Automorphic partial differential equations and spectral theory with applications to number theory

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Abstract

While proofs of the Riemann hypothesis and the Lindelöf hypothesis remain elusive, for some number-theoretic applications *any* bound that surpasses the "trivial" or "convex" bound for the growth of an L-function, i.e. any *subconvex bound*, suffices. In this paper, we construct a Poincaré series suitable for proving a subconvex bound for Rankin-Selberg convolutions for $GL_n \times GL_n$ over totally complex number fields. The Poincaré series, with transparent spectral expansion, is obtained by winding-up a free space fundamental solution for the operator $(\Delta - \lambda_z)^\nu$ on the free space G/K. As a sample application, not obviously related to subconvexity, a Perron transform extracts, from the Poincaré series, information about the number of lattice points in an expanding region in G/K, and from the spectral expansion, terms corresponding to the automorphic spectrum of the Laplacian. The result is an explicit formula relating the automorphic spectrum to the number of lattice points in an expanding region. A global automorphic Sobolev theory as well as a zonal spherical Sobolev theory legitimize derivations and manipulations of spectral expansions. This line of inquiry is relevant not only to the hoped-for subconvexity result but also to the development of techniques applicable to harmonic analysis of automorphic forms on higher rank groups.

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Chapter 1

Introduction

While proofs of the Riemann hypothesis and the Lindelöf hypothesis remain elusive, for some number-theoretic applications *any* bound that surpasses the "trivial" or "convex" bound for the growth of an L-function, i.e. any *subconvex bound*, suffices. For example, Cogdell's result on the representability of an integer as the sum of three squares in a ring of algebraic integers relies on a subconvex bound for an automorphic L-function in the conductor aspect [11], and Watson has shown that the quantum unique ergodicity conjecture for arithmetic surfaces would follow from subconvex bounds for certain degree eight L-functions [88].

In this paper, we construct a Poincaré series suitable for proving a subconvex bound for Rankin-Selberg convolutions for $GL_n \times GL_n$ over totally complex number fields.

The best-known method for obtaining subconvex bounds, going back to Selberg in the 1960's, relies on *shifted sums*, but this is difficult to apply to L-functions on higher rank groups. Other ideas include Michel and Venkatesh's application of *ergodic theory*, which succeeds nicely for GL_2 L-functions but is not clearly applicable in higher rank cases [69].

Another method, with origins in the work of Anton Good in the 1980's and resurrected by Diaconu and Goldfeld, uses spectral identities involving *integral moments*, proving subconvex bounds for GL_2 automorphic L-functions over $\mathbb{Q}$ and $\mathbb{Q}(i)$ in the t-aspect [20, 21]. Diaconu and Garrett's reformulation of this method, capturing its structural significance, enabled them to obtain subconvex bounds for GL_2 L-functions over an *arbitrary* number field in the t-aspect and soon after, with Goldfeld, to produce expressions involving moments for *arbitrary* Rankin-Selberg integral representations of L-functions *systematically* ([16, 17, 18]). This method has also been used by Letang, to obtain subconvex bounds for GL_2 L-functions over arbitrary number field in the depth aspect [63].

In particular, this general prescription for spectral identities enables treatment of higher rank groups, and the corresponding identities involve higher moments of GL_2 and GL_1 L-functions as well as second moments of L-functions attached to cusp forms on higher rank groups.

On the other hand, not every moment identity generated is suitable for extracting subconvex bounds. For example, in the case of the *doubling* integrals associated to classical groups, the spectral family is "too long," so the bound does not even recover the convexity bound. For $GL_n \times GL_{n-1}$, Diaconu and Garrett have verified that, as Venkatesh and others speculated, *conductor dropping* obstructs efforts to extract a subconvex bound.

Preliminary computations indicate that moment identities for Rankin-Selberg convolutions for $GL_n \times GL_n$ over *totally complex* number fields, apparently overlooked previously, *avoid* both these obstacles. Also, conveniently, the elementariness of spherical functions in this case makes the dependence on parameters clearer.

The long-term goal, admittedly ambitious, is to prove a subconvex bound for Rankin-Selberg L-functions for $GL_n \times GL_n$ over totally complex number fields. This provides motivation for research on several topics, including global automorphic Sobolev spaces and the extendibility of matrix coefficient functions to complexifications of Lie groups or symmetric spaces.

To obtain such a bound it is necessary (1) to produce a spectral identity involving sums of integral moments of such L-functions, (2) to estimate the growth in vertical strips of a corresponding generating function, and (3) finally to extract a subconvex bound from a suitable asymptotic with power saving in the error term.

The prescription of Diaconu and Garrett produces a suitable spectral identity: given a cusp form f, integrating a Poincaré series against $|f|^2$ yields a sum of integral moments of Rankin-Selberg L-functions [16]. With this mechanism for generating spectral identities, we avoid several issues that would be serious obstacles to a direct application of relative trace formula ideas.

In this paper, a Poincaré series for the $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ case, with transparent spectral expansion, is obtained by winding-up a free space fundamental solution for the operator $(\Delta - \lambda_z)^\nu$ on the symmetric space $SL_n(\mathbb{C})/SU(n)$.

The harmonic analysis of bi-K-invariant functions gives an integral representation of the free space fundamental solution. In fact, for any complex semi-simple Lie group G, we will compute the fundamental solution on G/K explicitly:

$$\begin{aligned} u_z &= \int_{\mathfrak{a}^*} \hat{u}_z(\lambda)\, \varphi_\lambda\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &= \frac{(-1)^\nu\, (-i)^d\, |W|}{\pi^+(\rho)\, \sum \operatorname{sgn} w\, e^{w\rho}} \cdot \int_{\mathfrak{a}^*} \frac{1}{(|\lambda|^2 + z^2)^\nu}\, \pi^+(\lambda)\, e^{i\lambda}\, d\lambda \end{aligned}$$

We will show that π^+ is a *harmonic* homogeneous polynomial, so *Hecke's identity* can be used to evaluate the integral. In the odd rank case, choose $\nu = d + \frac{n+1}{2}$, where d is the number of positive simple roots and n is the rank, for continuity of u_z, by Sobolev. Then

$$u_z(a) = \frac{(-1)^{d+(n+1)/2}\, |W|\, \pi^{(n+1)/2}}{\pi^+(\rho)\, \Gamma(d + (n+1)/2)} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\frac{\alpha(\log a)}{2})} \cdot \frac{e^{-z|\log a|}}{z}$$

In even rank, choose $\nu = d + \frac{n}{2} + 1$ for continuity. Then,

$$u_z(a) = \frac{(-1)^{d+(n/2)+1}\, \pi^{n/2}\, |W|}{\pi^+(\rho)\, \Gamma(d + (n/2) + 1)} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\frac{\alpha(\log a)}{2})} \cdot \frac{|\log a|}{z} \cdot K_1(z\, |\log a|)$$

The Poincaré series

$$\text{Pé}_z(g) = \sum_{\gamma \in \Gamma} u_z(\gamma \cdot g)$$

converges nicely for $\operatorname{Re}(z) \gg 1$.

As a sample application, not obviously related to subconvexity, a Perron transform extracts, from the Poincaré series, information about the number of lattice points in an expanding region in G/K, and from the spectral expansion, terms corresponding to the automorphic spectrum of the Laplacian. The result is an explicit formula relating the automorphic spectrum to the number of lattice points in an expanding region.

A global automorphic Sobolev theory as well as a zonal spherical Sobolev theory legitimize derivations and manipulations of spectral expansions.

Estimating the generating function will require significant extension of the methods used in the GL_2 case, which are certainly insufficient for higher rank. In particular, the GL_2 case required a spectral gap [58], a weak half of Weyl's law [24], and asymptotics of triple integrals of automorphic forms, e.g. an extention of the results of Sarnak and Bernstein-Reznikov to number fields [77, 7].

For a spectral gap result suitable for some higher rank cases see [57]. For a strong version of Weyl's law applicable in some higher rank cases see the papers of Lindenstrauss-Venkatesh and Lapid-Müller, the latter of which contains a good survey of the literature [64, 62].

The most significant issue is the asymptotics of triple integrals of automorphic forms for higher rank. To obtain a polynomial growth estimate for the generating function, we need precise exponential decay of certain matrix coefficient functions. The crucial step is showing that an analytic vector in the representation extends holomorphically to a certain domain in the complexification. In the spirit of Paley-Wiener theorems, the geometry of a maximal such domain in the complexification determines the precise exponential decay. The work of Krötz and Stanton treats $SL_n(\mathbb{R})$ inside $SL_n(\mathbb{C})$ [59, 60]. We will need to treat $SL_n(\mathbb{C})$ inside $SL_n(\mathbb{C}) \times SL_n(\mathbb{C})$.

A polynomial growth estimate for a suitable generating function on vertical strips yields an asymptotic with power saving in the error term and a subconvex bound, by standard analytic number theory methods.

To generate identities with high moments of zeta functions and other L-functions from smaller GL_m's, we use GL_n Eisenstein series instead of cusp forms. However, suitable regularization is necessary, as the Eisenstein series are not square integrable.

An additional complication is that the Eisenstein series appearing in spectral decompositions are those with cuspidal data, whereas the Eisenstein series necessary for the $GL_n \times GL_n$ Rankin-Selberg moment identity are degenerate, occuring as residues of cuspidal data Eisenstein series.

This line of inquiry is relevant not only to the hoped-for subconvexity result but also to the development of techniques applicable to harmonic analysis of automorphic forms on higher rank groups. A well-developed global automorphic Sobolev theory is crucial for legitimizing the application of harmonic analysis of automorphic forms on higher rank groups to number theory. Similarly, as Krötz and Stanton have remarked, understanding the extendibility of analytic vectors to complexifications is important for harmonic analysis of higher rank groups and symmetric spaces, just as complex analysis played a critical role in the development of Fourier analysis [59].

Chapter 2

Spectral theory of automorphic forms

The spectral theory of automorphic forms originates in the work of Maass, Roelcke, and Selberg in the 1950's; see for example [65, 76, 80]. Langlands' work in the 1960's was the decisive step in treating higher rank groups; see his notes [61], published later.

We follow the notation and exposition of Moeglin and Waldspurger [71], denoted by MW in this chapter.

2.1 General notation

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. Let G be a connected reductive algebraic group defined over k. Fix a maximal compact subgroup K of $G_{\mathbb{A}}$ such that $K = \prod K_v$, product over all places of k, where K_v is a maximal compact subgroup of $G(k_v)$.

Let $\mathbf{G}$ be a topological group which is a finite central covering of $G_{\mathbb{A}}$. There exists a finite set S of places of K such that for all $v \notin S$, K_v lifts into $\mathbf{G}$. Let $\mathbf{K}$ be the inverse image of K in $\mathbf{G}$. Suppose that G_k lifts to a subgroup of $\mathbf{G}$. Fix a lifting of it which we also denote by G_k. Let P be a parabolic subgroup of G (i.e. G/P is a complete algebraic variety), defined over k, and U its unipotent radical. Then $U_{\mathbb{A}}$ lifts canonically to $\mathbf{G}$. We still use $U_{\mathbb{A}}$ to denote the image of this lifting. Let M be a Levi subgroup of P, and let $\mathbf{M}$ denote the inverse image of $M_{\mathbb{A}}$ in $\mathbf{G}$.

Fix a parabolic subgroup P_0 of G, defined over k and minimal, and a Levi subgroup M_0 of P_0, defined over k. We use the phrase 'standard parabolic subgroup of G' to denote any parabolic subgroup of G defined over k and containing P_0. By a 'standard Levi subgroup of G' we mean any Levi subgroup, containing M_0, of a standard parabolic subgroup of G. Every standard parabolic subgroup possesses a unique standard Levi subgroup. Denote by Z_G the center of G, by $Z_{\mathbf{G}}$ the center of $\mathbf{G}$ and for M a Levi subgroup of G, denote by Z_M the center of M and by $Z_{\mathbf{M}}$ the center of $\mathbf{M}$.

Fix a standard Levi M of G; let $\mathrm{Rat}(M)$ denote the group of rational characters of M (i.e. the homomorphisms as algebraic groups of M into the mutiplicative group $\mathbf{G}_m$.) Set

$$\mathfrak{a}_M^* \;=\; \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{C}, \quad \mathfrak{a}_M \;=\; \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{C}).$$

We will also use the real spaces

$$\mathrm{Re}\, \mathfrak{a}_M^* \;=\; \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathrm{Re}\, \mathfrak{a}_M^* \;=\; \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{R}).$$

Let $\chi \in \mathrm{Rat}(M)$: for every place v of k, χ defines an algebraic character denoted by χ_v of $M(k_v)$ into k_v^*. We define $|\chi|$, a continuous homomorphism of $M_{\mathbb{A}}$ into $\mathbb{C}^*$, by

$$\text{For all } m = (m_v) \in M_{\mathbb{A}}, \;\; m^{|\chi|} \;=\; \prod_v |m_v^{\chi_v}|_v$$

where $|\cdot|_v$ is the absolute value of k_v and $m_v^{|\chi_v|}$ is the value of $|\chi_v|$ at the point m_v. Let

$$M^1 \;=\; \bigcap_{\chi \in \mathrm{Rat}(M)} \mathrm{Ker}\, |\chi|;$$

It is a normal subgroup of $M_{\mathbb{A}}$, and let X_M be the group of continuous homomorphisms of $M_{\mathbb{A}}$ into $\mathbb{C}^*$ which are trivial on M^1. The group X_M can be realized as a quotient of $\mathfrak{a}_M^*$, i.e. there exists a bijective morphism of groups

$$\kappa : \mathfrak{a}_M^* \to X_M$$

Set

$$\mathrm{Re}\, X_M \;=\; \kappa(\mathrm{Re}\, \mathfrak{a}_M^*), \;\; \mathrm{Im}\, X_M \;=\; \kappa(i\, \mathrm{Re}\, \mathfrak{a}_M^*)$$

In fact, κ induces an isomorphism

$$\operatorname{Re}\mathfrak{a}_M^* \simeq \operatorname{Re}X_M$$

and $\operatorname{Re}X_M$ is the group of characters of $M_{\mathbb{A}}\backslash M^1$ with values in $\mathbb{R}_+^*$. The group X_M can be identified with the group of characters of $\mathbf{M}$ trivial on $\mathbf{M}^1$, where $\mathbf{M}^1$ is the inverse image of M^1. The group $\operatorname{Re}X_M$ can then be identified with the group of characters of $\mathbf{M}/\mathbf{M}^1$ with values in $\mathbb{R}_+^*$. Further, $\operatorname{Re}X_M$ can be identified with the group of characters of $\mathbf{M}$ with values in $\mathbb{R}_+^*$ which are trivial on M_k, and similarly, it can be identified with the group of characters of $Z_{\mathbf{M}}$ with values in $\mathbb{R}_+^*$ which are trivial on $Z_{M,k} \cap Z_{\mathbf{M}}$. We write X_M^G for the subgroup of X_M consisting of the characters of $\mathbf{M}/\mathbf{M}^1$ trivial on $Z_{\mathbf{G}}$.

Let T_0 be the maximal split torus at the center of M_0. Let $R(T_0, G)$ be the set of roots of G relative to T_0 and $R^+(T_0, G)$ the positive roots of $R(T_0, G)$. Let $P = MU$ be a standard parabolic. Then $P_0 \cap M$ is a minimal parabolic of M. Let $R(T_0, M)$ be the set of roots of T_0 in M as well as the set of coroots of M, which are identified with subsets of roots, coroots, respectively, of G. Let T_M be the maximal split torus in the center of M.

Let ρ_0 be the half-sum of positive roots of T_0. More generally, let $P = MU$ be a standard parabolic subgroup of G, and denote by ρ_P the half-sum of positive roots of T_M.

We denote by W the Weyl group of G, i.e. the quotient

$$W = \operatorname{Norm}_{G_k} T_{0,k} / \operatorname{Cent}_{G_k} T_{0,k}$$

Let M be a standard Levi subgroup of G. Let W_M be the Weyl group of M (the analogue of W for the group M) and $W(M)$ the set of elements of W, of minimal length in their class wW_M, such that wMw^{-1} is again a standard Levi subgroup.

Fix an isomorphism $T_0 \to \mathbb{G}_m^R$ defined over k, where $\mathbb{G}_m$ is the multiplicative group. Embed $\mathbb{R}_+^*$ in $\mathbb{A}^*$, identifying $t \in \mathbb{R}_+^*$ with the element $(t_v) \in \mathbb{A}^*$ such that $t_v = t$ if v is archimedean, $t_v = 1$ if v is finite. Then ${\mathbb{R}_+^*}^R$ is identified with a subgroup of $T_{0,\mathbb{A}}$ which is split in the extension $\mathbf{T}_0$ since $\mathbb{R}_+^*$ is simply connected. We denote by $A_{\mathbf{M}_0}$ the unique connected subgroup of $\mathbf{T}_0$ which projects onto ${\mathbb{R}_+^*}^R$. More generally, if $P = MU$ is a standard parabolic subgroup of G, we set $A_{\mathbf{M}} = A_{\mathbf{M}_0} \cap Z_{\mathbf{M}}$. Let $\log^M$ be the composition of the map $\log : \mathbf{M}_0 \to \mathfrak{a}_{M_0}$ with the projection $\mathfrak{a}_{M_0} \to \mathfrak{a}_{M_0}^M$. We have the equality $A_{\mathbf{M}} = \{a \in A_{\mathbf{M}_0} : \log^M a = 0\}$. We define the group $A_{\mathbf{M}}^G = \{a \in A_{\mathbf{M}} : \log_G a = 0\}$.

2.2 Constant terms, automorphic forms, cuspidal forms

Let $P = MU$ be a standard parabolic subgroup of G, ϕ a measurable and locally L^1 function on $U_k \backslash \mathbf{G}$. We define the constant term along P to be the measurable, locally L^1 function ϕ_P on $U_{\mathbb{A}} \backslash \mathbf{G}$ given by

$$\phi_P(g) \quad = \quad \int_{U_k \backslash U_{\mathbb{A}}} \phi(ug)\, du$$

Let $\mathfrak{z}$ be the center of the universal enveloping algebra of the Lie algebra of $\mathbf{G}_\infty$. Let $\phi : U_{\mathbb{A}} M_k \backslash \mathbf{G} \to \mathbb{C}$ be a function. We say that ϕ is automorphic if it (i) has moderate growth, (ii) is smooth, (iii) is $\mathbf{K}$-finite, and (iv) is $\mathfrak{z}$-finite. For $\phi : U_{\mathbb{A}} M_k \backslash \mathbf{G} \to \mathbb{C}$ and $k \in \mathbf{K}$, define ϕ_k by

$$\phi_k(m) \quad = \quad m^{-\rho_P} \phi(mk)$$

where ρ_P is the half-sum of roots of M in $\operatorname{Lie} U$. Then ϕ is automorphic if and only if it is smooth, $\mathbf{K}$-finite and for all $k \in \mathbf{K}$, ϕ_k is automorphic over $M_k \backslash M$.

We denote by $A\big(U_{\mathbb{A}} M_k \backslash \mathbf{G}\big)$ the space of automorphic forms on $U_{\mathbb{A}} M_k \backslash \mathbf{G}$.

Let $P = MU$ be a standard parabolic subgroup of G, ϕ an automorphic form on $U_{\mathbb{A}} M_k \backslash \mathbf{G}$. We say that ϕ is cuspidal if for all parabolic subgroups P' such that $P_0 \subset P' \subsetneqq P$, we have $\phi_{P'} = 0$. We denote by $A_0\big(U_{\mathbb{A}} M_k \backslash \mathbf{G}\big)$ the space of cuspidal automorphic forms on $U_{\mathbb{A}} M_k \backslash \mathbf{G}$.

Proposition (See MW, I.3.2)**.** *Fix a character ξ of $Z_{\mathbf{G}}$ and denote by $A(U_{\mathbb{A}} M_k \backslash \mathbf{G})_\xi$ the subspace of $\phi \in A(U_{\mathbb{A}} M_k \backslash \mathbf{G})$ such that $\phi(zg) = \xi(z)\phi(g)$ for all $z \in Z_{\mathbf{G}}$, $g \in \mathbf{G}$. Set*

$$A(U_{\mathbb{A}} M_k \backslash \mathbf{G})_{Z,\xi} \quad = \quad \bigoplus_{\eta} A(U_{\mathbb{A}} M_k \backslash \mathbf{G})_\eta$$

where the sum is over the $\eta \in \mathrm{Hom}(Z_{\mathbf{M}}, \mathbb{C}^*)$ *such that* $\eta|_{Z_{\mathbf{G}}} = \xi$. *There is an isomorphism*

$$\mathbb{C}[\mathrm{Re}\,\mathfrak{a}_M^G] \otimes A(U_{\mathbb{A}} M_k \backslash \mathbf{G})_{Z,\xi} \longrightarrow A(U_{\mathbb{A}} M_k \backslash \mathbf{G})_{\xi}$$

Let π be an irreducible representation of $\mathbf{M}$. We denote by χ_π the central character of π. It is a character of $Z_{\mathbf{M}}$. Suppose χ_π is trivial on $Z_{M,k}$. Recall that with such a character χ we can associate an element denoted by $\mathrm{Re}\,\chi$ of $\mathrm{Re}\,X_M$: it suffices to identify the character $|\chi|$ with an element of $\mathrm{Re}\,X_M$. Simply set $\mathrm{Re}\,\pi = \mathrm{Re}\,\chi_\pi$. We define

$$\mathrm{Im}\,\pi = \pi \otimes (-\mathrm{Re}\,\pi).$$

Let $-\pi$ be the contragredient representation of π and $-\overline{\pi}$ be the conjugate of the contragredient.

2.3 Pseudo-Eisenstein series and decomposition along cuspidal support

Let V be an irreducible submodule in $A(M_k \backslash \mathbf{M})$. We denote by π_0 the representation of $\mathbf{M}_f \times \big((\mathbf{M} \cap \mathbf{K}), \mathrm{Lie}\,\mathbf{M}_\infty \otimes_{\mathbb{R}} \mathbb{C}\big)$ into V. We say that π_0 is an automorphic subrepresentation of $\mathbf{M}$. We denote by $\big(A(M_k \backslash \mathbf{M})\big)_{\pi_0}$ the isotypic submodule π_0. Suppose that π_0 is cuspidal, i.e. that $A\big(M_k \backslash \mathbf{M}\big)_{\pi_0}$ contains cuspidal automorphic forms. By convention, $A\big(M_k \backslash \mathbf{M}\big)_{\pi_0}$ is the isotypic subspace of $A_0\big(M_k \backslash \mathbf{M}\big)$ of type π_0. Let π be an irreducible automorphic representation of $\mathbf{M}$.

We say that π_0 is equivalent to π if there exists $\lambda \in X_M^{\mathbf{G}}$ such that $\pi \simeq \pi_0 \otimes \lambda$. We denote by $\mathfrak{P}$ the equivalence class of π_0, and we note that the subgroup of X_M^G given by $\{\nu \in X_M^G | \nu \otimes \pi \simeq \pi\}$ is independent of the chosen point π of $\mathfrak{P}$. We denote it by $\mathrm{Fix}_{X_M^G} \mathfrak{P}$.

A pair $(M, \mathfrak{P})$, where $\mathfrak{P}$ consists of cuspidal representations as above, will be called a cuspidal datum.

Denote by $P(X_M^G)$ the set of complex -valued functions on X_M^G which are Paley-Wiener, i.e. which are Fourier transforms of functions on $\mathbf{M}^1 Z_{\mathbf{G}} \backslash \mathbf{M}$, C^∞ with compact support, or else $f \in P(X_M^G)$ if for λ_0 fixes in $\mathrm{Re}\, X_M^G$, the function,

$$\hat{f}(m) \;=\; \int_{\lambda \in X_M^G, \mathrm{Re}\lambda = \lambda_0} |\mathrm{Fix}_{X_M^G}\mathfrak{P}|^{-1} f(\lambda) m^{\lambda}\, d\lambda$$

is a function on $\mathbf{M}^1 Z_{\mathbf{G}}\backslash\mathbf{M}$ with compact support (it does not depend on the choice of λ_0.)

Let ϕ be a $\mathbf{K}$-finite function on $\mathfrak{P}$ such that

$$\forall\, \pi \in \mathfrak{P},\;\; \phi(\pi) \;\in\; A(U_{\mathbb{A}} M_k\backslash\mathbf{G})_{\pi}$$

Fix $\pi_0 \in \mathfrak{P}$. We associate with ϕ a function $\tilde{\phi}$ on X_M^G with values in $A(U_{\mathbb{A}} M_k\backslash\mathbf{G})_{\pi_0}$ defined by:

$$\tilde{\phi}(\lambda) \;=\; \lambda^{-1}\, \phi(\pi_0 \otimes \lambda)$$

We say that ϕ is Paley-Wiener if $\tilde{\phi}$ can be identified with an element of

$$A(U_{\mathbb{A}} M_k\backslash\mathbf{G})_{\pi_0} \;\otimes_{\mathbb{C}}\; P(X_M^G)$$

We denote by $P_{(M,\mathfrak{P})}$ the set of Paley-Wiener functions.

Let $\phi \in P_{(M,\mathfrak{P})}$ and $\pi \in \mathfrak{P}$. We define the Fourier transform of ϕ. For all $g \in \mathbf{G}$, where $g = umk$ with $u \in U_{\mathbb{A}}$, $m \in \mathbf{M}$, $k \in \mathbf{K}$, and for all $m' \in \mathbf{M}$,

$$F_{\pi}(\phi)(g)(m') \;=\; |\mathrm{Fix}_{X_M^G}\,\mathfrak{P}|^{-1} \int_{\lambda \in \mathrm{Im} X_M^G} (m')^{-\lambda-\rho_P}\, \phi(\pi\otimes\lambda)(\pi\otimes\lambda)(m'mk)\, d\lambda$$

We denote by ϵ the evaluation of elements at the unit element of $\mathbf{M}$. Functions of the form $\epsilon F_{\pi}(\phi)$ which do not depend on the choice of π are written $\epsilon F(\phi)$.

Let R be a positive real number; denote by $P^R(X_M^G)$ the set of holomorphic functions F onto $D_R = \{\lambda \in X_M^G \,|\; \|\mathrm{Re}\,\lambda\| < R\}$ satisfying: for all $n \in \mathbf{N}$,

$$\sup_{\lambda \in D_R} |f(\lambda)|\, \|(1 + \|\lambda\|)^n \;<\; \infty$$

For $\pi \in \mathfrak{P}$, we denote by $(\mathrm{Re}\,\pi)^G$ the projection of $\mathrm{Re}\,\pi$ onto $\mathrm{Re}\, X_M^G$; see MW I.1.6(9). Considering the isomorphism $\phi \to \tilde{\phi}$ above, we define $P^R_{(M,\mathfrak{P})}$ to be the set of holomorphic functions on

$$\mathfrak{P}_R \;=\; \{\pi \in \mathfrak{P} \,|\;\; \|(\mathrm{Re}\,\pi)^G\| \;<\; R\}$$

which can be identified with an element of

$$\left(A(U_{\mathbb{A}}M_k\backslash\mathbf{G})_{\pi_0} \otimes P^R(X_M^G)\right)^{\mathrm{Fix}_{X_M^G}\mathfrak{P}}$$

where π_0 is fixed in $\mathfrak{P}$ and satisfies $(\mathrm{Re}\,\pi_0)^G = 0$.

Let M, M' be standard Levis of G corresponding to standard parabolics P and P' of unipotent radicals U and U'. Let $w \in G_k$. We suppose that:

$$wMw^{-1} = M'$$

Fix an equivalence class $\mathfrak{P}$ of irreducible automorphic subrepresentations of $\mathbf{M}$. Let $\pi \in \mathfrak{P}$ and ϕ_π be an element of $A(U_{\mathbb{A}}M_k\backslash\mathbf{G})_\pi$. For $g \in \mathbf{G}$, whenever the integral below is convergent, we define the intertwining operator:

$$(M(w,\pi)\phi_\pi)(g) \;=\; \int_{(U'_k\cap wU_kw^{-1})\backslash U_{\mathbb{A}}} \phi_\pi(w^{-1}ug)\,du$$

Let $\phi \in P_{(M,\mathfrak{P})}$, or, more generally, let $R > \|\rho_0\|$ and $\phi \in P^R_{(M,\mathfrak{P})}$. We denote by $\epsilon F(\phi)$ the function on $\mathbf{G}$ obtained by evaluating the Fourier transform of ϕ at the unit element of $\mathbf{M}$. If convergent, the series below defines the pseudo-Eisenstein series θ_ϕ, for $g \in \mathbf{G}$:

$$\theta_\phi(g) \;=\; \sum_{\gamma\in P_k\backslash G_k} \epsilon F(\phi)(\gamma g)$$

Proposition (See MW, II.2.4)**.** *Let ξ be a unitary character of $Z_{\mathbf{G}}$. We denote by $\mathfrak{E}$ the set of equivalence classes of a pair $(M,\mathfrak{P})$ formed by a standard Levi of G, denoted by M, and an equivalence class of cuspidal representations of $\mathbf{M}$ whose restriction to $Z_{\mathbf{G}}$ is ξ. For $\mathfrak{X} \in \mathfrak{E}$, we denote by $L^2(G_k\backslash\mathbf{G})_{\mathfrak{X}}$ the closed subspace of $L^2(G_k\backslash\mathbf{G})_\xi$ generated by the pseudo-Eisenstein series θ_ϕ where ϕ runs through $P_{(M',\mathfrak{P}')}$ and where $(M',\mathfrak{P}')$ describes $\mathfrak{X}$. Then we have*

$$L^2(G_k\backslash\mathbf{G})_\xi \;=\; \widehat{\bigoplus}_{\mathfrak{X}\in\mathfrak{E}} L^2(G_k\backslash\mathbf{G})_{\mathfrak{X}}$$

the sum being orthogonal.

2.4 Decomposing the scalar product of two pseudo-Eisenstein series

Fix a unitary character ξ of the center of $\mathbf{G}$ and an equivalence class $\mathfrak{X}$ of cuspidal data given by pairs $(M, \mathfrak{P})$ where $\mathfrak{P}$ consists of cuspidal representations of $\mathbf{M}$ whose restrictions to $Z_{\mathbf{G}}$ have character ξ. Let

$$\mathrm{Stab}(M, \mathfrak{P}) \quad = \quad \{w \in W(M) | wMw^{-1} = M, w\mathfrak{P} = \mathfrak{P}\}$$

Fix $(M, \mathfrak{P}) \in \mathfrak{X}$. For $\pi \in \mathfrak{P}$, we defined $\mathrm{Re}\,\pi$ which is an element of $\mathrm{Re}\, X_M^{\mathbf{G}}$ and $\mathrm{Im}\,\pi$ which is an element of $\mathfrak{P}$. Let $\mathfrak{S}$ be a subset of $\mathfrak{P}$ and set:

$$\mathfrak{S}^0 \quad = \quad \{\lambda \in X_M^G | \mathfrak{S} \otimes \lambda = \mathfrak{S}\}$$

We say that $\mathfrak{S}$ is an affine subspace of $\mathfrak{P}$ if there exists a vector subspace V of $\mathfrak{a}_M^{*\,G} (\simeq \mathrm{Re}\, X_M^G \otimes_{\mathbb{R}} \mathbb{C})$ such that $\mathfrak{S} = \pi \otimes \kappa(V)$, where π is any element of $\mathfrak{S}$. We say that the vector part of $\mathfrak{S}$ is defined over $\mathbb{R}$ if V is defined over $\mathbb{R}$.

We denote by $S^h_{M,\mathfrak{P}}$ the set of affine hyperplanes of $\mathfrak{P}$ which are singular for $M(w^{-1}, -w\overline{\pi})\phi(-w\overline{\pi})$ where w varies in $W(M)$ and where ϕ varies in $P_{(wM, -w\overline{\mathfrak{P}})}$, augmented by the set of conjugates of such hyperplanes under the action of $\mathrm{Stab}(M, \mathfrak{P})$. We denote by $S_{(M,\mathfrak{P})}$ the set of affine spaces of $\mathfrak{P}$ with vector part defined over $\mathbb{R}$ obtained as irreducible components of intersections of elements of $S^h_{M,\mathfrak{P}}$.

For every element $\mathfrak{S}$ of $S_{\mathfrak{X}}$, we denote by $d_{\mathfrak{S}}\pi$ the measure on $\mathfrak{S}$ obtained via the scalar product on $\mathrm{Re}\,\mathfrak{S}^0$.

Let $(M, \mathfrak{P}) \in \mathfrak{X}$, $\mathfrak{S}, \mathfrak{S}' \in S_{(M,\mathfrak{P})}$ with $\mathfrak{S}$ contained in $\mathfrak{S}'$. A residue datum for $\mathfrak{S}$ into $\mathfrak{S}'$ is a map of the set of meromorphic functions on $\mathfrak{S}'$ with polynomial singularities $S_{\mathfrak{X}}$ into the set of meromorphic functions on $\mathfrak{S}$ with polynomial singularities $S_{\mathfrak{X}}$ which are finite linear combinations with real coefficients of operators $\mathrm{Res}_i A_i$ of MW V.1.3.

