UNIVERSITY OF MINNESOTA
ST. ANTHONY FALLS HYDRAULIC LABORATORY
LORENZ G. STRAUB, Director

Technical Paper No. 39, Series B

A Note on the Linear Theory of Two-Dimensional Separated Flows about Thin Bodies

by

C. S. SONG

Prepared for
OFFICE OF NAVAL RESEARCH
Department of the Navy
Washington, D.C.

Contract Nonr 710(24), Task NR 062-052

August 1962
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ABSTRACT

By using a generalized method of solution for the mixed boundary value problem of analytic function theory, and by comparing the present method with the method of source-sink distribution and the method of analytic continuation, an attempt is made to unify the seemingly divergent development of the linear theories of thin foils with separating flows. It is shown that most of the mathematical models may be regarded as special cases of a generalized Riabouchinsky model.

The admission of a singularity, which is characteristic of the linear theory, introduces an arbitrary constant and hence the solution is generally non-unique. Therefore, it is always necessary to use additional conditions which are normally not required if exact theory is used. The number of the additional conditions required is equal to the number of singularities admitted. The solution can be made unique, however, by requiring that the solution must be sectionally continuous on the boundary and bounded at infinity.

By admitting a singularity at a separation point, the model will represent a flow wherein the free streamline separates normally from the solid boundary. Since it is known, in some instances, that a free streamline does not necessarily separate smoothly from the solid boundary, this model may have practical usefulness.

Finally, it is shown that all the three methods currently being widely used may be used to solve any mathematical model and yield identical solutions. For the sake of simplicity, only a symmetrical thin wedge is discussed in detail.

C O N T E N T S

L I S T   O F   I L L U S T R A T I O N S

L I S T O F S Y M B O L S

A_n, C_m - constants

C_D - drag coefficient

C_{D_0} - drag coefficient for open cavity model

C_{D_1} - drag coefficient due to tail body

C_{D_i} - drag coefficient for image model

$H(z)$ - function, homogeneous solution

$h(x)$ - tail body thickness

i - unit of imaginary number

ℓ - cavity length

$m(x)$ - strength of source-sink distribution

P - pressure

P_∞ - pressure at infinity

P_c - cavity pressure

$Q(z)$ - function of a complex variable

q - speed of flow

$\vec{q}$ - velocity of flow

r - radial distance

tw - half cavity thickness at tail

tw_o - half cavity thickness for an open cavity

T - tw/tw_o

T' - $1/T$

U - speed at infinity

u - x-component of perturbation velocity

v - y-component of perturbation velocity

$W(z)$ - complex conjugate of perturbation velocity

x,y - coordinates

$z = x + iy$ - complex variable

β - function of cavitation number

γ - half wedge angle

γ_1 - half wedge angle of tail body

μ - real part of a complex function

ν - negative of the imaginary part of a complex function

σ - cavitation number

τ, ξ - variable of integration

A NOTE ON THE LINEAR THEORY OF TWO-DIMENSIONAL SEPARATED FLOWS ABOUT THIN BODIES

I. INTRODUCTION

When a rigid body of finite size performs a steady motion in an otherwise quiescent, incompressible, infinite fluid, the resulting boundary layer is thin and the flow may be considered irrotational. For such a flow, ideal flow theory predicts zero drag. However, it is well known that the actual drag is usually much too large to be attributable to skin friction alone. This paradox may be resolved if one recalls that the flow within the boundary layer and the region immediately downstream of the body is rotational and that vorticity is transported indefinitely far downstream following the streamlines. That is to say, a realistic mathematical model should allow for a narrow but infinitely long region in which the flow must be considered rotational. Indeed, the solution of Kirchhoff and Rayleigh [1] for the flow about a two-dimensional plate with an infinite wake yielded a finite drag. Unfortunately, this solution applies when the wake pressure is constant and equal to the pressure at infinity, the zero cavitation number case, and this condition is not realized in real wakes.

For other cavitation numbers, Roshko [2] proposed an infinite wake model in which he assumed that the wake is divided into two regions--a finite constant pressure region with pressure less than the pressure at infinity, and an infinite parallel region wherein the pressure approaches that at infinity. The drag is considered to be the byproduct of energy dissipation in the parallel wake region. Although this model makes no provision for different degrees of energy dissipation, it is a good potential flow model.

The well-known Riabouchinsky model [3, 4] in which a hypothetical image is used as the source of drag also yields a satisfactory result when the image is not too close to the real body.

Tulin [5, 6] introduced a linear theory to cavity flow problems and greatly simplified the analysis. In the linear theory the body plus cavity and wake are represented by a cut on the x-axis of the physical plane. The linearized boundary condition is then satisfied by distributing source-sink or vortex singularities along the cut. Shortly after Tulin's original paper on the linear theory, Wu [7] solved a similar problem using the technique of conformal mapping and the principle of analytical continuation. Both Tulin and Wu assumed that the cavity is closed and, hence, a singularity is admitted at the tail of the cavity. Later, Fabula [8] proposed a partly open, linearized, cavity model. Fabula's model may be considered as a generalized Wu model because it contains the closed cavity model as a special case. Fabula was also able to obtain an open cavity model as another special case by setting the strength of the "closure singularity" equal to zero. It should be noted that Fabula's open cavity model corresponded to a linearized Roshko model and that the "non-singular solution" by Song [9] is also of a similar type. The purpose of this paper is to show that all the aforementioned models in the linear theory may be regarded as special cases of a linearized version of the "Generalized Riabouchinsky Model". By doing so, it is hoped that the relationship between the various existing models may be clarified and the possibilities of deducing more special models will become apparent.

Although most of the results discussed in this paper apply for thin bodies of arbitrary shape and small attack angle, only a thin wedge placed parallel to the main flow will be specifically considered. The thin wedge is used mainly because the solutions are simple and there exist many special solutions for comparison.

This research has been supported by the Office of Naval Research of the U.S. Department of the Navy under Contract Nonr 710(24), Task NR 062-052. The author is indebted to Professor Edward Silberman for his helpful discussions and review of the present work. The manuscript was prepared for printing by Carol Takyi under the general supervision of Mr. Loyal Johnson.

II. THE GENERALIZED RIABOUCHINSKY MODEL

Consider a wedge of unit chord and angle 2γ moving with speed U parallel to the chord. A cavity of constant pressure followed by a wake of variable pressure is assumed to form behind the wedge. To simulate the flow, Plesset and Shaffer [4] introduced a mirror image about the maximum diameter of the cavity, including a "tail body" at the end of the cavity identical with the wedge; the resulting flow is symmetrical about two perpendicular axes. As a result of symmetry the analysis is considerably simplified; simplification is very desirable for the non-linear theory. For the linear theory, however, symmetry is not so important and there is considerable freedom in choosing the "tail body"; the tail body need not be identical with the wedge. For the sake of simplicity let the tail body also be a wedge, of arbitrary chord, λ, and angle $2\gamma_1$. As will be seen later, it turns out that only one of the parameters λ and γ_1 may be chosen arbitrarily if a smooth junction of the cavity wall with the tail body is to be maintained. The resulting flow configuration is sketched in Fig. 1. As usual, the dimensionless conjugate perturbation velocity defined by the following formulas will be considered:

$$W(z) = u(x, y) - iv(x, y) \tag{1}$$

$$\vec{q} = q_c(1 + u, v), \quad u << 1, \quad v << 1 \tag{2}$$

$$z = x + iy \tag{3}$$

where q_c is the speed of flow on the cavity wall. The function $W(z)$ is an analytic function outside of the cavity-body region which is represented by a cut $\overline{oc}$ in the linearized plane shown in Fig. 1 (b). Since the flow is symmetrical with respect to the x-axis, only the upper half of the region needs to be considered.

