - L_5 - I_6)\,\sigma + 2\,(L_6 - L_5)\,(1 + \sigma)\,F_c^{-2} + \frac{2gS\bar{X}}{U^2} \tag{61}$$

where

$$C_M(0) = B_M C_M(S) + C_M(k) \tag{62}$$

$$B_M = \frac{32b^2}{5\pi^2 a^2}\left[\frac{1}{4} - a(2a + 1)\ \ell n\ \frac{2a + 1 + 2\ a(a + 1)}{4a} + 2a\sqrt{a(a + 1)}\ \ell n\ \frac{4(a + 1)}{2a + 1 + 2\sqrt{a(a + 1)}}\right] \tag{63}$$

$$C_M(S) = \frac{5\pi}{32}\,\alpha \tag{64}$$

$$C_M(k) = \left(\frac{5\pi}{32}\,B_M\,\Psi_o + h_o I_4 k^2\right) e^{j\omega t} \tag{65}$$

$$I_4 = \frac{b^3}{4\pi^2 a^3}\left\{1 - 6a + 8a^3 - 4a(2a - 1)\sqrt{a(a + 1)} + 8a(a + 1)\ \ell n\ \frac{4a}{2a + 1 + 2\sqrt{a(a + 1)}} + 8a^2(a + 1)\left[\ell n\frac{4a}{4a + 1 + 2\sqrt{a(a + 1)}} + \sqrt{\frac{a}{a + 1}}\ \ell n\frac{4(a + 1)}{2a + 1 + 2\sqrt{a(a + 1)}}\right]^2\right\} \tag{66}$$

$$L_5 = \frac{2b^2}{\pi^2 a^2}\left[\frac{1}{8} - a(a + \frac{1}{2})\ell n\ \frac{2a + 1 + 2\sqrt{a(a + 1)}}{4a}\right.$$

$$\left. + 4\sqrt{a(a + 1)}\ \ell n\ \frac{4(a + 1)}{2a + 1 + 2\sqrt{a(a + 1)}}\right] \quad \ell n\left(\sqrt{\ell_1} + \sqrt{\ell_1 - 1}\right) \tag{67}$$

$$L_6 = \frac{b^2}{\pi^3 a^2}\int_0^1 [\xi + a\ell n\ (\frac{a}{a + \zeta})]\ \frac{\xi}{a + \xi}\sqrt{\frac{1 - \xi}{\xi}}\int_0^{\ell_2}\sqrt{\frac{\tau}{1 + \tau}}\ \frac{d\tau}{\tau + \xi}\ d\xi \tag{68}$$

Here the submergence correction factor, B_M, has the limiting values of 1 and $\frac{8}{5}$ for the special cases of infinite submergence and zero submergence respectively. The submergence correction factor, B_M, is plotted together with B_F in Fig. 10.

V. FLAT PLATE WITH PITCHING OSCILLATION

When the hydrofoil is performing a rotational oscillation of amphitude α_o about the leading edge, the equation for the wetted surface is

$$Y = -\alpha x - \alpha_o x e^{j\omega t} \tag{15a}$$

and the boundary condition on the wetted surface is

$$\Psi_1 = \alpha + (\Psi_o + 2jk\Psi_o x - \tfrac{1}{2}k^2\alpha_o x^2)e^{j\omega t} \tag{17a}$$

All other boundary conditions remain the same as those for the heaving case discussed in the last section. However, the computation is somewhat complicated due to the additional term involving x^2 in Eq. (17a). The general procedure of solving this problem is identical to that used in the last section.

A. Cavity Length

For the calculation of cavity length it is required to replace the left hand side of Eq. (20) with

$$\alpha + (\Psi_o + \tfrac{5}{4} jk\alpha_o - \frac{63}{256} k^2 \alpha_o) e^{j\omega t} \tag{69}$$

The left hand side of Eq. (27) should be replaced by

$$[\alpha + (\alpha_o + \Psi_o) e^{j\omega t}]\, I_o + 2jk\alpha_o I_1 e^{j\omega t} - \tfrac{1}{2} k^2 \alpha_o I_3 e^{j\omega t} \tag{70}$$

where

$$I_3 = (\frac{b}{\pi a})^2 \int_0^1 (\tau + a) \left[\frac{\tau}{\tau + a} + \frac{a}{\tau + a} \ln (\frac{a}{a + \tau})\right]^2 \sqrt{\frac{\tau}{1 - \tau}}\, d\tau \tag{71}$$

B. The Frequency Response Function, Ψ_o

The constant Ψ_o is given by the following expression:

$$\Psi_o = \alpha_o \frac{1 + 2jkJ_2 - \frac{1}{2} k^2 J_3}{1 + jkJ_1} \tag{72}$$

where

$$J_3 = \frac{jk}{\pi} \int_0^\infty e^{-jkx} I_m \left\{ \sqrt{\frac{1 + \zeta}{-\zeta}} \int_{-1}^0 x^2(\tau) \sqrt{\frac{-\tau}{1 + \tau}} \frac{d\tau}{\tau - \zeta} \right\} dx \tag{73}$$

C. Force and Moment

The total force coefficient is also given by Eq. (51) and the subsequent equations. However, Eq. (55) should be replaced by the following expression:

$$C_F(k) = \left(\frac{\pi}{2}B_F\Psi_o + j\frac{4b}{\pi a}\alpha_o I_2 k - \frac{b}{\pi a}\alpha_o I_5 k^2\right)e^{j\omega t} \tag{55a}$$

where

$$I_5 = \left(\frac{b}{\pi a}\right)^2 \int_0^1 \left[\tau + a\ln\left(\frac{a}{a+\tau}\right)\right]^2 \left[\sqrt{\frac{a(a+1)}{\tau + a}} - 1\right] \sqrt{\frac{\tau}{1-\tau}}\,d\tau \tag{74}$$

The moment coefficient is given by Eq. (61) and the subsequent equations. Here also Eq. (65) should be replaced by

$$C_M(k) = \left(\frac{5\pi}{32}B_M\Psi_o - 2jk\alpha_o I_4 + \frac{1}{2}k^2\alpha_o I_6\right)e^{j\omega t} \tag{65a}$$

where

$$I_6 = \frac{b^3}{\pi^4 a^3} \int_0^1 \left[\xi + a\ln\left(\frac{a}{a+\xi}\right)\right] \frac{\xi}{a+\xi} \sqrt{\frac{1-\xi}{\xi}} \int_0^1 \left[\tau + a\ln\left(\frac{a}{a+\tau}\right)\right]^2 \sqrt{\frac{\tau}{1-\tau}}\,\frac{d\tau}{\tau-\xi}\,d\xi \tag{75}$$

in the calculation of $C_M(0)$ by Eq. (62).

