UNIVERSITY OF MINNESOTA
ST. ANTHONY FALLS HYDRAULIC LABORATORY
LORENZ G. STRAUB, Director

Technical Paper No. 43, Series B

A Quasi-Linear and Linear Theory for Non-Separated and Separated Two-Dimensional, Incompressible, Irrotational Flow about Lifting Bodies

by
C. S. SONG

Prepared for
OFFICE OF NAVAL RESEARCH
Department of the Navy
Washington, D.C.
Contract Nonr 710(24), Task NR 062-052

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ABSTRACT

A general theory is developed for calculating lift, form drag, and moment applicable to thin bodies at small angles of attack without separation or with separation at an arbitrary number of given points. The separated flows are related to fully cavitated and partially cavitated flows by making use of the concept of free streamlines. The closure condition of free-streamline theory is replaced by a boundedness condition. Unique solutions are thereby obtained for a large variety of problems.

The mathematical solution involves a Riemann-Hilbert mixed boundary value problem in an upper-half plane. The general solution for this problem is given in the Appendix and is applied to various kinds of mixed boundary conditions.

The method is exemplified by means of four illustrative calculations. As may be expected when the boundary profile is truly linear, the solution agrees with the classic exact solution.

C O N T E N T S

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

A_j - arbitrary constants

B_j - arbitrary constants

C_D - drag coefficient

C_F - total force coefficient

C_L - lift coefficient

C_M - moment coefficient

F - total force

$H(\zeta)$ - homogeneous solution

$h(\zeta)$ - a function related to homogeneous solution

i - unit imaginary number

j - dummy variable

ℓ - cavity length

ℓ_m - length of the m-th cavity

M - total moment

m - number of cavities

P - pressure at infinity

p - pressure at any point

p_c - pressure in a cavity

$Q(z)$ - an analytic function

q - speed of flow

q_c - reference speed

$\vec{q}$ - total velocity vector

U - speed at infinity

u - horizontal component of perturbation velocity

$\vec{V}$ - perturbation velocity vector

v - vertical component of perturbation velocity

$W(z)=\phi+i\Psi$ - complex velocity potential

x_o - distance between the leading edge and the stagnation point

$z=x+iy$ - complex coordinates

α - angle of attack

$\zeta=\xi+i\eta$ - complex coordinates

θ - direction of flow with respect to x-axis

λ - distance from leading edge to stagnation point in ζ-plane

ρ - density of fluid

σ - cavitation number

τ - variable of integration

ϕ - velocity potential

Ψ - stream function

A QUASI-LINEAR AND LINEAR THEORY FOR NON-SEPARATED AND SEPARATED TWO-DIMENSIONAL, INCOMPRESSIBLE, IRROTATIONAL FLOW ABOUT LIFTING BODIES

I. INTRODUCTION

The flow around a finite, rigid body in an incompressible, infinite fluid at large Reynolds number is characterized by the existence of a thin boundary layer joined to a relatively large potential flow region. The flow within the boundary layer is described by Prandtl's boundary layer equations, and the flow outside of the boundary layer can be treated by potential flow theory. Since the two solutions must be matched at the edge of the boundary layer, such a problem can be completely solved only by an iteration method. Unfortunately, the iteration method is often too complicated to be practical, and the boundary layer is usually disregarded entirely. Thus lifting force, form drag, and moment are usually determined from potential theory alone.

Since potential flow theory admits an arbitrary magnitude of circulation, an additional condition is required to determine a unique solution. This additional condition must be chosen so that the resulting flow pattern resembles that of a real fluid flow, and the essential part of the viscous effect is taken into account. The well-known Kutta-Joukowski condition, the condition of smooth separation with trailing vortex sheet, has been generally accepted as the best device for specifying the circulation for non-separated flow problems. However, this is by no means the only possibility. In the latter part of this paper an alternate method replacing the trailing vortex sheet by a trailing wake (or cavity) of constant pressure will be proposed.

When there is early separation, the situation is much more complicated. The location of the separation point and the pressure distribution in the separated flow region depend on viscosity as well as on other factors. Although the determination of the separation point is an important problem, it is not considered in this paper. Rather, point of separation is assumed to be given (by a sharp break in the profile, for example).

To specify the pressure within the separated region, a flow model will be constructed wherein the forward part of the separated region will be delineated by a free streamline, while the rear part will degenerate into a wake. Then there will be constant pressure in the forward part, and this will correspond to a cavity flow if the fluid inside the free streamline is disregarded (or, if it does not exist as in true cavity flows). For the potential flow solution the constant pressure in the cavity region enters as a parameter. For convenience this parameter is chosen as the cavitation number given by Eq. (9), which follows. When the flow is steady and two-dimensional, zero cavitation number implies an infinite cavity with constant pressure equal to the pressure at infinity, and positive cavitation number implies a finite cavity with constant pressure less than that at infinity. Since constant pressure and a closed streamline are incompatible, an additional device is required to "close" the cavity. To this end various models such as the Riabouchinsky model [1],* the re-entrant jet model [2], and the Roshko wake model [3] have been widely used. When Riabouchinsky's image body is replaced by a point, a "singularity model" is obtained. This extreme model has been used by Tulin [4] and others [5, 6] in connection with the linear theory of cavitating hydrofoils. Recently the author [7] has shown that another natural extension of the Riabouchinski model and the Roshko model is the "continuity model" in which the solution is required to be continuous in the cavity and wake. Only the special flow around a symmetrical wedge with one cavity was considered previously. Here the theory is extended to cover the most general case of unsymmetrical flow with an arbitrary number of cavities. It will be shown that a unique potential flow solution exists for a flow with an arbitrary number of cavities if the solution is required to be continuous in the cavities and wakes.

In linear theory it is customary to linearize the velocity potential as well as the boundary shape. In some instances, however, it is neither necessary nor desirable to linearize both the velocity potential and the boundary shape. Since the main difficulty in seeking the exact solution stems from the mapping of an unknown curved contour, very often it is sufficient to linearize the boundary shape only; this may be called a quasi-linear process. It is also possible to partially linearize the velocity potential. General linear and quasi-linear theories will be developed, and the theories will be applied to several well-known problems for the purpose of illustration.

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*Numbers in brackets refer to List of References on page 20.

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II. GENERAL THEORY

A. Complex Velocity and Its Linearization

When flow is two-dimensional, incompressible, and irrotational, the complex velocity potential

$$W(z) = \phi(x, y) + \iota\Psi(x, y) \tag{1}$$

is an analytic function of the complex coordinates

$$z = x + \iota y \tag{2}$$

It follows that a function defined as

$$Q(z) = \ln(-q_c^{-1} \frac{dW}{dz}) = \ln q/q_c - i\theta \tag{3}$$

is also an analytic function. Here

q = speed of flow,

θ = direction of flow,

q_c = a reference speed (usually the speed on the wall of the last cavity.)

A flow problem is thus reduced to a mixed boundary value problem for $Q(z)$, whose real part is known on a free surface and whose imaginary part is given on the solid surface. In particular, if the flow is about a slender body with small angle of attack, the resulting boundary profile consisting of the body, cavity, and wake is also slender. In such a case the boundary condition may be applied on a slot instead of on the actual boundary profile without causing too much error. The resulting solution is called quasi-linear since only the boundary profile is linearized.

In many cases the calculation may be further simplified if Eq. (3) is linearized to

$$Q(z) = u - \iota\theta \tag{4}$$

where u is the x-component of the perturbation velocity defined as

$$\vec{q} = q_c(1 + u, v) \tag{5}$$

When θ is small, Eq. (4) is approximately the same as the usual perturbation velocity

$$\vec{V} = u - \iota v \tag{6}$$

The major difference between Eqs. (4) and (6) occurs near a stagnation point where v and θ are of different order of magnitude; here Eq. (4) is to be preferred. Furthermore, by using Eq. (4) the boundary condition on the solid surface is exactly satisfied.

B. Statement of the Problem

Consider a slender two-dimensional body of known profile placed in a uniform infinite fluid of speed U and pressure P. The body may be fully wetted, or there may be one or more separated (or cavitated) regions. If there are cavities, then the separation points are assumed to be given. It is also assumed that the body-cavity-wake region is thin and the external flow is potential. A schematic drawing of such a flow pattern is shown in Fig. 1(a). Note that the number of cavities, m, may be any integer including zero, and the last cavity may end on the body or beyond the body.