The linearized boundary conditions are listed below:

(1) The vertical velocity component v is identical to zero on the lines $\overline{od_\infty}$ and $\overline{cd_\infty}$.

(2) The velocity vector on the wedge surface is parallel to the surface. That is,

$$v \simeq \frac{v}{1+u} = \tan \gamma, \quad \text{on } \overline{oa}$$

$$v \simeq - \tan \gamma_1, \quad \text{on } \overline{bc} \tag{4}$$

(3) The pressure in the cavity is assumed to be a constant, P_c, so that the speed q_c is also a constant. This leads to the condition that

$$u = 0, \quad \text{on } \overline{ab}$$

(4) The condition at infinity is, by Bernoulli's equation,

$$u_\infty = \frac{1}{\sqrt{1+\sigma}} - 1 = -\beta \tag{5}$$

$$v_\infty = 0$$

where σ is the cavitation number defined as

$$\sigma = \frac{P_\infty - P_c}{\frac{1}{2}\rho U^2} \tag{6}$$

(5) In order to guarantee a smooth junction between the cavity wall and the tail body, λ and γ_1 should be chosen so that the following relationship is fulfilled.

$$tw = \int_0^\ell v dx = - \operatorname{Im} \int_0^\ell W(z) dz = \lambda \tan \gamma_1 \tag{7}$$

where tw is the half cavity width at the point b.

To solve the mixed boundary value problem of this type the general method of solution reported by Chen and Rott [10] is useful. The key of the method is to change the problem into a regular boundary value problem by dividing (or multiplying) the original function by an analytic function which is alternately purely real or purely imaginary on the x-axis. Let such an analytic function be designated by $H(z)$ and the new function $Q(z)$ be defined as

$$Q(z) = \frac{W(z)}{H(z)} = \mu - i\nu \tag{8}$$

If $H(z)$ is purely real on $\overline{d_\infty a}$ and $\overline{bd_\infty}$ and purely imaginary on $\overline{ab}$, then the imaginary part of $Q(z)$ is completely specified on the whole x-axis. That is, the boundary condition on the x-axis for $Q(z)$ is

(1) $\nu(x) = 0$ on $\overline{od_\infty}$ and $\overline{cd_\infty}$

(2) $\nu(x) = \begin{cases} i \tan \gamma/H(x) & \text{on } \overline{oa} \\ -i \tan \gamma_1/H(x) & \text{on } \overline{bc} \end{cases}$

(3) $\nu(x) = 0$ on $\overline{ab}$

It is well known that the general piece-wise continuous solution satisfying the three given boundary conditions on the x-axis is [10].

$$Q(z) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\nu(\tau)d\tau}{\tau - z} + \frac{1}{\pi}\sum_{n=0}^{+\infty} A_n z^n \tag{9}$$

where A_n are real constants and n represents all positive integers. The addition of the second term on the right hand side of Eq. (9) is possible because it is real on the real axis.

By definition, it follows that

$$W(z) = \frac{1}{\pi} H(z)\left[-\int_{-\infty}^{+\infty} \frac{\nu(\tau)d\tau}{\tau - z} + \sum A_n z^n\right] \tag{10}$$

It is readily seen that any of the following functions has the required property of $H(z)$:

$$H(z) = \sqrt{(\ell - z)(1 - z)}, \sqrt{\frac{1 - z}{\ell - z}}, \sqrt{\frac{\ell - z}{1 - z}}, \frac{1}{\sqrt{(\ell - z)(1 - z)}} \tag{11}$$

In fact, it can be shown that use of any of the functions of Eq. (11) will lead to the same final result. For the purpose of illustration, however, only the second function will be considered in detail. A more rigorous proof of the uniqueness property is given in Appendix A.

If the second function of Eq. (11) is used, the solution may be written as

$$W(z) = \frac{1}{\pi}\sqrt{\frac{1-z}{\ell - z}}\left[-\tan\gamma \int_0^1 \sqrt{\frac{\ell-\tau}{1-\tau}} \frac{d\tau}{\tau - z} + \tan\gamma_1 \int_\ell^{\ell+\lambda} \sqrt{\frac{\ell-\tau}{1-\tau}} \frac{d\tau}{\tau - z} + A_o \right] \tag{10a}$$

Here only $n = 0$ is allowed because $W(z)$ must be finite everywhere (including the point at infinity) except at the origin where a logarithmic type singularity is inevitable. The constant A_o can be determined by the Kutta condition. That is, A_o must be chosen so that $W(z)$ is finite at $z = \ell$ which is the only apparent singularity in Eq. (10a). Therefore, it follows that

$$A_o = -\tan\gamma \int_0^1 \frac{d\tau}{\sqrt{(\ell-\tau)(1-\tau)}} + \tan\gamma_1 \int_\ell^{\ell+\lambda} \frac{d\tau}{\sqrt{(\ell-\tau)(1-\tau)}} \tag{12}$$

Substituting Eq. (12) into Eq. (10a) and simplifying, it follows that

$$W(z) = \frac{1}{\pi} \sqrt{(\ell - z)(1-z)} \left[-\tan\gamma \int_0^1 \frac{d\tau}{(\tau - z)\sqrt{(\ell-\tau)(1-\tau)}} \right.$$

$$\left. + \tan\gamma_1 \int_\ell^{\ell+\lambda} \frac{d\tau}{(\tau - z)\sqrt{(\ell-\tau)(1-\tau)}} \right] \tag{10b}$$

It is readily seen that if the first function of Eq. (11) were taken for $H(z)$, Eq. (10b) would still be the solution. In this case all the constants A_n have to vanish because $W(z)$ is finite at infinity. It may also be shown that the third choice in Eq. (10) will lead to the same conclusion. If the last function is chosen, then, two constants (A_o and A_1) are allowed and, hence, by applying the Kutta condition at $x = 1$ and $x = \ell$ the solution may also be reduced to Eq. (10b).

Performing the integration, Eq. (10b) leads to

$$W(z) = -\frac{1}{\pi}\tan\gamma \ln \frac{(1-\ell)z}{[\sqrt{\ell(1-z)}-\sqrt{\ell-z}]^2}$$

$$+\frac{1}{\pi}\tan\gamma_1 \ln \frac{[\sqrt{\lambda(z-1)}-\sqrt{(\ell-z)(1-\ell-\lambda)}]^2}{(\ell+\lambda-z)(1-\ell)} \tag{13}$$

Next, the "juncture condition", condition (5), will be applied to determine the $\lambda \sim \gamma_1$ relationship. By substituting Eq. (10b) into Eq. (7) and performing the integration, there results the following simple equation:

$$\tan\gamma_1 = \sqrt{\frac{\ell}{\lambda(\ell+\lambda-1)}}\tan\gamma \tag{14}$$

Finally, by applying the boundary condition at infinity, the cavity length, ℓ, may be determined as a function of the cavitation number, σ. The result is

$$\beta = \beta_o + \beta_1 \tag{15}$$

$$\beta_o = \frac{2\tan\gamma}{\pi}\ln\frac{\sqrt{\ell}+1}{\sqrt{\ell-1}} \tag{16}$$

$$\beta_1 = \frac{2\tan\gamma_1}{\pi}\ln\frac{\sqrt{\ell+\lambda-1}+\sqrt{\lambda}}{\sqrt{\ell-1}} \tag{17}$$

Equation (15) may also be written in the following alternate form:

$$\frac{\sigma}{\sigma_o} \approx \frac{\beta}{\beta_o} = 1 + \sqrt{\ell}\ln\frac{\sqrt{\ell+\lambda-1}+\sqrt{\lambda}}{\sqrt{\ell-1}} \Big/ \sqrt{\lambda(\ell+\lambda-1)}\ln\frac{\sqrt{\ell}+1}{\sqrt{\ell-1}} \tag{18}$$

where the relationship (14) has been incorporated in the equation and σ_o is the cavitation number corresponding to Fabula's open cavity model, as will be shown subsequently.