VI. ALTERNATE SOLUTION OF HEAVING MOTION IN INFINITE FLUID

This problem has been considered by several investigators, each using different method and different boundary conditions. One of the difficulties was pointed out by Wu [18] that if the cavity volume was allowed to change, the pressure at infinity would be unbounded. Recently, Wang and Wu [19] investigated the small-time behavior of an accelerating two-dimensional cavity in an incompressible infinite fluid. The cavity volume was allowed to change by admitting a source and a sink at infinity. In reality, there are three physical conditions which permit a variable cavity volume without allowing the pressure at infinity to be unbounded: 1) the existence of a free surface, 2) the effect of finite span and 3) the effect of compressibility. The first case is the main concern of the present paper and was discussed in detail in the previous sections. The second case may become important at a deep submergence (when submergence is much larger than the span). The effect of compressibility may be important only when both the submergence and the span are much larger than the wave length of the pressure wave emitted by the hydrofoil. The last two cases are beyond the scope of the present paper.

It is also pertinent to note that in all the above cases the cavity volume change is the result, rather than the condition, of the solutions. Moreover, the so-called cavity volume is not a real cavity, but rather, a space in which both gas and liquid exist. It appears that not only the "cavity volume" may change with time, but also the ratio of liquid to gas in the cavity may change with time.

In any event, Wang and Wu [19] have shown that the source and sink at infinity have little effect on the force acting on the body when the cavity is long.

The purpose of this section is to gain a better insight into the effect of the wake boundary condition imposed in previous sections. Since the infinite flow field is simply connected, there is no need to have a cut at the wake. The only assumption required at the wake is that the pressure must be continuous across the end of the cavity. The flow is assumed to be truly two-dimensional, incompressible, infinite and gravitation-free. The pressure at infinity is assumed to be bounded.

The following transformation formula transforms the linearized z-plane on to the upper half of the ζ-plane as shown in Fig. 11:

$$z = \frac{\ell\zeta^2}{\zeta^2 + \ell} \tag{76}$$

This formula transforms the end of the cavity to infinity, the point at infinity to $(0, i\sqrt{\ell})$ and the trailing edge of the plate to $(-x_o, 0)$, where

$$x_o = \sqrt{\frac{\ell}{\ell - 1}} \tag{77}$$

In this case, it is more convenient to redefine the acceleration potential to be

$$\phi = \frac{P_c - p}{\rho q_c^2} \tag{78}$$

The boundary condition on the cavity surface will be $\phi = 0$ and the condition at infinity is

$$F(z) = \frac{-\sigma}{2(1 + \sigma)}, \text{ at } \zeta = i\sqrt{\ell} \tag{79}$$

For a heaving oscillation, the boundary condition on the wetted part of the plate is given by Eq. (17).

The complex acceleration potential satisfying the boundary conditions on the ξ-axis and continuous at the end of the cavity $(\zeta = \pm\infty)$ is

$$F(\zeta) = \frac{1}{\pi}\sqrt{\frac{x_o + \zeta}{-\zeta}}\left[\int_{-x_o}^{0} \Psi_1(\tau)\sqrt{\frac{-\tau}{x_o + \tau}}\frac{d\tau}{\tau - \zeta} + A\right] \tag{80}$$

where A is a real constant.

Now the boundary condition at infinity $(\zeta = i\sqrt{\ell})$ may be used in Eq. (80) to obtain the following two equations by which the constant A and the cavity length ℓ may be determined.

$$A = - \int_{-x_o}^{0} \Psi_1(\tau)\sqrt{\frac{-\tau}{x_o + \tau}}\ \frac{\tau d\tau}{\tau^2 + \ell}$$

$$-\sqrt{\ell}\sqrt{\frac{\sqrt{\ell} - \sqrt{\ell - 1}}{\sqrt{\ell} + \sqrt{\ell - 1}}}\ \int_{-x_o}^{0} \Psi_1(\tau)\sqrt{\frac{-\tau}{x_o + \tau}}\ \frac{d\tau}{\tau^2 + \ell} \tag{81}$$

$$\frac{\pi\sigma}{2(1 + \sigma)} = \sqrt{\frac{2\ell^2}{\sqrt{\ell + 1}\ (\sqrt{\ell} + \sqrt{\ell - 1})}}\ \int_{-x_o}^{0} \Psi_1(\tau)\sqrt{\frac{-\tau}{x_o + \tau}}\ \frac{d\tau}{\tau^2 + \ell} \tag{82}$$

After performing the integration in Eq. (82), it is reduced to

$$\frac{\sigma}{2(1 + \sigma)}\left(\sqrt{\ell} + \sqrt{\ell - 1}\right) = \alpha + \Psi_o e^{j\omega t} - k^2 h_o\left[\ell + 2 - \sqrt{\ell(\ell - 1)}\right] e^{j\omega t} \tag{83}$$

When the flow is steady, Eq. (83) is reduced to the following simple equation:

$$\frac{\alpha(1 + \sigma)}{\sigma} = \frac{1}{2}\left(\sqrt{\ell} + \sqrt{\ell - 1}\right) \approx L_1(\ell) \tag{84}$$

To compare this result with the result obtained in Section IV, Eq. (84) is plotted in Fig. 4. Good agreement between the two results is indicated.

In order to compute the constant Ψ_o, the change in cavity length is again assumed to be small so that the cavity length ℓ in Eqs. (80) and (81) may be replaced by its average value. With this assumption, it follows that

$$\tilde{\Psi} = I_m\left\{\frac{1}{\pi}\sqrt{\frac{x_o + \zeta}{-\zeta}}\ \left[\int_{-x_o}^{0}[\Psi_o - k^2 h_o x(\tau)]\sqrt{\frac{-\tau}{x_o + \tau}}\ \frac{d\tau}{\tau - \zeta} + \tilde{A}\right]\right\} \tag{85}$$

$$\tilde{A} = \pi\left(1 - \sqrt{\frac{2}{x_o + 1}}\right)\Psi_o - \pi k^2 h_o\left\{\ell - \ell\sqrt{\frac{2}{x_o + 1}} - \left[2(x_o + 1)\sqrt{\ell - 1}\right]^{-\frac{3}{2}}\right\} \tag{86}$$

$$\Psi_o = jkh_o W'(k,\ell) \tag{87}$$

$$W'(k,\ell) = \frac{1 + jk\, J_2'(k,\ell)}{1 + jk\, J_1'(k,\ell)} \tag{88}$$

$$J_1'(k,\ell) = \int_0^\infty e^{-jkx}\left(\sqrt{\frac{1 + x_o r}{1 + x_o}} - 1\right)dx \tag{89}$$

$$J_2'(k,\ell) = -jk \int_0^\infty e^{-jkx}\left\{x\left[1 - \frac{(1 + r)}{2\sqrt{1 + x_o}}\sqrt{1 + x_o r}\right] + \frac{2x_o + 1}{4(x_o + 1)}\sqrt{\frac{1 + x_o r}{1 + x_o}}\right\}dx \tag{90}$$

$$r = \sqrt{\frac{1 + x}{x}}$$

Comparing Eq. (88) with Eq. (39), it may be readily shown that, when $\ell >> 1$,

$$W'(k,\ell) = W(k,\infty) + O(\sigma^2) \tag{91}$$

indicating the effect of the wake condition on Ψ_o is of order of σ^2.