It is well known that the pressure within the front part of a separated region or cavity is nearly constant; hence, the front part of the cavity wall may be assumed to be a free streamline. It should be noted, however, that there is a pressure recovery on the separation streamline at the rear part of the cavity, and the free streamline cannot be assumed to be a closed curve. Moreover, the flow in the cavity-wake region is of dissipative type. The aforementioned facts plus many other mathematical as well as physical considerations tend to argue strongly against the traditional use of the closed cavity model. Hence, the closure condition [4], frequently used as an additional condition for determining the cavity length, will be abandoned.

Accepting the fact that the free streamlines are open, the length of a cavity should be defined as the part of the cavity in which pressure is approximately constant. As stated before, the boundary conditions will be applied on a slot as shown in Fig. 1(b). The linearized z-plane will then be mapped on an upper-half plane by

$$z = \frac{x_o \ell_m \zeta^2}{x_o \zeta^2 + \ell_m - x_o} \tag{7}$$

when $\ell_m \geq 1$ and by

$$z = \frac{x_o \zeta^2}{x_o \zeta^2 + 1 - x_o} \tag{8}$$

when $\ell_m \leq 1$, where

$\zeta = \xi + \iota\eta$,

ℓ_m = distance from the leading edge to the end of the last cavity normalized with the chord,

x_o = the distance from leading edge to the front stagnation point, also normalized with the chord.

The ζ-plane is shown in Figs. 1(c) and 1(d).

The boundary conditions may be stated as follows:

i) Imaginary part of $Q(z)$, $-\theta(x)$, is given on the wetted part of the solid boundary.

ii) Real part of $Q(z)$, $\ln\sqrt{(1 + \sigma_j)/(1 + \sigma_m)}$, is given on the free stream boundary. Here, σ_j is the cavitation number of the j-th cavity defined as

$$\sigma_j = \frac{P - P_{c_j}}{\frac{1}{2}\rho U^2} \tag{9}$$

and P_{c_j} is the pressure in the j-th cavity.

iii) The value of $Q(z)$ at infinity is given as

$$Q(\infty) = \ln \frac{1}{\sqrt{1 + \sigma_m}} - \iota\alpha \tag{10}$$

where α is the angle of attack.

iv) In addition, the solution $Q(z)$ must be bounded everywhere except where the slope of a streamline is not continuous, i.e., at a point where θ is not continuous.

C. Solution of the Problem

When $\ell_m \geq 1$, the general solution satisfying the Hölder condition on the boundary and the first two boundary conditions indicated in the previous section is [also see Fig. 1(c)]

$$Q(\zeta) = -\frac{H(\zeta)}{\pi} \left\{ \int_{-\lambda}^{0} \frac{\theta(\tau)d\tau}{(\tau - \zeta)H(\tau)} + \sum_{j=1}^{m-1} \int_{c_j}^{0_{j+1}} \frac{\theta(\tau)d\tau}{(\tau - \zeta)H(\tau)} \right.$$

$$\left. + \sum_{j=1}^{m-1} \ln \sqrt{\frac{1 + \sigma_j}{1 + \sigma_m}} \int_{0_j}^{c_j} \frac{d\tau}{(\tau - \zeta)[\iota H(\tau)]} + \sum_{j=0}^{\infty} A_j \zeta^j \right\},$$

$$\lambda = \sqrt{\frac{\ell_m - x_o}{x_o(\ell_m - 1)}} \tag{11}$$

where

$$H(\zeta) = \sqrt{-\zeta(\zeta + \lambda)\Pi(\zeta - \ell_j)} \tag{12}$$

and A_j are real constants.

To satisfy the last boundary condition, the constants A_j must be chosen so that the infinite series converges for all finite ζ. The function $H(\zeta)$ is one of many homogeneous solutions. It may be noted that the choice of the homogeneous solution $H(\zeta)$ is quite arbitrary. In fact, it is shown in the Appendix that any possible solution satisfying the Hölder condition is reducible to Eq. (11) with $H(\zeta)$ given by Eq. (12). The general solution for the case $\ell_m \leq 1$ may also be written in a similar form.

To satisfy the third boundary condition, the condition at infinity, it is more convenient to consider three cases separately.

Case 1. $m = 0$

When there is no cavity, m is zero and the homogeneous solution $H(\zeta)$ is a constant which may be taken as unity without loss of generality. The solution, Eq. (11), bounded at infinity is

$$Q(\zeta) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\theta(\tau)d\tau}{\tau - \zeta} + A_o \tag{11a}$$

The boundary condition at infinity may now be applied to determine the unknown constants x_o and A_o. The required equation is

$$Q(\zeta_\infty) = -\iota\alpha = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\theta(\tau)d\tau}{\tau - \zeta_\infty} + A_o \tag{13}$$

where $\zeta_\infty = \iota\sqrt{(1 - x_o)/x_o}$ by virtue of the transformation formula, Eq. (8). Note that the speed at infinity, U, has been used as the reference speed. Equation (13) is an equation of complex variables which leads to two equations of real variables. The problem is, in principle, solved.

Case 2. $m = 1$

When the cavity is shorter than the chord, it is called a partial cavity, and when the cavity is longer than the chord, it is called a full cavity. Equation (7) is the proper mapping formula for a full cavity, whereas Eq. (8) should be used for a partial cavity. There is no other essential difference for the two cases, and only the full cavity case will be discussed.

The solution bounded at infinity is

$$Q(\zeta) = -\frac{\sqrt{-\zeta(\zeta+\lambda)}}{\pi} \int_{-\lambda}^{0} \frac{\theta(\tau)d\tau}{(\tau-\zeta)\sqrt{-\tau(\tau+\lambda)}}, \quad \lambda = \sqrt{\frac{\ell - x_o}{x_o(\ell - 1)}} \tag{11b}$$

Here again there are two constants, x_o and ℓ, to be determined. Applying the boundary condition at infinity, it follows that

$$\ln\frac{1}{\sqrt{1+\sigma}} - \iota\alpha = -\frac{\sqrt{-\zeta_\infty(\zeta_\infty+\lambda)}}{\pi} \int_{-\lambda}^{0} \frac{\theta(\tau)d\tau}{(\tau-\zeta_\infty)\sqrt{-\tau(\tau+\lambda)}} \tag{13a}$$

where $\zeta_\infty = \iota\sqrt{\ell - x_o/x_o}$. Separating the real part and the imaginary part, Eq. (13a) furnishes two equations required to solve for x_o and ℓ.

Case 3. $m \geq 2$

There are m+1 constants $(x_o, \ell_1, \ldots \ell_m)$ to be determined in this case. Since m+1 equations are required to determine m+1 constants, and since the condition at ζ_∞ furnishes only two equations, m-1 additional equations are required. The necessary equations are readily obtainable if the behavior of $Q(\zeta)$ given by Eq. (11) at $\zeta = \pm\infty\,(z = \ell_m)$ is considered. It is seen that the homogeneous solution $H(\zeta)$ grows indefinitely as the m-th power of ζ when ζ tends to infinity. Obviously, the boundedness of $Q(\zeta)$ at infinity cannot be guaranteed by setting all A_j equal to zero. It was shown in the Appendix of Ref. [7] that the necessary and sufficient conditions for $Q(\zeta)$ to be bounded at infinity are

$$\int_{-\lambda}^{0} \frac{\theta(\tau)d\tau}{H(\tau)} + \sum_{j=1}^{m-1} \int_{c_j}^{0_{j+1}} \frac{\theta(\tau)d\tau}{H(\tau)} + \sum_{j=1}^{m-1} \ln\sqrt{\frac{1+\sigma_j}{1+\sigma_m}} \int_{0_j}^{c_j} \frac{d\tau}{\iota H(\tau)} = 0 \tag{14-0}$$

$$\int_{-\lambda}^{0} \frac{\tau\theta(\tau)d\tau}{H(\tau)} + \sum_{j=1}^{m-1} \int_{c_j}^{0_{j+1}} \frac{\tau\theta(\tau)d\tau}{H(\tau)} + \sum_{j=1}^{m-1} \ln\sqrt{\frac{1+\sigma_j}{1+\sigma_m}} \int_{0_j}^{c_j} \frac{\tau d\tau}{\iota H(\tau)} = 0 \quad (14\text{-}1)$$

$$\cdots\cdots\cdots\cdots$$

$$\int_{-\lambda}^{0} \frac{\tau^{m-2}\theta(\tau)d\tau}{H(\tau)} + \sum_{j=1}^{m-1} \int_{c_j}^{0_{j+1}} \frac{\tau^{m-2}\theta(\tau)d\tau}{\iota H(\tau)}$$

$$+ \sum_{j=1}^{m-1} \ln\sqrt{\frac{1+\sigma_j}{1+\sigma_m}} \int_{0_j}^{c_j} \frac{\tau^{m-2}d\tau}{\iota H(\tau)} = 0 \quad (14\text{-}m\text{-}2)$$

The solution of the problem is, in theory, complete. Lift, form drag, and moment can be calculated by straightforward integrations.