The drag coefficient may be defined as

$$C_D = \frac{\text{Drag}}{\frac{1}{2}\rho U^2} = 2(1+\sigma)\sin\gamma\int_0^1 \frac{P-P_c}{\frac{1}{2}\rho q_c^2}\,dx$$

By linearizing the Bernoulli equation, the integrand may be shown to be approximately equal to $-2u$ and

$$C_D = -4(1+\sigma)\sin\gamma\int_0^1 \mathrm{Re}\, W(z)\,dz \tag{19}$$

where Re means "real part of".

Equation (10b) or (13) may now be substituted into Eq. (19) and there results

$$C_D = C_{D_0} + C_{D_1} \tag{19a}$$

where $$C_{D_0} = \frac{4}{\pi}(1+\sigma)\tan\gamma\sin\gamma\,\mathrm{Re}\int_0^1\int_0^1 \frac{H(z)\,d\tau\,dz}{(\tau - z)H(\tau)} \tag{20}$$

$$C_{D_1} = -\frac{4}{\pi}(1+\sigma)\tan\gamma_1\sin\gamma\,\mathrm{Re}\int_0^1\int_\ell^{\ell+\lambda} \frac{H(z)\,d\tau\,dz}{(\tau - z)\,H(\tau)} \tag{21}$$

and $H(z)$ is given by the first function of Eq. (11).

The double integral of Eq. (20) appeared in Reference [9] and its value was computed. The final result was

$$C_{D_0} = \frac{8}{\pi}(1+\sigma)\tan\gamma\sin\gamma\sqrt{\ell}\,\ln\frac{\sqrt{\ell}+1}{\sqrt{\ell}-1} \tag{20a}$$

The second double integral can be evaluated by direct integrations. The result is

$$C_{D_1} = \frac{8}{\pi}(1+\sigma)\tan\gamma_1\sin\gamma\Big[(\ell+\lambda)\ln\frac{\sqrt{\ell(\ell+\lambda-1)}+\sqrt{\lambda}}{\sqrt{(\ell+\lambda)(\ell-1)}}$$

$$-\sqrt{\lambda(\ell+\lambda-1)}\,\ln\frac{\sqrt{\ell}+1}{\sqrt{\ell}-1}\Big] \tag{21a}$$

The drag coefficient may also be written in the following alternate form:

$$\frac{C_D}{C_{D_0}} = (\ell + \lambda) \ln \frac{\sqrt{\ell(\ell + \lambda - 1)} + \sqrt{\lambda}}{\sqrt{(\ell + \lambda)(\ell - 1)}} \Big/ \sqrt{\lambda(\ell + \lambda - 1)} \ln \frac{\sqrt{\ell} + 1}{\sqrt{\ell} - 1} \tag{22}$$

where the relationship (14) has been incorporated in the equation.

The following special cases may be readily recognized by observing Eq. (14).

(1) When $\lambda \to \infty$, it is necessary that $\gamma_1 = 0$. It will be shown later that this is equivalent to Fabula's open cavity model (Song's non-singular model). In this case, $\beta = \beta_o$ and $C_D = C_{D_0}$. Consequently, it is seen that Eqs. (18) and (22) are the ratios of the cavitation number and the drag coefficient of a particular model with respect to those of the open cavity or non-singular model.

(2) When $\lambda = 1$, $\gamma_1 = \gamma$. This is the image model. In this case Eqs. (18) and (22) lead to

$$\frac{\beta_i}{\beta_o} = 2 \tag{23}$$

$$\frac{C_{D_i}}{C_{D_0}} = (\ell + 1) \ln \sqrt{\frac{\ell + 1}{\ell - 1}} \Big/ \sqrt{\ell} \ln \frac{\sqrt{\ell} + 1}{\sqrt{\ell} - 1} \tag{24}$$

The present result agrees with Fabula's result [8] obtained by a different method.

(3) When $\lambda = 0$, then, $\gamma_1 = \frac{\pi}{2}$. Since the velocity vector at $x = \ell$ is now parallel to the y-axis, this case corresponds to the tail-singularity model. The detailed discussion will be presented in the next section.

To show the effect of the size of the tail body on the cavitation number and the drag coefficient, Eqs. (18) and (22) are plotted in Figs. 2 and 3. It is readily seen that the effect of a tail body is large for small λ.

III. OTHER SPECIAL MODELS

Since there is considerable freedom in selecting the shape and the size of the tail body, a number of special models may be constructed by considering various possible limiting cases. For example, as indicated in the last section, the author's non-singular model [9] (equivalent to the linearized Roshko model and the open cavity model) may be considered as a limiting case of the general Riabouchinsky model as $\lambda \to \infty$ and $\gamma_1 = 0$. Tulin and Wu's singularity model and Fabula's partly closed model, on the other hand, may be regarded as another limiting case as $\lambda \to 0$ and $\gamma_1 \to \frac{\pi}{2}$.

A. Non-Singular Models

If the angle γ_1 is set equal to zero, then the second term of Eq. (10b) vanishes. The resulting velocity function is

$$W(z) = - \frac{\tan \gamma}{\pi} \sqrt{(\ell - z)(1 - z)} \int_0^1 \frac{d\tau}{(\tau - z)\sqrt{(\ell - \tau)(1 - \tau)}} \tag{10c}$$

This function can also be obtained directly from Eq. (10) by taking the first function of Eq. (11) for $H(z)$. The formula for the cavity length, by applying the condition at infinity, is given by Eq. (16). The drag coefficient, by substituting Eq. (10c) into Eq. (19), is given by Eq. (20a). It should be noted that for a small wedge angle both $\tan \gamma$ and $\sin \gamma$ may be approximated by γ and the equations for the cavity length and the drag coefficient agree with the equations given in Reference [9]. It is also interesting to note that when $\sigma = 0$ the drag coefficient is reduced to

$$C_{D_0} (\sigma = 0) = \frac{8\gamma^2}{\pi} \tag{20b}$$

whereas when $\gamma \to 0$ for a given σ the drag coefficient is reduced to

$$C_{D_0} (\gamma \to 0) = 2\sigma\gamma \tag{20c}$$

The half cavity width at the tail, adopting Fabula's notation, is

$$tw_o = \sqrt{\ell} \tan \gamma \tag{25}$$

where the subscript again means open cavity or non-singular model. It should be noted that, for a given body and a given cavitation number, ℓ is determined by Eq. (16) and, hence, the half cavity width is also determined. It is now clear that the cavity width is uniquely determined by the cavitation number if no singularity is admitted. A schematic drawing of the flow configuration corresponding to the present model is shown in Fig. 4 (a).

If a model with more realistic cavity-wake profile is desired, an infinite tail body of variable width as shown in Fig. 4 (b) may be used. In such a case, the velocity function would be written as

$$S(z) = \frac{1}{\pi} \sqrt{(\ell - z)(1 - z)} \left[-\tan\gamma \int_0^1 \frac{d\tau}{(\tau - z)\sqrt{(\ell - \tau)(1 - \tau)}} \right.$$

$$\left. + \int_\ell^\infty \frac{\frac{dh(\tau)}{d\tau}\, d\tau}{(\tau - z)\sqrt{(\ell - \tau)(1 - \tau)}} \right] \tag{10d}$$

For example, a hyperbolic profile may be assumed, and the tail body profile is given by

$$\frac{dh}{dx} = \sum C_m x^{-m} \tag{26}$$

The number of terms and the constants in Eq. (26) have to be determined from the assumed tail body profile. Of course, the best wake profile will not be known unless the viscous effect is taken into consideration.