We may canonically and $\mathrm{Stab}(M, \mathfrak{P})$-equivariantly identify $X_M^{\mathbf{G}}$ with $\mathfrak{P}$. The identification is denoted by

$$\lambda \in X_M^{\mathbf{G}} \;\longleftrightarrow\; \pi \simeq \pi_0 \otimes \lambda_\pi \in \mathfrak{P}$$

With this notation, for every $\mathfrak{S} \in S_{\mathfrak{X}}$ and for every $T \in \mathbb{R}_+^*$, we set:

$$\mathfrak{S}_{\leq T} \;\; = \;\; \{\pi \in \mathfrak{S} \mid \;\; \|\mathrm{Im}\lambda_\pi\|^2 \leq T + \|\mathrm{Re}\lambda_\pi\|^2\}$$

Let $\mathfrak{S}' \subset S_{\mathfrak{X}}$ with $\mathfrak{S}' \subset \mathfrak{S}$, U be a measurable set of $\mathfrak{S}'$ and $\mathrm{Res}^{\mathfrak{S}}_{\mathfrak{S}',U}$ a residue datum for $\mathfrak{S}'$ in $\mathfrak{S}$. We consider $\int_U \mathrm{Res}^{\mathfrak{S}}_{\mathfrak{S}',U}$ as a linear form defined on the set of meromorphic functions A on $\mathfrak{S}$ with polynomial singularities $S_{\mathfrak{X}}$ such that $\mathrm{Res}^{\mathfrak{S}}_{\mathfrak{S}',U}A$ is integrable on U. We consider the vector space V generated by these linear forms and the vector subspace V' generated by those where we impose the following conditions on U:

U must be in $\mathfrak{S}' - \mathfrak{S}'_{\leq T}$;

U must be relatively compact if $\mathrm{Re}\,\mathfrak{S}'$ is strictly contained in $\mathrm{Re}\,X_M^{\mathbf{G}}$.

Let $v, v' \in V$. We will say that $v(A) =_T v'(A)$ for a meromorphic function A on $\mathfrak{S}$ with polynomial singulartities $S_{\mathfrak{X}}$ in the domains of definition of v and v' if $(v - v') \in V'$ can be written in the form $\sum \int_U \mathrm{Res}^{\mathfrak{S}}_{\mathfrak{S}',U}$ where the U are sets satisfying the above conditions and such that $\mathrm{Res}^{\mathfrak{S}}_{\mathfrak{S}',U}A$ is holomorphic at every point of U and integrable over U.

We fix $(M, \mathfrak{P}) \in \mathfrak{X}$ and a finite set $\mathfrak{F}$ of $\mathbf{K}$-types; we write an $\mathfrak{F}$ in the exponent to denote the vector space generated by the functions on which $\mathbf{K}$ acts via one of the $\mathbf{K}$-types of $\mathfrak{F}$. Let $\mathfrak{S} \in S_{\mathfrak{X}}^{\mathfrak{F}}$; we define the origin of $\mathrm{Re}\,\mathfrak{S}$, denoted by $o(\mathfrak{S})$ by setting:

$$o(\mathfrak{S}) \; = \; \mathrm{Re}\,\mathfrak{S} \; \cap \; (\mathrm{Re}\,\mathfrak{S}^0)^{\perp}$$

where the orthogonal is taken in $\mathrm{Re}\,X_M^G$. Let $T \in \mathbb{R}_+^*$. Let $z(\mathfrak{S}) \in \mathrm{Re}\,\mathfrak{S}$; we say that $z(\mathfrak{S})$ is $T - \mathfrak{F}$-general but near $o(\mathfrak{S})$ if for every element $\mathfrak{S}'$ of $S_{\mathfrak{X}}^{\mathfrak{S}}$ strictly contained in $\mathfrak{S}$, we have:

$$z(\mathfrak{S}) \; \notin \; \{\mathrm{Re}\,\pi \mid \pi \; \in \; \mathfrak{S}'_{\leq T}\}$$

and the set $\{\mathrm{Re}\,\pi \mid \pi \in \mathfrak{S}'_{\leq T}\}$ does not intersect the ball of $\mathrm{Re}\,\mathfrak{S}$ of center $o(\mathfrak{S})$ and of radius $\|o(\mathfrak{S}) - z(\mathfrak{S})\|$ unless this set contains $o(\mathfrak{S})$.

For $w \in W(M)$, we have $(wMw^{-1}, w\mathfrak{P}) \in \mathfrak{X}$ and for $(M', \mathfrak{P}') \in \mathfrak{X}$, we set:

$$W((M, \mathfrak{P}), (M', \mathfrak{P}')) \;\; = \;\; \{w \in W(M) : (wMw^{-1}, w\mathfrak{P}) = (M', \mathfrak{P}')\}$$

Let $\phi' \in P_{\mathfrak{X}}^{\mathfrak{F}}$ and let $\phi \in P_{(M,\mathfrak{P})}^{\mathfrak{F}}$. We define a meromorphic function on $\mathfrak{P}$ with polynomial singularities on $S_{\mathfrak{X}}^{\mathfrak{F}}$, by setting for $\pi \in \mathfrak{P}$:

$$A(\phi', \phi)(\pi) \;=\; \langle \sum_{w \in W(M)} M(w^{-1}, -w\overline{\pi})\, \phi'(-w\overline{\pi}),\; \phi(\pi)\rangle$$

where $\sum_{w\in W(M)} M(w^{-1}, -w\overline{\pi})\, \phi'(-w\overline{\pi})$ signifies:

$$\sum_{(M',\mathfrak{P}')\in\mathfrak{X}} \left(\sum_{w \in W((M,\mathfrak{P}),(M',\mathfrak{P}'))} M(w^{-1}, -w\overline{\pi})\, \phi'_{(M',\mathfrak{P}')}(-w\overline{\pi})\right)$$

Let $R \in \mathbb{R}$. This definition can be extended to $\phi' \in P_{\mathfrak{X}}^{R,\mathfrak{F}}$ and to $\phi \in P_{M,\mathfrak{P}}^{R,\mathfrak{F}}$; in this case $A(\phi', \phi)$ is defined meromorphically on $\mathfrak{P}_R$. There exists $\lambda_0 \in \operatorname{Re} X_M^G$ (very positive) such that:

$$\langle \theta_{\phi'}, \theta_\phi \rangle \;=\; \int_{\pi\in\mathfrak{P},\, \operatorname{Re}\pi=\lambda_0} A(\phi', \phi)(\pi)\, d\pi$$

This follows from the computation of the scalar product of two pseudo-Eisenstein series. (See MW II.2.1). In what follows we fix $R \in \mathbb{R}$ such that there exists $\lambda_0 \in \operatorname{Re} X_M^G$ of norm less than R satisfying the above equality.

In addition to $(M,\mathfrak{P})$ and R, we fix a positive real number T and T' greater than $3T + 2R^2$. Let $\mathfrak{S} \in S_{(M,\mathfrak{P})}^{\mathfrak{F}}$; we set:

$$\operatorname{Norm}\mathfrak{S} \;=\; \{w \in \operatorname{Stab}(M,\mathfrak{P}) | w\mathfrak{S} = \mathfrak{S}\}$$

For every $\mathfrak{S} \in S_{(M,\mathfrak{P})}^{\mathfrak{F}}$, we fix $z(\mathfrak{S}) \in \operatorname{Re}(\mathfrak{S})$ such that for every $w \in \operatorname{Norm}\mathfrak{S}$, $wz(\mathfrak{S})$ is $T' - \mathfrak{F}$-general but near $o(\mathfrak{S})$. To simplify we also require that $z(\mathfrak{S}) = z(\mathfrak{S}')$ if $\operatorname{Re}\mathfrak{S} = \operatorname{Re}\mathfrak{S}'$.

Theorem (MW V.2.2)**.** *For every $\mathfrak{S} \in S_{(M,\mathfrak{P})}^{\mathfrak{F}}$, there exists a residue datum for $\mathfrak{S}$ in $\mathfrak{P}$ which is zero for almost all $\mathfrak{S}$ and in particular whenever $\|o(\mathfrak{S})\| > R$, denoted by* $\operatorname{Res}_{\mathfrak{S}}$*, such that:*

$$\forall\, \phi' \in P_{\mathfrak{X}}^{\mathfrak{F}},\;\; \forall\, \phi \in P_{(M,\mathfrak{P})}^{R,\mathfrak{F}}$$

$$\langle \theta_{\phi'}, \theta_\phi \rangle \;=_T\; \sum_{\mathfrak{S}\in S_{(M,\mathfrak{P}}^{\mathfrak{F}}} |\operatorname{Norm}\mathfrak{S}|^{-1} \int_{\pi\in\mathfrak{S}_{\le T},\, \operatorname{Re}\pi = z(\mathfrak{S})} \sum_{w\in\operatorname{Norm}\mathfrak{S}} \operatorname{Res}_{\mathfrak{S}} A(\phi', \phi)(w\pi)\;\, d_{\mathfrak{S}}\pi$$

2.5 Decomposing along the spectrum

We fix an equivalence class $\mathfrak{X}$ of cuspidal data and a finite set $\mathfrak{F}$ of $\mathbf{K}$-types. We denote by $L^2(G_k\backslash\mathbf{G})_{\mathfrak{X},\mathfrak{F}}$ the vector subspace of $L^2(G_k\backslash\mathbf{G})_{\mathfrak{X}}$ generated by the functions on which $\mathbf{K}$ acts via one of the $\mathbf{K}$-types contained in $\mathfrak{F}$ and we fix $T \in \mathbb{R}_+^*$; we define q_T to be the spectral projection $q_T = 1 - p_{-T}$ (where p_T is as in MW, Remark III.1.6.) and we begin by decomposing $q_T L^2(G_k\backslash\mathfrak{G})_{\mathfrak{X},\mathfrak{F}}$.

Let $\mathfrak{S}$, $\mathfrak{S}' \in S_{\mathfrak{X}}^{\mathfrak{F}}$; we denote by $(M, \mathfrak{P})$ and $(M', \mathfrak{P}')$ the cuspidal data attached to $\mathfrak{S}$ and $\mathfrak{S}'$. We say that $\mathfrak{S}$ and $\mathfrak{S}'$ are equivalent if there exists $w \in W((M,\mathfrak{P}),(M',\mathfrak{P}'))$ such that $w\mathfrak{S} = \mathfrak{S}'$. We denote by $[S_{\mathfrak{X}}^{\mathfrak{F}}]$ the set of these equivalence classes. For every $\mathfrak{C} \in [S_{\mathfrak{X}}^{\mathfrak{F}}]$, we fix $\mathfrak{S}_{\mathfrak{C}} \in \mathfrak{C}$ and, to simplify the expression, we suppose that all the elements $\mathfrak{S}_{\mathfrak{C}}$ have the same cuspidal data, $(M, \mathfrak{P})$.

For every $\mathfrak{C} \in [S_{\mathfrak{X}}^{\mathfrak{F}}]$, for every $\phi, \phi' \in P_{\mathfrak{X}}^{R,\mathfrak{F}}$, we define a meromorphic function on $\mathfrak{P}$ by:

$$r_{\mathfrak{C}}(\phi', \phi)(\pi) \quad = \quad |\mathrm{Norm}\,\mathfrak{S}_{\mathfrak{C}}|^{-1} \sum_{w \in W(M)} ((\mathrm{Res}_{w\mathfrak{S}_{\mathfrak{C}}}^{G} A)(\phi', \phi))(w\pi)$$

Let $\mathfrak{C} \in [S_{\mathfrak{X}}^{\mathfrak{F}}]$ and T be as above. For $\phi \in P_{\mathfrak{X}}^{R,\mathfrak{F}}$, we define the map $\mathrm{proj}_{\mathfrak{C},T}^{\mathfrak{F}}$ almost everywhere:

$$\pi \in \mathfrak{S}_{\mathfrak{C}} \quad \longrightarrow \quad e_{\mathfrak{C}}(\phi, \pi)$$

where

$$e_{\mathfrak{C}}(\phi, \pi) \quad = \quad |\mathrm{Norm}\,\mathfrak{S}_{\mathfrak{C}}|^{-1} \sum_{w \in W(M)} \mathrm{Res}_{w\mathfrak{S}_{\mathfrak{C}}}^{\mathbf{G}} E(\phi, w\pi)$$

where $E(\phi, \pi)$ is the Eisenstein series, which is a function meromorphically dependent on π, with polynomial singularities $S_{\mathfrak{X}}^{\mathfrak{F}}$ and with values in the set of automoprhic forms on $G_k\backslash\mathbf{G}$. The map $\mathrm{proj}_{\mathfrak{C},T}^{\mathfrak{F}}$ is an orthogonal projection $L^2(G_k\backslash\mathbf{G})_{\mathfrak{X},\mathfrak{F}}$ onto a closed subpace denoted $L_{\mathfrak{C},T}^{2\ \mathfrak{F}}$.

For every element π of

$$\{\pi \in \mathfrak{S}_{\mathfrak{C} \leq T} | \mathrm{Re}\,\pi = o(\mathfrak{S}_{\mathfrak{C}})\},$$

we denote by $A^{\mathfrak{F}}_{\mathfrak{C},\pi}$ the space of automorphic forms on $G_k\backslash\mathbf{G}$ generated by the elements $e_{\mathfrak{C}}(\phi,\pi)$ defined above.

Let $\overset{\circ}{\mathfrak{S}}_{\leq T}$ be

$$\overset{\circ}{\mathfrak{S}}_{\leq T} \;=\; \{\pi\in\mathfrak{S}_{\leq T}|\mathrm{Re}\,\pi = o(\mathfrak{S})\} \;-\; \bigcup_{\mathfrak{S}'\subsetneqq\mathfrak{S},\mathfrak{S}'\in S^{\mathfrak{F}}_{\mathfrak{X}}} \mathfrak{S}'$$

For $\phi\in P^{R,\mathfrak{F}}_{\mathfrak{X}}$, define a function on $\overset{\circ}{\mathfrak{S}}_{\leq T}$ denoted F_ϕ, by:

$$F_\phi(\pi) = e_{\mathfrak{C}}(\phi,\pi)$$

Theorem (MW V.3.13)**.** (i)*Fix $T\in\mathbb{R}^*_+$. For every $\mathfrak{S}\in S^{\mathfrak{F}}_{\mathfrak{X}}$, there exists a residue datum, denoted by $\mathrm{Res}^G_{\mathfrak{S}}$, whose value on the functions of the form $A(\phi',\phi)$, where $\phi',\phi\in P^{R,\mathfrak{F}}_{\mathfrak{X}}$ is uniquely determined whenever $\mathfrak{S}_{\leq T}$ contains an element of real part $o(\mathfrak{S})$ and which is zero for almost all $\mathfrak{S}\in S^{\mathfrak{F}}_{\mathfrak{X}}$, in particular for $\mathfrak{S}\in S^{\mathfrak{F}}_{\mathfrak{X}}$ if $\|o(\mathfrak{S})\|>R$, such that:*

$$\pi\in\mathfrak{S}_{\mathfrak{C}} \quad\longrightarrow\quad \sum_{w\in W(M)} (\mathrm{Res}^G_{w\mathfrak{S}_{\mathfrak{C}}} A(\phi',\phi))(w\pi)$$

is holomorphic at every point of $\mathfrak{S}_{\mathfrak{C},\leq T}$ of real part $o(\mathfrak{S})$ and

$$\langle\theta_{\phi'},q_T\theta_\phi\rangle \;=\; \sum_{\mathfrak{C}\in[S^{\mathfrak{F}}_{\mathfrak{X}}]} |\mathrm{Norm}\mathfrak{S}_{\mathfrak{C}}|^{-1} \times \int_{\pi\in\mathfrak{S}_{\mathfrak{C},\leq T},\,\mathrm{Re}\,\pi=o(\mathfrak{S}_{\mathfrak{C}})} \sum_{w\in W(M)} (\mathrm{Res}^G_{w\mathfrak{S}_{\mathfrak{C}}} A(\phi',\phi))(w\pi)\; d_{\mathfrak{S}_{\mathfrak{C}}}\pi$$

In particular, the family of projections $\mathrm{proj}^{\mathfrak{F}}_{\mathfrak{C},T}$ is uniquely determined by the property:

$$\mathrm{proj}^{\mathfrak{F}}_{\mathfrak{C},T}\theta_\phi \;=\; \int_{\pi\in\mathfrak{S}_{\mathfrak{C},\leq T},\,\mathrm{Re}\,\pi=o(\mathfrak{S}_{\mathfrak{C}})} e_{\mathfrak{C}}(\phi,\pi)\; d_{\mathfrak{S}_{\mathfrak{C}}}\pi$$

We denote by $\mathrm{Sing}^{G,\mathfrak{F}}_T$ the set of elements $\mathfrak{S}$ of $S^{\mathfrak{F}}_{\mathfrak{X}}$ such that $\{\pi\in\mathfrak{S}_{\leq T}|\mathrm{Re}\pi = o(\mathfrak{S})\}\neq\emptyset$ and there exists $\phi',\phi\in P^{R,\mathfrak{F}}_{\mathfrak{X}}$ satisfying $\mathrm{Res}^G_{\mathfrak{S}}A(\phi',\phi)\neq 0$.

(ii) *Let $T_1 > T$ be elements of $\mathbb{R}^*_+$. Then $\mathrm{Sing}^{G,\mathfrak{F}}_T$ is contained in $\mathrm{Sing}^{G,\mathfrak{F}}_{T_1}$ and consists of elements $\mathfrak{S}$ of $\mathrm{Sing}^{G,\mathfrak{F}}_{T_1}$ for which $\{\pi\in\mathfrak{S}_{\leq T}|\mathrm{Re}\pi = o(\mathfrak{S})\}$ is non-empty. We* *set* $\mathrm{Sing}^{G,\mathfrak{F}} = \bigcup_{T\in\mathbb{R}_+^*} \mathrm{Sing}_T^{G,\mathfrak{F}}$. *Let* $\mathfrak{S} \in \mathrm{Sing}^{G,\mathfrak{F}}$ *and let* $\phi', \phi \in P_{\mathfrak{X}}^{R,\mathfrak{F}}$*; then* $\mathrm{Res}_{\mathfrak{S}}^G A(\phi', \phi)$ *is independent of the choice of* $G \in \mathbb{R}_+^*$ *such that* $\mathfrak{S} \in \mathrm{Sing}_T^{G,\mathfrak{F}}$. *In particular,* $\mathfrak{S} \in \mathrm{Sing}^{G,\mathfrak{F}}$ *if and only if* $\mathfrak{S} \in \mathrm{Sing}_{T_{\mathfrak{S}}}^{G,\mathfrak{F}}$ *where* $T_{\mathfrak{S}}$ *is the smallest element of* $\mathbb{R}_+^*$ *such that* $\{\pi \in \mathfrak{S}_{\leq T_{\mathfrak{S}}} | \mathrm{Re}\,\pi = o(\mathfrak{S})\}$ *is non-empty.*

(iii) *The discrete spectrum of* $L^2(G_k\backslash\mathbf{G})_{\mathfrak{X},\mathfrak{F}}$ *is the closure of the vector space generated by the functions on* $G_k\backslash\mathbf{G}$ *of the form:*

$$\sum_{w\in W(M)} \mathrm{Res}_{w\mathfrak{S}}^G\, E(\phi, w\pi)$$

or equivalently,

$$\mathrm{Res}_{\mathfrak{S}}^G\, E(\phi, \pi)$$

where $\mathfrak{S} \in \mathrm{Sing}G$, $\mathfrak{S}$, *of cuspidal data* $(M, \mathfrak{B})$ *varying in* $\mathfrak{X}$, *is reduced to a point and* π *is this point. Each of the above functions in an eigenvector for the action of the center of the enveloping algebra at the archimedean places, and for the action of the Bernstein center at the finite places.*

Corollary (See MW V.3.13, V.2.2, V.3.2(5))**.** (i) *The assertions of MW V.2.2 (see theorem in 2.4, above) are true for every* T*; in particular* $e_{\mathfrak{C}}(\phi, \pi)$ *is holomorphic at every point of* $o(\mathfrak{S}) + \mathrm{Im}\mathfrak{S}$.

(ii) *Let* $\mathfrak{C} \in [S_{\mathfrak{X}}^{\mathfrak{F}}]$ *and let* $T < T_1$ *be positive numbers. Then* $L_{\mathfrak{C},T}^{2\,\mathfrak{F}}$ *is contained in* $L_{\mathfrak{C},T_1}^{2\,\mathfrak{F}}$ *and we denote by* $L^2(G_k\backslash\mathbf{G})_{\mathfrak{C},\mathfrak{F}}$ *the closure of the space* $\bigcup_{T\in\mathbb{R}_+^*} L_{\mathfrak{C},T}^{2\,\mathfrak{F}}$. *It is a Hilbert space which admits a spectral decomposition satisfying:*

> (a) *For every* $\pi \in \mathfrak{S}$ *of real part* $o(\mathfrak{S})$, $A_{\mathfrak{C},\pi}^{\mathfrak{F}}$ *is equipped with a positive definite scalar product satisfying*
>
> $$\forall\, \phi', \phi \;\in\; P_{\mathfrak{X}}^{R,\mathfrak{F}},\;\; \langle e_{\mathfrak{C}}(\phi', \pi), e_{\mathfrak{C}}(\phi, \pi)\rangle = r_{\mathfrak{C}}(\phi', \phi)(\pi)$$
>
> *This product is given by integration on* $G_k\backslash\mathbf{G}$ *only if* $A_{\mathfrak{C},\pi}^{\mathfrak{F}}$ *consists of square integrable automorphic forms, which is equivalent to saying that* $\mathfrak{C}$ *consists of points.*
>
> (b) *The map* $\pi \in \{\mathfrak{S}_{\mathfrak{C}} | \mathrm{Re}\,\pi = o(\mathfrak{S}_{\mathfrak{C}})\} \to A_{\mathfrak{C},\pi}^{\mathfrak{F}}$ *is a Hilbertian stack and* $L_{\mathfrak{C}}^{2\,\mathfrak{F}}$ *can be identified with the Hilbert space consisting of measurable functions* F

on $\{\mathfrak{S}_{\mathfrak{C}}|\mathrm{Re}\,\pi = o(\mathfrak{S}_{\mathfrak{C}})\}$ *such that:* $F(\pi) \in A^{\mathfrak{F}}_{\mathfrak{C},\pi}$ *almost everywhere,* $F(w\pi) = F(\pi)\ \forall\, w \in \mathrm{Norm}\mathfrak{S}_{\mathfrak{C}}$ *and*

$$\int_{\pi\in\mathfrak{S}_{\mathfrak{C}},\mathrm{Re}\,\pi=o(\mathfrak{S}_{\mathfrak{C}})} \|F(\pi)\|^2\, d_{\mathfrak{S}_{\mathfrak{C}}}\pi \quad < \quad \infty$$

where $\|\ \|^2$ *is the square of the norm in* $A^{\mathfrak{F}}_{\mathfrak{C},\pi}$ *defined in* (a).

(iii)

$$L^2(G_k\backslash\mathbf{G})_{\mathfrak{X},\mathfrak{F}} \quad = \quad \widehat{\bigoplus}_{\mathfrak{C}\in[S^{\mathfrak{F}}_{\mathfrak{X}}]} L^2(G_k\backslash\mathbf{G})_{\mathfrak{C},\mathfrak{F}}$$

the sum being orthogonal

For $\mathfrak{C} \in [S_{\mathfrak{X}}]$, $\mathfrak{S} \in \mathfrak{C}$ and for every $\pi \in \mathfrak{S}$ of real part $o(\mathfrak{S})$, set:

$$\begin{aligned} A_{\mathfrak{C},\pi} \quad &= \quad \bigcup_{\mathfrak{F}\ \ \mathfrak{C}\in[S^{\mathfrak{F}}_{\mathfrak{X}}]} A^{\mathfrak{F}}_{\mathfrak{C},\pi}, \\ L^2(G_k\backslash\mathbf{G}) \quad &= \quad \bigcup_{\mathfrak{F}\ \ \mathfrak{C}\in[S^{\mathfrak{F}}_{\mathfrak{X}}]} L^{2\ \mathfrak{F}}_{\mathfrak{C}}. \end{aligned}$$

The union can be completed to obtain a Hilbert space $\overline{A}_{\mathfrak{C},\pi}$.

Theorem (MW, Corollary V.3.14)**.** (i) *Let* $\mathfrak{C} \in [S_{\mathfrak{X}}]$*; fix* $\mathfrak{S} \in \mathfrak{C}$*. Then the map* $\pi \in o(\mathfrak{S}) + \mathrm{Im}\,\mathfrak{S} \to \overline{A}_{\mathfrak{C},\pi}$ *is a Hilbertian stack, and* $L^2(G_k\backslash\mathbf{G})_{\mathfrak{C}}$ *is isomorphic to the following Hilbert space, denoted by* $\mathrm{Hilb}_{\mathfrak{C}}$

> *The measurable functions* F *on* $o(\mathfrak{S}) + \mathrm{Im}\,\mathfrak{S}$ *such that: for almost every* $\pi \in o(\mathfrak{S}) + \mathrm{Im}\,\mathfrak{S}$*,* $F(\pi) \in \overline{A}_{\mathfrak{C},\pi}$ *and* $F(w\pi) = F(\pi)$ *for all* $w \in \mathrm{Norm}\,\mathfrak{S}$*,*
>
> $$\int_{\pi\in\mathfrak{S}+\mathrm{Im}\,\mathfrak{S}} \|F(\pi)\|^2 \quad < \quad +\infty$$
>
> *where the norm is that of* $\overline{A}_{\mathfrak{C},\pi}$.

This isomorphism, denoted by $j_{\mathfrak{C}}$*, is characterized by the equalities, for every* $\phi \in P_{\mathfrak{X}}$*:*

$$\mathrm{proj}_{L^2(G_k\backslash\mathbf{G})_{\mathfrak{C}}}\theta_\phi \quad = \quad j_{\mathfrak{C}} F_\phi,$$

Moreover $L^2(G_k\backslash\mathbf{G})_{\mathfrak{C}}$ *and* $\overline{A}_{\mathfrak{C},\pi}$ *are naturally equipped with a* $\mathbf{G}$*-action; so there is a* $\mathbf{G}$*-action on* $\mathrm{Hilb}_{\mathfrak{C}}$ *and the above isomorphism is* $\mathbf{G}$*-equivariant.*

(ii) $L^2(G_k\backslash\mathbf{G})_{\mathfrak{X}}$ *is the completion of the orthogonal sum:*

$$\bigoplus_{\mathfrak{C}\in[S_{\mathfrak{X}}]} L^2(G_k\backslash\mathbf{G})_{\mathfrak{C}}$$

2.6 Spectral decomposition in terms of cuspidal data on the Levi subgroups

Fix an equivalence class $\mathfrak{X}$ of cuspidal data. For every $\mathfrak{S}\in S_{\mathfrak{X}}$, we fix a residue datum $\mathrm{Res}_{\mathfrak{S}}^{G}$ for $\mathfrak{S}$ satisfying MW V.3.13(a) (see Theorem in 2.5, above). Moreover for every $(M,\mathfrak{P})\in\mathfrak{X}$ and for every standard Levi L containing M we fix a unitary character of the center of $\mathbf{L}$, denoted by ξ_L, extending ξ, trivial on $A_L^{\mathbf{G}}$ and such that:

$$\mathfrak{P}_L \;=\; \{\pi\in\mathfrak{P} \text{ having } \xi_L \text{ for restriction to } Z_L\} \;\neq\; \emptyset$$

A discrete parameter for $\mathfrak{X}$ is a pair (L,δ) where L is a standard Levi of G and where δ is a subrepresentation of the discrete spectrum of $\mathbf{L}$ having central character ξ_L extending ξ, trivial on $A_L^{\mathbf{G}}$, and such that δ is of the form

$$\delta \;=\; L^2(L_k\backslash\mathbf{L})_{\mathfrak{C}_L}$$

where $\mathfrak{X}_L$ is an equivalence class of cuspidal data (relative to L) 'contained' in $\mathfrak{X}$ and where $\mathfrak{C}_L$ is an element of $[S_{\mathfrak{X}_L}]$ consisting of points. We say that $\mathfrak{C}_L$ is the singular class attached to δ; we note that $\mathfrak{C}_L$ is uniquely determined by δ. We define the association class of (L,δ), denoted by $[(L,\delta)]$, to be the set of pairs $(wLw^{-1},w\delta)$ where w runs through $W(L)$.

We denote by $L^2(L_k\backslash\mathbf{L})_{d,\xi}$ the part of the discrete spectrum of L (modulo the center) whose central character is trivial on $A_L^{\mathbf{G}}$, and whose restriction to the center of $\mathbf{G}$ is the character ξ. We denote by $A(L_k\backslash\mathbf{L})_{d,\xi}$ the intersection of this space with the space of automorphic forms for L. Writing U_L for the unipotent radical of the standard parabolic of Levi L, we define:

$$A(U_{L,\mathbb{A}}L_k\backslash\mathbf{G})_{d,\xi} \;=\; \{f\in A(U_{L,\mathbb{A}}L_k\backslash\mathbf{G})\,|\,\forall\, k\in\mathbf{K}, f_k\in A(L_k\backslash\mathbf{L})_{d,\xi}\}$$

see Section 2.2, above, for the notation f_k. We denote by $P^R_{L,d}$ the space of holomorphic functions on $\{\mu \in X^{\mathbf{G}}_L \mid \|\mathrm{Re}\,\mu \ < \ R\}$ with values in $A(U_{L,\mathbb{A}}L_k\backslash\mathbf{G})_{d,\xi}$ satisfying the growth condition of the Paley-Wiener space in Section 2.3.

Theorem (MW VI.2.1)**.** *Let L be a standard Levi of G.*

(i) *Let $\phi \in P^R_{L,d}$ and let $\mu \in X^{\mathbf{G}}_L$. The Eisenstein series $E(\phi,\mu)$ (or more precisely $E^G_L(\phi,\mu)$), defined by a series which converges for very positive* $\mathrm{Re}\,\mu$ *can be continued to a meromorphic operator on μ holomorphic on* $\mathrm{Im}\,X^{\mathbf{G}}_L$*. This is also true for the intertwining operator $M(t^{-1},t\mu)$ for $t \in W$ of minimal length in its right class modulo W_L such that tLt^{-1} is still a standard Levi. We have the functional equation:*

$$E^G_L(\phi,\mu) \;\; = \;\; E^G_{tLt^{-1}}(M(t,\mu)\phi, t\mu)$$

(ii) *The limit*

$$\lim_{T\to\infty} \int_{\mu\in \mathrm{Im}\,X^{\mathbf{G}}_L,\ \|\mu\|\le T} E(\phi,\mu)d\mu$$

exists in the L^2, i.e. defines an element of $L^2(G_k\backslash\mathbf{G})_\xi$. We denote by $L^2(G_k\backslash\mathbf{G})_{L,\xi}$ the closed subspace of $L^2(G_k\backslash\mathbf{G})$ generated by these elements. Let L' be another standard Levi of G; suppose that L and L' are conjugate. Then $L^2(G_k\backslash\mathbf{G})_{L,\xi} = L^2(G_k\backslash\mathbf{G})_{L',\xi}$. We denote this space by $L^2(G_k\backslash\mathbf{G})_{[L],\xi}$ where $[L]$ is the association class of L.

(iii) *We have the orthogonal decomposition:*

$$L^2(G_k\backslash\mathbf{G})_\xi \;\; = \;\; \bigoplus_{[L]} L^2(G_k\backslash\mathbf{G})_{[L],\xi}$$

where $[L]$ runs through the association classes of standard Levis of G.

(iv) *Let W^L be the set of elements of W of minimal length modulo W_L and*

$$\mathrm{Stab}_{W^L}L \;\; = \;\; \{w \in W^L \mid wLw^{-1} = L\}$$

Then $L^2(G_k\backslash\mathbf{G})_{d,\xi}$ is isometric to the following Hilbert space:

> *Measurable functions F on* $\mathrm{Im}\,X^{\mathbf{G}}_L$ *with values in* $\mathrm{ind}^{\mathbf{K}}_{\mathbf{K}\cap\mathbf{L}}\, L^2(L_k\backslash\mathbf{L})_{d,\xi}$ *such that:*
>
> $$M(t^{-1},t\mu)F(t\mu) \;\; = \;\; F(\mu), \tag{2.1}$$

almost everywhere, for every $t \in \mathrm{Stab}_{W^L} L$*, and*

$$\lim_{T\to\infty} \int_{\mu \in \mathrm{Im} X_L^{\mathbf{G}},\ \|\mu\| \leq T} |\mathrm{Stab}_{W^L} L|^{-1} \, \|F(\mu)\|^2 \, d\mu \quad < \quad \infty \tag{2.2}$$

where the norm is that of $\mathrm{ind}_{\mathbf{K}\cap\mathbf{L}}^{\mathbf{K}} L^2(L_k \backslash \mathbf{L})_{d,\xi}$*. The norm of this Hilbert space is given by Equation (2.2).*

Chapter 3

Automorphic spectral expansions and Sobolev spaces

3.1 Parametrization of spectrum, spectral transform and inversion

The spectral theory of automorphic forms gives a decomposition of the space of square integrable automorphic forms in terms of eigenfunctions. For succinctness, we restrict our attention to automorphic forms that are *spherical* at infinity and have *trivial central character*.