III. ILLUSTRATIVE EXAMPLES

A. Fully Wetted Flat Plate

The exact solution of this simple aerodynamic problem has been known for a long time. The boundary profile is now truly a slot, and the quasi-linear theory is expected to yield the exact solution.

The mapping formula, Eq. (8) maps the flow in the z-plane into the upper half ζ-plane, as shown in Fig. 2. The boundary conditions are

i) $\theta = 0, \quad 0 < \xi \quad$ and $\quad \xi < -1$

ii) $\theta = \pi, \quad -1 < \xi < 0$

iii) $Q = -\iota\alpha \quad$ at $\quad \zeta = \iota\sqrt{\frac{1 - x_o}{x_o}}$

The solution satisfying the first two boundary conditions is, after integrating Eq. (11a),

$$Q(\zeta) = \ln\frac{1+\zeta}{\zeta} + A_o \quad (15)$$

Applying the boundary condition at ζ_∞, there is obtained

$$\ln \frac{1}{\sqrt{1 - x_o}} + A_o = 0 \tag{16}$$

$$\tan^{-1} \sqrt{\frac{x_o}{1 - x_o}} = \alpha \tag{17}$$

Solving the simultaneous equations, there is obtained

$$x_o = \sin^2 \alpha \tag{18}$$

$$Q(\zeta) = \ln \left(\frac{1 + \zeta}{\zeta} \cos \alpha\right) \tag{19}$$

The total force acting on the plate is expressible as

$$F = \int_{-\infty}^{+\infty} (P - p) \frac{dx}{d\xi} \, d\xi$$

which, upon substitution and application of Bernoulli's equation, leads to

$$F = \rho U^2 \cot^2\alpha \int_{-\infty}^{+\infty} \left[\left(\frac{1 + \xi}{\xi}\right)^2 \cos^2\alpha - 1\right] \frac{\xi \, d\xi}{(\xi^2 + \cot^2\alpha)^2} \tag{20}$$

In performing the above integration, it should be noted that the integrand has a simple pole at the origin and, hence, the Cauchy principal value has to be taken there. The result is

$$F = \pi\rho U^2 e^{-\iota\alpha} \sin \alpha \tag{21}$$

The imaginary part of Eq. (21) is the force parallel to the x-axis and is known as the suction drag. It is readily seen that the complex force given by Eq. (21) is a force of magnitude $\pi\rho U^2 \sin \alpha$ and perpendicular to the flow at infinity. The lift and drag coefficients are, respectively,

$$C_L = 2\pi \sin \alpha \tag{22}$$

$$C_D = 0 \tag{23}$$

where C_L measures the force normal to the flow at infinity.

The moment about the leading edge is expressible as

$$M = \int_{-\infty}^{+\infty} (P - p)\, x\, \frac{dx}{d\xi}\, d\xi$$

and upon substitution it leads to

$$M = \rho U^2 \cot^2\alpha \int_{-\infty}^{+\infty} \left[\left(\frac{1+\xi}{\xi}\right)^2 \cos\alpha - 1\right] \frac{\xi^3 d\xi}{(\xi^2 + \cot^2\alpha)^3} \tag{24}$$

There is no singularity in the integrand of Eq. (24) and a straightforward integration yields

$$M = \frac{1}{4}\pi\rho U^2 \sin\alpha \cos\alpha \tag{25}$$

The moment arm is equal to one-fourth of the chord measured from the leading edge. All the results agree exactly with the well-known solution which may be found in any book on aerodynamics.

B. An Alternate Solution for Fully Wetted Flat Plate

The problem considered in the previous article will be solved by using a slightly different boundary condition. Here the streamlines leaving the trailing edge are considered as free streamlines creating a separated region (wake or cavity) of infinitesimal thickness and finite length as shown in Fig. 3(a). The mapping function, Eq. (7), maps the z-plane into the upper half ζ-plane as shown in Fig. 3(b). The boundary conditions are

i) $\theta = \begin{cases} 0, & 0 < \xi < \lambda \text{ and } -\lambda < \xi < -1 \\ \pi, & -1 < \xi < 0 \end{cases}$

ii) $\mathrm{Re}Q = 0, \quad \lambda < \xi \text{ and } \xi < -\lambda$

iii) $Q = \ln \dfrac{1}{\sqrt{1+\sigma}} - \iota\alpha$ at $\zeta = \iota\sqrt{\dfrac{\ell - x_o}{x_o}} = \iota\eta_\infty$

The solution satisfying the first two boundary conditions is

$$Q(\zeta) = -\sqrt{\lambda^2 - \zeta^2} \int_{-1}^{0} \frac{d\tau}{(\tau - \zeta)\sqrt{\lambda^2 - \tau^2}}, \quad \lambda = \sqrt{\frac{\ell - x_o}{x_o(\ell - 1)}} \tag{26}$$

The boundary condition at infinity leads to the following equations.

$$\ln\sqrt{1+\sigma} = -\sqrt{\lambda^2 + \eta_\infty^2} \int_0^1 \frac{\tau d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\lambda^2 - \tau^2}} \tag{27}$$

$$\alpha = \sqrt{\eta_\infty^2(\lambda^2 + \eta_\infty^2)} \int_0^1 \frac{d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\lambda^2 - \tau^2}} \tag{28}$$

Equations (27) and (28) have a unique set of real solutions for values of ℓ between one and infinity.

The real part of $Q(\xi)$ for $-\lambda < \xi < \lambda$ is, by integrating Eq. (26)

$$\ln(q/q_c) = \ln \frac{(1+\xi)[\sqrt{\lambda(\lambda - \xi)} + \sqrt{\lambda(\lambda + \xi)}\,]^2}{\xi[\sqrt{(\lambda - 1)(\lambda - \xi)} + \sqrt{(\lambda + 1)(\lambda + \xi)}\,]^2} \tag{29}$$

The total force acting on the plate is

$$F = \rho q_c^2 \ell \eta_\infty^2 \int_{-\lambda}^{\lambda} \left(\frac{q}{q_c}\right)^2 \frac{\xi d\xi}{(\xi^2 + \eta_\infty^2)^2} \tag{30}$$

Substituting Eq. (29) into Eq. (30) and simplifying, it can be shown that

$$F = 4\rho q_c^2 \ell \eta_\infty^2 \frac{\sqrt{\lambda^2 - 1}}{\lambda^2} \int_0^\lambda \frac{(\lambda + \sqrt{\lambda^2 - \xi^2})^2 \sqrt{\lambda^2 - \xi^2}\, d\xi}{(\sqrt{\lambda^2 - 1} + \sqrt{\lambda^2 - \xi^2})^2 (\xi^2 + \eta_\infty^2)^2}$$

$$+ 4\pi \iota \rho q_c^2 \frac{\lambda^2}{(\lambda + \sqrt{\lambda^2 - 1})^2} \qquad (31)$$

The last term in Eq. (31) is the suction drag due to the singularity at the leading edge. The moment about the leading edge is

$$M = 4\rho q_c^2 \ell^2 \eta_\infty^2 \frac{\sqrt{\lambda^2 - 1}}{\lambda^2} \int_0^\lambda \frac{(\lambda + \sqrt{\lambda^2 - \xi^2})^2 \xi^2 \sqrt{\lambda^2 - \xi^2}\, d\xi}{(\sqrt{\lambda^2 - 1} + \sqrt{\lambda^2 - \xi^2})^2 (\xi^2 + \eta_\infty^2)^3} \qquad (32)$$

Two extreme cases when $\ell = 1$ and when $\ell \to \infty$ are of special interest.

Case 1. $\ell = 1$

This is the case when there is no cavity, and the flow model is the same as that of the previous article. In this case $\lambda \to \infty$, and Eqs. (27) and (28) yield $x_o = -\sigma = \sin^2\alpha$. It is readily checked that the solution is identical to the previous solution.