The total force coefficient is computed and the result is given by the following expression:

$$\overline{C_F} = \frac{\pi}{2}\left(\alpha + \Psi_o e^{j\omega t} - \frac{15}{16}k^2 h_o e^{j\omega t}\right)(1+\sigma)$$

$$+ \frac{\pi}{2}\left\{(\alpha + \Psi_o e^{j\omega t}) - \frac{1}{4}k^2 h_o\left[3\ell + 1 - \frac{3}{4}\left(\sqrt{\ell} + \sqrt{\ell - 1}\right)^2\right] e^{j\omega t}\right\}\frac{(1+\sigma)}{(\sqrt{\ell} + \sqrt{\ell - 1})^2} \tag{92}$$

It is readily seen that the first term of Eq. (92) agrees with the first term of Eq. (51) when $b \to \infty$ and the second term of Eq. (92) is of order of σ^2 for long cavities. It is now clear that the effect of the wake condition on the total force coefficient is, at most, of order of σ^2.

Since the present results are applicable for a super-cavity of arbitrary length, it may be interesting to consider a limiting case of $\ell = 1$. For the cavity length calculation, Eq. (83) is reduced to

$$\frac{\sigma}{2(1+\sigma)} = \alpha + \Psi_o e^{j\omega t} - 3k^2 h_o e^{j\omega t} \tag{83a}$$

when $\ell = 1$, $x_o \to \infty$ and Eqs. (89) and (90) are reduced to

$$J_1'(k,\ell) = \int_0^\infty e^{jkx}\left(\sqrt[4]{\frac{x+1}{x}} - 1\right)dx \tag{89a}$$

$$J_2'(k,\ell) = -jk \int_0^{\infty} e^{-jkx} \left\{ x\left[1 - \frac{1}{2}\left(1 + \sqrt{\frac{x+1}{x}}\right) \sqrt[4]{\frac{x+1}{x}}\right] + \frac{1}{2} \sqrt[4]{\frac{x+1}{x}} \right\} dx \tag{90a}$$

The total force coefficient is

$$C_F(1) = \pi(\alpha + \Psi_o e^{j\omega t} - \tfrac{7}{8}k^2 h_o e^{j\omega t})(1 + \sigma) \tag{92a}$$

$$\approx 2(1 + 2\alpha)\ C_F(\infty)$$

This means the total force coefficient acting on a plate with $\ell = 1$ is approximately twice the total force coefficient with infinite cavity.

VII. CONCLUSIONS

The effect of the presence of a free surface near the suction side of an oscillating and supercavitating flat plate on the force and moment characteristics was analyzed in the main part of the paper. To study the effect of wake conditions the infinite fluid case was discussed further in the last section. In summary, some of the more important results are listed below.

(1) The effect of a free-surface at the suction side of a supercavitating hydrofoil is to shorten the cavity and to increase the total force coefficient. The force coefficient at $\sigma = 0$ is increased as the free surface is approached in proportion to the factor B_F in Fig. 10 for the steady case.

(2) In the unsteady oscillatory case, the cavity length varies with the oscillation as given by Eqs. (20) and (27). The total force coefficient also varies with the oscillation as given by Eq. (51).

(3) Even with the presence of a free-surface and oscillations, the total force coefficient at small cavitation number greater than zero is approximately equal to the corresponding value at zero cavitation number multiplied by a factor $(1 + \sigma)$.

(4) In the limiting case of infinite fluid and infinite cavity, the present theory agrees with the existing theory of Parkin.

(5) In the limiting case of zero submergence, the cavitation number is identical to zero and the total force coefficient is one half the value for fully wetted flow in an infinite fluid for both steady and unsteady flow.

(6) The effect of a gravity field is to shorten the cavity; zero cavitation number implies an infinite cavity only when there is no gravity.

(7) For a long cavity, the effect of the wake conditions studied here is of order of σ^2.

(8) The effect of finite cavity length on the frequency response functions $\Psi_o(k, \ell)$ and $W(k, \ell)$ is given by Eqs. (88), (89) and (90) for the range of cavity length between one and infinity. Although the equations are not much more complicated than those of the infinite cavity case, their numerical computations have not been carried out.

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F I G U R E S

(1 through 11)

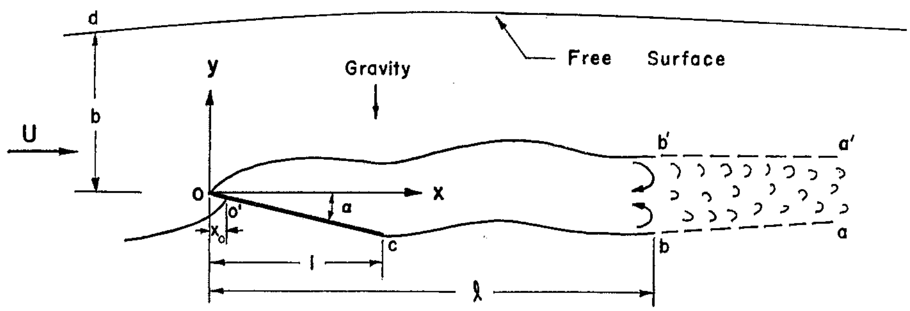
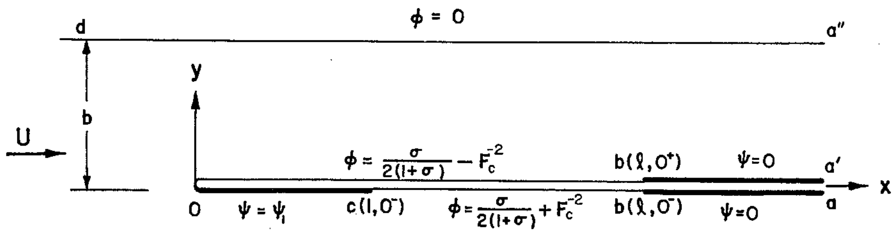
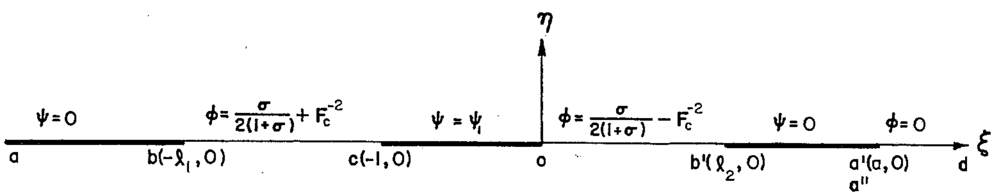