There remains much room for further study on the effect of viscous dissipation, perhaps through consideration of the momentum thickness or the displacement thickness. However, this is beyond the scope of the present paper and will not be discussed further.

B. Singularity Models

It is now clear that if a closed or partly closed cavity is required, an additional free constant should be introduced in the function $W(z)$.

Besides using a hyperbolic tail body as discussed in the last section, the same object may also be achieved by admitting a singularity at a suitable point. It should be observed that there are two points ($x = 1$, $x = \ell$) on the x-axis where a singularity may be admitted. Since u is assumed to be constant on the cavity profile, a singularity means infinite v which implies a vertical velocity vector.

1. Tail Singularity

This case may also be considered as the limiting case of the general Riabouchinsky model as $\gamma_1 \to \frac{\pi}{2}$ and $\lambda \to 0$. For this condition, the second term of Eq. (10a) is a product of infinity and zero which is indeterminate. Hence, this term may be treated as a free constant and may be absorbed into the constant A_o. The general form of $W(z)$ is

$$W(z) = \frac{1}{\pi}\sqrt{\frac{1 - z}{\ell - z}}\ [-\tan\gamma \int_0^1 \sqrt{\frac{\ell - \tau}{1 - \tau}}\ \frac{d\tau}{\tau - z} + A_o] \tag{10e}$$

Since the first term of Eq. (10e) vanishes for $z \to \infty$, the condition at infinity is

$$A_o = -\pi\beta \tag{27}$$

The half cavity width is

$$tw = \tan\gamma\ [\sqrt{\ell} + (\ell - 1)\ \ln \frac{\sqrt{\ell - 1}}{\sqrt{\ell} - 1}\ ] - \frac{\pi}{2}\ (\ell - 1)\ \beta \tag{28}$$

By using Eq. (25), Eq. (28) may also be written as

$$\frac{\pi\beta}{\tan\gamma} = (1 - T)\ \frac{2\sqrt{\ell}}{\ell - 1} + \ln \frac{\sqrt{\ell} + 1}{\sqrt{\ell} - 1} \tag{28a}$$

where T is the ratio, tw/tw_o. Equation (28a) was first given by Fabula [8].

The drag coefficient is

$$C_D = \frac{8}{\pi} (1 + \sigma) \sin\gamma \tan\gamma \left\{ \frac{\pi \beta}{2 \tan\gamma} [\sqrt{\ell} - (\ell - 1) \ln \frac{\sqrt{\ell - 1}}{\sqrt{\ell} - 1}] \right.$$

$$\left. + (\ell - 1) [\ln \frac{\sqrt{\ell - 1}}{\sqrt{\ell} - 1}]^2 \right\} \tag{29}$$

An alternate form is, by using Eq. (28a)

$$C_D = \frac{8}{\pi} (1 + \sigma) \sin\gamma \tan\gamma [T\sqrt{\ell} \ \ln \frac{\sqrt{\ell - 1}}{\sqrt{\ell} - 1} + (1 - T) \frac{\ell}{\ell - 1}] \tag{29a}$$

The equations for the closed cavity model are readily obtained by setting $tw = T = 0$ in Eq. (28a) and Eq. (29a). The results are

$$\beta_c = \frac{2 \tan\gamma}{\pi} (\frac{\sqrt{\ell}}{\ell - 1} + \ln \frac{\sqrt{\ell - 1}}{\sqrt{\ell} - 1}) \tag{30}$$

$$C_{DC} = \frac{8}{\pi} (1 + \sigma) \sin\gamma \tan\gamma (\frac{\ell}{\ell - 1}) \tag{31}$$

By replacing $\sin\gamma$ and $\tan\gamma$ by γ, Eqs. (30) and (31) become identical to the corresponding formulas given by Wu [7]. For the other special case, if $tw = tw_o$, or if A_o is chosen to be equal to that of the non-singular model,

$$A_o = -2 \tan\gamma \ln \frac{\sqrt{\ell - 1}}{\sqrt{\ell} - 1} \tag{32}$$

Then, Eqs. (28a) and (29) are reduced to Eqs. (16) and (20a) of the non-singular model respectively.

Flow configurations corresponding to the various singularity models are indicated in Fig. 5. Fig. 5 (a) is the closed-cavity model which has the strongest tail-singularity and Fig. 5 (b) is a partly closed-cavity model. Fig. 5 (c) is the limiting condition of the tail-singularity model as the strength of the tail-singularity is reduced to zero. It should be noted that this model gives the same flow as that of the open cavity model.

2. Separation Singularity

Separation of flow due to gas injection or a cavitation flow due to roughness, etc., may be simulated by this model. The resulting flow configuration is shown in Fig. 6. The singularity occurs at the trailing edge of the wedge.

Using the third function of Eq. (11), there results a velocity function

$$W(z) = \frac{1}{\pi}\sqrt{\frac{\ell - z}{1 - z}}\,[-\tan\gamma\int_0^1 \sqrt{\frac{1-\tau}{\ell-\tau}}\,\frac{d\tau}{\tau - z} + A_o] \tag{10d}$$

The condition at infinity leads to Eq. (27) and the cavity width is computed to be

$$tw = \tan\gamma[\sqrt{\ell} - (\ell - 1)\ \ln\frac{\sqrt{\ell - 1}}{\sqrt{\ell} - 1}] + \frac{\pi}{2}(\ell - 1)\,\beta \tag{33}$$

The drag coefficient is

$$C_D = \frac{8}{\pi}(1+\sigma)\ \sin\gamma\tan\gamma[T'\sqrt{\ell}\ \ln\frac{\sqrt{\ell-1}}{\sqrt{\ell}-1} + (T' - 1)\frac{\ell}{\ell - 1}] \tag{34}$$

where $T' = 1/T$

It is readily seen that, for a given cavitation number, this model gives wider cavity and larger drag coefficient than the cavity with smooth flow at the wedge. It should also be noted that, since the right hand side of Eq. (33) is always positive, the cavity can never close.

3. Double Singularities

The final, and most general, case is to admit singularities at both points, the trailing edge of the wedge and the end of the cavity. By properly adjusting the strength of the singularities, a cavity of arbitrary opening with arbitrary deflection at the separation point may be obtained.

In this case, two free constants are allowed and the velocity function is

$$W(z) = \frac{1}{\pi\sqrt{(\ell - z)(1 - z)}}\left[- \tan\gamma \int_0^1 \frac{\sqrt{(\ell - \tau)(1 - \tau)}\,d\tau}{\tau - z} + A_o + A_1 z\right] \quad (10f)$$

Applying the condition at infinity, there results

$$A_1 = - \pi \beta \quad (35)$$

The half cavity width at the tail is

$$tw = 2 \tan\gamma - A_o + \frac{\pi}{2} (\ell + 1) \beta \quad (36)$$

and the drag coefficient is

$$C_D = \frac{8(1 + \sigma) \sin\gamma}{\pi}\left\{\frac{\pi\beta}{2}\left[(\ell + 1) \ln \frac{\sqrt{\ell} + 1}{\sqrt{\ell} - 1} - \sqrt{\ell}\right] - A_o \ln \frac{\sqrt{\ell} + 1}{\sqrt{\ell} - 1} - \tan\gamma \sqrt{\ell} \ln \frac{\sqrt{\ell} + 1}{\sqrt{\ell} - 1}\right\} \quad (37)$$

It is readily seen that there is a free constant, A_o, which cannot be determined unless one more condition is given. This additional condition could be the momentum of the jet, in case of a jet flap, or the roughness drag, in case the separation is due to roughness or a solid flap.

The flow configuration of the general case is sketched in Fig. 7.