Case 2. $\ell \to \infty$

Solving Eqs. (27) and (28) it follows that $\sigma = 0$ and $x_o = \sin^2\alpha$. For a small α

$$\lambda^2 = \frac{1}{x_o} \gg 1$$

and $\lambda^2 - 1$ is approximately equal to λ^2. In this case Eqs. (31) and (32) may readily be integrated. The resulting lift, drag, and moment coefficients are

$$C_L = 2\pi \sin\alpha + O(\sin^3\alpha) \qquad (33)$$

$$C_D = O(\sin^3\alpha) \qquad (34)$$

$$C_M = \frac{1}{2}\pi \sin\alpha + O(\sin^3\alpha) \tag{35}$$

The results indicate that for a small angle of attack the two models yield almost identical solutions. When $\ell = 1$, the wake model yields identical results with the classic solution, and when $\ell \rightarrow \infty$, the change in the cavitation number (negative of the trailing edge pressure coefficient) is $\sin^2\alpha$, and the change in C_L, C_D, and C_M is of order of $\sin^3\alpha$.

C. Fully Cavitated Flat Plate

Because of its potential application to high speed ships and hydraulic machines, the problem of fully cavitated flow about slender bodies has been intensely studied by many investigators.

The cavity is assumed to separate from the leading edge and the trailing edge, forming a cavity of length ℓ exceeding the chord as shown in Fig. 4(a). The boundary profile is linearized to a slot $\overline{oc}$ on the x-axis as shown in Fig. 4(b). The transformation formula, Eq. (7), maps the linearized z-plane on the upper half of the ζ-plane as shown in Fig. 4(c). The boundary conditions are

i) $\theta = \begin{cases} 0, & -\lambda < \xi < -1 \\ \pi, & -1 < \xi < 0 \end{cases}$

ii) $\mathrm{Re}Q = 0, \quad 0 < \xi \text{ and } \xi < -\lambda$

iii) $Q = \ln \dfrac{1}{\sqrt{1+\sigma}} - \iota\alpha \quad \text{at } \zeta = \iota\sqrt{\dfrac{\ell - x_o}{x_o}} = \iota\eta_\infty$

The solution satisfying the first two boundary conditions is

$$Q(\zeta) = -\sqrt{-\zeta(\lambda+\zeta)} \int_{-1}^{0} \frac{d\tau}{(\tau-\zeta)\sqrt{-\tau(\lambda+\tau)}}, \quad \lambda = \sqrt{\frac{\ell - x_o}{x_o(\ell - 1)}} \tag{36}$$

The boundary condition at infinity leads to the following equations:

$$\ln\sqrt{1+\sigma} = \eta_\infty \sqrt{\frac{\eta_\infty\sqrt{\eta_\infty^2 + \lambda^2} - \eta_\infty^2}{2}} \int_0^1 \frac{d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\tau(\lambda - \tau)}}$$

$$- \sqrt{\frac{\eta_\infty \sqrt{\eta_\infty^2 + \lambda^2} + \eta_\infty^2}{2}} \int_0^1 \frac{\tau d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\tau(\lambda - \tau)}} \tag{37}$$

$$\alpha = \sqrt{\frac{\eta_\infty \sqrt{\eta_\infty^2 + \lambda^2} - \eta_\infty^2}{2}} \int_0^1 \frac{\tau d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\tau(\lambda - \tau)}}$$

$$+ \eta_\infty \sqrt{\frac{\eta_\infty \sqrt{\eta_\infty^2 + \lambda^2} + \eta_\infty^2}{2}} \int_0^1 \frac{d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\tau(\lambda - \tau)}} \tag{38}$$

Here again the equations have a unique set of real solutions for values of ℓ between one and infinity.

The real part of $Q(\zeta)$ for $-\lambda < \xi < \lambda$ is, in this case

$$\ln(q/q_c) = \ln \frac{[\sqrt{\lambda + \xi} - \sqrt{\xi(1 - \lambda)}]^2}{\lambda(1 + \xi)} \tag{39}$$

Unlike the fully wetted case there is no suction drag at the leading edge, and the total force is normal to the plate. It can be shown that the force coefficient is

$$C_F = 8(1 + \sigma)\, \ell\eta_\infty^2 \sqrt{\lambda - 1} \int_0^1 \frac{\xi \sqrt{\xi(\lambda - \xi)}\, d\xi}{[\sqrt{\lambda - \xi} + \sqrt{(\lambda - 1)\xi}]^2(\xi^2 + \eta_\infty^2)^2} \tag{40}$$

and the moment coefficient referred to the leading edge is

$$C_M = 8(1 + \sigma)\, \ell^2\eta_\infty^2 \sqrt{\lambda - 1} \int_0^1 \frac{\xi^3 \sqrt{\xi(\lambda - \xi)}\, d\xi}{[\sqrt{\lambda - \xi} + \sqrt{(\lambda - 1)\xi}]^2(\xi^2 + \eta_\infty^2)^3} \tag{41}$$

The integrals in Eqs. (40) and (41) converge for all ℓ greater than or equal to one. The following two limiting cases are of special interest.

Case 1. $\ell \to \infty$

When the cavity is infinitely long, the solution of Eqs. (37) and (38) is

$$\sigma = 0, \qquad x_o = \sin^4 \frac{1}{2} \alpha \tag{42}$$

The force coefficient, Eq. (40), is simplified to

$$C_F = 8\sqrt{\lambda - 1} \int_0^1 \frac{\xi \sqrt{\xi(1 - \xi)}\, d\xi}{[\sqrt{1 - \xi} + \sqrt{(\lambda - 1)\xi}]^2} \tag{40a}$$

Similarly, the moment coefficient is

$$C_M = 8\sqrt{\lambda - 1} \int_0^1 \frac{\xi^3 \sqrt{\xi(1 - \xi)}\, d\xi}{[\sqrt{1 - \xi} + \sqrt{(\lambda - 1)\xi}]^2} \tag{41a}$$

To show the accuracy of the method, the present solution is compared graphically with the exact solution and the linear solution in Fig. 5. As may be expected, the quasi-linear theory yields more accurate results than the linear theory. It is interesting to note that the quasi-linear theory under-predicts both the force coefficient and the moment coefficient, whereas the linear theory over-predicts them.

Case 2. $\ell \to 1$

In this case $\lambda \to \infty$, and the solutions of Eqs. (37) and (38) are

$$x_o \approx \frac{1}{4} \tan^4 \alpha, \qquad \sigma \approx \frac{2 \tan^2 \alpha}{1 - \tan^2 \alpha} \tag{43}$$

The force coefficient and the moment coefficient are

$$C_F = 8(1 + \sigma)\,\frac{1 - x_o}{x_o} \int_0^\infty \frac{\xi\sqrt{\xi}\,d\xi}{(1 + \sqrt{\xi})^2\left(\xi^2 + \frac{1 - x_o}{x_o}\right)^2} \tag{40b}$$

$$C_M = 8(1 + \sigma)\,\frac{1 - x_o}{x_o} \int_0^\infty \frac{\xi^3\sqrt{\xi}\,d\xi}{(1 + \sqrt{\xi})^2\left(\xi^2 + \frac{1 - x_o}{x_o}\right)^3} \tag{41b}$$

D. Partially Cavitated Flat Plate

A plate is called partially cavitated when the region of constant cavity pressure terminates on the plate. Flow configurations in the physical and in the transformed plane are depicted in Fig. 6. Equation (8) is the required mapping formula. The free streamline is mapped on the ξ-axis between $\xi = 0$ and $\xi = \ell'$ where

$$\ell' = \sqrt{\frac{(1 - x_o)\ell}{x_o(1 - \ell)}} \tag{44}$$

The boundary conditions are

i) $\Theta = \begin{cases} 0, & \xi < -1 \text{ and } \ell' < \xi \\ \pi, & -1 < \xi < 0 \end{cases}$

ii) $\mathrm{Re}Q = 0, \qquad 0 < \xi < \ell'$

iii) $Q(\zeta_\infty) = \ln \dfrac{1}{\sqrt{1 + \sigma}} - \iota\alpha, \qquad \zeta_\infty = \iota\sqrt{\dfrac{1 - x_o}{x_o}} = \iota\eta_\infty$

The solution satisfying the first two boundary conditions bounded at the trailing edge is

$$Q(\zeta) = -\sqrt{-\zeta(\ell' - \zeta)} \int_{-1}^{0} \frac{d\tau}{(\tau - \zeta)\sqrt{-\tau(\ell' - \tau)}} \tag{45}$$

The boundary condition at infinity leads to the following equations.