Though the automorphic spectrum consists of disparate pieces (cusp forms, Eisenstein series, residues of Eisenstein series) it will be useful to have a uniform notation. We define a parameter space Ξ with spectral (Plancherel) measure $d\xi$ and let Φ_ξ denote the elements of the spectrum.

Remark 3.1.1. Langlands functoriality would imply that every L-function of number theoretic interest is attached to some GL_n. The automorphic spectral theory for GL_n, articulated in Jacquet's conjectures and proven by Moeglin and Waldspurger [70], is easier to state than the theory for arbitrary reductive groups. For example, for $GL_n(\mathbb{R})$, the spectral "basis" $\{\Phi_\xi\}$ is parametrized by

$$\xi \;=\; \phi_1 \otimes \cdots \otimes \phi_\ell \qquad 1 \leq \ell \leq n$$

where ϕ_i is an automorphic form on $GL_{m_i}(\mathbb{R})$, $\quad 1 \leq m_i \leq n$. If $m_i = 1$, then ϕ_i is a character, and if $m_i > 1$, then ϕ_i is a cusp form. Since characters on $GL_1(\mathbb{R})$ are parametrized by $(0, +\infty)$ and since cusp forms decompose discretely with finite multiplicity, the spectral measure on Ξ is essentially given by products of Lebesgue measures on $(0, +\infty)$ with counting measures on orthonormal bases of cusp forms.

The general spectral theory discussed in Chapter 2 implies the following:

For test functions f on $G_k \backslash G_{\mathbb{A}}$, the spectral transform $\mathcal{F} : C_c^\infty(G_k \backslash G_{\mathbb{A}}) \to C^0(\Xi)$ by $f \to \langle f, \Phi_\xi \rangle$, where $\langle \, , \, \rangle$ is

$$\langle f_1, f_2 \rangle \;=\; \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} f_1(g) \, \overline{f}_2(g) \, dg$$

extends to an isometry $\mathcal{F} : L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}} / K_\infty) \to L^2(\Xi)$. On compactly supported continuous functions, the inverse map is given by an integral formula,

$$v \longrightarrow \int_\Xi v(\xi) \, \Phi_\xi \, d\xi$$

In fact, the inverse transform is given by this (L^2-convergent) integral for v in the image $\mathcal{F}\big(C_c^\infty(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}})\big)$, so an inversion formula

$$f \;\stackrel{L^2}{=}\; \int_\Xi \mathcal{F}f(\xi) \, \Phi_\xi \, d\xi$$

holds (at least) for test functions on $Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}$.

An automorphic Sobolev theory is necessary to justify construction of a $GL_n \times GL_n$ Poincaré series suitable for applications. Since the critical issues arise at the *archimedean* place, we will consider global *archimedean* spherical automorphic Sobolev spaces. Let X be the arithmetic quotient $\Gamma \backslash G / K$, where G is a semi-simple or reductive Lie group with discrete subgroup Γ and maximal compact subgroup K. Let Ξ be a parameter space for the spectrum with spectral (Plancherel) measure $d\xi$ and let Φ_ξ denote the elements of the spectrum. Let $\mathcal{F}$ be the spectral transform. Let λ_ξ be the eigenvalue of Casimir on Φ_ξ.

Sobolev theory provides a framework for discussing the interaction of differential operators and spectral transforms/inversions. Let Δ denote the Laplacian, i.e. the Casimir operator on right-K-invariant functions. For *test functions* f on $\Gamma\backslash G/K$, integration by parts yields

$$\mathcal{F}(\Delta f)\,(\xi) \;=\; \int_G \Delta f\;\overline{\Phi}_\xi \;=\; \int_G f\;\Delta\overline{\Phi}_\xi \;=\; \lambda_\xi \cdot \int_G f\,\overline{\Phi}_\xi \;=\; \lambda_\xi \cdot \mathcal{F}f\,(\xi)$$

Sobolev theory for $\Gamma\backslash G/K$ enables us to extend this relation by continuity to larger spaces of functions.

For a discussion of global automorphic Sobolev spaces for $SL_2(\mathbb{C})$, see [35].

3.2 Characterizations of Sobolev spaces

We define positive index global archimedean spherical automorphic Sobolev spaces as the right K-invariant subspaces of completions of $C_c^\infty(\Gamma\backslash G)$ with respect to a topology induced by seminorms associated to derivatives from the universal enveloping algebra, as follows.

The universal enveloping algebra $\mathcal{U}\mathfrak{g}$ acting on functions on G can be identified with G-invariant differential operators on G. The *right* action of $\mathcal{U}\mathfrak{g}$ descends to functions on $\Gamma\backslash G$. Let $\mathcal{U}\mathfrak{g}^{\leq \ell}$ be the finite dimensional subspace of $\mathcal{U}\mathfrak{g}$ consisting of elements of degree less than or equal to ℓ.

Each $\alpha \in \mathcal{U}\mathfrak{g}$ gives a seminorm ν_α on $C_c^\infty(\Gamma\backslash G)$.

$$\nu_\alpha(f) \;=\; \|\alpha f\|^2_{L^2(\Gamma\backslash G)}$$

The topology induced by supremums of finite linear combinations of these is equivalent to the topology induced by the seminorms

$$\nu_B(f) \;=\; \sup_{\gamma \in B} \|\gamma f\|^2_{L^2(\Gamma\backslash G)} \qquad \text{bounded } B \subset (\mathcal{U}\mathfrak{g})^{\leq \ell}$$

Definition 3.2.1. Consider the space of smooth functions that are bounded with respect to these seminorms:

$$\{f \in C^\infty(\Gamma\backslash G) : \nu_\alpha f \;<\; \infty \;\text{ for all } \alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}\}$$

Let $H^\ell(\Gamma\backslash G)$ be the completion of this space with respect to the topology induced by the family $\{\nu_\alpha : \alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}\}$. The *global archimedean spherical automorphic Sobolev space* $H^\ell(X) = H^\ell(\Gamma\backslash G)^K$ is the subspace of right-K-invariant functions.

Proposition 3.2.1. *The space of test functions $C_c^\infty(X)$ is dense in $H^\ell(X)$.*

Proof. We approximate a smooth function $f \in H^\ell(X)$ by pointwise products with smooth cut-off functions, as follows. Let $\{\eta_n\} \subset C_c^\infty(\Gamma\backslash G)$ be a family of left N-invariant, right K-invariant smooth cut-off functions with $\cup_n \mathrm{spt}(\eta_n) = \Gamma\backslash G$ and

$$\sup_{g\in\Gamma\backslash G} |\alpha\eta_n(g)| \;\ll\; 1$$

for all $\alpha \in \mathcal{U}\mathfrak{g}^{\leq\ell}$, where the implied constant does not depend on the support of η, but may depend on ℓ.

By definition,

$$\nu_\gamma\big(\eta_n \cdot f - f\big) \;=\; \|\gamma\big(\eta_n \cdot f - f\big)\|_{L^2(\Gamma\backslash G)}$$

Leibnitz' rule implies that $\gamma\big(\eta_n \cdot f - f\big)$ is a finite linear combination of terms of the form $\alpha(\eta_n - 1)\cdot\beta f$ where α, $\beta \in \mathcal{U}\mathfrak{g}^{\leq\ell}$. When $\alpha = 0$,

$$\|\alpha(\eta_n - 1)\cdot\beta f\|_{L^2(\Gamma\backslash G)} \;=\; \|(\eta_n - 1)\cdot\beta f\|_{L^2(\Gamma\backslash G)} \;\leq\; \int_{\Gamma\backslash G - \mathrm{spt}\eta_n} |(\beta\, f)(g)|^2\, dg$$

Otherwise, $\alpha(\eta_n - 1) = \alpha\eta_n$, and

$$\begin{aligned}
\|\alpha(\eta_n - 1)\cdot\beta f\|_{L^2(\Gamma\backslash G)} \;&=\; \|\alpha\eta_n\cdot\beta f\|_{L^2(\Gamma\backslash G)} \\
&\ll\; \sup_{g\in G} |\alpha\,\eta_n(g)| \;\cdot\; \int_{\Gamma\backslash G - \mathrm{spt}\eta_n} |(\beta\, f)(g)|^2\, dg \\
&\ll\; \int_{\Gamma\backslash G - \mathrm{spt}\eta_n} |(\beta\, f)(g)|^2\, dg
\end{aligned}$$

Let B be any bounded set containing all of the (finitely many) β that appear as a result of applying Leibniz' rule. Then

$$\nu_\gamma\big(\eta_n \cdot f - f\big) \;\ll\; \sup_{\beta\in B} \int_{\Gamma\backslash G - \mathrm{spt}\eta_n} |(\beta\, f)(g)|^2\, dg$$

Since B is bounded and $f \in H^\ell(X)$,

$$\sup_{\beta\in B} \int_{\Gamma\backslash G - \mathrm{spt}\eta_n} |(\beta\, f)(g)|^2\, dg \quad\longrightarrow\quad 0 \qquad \text{as } \; n\to\infty$$

□

Proposition 3.2.2. *Let Ω be the Casimir operator in the center of $\mathcal{U}\mathfrak{g}$. The norm $\|\cdot\|_{2\ell}$ on $C_c^\infty(\Gamma\backslash G)^K$ given by*

$$\|f\|_{2\ell}^2 \;=\; \|f\|^2 \;+\; \|(1-\Omega)\, f\|^2 \;+\; \|(1-\Omega)^2\, f\|^2 \;+\; \ldots \;+\; \|(1-\Omega)^\ell\, f\|^2$$

where $\|\cdot\|$ is the usual norm on $L^2(\Gamma\backslash G)$, induces a topology on $C_c^\infty(\Gamma\backslash G)^K$ that is equivalent to the topology induced by the family $\{\nu_\alpha : \alpha \in \mathcal{U}\mathfrak{g}^{\leq 2\ell}\}$ of seminorms and with respect to which $H^{2\ell}(X)$ is a Hilbert space.

Proof. We start by comparing ν_α to a seminorm involving only the Casimir operator.

Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g}$, and let $\{X_i\}$ be a basis for $\mathfrak{g}$. Then

$$\Omega \;=\; \sum_{X_i} X_i\, X_i^*$$

Let $\Omega_{\mathfrak{p}}$ and $\Omega_{\mathfrak{k}}$ denote the subsums corresponding to the subspaces $\mathfrak{p}$ and $\mathfrak{k}$ respectively. The set Σ of possible K-types of $\gamma\, f$, for $\gamma \in \mathcal{U}\mathfrak{g}^{\leq \ell}$, is finite. Let λ_σ denote the $\Omega_{\mathfrak{k}}$-eigenvalue of a function of K-type σ, and let C be a number greater than the maximum value of $\{\lambda_\sigma : \sigma \in \Sigma\}$.

By the Poincaré-Birkhoff-Witt theorem we may assume α is a monomial of the form

$$\alpha \;=\; x_1 \ldots x_n\, y_1 \ldots y_m$$

where $x_i \in \mathfrak{p}$ and $y_i \in \mathfrak{k}$. On right-K-invariant functions , each y_i acts by zero. Thus, for $f \in C_c^\infty(\Gamma\backslash G)^K$,

$$\nu_\alpha f \;=\; \langle \alpha f, \alpha f\rangle_{L^2(\Gamma\backslash G)} \;=\; \langle x_1 \ldots x_n\, f,\; x_1 \ldots x_n f\rangle_{L^2(\Gamma\backslash G)} \qquad (x_i \in \mathfrak{p})$$

Lemma 3.2.1. *For $\varphi \in C_c^\infty(\Gamma\backslash G)$ and $\alpha = x_1 \ldots x_n$ a monomial in $\mathcal{U}\mathfrak{g}$ with $x_i \in \mathfrak{p}$,*

$$\langle \alpha\, \varphi, \alpha\, \varphi\rangle \;\leq\; \langle(-\Omega + C)^n\, \varphi,\, \varphi\rangle$$

where $\langle\, ,\, \rangle$ is the usual inner product on $L^2(\Gamma\backslash G)$.

Proof. We proceed by induction on $n = \deg \alpha$.

For $n = 1$, $\alpha = x \in \mathfrak{p}$. Since the Killing form is positive definite on $\mathfrak{p}$, we may choose a self-dual basis $\{X_i\}$ for $\mathfrak{p}$ such that $X_1 = x$. Then,

$$\begin{aligned} \langle x\varphi,\, x\varphi\rangle \;\; &\leq \;\; \sum_i \langle X_i\,\varphi,\, X_i\,\varphi\rangle \;\; = \;\; -\sum_i \langle X_i^2\,\varphi, \varphi\rangle \\ &= \;\langle -\Omega_{\mathfrak{p}}\,\varphi, \varphi\rangle \;\; = \;\; \langle (-\Omega + \Omega_{\mathfrak{k}})\,\varphi, \varphi\rangle \;\; \leq \;\; \langle (-\Omega + C)\,\varphi, \varphi\rangle \end{aligned}$$

For $n > 1$, write $\alpha = x\gamma$, where $x = x_1$ and $\gamma = x_2 \ldots x_n$. Then, by the above argument,

$$\langle x\,\gamma\varphi,\; x\,\gamma\varphi\rangle \;\; \leq \;\; \langle (-\Omega + C)\,\gamma\varphi,\, \gamma\varphi\rangle$$

Claim 1. *The operator $-\Omega + C$ has a positive symmetric square root in the center of $\mathcal{U}\mathfrak{g}$.*

Proof. Let $T = (-\Omega + C)$. The operator $-\Omega_{\mathfrak{k}} + C$ is a non-negative symmetric operator on images of (smooth) spherical vectors under $\mathcal{U}\mathfrak{g}^{\leq \ell}$, so

$$T = (-\Omega + C) = -\Omega_{\mathfrak{p}} - \Omega_{\mathfrak{k}} + C$$

is a strictly positive symmetric unbounded operator. By Friedrichs [28, 29] there is an everywhere defined inverse R, which is a positive symmetric *bounded* operator, and which, by the spectral theory for bounded symmetric operators, has a positive symmetric square root $\sqrt{R}$ in the closure of the polynomial algebra $\mathbb{C}[R]$. Then $(1 - \sqrt{R})$ is a symmetric positive square root of T commuting with all operators that commute with T, i.e. all of $\mathcal{U}\mathfrak{g}$, since $T = (-\Omega + C)$ is in the center of $\mathcal{U}\mathfrak{g}$.

□

Thus, $\langle (-\Omega + C)\,\gamma\varphi,\, \gamma\varphi\rangle$ is

$$\begin{aligned} \langle \sqrt{-\Omega + C}\,\sqrt{-\Omega + C}\;\gamma\varphi,\; \gamma\varphi\rangle \;\; &= \;\; \langle \sqrt{-\Omega + C}\;\gamma\varphi,\; \sqrt{-\Omega + C}\;\gamma\varphi\rangle \\ &= \;\; \langle \gamma\,\sqrt{-\Omega + C}\;\varphi,\; \gamma\,\sqrt{-\Omega + C}\;\varphi\rangle \end{aligned}$$

By inductive hypothesis,

$$\begin{aligned}\langle \gamma \sqrt{-\Omega+C}\,\varphi,\ \gamma \sqrt{-\Omega+C}\,\varphi\rangle &\le \langle (-\Omega+C)^{n-1}\sqrt{-\Omega+C}\,\varphi,\ \sqrt{-\Omega+C}\,\varphi\rangle \\ &= \langle (-\Omega+C)^{n}\,\varphi,\ \varphi\rangle\end{aligned}$$

□

Thus, for any $\alpha \in \mathcal{U}\mathfrak{g}$, there is a constant C, possibly depending on the degree of α, such that

$$\nu_\alpha(f) \ \ll\ \langle (-\Omega+C)^{\deg\alpha} f, f\rangle \qquad \text{for all } f \in C_c^\infty(\Gamma\backslash G)^K$$

In fact, for right K-invariant functions,

$$(-\Omega+C)^{\deg\alpha} f \ = \ (-\Omega_{\mathfrak{p}}+C)^{\deg\alpha} f$$

Since $\Omega_{\mathfrak{p}}$ is positive semi-definite, multiplying by a positive constant does not change the topology. Thus, we may take $C = 1$.

That is, the subfamily $\{\nu_\alpha : \alpha = (1-\Omega)^k, k \le \ell\}$ of seminorms on $C_c^\infty(\Gamma\backslash G)^K$ dominates the family $\{\nu_\alpha : \alpha \in \mathcal{U}\mathfrak{g}^{\le 2\ell}\}$ and thus induces an equivalent topology.

□

It will be necessary to have another description of Sobolev spaces. Consider

$$W^{2,\ell}(\Gamma\backslash G) \ = \ \{f \in L^2(\Gamma\backslash G) : \alpha f \in L^2(\Gamma\backslash G) \text{ for all } \alpha \in \mathcal{U}\mathfrak{g}^{\le\ell}\}$$

where the action of $\mathcal{U}\mathfrak{g}$ on $L^2(\Gamma\backslash G)$ is by distributional differentiation: f in $L^2(\Gamma\backslash G)$ may identified with the distribution Λ_f on $\Gamma\backslash G$

$$\Lambda_f(\varphi) \ = \ \int_{\Gamma\backslash G} f(g)\,\varphi(g)\,dg$$

and the distributional derivative $\alpha\,\Lambda_f$ is

$$(\alpha\,\Lambda_f)(\varphi) \ = \ \Lambda_f(\alpha\,\varphi)$$

Give $W^{2,\ell}(\Gamma\backslash G)$ the topology induced by the seminorms

$$\nu_\alpha f \ = \ \|\alpha\, f\|^2_{L^2(\Gamma\backslash G)} \qquad \alpha \in \mathcal{U}\mathfrak{g}^{\le\ell}$$

where, again, αf is a distributional derivative.

Let $W^{2,\ell}(X) = W^{2,\ell}(\Gamma\backslash G)^K$.

Proposition 3.2.3. *These spaces are equal to the corresponding Sobolev spaces:*

$$W^{2,\ell}(\Gamma\backslash G) \;=\; H^\ell(\Gamma\backslash G) \qquad \textit{and} \qquad W^{2,\ell}(X) \;=\; H^\ell(X)$$

Proof. Since $W^{2,\ell}(X)$ and $H^\ell(X)$ are simply the right-K-invariants in $W^{2,\ell}(\Gamma\backslash G)$ and $H^\ell(\Gamma\backslash G)$ respectively, it suffices to prove the equality of $W^{2,\ell}(\Gamma\backslash G)$ and $H^\ell(\Gamma\backslash G)$.

Since test functions are dense in $H^\ell(\Gamma\backslash G)$, it suffices to show the density of test functions in $W^{2,\ell}(\Gamma\backslash G)$. Since G acts continuously on $W^{2,\ell}(\Gamma\backslash G)$ by right translation, Theorem 3.5.2 implies that *mollifications* are dense in $W^{2,\ell}(\Gamma\backslash G)$, i.e. for a smooth approximate identity $\{\eta_i\}$ on G and $f \in W^{2,\ell}(\Gamma\backslash G)$, the mollifications $\eta \cdot f$, given by the following $W^{2,\ell}(\Gamma\backslash G)$-valued Gelfand-Pettis integrals,

$$(\eta_i \cdot f)(g) \;=\; \Big(\int_G \eta_i(h)\, R_h\, f\, dh\Big)(g) \;=\; \int_G \eta_i(h)\, (R_h\, f)(g)\, dh \;=\; \int_G \eta_i(h)\, f(gh)\, dh$$

approach f in the topology of $W^{\ell,2}(\Gamma\backslash G)$. Thus it suffices to show that mollifications can be approximated by test functions.

Urysohn's Lemma implies that $C_c^0(\Gamma\backslash G)$ is dense in $L^2(\Gamma\backslash G)$ and thus in $W^{2,\ell}(\Gamma\backslash G)$. Thus it suffices to consider mollifications of continuous, compactly supported functions.

Let $\eta \in C_c^\infty(G)$ and $f \in C_c^0(\Gamma\backslash G)$. By the proof of Theorem 3.5.2, $\eta \cdot f$ is a smooth vector, and for all $\alpha \in \mathcal{U}\mathfrak{g}$,

$$\alpha \cdot (\eta \cdot f) \;=\; (L_\alpha \eta) \cdot f$$

Since f is a continuous *function* and the action of the group is by translations, the abstract differentiation of $\eta \cdot f$ as a *vector* is the same as differentiation of $\eta \cdot f$ as a *function*, as follows. For $X \in \mathfrak{g}$, the action on $\eta \cdot f$ as a vector is

$$\frac{\partial}{\partial t}\Big|_{t=0} e^{tX} \cdot (\eta \cdot f) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} e^{tX} \cdot \int_G \eta(g)\;\, g \cdot f\, dg \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \int_G \eta(g)\;\, (g\, e^{tX}) \cdot f\, dg$$

Now using the fact that f is a *function* and the group action on f is by translation,

$$\big(X \cdot (\eta \cdot f)\big)(h) \;=\; \Big(\frac{\partial}{\partial t}\Big|_{t=0} \int_G \eta(g)\;\, R_{e^{tX}g} f\, dg\Big)(h) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \int_G \eta(g)\; R_{e^{tX}g} f(h)\, dg$$

$$= \frac{\partial}{\partial t}\bigg|_{t=0} \int_G \eta(g)\, f(he^{tX} g)\, dg \;=\; \frac{\partial}{\partial t}\bigg|_{t=0} \int_G \eta(g)\, R_g f(he^{tX})\, dg$$

$$= \frac{\partial}{\partial t}\bigg|_{t=0} \left(\int_G \eta(g)\, R_g f\, dg\right)(he^{tX}) \;=\; \frac{\partial}{\partial t}\bigg|_{t=0} (\eta \cdot f)(he^{tX}) \;=\; R_X(\eta \cdot f)(h)$$

This is the action of X on $(\eta \cdot f)$ as a *function* on $\Gamma\backslash G$. Thus the smoothness of $(\eta \cdot f)$ as a vector implies that it is a genuine smooth function.

The support of $\eta \cdot f$ is contained in the product of the compact supports of η and f. Since the product of two compact sets is again compact, $\eta \cdot f$ is compactly supported.

□

Remark 3.2.1. By Proposition 3.2.2, $H^{2\ell}(X) = W^{2,2\ell}(X)$ is a Hilbert space with norm

$$\|f\|_{2\ell}^2 \;=\; \|f\|^2 \;+\; \|(1-\Omega)\, f\|^2 \;+\; \ldots \;+\; \|(1-\Omega)^\ell\, f\|^2$$

where $\|\cdot\|$ is the usual norm on $L^2(\Gamma\backslash G)$, and $(1-\Omega)^k f$ is a distributional derivative.

3.3 Spectral transform, inversion, and differentiation on Sobolev spaces

Proposition 3.3.1. *For $\ell \geq 0$, the Laplacian extends to a continuous linear map $H^{2\ell+2}(X) \to H^{2\ell}(X)$, the spectral transform extends to a map on $H^{2\ell}(X)$, and*

$$\mathcal{F}\big((1-\Delta)f\big) \;=\; (1-\lambda_\xi)\cdot \mathcal{F}f \qquad \textit{for all } f \in H^{2\ell+2}(X)$$

Proof. The Laplacian is a continuous map

$$\Delta : C^\infty(\Gamma\backslash G) \cap H^{2\ell+2}(\Gamma\backslash G) \to C^\infty(\Gamma\backslash G) \cap H^{2\ell}(\Gamma\backslash G)$$

by the construction of the Sobolev topology. Since the Laplacian is the image of the Casimir operator, which is in the center of the universal enveloping algebra, it preserves right-K-invariance. Thus the Laplacian extends to a (continuous linear) map, also denoted Δ, from $H^{2\ell+2}(X)$ to $H^{2\ell}(X)$.

The spectral transform, defined on $C_c^\infty(\Gamma\backslash G)^K$ by the integral transform, extends by continuity to $H^{2\ell}(X)$. This extension agrees with the extension to $L^2(X)$ coming from Plancherel.

Since $\mathcal{F}\big(\Delta\varphi\big) = \lambda_\xi \cdot \mathcal{F}\varphi$, for $\varphi \in C_c^\infty(\Gamma\backslash G)^K$, by integration by parts,

$$\mathcal{F}\big((1-\Delta)f\big) \quad = \quad (1-\lambda_\xi)\cdot \mathcal{F}f \qquad \text{for all } f \in H^{2\ell+2}(X)$$

by continuity.

$\square$

Let μ be the following multiplication map on functions on Ξ

$$\mu(v)(\xi) \quad = \quad (1-\lambda_\xi)\cdot v(\xi)$$

For $\ell \in \mathbb{Z}$, the weighted L^2-spaces

$$V^{2\ell} \quad = \quad \{v \text{ measurable } : \mu^\ell(v) \in L^2(\Xi)\}$$

with norms

$$\|v\|_{V^{2\ell}} \quad = \quad \|\mu^\ell(v)\|_{L^2(\Xi)} \quad = \quad \int_\Xi (1-\lambda_\xi)^\ell \, |v(\xi)|^2 \, d\xi$$

are Hilbert spaces with $V^{2\ell+2} \subset V^{2\ell}$ for all ℓ. In fact, these are dense inclusions, since truncations are dense in all $V^{2\ell}$-spaces.

The multiplication map μ is a Hilbert space isomorphism:

$$V^{2\ell+2} \xrightarrow{\mu} V^{2\ell}$$

since it is clearly continuous, linear and bijective, and for $v \in V^{2\ell+2}$,

$$\|\mu(v)\|_{V^{2\ell}} \quad = \quad \|\mu^{\ell+1}(v)\|_{L^2(\Xi)} \quad = \quad \|v\|_{V^{2\ell+2}}$$

The negatively indexed spaces are (naturally identified with) the Hilbert space duals of their positively indexed counterparts:

$$(V^{2\ell})^* \quad \approx \quad V^{-2\ell}$$

by integration: $u \in V^{-2\ell}$ gives rise to $\Lambda_u \in (V^{2\ell})^*$ by

$$\Lambda_u(v) \quad = \quad \int_\Xi u(\xi)\cdot v(\xi)\, d\xi$$

The adjoints to inclusion maps are genuine inclusions, since $V^{2\ell+2} \hookrightarrow V^{2\ell}$ is dense for all $\ell \geq 0$, and, under the identification $(V^{2\ell})^* = V^{-2\ell}$ the adjoint map $\mu^* : (V^{2\ell})^* \to (V^{2\ell+2})^*$ is the multiplication map $\mu : V^{-2\ell} \to V^{-2\ell-2}$, since

$$\begin{aligned} \mu^*(\Lambda_u)(v) &= \Lambda_u(\mu(v)) = \int_\Xi u(\xi) \cdot (1-\lambda_\xi) v(\xi)\, d\xi \\ &= \int_\Xi (1-\lambda_\xi) u(\xi) \cdot v(\xi)\; d\xi = \Lambda_{\mu(u)}(v) \end{aligned}$$

Proposition 3.3.2. *For $\ell \geq 0$, the spectral transform $\mathcal{F}$ is an isometric isomorphism $H^{2\ell}(X) \to V^{2\ell}$*

Proof. On compactly supported functions, the spectral transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are given by integrals, which are certainly continuous linear maps. The Plancherel theorem extends $\mathcal{F}$ and $\mathcal{F}^{-1}$ to isometries between $L^2(X)$ and $L^2(\Xi)$. Thus $\mathcal{F}$ on $H^{2\ell}(X) \subset L^2(X)$ is a continuous linear L^2-isometry onto its image.

Let $f \in H^{2\ell}(X)$. By Proposition 3.2.3, the distributional derivatives $(1-\Delta)^k f \in L^2(X)$ for all $k \leq \ell$. By the Plancherel theorem and Proposition 3.3.1,

$$\|(1-\Delta)^\ell f\|_{L^2(X)} = \|\mathcal{F}\big((1-\Delta)^\ell f\big)\|_{L^2(\Xi)} = \|(1-\lambda_\xi)^\ell \cdot \mathcal{F}f\|_{L^2(\Xi)}$$

Thus $\mathcal{F}\big(H^{2\ell}\big) \subset V^{2\ell}$.

Since $V^{2\ell} \subset L^2(\Xi)$, $\mathcal{F}^{-1}$ is defined on $V^{2\ell}$ and $\mathcal{F}^{-1}\big(V^{2\ell}\big) \subset L^2(X)$. The following claim shows that $\mathcal{F}^{-1}\big(V^{2\ell}\big) \subset H^{2\ell}(X)$.

Claim 2. *For $v \in V^{2\ell}$, the distributional derivatives $(1-\Delta)^k \mathcal{F}^{-1} v \in L^2(X)$, for all $0 \leq k \leq \ell$.*

Proof. Evaluating at a test function φ,

$$\big((1-\Delta)\, \mathcal{F}^{-1} v\big)\varphi = \mathcal{F}^{-1} v\, \big((1-\Delta)\varphi\big)$$

By Plancherel,

$$\mathcal{F}^{-1} v\, \big((1-\Delta)\varphi\big) = v\big(\mathcal{F}\,(1-\Delta)\varphi\big)$$

By Proposition 3.3.1 and using Plancherel again,

$$v\big(\mathcal{F}\,(1-\Delta)\varphi\big) \;=\; v\big((1-\lambda_\xi)\cdot\mathcal{F}\varphi\big) \;=\; \big((1-\lambda_\xi)\cdot v\big)\,(\mathcal{F}\varphi) \;=\; \mathcal{F}^{-1}\big((1-\lambda_\xi)\cdot v\big)\,\varphi$$

By induction, we have the following identity of distributions:

$$(1-\Delta)^k\,\mathcal{F}^{-1}v \;=\; \mathcal{F}^{-1}\big((1-\lambda_\xi)^k\,v\big)$$

Since $\mathcal{F}$ is an L^2-isometry and $(1-\lambda_\xi)^k\,v \in L^2(\Xi)$ for all $0 \le k \le \ell$,

$$(1-\Delta)^k\,\mathcal{F}^{-1}v \;=\; \mathcal{F}^{-1}\big((1-\lambda_\xi)^k\,v\big) \;\in\; L^2(X) \qquad \text{for } 0 \le k \le \ell$$

□

This concludes the proof of the theorem.

□

Remark 3.3.1. This Hilbert space isomorphism $\mathcal{F} : H^{2\ell}(X) \to V^{2\ell}$ gives a *spectral* characterization of the $2\ell^{\text{th}}$ Sobolev space, namely the preimage of $V^{2\ell}$ under $\mathcal{F}$.

$$H^{2\ell}(X) = \{f \in L^2(X) : (1-\lambda_\xi)^\ell \cdot \mathcal{F}f(\xi) \in L^2(\Xi)\}$$

3.4 Negatively indexed Sobolev spaces and distributions

The theory of negatively indexed Sobolev spaces allows application of spectral theory to solving differential equations involving certain *distributions.*

Definition 3.4.1. For $\ell > 0$, the negatively indexed Sobolev space $H^{-\ell}(X)$ is the Hilbert space dual of $H^\ell(X)$.

Remark 3.4.1. Since the space of test functions is a dense subspace of $H^\ell(X)$ with $\ell > 0$, dualizing gives an inclusion of $H^{-\ell}(X)$ into the space of distributions. Indeed the adjoint map $\text{inc}^* : H^{-\ell}(X) \to \big(C_c^\infty(X)\big)^*$ to the inclusion $\text{inc} : C_c^\infty(X) \hookrightarrow H^\ell(X)$ is a literal inclusion since, for any $u \in H^{-\ell}(X)$,

$$\text{inc}^*(u)(\varphi) \;=\; u(\varphi) \qquad \text{for all } \varphi \in C_c^\infty(X)$$

i.e. $\text{inc}^*u \;=\; u|_{C_c^\infty(X)}$, and the injectivity follows from density of test functions in $H^\ell(X)$: if two elements of $H^{-\ell}(X)$ agree on test functions they agree on $H^\ell(X)$, by continuity. Similarly, the adjoints of the dense inclusions $H^\ell \hookrightarrow H^{\ell-1}$ are literal inclusions $H^{-\ell+1}(X) \hookrightarrow H^{-\ell}(X)$. In fact, the self-duality of $H^0(X) = L^2(X)$ implies that $H^\ell(X) \hookrightarrow H^{\ell-1}$ for all $\ell \in \mathbb{Z}$, when $L^2(X)$ is *identified* with its dual by

$$f \in L^2(X) \quad \longleftrightarrow \quad \Lambda_f \;=\; \langle f, - \rangle_{L^2(X)} \;\in\; \big(L^2(X)\big)^*$$

Proposition 3.4.1. *The spectral transform extends to an isometric isomorphism on negatively indexed Sobolev spaces $\mathcal{F} : H^{-2\ell} \to V^{-2\ell}$, and*

$$\mathcal{F}((1-\Delta)\,u) \;=\; (1-\lambda_\xi)\cdot \mathcal{F}u$$

holds for $u \in H^{2\ell}(X)$, for any $\ell \in \mathbb{Z}$.