IV. COMPARISON OF DIFFERENT METHODS

It has been shown that linearized cavitation problems can be attacked by three different methods; namely, source-sink distribution method, method of analytic continuation, and the present method using a variable tail body. It was also shown that all three methods lead to identical solutions, at least when the closed tail-singularity model is used. Moreover, for the open cavity model and partly open cavity models with tail singularity, the last two methods gave identical answers, also. In view of these agreements, it is natural to ask if the solutions for all models would be independent of the method used. To show this would be too time consuming. It seems sufficient for the present purpose to show that the open cavity model (the non-singular solution) may also be solved by the classical source-sink distribution method which was first introduced to cavitation problems by Tulin [5].

Tulin's method is outlined as follows:

By defining a perturbation velocity potential as

$$\phi(x, y) = \int_0^{\ell} \frac{m(x')\ln r dx'}{2\pi} \qquad \text{where} \quad r = [(x - x')^2 + y^2]^{\frac{1}{2}} \tag{38}$$

it can be shown that the perturbation velocity components on the x-axis are

$$u(x, o) = \frac{1}{2\pi} \int_0^{\ell} \frac{m(x')dx'}{(x - x')} \tag{39}$$

$$v(x, o) = \frac{m(x)}{2} \tag{40}$$

It should be noted that the perturbation velocity here is referred to U rather than q_c. Since $v(x, o)$ is known in the interval $o \leq x \leq 1$ and $u(x, o)$ is known in the region $1 \leq x \leq \ell$, by substituting Eq. (40) into Eq. (39), the following integral equation for $m(x)$ is obtained:

$$\int_1^{\ell} \frac{m(x')dx'}{x - x'} = \pi\sigma U - 2q_c \tan\gamma \int_0^1 \frac{dx'}{x - x'} = f(x), \quad 1 < x < \ell \tag{41}$$

The general solution of the integral equation (41) is [11]

$$m(x) = - \frac{1}{\pi^2 \sqrt{(x - 1)(\ell - x)}} \left[\int_1^{\ell} \frac{\sqrt{(x' - 1)(\ell - x')}}{x - x'} f(x') dx' + \Gamma \right],$$

$$1 < x < \ell \tag{42}$$

where Γ is an arbitrary constant which may be determined by applying the Kutta condition at $x = 1$. After the Kutta condition is applied the solution is reduced to

$$m(x) = - \frac{\sqrt{x - 1}}{\pi^2 \sqrt{\ell - x}} \int_1^{\ell} \frac{\sqrt{\ell - x'}}{(x - x')\sqrt{x' - 1}} f(x') dx', \quad 1 \leq x < \ell \tag{43}$$

Note that the solution (42) applies on an open interval $(1, \ell)$ whereas the solution (43) applies on a half-closed interval $[1, \ell)$.

At this point, Tulin applied a closure condition to determine, uniquely, the relationship between the cavity length and the cavitation number. The required additional condition can also be obtained, without using a closure condition, if one insists on a continuous solution in the closed interval $[1, \ell]$. The singularity at $x = \ell$ may be removed if

$$\int_1^{\ell} \frac{f(x') dx'}{\sqrt{(x' - 1)(\ell - x')}} = 0 \tag{44}$$

That is, by multiplying (44) by $\dfrac{\sqrt{x - 1}}{\pi^2 \sqrt{\ell - x}}$ and adding the result to the righthand side of (43), there results

$$m(x) = -\frac{\sqrt{(x-1)(\ell-x)}}{\pi^2}\int_1^{\ell}\frac{f(x')dx'}{(x-x')\sqrt{(x'-1)(\ell-x')}}, \quad 1 \le x \le \ell \quad (45)$$

Equation (44) gives the required relationship between the cavity length and the cavitation number. After performing the integration, Eq. (44) leads to Eq. (16) with β_o linearized to $\frac{1}{2}\sigma_o$. Needless to say, the drag coefficient will be identical to Eq. (20a) which is the result from the non-singular solution.

More detailed derivation of the solution (45) is given in Appendix B.

V. CONCLUSION

An attempt has been made to unify most of the existing mathematical models for the linearized theory of two-dimensional flow with cavity and wake behind a thin body. For the sake of simplicity, only a symmetrical wedge is discussed in detail. However, it is obvious that many of the conclusions drawn here can also be applied to other bodies. It was first shown that all the existing singular and non-singular models can be treated as special cases of a generalized Riabouchinsky model. The existence of a singularity in the solution is a characteristic of the linear theory. For the singular cavity problem, there are two juncture points where a singularity may occur--the separation point from the body and the tail of cavity. The strength of the singularity at the separation point may be used to determine the maximum cavity thickness, whereas the strength of the tail singularity may be used to determine the thickness of the cavity at the tail.

Existence of a singularity means the existence of an undetermined free constant. This is why the admittance of the tail singularity demands an additional condition, "the closure condition". The closure condition is used to determine the strength of the tail singularity and the corresponding free constant. The cavity length-cavitation number relationship is, just like in the non-linear theory, determined by the condition at infinity. It is natural to conclude that if a singularity is admitted at the separation point, there will be another undetermined constant and another additional condition, an "opening condition", will be required. It is thus clear that the solution is essentially non-unique if one or more singularities are admitted. However, a unique solution may be obtained by demanding that the solution must be sectionally continuous. This is the so-called "non-singular solution" presented in Reference [9].

Finally, it can be concluded that the three methods (source-sink distribution method, method of analytic continuation, and the present general method of solution for the mixed boundary value problem) give identical solutions. The choice of method is, therefore, more or less up to the individual investigator. However, when the problem involves more than one cavity, then it would seem that the present method should be superior to the other two methods.

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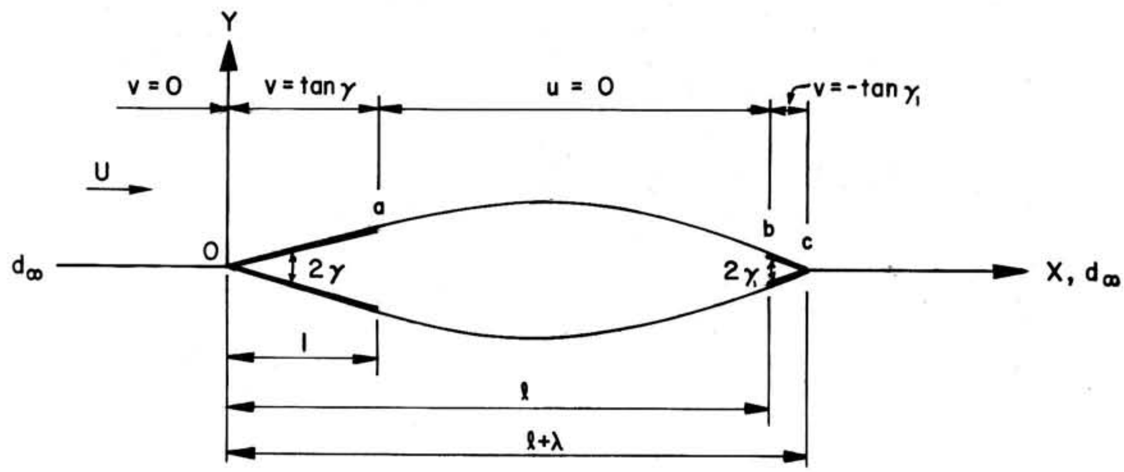
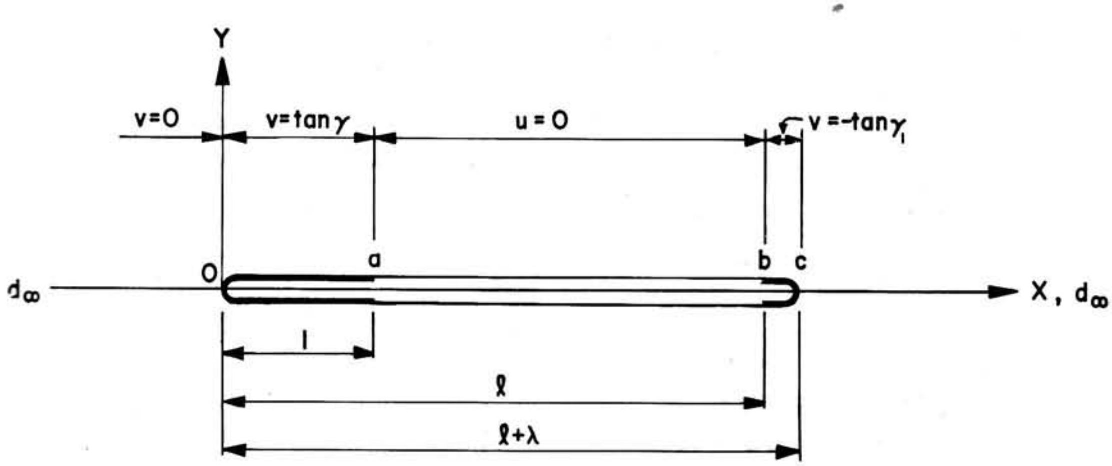
FIGURES