$$\ln\sqrt{1+\sigma} = \sqrt{\frac{\eta_\infty}{2}}\left[\eta_\infty\sqrt{\sqrt{\eta_\infty^2 + \ell'^2} + \eta_\infty}\int_0^1 \frac{d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\tau(\ell' + \tau)}}\right.$$

$$\left. - \sqrt{\sqrt{\eta_\infty^2 + \ell'^2} - \eta_\infty}\int_0^1 \frac{\tau d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\tau(\ell' + \tau)}}\right] \tag{46}$$

$$\alpha = \sqrt{\frac{\eta_\infty}{2}}\left[\eta_\infty\sqrt{\sqrt{\eta_\infty^2 + \ell'^2} - \eta_\infty}\int_0^1 \frac{d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\tau(\ell' + \tau)}}\right.$$

$$\left. + \sqrt{\sqrt{\eta_\infty^2 + \ell'^2} + \eta_\infty}\int_0^1 \frac{\tau d\tau}{(\tau^2 + \eta_\infty^2)\sqrt{\tau(\ell' + \tau)}}\right] \tag{47}$$

The above equations have a unique set of real solutions for $0 < \ell \leq 1$. The force coefficient, when $\ell < 1$, is

$$C_F = 8(1+\sigma)\eta_\infty^2\sqrt{1+\ell'}\left\{\int_0^\infty \frac{\xi\sqrt{\xi(\xi + \ell')}\, d\xi}{[\sqrt{\xi + \ell'} + \sqrt{\xi(1+\ell')}\,]^2(\xi^2 + \eta_\infty^2)^2}\right.$$

$$\left. - \int_{\ell'}^\infty \frac{\xi\sqrt{\xi(\xi - \ell')}\, d\xi}{[\sqrt{\xi - \ell'} + \sqrt{\xi(1+\ell')}\,]^2(\xi^2 + \eta_\infty^2)^2}\right\} \tag{48}$$

and the moment coefficient is

$$C_M = 8(1 + \sigma)\eta_\infty^2 \sqrt{1 + \ell'} \left\{ \int_0^\infty \frac{\xi^3\sqrt{\xi(\xi + \ell')}\, d\xi}{[\sqrt{\xi + \ell'} + \sqrt{\xi(1 + \ell')}\,]^2(\xi^2 + \eta_\infty^2)^3} \right.$$

$$\left. - \int_{\ell'}^\infty \frac{\xi^3\sqrt{\xi(\xi - \ell')}\, d\xi}{[\sqrt{\xi - \ell'} + \sqrt{\xi(1 + \ell')}\,]^2(\xi^2 + \eta_\infty^2)^3} \right\} \qquad (49)$$

It is readily seen that $\ell' \to \infty$ when $\ell = 1$, and then the solution agrees with the limiting case of the full cavity solution discussed in the previous article. For the other limiting case, when $\ell \to 0$, Eq. (48) implies $\sigma \to \infty$. This means $q_c \to \infty$, and the leading edge is a singular point, as may be expected.

IV. CONCLUSIONS

A quasi-linear and linear theory has been developed for two-dimensional, incompressible, irrotational flows. The method applies for the type of problems wherein the boundary profiles are almost linear. Whereas the linearization of the boundary profile is essential for simplicity, the linearization of the complex velocity is not always necessary. A general theory, applicable to a flow without separation or with an arbitrary number of separated regions, was developed. The well-known closure condition was replaced by the condition of the boundedness of the solution. It was shown that the boundedness condition yields unique solutions for all flows considered. The boundedness condition is, in general, simpler than the closure condition. It appears that the present method yields more accurate results than some well-known linear theories.

The method was exemplified by way of illustrative examples. The following additional conclusions may be drawn from the results of illustrative examples.

1. When the boundary profile is truly linear, the solution agrees with the classic exact solution.

2. The trailing streamline behind a fully wetted body may be regarded as a wake of constant pressure as well as a vortex sheet. Since no singularity is admitted at the end of the wake, the limiting case of zero wake length corresponds to the case of a vortex sheet.

3. Unlike most of the linear theories the present theory yields finite force when the cavity length is equal to the chord. This yields a solution which permits a continuous change from a partial cavity to a full cavity.

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F I G U R E S

(1 through 6)