Proof. To simplify notation, for this proof let $H^{2\ell} = H^{2\ell}(X)$.

Propositions 3.3.1 and 3.3.2 give the result for positively indexed Sobolev spaces, expressed in the following commutative diagram,

$$\begin{tikzcd}
\cdots \arrow[r, "(1-\Delta)"] & H^4 \arrow[r, "(1-\Delta)"] \arrow[d, "\mathcal{F}"', "\approx"] & H^2 \arrow[r, "(1-\Delta)"] \arrow[d, "\mathcal{F}"', "\approx"] & H^0 \arrow[d, "\mathcal{F}"', "\approx"] \\
\cdots \arrow[r, "\mu"] & V^4 \arrow[r, "\mu"] & V^2 \arrow[r, "\mu"] & V^0
\end{tikzcd}$$

where, again, $\mu(v)(\xi) \;=\; (1-\lambda_\xi)\cdot v(\xi)$.

Dualizing, we immediately have the commutativity of the adjoint diagram.

$$\begin{tikzcd}
(H^0)^* \arrow[r, "(1-\Delta)^*"] & (H^2)^* \arrow[r, "(1-\Delta)^*"] & (H^4)^* \arrow[r, "(1-\Delta)^*"] & \cdots \\
(V^0)^* \arrow[r, "\mu^*"] \arrow[u, "\mathcal{F}^*", "\approx"'] & (V^{-2})^* \arrow[r, "\mu^*"] \arrow[u, "\mathcal{F}^*", "\approx"'] & (V^{-4})^* \arrow[r, "\mu^*"] \arrow[u, "\mathcal{F}^*", "\approx"'] & \cdots
\end{tikzcd}$$

The self-duality of L^2 and the Plancherel theorem allow the two diagrams to be connected. Identifying $(V^{2\ell})^*$ with $V^{-2\ell}$, and recalling the definition $H^{-2\ell} = (H^{2\ell})^*$, yields the commutativity of

$$\begin{tikzcd}
\cdots \arrow[r, "(1-\Delta)"] & H^4 \arrow[r, "(1-\Delta)"] \arrow[d, "\mathcal{F}"', "\approx"] & H^2 \arrow[r, "(1-\Delta)"] \arrow[d, "\mathcal{F}"', "\approx"] & H^0 \arrow[r, "(1-\Delta)^*"] \arrow[d, bend right, "\mathcal{F}"', "\approx"] & H^{-2} \arrow[r, "(1-\Delta)^*"] & H^{-4} \arrow[r, "(1-\Delta)^*"] & \cdots \\
\cdots \arrow[r, "\mu"] & V^4 \arrow[r, "\mu"] & V^2 \arrow[r, "\mu"] & V^0 \arrow[r, "\mu"] \arrow[u, bend right, "\mathcal{F}^{-1}"', "\approx"] & V^2 \arrow[r, "\mu"] \arrow[u, "\mathcal{F}^*", "\approx"'] & V^4 \arrow[r, "\mu"] \arrow[u, "\mathcal{F}^*", "\approx"'] & \cdots
\end{tikzcd}$$

Since $V^{2\ell+2}$ is dense in $V^{2\ell}$ for all $\ell \in \mathbb{Z}$, and $H^{2\ell} \approx V^{2\ell}$ for all $\ell \in \mathbb{Z}$, $H^{2\ell+2}$ is dense in $H^{2\ell}$ for all $\ell \in \mathbb{Z}$. Thus test functions are dense in *all* the Sobolev spaces.

The adjoint map $(1-\Delta)^* : H^{-2\ell} \to H^{-2\ell-2}$ is the continuous extension of $(1-\Delta)$ from the space of test functions, since, for a test function φ, identified with an element of $H^{-2\ell}$ by Λ_φ, integration by parts yields

$$\big((1-\Delta)^* \Lambda_\varphi\big)(f) \;=\; \Lambda_\varphi\big((1-\Delta) f\big) \;=\; \langle \varphi, (1-\Delta) f\rangle \;=\; \langle (1-\Delta)\varphi, f\rangle \;=\; \Lambda_{(1-\Delta)\varphi}(f)$$

for all f in $H^{2\ell+2}$, where $\langle\, ,\,\rangle$ denotes the usual inner product on $L^2(\Gamma\backslash G)$.

The map $\big(\mathcal{F}^*\big)^{-1}$ on $H^{-2\ell}$ is the continuous extension of $\mathcal{F}$ from the space of test functions, since for a test function φ, identified with an element of $H^{-2\ell}$ by Λ_φ,

$$\big(\mathcal{F}^* \Lambda_{\mathcal{F}\varphi}\big)(f) \;=\; \Lambda_{\mathcal{F}\varphi}\big(\mathcal{F} f\big) \;=\; \langle \mathcal{F}\varphi, \mathcal{F} f\rangle_{V^{2\ell}} \;=\; \langle \varphi, f\rangle_{H^{2\ell}} \;=\; \Lambda_\varphi(f)$$

for all $f \in H^{2\ell}$.

Thus, the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r, "(1-\Delta)"] & H^4 \arrow[r, "(1-\Delta)"] \arrow[d, "\mathcal{F}"', "\approx"] & H^2 \arrow[r, "(1-\Delta)"] \arrow[d, "\mathcal{F}"', "\approx"] & H^0 \arrow[r, "(1-\Delta)"] \arrow[d, "\mathcal{F}"', "\approx"] & H^{-2} \arrow[r, "(1-\Delta)"] \arrow[d, "\mathcal{F}"', "\approx"] & H^{-4} \arrow[r, "(1-\Delta)"] \arrow[d, "\mathcal{F}"', "\approx"] & \cdots \\
\cdots \arrow[r, "\mu"] & V^4 \arrow[r, "\mu"] & V^2 \arrow[r, "\mu"] & V^0 \arrow[r, "\mu"] & V^{-2} \arrow[r, "\mu"] & V^{-4} \arrow[r, "\mu"] & \cdots
\end{tikzcd}$$

In other words, the relation

$$\mathcal{F}\big((1-\Delta)\, u\big) = (1-\lambda_\xi) \cdot \mathcal{F} u$$

holds for any u contained in a Sobolev space.

$\square$

Definition 3.4.2. For a smooth manifold M, the positively indexed local Sobolev spaces $H^\ell_{\text{loc}}(M)$ consist of functions f on M such that for all points $x \in M$, all open neighborhoods U of x small enough that there is a diffeomorphism $\Phi : U \to \mathbb{R}^n$ with $\Omega = \Phi(U)$ having compact closure, and all test functions φ with support in U, the function

$$(f \cdot \varphi) \circ \Phi^{-1} : \Omega \longrightarrow \mathbb{C}$$

is in the Euclidean Sobolev space $H^\ell(\Omega)$.

Proposition 3.4.2. *Positively indexed global Sobolev spaces $H^\ell(X)$ lie inside the corresponding local Sobolev spaces:*

$$H^\ell(X) \; \subset \; H^\ell_{loc}(G/K) \qquad \textit{for} \;\; \ell \geq 0$$

Proof. Let x be a point in $\Gamma\backslash G/K$, U an open neighborhood of x, Φ be a diffeomorphism $U \to \Omega \subset \mathbb{R}^n$ where Ω has compact closure, and let φ be a test function whose support is contained in U. We must show that $(f \cdot \varphi) \circ \Phi$ and all its derivatives up to order ℓ are square integrable on Ω. By Leibnitz' rule, $\partial^\mu \, (f \cdot \varphi)$ is a linear combination of terms of the form

$$\partial^\alpha f \, \cdot \, \partial^\beta \varphi \qquad \text{where} \;\; |\alpha| + |\beta| \leq |\mu|$$

But since

$$\sup_{|\alpha| \leq \ell} \|\partial^\alpha f\|_{L^2(\Omega)} \;\; \ll \;\; \|f\|_{H^\ell(X)} \qquad \text{and} \qquad \sup_{|\beta| \leq \ell} \|\partial^\beta \varphi\|_{L^2(\Omega)} \;\; \ll \;\; \|\varphi\|_{C^k(\Omega)}$$

and both of these are finite,

$$\|\partial^\mu \, (f \cdot \varphi)\|_{L^2(\Omega)} \;\; \ll \;\; \|f\|_{H^\ell(X)} \cdot \|\varphi\|_{C^k(\Omega)} \;\; < \;\; \infty$$

$\square$

Thus a comparison may be made between the global Sobolev topologies and (local) C^k-topologies, via the Sobolev embedding theorem for local Sobolev spaces.

Theorem 3.4.1 (Sobolev embedding)**.** *For a smooth manifold M,*

$$H^{\ell+k}_{loc}(M) \; \subset \; C^k(M) \qquad \textit{for} \;\; \ell > \dim(M)/2$$

Corollary 3.4.1. *For* $\ell > \dim(G/K)/2$,

$$H^{\ell+k}(X) \ \subset \ C^k(G/K)$$

Proof. By Proposition 3.4.2, $H^\ell(X) \subset H^\ell_{\mathrm{loc}}(G/K)$, and by Theorem 3.4.1, $H^\ell_{\mathrm{loc}}(G/K) \subset C^k(G/K)$.

□

This embedding of global Sobolev spaces into C^k-spaces is used to prove that the integral defining spectral inversion for test functions can be extended to sufficiently highly indexed Sobolev spaces, i.e. the abstract isometric isomorphism $\mathcal{F}^{-1} \circ \mathcal{F} : H^\ell(X) \to H^\ell(X)$ is given by an integral that is convergent uniformly pointwise, when $\ell > \dim(G/K)/2$.

Proposition 3.4.3. *For* $f \in H^s(X)$, $s > k + \dim(G/K)/2$,

$$f \ = \ \int_\Xi \mathcal{F}f(\xi)\, \Phi_\xi \; d\xi \qquad \textit{in } C^k(X)$$

Proof. Let $\{\Xi_n\}$ be a nested family of compact sets in Ξ whose union is all of Ξ, χ_n be the characteristic function of Ξ_n, and f_n be given by the $C^\infty(X)$-valued Gelfand-Pettis integral

$$f_n \ = \ \int_\Xi \chi_n(\xi)\, \mathcal{F}f(\xi)\, \Phi_\xi \, d\xi$$

Since $\chi_n(\xi)\, \mathcal{F}f(\xi)$ is compactly supported,

$$f_n \ = \ \mathcal{F}^{-1}(\chi_n \cdot \mathcal{F}f)$$

Thus, by Proposition 3.3.2,

$$\|f_n - f_m\|_{H^s(X)} \ = \ \|(\chi_n - \chi_m) \cdot \mathcal{F}f\|_{V^s}$$

Since $\mathcal{F}f$ lies in V^s, these tails certainly approach zero as $n, m \to \infty$. Similarly,

$$\|f_n - f\|_{H^s(X)} \ = \ \|(\chi_n - 1) \cdot \mathcal{F}f\|_{V^s} \ \longrightarrow \ 0 \qquad \text{as } n \to \infty$$

By Corollary 3.4.1, f_n approaches f in $C^k(X)$.

□

The embedding of global Sobolev spaces into C^k-spaces also implies that *compactly supported* distributions lie in global Sobolev spaces, as follows.

Proposition 3.4.4. *Any compactly supported distribution on X lies in a global archimedian spherical automorphic Sobolev space. Specifically, a compactly supported distribution of order k lies in $H^{-s}(X)$ for all $s > k + \dim(G/K)/2$.*

Proof. A compactly supported distribution u lies in $\big(C^\infty(G/K)\big)^*$. Since compactly supported distributions are of finite order, u extends continuously to $C^k(G/K)$ for some $k \geq 0$. Using Corollary 3.4.1 and dualizing, u lies in $H^{-(\ell+k)}(X)$, for $\ell > \dim(G/K)/2$. □

Remark 3.4.2. In particular, this implies that the Dirac delta distribution at the base point $x_o = \Gamma \cdot 1 \cdot K$ in $\Gamma\backslash G/K$ lies in $H^{-\ell}(X)$ for all $\ell > \dim(G/K)/2$.

Proposition 3.4.5. *For a compactly supported distribution u of order k,*

$$\mathcal{F}u \;=\; u(\Phi_\xi) \qquad \text{in } V^{-s} \quad \text{where } s > k + \dim(G/K)/2$$

Proof. A compactly supported distribution u of order k lies in H^{-s} for any $s > k + \dim(G/K)/2$. Let f be any element in $H^s(X)$. Then,

$$\langle \mathcal{F}f, \mathcal{F}u\rangle_{V^s \times V^{-s}} \;=\; \langle f, u\rangle_{H^s \times H^{-s}} \;=\; u(f)$$

Since the spectral expansion of f converges to it in the $H^s(X)$ topology by Proposition 3.4.3,

$$u(f) \;=\; u\bigg(\lim_n \int_{\Xi_n} \mathcal{F}f(\xi)\,\Phi_\xi\; d\xi\bigg) \;=\; \lim_n\, u\bigg(\int_{\Xi_n} \mathcal{F}f(\xi)\,\Phi_\xi\; d\xi\bigg)$$

Since the integral is a $C^\infty(X)$-valued Gelfand-Pettis integral and u is an element of $(C^\infty(X))^*$,

$$u\bigg(\int_{\Xi_n} \mathcal{F}f(\xi)\,\Phi_\xi\; d\xi\bigg) \;=\; \int_{\Xi_n} \mathcal{F}f(\xi)\,u(\Phi_\xi)\; d\xi$$

The limit as $n \to \infty$ is finite, by comparison with the original expression which surely is finite, and thus

$$\langle \mathcal{F}f, \mathcal{F}u\rangle_{V^s \times V^{-s}} \;=\; \int_{\Xi} \mathcal{F}f(\xi)\,u(\Phi_\xi)\; d\xi \;=\; \langle \mathcal{F}f, u(\Phi_\xi)\rangle_{V^s \times V^{-s}}$$

Thus, $\mathcal{F}u = u(\Phi_\xi)$ as elements of V^{-s}

□

Remark 3.4.3. This implies that the spectral transform of the Dirac delta distribution is $\mathcal{F}\delta = \Phi_\xi(x_o)$.

3.5 Gelfand-Pettis integrals and mollification

We describe the vector-valued (weak) integrals of Gelfand [39] and Pettis [75]. See Garrett's essay [38].

For X, μ a measure space and V a locally convex, quasi-complete topological vector space, a Gelfand-Pettis (or weak) integral is a vector-valued integral $C_c^o(X, V) \to V$ denoted

$$f \quad \longrightarrow \quad I_f \;=\; \int_X f\, d\mu$$

such that, for all $\alpha \in V^*$,

$$\alpha(I_f) \;=\; \int_X \alpha \circ f\; d\mu$$

where this latter integral is the usual scalar-valued Lebesgue integral.

Remark 3.5.1. Hilbert, Banach, Frechet, LF spaces, and their weak duals are locally convex, quasi-complete topological vector spaces; see [38].

Theorem 3.5.1 (Gelfand-Pettis integrals)**.** *Gelfand-Pettis integrals* exist*, are* unique*, and satisfy the following* estimate*:*

$$I_f \;\in\; \mu(\mathrm{spt} f) \cdot \big(\textit{closure of compact hull of } f(X)\big)$$

Proposition 3.5.1. *Any continuous linear operator between locally convex, quasi-complete topological vetor spaces $T : V \to W$ commutes with the Gelfand-Pettis integral:*

$$T(I_f) \quad = \quad I_{Tf}$$

For a locally compact Hausdorff topological group G, with Haar measure dg, acting continuously on a locally convex, quasi-complete vector space V, the group algebra $C_c^o(G)$ acts on V by *averaging*:

$$\eta \cdot v \;=\; \int_G \eta(g)\, g \cdot v\, dg$$

An *approximate identity* on G is a family $\{\psi_i\}$ of non-negative continuous compactly-supported functions on G, each of mass one, such that the supports shrink to the identity, i.e. for every neighborhood U of the identity, there is $N > 0$ such that for $i \geq N$, the support of ψ_i is inside U.

Proposition 3.5.2. *Let G be a locally compact Hausdorff topological group acting continuously on a locally convex, quasi-complete vector space V. Let $\{\psi_i\}$ be an approximate identity on G. Then, for any $v \in V$, $\psi_i \cdot v \to v$ in the topology of V.*

Let G be a Lie group acting continuously on a quasi-complete locally convex vector space V. The space V^∞ of *smooth vectors* in V consists of those vectors on which G acts smoothly:

$$V^\infty \quad = \quad \{v \in V : g \to g \cdot v \text{ is a smooth } V\text{-valued function on } G\}$$

The subspace $V^{(1)}$ of differentiable vectors of V consists of vectors v such that

$$X \cdot v \quad = \quad \frac{d}{dt}\bigg|_{t=0} \exp(tX) \cdot v$$

exists for all X in the Lie algebra $\mathfrak{g}$. Inductively, a vector in v is in the space $V^{(k+1)}$ of k-times differentiable vectors if it lies in $V^{(1)}$ and $X \cdot v$ lies in $V^{(k)}$ for all $X \in \mathfrak{g}$. The space of smooth vectors V^∞ is the intersection of all $V^{(k)}$, $k \geq 0$.

Theorem 3.5.2. *For a Lie group G acting continuously on a quasi-complete locally convex topological vector space V, the space V^∞ of smooth vectors is dense in V.*

Proof. Let $\{\eta_i\}$ be a *smooth* approximate identity on G. The vector subspace spanned by *mollifcations* $\eta_i \cdot v$ of vectors $v \in V$, the *Gårding subspace* of V, is dense in V, by Proposition 3.5.2.

The mollification $\eta \cdot v$ is a smooth vector since,

$$\begin{aligned} X \cdot (\eta \cdot v) \quad &= \quad \frac{\partial}{\partial t}\bigg|_{t=0} e^{tX} \cdot (\eta \cdot v) \quad = \quad \frac{\partial}{\partial t}\bigg|_{t=0} e^{tX} \cdot \int_G \eta(g)\, g \cdot v \, dg \\ &= \quad \frac{\partial}{\partial t}\bigg|_{t=0} \int_G \eta(g)\, (e^{tX} g) \cdot v \, dg \quad = \quad \frac{\partial}{\partial t}\bigg|_{t=0} \int_G \eta(e^{-tX} g)\;\; g \cdot v \, dg \end{aligned}$$

Since η is a test function, the integrand is bounded for $|t| \leq 1$, and the differentiation may be moved inside the integral.

$$\frac{\partial}{\partial t}\bigg|_{t=0} \int_G \eta(e^{-tX}g)\ \ g \cdot v\ dg \ \ = \ \ \int_G L_X\eta(g)\ \ g \cdot v\ dg \ \ = \ \ (L_X\eta) \cdot v$$

Iterating,

$$Y \cdot \big(X \cdot (\eta \cdot v)\big) \ \ = \ \ Y \cdot \big((L_X\eta) \cdot v\big) \ \ = \ \ \big(L_Y(L_X\eta)\big) \cdot v$$

Since the universal enveloping algebra is spanned by monomials, by the Poincaré-Birkhoff-Witt theorem, this shows that $\alpha \cdot (\eta \cdot v)$ exists for all $\alpha \in \mathcal{U}\mathfrak{g}$. Thus $\eta \cdot v$ is a smooth vector.

$\square$

Remark 3.5.2. The theorem of Dixmier-Malliavin [22] proves that the Gårding subspace is equal to the space of smooth vectors, i.e. *all* smooth vectors are finite linear combinations of mollifications.

Remark 3.5.3. For a *function space* V, the space of smooth vectors V^∞ is *not necessarily* the subspace of *smooth functions* in V. Thus, even with Dixmier-Malliavin, Proposition 3.5.2 does *not* prove the density of smooth *functions* in V.

Chapter 4

Standard Estimates

We discuss standard estimates for $\Gamma\backslash G/K$. For the case $SL_2(\mathbb{Z}[i])\backslash SL_2(\mathbb{C})/SU(2)$, see [37].

Let Φ_ξ be a spectral basis for $L^2(\Gamma\backslash G)$ in the sense of 3.1.

Each Φ_ξ generates an irreducible representation of G under right translation. In particular, at archimedean places, each generates an unramified principal series I_μ $(\mu \in \mathfrak{a}^*_{\mathbb{C}})$, by [10]. We index the characters such that the standard intertwining operators send $I_\mu \to I_{w\cdot\mu}$, where w is an element of the Weyl group.

By the Iwasawa decomposition, each Φ_ξ is the unique spherical vector in the copy of the unramified principal series it generates at archimedean places, up to a constant, and so, for any bi-K-invariant compactly supported measure η, acting by (Gelfand-Pettis) integral operators, $\eta \cdot \Phi_\xi$ is another right K-invariant vector in the representation space of Φ_ξ, so necessarily a scalar multiple of Φ_ξ. Let $\chi_\xi(\eta) = \chi_\mu(\eta)$ denote the eigenvalue

$$\eta \cdot \Phi_\xi \quad = \quad \chi_\xi(\eta) \cdot \Phi_\xi$$

A model for I_μ is the induced representation

$$I_\mu \quad = \quad \{f \text{ smooth, } K\text{-finite } : f(na \cdot g) = e^{\mu(\log a)} \cdot f(g), \ \ \forall n \in N, a \in A, g \in G\}$$

Casimir acts by a negative scalar λ_μ on I_μ, and $|\lambda_\mu| \asymp |\mu|^2$.

The action of $\eta \in C_c^0(G)$ on functions on $\Gamma\backslash G$ is given by a kernel,

$$q_x(y) \;=\; \sum_{\gamma\in\Gamma} \eta(x^{-1}\gamma y) \tag{4.1}$$

Indeed:

$$\begin{aligned}(\eta \cdot f)(x) &= \int_G \eta(x)\, f(xy)\, dy \;=\; \int_G \eta(x^{-1}y)\, f(y)\, dy \\ &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \eta(x^{-1}\gamma y)\; f(y)\, dy \;=\; \langle q_x, f\rangle\end{aligned}$$

Lemma 4.0.1. *Let C be a compact set in G, δ a positive real number, and η the characteristic function of the ball of radius δ in G, using the bi-K-invariant metric arising from the group norm. Then, for all $x \in C$,*

$$\|q_x\|^2_{L^2(\Gamma\backslash G)} \;\ll_C\; \delta^{\dim G/K}$$

where q_x is the associated kernel (see Equation (4.1)).

Proof. If q_x is square-integrable as a function on $\Gamma\backslash G$, its norm is

$$\|q_x\|^2 \;=\; \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \eta(x^{-1}\gamma y) \sum_{\gamma'\in\Gamma} \eta(x^{-1}\gamma' y)\;\; dy \;=\; \int_G \sum_{\gamma\in\Gamma} \eta(x^{-1}\gamma y)\, \eta(x^{-1}y)\;\; dy$$

Since η is the characteristic function of the ball $B_\delta(1)$,

$$\eta(x^{-1}\gamma y)\, \eta(x^{-1}y) \;=\; \begin{cases} 1 & \text{if } y \in B_\delta(x) \text{ and } \gamma y \in B_\delta(x) \\ 0 & \text{otherwise} \end{cases}$$

For fixed $y \in B_\delta(x)$, the sum over $\gamma \in \Gamma$, if finite, is equal to the cardinality of the set

$$\{\gamma \in \Gamma : \gamma y \in B_\delta(x)\}$$

which is contained in a finite set, independent of x and y, as follows

$$\begin{aligned}\{\gamma \in \Gamma : \gamma y \in B_\delta(x)\} &\subset \{\gamma \in \Gamma : \gamma y \in \overline{B_\delta(1)\cdot C} \text{ for some } y \in C\} \\ &\subset \Gamma \,\cap\, \left(\overline{B_\delta(1)\cdot C} \cdot C^{-1}\right)\end{aligned}$$

The finiteness follows from the fact that C is compact so $\overline{B_\delta(1)\cdot C}\cdot C^{-1}$ is compact as well, and Γ is a discrete subgroup. Thus,

$$\|q_x\|^2 \;=\; \int_{B_\delta(x)} \#\{\gamma \in \Gamma : \gamma y \in B_\delta(x)\}\;\; dy \;\ll_C\; \int_{B_\delta(x)} 1\, dy \;\asymp\; \delta^{\dim(G/K)}$$

$\square$

Lemma 4.0.2. *Let δ be a positive real number and η the characteristic function of the ball of radius δ in G, as above. Let $\chi_\mu(\eta)$ be the eigenvalue of the action of η on the principal series I_μ. Then, for $\delta \ll |\mu|^{-2}$,*

$$|\chi_\mu(\eta)| \;\gg\; \delta^{\dim G/K}$$

Proof. Let φ° be a spherical function in I_μ. Then

$$\chi_\mu(\eta) \;=\; \varphi^\circ(1)^{-1} \cdot \int_G \eta(g)\,\varphi^\circ(g)\,dg \;=\; \varphi^\circ(1)^{-1} \cdot \int_{B_\delta(1)} e^{\mu(\log A(g))}\,dg$$

By continuity of the exponential at zero, there is $\varepsilon > 0$ such that

$$|z| < \varepsilon \;\Rightarrow\; |e^z - 1| < \tfrac{1}{2}$$

By continuity of μ, there is a $\theta > 0$ such that

$$|\log a| < \theta \quad\Rightarrow\quad |\mu(\log a)| < \varepsilon \;\text{ and }\; |e^{\mu(\log a)} - 1| \;<\; \tfrac{1}{2}$$

i.e. for $|\log a| < \theta$, $|e^{\mu(\log a)}| \;>\; \frac{1}{2}$. In fact, taking $|\log a| < \delta$ suffices, so

$$\chi_\mu(\eta) \;>\; \tfrac{1}{2}\,\varphi^\circ(1)^{-1} \cdot \mathrm{vol}\, B_\delta(1) \;\gg\; \delta^{\dim G/K}$$

$\square$

Theorem 4.0.3 (Standard estimate)**.** *Let C be a compact set in G. For $x \in C$, and for T a large positive number,*

$$\int_{\Xi_T} |\Phi_\xi(x)|^2\,d\xi \;\ll_C\; T^{\,\dim G/K}$$

where $\Xi_T \;=\; \{\xi \in \Xi :\, |\lambda_\xi| \le T\}$.

Proof. Let $\delta = 1/T$, and let η be the characteristic function of the ball of radius δ in G. Let q_x be the associated kernel (see Equation (4.1)). By the Plancherel theorem and Lemma 4.0.1,

$$\int_\Xi |\langle q_x, \Phi_\xi\rangle|^2\,d\xi \;=\; \|q_x\|^2 \;\ll_C\; T^{-\dim G/K}$$

Recall

$$\langle q_x, f\rangle \;=\; (\eta \cdot f)(x) \;=\; \chi_f(\eta)\cdot f(x)$$

Thus, by Lemma 4.0.2,

$$|\Phi_\xi(x)|^2 \;=\; \frac{|\langle q_x, \Phi_\xi\rangle|^2}{|\chi_\mu(\eta)|^2} \;\ll\; |\langle q_x, \Phi_\xi\rangle|^2 \cdot T^{\,2\dim G/K}$$

for $|\lambda_\xi| \leq T$, i.e. $|\mu|^2 \ll \delta^{-1}$. Thus,

$$\int_{\Xi_T} |\Phi_\xi(x)|^2 \, d\xi \;\ll_C\; T^{\,2\dim G/K} \cdot \int_{\Xi_T} |\langle q_x, \Phi_\xi\rangle|^2 \;\ll_C\; T^{\,\dim G/K}$$

$\square$

Chapter 5

Spherical transforms, global zonal spherical Sobolev spaces, and differential equations on G/K

5.1 Spherical transform and inversion

We use the well-known harmonic analysis of spherical functions, developed by Gelfand and Naimark, Berezin, Harish-Chandra, Helgason, Bhantu-Murthy, Gindikin and Karpelevič, Gangolli, and others. See especially [49, 50, 43, 51] for the formula for the spherical transform, spherical inversion, and Plancherel theorem.

Let G be a semi-simple Lie group G, with maximal compact subgroup K. Let $X = K\backslash G/K$ and $\Xi = \mathfrak{a}^*/W \approx \mathfrak{a}_+$. The spherical transform of Harish-Chandra and Berezin integrates a bi-K-invariant against a zonal spherical function:

$$\mathcal{F}f\,(\xi) \;=\; \int_G f(g)\,\overline{\varphi}_{\rho+i\xi}(g)\,dg$$

Zonal spherical functions $\varphi_{\rho+i\xi}$ are eigenfunctions for Casimir (restricted to bi-K-invariant functions) with eigenvalue $\lambda_\xi = -(|\xi|^2 + |\rho|^2)$. The inverse transform is

$$\mathcal{F}^{-1}f \;=\; \int_\Xi f(\xi)\,\varphi_{\rho+i\xi}\;|\mathbf{c}(\xi)|^{-2}d\xi$$

where $\mathbf{c}(\xi)$ is the Harish-Chandra $\mathbf{c}$-function. For brevity, denote $L^2(\Xi, |\mathbf{c}(\xi)|^{-2})$ by $L^2(\Xi)$.

Plancherel ensures that the spectral transform and its inverse are isometries between $L^2(X)$ and $L^2(\Xi)$.

Sobolev theory provides a framework for discussing the interaction of differential operators and spectral transforms/inversions. Let Δ denote the Laplacian, i.e. the Casimir operator on right-K-invariant functions. For bi-K-invariant *test functions* f, integration by parts yields

$$\mathcal{F}(\Delta f)\,(\xi) \;=\; \int_G \Delta f\;\overline{\varphi}_{\rho+i\xi} \;=\; \int_G f\;\Delta\overline{\varphi}_{\rho+i\xi} \;=\; \lambda_\xi \cdot \int_G f\,\overline{\varphi}_{\rho+i\xi} \;=\; \lambda_\xi \cdot \mathcal{F}f\,(\xi)$$

Sobolev theory for $K\backslash G/K$ enables us to extend this relation by continuity to larger spaces of functions.

5.2 Characterizations of Sobolev spaces

We define positive index zonal spherical Sobolev spaces as the left K-invariant subspaces of completions of $C_c^\infty(G/K)$ with respect to a topology induced by seminorms associated to derivatives from the universal enveloping algebra, as follows.

The universal enveloping algebra $\mathcal{U}\mathfrak{g}$ acting on functions on G can be identified with G-invariant differential operators on G. The *left* action of $\mathcal{U}\mathfrak{g}$ descends to functions on G/K. Let $\mathcal{U}\mathfrak{g}^{\leq \ell}$ be the finite dimensional subspace of $\mathcal{U}\mathfrak{g}$ consisting of elements of degree less than or equal to ℓ.

Each $\alpha \in \mathcal{U}\mathfrak{g}$ gives a seminorm ν_α on $C_c^\infty(G/K)$.

$$\nu_\alpha(f) \;=\; \|\alpha f\|^2_{L^2(G/K)}$$

The topology induced by supremums of finite linear combinations of these is equivalent to the topology induced by the seminorms

$$\nu_B(f) \;=\; \sup_{\gamma \in B} \|\gamma f\|^2_{L^2(G/K)} \qquad \text{bounded } B \subset (\mathcal{U}\mathfrak{g})^{\leq \ell}$$

Definition 5.2.1. Consider the space of smooth functions that are bounded with respect to these seminorms:

$$\{f \in C^\infty(G/K) : \nu_\alpha f \ < \ \infty \ \text{ for all } \alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}\}$$

Let $H^\ell(G/K)$ be the completion of this space with respect to the topology induced by the family $\{\nu_\alpha : \alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}\}$. The *global zonal spherical Sobolev space* $H^\ell(X) = H^\ell(G/K)^K$ is the subspace of left-K-invariant functions.

Proposition 5.2.1. *The space of test functions $C_c^\infty(X)$ is dense in $H^\ell(X)$.*

Proof. We approximate a smooth function $f \in H^\ell(X)$ by pointwise products with smooth cut-off functions, as follows. Let $\varphi_R \in C_c^\infty(\mathbb{R}^n)$ such that φ_R is identically one on the ball of radius R, identically zero outside the ball of radius $R+1$, and

$$\sup_{x \in \mathbb{R}^n} |\partial^\alpha \varphi_R(x)| \ \ll \ 1$$

for all multi-indices α with $|\alpha| \leq \ell$, where the implied constant does not depend on R, but may depend on ℓ.

Let $\eta_R = \varphi_R \circ H$ where $H : G \to \mathfrak{a} \approx \mathbb{R}^n$ is the function defined by $g = k' \cdot \exp(H(g)) \cdot k$.

Claim 3. *For any $\gamma \in \mathcal{U}\mathfrak{g}^{\leq \ell}$,*

$$\sup_{g \in G} |\gamma\, \eta_R(g)| \ \ll \ 1 \qquad \textit{(uniform in } R\textit{)}$$

where the implied constant does not depend on R, but may depend on ℓ.