Fig. 1 - Unsteady Cavity Flow over Foil under Free-Surface



a) Z– plane



b) ζ– plane

Fig. 2 - Linearized Flow Configurations

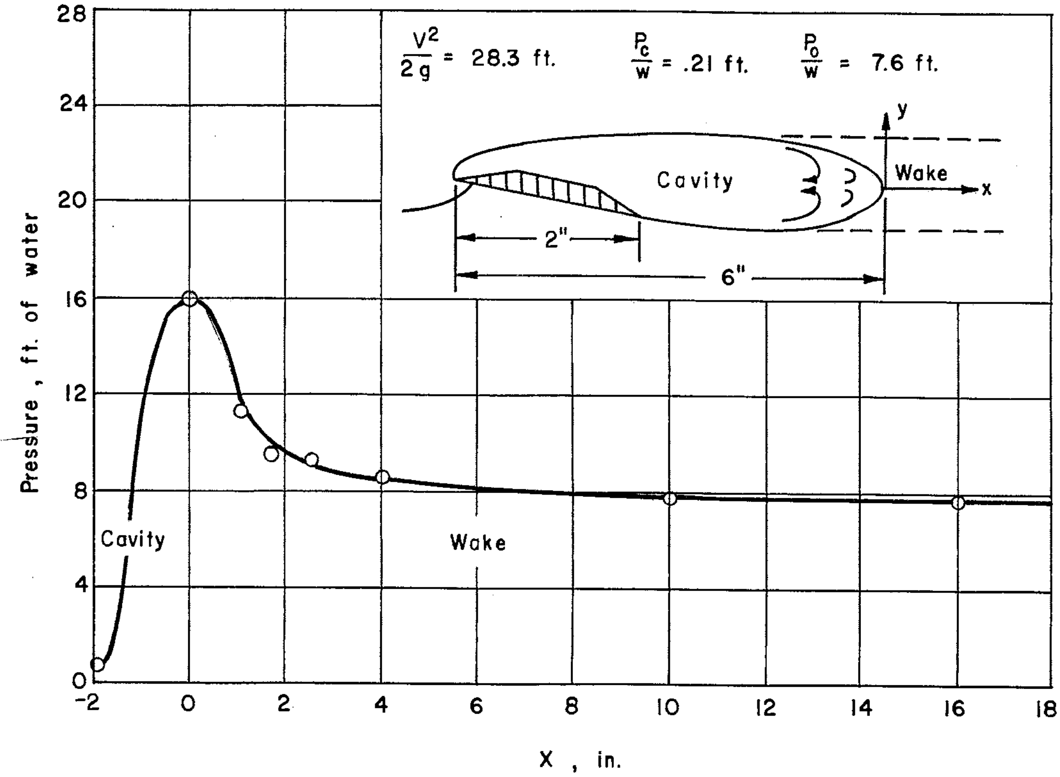


Fig. 3 - Measured Pressure Distribution in a Wake

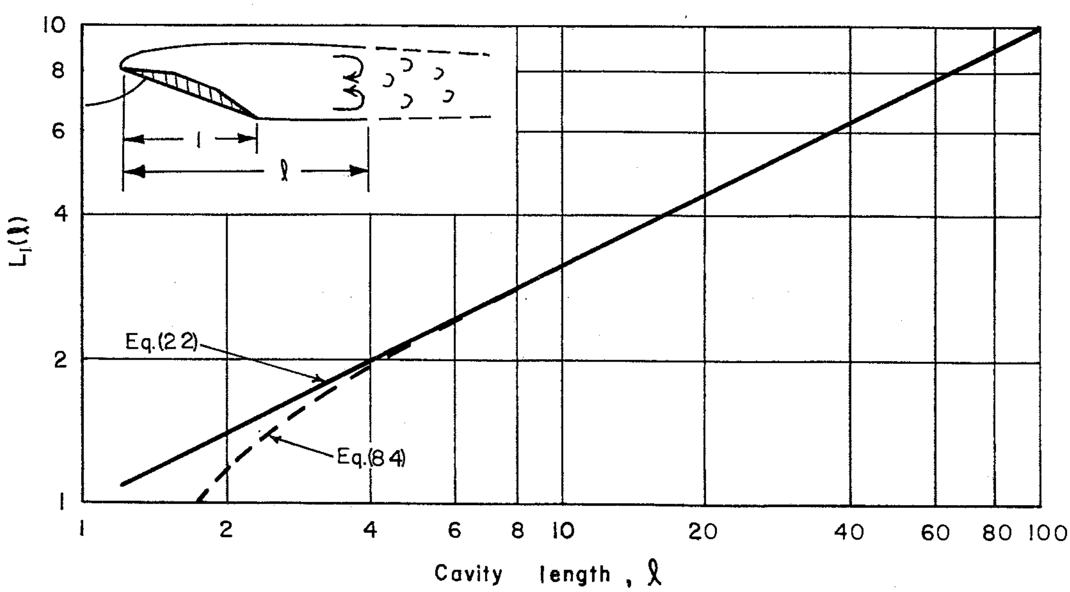