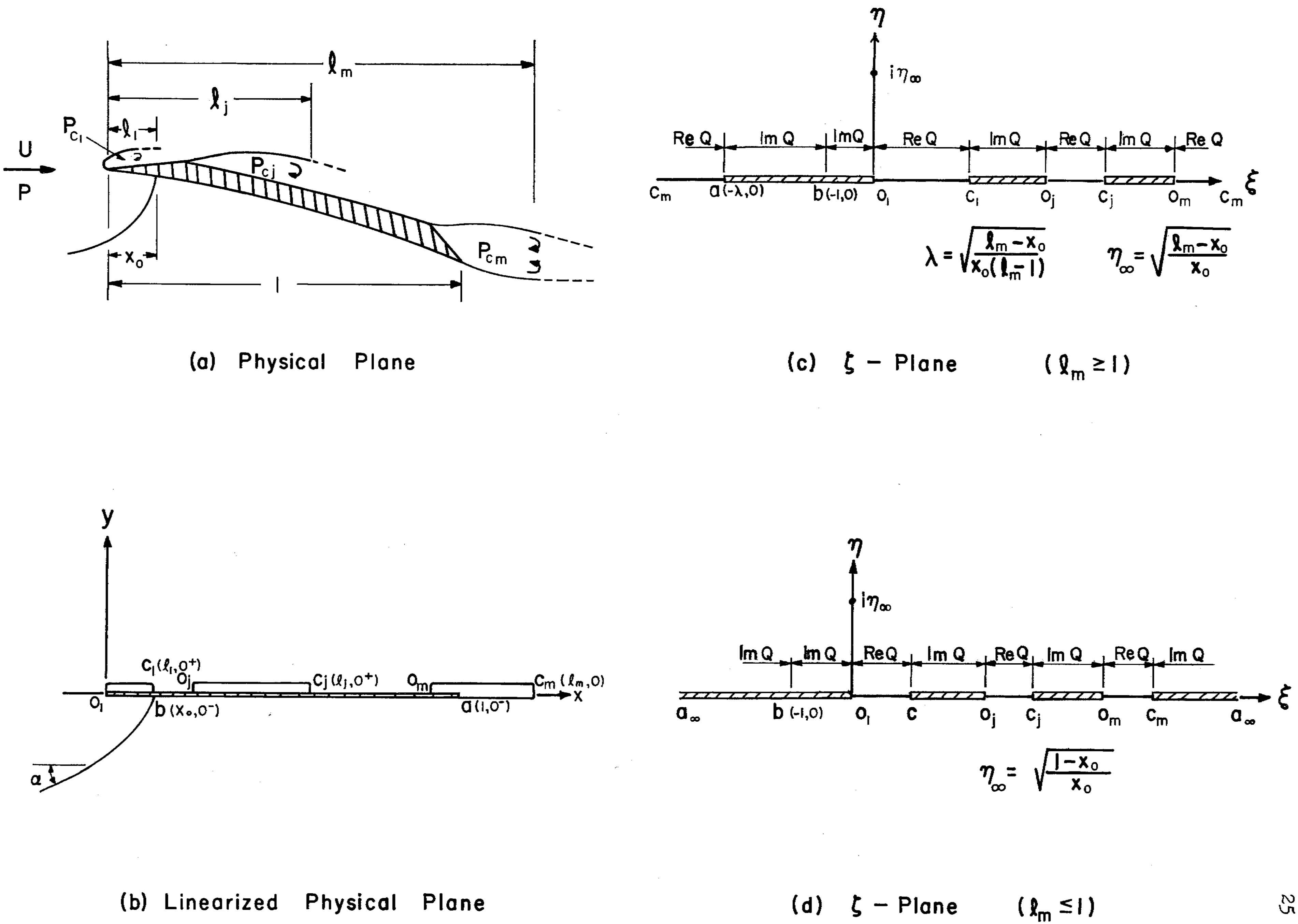

Fig. 1 – Flow around a Lifting Body with m Cavities

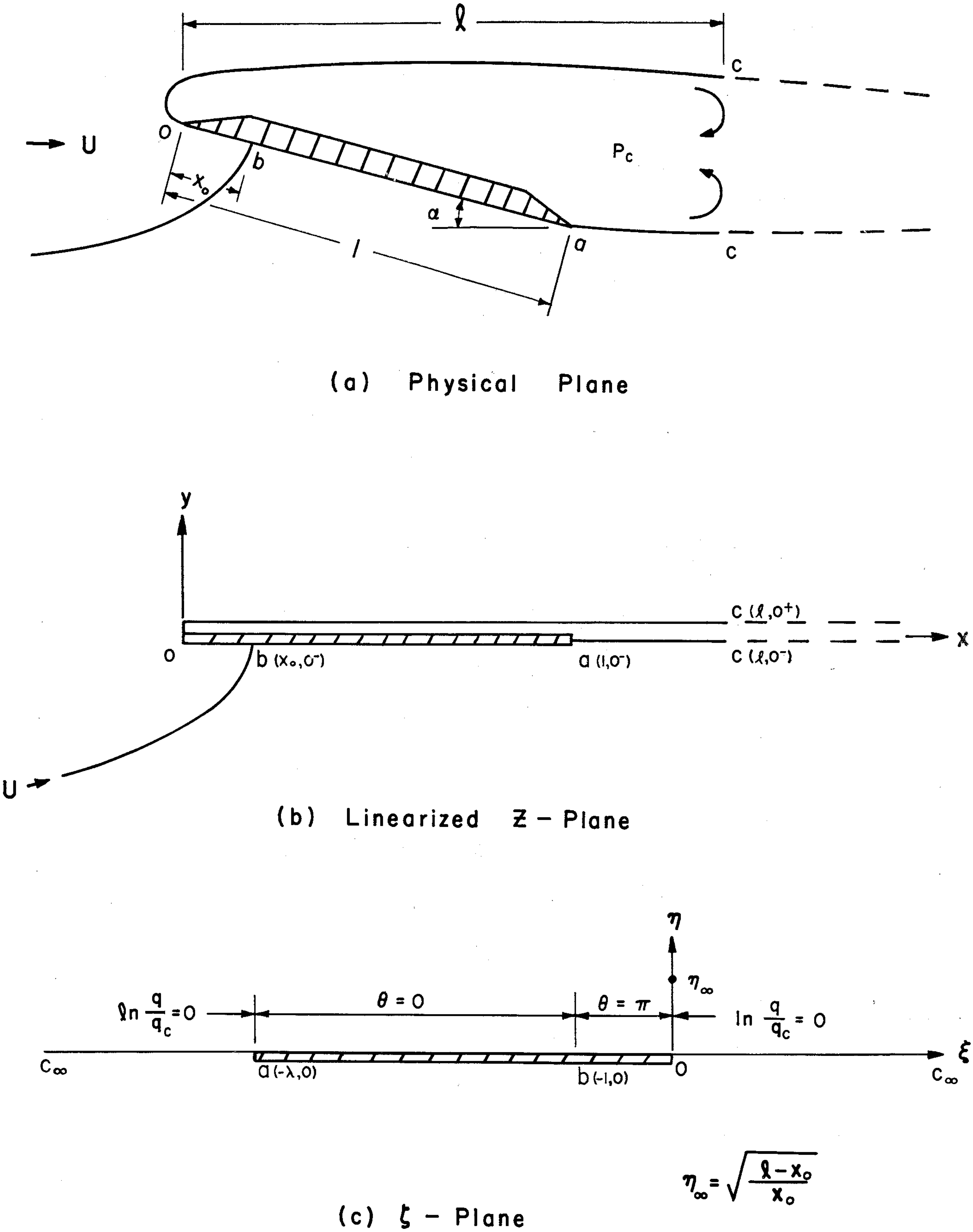

Fig. 4 - Fully Cavitated Flat Plate

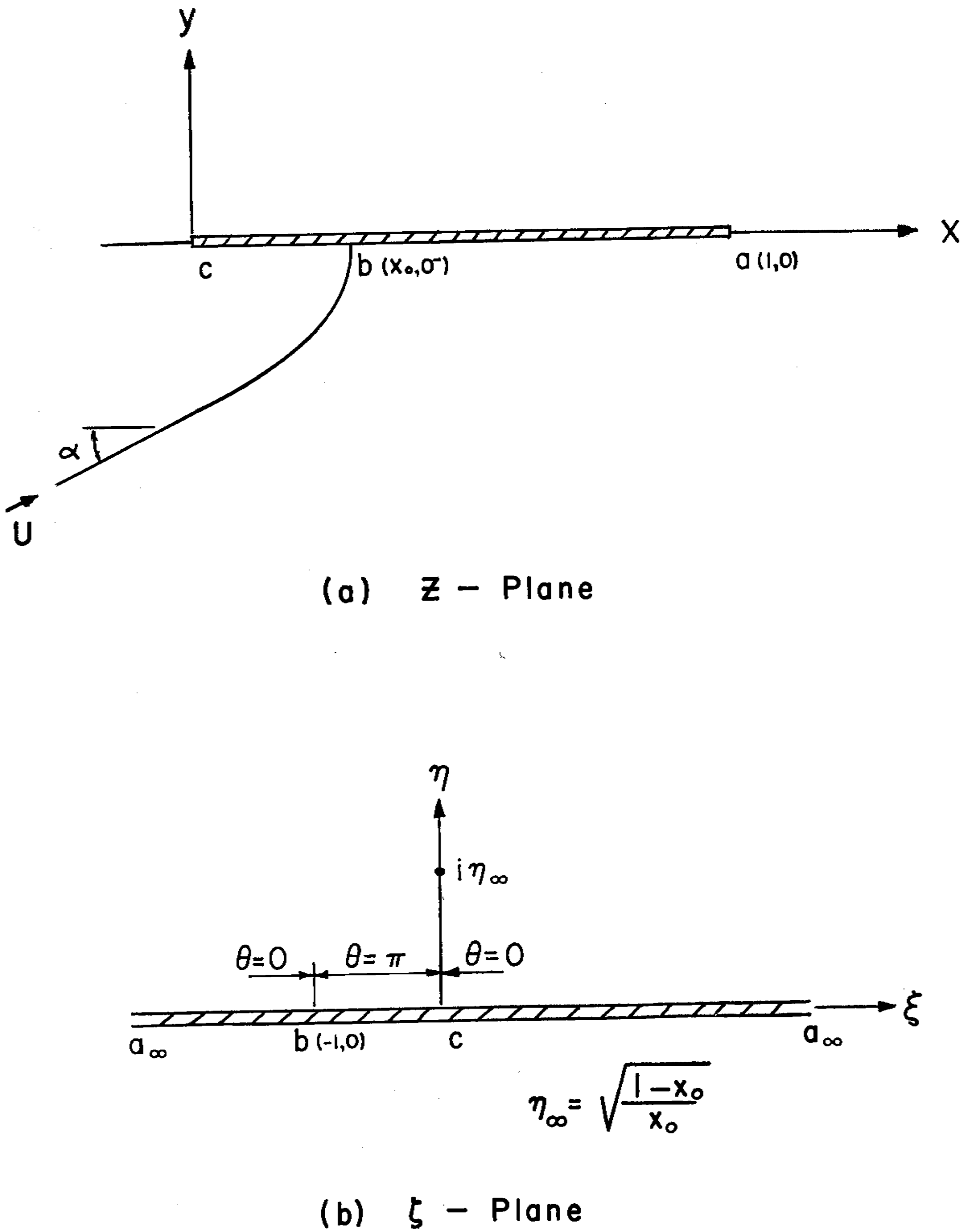

Fig. 2 - Flow around a Flat Plate without Separation