Proof. For $\gamma \in \mathcal{U}\mathfrak{g}^{\leq \ell}$, $\gamma\, \eta_R$ is a finite linear combination of terms of the form

$$(\partial^\alpha \varphi_R \circ H)(\beta H_i)$$

where α is a multi-index with $|\alpha| \leq \ell$, $\beta \in \mathcal{U}\mathfrak{g}^{\leq \ell}$, and and $H_i(g)$ is the i^{th} component of $H(g) \in \mathbb{R}^n$. Thus,

$$\sup_{g \in G} |\gamma\, \eta_R(g)| \ \ll \ \sup_{g \in G} \sup_{|\alpha| \leq \ell} \sup_{\beta \in \mathcal{U}\mathfrak{g}^{\leq \ell}} (\partial^\alpha \varphi_R \circ H)(\beta H_i) \ \ll \ \sup_{|\alpha| \leq \ell} \sup_{x \in R^n} |\partial^\alpha \varphi_R(x)| \ \ll \ 1$$

where the implied constant does not depend on R but may depend on ℓ.

$\square$

Claim 4. *For any* $\gamma \in \mathcal{U}\mathfrak{g}^{\leq \ell}$

$$\nu_\gamma\big(\eta_R \cdot f - f\big) \to 0 \qquad as \;\; R \to \infty$$

Proof. By definition,

$$\nu_\gamma\big(\eta_R \cdot f - f\big) \;\;=\;\; \|\gamma\big(\eta_R \cdot f - f\big)\|_{L^2(G/K)}$$

Leibnitz' rule implies that $\gamma\big(\eta_R \cdot f - f\big)$ is a finite linear combination of terms of the form $\alpha(\eta_R - 1) \cdot \beta f$ where α, $\beta \in \mathcal{U}\mathfrak{g}^{\leq \ell}$. When $\alpha = 0$,

$$\|\alpha(\eta_R - 1) \cdot \beta f\|_{L^2(G/K)} \;\;=\;\; \|(\eta_R - 1) \cdot \beta f\|_{L^2(G/K)} \;\;\leq\;\; \int_{|H(g)|\,\geq\, R} |(\beta\, f)(g)|^2\, dg$$

Otherwise, $\alpha(\eta_R - 1) = \alpha\eta_R$, and

$$\begin{aligned}
\|\alpha(\eta_R - 1) \cdot \beta f\|_{L^2(G/K)} \;\;&=\;\; \|\alpha\eta_R \cdot \beta f\|_{L^2(G/K)} \\
&\ll\;\; \sup_{g \in G}\, |\alpha\, \eta_R(g)| \;\cdot\; \int_{|H(g)|\,\geq\, R} |(\beta\, f)(g)|^2\, dg \\
&\ll\;\; \int_{|H(g)|\,\geq\, R} |(\beta\, f)(g)|^2\, dg
\end{aligned}$$

Let B be any bounded set containing all of the (finitely many) β that appear as a result of applying Leibniz' rule. Then

$$\nu_\gamma\big(\eta_R \cdot f - f\big) \;\;\ll\;\; \sup_{\beta \in B} \int_{|H(g)|\,\geq\, R} |(\beta\, f)(g)|^2\, dg$$

Since B is bounded and $f \in H^\ell(X)$,

$$\sup_{\beta \in B} \int_{|H(g)|\,\geq\, R} |(\beta\, f)(g)|^2\, dg \quad\longrightarrow\quad 0 \qquad \text{as} \;\; R \to \infty$$

$\square$

$\square$

Proposition 5.2.2. *Let* Ω *be the Casimir operator in the center of* $\mathcal{U}\mathfrak{g}$*. The norm* $\|\cdot\|_{2\ell}$ *on* $C_c^\infty(G/K)^K$ *given by*

$$\|f\|_{2\ell}^2 \;\;=\;\; \|f\|^2 \;+\; \|(1-\Omega)\, f\|^2 \;+\; \|(1-\Omega)^2\, f\|^2 \;+\; \ldots \;+\; \|(1-\Omega)^\ell\, f\|^2$$

where $\|\cdot\|$ *is the usual norm on* $L^2(G/K)$*, induces a topology on* $C_c^\infty(G/K)^K$ *that is equivalent to the topology induced by the family* $\{\nu_\alpha : \alpha \in \mathcal{U}\mathfrak{g}^{\leq 2\ell}\}$ *of seminorms and with respect to which* $H^{2\ell}(X)$ *is a Hilbert space.*

Proof. We start by comparing ν_α to a seminorm involving only the Casimir operator.

Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g}$, and let $\{X_i\}$ be a basis for $\mathfrak{g}$. Then

$$\Omega \;=\; \sum_{X_i} X_i\, X_i^*$$

Let $\Omega_{\mathfrak{p}}$ and $\Omega_{\mathfrak{k}}$ denote the subsums corresponding to the subspaces $\mathfrak{p}$ and $\mathfrak{k}$ respectively. The set Σ of possible K-types of $\gamma\, f$, for $\gamma \in \mathcal{U}\mathfrak{g}^{\leq \ell}$, is finite. Let λ_σ denote the $\Omega_{\mathfrak{k}}$-eigenvalue of a function of K-type σ, and let C be a number greater than the maximum value of $\{\lambda_\sigma : \sigma \in \Sigma\}$.

By the Poincaré-Birkhoff-Witt theorem we may assume α is a monomial of the form

$$\alpha \;=\; x_1 \ldots x_n\, y_1 \ldots y_m$$

where $x_i \in \mathfrak{p}$ and $y_i \in \mathfrak{k}$. On *bi-K-invariant* functions , each y_i acts by zero. Thus, for $f \in C_c^\infty(G/K)^K$,

$$\nu_\alpha f \;=\; \langle \alpha f, \alpha f\rangle_{L^2(G/K)} \;=\; \langle x_1 \ldots x_n\, f,\; x_1 \ldots x_n f\rangle_{L^2(G/K)} \qquad (x_i \in \mathfrak{p})$$

Lemma 5.2.1. *For* $\varphi \in C_c^\infty(G/K)$ *and* $\alpha = x_1 \ldots x_n$ *a monomial in* $\mathcal{U}\mathfrak{g}$ *with* $x_i \in \mathfrak{p}$*,*

$$\langle \alpha\, \varphi, \alpha\, \varphi\rangle \;\leq\; \langle (-\Omega + C)^n\, \varphi,\, \varphi\rangle$$

where $\langle\, ,\, \rangle$ *is the usual inner product on* $L^2(G/K)$*.*

Proof. We proceed by induction on $n = \deg \alpha$.

For $n = 1$, $\alpha = x \in \mathfrak{p}$. Since the Killing form is positive definite on $\mathfrak{p}$, we may choose a self-dual basis $\{X_i\}$ for $\mathfrak{p}$ such that $X_1 = x$. Then,

$$\langle x\varphi,\, x\varphi\rangle \;\leq\; \sum_i \langle X_i\, \varphi,\, X_i\, \varphi\rangle \;=\; -\sum_i \langle X_i^2\, \varphi, \varphi\rangle \;=\; \langle -\Omega_{\mathfrak{p}}\, \varphi, \varphi\rangle$$

$$= \langle (-\Omega + \Omega_{\mathfrak{k}})\,\varphi, \varphi\rangle \;\; \leq \;\; \langle (-\Omega + C)\,\varphi, \varphi\rangle$$

For $n > 1$, write $\alpha = x\gamma$, where $x = x_1$ and $\gamma = x_2 \ldots x_n$. Then, by the above argument,

$$\langle x\,\gamma\varphi,\; x\,\gamma\varphi\rangle \;\; \leq \;\; \langle (-\Omega + C)\,\gamma\varphi,\; \gamma\varphi\rangle$$

The operator $-\Omega + C$ has a positive symmetric square root in the center of $\mathcal{U}\mathfrak{g}$. (See the proof of Proposition 3.2.2.) Thus, $\langle (-\Omega + C)\,\gamma\varphi,\; \gamma\varphi\rangle$ is

$$\begin{aligned} \langle \sqrt{-\Omega + C}\,\sqrt{-\Omega + C}\;\gamma\varphi,\; \gamma\varphi\rangle \;\; &= \;\; \langle \sqrt{-\Omega + C}\;\gamma\varphi,\; \sqrt{-\Omega + C}\;\gamma\varphi\rangle \\ &= \;\; \langle \gamma\,\sqrt{-\Omega + C}\;\varphi,\; \gamma\,\sqrt{-\Omega + C}\;\varphi\rangle \end{aligned}$$

By inductive hypothesis,

$$\begin{aligned} \langle \gamma\,\sqrt{-\Omega + C}\;\varphi,\; \gamma\,\sqrt{-\Omega + C}\;\varphi\rangle \;\; &\leq \;\; \langle (-\Omega + C)^{n-1}\,\sqrt{-\Omega + C}\;\varphi,\; \sqrt{-\Omega + C}\;\varphi\rangle \\ &= \;\; \langle (-\Omega + C)^{n}\,\varphi,\; \varphi\rangle \end{aligned}$$

□

Thus, for any $\alpha \in \mathcal{U}\mathfrak{g}$, there is a constant C, possibly depending on the degree of α, such that

$$\nu_\alpha(f) \;\; \ll \;\; \langle (-\Omega + C)^{\deg\alpha}\, f, f\rangle \qquad \text{for all } f \in C_c^\infty(G/K)^K$$

In fact, for bi-K-invariant functions,

$$(-\Omega + C)^{\deg\alpha}\, f \;\; = \;\; (-\Omega_{\mathfrak{p}} + C)^{\deg\alpha}\, f$$

Since $\Omega_{\mathfrak{p}}$ is positive semi-definite, multiplying by a positive constant does not change the topology. Thus, we may take $C = 1$.

That is, the subfamily $\{\nu_\alpha : \alpha = (1 - \Omega)^k, k \leq \ell\}$ of seminorms on $C_c^\infty(G/K)^K$ dominates the family $\{\nu_\alpha : \alpha \in \mathcal{U}\mathfrak{g}^{\leq 2\ell}\}$ and thus induces an equivalent topology.

□

It will be necessary to have another description of Sobolev spaces. Consider

$$W^{2,\ell}(G/K) \;=\; \{f \in L^2(G/K) : \alpha\, f \in L^2(G/K) \text{ for all } \alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}\}$$

where the action of $\mathcal{U}\mathfrak{g}$ on $L^2(G/K)$ is by distributional differentiation: f in $L^2(G/K)$ may identified with the distribution Λ_f on G/K

$$\Lambda_f(\varphi) \;=\; \int_{G/K} f(g)\,\varphi(g)\,dg$$

and the distributional derivative $\alpha\,\Lambda_f$ is

$$(\alpha\,\Lambda_f)(\varphi) \;=\; \Lambda_f(\alpha\,\varphi)$$

Give $W^{2,\ell}(G/K)$ the topology induced by the seminorms

$$\nu_\alpha f \;=\; \|\alpha\, f\|^2_{L^2(G/K)} \qquad \alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}$$

where, again, $\alpha\, f$ is a distributional derivative.

Let $W^{2,\ell}(X) = W^{2,\ell}(G/K)^K$.

Proposition 5.2.3. *These spaces are equal to the corresponding Sobolev spaces:*

$$W^{2,\ell}(G/K) \;=\; H^\ell(G/K) \qquad \textit{and} \qquad W^{2,\ell}(X) \;=\; H^\ell(X)$$

Proof. Since $W^{2,\ell}(X)$ and $H^\ell(X)$ are simply the left-K-invariants in $W^{2,\ell}(G/K)$ and $H^\ell(G/K)$ respectively, it suffices to prove the equality of $W^{2,\ell}(G/K)$ and $H^\ell(G/K)$.

Since test functions are dense in $H^\ell(G/K)$, it suffices to show the density of test functions in $W^{2,\ell}(G/K)$. Since G acts continuously on $W^{2,\ell}(G/K)$ by left translation, Theorem 3.5.2 implies that *mollifications* are dense in $W^{2,\ell}(G/K)$, i.e. for a smooth approximate identity $\{\eta_i\}$ on G and $f \in W^{2,\ell}(G/K)$, the mollifications $\eta \cdot f$, given by the following $W^{2,\ell}(G/K)$-valued Gelfand-Pettis integrals,

$$(\eta_i{\cdot} f)(g) \;=\; \Big(\int_G \eta_i(h)\, L_h\, f\, dh\Big)(g) \;=\; \int_G \eta_i(h)\,(L_h\, f)(g)\,dh \;=\; \int_G \eta_i(h)\, f(h^{-1}g)\,dh$$

approach f in the topology of $W^{\ell,2}(G/K)$. Thus it suffices to show that mollifications can be approximated by test functions.

Urysohn's Lemma implies that $C_c^0(G/K)$ is dense in $L^2(G/K)$ and thus in $W^{2,\ell}(G/K)$. Thus it suffices to consider mollifications of continuous, compactly supported functions.

Let $\eta \in C_c^\infty(G)$ and $f \in C_c^0(G/K)$. By the proof of Theorem 3.5.2, $\eta \cdot f$ is a smooth vector, and for all $\alpha \in \mathcal{U}\mathfrak{g}$,

$$\alpha \cdot (\eta \cdot f) \quad = \quad (L_\alpha \eta) \cdot f$$

Since f is a continuous function and the action of the group is by translations, the abstract differentiation of $\eta \cdot f$ as a *vector* is the same as differentiation of $\eta \cdot f$ as a *function*. (See the proof of Proposition 3.2.3.)

The support of $\eta \cdot f$ is contained in the product of the compact supports of η and f. Since the product of two compact sets is again compact, $\eta \cdot f$ is compactly supported.

□

Remark 5.2.1. By Proposition 5.2.2, $H^{2\ell}(X) = W^{2,2\ell}(X)$ is a Hilbert space with norm

$$\|f\|_{2\ell}^2 \quad = \quad \|f\|^2 \; + \; \|(1-\Omega)\, f\|^2 \; + \; \ldots \; + \; \|(1-\Omega)^\ell\, f\|^2$$

where $\|\cdot\|$ is the usual norm on $L^2(G/K)$, and $(1-\Omega)^k f$ is a distributional derivative.

5.3 Spectral transform, inversion, and differentiation on Sobolev spaces

Proposition 5.3.1. *For $\ell \geq 0$, the Laplacian extends to a continuous linear map $H^{2\ell+2}(X) \to H^{2\ell}(X)$, the spherical transform extends to a map on $H^{2\ell}(X)$, and*

$$\mathcal{F}\big((1-\Delta)f\big) \quad = \quad (1-\lambda_\xi) \cdot \mathcal{F}f \qquad \textit{for all } f \in H^{2\ell+2}(X)$$

Proof. The Laplacian is a continuous map

$$\Delta : C^\infty(G/K) \cap H^{2\ell+2}(G/K) \to C^\infty(G/K) \cap H^{2\ell}(G/K)$$

by the construction of the Sobolev topology. Since the Laplacian is the image of the Casimir operator, which is in the center of the universal enveloping algebra, it preserves bi-K-invariance. Thus the Laplacian extends to a (continuous linear) map, also denoted Δ, from $H^{2\ell+2}(X)$ to $H^{2\ell}(X)$.

The spherical transform, defined on $C_c^\infty(G/K)^K$ by the integral transform of Harish-Chandra and Berezin, extends by continuity to $H^{2\ell}(X)$. This extension agrees with the extension to $L^2(X)$ coming from Plancherel.

Since $\mathcal{F}\big(\Delta\varphi\big) = \lambda_\xi \cdot \mathcal{F}\varphi$, for $\varphi \in C_c^\infty(G/K)^K$, by integration by parts,

$$\mathcal{F}\big((1-\Delta)f\big) \;=\; (1-\lambda_\xi)\cdot\mathcal{F}f \qquad \text{for all } f \in H^{2\ell+2}(X)$$

by continuity.

$\square$

Let μ be the following multiplication map on functions on Ξ

$$\mu(v)(\xi) \;=\; (1-\lambda_\xi)\cdot v(\xi) \;=\; (1+|\rho|^2+|\xi|^2)\cdot v(\xi)$$

where ρ is the half sum of positive roots. For $\ell \in \mathbb{Z}$, the weighted L^2-spaces

$$V^{2\ell} \;=\; \{v \text{ measurable } : \mu^\ell(v) \in L^2(\Xi)\}$$

with norms

$$\|v\|_{V^{2\ell}} \;=\; \|\mu^\ell(v)\|_{L^2(\Xi)} \;=\; \int_\Xi (1+|\rho|^2+|\xi|^2)^\ell \, |v(\xi)|^2 \, |\mathbf{c}(\xi)|^{-2}\, d\xi$$

are Hilbert spaces with $V^{2\ell+2} \subset V^{2\ell}$ for all ℓ. In fact, these are dense inclusions, since truncations are dense in all $V^{2\ell}$-spaces.

The multiplication map μ is a Hilbert space isomorphism:

$$V^{2\ell+2} \xrightarrow{\mu} V^{2\ell}$$

since it is clearly continuous, linear and bijective, and for $v \in V^{2\ell+2}$,

$$\|\mu(v)\|_{V^{2\ell}} \;=\; \|\mu^{\ell+1}(v)\|_{L^2(\Xi)} \;=\; \|v\|_{V^{2\ell+2}}$$

The negatively indexed spaces are (naturally identified with) the Hilbert space duals of their positively indexed counterparts:

$$(V^{2\ell})^* \;\approx\; V^{-2\ell}$$

by integration: $u \in V^{-2\ell}$ gives rise to $\Lambda_u \in (V^{2\ell})^*$ by

$$\Lambda_u(v) \quad = \quad \int_\Xi u(\xi) \cdot v(\xi)\, |\mathbf{c}(\xi)|^{-2}\, d\xi$$

The adjoints to inclusion maps are genuine inclusions, since $V^{2\ell+2} \hookrightarrow V^{2\ell}$ is dense for all $\ell \geq 0$, and, under the identification $(V^{2\ell})^* = V^{-2\ell}$ the adjoint map $\mu^* : (V^{2\ell})^* \to (V^{2\ell+2})^*$ is the multiplication map $\mu : V^{-2\ell} \to V^{-2\ell-2}$, since

$$\begin{aligned} \mu^*(\Lambda_u)(v) \quad = \quad \Lambda_u(\mu(v)) \quad &= \quad \int_\Xi u(\xi) \cdot (1 + |\rho|^2 + |\xi|^2) v(\xi)\, |\mathbf{c}(\xi)|^{-2}\, d\xi \\ &= \quad \int_\Xi (1 + |\rho|^2 + |\xi|^2) u(\xi) \cdot v(\xi)\, |\mathbf{c}(\xi)|^{-2}\, d\xi \quad = \quad \Lambda_{\mu(u)}(v) \end{aligned}$$

Proposition 5.3.2. *For $\ell \geq 0$, the spherical transform $\mathcal{F}$ is an isometric isomorphism $H^{2\ell}(X) \to V^{2\ell}$.*

Proof. On compactly supported functions, the spherical transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are given by integrals, which are certainly continuous linear maps. The Plancherel theorem extends $\mathcal{F}$ and $\mathcal{F}^{-1}$ to isometries between $L^2(X)$ and $L^2(\Xi)$. Thus $\mathcal{F}$ on $H^{2\ell}(X) \subset L^2(X)$ is a continuous linear L^2-isometry onto its image.

Let $f \in H^{2\ell}(X)$. By Proposition 5.2.3, the distributional derivatives $(1 - \Delta)^k f$ lie in $L^2(X)$ for all $k \leq \ell$. By the Plancherel theorem and Proposition 5.3.1,

$$\|(1 - \Delta)^\ell f\|_{L^2(X)} \quad = \quad \|\mathcal{F}\big((1 - \Delta)^\ell f\big)\|_{L^2(\Xi)} \quad = \quad \|(1 - \lambda_\xi)^\ell \cdot \mathcal{F} f\|_{L^2(\Xi)}$$

Thus $\mathcal{F}\big(H^{2\ell}\big) \subset V^{2\ell}$. Since $V^{2\ell} \subset L^2(\Xi)$, $\mathcal{F}^{-1}$ is defined on $V^{2\ell}$ and $\mathcal{F}^{-1}\big(V^{2\ell}\big) \subset L^2(X)$. The following claim shows that $\mathcal{F}^{-1}\big(V^{2\ell}\big) \subset H^{2\ell}(X)$.

Claim 5. *For $v \in V^{2\ell}$, the distributional derivatives $(1 - \Delta)^k \mathcal{F}^{-1} v \in L^2(X)$, for all $0 \leq k \leq \ell$.*

Proof. Evaluating at a test function φ,

$$\big((1 - \Delta)\, \mathcal{F}^{-1} v\big)\varphi \quad = \quad \mathcal{F}^{-1} v\, \big((1 - \Delta)\varphi\big)$$

By Plancherel,

$$\mathcal{F}^{-1}v\left((1-\Delta)\varphi\right) \;=\; v\big(\mathcal{F}\,(1-\Delta)\varphi\big)$$

By Proposition 5.3.1 and using Plancherel again,

$$v\big(\mathcal{F}\,(1-\Delta)\varphi\big) \;=\; v\big((1-\lambda_\xi)\cdot\mathcal{F}\varphi\big) \;=\; \big((1-\lambda_\xi)\cdot v\big)\,(\mathcal{F}\varphi) \;=\; \mathcal{F}^{-1}\big((1-\lambda_\xi)\cdot v\big)\,\varphi$$

By induction, we have the following identity of distributions:

$$(1-\Delta)^k\,\mathcal{F}^{-1}v \;=\; \mathcal{F}^{-1}\big((1-\lambda_\xi)^k\,v\big)$$

Since $\mathcal{F}$ is an L^2-isometry and $(1-\lambda_\xi)^k\,v \in L^2(\Xi)$ for all $0 \le k \le \ell$,

$$(1-\Delta)^k\,\mathcal{F}^{-1}v \;=\; \mathcal{F}^{-1}\big((1-\lambda_\xi)^k\,v\big) \;\in\; L^2(X) \qquad \text{for } 0 \le k \le \ell$$

□

This concludes the proof of the theorem.

□

Remark 5.3.1. This Hilbert space isomorphism $\mathcal{F}: H^{2\ell} \to V^{2\ell}$ gives a *spectral* characterization of the $2\ell^{\text{th}}$ Sobolev space, namely the preimage of $V^{2\ell}$ under $\mathcal{F}$.

$$H^{2\ell}(X) = \{f \in L^2(X) : (1-\lambda_\xi)^\ell \cdot \mathcal{F}f(\xi) \in L^2(\Xi)\}$$

5.4 Negatively indexed Sobolev spaces and distributions

The theory of negatively indexed Sobolev spaces allows application of spectral theory to solving differential equations involving certain *distributions.*

Definition 5.4.1. For $\ell > 0$, the negatively indexed Sobolev space $H^{-\ell}(X)$ is the Hilbert space dual of $H^\ell(X)$.

Remark 5.4.1. Since the space of test functions is a dense subspace of $H^\ell(X)$ with $\ell > 0$, dualizing gives an inclusion of $H^{-\ell}(X)$ into the space of distributions. Indeed the adjoint map $\mathrm{inc}^* : H^{-\ell}(X) \to \big(C_c^\infty(X)\big)^*$ to the inclusion $\mathrm{inc} : C_c^\infty(X) \hookrightarrow H^\ell(X)$ is a literal inclusion since, for any $u \in H^{-\ell}(X)$,

$$\mathrm{inc}^*(u)(\varphi) \;=\; u(\varphi) \qquad \text{for all } \varphi \in C_c^\infty(X)$$

i.e. $\mathrm{inc}^* u = u|_{C_c^\infty(X)}$, and the injectivity follows from density of test functions in $H^\ell(X)$: if two elements of $H^{-\ell}(X)$ agree on test functions they agree on $H^\ell(X)$, by continuity. Similarly, the adjoints of the dense inclusions $H^\ell \hookrightarrow H^{\ell-1}$ are literal inclusions $H^{-\ell+1}(X) \hookrightarrow H^{-\ell}(X)$. In fact, the self-duality of $H^0(X) = L^2(X)$ implies that $H^\ell(X) \hookrightarrow H^{\ell-1}$ for all $\ell \in \mathbb{Z}$, when $L^2(X)$ is *identified* with its dual by

$$f \in L^2(X) \quad \longleftrightarrow \quad \Lambda_f \; = \; \langle f, - \rangle_{L^2(X)} \; \in \; \big(L^2(X)\big)^*$$

Proposition 5.4.1. *The spectral transform extends to an isometric isomorphism on negatively indexed Sobolev spaces $\mathcal{F} : H^{-2\ell} \to V^{-2\ell}$, and*

$$\mathcal{F}((1-\Delta)\, u) \;\; = \;\; (1-\lambda_\xi) \cdot \mathcal{F}u$$

holds for $u \in H^{2\ell}(X)$, for any $\ell \in \mathbb{Z}$.

Proof. Propositions 5.3.1 and 5.3.2 give the result for positively indexed Sobolev spaces. As in the proof of Proposition 3.4.1, we obtain the extension to negatively indexed Sobolev spaces using the self-duality of L^2 and the Plancherel theorem, and the density of test functions in negatvely indexed Sobolev spaces, which follows from the density of test functions in the $V^{2\ell}$ spaces.

□

Recall the definition (3.4.2) of the local Sobolev space of a smooth manifold M.

Proposition 5.4.2. *Positively indexed global Sobolev spaces on G/K lie inside the corresponding local Sobolev spaces:*

$$H^\ell(G/K) \; \subset \; H^\ell_{loc}(G/K) \qquad \text{for } \; \ell \geq 0$$

Proof. Let x be a point in G/K, U an open neighborhood of x, Φ be a diffeomorphism $U \to \Omega \subset \mathbb{R}^n$ where Ω has compact closure, and let φ be a test function whose support is contained in U. We must show that $(f \cdot \varphi) \circ \Phi$ and all its derivatives up to order ℓ are square integrable on Ω. By Leibnitz' rule, $\partial^\mu\,(f \cdot \varphi)$ is a linear combination of terms of the form

$$\partial^\alpha f \, \cdot \, \partial^\beta \varphi \qquad \text{where } \; |\alpha| + |\beta| \leq |\mu|$$

But since

$$\sup_{|\alpha|\le\ell} \|\partial^\alpha f\|_{L^2(\Omega)} \;\ll\; \|f\|_{H^\ell(G/K)} \qquad \text{and} \qquad \sup_{|\beta|\le\ell} \|\partial^\beta \varphi\|_{L^2(\Omega)} \;\ll\; \|\varphi\|_{C^k(\Omega)}$$

and both of these are finite,

$$\|\partial^\mu (f\cdot\varphi)\|_{L^2(\Omega)} \;\ll\; \|f\|_{H^\ell(G/K)}\cdot\|\varphi\|_{C^k(\Omega)} \;<\; \infty$$

□

Thus a comparison may be made between the global Sobolev topologies and (local) C^k-topologies, via the Sobolev embedding theorem for local Sobolev spaces.

Corollary 5.4.1. *For $\ell > \dim(G/K)/2$,*

$$H^{\ell+k}(X) \;\subset\; H^{\ell+k}(G/K) \;\subset\; C^k(G/K)$$

Proof. By Proposition 5.4.2, $H^\ell(G/K) \subset H^\ell_{\text{loc}}(G/K)$, and by Theorem 3.4.1, $H^\ell_{\text{loc}}(G/K) \subset C^k(G/K)$. □

This embedding of global Sobolev spaces into C^k-spaces is used to prove that the integral defining spectral inversion for test functions can be extended to sufficiently highly indexed Sobolev spaces, i.e. the abstract isometric isomorphism $\mathcal{F}^{-1}\circ\mathcal{F} : H^\ell(X) \to H^\ell(X)$ is given by an integral that is convergent uniformly pointwise, when $\ell > \dim(G/K)/2$.

Proposition 5.4.3. *For $f \in H^s(X)$, $s > k + \dim(G/K)/2$,*

$$f \;=\; \int_\Xi \mathcal{F}f(\xi)\,\varphi_{\rho+i\xi}\,|\mathbf{c}(\xi)|^{-2}\,d\xi \qquad \textit{in } C^k(X)$$

Proof. Let $\{\Xi_n\}$ be a nested family of compact sets in Ξ whose union is all of Ξ, χ_n be the characteristic function of Ξ_n, and f_n be given by the $C^\infty(X)$-valued Gelfand-Pettis integral

$$f_n \;=\; \int_\Xi \chi_n(\xi)\,\mathcal{F}f(\xi)\,\varphi_{\rho+i\xi}\,|\mathbf{c}(\xi)|^{-2}\,d\xi$$

Since $\chi_n(\xi)\,\mathcal{F}f(\xi)$ is compactly supported,

$$f_n \;=\; \mathcal{F}^{-1}(\chi_n\cdot\mathcal{F}f)$$

Thus,

$$\|f_n - f_m\|_{H^s(X)} \;=\; \|(\chi_n - \chi_m)\cdot \mathcal{F}f\|_{V^s}$$

Since $\mathcal{F}f$ lies in V^s, these tails certainly approach zero as $n, m \to \infty$. Similarly,

$$\|f_n - f\|_{H^s(X)} \;=\; \|(\chi_n - 1)\cdot \mathcal{F}f\|_{V^s} \;\longrightarrow\; 0 \qquad \text{as } n \to \infty$$

By Corollary 5.4.1, f_n approaches f in $C^k(X)$.

□

The embedding of global Sobolev spaces into C^k-spaces also implies that *compactly supported* distributions lie in global Sobolev spaces, as follows.

Proposition 5.4.4. *Any compactly supported distribution on X lies in a global zonal spherical Sobolev space. Specifically, a compactly supported distribution of order k lies in $H^{-s}(X)$ for all $s > k + \dim(G/K)/2$.*

Proof. A compactly supported distribution u lies in $\big(C^\infty(G/K)\big)^*$. Since compactly supported distributions are of finite order, u extends continuously to $C^k(G/K)$ for some $k \geq 0$. Using Corollary 5.4.1 and dualizing, u lies in $H^{-(\ell+k)}(X)$, for $\ell > \dim(G/K)/2$.

□

Remark 5.4.2. In particular, this implies that the Dirac delta distribution at the base point $x_o = 1 \cdot K$ in G/K lies in $H^{-\ell}(X)$ for all $\ell > \dim(G/K)/2$.

Proposition 5.4.5. *For a compactly supported distribution u of order k,*

$$\mathcal{F}u \;=\; u(\varphi_{\rho+i\xi}) \qquad \textit{in } V^{-s} \quad \textit{where } s > k + \dim(G/K)/2$$

Proof. A compactly supported distribution u of order k lies in H^{-s} for any $s > k + \dim(G/K)/2$. Let f be any element in $H^s(X)$. Then,

$$\langle \mathcal{F}f, \mathcal{F}u\rangle_{V^s\times V^{-s}} \;=\; \langle f, u\rangle_{H^s\times V^s} \;=\; u(f)$$

Since the spectral expansion of f converges to it in the $H^s(X)$ topology by Proposition 5.4.3,

$$u(f) \;=\; u\bigg(\lim_n \int_{\Xi_n} \mathcal{F}f(\xi)\,\varphi_{\rho+i\xi}\,|\mathbf{c}(\xi)|^{-2}\,d\xi\bigg) \;=\; \lim_n u\bigg(\int_{\Xi_n} \mathcal{F}f(\xi)\,\varphi_{\rho+i\xi}\,|\mathbf{c}(\xi)|^{-2}\,d\xi\bigg)$$

Since the integral is a $C^\infty(X)$-valued Gelfand-Pettis integral and u is an element of $(C^\infty(X))^*$,

$$u\left(\int_{\Xi_n} \mathcal{F}f(\xi)\,\varphi_{\rho+i\xi}\;|\mathbf{c}(\xi)|^{-2}\,d\xi\right) \;=\; \int_{\Xi_n} \mathcal{F}f(\xi)\,u(\varphi_{\rho+i\xi})\;|\mathbf{c}(\xi)|^{-2}\,d\xi$$

The limit as $n \to \infty$ is finite, by comparison with the original expression which surely is finite, and thus

$$\langle \mathcal{F}f, \mathcal{F}u\rangle_{V^s\times V^{-s}} \;=\; \int_{\Xi} \mathcal{F}f(\xi)\,u(\varphi_{\rho+i\xi})\;|\mathbf{c}(\xi)|^{-2}\,d\xi \;=\; \langle \mathcal{F}f, u(\varphi_{\rho+i\xi})\rangle_{V^s\times V^{-s}}$$

Thus, $\mathcal{F}u = u(\varphi_{\rho+i\xi})$ as elements of V^{-s}

□

Remark 5.4.3. This implies that the spherical transform of the Dirac delta distribution is $\mathcal{F}\delta = \varphi_{\rho+i\xi}(1) = 1$.