(1 through 7)



(a) Physical Plane



(b) Linearized Physical Plane

Fig. 1 - Generalized Riabouchinsky Model

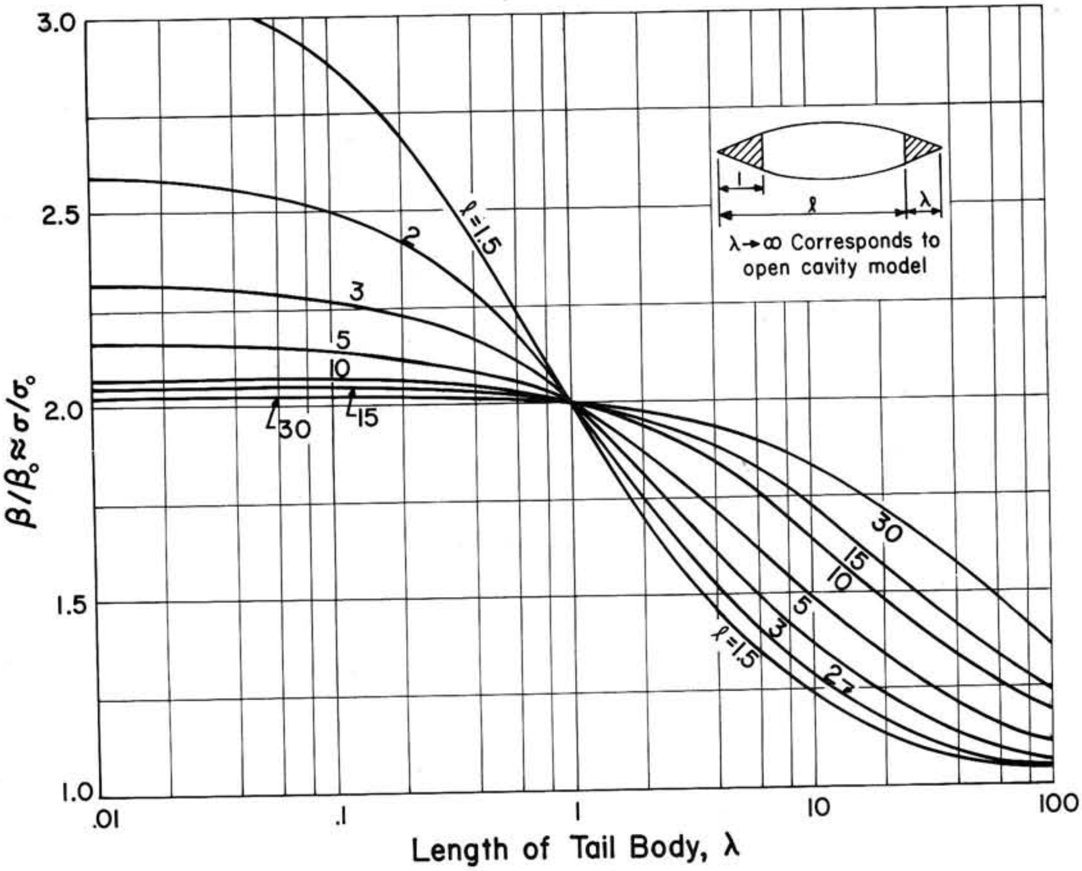


Fig. 2 - The Effect of Tail-Body on Cavitation Number

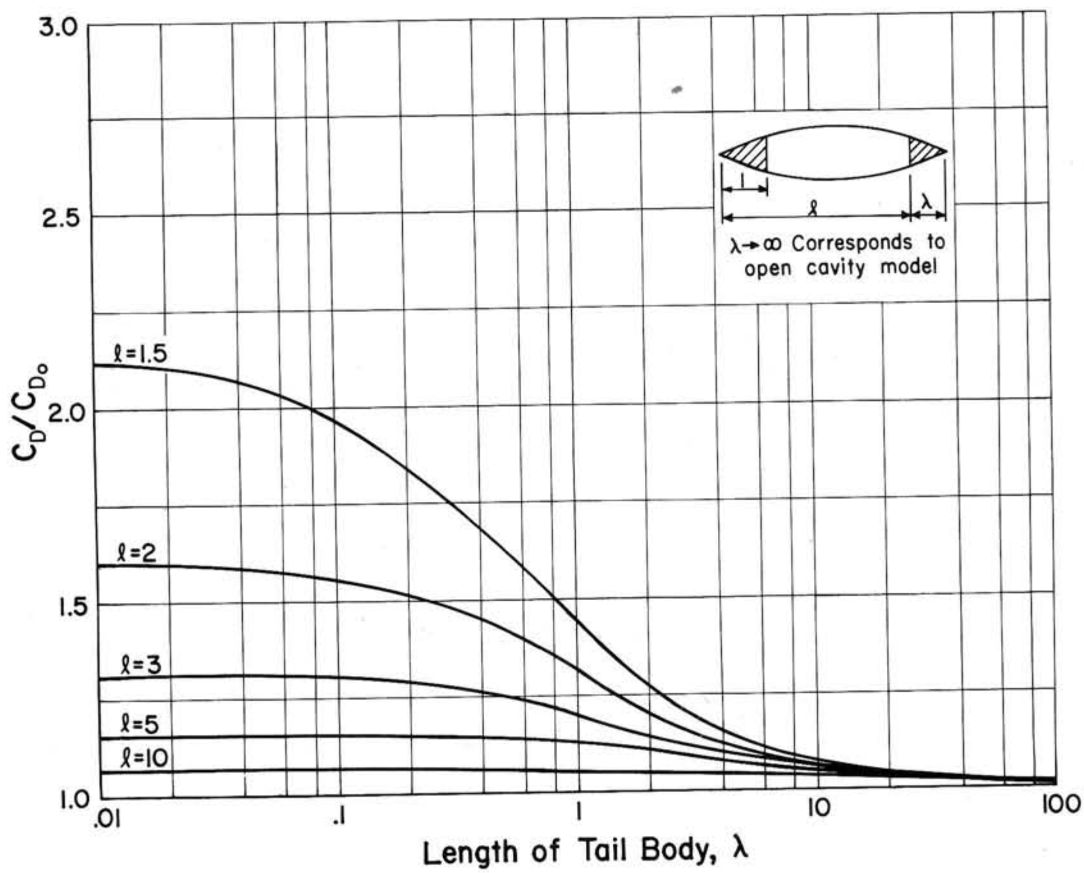
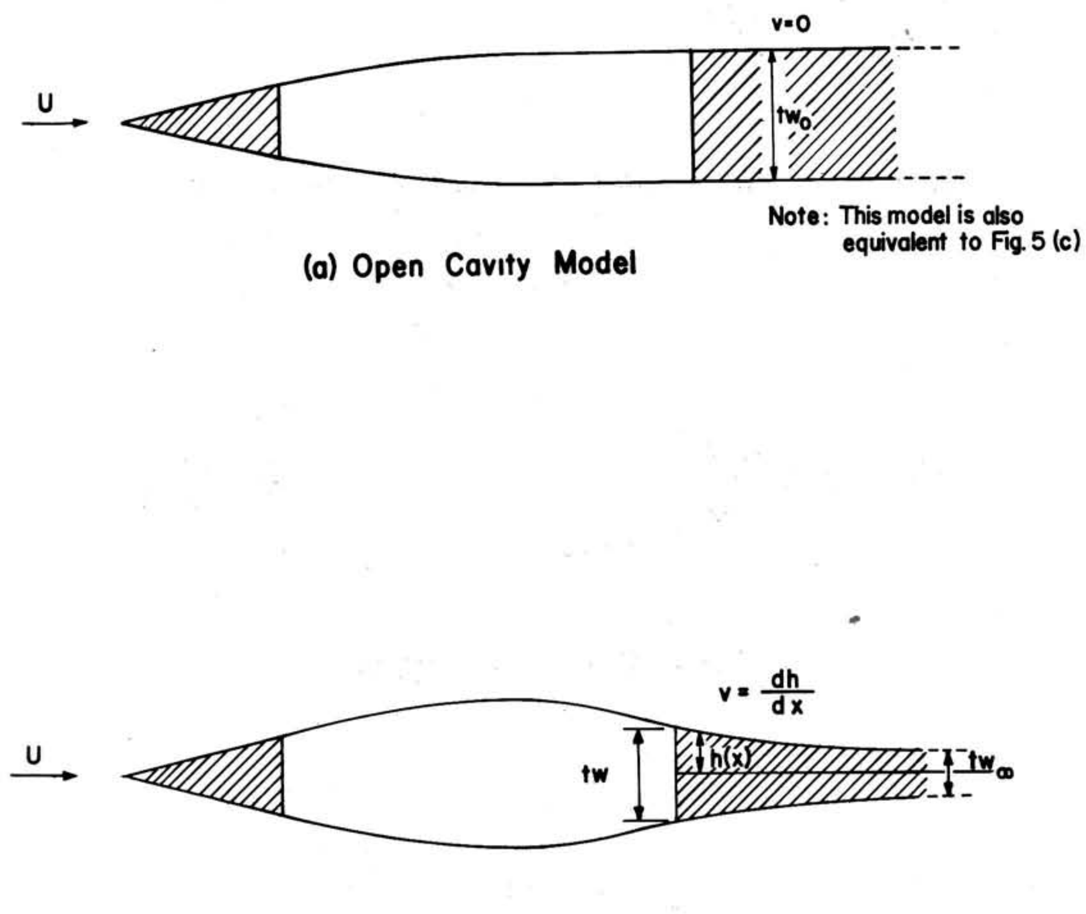
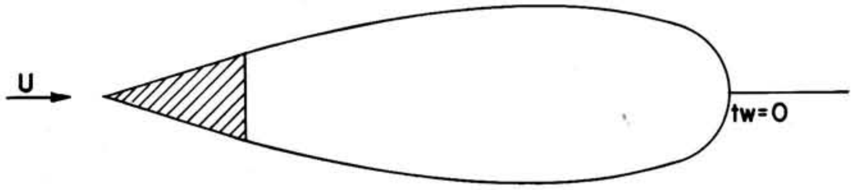


Fig. 3 - The Effect of Tail-Body on Drag Coefficient

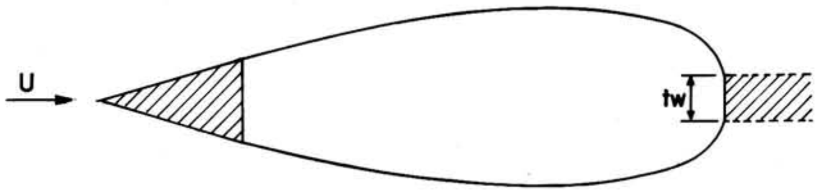


(b) Partly Closed Model

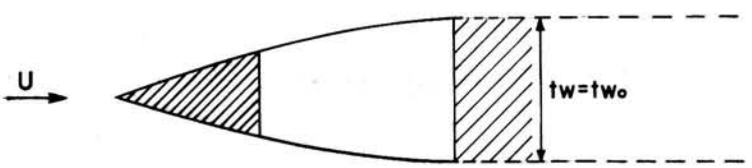
Fig. 4 - Non-Singular Models



(a) Closed Cavity Model (Maximum Tail-Singularity)



(b) Partly Open Cavity Model



Note: This model is also equivalent to Fig.4(a)

(c) Open Cavity Model (Zero Tail-Singularity)

Fig. 5 - Tail-Singularity Models

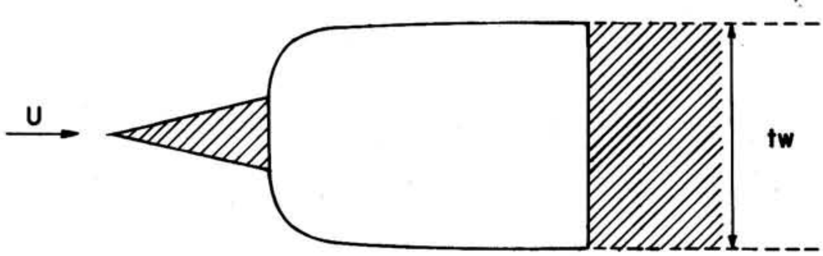


Fig. 6 – Separation–Singularity Model

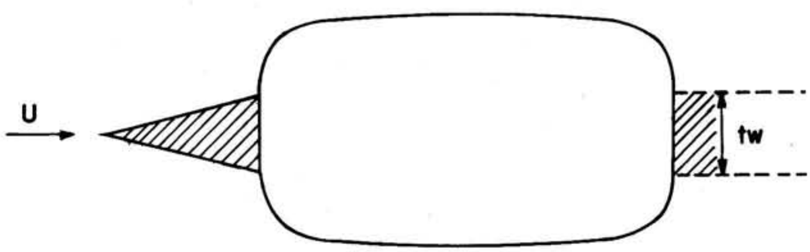


Fig. 7 – Double–Singularity Model

A P P E N D I X   A

A UNIQUENESS THEOREM OF MIXED BOUNDARY VALUE PROBLEM

APPENDIX A

A UNIQUENESS THEOREM OF MIXED BOUNDARY VALUE PROBLEM

The uniqueness of the following boundary value problem will be proved. Thé problem is to find a function $W(z) = u(x, y) - iv(x, y)$, $z = x + iy$, which is analytic on the upper half z-plane, bounded at infinity, sectionally continuous on the real axis except at the origin where a logarithmic type singularity may be required, and satisfying the following boundary conditions on the real axis.

$$
\begin{array}{ll}
x \leq a_1 & v = v_1(x) \\
a_1 \leq x \leq a_2 & u = u_2(x) \\
a_2 \leq x \leq a_3 & v = v_3(x) \\
\vdots & \vdots \\
a_n \leq x & \begin{cases} v = v_{n+1}(x) & \text{if } n \text{ is even} \\ u = u_{n+1}(x) & \text{if } n \text{ is odd} \end{cases}
\end{array}
$$

where v_1, u_2, etc. are sectionally continuous functions of x and $a_1 < a_2 < a_3 \; . \; . \; . \; . \; . < a_n$.

We shall consider the cases with n even and odd separately.