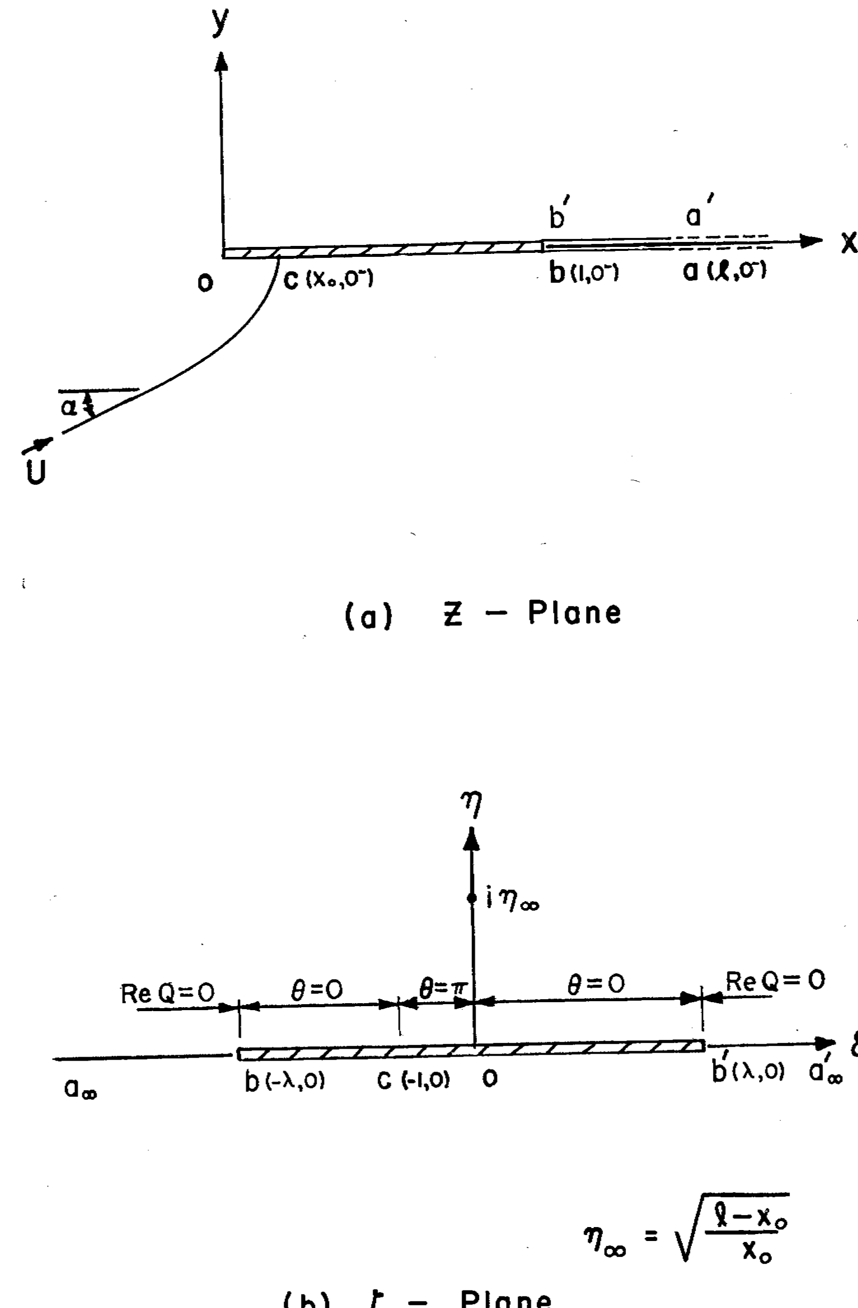

Fig. 3 - An Alternate Model of Fully Wetted Flow around a Flat Plate

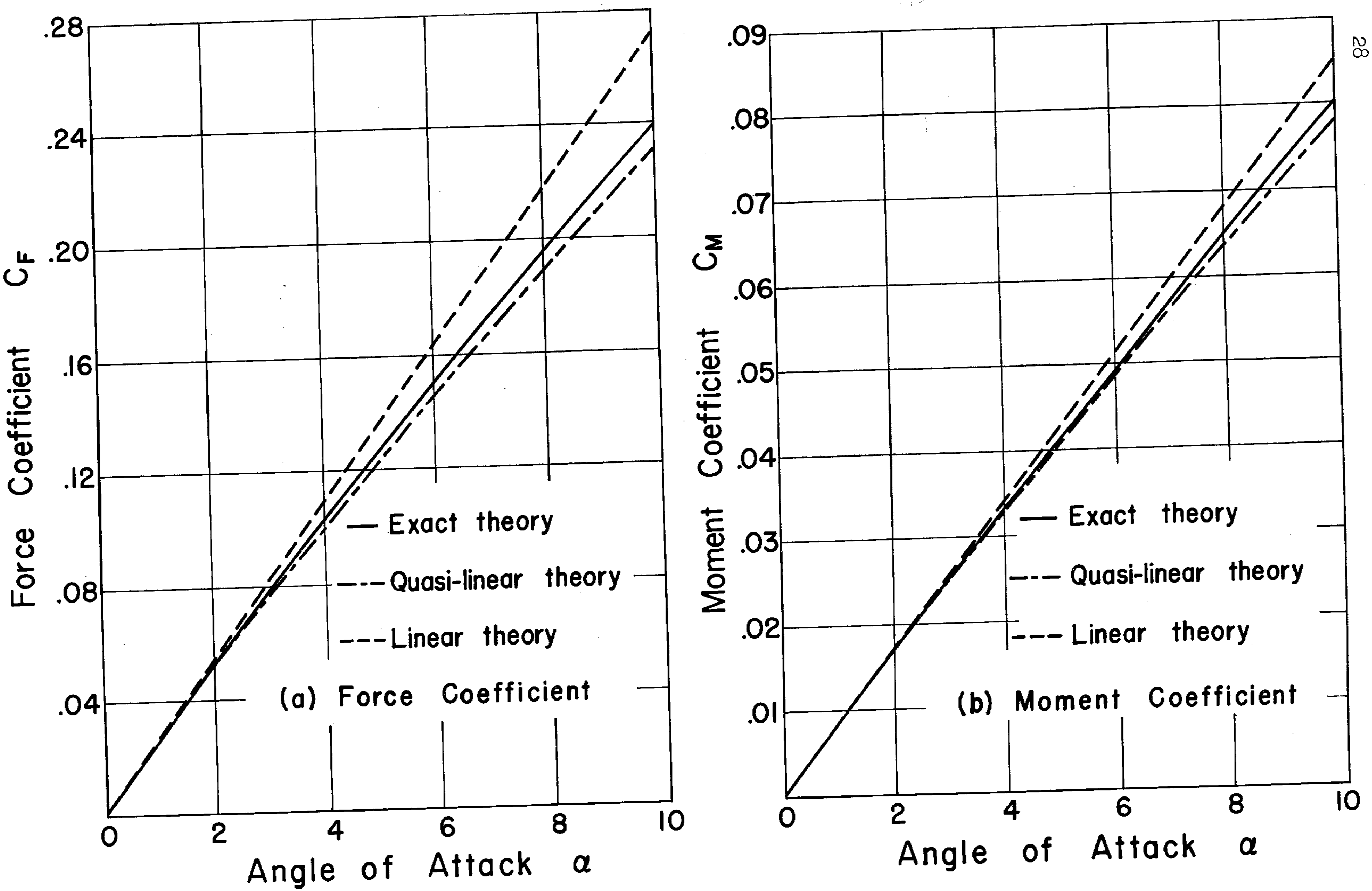

Fig. 5 - Force Coefficient and Moment Coefficient for a Flat Plate with Infinite Cavity