Chapter 6

Fundamental Solution for $(\Delta - \lambda_z)^\nu$ on G/K

6.1 Introduction

We compute a fundamental solution u_z of the differential operator $(\Delta - \lambda_z)^\nu$ on the Riemannian symmetric space G/K, where G is any complex semi-simple Lie group and K is a maximal compact subgroup. Since the delta function $\delta_{1 \cdot K}$ at the base point is actually *bi-K-invariant*, we can use the harmonic analysis of spherical functions.

Instead of using the *existence* of a fundamental solution to prove *solvability* of a differential operator, we obtain an explicit *expression* for the fundmantal solution, with eye towards further applications involving the Poincaré series obtained by *winding up* the fundamental solution.

In particular, the presence of a complex (eigenvalue) parameter z in the differential operator makes the fundamental solution suitable for further applications, and the simple, explicit nature of the fundamental solution allows relatively easy estimation of its behavior in the eigenvalue parameter, proving convergence of the Poincare series in L^2 and, in fact, in a Sobolev space sufficient to prove continuity. Further, this makes it possible to determine the vertical growth of the Poincare series in the eigenvalue parameter.

See, for example, the work of Andersen [2, 3], van den Ban and Schlichtkrull [84], Benabdallah and Rouvière [5], and Bopp and Harinck [8], for papers proving existence of fundamental solutions for differential operators on semi-simple symmetric space.

Our main result is the following theorem, whose proof is given in Section 6.2.

Theorem. *When G is of odd rank, let $\nu = d + \frac{n+1}{2}$, where d is the number of positive roots, not counting multiplicities, and $n = \dim \mathfrak{a}$ the rank. Then the fundamental solution u_z for the operator $(\Delta - \lambda_z)^\nu$ on G/K is given by:*

$$u_z(a) \quad = \quad \frac{(-1)^{d+(n+1)/2}\,|W|\,\pi^{(n+1)/2}}{\pi^+(\rho)\,\Gamma(d+(n+1)/2)} \cdot \prod_{\alpha\in\Sigma^+} \frac{\alpha(\log a)}{2\sinh(\frac{\alpha(\log a)}{2})} \cdot \frac{e^{-z|\log a|}}{z}$$

When G is of even rank, let $\nu = d + \frac{n}{2} + 1$. Then, with K_1 the usual Bessel function,

$$u_z(a) \quad = \quad \frac{(-1)^{d+(n/2)+1}\,\pi^{n/2}\,|W|}{\pi^+(\rho)\,\Gamma(d+(n/2)+1)} \cdot \prod_{\alpha\in\Sigma^+} \frac{\alpha(\log a)}{2\sinh(\frac{\alpha(\log a)}{2})} \cdot \frac{|\log a|}{z} \cdot K_1(z\,|\log a|)$$

Remark 6.1.1. Here $|\log a|$ is the norm, in the Euclidean sense, of $\log a \in \mathfrak{a} \approx \mathbb{R}^n$. In rank higher than one, it is *not* simply a "Cartan radius," since an element of $G = K \cdot \exp(\mathfrak{a}_+) \cdot K$ has "length" in several directions, which can be described, for example, by evaluation of the positive simple roots on $\log a$. It is perhaps more accurate to describe $\log a$ as the "multi-radius" of $g = k \cdot a \cdot k'$ and $|\log a|$ as the mean square of the multi-radius.

For a derivation of the fundamental solution in the case $G = SL_2(\mathbb{C})$, assuming a suitable global zonal spherical Sobolev theory, see Garrett's Newark talk [33] and the supplementary notes [36]. Our results for the general case are sketched in Garrett's Durham talk [34].

6.2 Proof of theorem

Let G be a complex semi-simple Lie group with finite center and K a maximal compact subgroup. Let $G = NAK$, $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ be corresponding Iwasawa decompositions. Let Σ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, let Σ^+ denote the subset of positive roots (for the ordering corresponding to $\mathfrak{n}$), and let $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$, m_α denoting the multiplicity of α. Let $\mathfrak{a}^*_{\mathbb{C}}$ denote the set of complex-valued linear functions on $\mathfrak{a}$.

Consider the differential equation on the symmetric space $X = G/K$:

$$(\Delta - \lambda_z)^\nu \, u_z \;\; = \;\; \delta_{1\cdot K}$$

where the Laplacian Δ is the image of the Casimir operator for $\mathfrak{g}$, λ_z is $z^2 - |\rho|^2$ for a complex parameter z, ν is an integral power, and $\delta_{1\cdot K}$ is Dirac delta at the basepoint $x_o = 1 \cdot K \in G/K$. Since $\delta_{1\cdot K}$ is also *left-K*-invariant, we can look for a left-K-invariant solution on G/K. Zonal spherical Sobolev theory justifies the use of the harmonic analysis of bi-K-invariant functions to produce a solution.

Proposition 6.2.1. *For integral $\nu > \dim(G/K)/2$, u_z is a continuous left-K-invariant function on G/K with the following integral representation:*

$$u_z(g) \;\; = \;\; \int_\Xi \frac{(-1)^\nu}{(|\xi|^2 + z^2)^\nu} \;\; \varphi_{\rho+i\xi}(g)\, |\mathbf{c}(\xi)|^{-2} \, d\xi$$

Proof. Since $\delta_{1\cdot K}$ is a compactly supported distribution of order zero, by Proposition 5.4.4, it lies in the global zonal spherical Sobolev spaces $H^{-\ell}(X)$ for all $\ell > \dim(G/K)/2$. Thus there is an element u_z of $H^{-\ell+2\nu}(X)$ satisfying this equation. The solution u_z is unique in Sobolev spaces, since any u_z' satisfying

$$\big(\Delta - (z^2 - |\rho|^2)\big)^\nu \, u_z' \;\; = \;\; \delta_{1\cdot K}$$

must necessarily have the same spherical transform. Indeed,

$$(-1)^\nu \, (|\xi|^2 + z^2)^{-\nu} \cdot \mathcal{F} u_z \;\; = \;\; \mathcal{F}\, \big(\Delta - (z^2 - |\rho|^2)\big)^\nu \, u_z' \;\; = \;\; \mathcal{F}\, \delta_{1\cdot K} \;\; = \;\; 1$$

For $\nu > \dim(G/K)/2$, the solution is continuous, by Corollary 5.4.1, and by Proposition 5.4.3,

$$u_z(g) \;\; = \;\; \int_\Xi \mathcal{F} u_z(\xi)\, \varphi_{\rho+i\xi}(g)\, |\mathbf{c}(\xi)|^{-2} \, d\xi \;\; = \;\; \int_\Xi \frac{(-1)^\nu}{(|\xi|^2 + z^2)^\nu} \;\; \varphi_{\rho+i\xi}(g)\, |\mathbf{c}(\xi)|^{-2} \, d\xi$$

□

Proposition 6.2.2. *For a complex semi-simple Lie group, spherical inversion is given by*

$$\mathcal{F}^{-1} f \;\; = \;\; \frac{(-i)^d \, |W|}{\pi^+(\rho) \, \sum \operatorname{sgn}(w)\, e^{w\rho}} \cdot \int_{\mathfrak{a}^*} f(\lambda)\, \pi^+(\lambda)\, e^{i\lambda} \, d\lambda$$

where d is the number of positive roots, counted without multiplicity.

Proof. For a *complex* semi-simple Lie group, the zonal spherical functions are *elementary.* The spherical function associated with the principal series I_χ with $\chi = e^{\rho+i\lambda}$ is

$$\varphi_{\rho+i\lambda} \;=\; \frac{\pi^+(\rho)}{\pi^+(i\lambda)}\,\frac{\sum \mathrm{sgn}(w)\, e^{i\,w\lambda}}{\sum \mathrm{sgn}(w)\, e^{w\rho}}$$

where the sums are taken over the elements w of the Weyl group, and the function π^+ is the product

$$\pi^+(\mu) \;=\; \prod_{\alpha>0} \langle \alpha, \mu \rangle$$

over positive roots, *without* multiplicities. The ratio of $\pi^+(\rho)$ to $\pi^+(i\lambda)$ is the $\mathbf{c}$-function. The inverse spherical transform is an integral over the Euclidean space $\mathfrak{a}^* \approx \mathbb{R}^n$.

$$\mathcal{F}^{-1} f \;=\; \int_{\mathfrak{a}^*} f(\lambda)\, \varphi_{\rho+i\lambda}\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda$$

Again, in the case of complex semi-simple Lie groups, this has an elementary form

$$\begin{aligned}
\mathcal{F}^{-1} f \;&=\; \int_{\mathfrak{a}^*} f(\lambda)\, \frac{\pi^+(\rho)}{\pi^+(i\lambda)}\, \frac{\sum \mathrm{sgn}(w)\, e^{i\,w\lambda}}{\sum \mathrm{sgn}(w)\, e^{w\rho}} \left| \frac{\pi^+(i\lambda)}{\pi^+(\rho)} \right|^2 d\lambda \\
&=\; \frac{1}{\pi^+(\rho)\, \sum \mathrm{sgn}(w)\, e^{w\rho}} \int_{\mathfrak{a}^*} f(\lambda) \left(\sum_{w \in W} \mathrm{sgn}(w)\, e^{i\,w\lambda} \right) \overline{\pi^+}(i\lambda)\, d\lambda
\end{aligned}$$

Properties of π^+, proven in Section 6.3, enable us to simplify this integral further. Since it is a homogeneous polynomial of degree d, equal to the number of positive roots, counted without multiplicity,

$$\overline{\pi^+}(i\lambda) \;=\; \pi^+(-i\lambda) \;=\; (-i)^d \cdot \pi^+(\lambda)$$

Also, π^+ is W-equivariant by the sign character, so the change of variables $\lambda \to w^{-1}\lambda$ yields

$$\mathcal{F}^{-1} f \;=\; \frac{(-i)^d\, |W|}{\pi^+(\rho)\, \sum \mathrm{sgn}(w)\, e^{w\rho}} \cdot \int_{\mathfrak{a}^*} f(\lambda)\, \pi^+(\lambda)\, e^{i\lambda}\, d\lambda$$

□

Remark 6.2.1. According to Helgason [54], the formula for zonal spherical functions given in the proof above was proven for $G = SL_n(\mathbb{C})$ by Gelfand and Naimark [42] and for the general case by Harish-Chandra [48, 49] and Berezin [6].

A less obvious property (see Theorem 6.3.1) is that π^+ is *harmonic* with respect to the Laplacian naturally associated to the pairing on $\mathfrak{a}^*$. This fact will be crucial for evaluating the integral representation of the fundamental solution.

Corollary 6.2.1. *The fundamental solution u_z has the following integral representation:*

$$u_z \;=\; \frac{(-1)^\nu\,(-i)^d\,|W|}{\pi^+(\rho)\;\sum \operatorname{sgn} w\, e^{w\rho}} \;\cdot\; \int_{\mathfrak{a}^*} \frac{1}{(|\lambda|^2+z^2)^\nu}\;\pi^+(\lambda)\;e^{i\lambda}\;d\lambda$$

The key step in evaluating this integral is the application of *Hecke's identity*, which evaluates the Fourier transform of the product f of a normalized Gaussian and a harmonic homogeneous polynomial P_d of degree d on a Euclidean space:

$$\int_{\mathbb{R}^n} e^{-|x|^2}\cdot P_d(x)\;e^{-i\langle x,\xi\rangle}\;dx \;=\; \hat{f}(\xi) \;=\; (-i)^d\,f(\xi) \;=\; (-i)^d\,e^{-|\xi|^2}\;P_d(\xi)$$

Let $I(\log a)$ denote the integral we need to compute:

$$I(\log a) \;=\; \int_{\mathfrak{a}^*} \frac{1}{(|\lambda|^2+z^2)^\nu}\;\pi^+(\lambda)\;e^{i\langle\lambda,\log a\rangle}\;d\lambda$$

Proposition 6.2.3. *The integral $I(\log a)$ can be reduced to an integral over the real line:*

$$I(\log a) \;=\; i^d\,\pi^+(\log a)\cdot\frac{\Gamma(\nu-d)}{\Gamma(\nu)}\cdot\frac{\pi^{(n-1)/2}}{\Gamma(\nu-d)}\cdot\Gamma\big(\nu-d-\tfrac{n-1}{2}\big)\cdot\int_{\mathbb{R}} \frac{e^{i\lambda_1|\log a|}}{(\lambda_1^2+z^2)^{\nu-d-(n-1)/2}}\;d\lambda_1$$

where $n=\dim\mathfrak{a}$ and d is the number of positive roots, counted without multiplicity.

Proof. Apply the identity

$$\frac{\Gamma(s)}{z^s} \;=\; \int_0^\infty t^s\,e^{-tz}\,\frac{dt}{t}$$

to $(|\lambda|^2+z^2)^{-\nu}$ in the integrand of $I(\log a)$:

$$\begin{aligned} I(\log a) \;&=\; \frac{1}{\Gamma(\nu)}\cdot\int_0^\infty \int_{\mathfrak{a}^*} t^\nu\;e^{-t(|\lambda|^2+z^2)}\;\pi^+(\lambda)\;e^{i\lambda}\;d\lambda\,\frac{dt}{t} \\ &=\; \frac{1}{\Gamma(\nu)}\cdot\int_0^\infty t^\nu\;e^{-tz^2}\;\int_{\mathfrak{a}^*} e^{-t|\lambda|^2}\;\pi^+(\lambda)\;e^{i\langle\lambda,\log a\rangle}\;d\lambda\,\frac{dt}{t} \end{aligned}$$

Change variables $\lambda \to \lambda/\sqrt{t}$.

$$\begin{aligned}
I(\log a) &= \frac{1}{\Gamma(\nu)} \cdot \int_0^\infty t^\nu \, e^{-tz^2} \int_{\mathfrak{a}^*} e^{-|\lambda|^2} \, t^{-d/2} \cdot \pi^+(\lambda) \, e^{i\langle \lambda/\sqrt{t}\,, \log a\rangle} \; t^{-n/2} \, d\lambda \, \frac{dt}{t} \\
&= \frac{1}{\Gamma(\nu)} \cdot \int_0^\infty t^{\nu-(d+r)/2} \, e^{-tz^2} \int_{\mathfrak{a}^*} e^{-|\lambda|^2} \, \pi^+(\lambda) \, e^{-i\langle \lambda\,, -\log a/\sqrt{t}\rangle} \; d\lambda \, \frac{dt}{t}
\end{aligned}$$

By Theorem 6.3.1, π^+ is a harmonic polynomial, and thus the integral over $\mathfrak{a}^*$ is the Fourier transform of the product of a Gaussian and a harmonic polynomial. By Hecke's identity,

$$\begin{aligned}
\int_{\mathfrak{a}^*} e^{-|\lambda|^2} \, \pi^+(\lambda) \, e^{-i\langle \lambda\,, -\log a/\sqrt{t}\rangle} \; d\lambda &= (-i)^d \, \pi^+\big(-\log a/\sqrt{t}\big) \, e^{-|\log a|^2/t} \\
&= i^d \, t^{-d/2} \, \pi^+(\log a) \, e^{-|\log a|^2/t}
\end{aligned}$$

Returning to the main integral,

$$\begin{aligned}
I(\log a) &= \frac{i^d \, \pi^+(\log a)}{\Gamma(\nu)} \cdot \int_0^\infty t^{\nu-d-n/2} \, e^{-tz^2} \, e^{-|\log a|^2/t} \, \frac{dt}{t} \\
&= \frac{i^d \, \pi^+(\log a)}{\Gamma(\nu)} \cdot \int_0^\infty t^{\nu-d} \, e^{-tz^2} \left(t^{-n/2} e^{-|\log a|^2/t}\right) \frac{dt}{t}
\end{aligned}$$

Replacing the Gaussian by its Fourier transform,

$$t^{-n/2} e^{-|\log a|^2/t} = \int_{\mathfrak{a}^*} e^{-t|\lambda|^2} \, e^{i\langle \lambda, \log a\rangle} \, d\lambda$$

the integral becomes

$$\begin{aligned}
I(\log a) &= \frac{i^d \, \pi^+(\log a)}{\Gamma(\nu)} \cdot \int_0^\infty t^{\nu-d} \, e^{-tz^2} \int_{\mathfrak{a}^*} e^{i\langle \lambda, \log a\rangle} \, e^{-t|\lambda|^2} \, d\lambda \, \frac{dt}{t} \\
&= \frac{i^d \, \pi^+(\log a)}{\Gamma(\nu)} \cdot \int_{\mathfrak{a}^*} \int_0^\infty t^{\nu-d} \, e^{-t(|\lambda|^2+z^2)} \, \frac{dt}{t} \; e^{i\langle \lambda, \log a\rangle} \, d\lambda \\
&= \frac{i^d \, \pi^+(\log a)}{\Gamma(\nu)} \cdot \int_{\mathfrak{a}^*} \frac{\Gamma(\nu-d)}{(|\lambda|^2+z^2)^{\nu-d}} \, e^{i\langle \lambda, \log a\rangle} \, d\lambda \\
&= i^d \, \pi^+(\log a) \cdot \frac{\Gamma(\nu-d)}{\Gamma(\nu)} \cdot \int_{\mathfrak{a}^*} \frac{e^{i\langle \lambda, \log a\rangle}}{(|\lambda|^2+z^2)^{\nu-d}} \, d\lambda
\end{aligned}$$

This integral is rotation-invariant as a function of $\log a$, so, writing $\lambda = (\lambda_1, \ldots, \lambda_n)$, we may assume $\langle \lambda, \log a\rangle = \lambda_1 \cdot |\log a|$. We denote this integral by $J(|\log a|)$. Then

$$J(|\log a|) = \int_{\mathfrak{a}^*} \frac{e^{i\lambda_1 \cdot |\log a|}}{(|\lambda|^2+z^2)^{\nu-d}} \, d\lambda$$

To evaluate this integral, use the Γ-function identity again, and identify $\mathfrak{a}^*$ with $\mathbb{R}^n$. Then $J(|\log a|)$ is

$$
\begin{aligned}
&\frac{1}{\Gamma(\nu-d)} \cdot \int_0^\infty t^{\nu-d} \int_{\mathbb{R}^n} e^{-t(|\lambda|^2+z^2)}\, e^{i\lambda_1|\log a|}\, d\lambda_1\, d\lambda_2\, \ldots\, \lambda_n\, \frac{dt}{t} \\
&= \frac{1}{\Gamma(\nu-d)} \cdot \int_0^\infty t^{\nu-d} \int_{\mathbb{R}} e^{-t(\lambda_1^2+z^2)}\, e^{i\lambda_1|\log a|}\, d\lambda_1 \int_{\mathbb{R}^{n-1}} e^{-t(\lambda_2^2+\cdots+\lambda_n^2)}\, d\lambda_2\, \ldots\, \lambda_n\, \frac{dt}{t} \\
&= \frac{\pi^{(n-1)/2}}{\Gamma(\nu-d)} \cdot \int_0^\infty t^{\nu-d-(n-1)/2} \int_{\mathbb{R}} e^{-t(\lambda_1^2+z^2)}\, e^{i\lambda_1|\log a|}\, d\lambda_1\, \frac{dt}{t} \\
&= \frac{\pi^{(n-1)/2}}{\Gamma(\nu-d)} \cdot \Gamma\big(\nu-d-\tfrac{n-1}{2}\big) \cdot \int_{\mathbb{R}} \frac{e^{i\lambda_1|\log a|}}{(\lambda_1^2+z^2)^{\nu-d-(n-1)/2}}\, d\lambda_1
\end{aligned}
$$

Since

$$
I(\log a) \;=\; i^d\, \pi^+(\log a) \cdot \frac{\Gamma(\nu-d)}{\Gamma(\nu)} \cdot J(|\log a|)
$$

we have the desired conclusion. □

The remaining integral can be evaluated by *residues* when the exponent in the denominator is a sufficiently large *integer*, i.e. when G is of *odd* rank. For the even rank case, the integral can be expressed in terms of a *K-Bessel function*.

Proposition 6.2.4. *For n odd, let $\nu \;=\; d+\frac{n+1}{2}$. Then*

$$
I(\log a) \;=\; i^d\, \pi^+(\log a) \cdot \frac{\pi^{(n+1)/2}}{\Gamma(d+(n+1)/2)} \;\cdot\; \frac{e^{-z\,|\log a|}}{z}
$$

Proof. Recall from Proposition 6.2.3 that

$$
\begin{aligned}
&I(\log a) \\
&= i^d\, \pi^+(\log a) \cdot \frac{\Gamma(\nu-d)}{\Gamma(\nu)} \cdot \frac{\pi^{(n-1)/2}}{\Gamma(\nu-d)} \cdot \Gamma\big(\nu-d-\tfrac{n-1}{2}\big) \cdot \int_{\mathbb{R}} \frac{e^{i\lambda_1|\log a|}}{(\lambda_1^2+z^2)^{\nu-d-(n-1)/2}}\, d\lambda_1
\end{aligned}
$$

The integral over $\mathbb{R}$ can be evaluated by residues.

$$
\int_{\mathbb{R}} \frac{e^{i\lambda_1|\log a|}}{(\lambda_1^2+z^2)^{\nu-d-(n-1)/2}}\, d\lambda_1 \;=\; 2\pi i \cdot \operatorname*{Res}_{\lambda_1=iz} \left(\frac{e^{i\lambda_1|\log a|}}{(\lambda_1^2+z^2)^{\nu-d-(n-1)/2}}\right)
$$

The choice of ν simplifies the integral:

$$
\int_{\mathbb{R}} \frac{e^{i\lambda_1|\log a|}}{(\lambda_1^2+z^2)^{\nu-d-(n-1)/2}}\, d\lambda_1 \;=\; \int_{\mathbb{R}} \frac{e^{i\lambda_1|\log a|}}{\lambda_1^2+z^2}\, d\lambda_1
$$

Since the residue at $\lambda_1 = iz$ is

$$\frac{e^{i(iz)\cdot|\log a|}}{2iz} \;=\; \frac{e^{-z\,|\log a|}}{2iz}$$

the integral is

$$\int_{\mathbb{R}} \frac{e^{i\lambda_1|\log a|}}{\lambda_1^2 + z^2}\, d\lambda_1 \;=\; \frac{\pi\, e^{-z\,|\log a|}}{z}$$

Thus,

$$\begin{aligned}
&I(\log a) \\
&= \; i^d\,\pi^+(\log a)\cdot\frac{\Gamma(\nu-d)}{\Gamma(\nu)}\cdot\frac{\pi^{(n-1)/2}}{\Gamma(\nu-d)}\cdot\Gamma\big(\nu-d-\tfrac{n-1}{2}\big)\cdot\int_{\mathbb{R}}\frac{e^{i\lambda_1|\log a|}}{(\lambda_1^2+z^2)^{\nu-d-(n-1)/2}}\,d\lambda_1 \\
&= \; i^d\,\pi^+(\log a)\cdot\frac{\Gamma((n+1)/2)}{\Gamma(d+(n+1)/2)}\;\cdot\;\frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)}\cdot\frac{\pi\, e^{-z\,|\log a|}}{z} \\
&= \; i^d\,\pi^+(\log a)\cdot\frac{\pi^{(n+1)/2}}{\Gamma(d+(n+1)/2)}\;\cdot\;\frac{e^{-z\,|\log a|}}{z}
\end{aligned}$$

□

Proposition 6.2.5. *For n even, let $\nu \;=\; d+\frac{n}{2}+1$. Then,*

$$I(\log a) \;=\; i^d\,\pi^+(\log a)\cdot\frac{\Gamma((n/2)+1)}{\Gamma(d+(n/2)+1)}\;\cdot\;\frac{\pi^{n/2}}{\Gamma((n/2)+1)}\cdot\frac{|\log a|}{z}\cdot K_1(z\,|\log a|)$$

Proof. Recall from Proposition 6.2.3 that

$$I(\log a) \;=\; i^d\,\pi^+(\log a)\cdot\frac{\Gamma(\nu-d)}{\Gamma(\nu)}\cdot\frac{\pi^{(n-1)/2}}{\Gamma(\nu-d)}\cdot\Gamma\big(\nu-d-\tfrac{n-1}{2}\big)\cdot\int_{\mathbb{R}}\frac{e^{i\lambda_1|\log a|}}{(\lambda_1^2+z^2)^{\nu-d-(n-1)/2}}\,d\lambda_1$$

The integral over $\mathbb{R}$ is

$$\begin{aligned}
\int_{\mathbb{R}}\frac{e^{i\lambda_1|\log a|}}{(\lambda_1^2+z^2)^{3/2}}\,d\lambda_1 \;&=\; \int_{\mathbb{R}}\frac{\cos(\lambda_1|\log a|)}{(\lambda_1^2+z^2)^{3/2}}\,d\lambda_1 + \int_{\mathbb{R}}\frac{i\sin(\lambda_1|\log a|)}{(\lambda_1^2+z^2)^{3/2}}\,d\lambda_1 \\
&=\; 2\cdot\int_0^\infty\frac{\cos(\lambda_1|\log a|)}{(\lambda_1^2+z^2)^{3/2}}\,d\lambda_1 \\
&=\; \frac{2\sqrt{\pi}\,|\log a|}{\Gamma(3/2)\cdot 2z}\;\cdot\;K_1(z\,|\log a|) \\
&=\; \frac{\sqrt{\pi}\,|\log a|}{\Gamma(3/2)\,z}\;\cdot\;K_1(z\,|\log a|)
\end{aligned}$$

where K_1 is a modified Bessel function K_α of the second kind, which has the following integral representation (see [1], 9.6.25).

$$K_\alpha(xz) \;=\; \frac{\Gamma(\alpha+\frac{1}{2})\,(2z)^\alpha}{\pi^{1/2}\,x^\alpha}\cdot\int_0^\infty \frac{\cos(xt)}{(t^2+z^2)^{\alpha+1/2}}\,dt \qquad \mathrm{Re}(\alpha)>-\tfrac{1}{2},\ x>0,\ |\arg z|<\tfrac{\pi}{2}$$

Thus,

$$\begin{aligned} I(\log a) & \\ = {} & i^d\,\pi^+(\log a)\cdot\frac{\Gamma((n/2)+1)}{\Gamma(d+(n/2)+1)}\cdot\frac{\pi^{(n-1)/2}}{\Gamma((n/2)+1)}\cdot\Gamma(3/2)\cdot\frac{\sqrt{\pi}\,|\log a|}{\Gamma(3/2)\,z}\cdot K_1(z\,|\log a|) \\ = {} & i^d\,\pi^+(\log a)\cdot\frac{\Gamma((n/2)+1)}{\Gamma(d+(n/2)+1)}\cdot\frac{\pi^{n/2}}{\Gamma((n/2)+1)}\cdot\frac{|\log a|}{z}\cdot K_1(z\,|\log a|)\end{aligned}$$

□

Now we prove the theorem stated in the introduction:

Theorem 6.2.1. *When G is of odd rank, let $\nu = d+\frac{n+1}{2}$, where d is the number of positive roots, counted without multiplicities, and n is the rank. Then the fundamental solution u_z for the operator $(\Delta-\lambda_z)^\nu$ on G/K is given by:*

$$u_z(a) \;=\; \frac{(-1)^{d+(n+1)/2}\,|W|\,\pi^{(n+1)/2}}{\pi^+(\rho)\,\Gamma(d+(n+1)/2)}\cdot\prod_{\alpha\in\Sigma^+}\frac{\alpha(\log a)}{2\sinh(\frac{\alpha(\log a)}{2})}\cdot\frac{e^{-z|\log a|}}{z}$$

When G is of even rank, let $\nu = d+\frac{n}{2}+1$. Then, with K_1 the usual Bessel function,

$$u_z(a) \;=\; \frac{(-1)^{d+(n/2)+1}\,\pi^{n/2}\,|W|}{\pi^+(\rho)\;\Gamma(d+(n/2)+1)}\cdot\prod_{\alpha\in\Sigma^+}\frac{\alpha(\log a)}{2\sinh(\frac{\alpha(\log a)}{2})}\cdot\frac{|\log a|}{z}\cdot K_1(z\,|\log a|)$$

Proof. Recall that the fundamental solution is

$$u_z(a) \;=\; \frac{(-1)^\nu\,(-i)^d\,|W|}{\pi^+(\rho)\;\sum \mathrm{sgn}\,w\,e^{w\rho(\log a)}}\cdot I(\log a)$$

By Proposition 6.2.4, when G is of odd rank and $\nu = d+\frac{n+1}{2}$,

$$\begin{aligned} u_z(a) \;=\; & \frac{(-1)^{d+(n+1)/2}\,(-i)^d\,|W|}{\pi^+(\rho)\;\sum \mathrm{sgn}\,w\,e^{w\rho(\log a)}}\cdot i^d\,\pi^+(\log a)\cdot\frac{\pi^{(n+1)/2}}{\Gamma(d+(n+1)/2)}\cdot\frac{e^{-z\,|\log a|}}{z} \\ = \; & \frac{(-1)^{d+(n+1)/2}\,|W|\,\pi^{(n+1)/2}}{\pi^+(\rho)\,\Gamma(d+(n+1)/2)}\cdot\frac{\pi^+(\log a)}{\sum \mathrm{sgn}\,w\;e^{w\rho(\log a)}}\cdot\frac{e^{-z|\log a|}}{z}\end{aligned}$$

By Proposition 6.2.5, when G is of even rank and $\nu = d + \frac{n}{2} + 1$,

$$\begin{aligned} u_z(a) &= \frac{(-1)^{d+(n/2)+1}\,|W|}{\pi^+(\rho)} \cdot \frac{\pi^+(\log a)}{\sum \operatorname{sgn} w\, e^{w\rho(\log a)}} \cdot \frac{\Gamma((n/2)+1)}{\Gamma(d+(n/2)+1)} \cdot J(|\log a|) \\ &= \frac{(-1)^{d+(n/2)+1}\,\pi^{n/2}\,|W|}{\pi^+(\rho)\,\Gamma(d+(n/2)+1)} \cdot \frac{\pi^+(\log a)}{\sum \operatorname{sgn} w\, e^{w\rho(\log a)}} \cdot \frac{|\log a|}{z} \cdot K_1(z\,|\log a|) \end{aligned}$$

The factorization

$$\frac{\pi^+(\log a)}{\sum \operatorname{sgn} w\, e^{w\rho(\log a)}} = \prod_{\alpha>0} \frac{\alpha(\log a)}{2\sinh\left(\frac{\alpha(\log a)}{2}\right)}$$

follows from the definition of π^+ and the fact that

$$\sum_{w\in W} \operatorname{sgn}(w)\, e^{w\rho} = \prod_{\alpha>0} 2\sinh\left(\tfrac{\alpha}{2}\right)$$

(See [54], Prop. 5.15.)

□

Remark 6.2.2. For fixed α, large $|z|$, and $\mu = 4\alpha^2$ (see [1], 9.7.2),

$$K_\alpha(z) \approx \sqrt{\tfrac{\pi}{2z}} e^{-z}\left(1 + \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{(2!\,(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!\,(8z)^3}\right) \quad \left(|\arg z| < \tfrac{3\pi}{2}\right)$$

Thus in the even rank case the fundamental solution has the following asymptotic:

$$u_z(a) \approx \frac{(-1)^{d+(n/2)+1}\,\pi^{(n+1)/2}\,|W|}{\sqrt{2}\,\pi^+(\rho)\,\Gamma(d+(n/2)+1)} \cdot \frac{\pi^+(\log a)}{\sum \operatorname{sgn} w\, e^{w\rho(\log a)}} \cdot \sqrt{\frac{|\log a|}{z}} \cdot \frac{e^{-z|\log a|}}{z}$$

Remark 6.2.3. Recall from Proposition 6.2.1 that zonal spherical Sobolev theory ensures the continuity of u_z for ν chosen as in the theorem. For $G = SL_2(\mathbb{C})$, the continuity is visible, since fundamental solution is, up to a constant,

$$u_z(a_r) = \frac{r\, e^{-(2z-1)r}}{(2z-1)\,\sinh r} \quad \text{where } a_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}$$

6.3 Properties of π^+

We prove several properties of the function π^+ appearing in the formula for the Harish-Chandra **c**-function for a complex semi-simple group, which were necessary for the computations above.

First we dispense with the elementary facts, that π^+ is W-equivariant by the sign character and is a homogeneous polynomial.

A simple reflection σ_α sends the positive root α to $-\alpha$ and permutes the other positive roots, so

$$\pi^+(\sigma_\alpha \mu) \;=\; \prod_{\gamma>0} \langle \gamma, \sigma_\alpha \mu\rangle \;=\; \prod_{\gamma>0} \langle \sigma_\alpha \gamma, \mu\rangle \;=\; -\pi^+(\mu)$$

Any element w of the Weyl group can be written as a product of simple reflections $\sigma_{\alpha_1} \dots \sigma_{\alpha_p}$, and $\mathrm{sgn}(w) = (-1)^p$, so

$$\pi^+(w\mu) \;=\; (-1)^p\, \pi^+(\mu) \;=\; \mathrm{sgn}(w)\, \pi^+(\mu)$$

To see that π^+ is a homogeneous polynomial, observe that it is a product of linear factors and

$$\pi^+(c \cdot \mu) \;=\; \prod_{\alpha>0} \langle \alpha, c \cdot \mu\rangle \;=\; \prod_{\alpha>0} c \cdot \langle \alpha, \mu\rangle \;=\; c^d \prod_{\alpha>0} \langle \alpha, \mu\rangle \;=\; c^d \cdot \pi^+(\mu)$$

where d is the number of positive roots.