1. n even $(n = 2m, \; m = 1, 2, 3)$

Theorem: The solution of the problem stated above is

$$W(z) = -\frac{H(z)}{\pi}\int_{-\infty}^{+\infty} \frac{q(\tau)d\tau}{\tau - z} \tag{A-1}$$

if

$$\int_{-\infty}^{+\infty} q(\tau)d\tau = \int_{-\infty}^{+\infty} \tau q(\tau)d\tau = \; . \; . \; . = \int_{-\infty}^{+\infty} \tau^{m-2} q(\tau)d\tau = 0$$

and

$$\int_{-\infty}^{+\infty} \tau^{m-1} q(\tau)\, d\tau \quad \text{is bounded} \tag{A-2}$$

where

$$H(z) = \sqrt{(z - a_1)(z - a_2) \ldots (z - a_n)} \tag{A-3}$$

$$q(\tau) = \frac{v_i(\tau)}{H(\tau)} \qquad \text{if } i \text{ is odd}$$

$$q(\tau) = i\, \frac{v_i(\tau)}{H(\tau)} \qquad \text{if } i \text{ is even} \tag{A-4}$$

Moreover, the solution is unique. It should be noted that the integral of (A-1) is improper at $\tau = z$ and the Cauchy principal value must be taken at this point.

Proof: It is obvious that the function (A-1) is analytic on the upper half of the z-plane, sectionally continuous on the real axis except at the origin, and gives the correct boundary value on the real axis. Therefore, for the existence part of the proof, it is sufficient to prove that the function is bounded at infinity.

Since, for large z, the function $\frac{H(z)}{z - \tau}$ behaves like $z^{m-1} + z^{m-2}\tau + \ldots + z\tau^{m-2} + \tau^{m-1} + \ldots$, the function (A-1) behaves like

$$\frac{z^{m-1}}{\pi} \int_{-\infty}^{+\infty} q(\tau)\, d\tau + \frac{z^{m-2}}{\pi} \int_{-\infty}^{+\infty} \tau\, q(\tau)\, d\tau + \ldots + \frac{z}{\pi} \int_{-\infty}^{+\infty} \tau^{m-2} q(\tau)\, d\tau$$

$$+ \frac{1}{\pi} \int_{-\infty}^{+\infty} \tau^{m-1} q(\tau)\, d\tau + O\left(\frac{1}{z}\right) \tag{A-5}$$

Now by the condition (A-2), it is evident that (A-5) is bounded at infinity and so is (A-1). Therefore, it is proved that (A-1) is a solution.

Suppose there is another solution, $W_1(z)$, then

$$w(z) = W(z) - W_1(z) \tag{A-6}$$

must also be analytic on the upper z-plane, be bounded at infinity, be sectionally continuous on the real axis except at the origin where a logarithmic type singularity may be admitted, and satisfy the following boundary conditions on the real axis:

$$\begin{array}{ll} x \leq a_1 & v = 0 \\ a_1 \leq x \leq a_2 & u = 0 \\ a_2 \leq x \leq a_3 & v = 0 \\ \vdots & \vdots \\ a_n \leq x & v = 0 \end{array}$$

It is now sufficient to show that the only solution for $w(z)$ is zero. To show this, first we note that the function may be expressed as

$$w(z) = AH(z) \tag{A-8}$$

Here, A is a constant. Since $H(z)$ is not bounded at infinity the constant A must be zero. This means $w(z) \equiv 0$ and

$$W_1(z) = W(z) \tag{A-9}$$

and the solution is unique.

2. n odd ($n = 2m + 1$, m includes all positive integers including zero)

Theorem: The function (A-1) is the unique solution if

$$\int_{-\infty}^{+\infty} q(\tau)d\tau = \int_{-\infty}^{+\infty} \tau q(\tau)d\tau = \ . \ . \ . = \int_{-\infty}^{+\infty} \tau^{m-1}q(\tau) = 0 \qquad \text{(A-10)}$$

where

$$H(z) = \sqrt{(z - a_1)(z - a_2) \ . \ . \ . (z - a_{n-1})(a_n - z)} \qquad \text{(A-11)}$$

and $q(\tau)$ is defined by (A-4).

The proof for this case is analogous to the previous case and will not be repeated.

APPENDIX B

THE SECTIONALLY CONTINUOUS SOLUTION OF A SINGULAR INTEGRAL EQUATION

A P P E N D I X B

THE SECTIONALLY CONTINUOUS SOLUTION OF A SINGULAR INTEGRAL EQUATION

Here the solution of the following integral equation which appears very frequently in thin airfoil and thin hydrofoil theories is to be discussed.

$$\int_a^b \frac{m(\xi)d\xi}{x - \xi} = f(x), \qquad a < b \tag{B-1}$$

It is well known that the general solution of the integral equation (B-1) valid in an open interval (a, b) is

$$m(x) = -\frac{1}{\pi^2\sqrt{(b - x)(x - a)}}\left[\int_a^b \frac{f(\xi)\sqrt{(b - \xi)(\xi - a)}d\xi}{\xi - x} + \Gamma\right] \tag{B-2}$$

where Γ is an arbitrary constant.

The solution (B-2) is not valid at the two end points. However, it can be made valid at $x = a$ if we set

$$\Gamma = -\int_a^b \frac{f(\xi)\sqrt{b - \xi}}{\sqrt{\xi - a}}\,d\xi \tag{B-3}$$

since, then

$$m(x) = -\frac{x - a}{\pi^2\sqrt{(b - x)(x - a)}}\int_a^b \frac{f(\xi)\sqrt{b - \xi}}{(\xi - x)\sqrt{\xi - a}}\,d\xi \tag{B-4}$$

and it is possible to define

$$m(a) = \lim_{x \to a} m(x) = 0 \tag{B-5}$$

The resulting function

$$m(x) = -\frac{\sqrt{x-a}}{\pi^2\sqrt{b-x}} \int_a^b \frac{f(\xi)\sqrt{b-\xi}}{(\xi - x)\sqrt{\xi - a}}\, d\xi, \quad a \le x < b \tag{B-6}$$

is then valid in the half closed interval $[a, b)$. Using a similar procedure, a solution which is valid in $(a, b]$ may also be derived.

In the process of closing one end of the interval, the only arbitrary constant in Eq. (B-2) has been used up. To close the other end of the interval, therefore, we must look for another means. It is not difficult to see, however, that to make Eq. (B-6) finite at $x = b$, the integral must vanish at $x = b$. Indeed, if

$$\int_a^b \frac{f(\xi)d\xi}{\sqrt{(b-\xi)(\xi-a)}} = 0 \tag{B-7}$$

Then, by multiplying the factor $-\sqrt{x-a}/\pi^2\ \sqrt{b-x}$ by Eq. (B-7) and adding the result to the righthand side of Eq. (B-6) we get

$$m(x) = -\frac{(b-x)\sqrt{x-a}}{\pi^2\sqrt{b-x}} \int_a^b \frac{f(\xi)d\xi}{(\xi - x)\sqrt{(b-\xi)(\xi-a)}}, \quad a \le x < b \tag{B-8}$$

Equation (B-8) is still valid in the half closed interval $[a, b)$ and it is the same as

$$m(x) = -\frac{\sqrt{(b-x)(x-a)}}{\pi^2} \int_a^b \frac{f(\xi)d\xi}{(\xi - x)\sqrt{(b-\xi)(\xi-a)}} \tag{B-9}$$

Since Eq. (B-8) is a solution whose limiting value as $x \to b$ is zero and this coincides with $m(b)$ of Eq. (B-9), by defining

$$m(b) = \lim_{x \to b} m(x) = 0$$

(B-9) is a solution which is sectionally continuous in the closed interval $[a, b]$ if $f(x)$ is also sectionally continuous in the same interval.

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