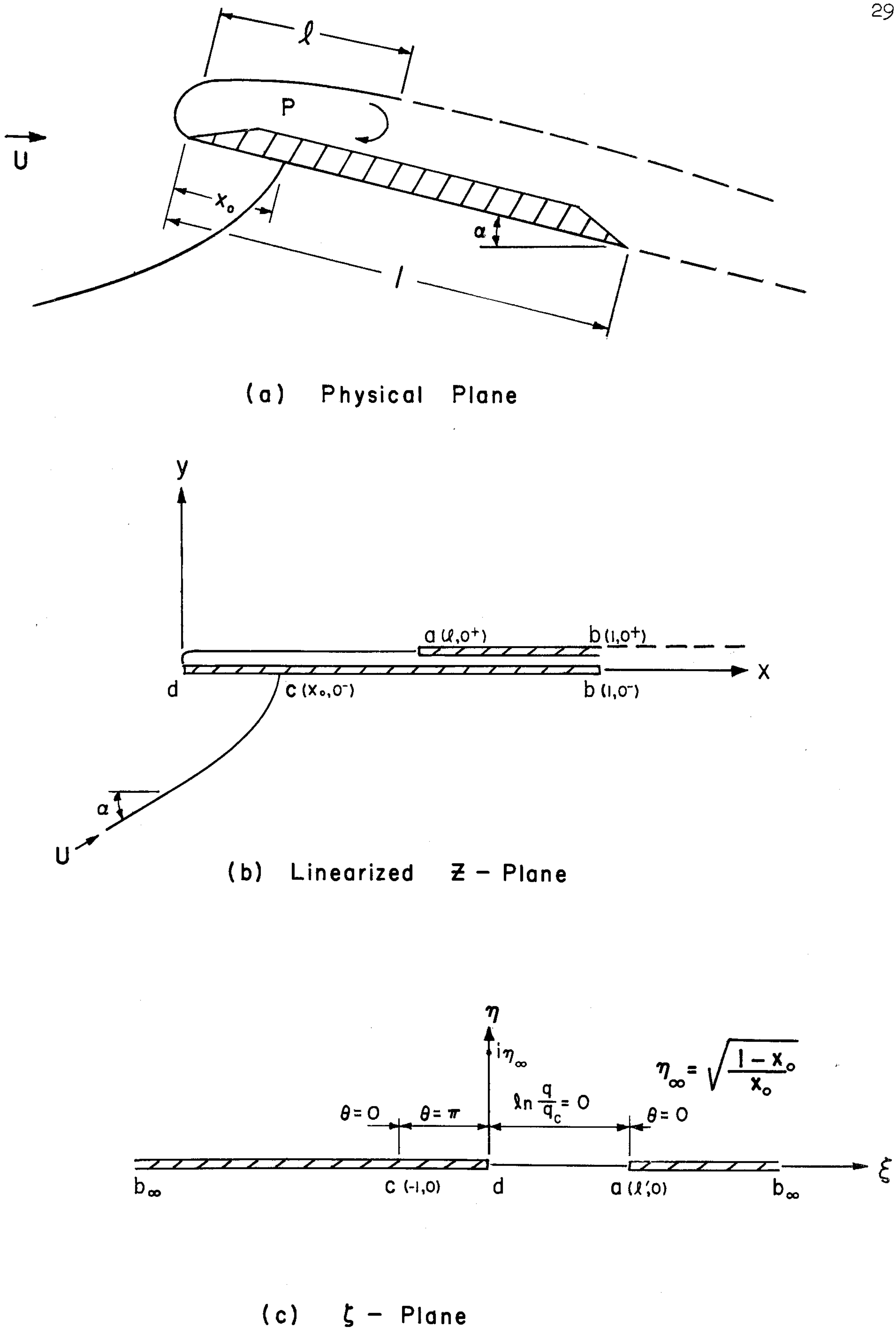

Fig. 6 - Partially Cavitated Flat Plate

APPENDIX

The General Solution of the Riemann-Hilbert Mixed Boundary Value Problem in an Upper-Half Plane

APPENDIX

The General Solution of the Riemann-Hilbert Mixed Boundary Value Problem in an Upper-Half Plane

It is to be shown that the analytic function $Q(\zeta)$ regular in the domain $\mathrm{Re}\{\zeta\} > 0$ satisfying boundary conditions on $\mathrm{Re}\{\zeta\} = 0$ as indicated in Fig. 1(c) is given by Eqs. (11) and (12). The function may have isolated singularities on the boundary, provided that the Hölder condition is satisfied at the singularities. Here, a function $f(x)$ is said to satisfy the Hölder condition at x_o if

$$\left|f(x) - f(x_o)\right| \leq A\left|x - x_o\right|^{a}, \quad 0 < A < \infty, \quad 0 < a < 1 \tag{A-1}$$

for all x sufficiently close to x_o.

If $H(\zeta)$ is a homogeneous solution, then the imaginary part of the new function $W(\zeta) = Q(\zeta)/H(\zeta)$ is completely specified on the real axis. It can be readily shown that

$$W(\zeta) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{Im}\{W(\tau)\}\, d\tau}{\tau - \zeta} + \sum_{j=0}^{\infty} A_j \zeta^j, \quad A_j = \text{real constants} \tag{A-2}$$

is a solution of the reduced boundary value problem. The addition of the infinite series in Eq. (A-2) is possible because it is regular for all $\zeta < \infty$ and real on the real axis. By definition, Eq. (A-2) is reduced to Eq. (11) if the former is multiplied by $H(\zeta)$. This shows that Eq. (11) is a solution. To show that Eq. (11) with $H(\zeta)$ defined by Eq. (12) is the general solution, we note that the most general homogeneous solution may be written as

$$H(\zeta) = \iota \frac{p(\zeta)}{q(\zeta)} \Pi(\zeta - b_j)^{\pm \frac{1}{2}} \tag{A-3}$$

where $p(\zeta)$ and $q(\zeta)$ are polynomials of arbitrary order, b_j are zero, $-\lambda$, or c_j as the case may be.

Since zeros of $q(\zeta)$ are poles of $H(\zeta)$ and $Q(\zeta)$, the polynomial $q(\zeta)$ must be a constant or of zero order. For otherwise, if $Q(\zeta)$ has a pole in the domain $\mathrm{Re}\{\zeta\} > 0$ then the function is not regular, and if $Q(\zeta)$ has a pole on the boundary, the Hölder condition is violated.

The polynomial $p(\zeta)$ may have any number of real roots and an even number of complex roots. When $p(\zeta)$ has a real root, say at $\zeta = a$, then we may write

$$H(\zeta) = (\zeta - a)h(\zeta) \tag{A-4}$$

In this case the solution $Q(\zeta)$ may be written as

$$Q(\zeta) = \frac{(\zeta - a)h(\zeta)}{\pi} \left\{ \int_{-\infty}^{+\infty} \frac{Q_o \, d\tau}{(\tau - \zeta)(\tau - a)h(\tau)} + \sum A_j \zeta^j \right\} \tag{A-5}$$

where Q_o is a known function of τ depending on the boundary condition. Since $\frac{1}{(\tau - \zeta)(\tau - a)} = \frac{1}{(\zeta - a)(\tau - \zeta)} - \frac{1}{(\zeta - a)(\tau - a)}$, Eq. (A-5) is reduced to

$$Q(\zeta) = \frac{h(\zeta)}{\pi} \left\{ \int_{-\infty}^{+\infty} \frac{Q_o \, d\tau}{(\tau - \zeta)h(\tau)} - \int_{-\infty}^{+\infty} \frac{Q_o \, d\tau}{(\tau - a)h(\tau)} + (\zeta - a)\sum A_j \zeta^j \right\}$$

or

$$Q(\zeta) = \frac{h(\zeta)}{\pi} \left\{ \int_{-\infty}^{+\infty} \frac{Q_o \, d\tau}{(\tau - \zeta)h(\tau)} + \sum B_j \zeta^j \right\} \tag{A-6}$$

This equation is identical to Eq. (11) except $H(\zeta)$ is replaced by $h(\zeta)$. This means that the order of the polynomial $p(\zeta)$ may be reduced by one if there is a real root. Repeating the same argument, the order of the polynomial may be reduced by n if there are n real roots.

The next step is to consider the case when $p(\zeta)$ has a set of complex roots. In this case the homogeneous solution may be written as

$$H(\zeta) = (\zeta^2 + a^2)h(\zeta) \tag{A-7}$$

The solution $Q(\zeta)$ takes the following form

$$Q(\zeta) = \frac{(\zeta^2 + a^2)h(\zeta)}{\pi} \left\{ \int_{-\infty}^{+\infty} \frac{Q_o \, d\tau}{(\tau - \zeta)(\tau^2 + a^2)h(\tau)} + \sum A_j \zeta^j \right\} \tag{A-8}$$

Again, by using the method of partial fractions, it may be readily seen that the solution is reducible to Eq. (A-6). Summing up the result, it is now clear that the order of the polynomial $p(\zeta)$ may be reduced to zero without losing the generality of the solution $Q(\zeta)$.

Finally, we shall consider the double signs ($\pm$) occurring in Eq. (A-3). If there is a negative sign, we may write

$$H(\zeta) = \frac{h(\zeta)}{\sqrt{\zeta - b}} \tag{A-9}$$

and the solution for $Q(\zeta)$ is

$$Q(\zeta) = \frac{h(\zeta)}{\pi\sqrt{\zeta - b}} \left\{ \int_{-\infty}^{+\infty} \frac{Q_o \sqrt{\tau - b} \, d\tau}{(\tau - \zeta)h(\tau)} + \sum A_j \zeta^j \right\} \tag{A-10}$$

Since all the coefficients of the infinite series are arbitrary constants, we may write

$$\sum A_j \zeta^j = - \int_{-\infty}^{\infty} \frac{Q_o \, d\tau}{\sqrt{\tau - b} \;\; h(\tau)} + (\zeta - b) \sum B_j \zeta^j \tag{A-11}$$

Substituting Eq. (A-11) into Eq. (A-10) and simplifying, there results

$$Q(\zeta) = \frac{\sqrt{\zeta - b} \;\; h(\tau)}{\pi} \left\{ \int_{-\infty}^{\infty} \frac{Q_o \, d\tau}{(\tau - \zeta)\sqrt{\tau - b} \;\; h(\tau)} + \sum B_j \zeta^j \right\} \tag{A-12}$$

This proves that the choice of the double signs is immaterial. This also completes the proof that the general solution of the Riemann-Hilbert problem is Eq. (11) with the homogeneous solution given by Eq. (12).

It should be noted that the infinite series must satisfy certain convergence requirements in order to guarantee that the Hölder condition is satisfied. For example, the infinite series in Eq. (A-10) must converge at $\zeta = b$ whereas the infinite series in Eq. (A-12) may diverge such that

$$\lim_{\zeta \to b} (\zeta - b) \sum B_j \zeta^j$$

exists.

If $\zeta = b$ is a regular point, which is the requirement imposed by the last boundary condition in Section II, then the infinite series in Eq. (A-12) must converge at $\zeta = b$.

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