The property that enables us to use *Hecke's identity* in the computations above is

Theorem 6.3.1. *For a complex semi-simple Lie group G, the function $\pi^+ : \mathfrak{a}^* \to \mathbb{R}$ given by*

$$\pi^+(\mu) \;=\; \prod_{\alpha>0} \langle \alpha, \mu\rangle$$

where the product is taken over all postive roots, counted without multiplicity, is harmonic with respect to the Laplacian naturally associated to the pairing on $\mathfrak{a}^*$.

Proof. We will use the following lemma.

Lemma 6.3.1. *Let I be the set of all non-orthogonal pairs of distinct positive roots, as functions on $\mathfrak{a}$. Then π^+ is harmonic if*

$$\sum_{(\beta,\gamma)\,\in\, I} \frac{\langle \beta, \gamma\rangle}{\beta\, \gamma} \;=\; 0$$

Proof. Observe that $\mathfrak{a}^*$ is a Euclidean space, and its Lie algebra can be identified with itself. For any basis $\{x_i\}$ of $\mathfrak{a}^*$, the Casimir operator (Laplacian) is

$$\Delta \;=\; \sum_i x_i\, x_i^*$$

First consider the product of two linear functionals. For any α, β in $\mathfrak{a}^*$,

$$\begin{aligned}\Delta\,\langle\alpha,\ \rangle\langle\beta,\ \rangle \;&=\; \sum_i x_i\left(\langle\alpha, x_i^*\rangle\,\langle\beta,\ \rangle + \langle\alpha,\ \rangle\,\langle\beta, x_i^*\rangle\right)\\ &=\; \sum_i\left(\langle\alpha, x_i^*\rangle\,\langle\beta, x_i\rangle + \langle\alpha, x_i\rangle\,\langle\beta, x_i^*\rangle\right) \;=\; 2\langle\alpha,\beta\rangle\end{aligned}$$

And so

$$\begin{aligned}\Delta\pi^+ \;&=\; \sum_i x_i x_i^* \pi^+ \;=\; \sum_i x_i \sum_{\beta>0} \alpha(x_i^*)\cdot\frac{\pi^+}{\beta}\\ &=\; \sum_i\sum_{\beta>0}\beta(x_i^*)\cdot\Big(\sum_{\gamma\neq\beta}\gamma(x_i)\cdot\frac{\pi^+}{\beta\gamma}\Big) \;=\; \sum_{\beta\neq\gamma}\frac{\langle\beta,\gamma\rangle}{\beta\,\gamma}\cdot\pi^+\end{aligned}$$

So it suffices to show that

$$\sum_{\beta\neq\gamma}\frac{\langle\beta,\gamma\rangle}{\beta\,\gamma} \;=\; 0$$

$\square$

Remark 6.3.1. When the Lie algebra $\mathfrak{g}$ is not simple, but merely semi-simple, i.e. $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, any pair β, γ of roots with $\beta \in \mathfrak{g}_1$ and $\gamma \in \mathfrak{g}_2$ will have $\langle\beta,\gamma\rangle = 0$, so it suffices to consider $\mathfrak{g}$ *simple.*

First we treat the rank two cases. The complex simple Lie algebras of rank two are $\mathfrak{sl}_3$, $\mathfrak{sp}_2$, and $\mathfrak{g}_2$. We treat each case separately.

Proposition 6.3.1. *For $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{sp}_2$, or $\mathfrak{g}_2$, the following sum over all pairs (β, γ) of distinct positive roots is zero:*

$$\sum_{\beta\neq\gamma}\frac{\langle\beta,\gamma\rangle}{\beta\,\gamma} \;=\; 0$$

Proof. The positive roots in $\mathfrak{sl}_3$ are α, β, and $(\alpha+\beta)$ with

$$\langle \alpha, \alpha \rangle \;=\; 2 \qquad \langle \beta, \beta \rangle \;=\; 2 \qquad \langle \alpha, \beta \rangle = -1$$

In other words, the two simple roots have the same length and have an angle of $2\pi/3$ between them. The pairs of distinct positive roots are (α, β), $(\alpha, \alpha+\beta)$ and $(\beta, \alpha+\beta)$, so the sum we must compute is

$$\frac{\langle \alpha, \beta \rangle}{\alpha\,\beta} \;+\; \frac{\langle \alpha, \alpha+\beta \rangle}{\alpha\,(\alpha+\beta)} \;+\; \frac{\langle \beta, \alpha+\beta \rangle}{\beta\,(\alpha+\beta)}$$

Clearing denominators and evaluating the parings,

$$\langle \alpha, \beta \rangle \cdot (\alpha+\beta) \;+\; \langle \alpha, \alpha+\beta \rangle \cdot \beta \;+\; \langle \beta, \alpha+\beta \rangle \cdot \alpha \quad = \quad -(\alpha+\beta) \;+\; \beta \;+\; \alpha \quad = \quad 0$$

For $\mathfrak{sp}_2$, the simple roots have lengths 1 and $\sqrt{2}$ and have an angle of $3\pi/4$ between them:

$$\langle \alpha, \alpha \rangle \;=\; 1 \qquad \langle \beta, \beta \rangle \;=\; 2 \qquad \langle \alpha, \beta \rangle \;=\; -1$$

The other positive roots are $(\alpha+\beta)$ and $(2\alpha+\beta)$. The non-orthogonal pairs of distinct positive roots are (α, β), $(\alpha, 2\alpha+\beta)$, $(\beta, \alpha+\beta)$, and $(\alpha+\beta, 2\alpha+\beta)$. So the sum we must compute is

$$\frac{\langle \alpha, \beta \rangle}{\alpha\,\beta} \;+\; \frac{\langle \alpha, 2\alpha+\beta \rangle}{\alpha\,(2\alpha+\beta)} \;+\; \frac{\langle \beta, \alpha+\beta \rangle}{\beta\,(\alpha+\beta)} \;+\; \frac{\langle \alpha+\beta, 2\alpha+\beta \rangle}{(\alpha+\beta)(2\alpha+\beta)}$$

Again, clearing denominators,

$$\langle \alpha, \beta \rangle \cdot (\alpha+\beta)(2\alpha+\beta) \;+\; \langle \alpha, 2\alpha+\beta \rangle \cdot \beta(\alpha+\beta) \;+\; \langle \beta, \alpha+\beta \rangle \cdot \alpha(2\alpha+\beta) \;+\; \langle \alpha+\beta, 2\alpha+\beta \rangle \cdot \alpha\beta$$

and evaluating the pairings,

$$\begin{aligned} & -(\alpha+\beta)(2\alpha+\beta) \;+\; \beta(\alpha+\beta) \;+\; \alpha(2\alpha+\beta) \;+\; \alpha\beta \\ &= \;\; -(2\alpha^2+3\alpha\beta+\beta^2) \;+\; \alpha\beta+\beta^2 \;+\; 2\alpha^2+\alpha\beta \;+\; \alpha\beta \quad = \quad 0 \end{aligned}$$

Finally we consider the exceptional Lie algebra $\mathfrak{g}_2$. The simple roots have lengths 1 and $\sqrt{3}$ and have an angle of $5\pi/6$ between them:

$$\langle \alpha, \alpha \rangle = 1 \qquad \langle \beta, \beta \rangle \;=\; 3 \qquad \langle \alpha, \beta \rangle \;=\; -3/2$$

The other positive roots are $(\alpha+\beta)$, $(2\alpha+\beta)$, $(3\alpha+\beta)$, and $(3\alpha+2\beta)$. Notice that the roots α and $\alpha+\beta$ have the same length and have an angle of $3\pi/2$ between them. So together with their sum $2\alpha+\beta$, they form a copy of the $\mathfrak{sl}_3$ root system. The three terms corresponding to the three pairs of roots among these roots will cancel, as in the $\mathfrak{sl}_3$ case. Similarly, the roots $(3\alpha+\beta)$ and β have the same length and have an angle of $3\pi/2$ between them, so, together with their sum, $(3\alpha+2\beta)$ they form a copy of the $\mathfrak{sl}_3$ root system, and the three terms in the sum corresponding to the three pairs among these roots will also cancel. The remaining six pairs of distinct, non-orthogonal postitive roots are $(\alpha, 3\alpha+\beta)$, (α,β), $(3\alpha+\beta, 2\alpha+\beta)$, $(2\alpha+\beta, 3\alpha+2\beta)$, $(3\alpha+2\beta, \alpha+\beta)$, and $(\alpha+\beta,\beta)$. We shall see that the six terms corresponding to these pairs cancel as a group.

After clearing denominators, the relevant sum is

$$\begin{aligned}
&\langle\alpha,\beta\rangle\cdot(\alpha+\beta)(2\alpha+\beta)(3\alpha+\beta)(3\alpha+2\beta) \;+\; \langle\alpha,3\alpha+\beta\rangle\cdot\beta(\alpha+\beta)(2\alpha+\beta)(3\alpha+2\beta)\\
&\;+\;\; \langle 3\alpha+\beta,2\alpha+\beta\rangle\cdot\alpha\beta(\alpha+\beta)(3\alpha+2\beta)+\langle 2\alpha+\beta,3\alpha+2\beta\rangle\cdot\alpha\beta(\alpha+\beta)(3\alpha+\beta)\\
&\;+\;\; \langle 3\alpha+2\beta,\alpha+\beta\rangle\cdot\alpha\beta(2\alpha+\beta)(3\alpha+\beta)+\langle\alpha+\beta,\beta\rangle\cdot\alpha(2\alpha+\beta)(3\alpha+\beta)(3\alpha+2\beta)
\end{aligned}$$

Evaluating the pairings and factoring out $(3/2)$, this is

$$\begin{aligned}
&-(\alpha+\beta)(2\alpha+\beta)(3\alpha+\beta)(3\alpha+2\beta) \;+\; \beta(\alpha+\beta)(2\alpha+\beta)(3\alpha+2\beta)\\
&\;+\;\; \alpha\beta(\alpha+\beta)(3\alpha+2\beta) \;+\; \alpha\beta(\alpha+\beta)(3\alpha+\beta)\\
&\;+\;\; \alpha\beta(2\alpha+\beta)(3\alpha+\beta) \;+\; \alpha(2\alpha+\beta)(3\alpha+\beta)(3\alpha+2\beta)
\end{aligned}$$

Multiplying out,

$$\begin{array}{rcrcrcrcr}
-18\alpha^4 & - & 45\,\alpha^3\beta & - & 40\,\alpha^2\beta^2 & - & 15\,\alpha\beta^3 & - & 2\beta^4\\
 & + & 6\,\alpha^3\beta & + & 13\,\alpha^2\beta^2 & + & 9\,\alpha\beta^3 & + & 2\beta^4\\
 & + & 3\,\alpha^3\beta & + & 5\,\alpha^2\beta^2 & + & 2\,\alpha\beta^3 & & \\
 & + & 3\,\alpha^3\beta & + & 4\,\alpha^2\beta^2 & + & \alpha\beta^3 & & \\
 & + & 6\,\alpha^3\beta & + & 5\,\alpha^2\beta^2 & + & \alpha\beta^3 & & \\
+18\alpha^4 & + & 27\,\alpha^3\beta & + & 13\,\alpha^2\beta^2 & + & 2\,\alpha\beta^3 & &
\end{array}$$

And this sum is zero. $\square$

Now we treat the case of arbitrary rank by reducing to the rank two case.

Proposition 6.3.2. *For any complex simple Lie algebra $\mathfrak{g}$, the following sum over all pairs (β, γ) of distinct positive roots is zero:*

$$\sum_{\beta \neq \gamma} \frac{\langle \beta, \gamma \rangle}{\beta\, \gamma} \;=\; 0$$

Proof. Let I be the indexing set $\{(\beta, \gamma)\}$ of pairs of distinct, non-orthogonal positive roots. For each $(\beta, \gamma) \in I$, let $\mathcal{R}_{\beta,\gamma}$ be the two-dimensional root system generated by β and γ. For such a root system $\mathcal{R}$, let $I_{\mathcal{R}}$ be the set of pairs of distict, non-orthogonal positive roots, where positivity is inherited from the ambient $\mathfrak{g}$. The collection J of all such $I_{\mathcal{R}}$ is a cover of I. We refine J to a subcover J' of disjoint sets, in the following way.

For any pair $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$ of sets in J with non-empty intersection, there is a two-dimensional root system $\mathcal{R}''$ such that $I_{\mathcal{R}''}$ contains $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$. Indeed, letting (β, γ) and (β', γ') be pairs in I generating $\mathcal{R}$ and $\mathcal{R}'$ respectively, the non-empty intersection of $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$ implies that there is a pair (β'', γ'') lying in both $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$. Since $\mathcal{R}$ and $\mathcal{R}'$ are two-dimensional and β'' and γ'' are linearly independent, all six roots lie in a plane. Since all six roots lie in the root system for $\mathfrak{g}$, they generate a two-dimensional root system $\mathcal{R}''$ containing $\mathcal{R}$ and $\mathcal{R}'$, and $I_{\mathcal{R}''} \supset I_{\mathcal{R}}, I_{\mathcal{R}'}$.

Thus we refine J to a subcover J': if $I_{\mathcal{R}}$ in J intersects any $I_{\mathcal{R}'}$ in J, replace $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$ with the set $I_{\mathcal{R}''}$ described above.

The sets $I_{\mathcal{R}}$ in J' are mutually disjoint, and, for any $(\beta, \gamma) \in I$, there is a root system $\mathcal{R}$ such that $(\beta, \gamma) \in I_{\mathcal{R}} \in J'$, thus

$$\sum_{(\beta,\gamma) \in I} \frac{\langle \beta, \gamma \rangle}{\beta\, \gamma} \;=\; \sum_{I_{\mathcal{R}} \in J'} \; \sum_{(\beta,\gamma) \in I_{\mathcal{R}}} \frac{\langle \beta, \gamma \rangle}{\beta\, \gamma}$$

By the classification of complex simple Lie algebras of rank two, $\mathcal{R}$ is isomorphic to the root system of $\mathfrak{sl}_3$, $\mathfrak{sp}_2$, or $\mathfrak{g}_2$. Thus, by Proposition 6.3.1, the inner sum over $I_{\mathcal{R}}$ is zero, proving that the whole sum is zero.

Note that the refinement *is* necessary, as there are copies of $\mathfrak{sl}_3$ inside $\mathfrak{g}_2$.

Note that the only time the root system of $\mathfrak{g}_2$ appears is in the case of $\mathfrak{g}_2$ itself, since, by the classification, $\mathfrak{g}_2$ is the only root system containing roots that have an angle of $\pi/6$ or $5\pi/6$ between them.

$\square$

By Lemma 6.3.1 and Remark 6.3.1, this completes the proof of the theorem. $\square$

Chapter 7

Lattice point application

7.1 Introduction

As a sample application, not obviously related to subconvexity, we discuss lattice points in symmetric spaces.

Certainly the simplest non-trivial lattice-point counting problem is the *Gauss circle problem*, counting lattice points within a circle in the Euclidean plane. Elementary packing arguments yield

$$N(T) \;=\; \#\{\xi \in \mathbb{Z}^2 : |\xi| \le T\} \;=\; 2\pi \cdot T^2 \;+\; \mathrm{O}(T)$$

The error term has been improved, in the early 20th century by Voronoi [85], Sierpiński [81], and van der Corput [13, 14, 15], and more recently by Iwaniec and Mozzochi [56], and Huxley [55]. The optimal error term, conjectured to be $\mathrm{O}(T^{1/2+\varepsilon})$, is still undetermined.

In the hyperbolic plane, where the circumference of a circle is proportionate to its area, packing arguments fail to produce an asymptotic with error term. Subtler methods have produced asymptotics for lattice-point counting in hyperbolic spaces; for a sampling, see [74, 67, 9]. In affine symmetric spaces, ergodic methods have been used to produce asymptotics for lattice point counting; see the papers by Bartels [4], Duke, Rudnick, and Sarnak [25], Eskin and McMullen [27], and Maucourant [66]. The book of Gorodnik and Nevo gives a good exposition of ergodic methods and lattice-point counting [47].

In contrast, we use spectral methods, which allow treatment of symmetric spaces of higher rank, to produce an *exact formula* relating the number of lattice points in an expanding region and the automorphic spectrum. Instead of expressing something mysterious in terms of something familiar, this formula, like the explicit formula of Riemann-von Mangoldt, which relates the prime numbers to the zeros of zeta, gives a relationship between two mysterious things; its appeal lies not in its utility for evaluation of one side, but rather in the fact that it reveals a connection between seemingly disparate things.

For a sketch of this discussion, in the case of $SL_2(\mathbb{C})/SU(2)$, see Garrett's Newark talk [33].

7.2 Spectral identity: expressions for the automorphic fundamental solution for $(\Delta - \lambda_z)^\nu$

Let G be a complex semi-simple Lie group with finite center and K a maximal compact subgroup. Let $G = NAK$, $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ be corresponding Iwasawa decompositions. Let Σ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, let Σ^+ denote the subset of positive roots (for the ordering corresponding to $\mathfrak{n}$), and let $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$, m_α denoting the multiplicity of α. Let $\mathfrak{a}^*_{\mathbb{C}}$ denote the set of complex-valued linear functions on $\mathfrak{a}$.

Let Γ be a arithmetic subgroup. Let Φ_ξ be a spectral basis for $L^2(\Gamma\backslash G)$ in the sense of 3.1.

Consider the differential equation on the arithmetic quotient $X = \Gamma\backslash G/K$:

$$(\Delta - \lambda_z)^\nu \, v_z \;\; = \;\; \delta_{x_o}$$

where the Laplacian Δ is the image of the Casimir operator for $\mathfrak{g}$, $\lambda_z = (z^2 - \langle \rho, \rho\rangle)$ for a complex parameter z, ν is an integral power, and $\delta_{x_o} = \delta_{\Gamma\cdot 1\cdot K}$ is Dirac delta at the basepoint in $\Gamma\backslash G/K$. Global automorphic Sobolev theory justifies the use of the harmonic analysis of automorphic forms to produce a solution, as follows.

Proposition 7.2.1. *For integral $\nu > (\dim X)/2$, v_z is a continuous right-K-invariant function on $\Gamma\backslash G$ with the following integral representation:*

$$v_z(g) \quad = \quad \int_\Xi \frac{\overline{\Phi}_\xi(x_o)}{(\lambda_\xi - \lambda_z)^\nu} \ \Phi_\xi(g)\, d\xi$$

Proof. Since δ_{x_o} is a compactly supported distribution of order zero, by Proposition 3.4.4, it lies in the global automorphic Sobolev spaces $H^{-\ell}(X)$ for all $\ell > (\dim X)/2$. Thus there is an element v_z of $H^{-\ell+2\nu}(X)$ satisfying this equation. The solution v_z is unique in Sobolev spaces, since any w_z satisfying

$$(\Delta - \lambda_z)^\nu \ w_z \quad = \quad \delta_{x_o}$$

must necessarily have the same spectral transform. Indeed,

$$(\lambda_\xi - \lambda_z)^{-\nu} \cdot \mathcal{F} v_z \quad = \quad \mathcal{F}\,(\Delta - \lambda_z)^\nu \ w_z \quad = \quad \mathcal{F}\,\delta_{x_o} \quad = \quad \overline{\Phi}_\xi(x_o)$$

For $\nu > (\dim X)/2$, the solution is continuous, by Corollary 3.4.1, and by Proposition 3.4.3,

$$v_z(g) \quad = \quad \int_\Xi \mathcal{F}\,v_z(\xi)\,\Phi_\xi(g)\,d\xi \quad = \quad \int_\Xi \frac{\overline{\Phi}_\xi(x_o)}{(\lambda_\xi - \lambda_z)^\nu} \ \Phi_\xi(g)\,d\xi$$

□

Let u_z denote the solution to the differential equation on the free space $X = G/K$:

$$(\Delta - \lambda_z)^\nu \ u_z \quad = \quad \delta_{1\cdot K}$$

as in Chapter 6. Recall (see Theorem 6.2.1), when G is of odd rank and $\nu = (n+1)/2+d$, where d is the number of positive roots, counted without multiplicities, and n is the rank,

$$u_z(a) \quad = \quad C_G \ \cdot \ \prod_{\alpha\in\Sigma^+} \frac{\alpha(\log a)}{2\sinh(\frac{\alpha(\log a)}{2})} \ \cdot \ \frac{e^{-z|\log a|}}{z}$$

where C_G is an explicit constant depending on the group G. When G is of even rank and $\nu = (n/2) + d + 1$,

$$u_z(a) \quad = \quad C_G \ \cdot \ \prod_{\alpha\in\Sigma^+} \frac{\alpha(\log a)}{2\sinh(\frac{\alpha(\log a)}{2})} \ \cdot \ \frac{|\log a|}{z} \ \cdot \ K_1(z\,|\log a|)$$

where K_1 is the usual Bessel function.

Proposition 7.2.2. *For* $\mathrm{Re}(z) \gg 1$*, the Poincaré series*

$$\mathrm{Pé}_z(g) \;=\; \sum_{\gamma\in\Gamma} u_z(\gamma\cdot g)$$

converges absolutely and uniformly on compacts to a continuous function on $\Gamma\backslash G/K$*. Moreover, it is of moderate growth, and it is square-integrable modulo* Γ*.*

Proof. By Propositions 7.5.2 and 7.5.3, it suffices to show that the free-space fundamental solution u_z is of sufficient rapid decay. Let $\|\cdot\|$ be the gauge on G with $\sigma_o > 0$ such that

$$\int_G \frac{1}{\|g\|^\sigma}\, dg \;<\; \infty \qquad (\text{for } \sigma > \sigma_o)$$

Note that the product over positive simple roots is bounded, so it suffices to show that there is a $\sigma > \sigma_o$ such that

$$|\log a|\cdot e^{-\mathrm{Re}(z)|\log a|} \;\ll\; \|a\|^{-\sigma}$$

We claim that $|\log_A(a)|$ and $\log\big(\|a\|\big)$ are comparable. On diagonal matrices (a_i), the gauge is

$$\|(a_i)\| \;=\; \max_{1\le i\le n}\{a_i, a_i^{-1}\}$$

Taking logarithms,

$$\log\big(\|(a_i)\|\big) \;=\; \max_{1\le i\le n}\{|\log a_i|\} \qquad (\ell^\infty\text{-norm on } \mathfrak{a})$$

On the other hand,

$$|\log_A a| \;=\; \Big(\sum_{i=1}^n (\log a_i)^2\Big)^{1/2} \qquad (\ell^2\text{-norm on } \mathfrak{a})$$

The usual comparison:

$$\frac{1}{\sqrt{n}}\cdot \max_{1\le i\le n}\{|\log a_i|\} \;\le\; \Big(\sum_{i=1}^n (\log a_i)^2\Big)^{1/2} \;\le\; \max_{1\le i\le n}\{|\log a_i|\}$$

allows us to conclude that u_z is of sufficient rapid decay, as follows:

$$|u_z(a)| \;\ll\; |\log a|\cdot e^{-\mathrm{Re}(z)|\log a|} \;=\; \frac{|\log a|}{\big(e^{|\log a|}\big)^{\mathrm{Re}(z)}} \;\ll\; \frac{\log\big(\|a\|\big)}{\|a\|^{\mathrm{Re}(z)}} \;\ll\; \frac{1}{\|a\|^{\mathrm{Re}(z)-1}}$$

□

Remark 7.2.1. For the case $G = SL_2(\mathbb{C})$,

$$\text{Pé}_z(1) \;=\; \sum_{\gamma\in\Gamma} \frac{r_\gamma\, e^{-(2z-1)r_\gamma}}{(2z-1)\,\sinh r_\gamma}$$

where r_γ is the Cartan radius of γ. Thus $e^{-(2z-1)r_\gamma}$ is the $-(2z-1)^{\text{th}}$ power of the length of the arc from the basepoint $x_o = 1\cdot K$ to its image $\gamma\cdot x_o$. On the quotient, this arc becomes a closed geodesic, and the sum over Γ would essentially be the Selberg zeta function associated to Γ.

Theorem 7.2.1 (Spectral identity)**.** *For* $\text{Re}(z) \gg 1$*,*

$$\text{Pé}_z(g) \;=\; \int_\Xi \frac{\overline{\Phi}_\xi(x_o)\cdot\Phi_\xi(g)}{(\lambda_\xi-\lambda_z)^\nu}\; d\xi \qquad (\textit{uniformly pointwise})$$

where u_z is the free-space fundamental solution in Theorem 6.2.1. In particular, when G is of odd rank,

$$\sum_{\gamma\in\Gamma} C_G \cdot \prod_{\alpha\in\Sigma^+} \frac{\alpha(H(\gamma\cdot g))}{2\sinh(\frac{\alpha(H(\gamma\cdot g))}{2})} \cdot \frac{e^{-z|H(\gamma\cdot g)|}}{z} \;=\; \int_\Xi \frac{\overline{\Phi}_\xi(x_o)\cdot\Phi_\xi(g)}{(\lambda_\xi-\lambda_z)^\nu}\; d\xi$$

and when G is of even rank,

$$\sum_{\gamma\in\Gamma} C_G\cdot \prod_{\alpha\in\Sigma^+} \frac{\alpha(H(\gamma\cdot g))}{2\sinh(\frac{\alpha(H(\gamma\cdot g))}{2})} \cdot \frac{|H(\gamma\cdot g)|}{z} \cdot K_1(z\,|H(\gamma\cdot g)|) \;=\; \int_\Xi \frac{\overline{\Phi}_\xi(x_o)\cdot\Phi_\xi(g)}{(\lambda_\xi-\lambda_z)^\nu}\; d\xi$$

where $H(g)$ is defined by $g = k\cdot exp(H(g))\cdot k'$, for $k, k' \in K$.

Proof. By Proposition 7.2.2, the Poincaré series Pé_z is an automorphic fundamental solution for $(\Delta-\lambda_z)^\nu$ in $L^2(\Gamma\backslash G/K) = H^0(\Gamma\backslash G/K)$. By the uniqueness of solutions in Sobolev spaces, $\text{Pé}_z = v_z$ in an L^2-sense. But both functions are continuous, so by the uniqueness of continuous functions in an L^2-equivalence class, $\text{Pé}_z = v_z$ in $C^0(\Gamma\backslash G/K)$. $\square$

We refer to the sum over Γ as the *geometric expression* of the automorphic fundamental solution and the integral over Ξ as the *spectral expression* of the same.

To extract tangible information from this spectral identity we use a variant on the classical Perron method.

7.3 Perron integral operators

Perron integral operators extract tangible information from identities, e.g. from the identity of the Euler product and the Hadamard product of the Riemann zeta function a Perron integral operator produces the Riemann-von Mangoldt explicit formula relating the prime numbers to the zeros of zeta.

The key idea, though just a heuristic, is that the Heaviside step function can be expressed as a certain contour integral:

$$\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \frac{e^{sx}}{s}\,ds \overset{\text{heuristically}}{=} H(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases} \qquad (\mathrm{Re}(s)>0)$$

though this cannot possibly be quite true as stated, since the contour integral is not absolutely convergent. More precisely, the *finite* contour integrals satisfy

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{e^{sx}}{s}\,ds = \begin{cases} 1+O_\sigma\left(\frac{e^{\sigma x}}{T\cdot x}\right) & \text{if } x>0 \\ O_\sigma\left(\frac{e^{\sigma x}}{T\cdot |x|}\right) & \text{if } x<0 \end{cases}$$

so, in particular, taking the limit as $T\to\infty$ does give the Heaviside step function.

The following variant has the virtue of absolute convergence.

Lemma 7.3.1.

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{e^{sX}}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)}\,ds = \begin{cases} \frac{(1-e^{-\theta X})^\ell}{\ell!\,\theta^\ell}+O_\sigma\left(\frac{e^{\sigma X}}{T^{\ell+1}\cdot X}\right) & \textit{if } X>0 \\ O_\sigma\left(\frac{e^{\sigma X}}{T^{\ell+1}\cdot |X|}\right) & \textit{if } X<0 \end{cases}$$

Proof. First consider the case $X>0$. Let B be a large positive number, certainly greater than $\ell\theta$, and let $\mathcal{R}$ be the rectangle with vertices $\sigma\pm iT$, $-B\pm iT$. By the residue formula, the integral over $\mathcal{R}$ is equal to the sum of the residues at the simple poles $s=-n\theta$, $0\le n\le\ell$, which are given by

$$\begin{aligned}\operatorname*{Res}_{s=-n\theta} \frac{e^{sX}}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)} &= \frac{e^{-n\theta X}}{(-n\theta)((1-n)\theta)\dots(-\theta)(\theta)\dots((\ell-n)\theta)} \\ &= \frac{\left(-e^{-\theta X}\right)^n}{n!\cdot(\ell-n)!\cdot\theta^\ell}\end{aligned}$$

So the integral over $\mathcal{R}$ is equal to the sum

$$\sum_{n=0}^{\ell} \frac{(-e^{-\theta X})^n}{n! \cdot (\ell-n)! \cdot \theta^\ell} \quad = \quad \frac{1}{\ell!\,\theta^\ell} \sum_{n=0}^{\ell} \binom{\ell}{n} \big(- e^{-\theta X}\big)^n \quad = \quad \frac{\big(1-e^{-\theta X}\big)^\ell}{\ell!\,\theta^\ell}$$

On the other hand, the integral over $\mathcal{R}$ is given by the sum of the original integral, over the right side of the rectangle, with the auxilliary integrals, over the top, bottom, and left sides of the rectangle. We estimate the auxilliary integrals, starting with the integral over the left side of the rectangle.

$$\left| \int_{-B+iT}^{-B-iT} \frac{e^{sX}\,ds}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)} \right| \quad \leq \quad \frac{2T\cdot e^{-BX}}{B(B-\theta)(B-2\theta)\dots(B-\ell\theta)}$$

Now consider the integrals over the top and bottom of the rectangle.

$$\pm \int_{-B\pm iT}^{\sigma\pm iT} \frac{e^{sX}\,ds}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)} \quad = \quad \pm e^{\sigma X} \int_{-B\pm iT}^{\sigma\pm iT} \frac{e^{(s-\sigma)X}\,ds}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)}$$

We estimate the magnitude:

$$\left| \int_{-B\pm iT}^{\sigma\pm iT} \frac{e^{sX}\,ds}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)} \right| \quad \leq \quad \frac{e^{\sigma X}}{T^{\ell+1}} \int_0^B e^{-uX}\,du \quad = \quad \frac{e^{\sigma X}\,(1-e^{-BX})}{T^{\ell+1}\cdot X}$$

The original integral is equal to the sum of residues of poles inside $\mathcal{R}$ minus the auxilliary integrals, which we have estimated. As $B\to\infty$, the integral along the left side of the rectangle vanishes, and for T large, the term coming from the sum of residues is of higher order than the terms coming from the integrals over the top and bottom of the rectangle. In particular,

$$\frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \frac{e^{sX}}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)}\,ds \quad = \quad \frac{\big(1-e^{-\theta X}\big)^\ell}{\ell!\,\theta^\ell} \;+\; O_\sigma\!\left(\frac{e^{\sigma X}}{T^{\ell+1}\cdot X}\right)$$

The proof of the $X<0$ case is very similar. Notice that now the integrand has exponential decay as $\mathrm{Re}(s)\to\infty$, so we will move the contour to the *right*, where there are no poles. Let B be a large positive number, certainly greater than σ, and $\mathcal{R}$ be the rectangle with vertices $\sigma\pm iT$, $B\pm iT$. The integral over $\mathcal{R}$ is zero, since the integrand is holomorphic for $\mathrm{Re}(s)>0$. The estimates for the auxilliary integrals are very similar to the previous case. The integral over the right side of the rectangle can be estimated by

$$\left| \int_{B+iT}^{B-iT} \frac{e^{sX}\, ds}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)} \right| \;\leq\; 2T \cdot \frac{e^{-B|X|}}{B(B+\theta)(B+2\theta)\dots(B+\ell\theta)}$$

and the integrals over the top and bottom of the rectangle can be estimated by

$$\left| \int_{B\pm iT}^{\sigma\pm iT} \frac{e^{sX}\, ds}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)} \right| \;\leq\; \frac{e^{\sigma X}}{T^{\ell+1}} \int_0^B e^{-u|X|}\, du \;=\; \frac{e^{\sigma X}\,(1-e^{-B|X|})}{T^{\ell+1}\cdot|X|}$$

The original integral, over the left side of the rectangle, is the negative of the sum of the auxilliary integrals. As $B \to \infty$ the integral over the right side of the rectangle vanishes, so the original integral is of the same order as the integrals over the top and bottom integrals, i.e.

$$\frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \frac{e^{sX}}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)}\, ds \;=\; O_\sigma\!\left(\frac{e^{\sigma X}}{T^{\ell+1}\cdot|X|} \right)$$

□

7.4 Explicit formula for lattice-point counting

For simplicity, assume G has odd rank and $\Gamma\backslash G$ is compact.