Fig. 4 - Cavity Length in an Infinite Gravitation-Free Field

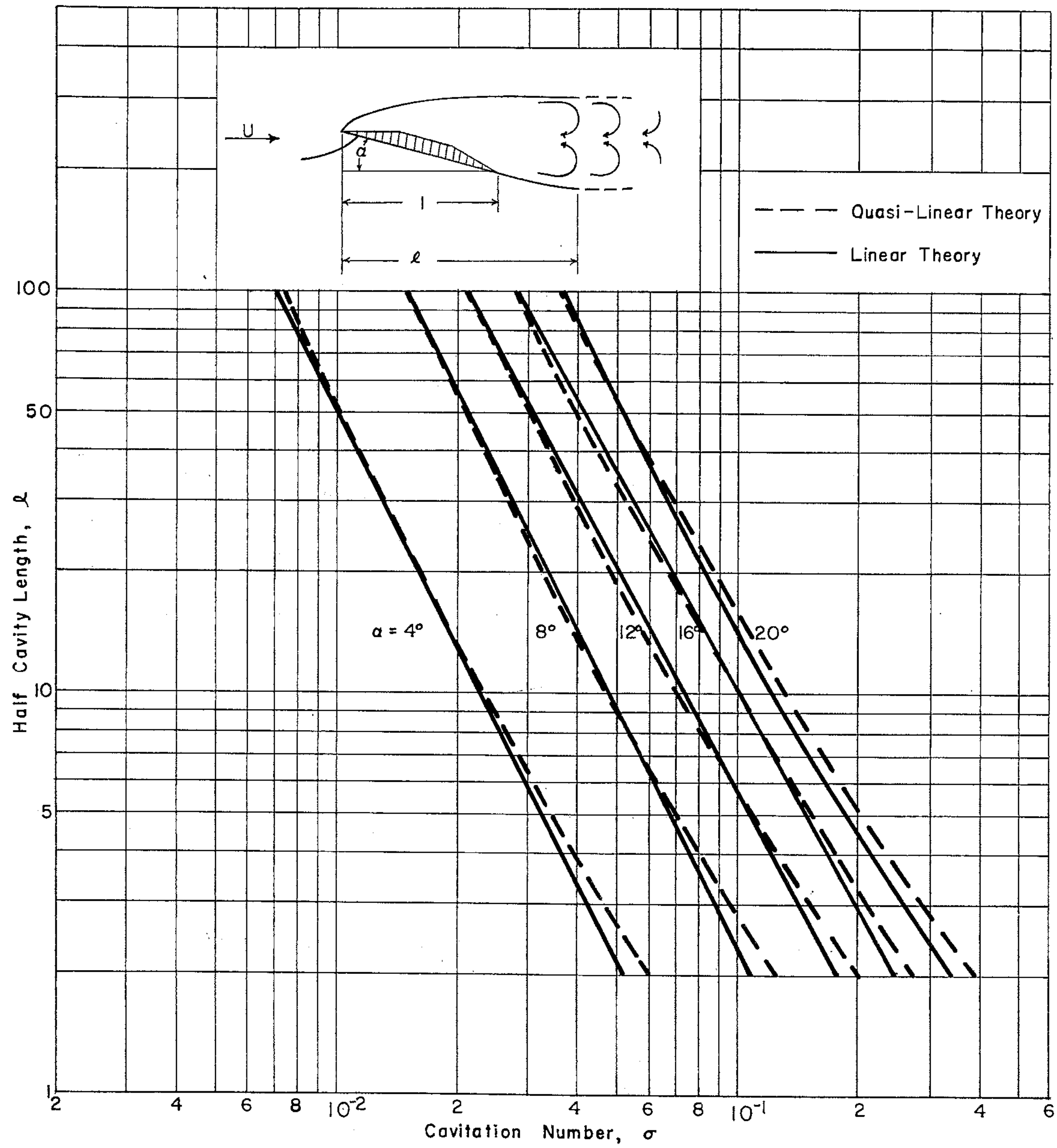


Fig. 5 – Steady Flow Cavity Length in an Infinite Gravitation-Free Field

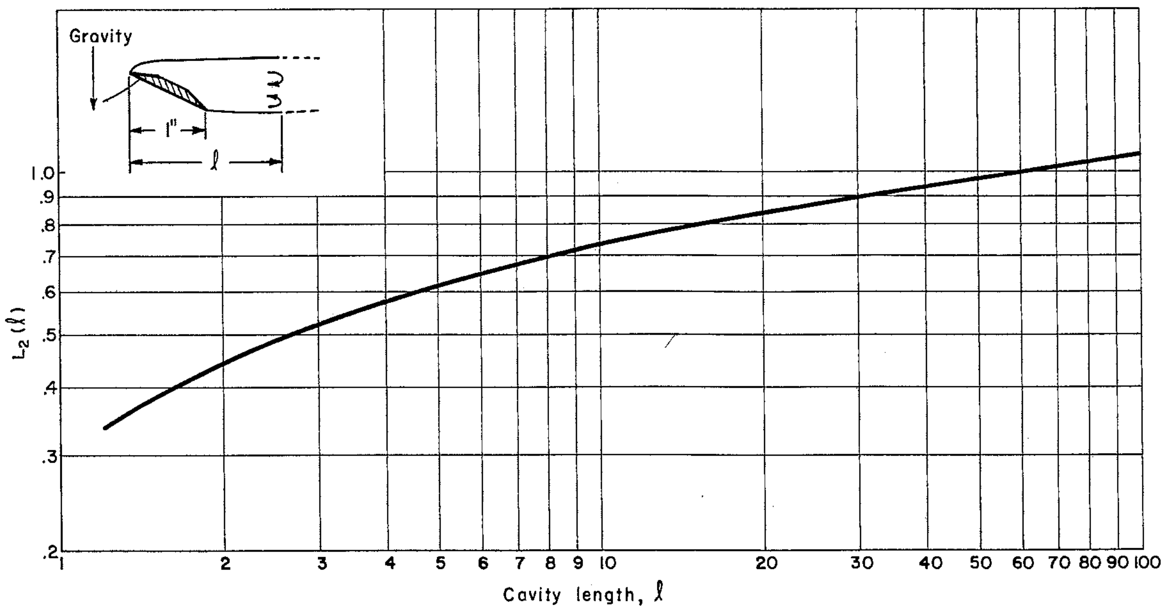


Fig. 6 - Cavity Length in an Infinite Fluid with Zero Cavitiation Number in a Gravitational Field