Theorem 7.4.1. *For a complex semi-simple Lie group G of odd rank, with maximal compact K and a co-compact lattice Γ, the number of lattice points within an expanding region of the basepoint $x_o = 1\cdot K$ is related to the automorphic spectrum by the following explicit formula:*

$$\begin{aligned} &\tilde{C}_G \cdot \sum_{\gamma:\,|\log a_\gamma|<X} \; \prod_{\alpha\in\Sigma^+} \frac{\alpha(\log a_\gamma)}{2\sinh\left(\frac{\alpha(\log a_\gamma)}{2}\right)} \cdot \frac{1}{\ell!\,\theta^\ell}\left(1 - e^{-\theta(X-|\log a_\gamma|)}\right)^\ell \\ &= \left(A_{|\rho|}^{\ell,\theta}(X)\cdot e^{|\rho|X} + B_{|\rho|}^{\ell,\theta}(X)\cdot e^{-|\rho|X}\right)\cdot|\Phi_1(x_o)|^2 \\ &\qquad + \sum_{\Xi-\{\Phi_1\}} |\Phi_\xi(x_o)|^2 \cdot \left(A_{z_\xi}^{\ell,\theta}(X)\, e^{z_\xi X} + B_{z_\xi}^{\ell,\theta}\, e^{-z_\xi X} + \mathrm{Per}_{z_\xi}^{\ell,\theta}(X)\right) \end{aligned}$$

where $\tilde{C}_G$ is an explicit constant depending only on the group, Φ_1 is the constant automorphic form, z_ξ is given by $\lambda_\xi = (z_\xi^2 - |\rho|^2)$, $A_{z_\xi}^{\ell,\theta}(X)$ and $B_{z_\xi}^{\ell,\theta}(X)$ are polynomial in X, of degree $(\nu-1)$, and rational in z_ξ, and $\mathrm{Per}_{z_\xi}^{\ell,\theta}(X)$ is of exponential decay in X and rational in z_ξ.

Proof. The compactness of $\Gamma\backslash G$ implies that the spectrum Ξ is discrete, and for G of odd rank, the spectral identity of Theorem 7.2.1, evaluated at the basepoint, becomes

$$C_G \cdot \sum_{\gamma\in\Gamma} \prod_{\alpha\in\Sigma^+} \frac{\alpha(\log a_\gamma)}{2\sinh\left(\frac{\alpha(\log a_\gamma)}{2}\right)} \cdot \frac{e^{-z\,|\log a_\gamma|}}{z} \quad = \quad \sum_{\xi\in\Xi} \frac{|\Phi_\xi(x_o)|^2}{(\lambda_\xi - \lambda_z)^\nu}$$

where C_G is an explicit constant (see Theorem 6.2.1) depending only on the group G, and a_γ is given by $\gamma = k \cdot a_\gamma \cdot k'$.

We apply a Perron integral transform

$$P_{\ell,\theta}(f)(X) \quad = \quad \frac{1}{2\pi i} \int_{\sigma+i\mathbb{R}} f(z) \cdot \frac{z \cdot e^{zX}}{z(z+\theta)(z+2\theta)\dots(z+\ell\theta)}\, dz$$

to both sides of the identity.

On the geometric side,

$$P_{\ell,\theta}\left(\frac{e^{z\,|\log a_\gamma|}}{z}\right) \quad = \quad \int_{\sigma+i\mathbb{R}} \frac{e^{z(X-|\log a_\gamma|)}}{z(z+\theta)(z+2\theta)\dots(z+\ell\theta)}\, dz$$

$$= \quad \begin{cases} (1 - e^{-\theta(X-|\log a_\gamma|)})^\ell/(\ell!\,\theta^\ell) & \text{if } X \;>\; |\log a_\gamma| \\ 0 & \text{if } X \;<\; |\log a_\gamma| \end{cases}$$

by Lemma 7.3.1. Thus

$$P_{\ell,\theta}\big(v_z(x_o)\big) \quad = \quad C_G \cdot \sum_{\gamma:\,|\log a_\gamma|<X} \prod_{\alpha\in\Sigma^+} \frac{\alpha(\log a_\gamma)}{2\sinh\left(\frac{\alpha(\log a_\gamma)}{2}\right)} \cdot \frac{1}{\ell!\,\theta^\ell}\big(1 \;-\; e^{-\theta(X-|\log a_\gamma|)}\big)^\ell$$

On the spectral side, write $\lambda_\xi = (z_\xi^2 - |\rho|^2)$. Then $\lambda_\xi - \lambda_z = -(z-z_\xi)(z+z_\xi)$, and

$$P_{\ell,\theta}\left(\frac{|\Phi_\xi(x_o)|^2}{(\lambda_\xi - \lambda_z)^\nu}\right) \quad = \quad \frac{1}{2\pi i} \int_{\sigma+i\mathbb{R}} \frac{(-1)^\nu\,|\Phi_\xi(x_o)|^2}{(z-z_\xi)^\nu(z+z_\xi)^\nu} \cdot \frac{e^{zX}}{(z+\theta)(z+2\theta)\dots(z+\ell\theta)}\, dz$$

Move $(-1)^\nu|\Phi_\xi(x_o)|^2$ outside the integral, and evaluate by residues. The residues of the poles at the spectrum $z = \pm z_\xi$ are

$$A_{z_\xi}^{\ell,\theta}(X) \quad = \quad \frac{1}{(\nu-1)!} \lim_{z\to z_\xi} \frac{\partial^{\nu-1}}{\partial z^{\nu-1}} \left((z+z_\xi)^{-\nu}\; \frac{e^{zX}}{(z+\theta)\dots(z+\ell\theta)} \right)$$

$$B_{z_\xi}^{\ell,\theta}(X) \;=\; \frac{1}{(\nu-1)!} \lim_{z\to -z_\xi} \frac{\partial^{\nu-1}}{\partial z^{\nu-1}} \left((z-z_\xi)^{-\nu} \frac{e^{zX}}{(z+\theta)\ldots(z+\ell\theta)} \right)$$

Visibly, these are polynomial in X and rational in z_ξ. When $\Phi_\xi = \Phi_1$ is the constant automorphic form, $\lambda_\xi = 0 \Rightarrow z_\xi = \pm|\rho|$. The sum of the residues of the simple poles at $z = -m\theta$ is:

$$\mathrm{Per}_{z_\xi}^{\ell,\theta}(X) \;=\; \frac{1}{\theta^{\ell-1}} \sum_{m=1}^{\ell} \frac{(-1)^{m-1}\, e^{-m\theta X}}{(m-1)!\,(\ell-m)!\,(z_\xi^2 - m^2\theta^2)^\nu}$$

Thus,

$$\begin{aligned} P_{\ell,\theta}\big(\text{Pé}_z(x_o)\big) \;&=\; \big(A_{|\rho|}^{\ell,\theta}(X)\cdot e^{|\rho|X} \;+\; B_{|\rho|}^{\ell,\theta}(X)\cdot e^{-|\rho|X}\big)\cdot|\Phi_1(x_o)|^2 \\ &\quad + \sum_{\Xi-\{\Phi_1\}} |\Phi_\xi(x_o)|^2\cdot\big(A_{z_\xi}^{\ell,\theta}(X)\, e^{z_\xi X} \;+\; B_{z_\xi}^{\ell,\theta}\, e^{-z_\xi X} \;+\; \mathrm{Per}_{z_\xi}^{\ell,\theta}(X)\big) \end{aligned}$$

Since $v_z(x_o) = \text{Pé}_z(x_o)$, by Theorem 7.2.1, we have the desired equality with $\tilde{C}_G = (-1)^\nu C_G$. $\square$

7.5 Gauges on groups

We recall some general facts about gauges on groups and convergence of Poincaré series. See Wallach's book [86] or Appendix 1 of [16].

For a countably-based, locally compact Hausdorff, unimodular group G with compact subgroup K, a *gauge* $g \to \|g\|$ is a continuous positive real-valued function on G such that:

1. $\|e\| = 1$, $\|g\| \ge 1$, and $\|g^{-1}\| = \|g\|$

2. *Submultiplicativity*: $\|gh\| \le \|g\|\cdot\|h\|$

3. *K-invariance*: $\|k\cdot g\| = \|g\| = \|g\cdot k\|$

4. *Integrability*: for some $\sigma_o > 0$,

$$\int_G \frac{1}{\|g\|^\sigma}\, dg \;<\; \infty \qquad (\text{for } \sigma > \sigma_o)$$

General reductive groups have gauges, and on GL_n they admit a particularly simple description, in terms of the operator norm:

$$\|g\| \;=\; \max\left(|\,g\,|_{\mathrm{op}},\, |g^{-1}|_{\mathrm{op}}\right) \qquad \text{where} \qquad |\,g\,|_{\mathrm{op}} \;=\; \sup_{|x|\le 1}\, |g\cdot x|$$

Proposition 7.5.1 (Summability)**.** *For a discrete subgroup $\Gamma \subset G$,*

$$\sum_{\gamma\in\Gamma} \frac{1}{\|\gamma\|^\sigma} \;<\; \infty \qquad (\textit{for } \sigma > \sigma_o)$$

where σ_o is the power describing the integrability of the gauge.

Proof. Let U be a small open neighborhood of $e \in G$ such that $U \cap \Gamma = \{e\}$. Then

$$\int_U \frac{dg}{\|g\|^\sigma} \cdot \sum_{\gamma\in\Gamma} \frac{1}{\|\gamma\|^\sigma} \;=\; \int_U \sum_{\gamma\in\Gamma} \frac{dg}{\|g\|^\sigma \cdot \|\gamma\|^\sigma} \;\le\; \int_U \sum_{\gamma\in\Gamma} \frac{dg}{\|\gamma\cdot g\|^\sigma}$$

Interchange sum and integral, and change variables, $g \to \gamma^{-1}g$,

$$\int_U \sum_{\gamma\in\Gamma} \frac{dg}{\|\gamma\cdot g\|^\sigma} \;=\; \sum_{\gamma\in\Gamma} \int_{\gamma^{-1}\cdot U} \frac{dg}{\|g\|^\sigma} \;\le\; \int_G \frac{dg}{\|g\|^\sigma} \;<\; \infty$$

□

If convergent, *Poincaré series* associated to a function f on G is

$$\mathrm{Pé}_f(g) \;=\; \sum_{\gamma\in\Gamma} f(\gamma\cdot g)$$

Proposition 7.5.2. *If there is $\sigma > \sigma_o$, such that*

$$|f(g)| \;\ll\; \|g\|^{-\sigma}$$

Then the Poincaré series associated to f converges absolutely and uniformly on compact sets. Moreover,

$$|\mathrm{Pé}_f(g)| \;\ll\; \|g\|^\sigma \qquad (\textit{for all } \sigma > \sigma_o)$$

Proof. The translate by γ satisfies:

$$|f(\gamma\cdot g)| \;\ll\; \frac{1}{\|\gamma\cdot g\|^\sigma} \;\le\; \frac{\|g\|^\sigma}{\|\gamma\|^\sigma}$$

since

$$\|\gamma\| \;=\; \|\gamma \cdot gg^{-1}\| \;\leq \|\gamma \cdot g\| \cdot \|g\|$$

Thus,

$$\sum_{\gamma \in \Gamma} |f(\gamma \cdot g)| \;\ll\; \sum_{\gamma \in \Gamma} \frac{\|g\|^\sigma}{\|\gamma\|^\sigma} \;\ll\; \|g\|^\sigma$$

□

Proposition 7.5.3. *If there is a $\sigma > \sigma_o$ such that*

$$|f(g)| \;\ll\; \|g\|^{-2\sigma}$$

then the Poincaré series associated to f is square integrable modulo Γ, i.e.

$$\int_{\Gamma\backslash G} |\mathrm{Pé}_f(g)|^2 \, dg \;<\; \infty$$

Proof. Unwinding,

$$\int_{\Gamma\backslash G} |\mathrm{Pé}(g)|^2 \, dg \;=\; \int_{\Gamma\backslash G} \mathrm{Pé}(g) \cdot \overline{\mathrm{Pé}}(g) \, dg$$

$$= \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma} f(\gamma \cdot g) \;\; \overline{\mathrm{Pé}}(g) \, dg \;=\; \int_G f(g) \;\; \overline{\mathrm{Pé}}(g) \, dg$$

Since $|f(g)| \ll \|g\|^{-2\sigma}$, by hypothesis, and $|\overline{\mathrm{Pé}}(g)| \ll \|g\|^\sigma$, by the previous proposition,

$$\int_{\Gamma\backslash G} |\mathrm{Pé}(g)|^2 \, dg \;\ll\; \int_G \frac{dg}{\|g\|^\sigma} \;<\; \infty$$

□

Chapter 8

Generating identities involving moments of L-functions

We return to the adelic viewpoint to discuss applications to L-functions. The adelic Poincaré series is easily obtainable from the Poincaré series constructed in Proposition 7.2.2, since the adelic Poincaré series is essentially trivial at finite places.

8.1 Spectral identities involving moments of GL_2 L-functions

We discuss the Diaconu-Garrett-Goldfeld prescription for spectral identities involving moments, which was used to obtain subconvex bounds for GL_2 L-functions over an arbitrary number field in the t-aspect [16, 17, 18]) and in the depth aspect [63]. In particular, integrating a Poincaré series against the norm-square of a cusp form f yields a sum of moments of L-functions:

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé} \cdot |f|^2 \;=\; \sum_{\chi \in \widehat{C}_0} \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} L(1-s+v, f \otimes \bar{\chi}) \cdot L(s, \bar{f} \otimes \chi) \cdot \mathcal{K}_\infty(s, \chi_0, \chi)\, ds$$

where $\mathcal{K}_\infty$ is a weight function. We sketch the proof. Start by unwinding the Poincaré series, then using the Fourier-Whittaker expansion of f and Mellin inversion for $\bar{f}$:

$$\begin{aligned}
\int_{Z_\mathbb{A} G_k \backslash G_\mathbb{A}} \text{Pé} \cdot |f|^2 &= \int_{Z_\mathbb{A} M_k \backslash G_\mathbb{A}} \varphi \cdot f \cdot \bar{f} \\
&= \int_{Z_\mathbb{A} \backslash G_\mathbb{A}} \varphi \cdot W_f \cdot \bar{f} \\
&= \frac{1}{2\pi i} \int_{Z_\mathbb{A} \backslash G_\mathbb{A}} \varphi \cdot W_f \cdot \int_{\sigma + i\mathbb{R}} \left(\int_{M_k \backslash M_\mathbb{A}} \bar{f} \cdot \bar{\chi}_s \right) \chi_s \, ds \\
&= \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} \int_{Z_\mathbb{A} \backslash G_\mathbb{A}} \varphi \cdot W_f \cdot \left(\int_{Z_k \backslash M_\mathbb{A}} W_{\bar{f}} \cdot \bar{\chi}_s \right) \chi_s \, ds
\end{aligned}$$

Factoring this integral and using Iwasawa coordinates, this becomes

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} \prod_v \Bigg(\int_{Z_v \backslash M_v} W_f(m) \chi_s(m) \int_{Z_v \backslash M_v} W_{\bar{f}}(m') \bar{\chi}_s(m') \\
\cdot \int_{N_v} \varphi(mn) \psi(mnm^{-1}) \psi(m'nm'^{-1}) dn \, dm' dm \Bigg) \, ds
\end{aligned}$$

The data φ for the Poincaré series is a deformation at an archimedean place of a character on M, so for m, m' in the supports of the Whittaker functions the inner integral is constant, and equal to 1 at all finite primes. Then the local integral is the product of local L-factors.

On the other hand, the automorphic spectral decomposition of the Poincaré series gives an alternate expression.

$$\begin{aligned}
\text{Pé} = & \left(\int_{N_\infty} \varphi_\infty \right) \cdot E_{v+1} \\
& + \sum_F \left(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \overline{W}_{F,\infty} \right) \cdot L(v + \tfrac{1}{2}, \overline{F}) \cdot F \\
& + \sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i \kappa} \int_{\mathrm{Re}(s) = \frac{1}{2}} \left(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot W^E_{1-s,\overline{\chi},\infty} \right) \\
& \qquad \cdot \frac{L(v + 1 - s, \overline{\chi}) L(v + s, \chi)}{L(2 - 2s, \overline{\chi}^2)} |\mathfrak{d}|^{-v+s-1/2} \cdot E_{s,\chi} \, ds
\end{aligned}$$

where $\{F\}$ is an orthonormal basis of everywhere locally spherical cusp forms and $\mathfrak{d}$ is the idele with ν^{th} component $\mathfrak{d}_\nu$ at finite places and component 1 at archimedean places. This gives the meromorphic continuation of the Poincaré series.

Diaconu and Garrett also described the procedure outlined above, for obtaining a spectral identity involving moments of L-functions, in more structural terms, enabling application to L-functions on higher rank groups, as we discuss in the following section.

8.2 General prescription for moment identities

For a reductive group G defined over a number field k and H a k-subgroup of G containing the center Z of G, consider the two chains of subgroups in $G \times G$,

$$\begin{array}{ccccc} H^{\Delta} & \subset & H \times H & \subset & G \times G \\ H^{\Delta} & \subset & G^{\Delta} & \subset & G \times G \end{array}$$

where the superscript Δ indicates the diagonal copy. Diagrammatically,

$$\begin{tikzcd} & G \times G & \\ H \times H \arrow[ur] & & G^{\Delta} \arrow[ul] \\ & H^{\Delta} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Consider the distribution u on $G \times G$ given by

$$u(f_1 \times f_2) \quad = \quad \int_{H^{\Delta}} f_1(h) \cdot f_2(h)\, dh$$

In specific examples, this may be either an integral along $H_{\mathbb{A}}$ or along $H_k \backslash H_{\mathbb{A}}$. Precomposing u with restriction from $G \times G$ to $H \times H$ yields the same distribution as precomposing with restriction from $G \times G$ to G^{Δ}:

$$u \circ \mathrm{Res}^{G \times G}_{H \times H} \quad = \quad u \circ \mathrm{Res}^{G \times G}_{G^{\Delta}}$$

Given an abstract spectral decomposition for (suitable) functions and distributions on H and G, we decompose. Again, we use the notation in 3.1. For $\Phi^H_{\xi_1}, \Phi^H_{\xi_2}$ in a spectral family, parametrized by Ξ_H, which would be an orthonormal basis if the decomposition were discrete, but in general must include continuous spectrum, we compute the spectral coefficient of $u \circ \mathrm{Res}^{G\times G}_{H\times H}$:

$$\langle\, u \circ \mathrm{Res}^{G\times G}_{H\times H}\,,\;\; \Phi^H_{\xi_1} \times \Phi^H_{\xi_2}\,\rangle_{H\times H} \;\;=\;\; \int_H \bar{\Phi}^H_{\xi_1}(h)\,\bar{\Phi}^H_{\xi_2}(h)\,dh \;\;=\;\; \begin{cases} 1 & \text{if } \Phi^H_{\xi_2} = \bar{\Phi}^H_{\xi_1} \\ 0 & \text{else} \end{cases}$$

yielding the spectral decomposition

$$u \circ \mathrm{Res}^{G\times G}_{H\times H} \;\;=\;\; \int_{\Xi_H} \;\Phi^H_\xi \otimes \bar{\Phi}^H_\xi$$

On the other hand, for Φ^G_ξ in a spectral family parametrized by Ξ_G,

$$\langle\, u \circ \mathrm{Res}^{G\times G}_{G^\Delta},\;\; \Phi^G_\xi\,\rangle_{G^\Delta} \;\;=\;\; \int_H \Phi^G_\xi(h)\,dh \;\;=\;\; (\Phi^G_\xi)_H$$

where the subscript H denotes the period over $H_k\backslash H_{\mathbb{A}}$. Thus we also have the spectral expansion

$$u \circ \mathrm{Res}^{G\times G}_{G^\Delta} \;\;=\;\; \int_{\Xi_G} \;(\Phi^G_\xi)_H \cdot \Phi^G_\xi$$

Applying both to $f \otimes \check{f}$ yields the identity:

$$\int_{\Xi_H} \;|\langle \Phi^H_\xi, f\rangle_H|^2 \;\;=\;\; \int_{\Xi_G} \;(\Phi^G_\xi)_H \cdot \langle \Phi^G_\xi, |f|^2\rangle_G$$

When $H \subset G$ is an Euler-Gelfand pair, the integrals $\langle \Phi^H_\xi, f\rangle_H$ often have Euler products. By contrast, the right hand side produces integrals of three eigenfunctions. The *positivity* of the left hand side is important for applications.

Diagrammatically this is,

$$\begin{tikzcd}
\int_{\Xi_H} |\langle \Phi^H_\xi, f\rangle_H|^2 & & f \otimes \check{f} \arrow[ll] \arrow[rr] & & \int_{\Xi_G} (\Phi^G_\xi)_H \cdot \langle \Phi^G_\xi, |f|^2\rangle_G \\
& & G \times G & & \\
\int_{\Xi_H} \Phi^H_\xi \otimes \bar{\Phi}^H_\xi \arrow[uu] & H \times H \arrow[ur] & & G^\Delta \arrow[ul] & \int_{\Xi_G} (\Phi^G_\xi)_H \cdot \Phi^G_\xi \arrow[uu] \\
& & H^\Delta \arrow[ul] \arrow[ur] & & \\
& & u \sim 1 \arrow[uull] \arrow[uurr] & &
\end{tikzcd}$$

For the case $GL_1 \subset GL_2$,

$$\langle \Phi_\xi^H, f \rangle_H \;\;=\;\; \int_{k^\times \backslash \mathbf{J}} \chi(y)\, |y|^s\, \bar{f}\begin{pmatrix} y & \\ & 1 \end{pmatrix} \frac{dy}{y} \;\;=\;\; \Lambda(f, s, \chi)$$

and the left hand side of the identity is a sum/integral moment.

This spectral identity is too harsh for applications, because the exponential decay of archimedean factors will mask sublter contributions. We *deform* the spectral identity in the following sense. Let v_o be an archimedean place and let Ω be the Casimir element on G_{v_o}. For $\lambda \in \mathbb{C}$, the solution β_λ to the differential equation on $H_{v_o} \backslash G_{v_o}$

$$(\Omega - \lambda)^\nu \, \beta_\lambda \;\;=\;\; u$$

is a smoothed deformation of the distribution u. Let φ_λ be the product of local factors

$$\varphi_{\lambda,v} \;\;=\;\; \begin{cases} \beta_\lambda & \text{if } v = v_o \\ \mathrm{char}_{H_v \cdot K_v} & \text{else} \end{cases}$$

Form a Poincaré series

$$\mathrm{Pé}_\lambda(g) \;\;=\;\; \sum_{H_k \backslash G_k} \varphi_\lambda(\gamma \cdot g)$$

Integrating the Poincaré series against $|f|^2$,

$$\langle \mathrm{Pé}, |f|^2 \rangle_G \;\;=\;\; \int_{Z_\mathbb{A} G_k \backslash G_\mathbb{A}} \mathrm{Pé}_\lambda(g) \cdot |f(g)|^2 \, dg \;\;=\;\; \int_{Z_\mathbb{A} H_k \backslash G_\mathbb{A}} \varphi_\lambda(g) \cdot f(g) \cdot \check{f}(g) \, dg$$

By construction φ_λ is left H-invariant, so

$$\int_{Z_\mathbb{A} H_k \backslash G_\mathbb{A}} \varphi_\lambda(g) \cdot f(g) \cdot \check{f}(g) \, dg \;\;=\;\; \int_{H_\mathbb{A} \backslash G_\mathbb{A}} \varphi_\lambda(g) \;\; \int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} f(hg) \cdot \check{f}(hg) \, dh \;\, dg$$

Expand $f(hg) = R_g f(h)$ along $H_k \backslash H_\mathbb{A}$:

$$f(hg) \;\;=\;\; \int_{\Xi_H} F(h) \cdot \left(\int_{H_k \backslash H_\mathbb{A}} f(\eta g) \cdot \bar{\Phi}_\xi^H(\eta) \, d\eta \right)$$

Substituting in yields

$$\int_{\Xi_H} \int_{H_\mathbb{A} \backslash G_\mathbb{A}} \varphi_\lambda(g) \left(\int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} f(\eta g) \cdot \bar{\Phi}_\xi^H(\eta) \, d\eta \right) \left(\int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} \Phi_\xi^H(h) \cdot \check{f}(hg) \, dh \right) dg$$

$$= \int_{\Xi_H} \int_{H_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi_\lambda(g) \left| \langle R_g f, \Phi_\xi^H \rangle_H \right|^2 dg$$

Because φ_λ is significantly deformed only at the single archimedean place v_o, in the integral over $H_A \backslash G_{\mathbb{A}}$ the adele group element $g = \{g_v\}$ can be taken in H_v except at the single place v_o. Thus,

$$\langle \text{Pé}, |f|^2 \rangle_G = \int_{\Xi_H} \int_{H_{v_o} \backslash G_{v_o}} \beta_\lambda(g) \left| \langle R_g f, \Phi_\xi^H \rangle_H \right|^2 dg$$

Using the spectral expansion of the Poincaré series,

$$\text{Pé} = \int_{\Xi_G} \frac{(\Phi_\xi^G)_H}{(\lambda_\xi - \lambda)^\nu} \cdot \Phi_\xi^G$$

we obtain an alternate expression for the integral against $|f|^2$,

$$\langle \text{Pé}, |f|^2 \rangle_G = \int_{\Xi_G} \frac{(\Phi_\xi^G)_H}{(\lambda_\xi - \lambda)^\nu} \cdot \langle \Phi_\xi^G, |f|^2 \rangle_G$$

$$= \int_{\Xi_G} \frac{(\Phi_\xi^G)_H}{(\lambda_\xi - \lambda)^\nu} \cdot \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \Phi_\xi^G(g) \cdot |f(g)|^2 \, dg$$

Equating the two expressions yields the deformed spectral identity:

$$\int_{\Xi_H} \int_{H_{v_o} \backslash G_{v_o}} \beta_\lambda(g) \left| \langle R_g f, \Phi_\xi^H \rangle_H \right|^2 dg = \int_{\Xi_G} \frac{(\Phi_\xi^G)_H}{(\lambda_\xi - \lambda)^\nu} \cdot \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \Phi_\xi^G(g) \cdot |f(g)|^2 \, dg$$

When $H \subset G$ is an Euler-Gelfand pair, L-functions often arise, since the spectral components often have Euler products.

Specializing to the case $GL_1 \subset GL_2$ yields the spectral identity involving second moments of GL_2 automorphic L-functions over an arbitrary number field discussed above, from which Diaconu and Garrett extracted a subconvex bound in the t-aspect.

Diaconu and Garrett have also obtained spectral identites for classical doubling integrals and for arbitrary $GL_{n-1} \times GL_n$ Rankin-Selberg L-functions by specializing to $Sp_{2n} \times Sp_{2n} \subset Sp_{4n}$, for example, and $GL_{n-1} \subset GL_n$, respectively. However, these spectral identities do *not* yield subconvex bounds, in the doubling case because the spectral family is too *long*: the bound obtained after averaging over the spectral family fails to match convexity, and in the Rankin-Selberg case due to *conductor dropping.*

However, the case $GL_n^\Delta \subset GL_n \times GL_n$, over a *totally complex* number field, which yields moments of $GL_n \times GL_n$ Rankin-Selberg L-functions, *avoids* both of these obstacles. Further, the relative simplicity of the harmonic analysis of complex semi-simple Lie groups allows explicit computations, making the dependence on parameters less inaccessible.

8.3 Identities involving moments of $GL_n \times GL_n$ Rankin-Selberg L-functions

To produce moments of $GL_n \times GL_n$ L-functions, consider the (harsh) version of the spectral identity

$$\int_{\Xi_H} |\langle \Phi_\xi^H, f\rangle_H|^2 \;=\; \int_{\Xi_G} (\Phi_\xi^G)_H \cdot \langle \Phi_\xi^G, |f|^2\rangle_G$$

when $G = GL_n \times GL_n$ and $H = GL_n^\Delta$. In this case, Φ_ξ^H is a GL_n automorphic form, and $f = f_1 \times f_2$ for f_1, f_2 cusp forms on GL_n. Thus

$$\langle \Phi_\xi^H, f\rangle_H \;=\; \int_{GL_n(k)\backslash GL_n(\mathbb{A})} \Phi_\xi^{GL_n}(g)\, \bar{f}_1(g)\, \bar{f}_2(g)\, dg$$

When $n = 2$ and $\Phi_\xi^{GL_n}$ is an Eisenstein series, this is the $GL_2 \times GL_2$ Rankin-Selberg L-function. For $n > 2$, the Eisenstein series appearing in spectral decompositions are those with cuspidal data, whereas the Eisenstein series necessary for the $GL_n \times GL_n$ Rankin-Selberg moment identity are degenerate, occuring as residues of cuspidal data Eisenstein series. For the right hand side, we compute the period integral $(\Phi_\xi^G)_H$, where $\Phi_\xi^G = \Phi_{\xi_1}^{GL_n} \times \Phi_{\xi_2}^{GL_n}$ on $GL_n \times GL_n$

$$(\Phi_\xi^G)_H \;=\; \int_{GL_n(k)\backslash GL_n(\mathbb{A})} \Phi_{\xi_1}^{GL_n}(g) \cdot \Phi_{\xi_2}^{GL_n}(g)\, dg \;=\; \begin{cases} 1 & \text{if } \Phi_{\xi_2}^{GL_n} = \bar{\Phi}_{\xi_1}^{GL_n} \\ 0 & \text{else} \end{cases}$$

Thus the right hand side is

$$\int_{\Xi_{GL_n}} \left(\int_{GL_n(k)\backslash GL_n(\mathbb{A})} \Phi_\xi^{GL_n}(g) \cdot |f_1(g)|^2\, dg \right) \cdot \left(\int_{GL_n(k)\backslash GL_n(\mathbb{A})} \bar{\Phi}_\xi^{GL_n}(g) \cdot |f_2(g)|^2\, dg \right)$$

While this harsh spectral identity clearly produces moments of L-functions, we will need the more delicate, *deformed* spectral identity, for applications. Integrate the

Poincaré series on $GL_n \times GL_n$:

$$\text{Pé}(x_1, x_2) \;=\; \sum_{\gamma \in \; GL_n(k)^\Delta \backslash GL_n(k) \times GL_n(k)} \varphi(\gamma \cdot (x_1, x_2))$$

against $|f_1 \times f_2|^2$, for f_1, f_2 cusp forms on GL_n, to obtain

$$\int_{\Xi_{GL_n}} \int_{GL_n(\mathbb{C})^\Delta \backslash (GL_n(\mathbb{C}) \times GL_n(\mathbb{C}))} \beta_\lambda(x_1, x_2) \cdot \left|\langle\, R_{x_1} f_1 \;\cdot\; R_{x_2} f_2, \;\; \Phi_\xi^{GL_n} \,\rangle_{GL_n}\right|^2 dx_1\, dx_1$$

$$= \;\; \int_{\Xi_{GL_n}} \frac{1}{(\lambda_{\xi \times \bar{\xi}} - \lambda)^\nu} \;\cdot\; \langle \Phi_\xi^{GL_n}, |f_1|^2 \rangle_{GL_n} \;\cdot\; \langle \bar{\Phi}_\xi^{GL_n}, |f_2|^2 \rangle_{GL_n}$$

The double integral on the left hand side is approximately an integral moment of a $GL_n \times GL_n$ Rankin-Selberg L-function.

Note that a Poincaré series on $GL_n \times GL_n$ is obtained from a one-variable Poincaré series (essentially the one constructed in Proposition 7.2.2) on GL_n, as follows.

Let k be a totally complex number field. We obtain a one-variable Poincaré series on $G = GL_n(\mathbb{A}_k)$ by winding-up

$$\text{Pé}(g) \;=\; \sum_{\gamma \in G_k} \varphi(\gamma \cdot g)$$

where the data φ is a deformation of the Dirac delta function in the following sense. For the moment assume that there is a unique archimedean place, v_o. Consider the differential equation

$$(\Omega - \lambda_z)^\nu \, u_z \;=\; \delta_{1 \cdot K}$$

on the symmetric space G_{v_o}/K_{v_o}, where λ_z is a complex parameter, Ω is the Casimir operator for G_{v_o} and $\delta_{1 \cdot K}$ is the Dirac delta function at the base point. Define φ_z to be the product of local factors $\varphi_{z,v}$, given by

$$\varphi_{z,v} \;=\; \begin{cases} u_z & \text{if } v = v_o \\ \text{char}_{K_v} & \text{if } v \neq v_o \end{cases}$$

The two-variable function $\text{Pé}(x^{-1}y)$ is a left-GL_n^Δ-invariant function on $GL_n \times GL_n$ suitable for producing the desired moment identities.

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