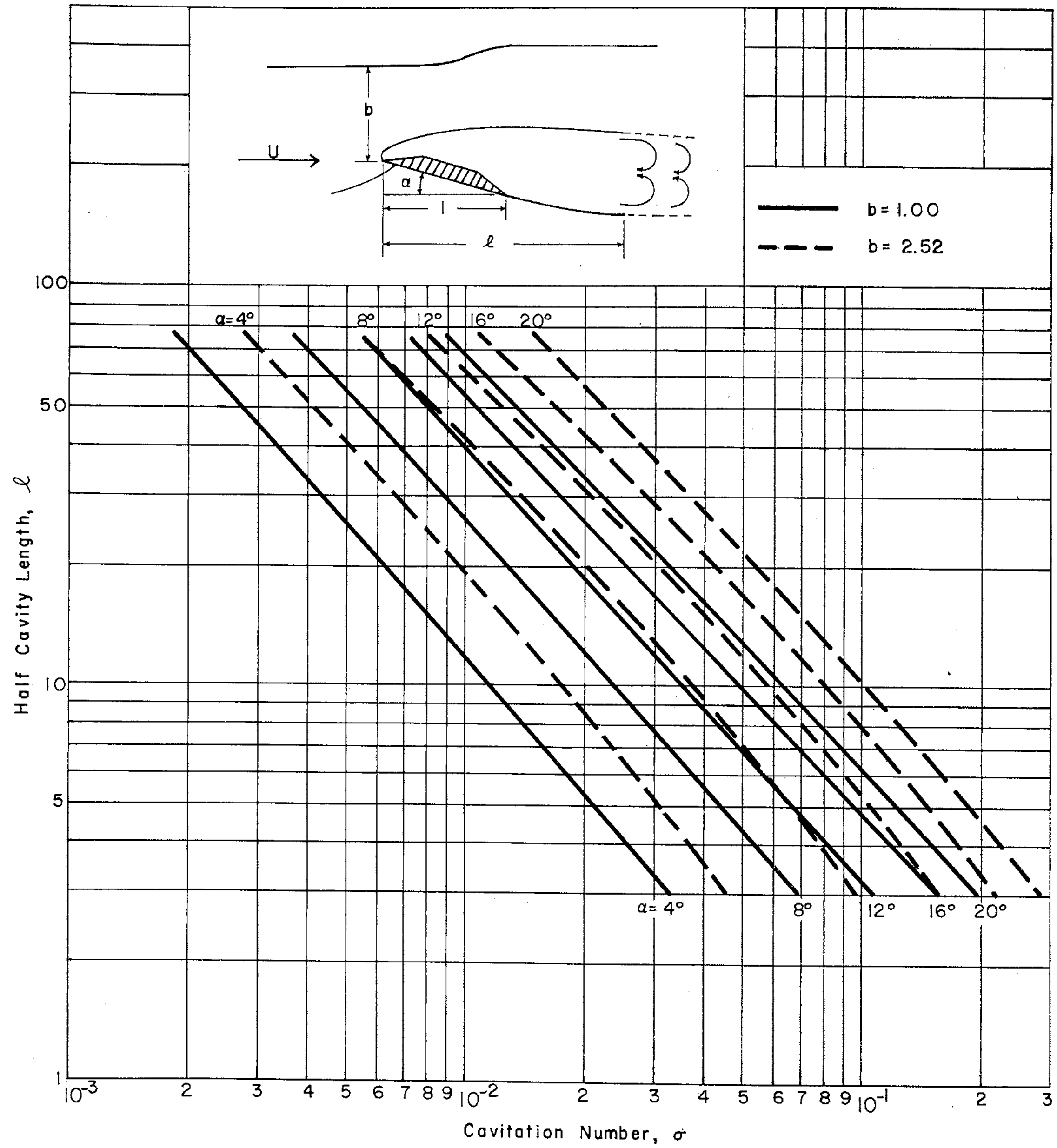


Fig. 7 - Steady Cavity Length under a Free-Surface but without Gravity Force

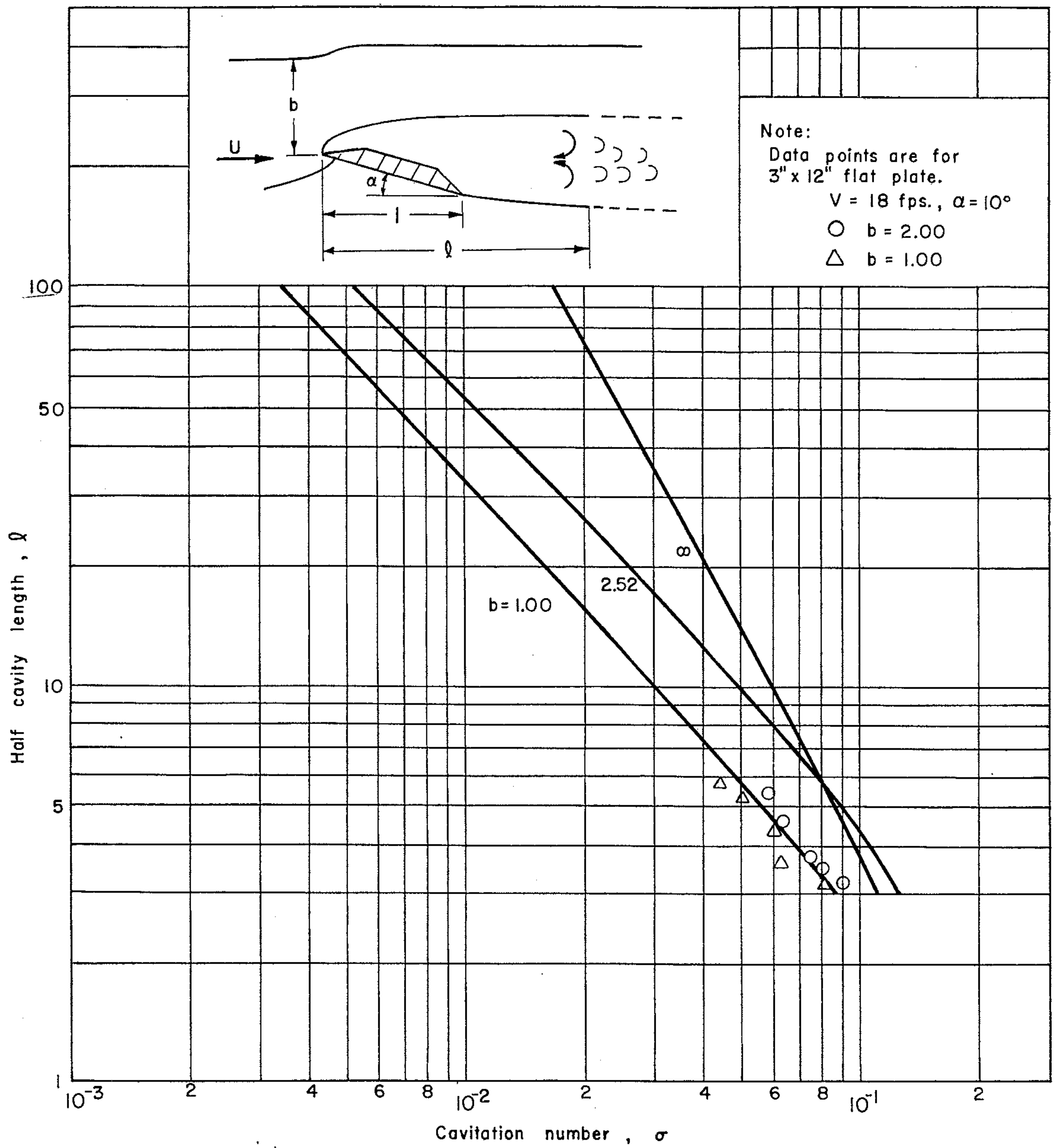


Fig. 8 - Comparison of Theory and Data for Steady Cavity Length at Finite Submergence

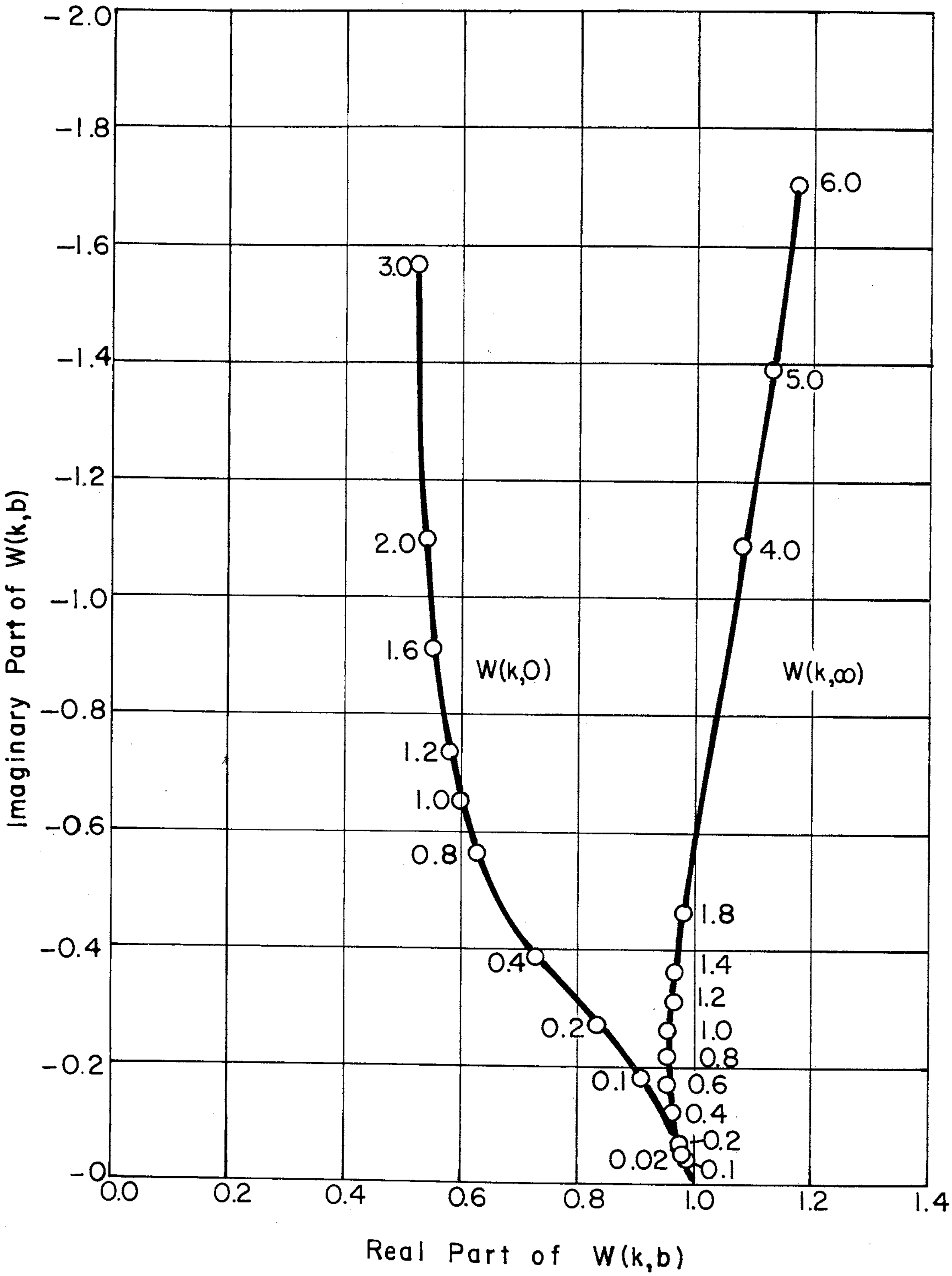


Fig. 9 - Hydrofoil Frequency Response to Harmonic Heaving Oscillation

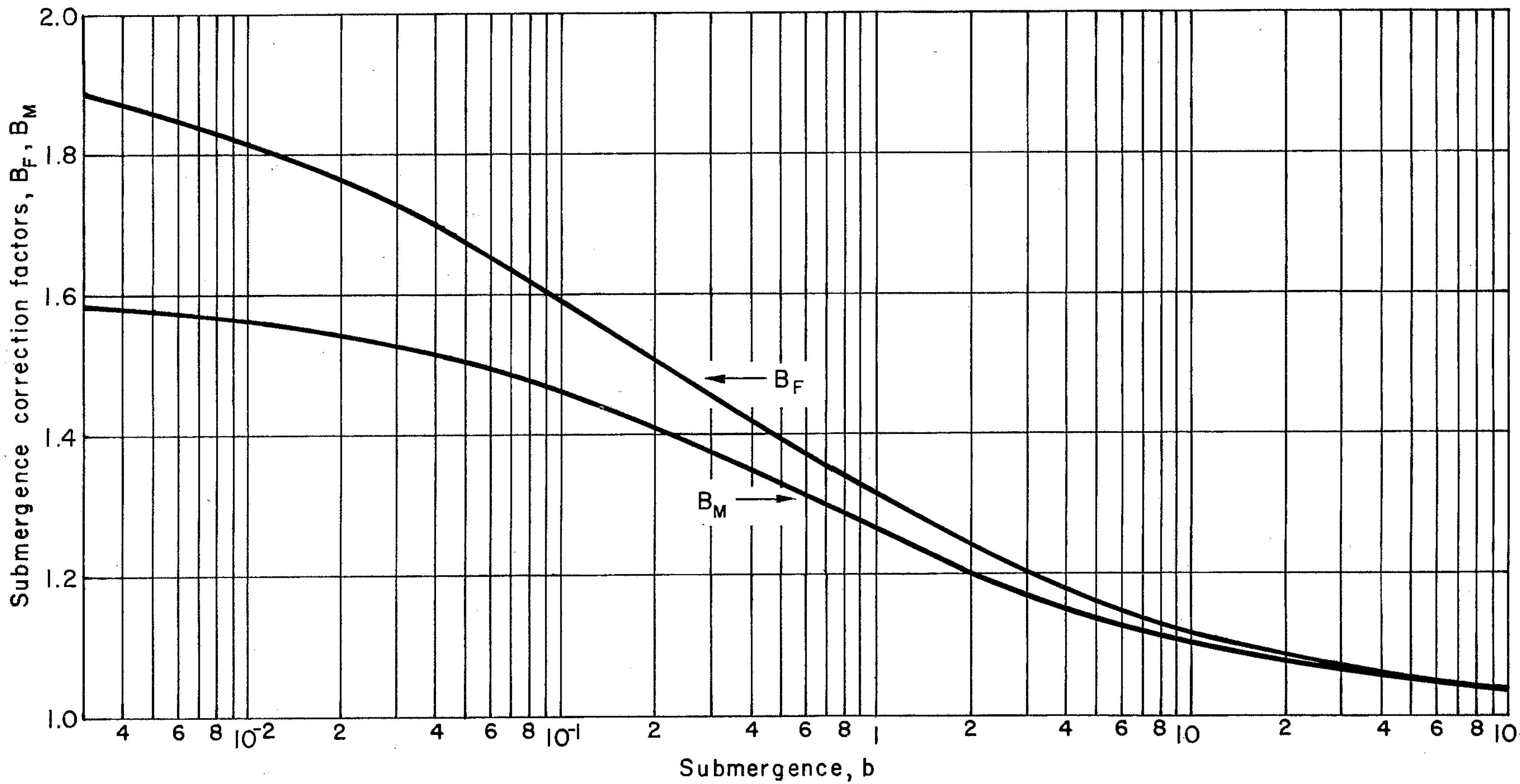


Fig. 10 – Submergence Correction Factor

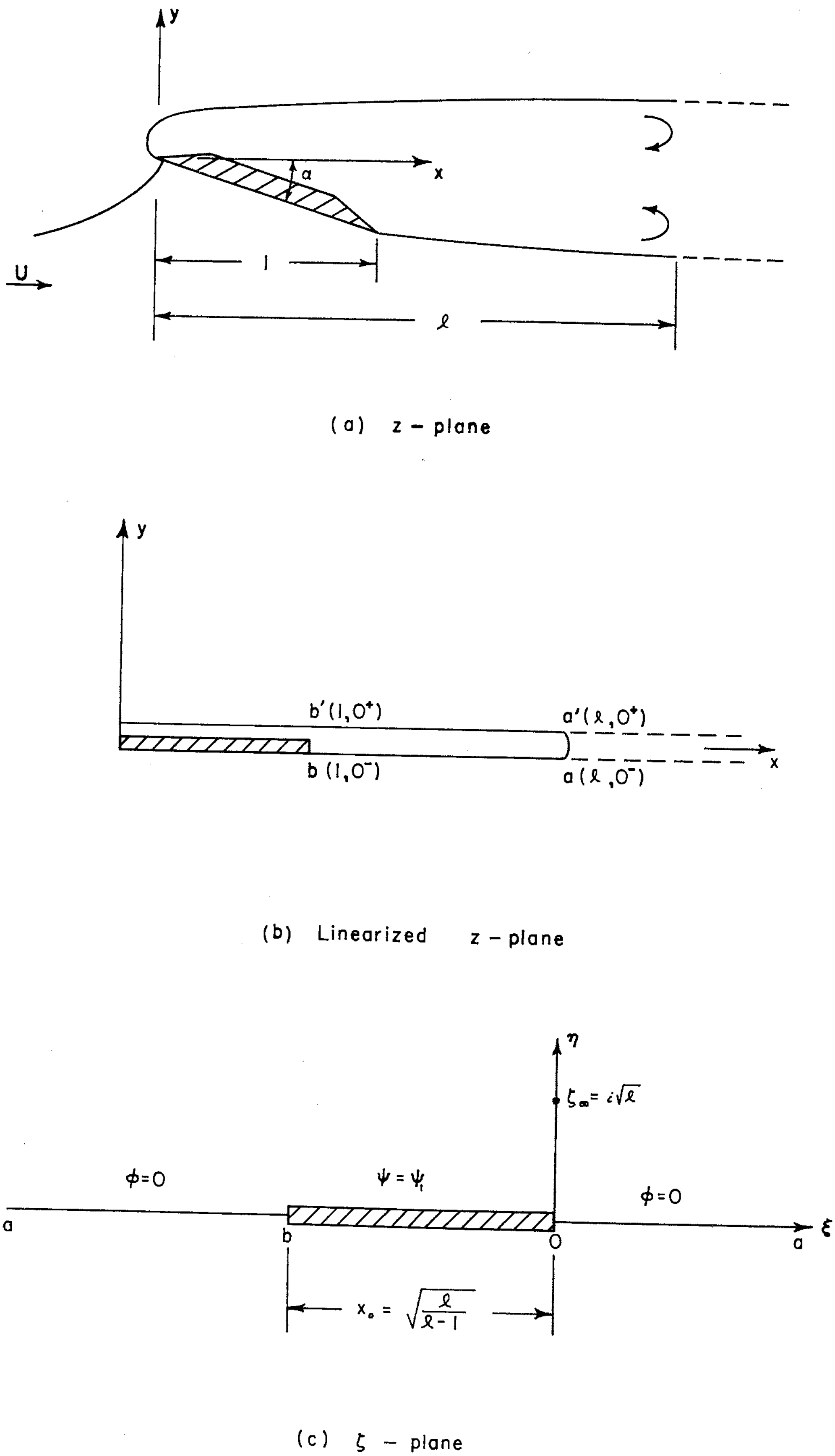


Fig. 11 - Infinite Fluid Flow Configurations

A P P E N D I X

Evaluation of I_1 and Other Related Integrals

APPENDIX

Evaluation of I_1 and Other Related Integrals

In evaluating I_1 and several other related integrals, it is necessary to calculate the following two integrals:

$$Y_o(\lambda) = \int_0^1 \frac{1}{\sqrt{\tau(1-\tau)}} \ell n(1+\lambda\tau)d\tau \tag{A1}$$

and

$$Y_1(\lambda,\beta) = \int_0^1 \frac{1}{(\tau+\beta)\sqrt{\tau(1-\tau)}} \ell n(1+\lambda\tau)d\tau \tag{A2}$$

Although it is possible to use the method of contour integration, the integrals $Y_o(\lambda)$ and $Y_1(\lambda, \beta)$ may be integrated more easily in the following way:

First, we differentiate $Y_o(\lambda)$ with respect to λ and obtain

$$Y_o' = \int_0^1 \frac{\tau}{(1+\lambda\tau)\sqrt{\tau(1-\tau)}} d\tau = \pi\left(\frac{1}{\lambda} - \frac{1}{\lambda\sqrt{1+\lambda}}\right) \tag{A3}$$

since $Y_o(0) = 0$ it is clear that

$$Y_o(\lambda) = \int_0^\lambda Y_o'(x)dx$$

After integration, it follows that

$$Y_o(\lambda) = \pi\ell n\left[\frac{1}{4}(2+\lambda+2\sqrt{\lambda+1})\right] \tag{A4}$$

To evaluate $Y_1(\lambda, \ )$, we first differentiate it with respect to λ, and

$$Y_1' = \frac{\partial Y_1}{\partial \lambda} = \int_0^1 \frac{\tau}{(1 + \lambda\tau)(\tau + \beta)\sqrt{\tau(1 - \tau)}} d\tau = \frac{\pi}{1 - \lambda\beta}\left(\sqrt{\frac{1}{1 + \lambda}} - \sqrt{\frac{\beta}{1 + \beta}}\right) \tag{A5}$$

Again we have $Y_1(0, \ ) = 0$ and, hence

$$Y_1(\lambda,\beta) = \int_0^\lambda Y_1'(x,\beta)dx$$

or

$$Y_1(\lambda,\beta) = \frac{2\pi}{\sqrt{\beta(1 + \beta)}} \ln\left[\frac{\sqrt{\beta(1 + \lambda)} + \sqrt{\beta + 1}}{\sqrt{\beta + 1} + \sqrt{\beta}}\right] \tag{A6}$$

The integral I_1 may be written as

$$I_1 = \frac{b}{\pi a}\int_0^1 \left[\frac{\tau}{\tau + a}\sqrt{\frac{\tau}{1 - \tau}} - \frac{a}{\sqrt{\tau(1 - \tau)}} \ln(1 + \tau/a)\right.$$

$$\left. + \frac{a^2}{(\tau + a)\sqrt{\tau(1 - \tau)}} \ln(1 + \tau/a)\right] d\tau$$

Using Eqs. (A4) and (A6) and setting $\lambda = 1/a$ and $\beta = a$, it is readily obtained that

$$I_1 = \frac{b}{a}\left[\frac{1}{2} - a + a\sqrt{\frac{a}{a+1}} + a\ln\frac{4a}{2a+1+2\sqrt{a(a+1)}}\right.$$

$$\left. + \frac{a^2}{2\sqrt{a(a+1)}}\ln\frac{4(a+1)}{2a+1+2\sqrt{a(a+1)}}\right] \qquad \text{(A7)}$$

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