Real Root Counting for Parametric Polynomial Systems and Applications

A dissertation
submitted to the faculty of the graduate school
of the University of Minnesota
by

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In partial fulfillment of the requirements
for the degree of
doctor of philosophy

Richard Moeckel, Advisor

April 2011

Acknowledgments

Above all I am very grateful to my advisor, Professor Richard Moeckel, for all his great help during my studies in Minnesota. I thank him for spending time discussing problems with me and giving me many insightful ideas of solving them. I thank him for sharing his Mathematica notebooks. Also I thank him for reading my papers and his patient corrections of my writings. Finally, I thank him for being always supportive and encouraging so that I gain my confidence and correct attitude toward mathematical researches.

I am also thankful to Professors Tryphon Georgiou, Richard McGehee, and Joel Roberts for showing interests in my work and willing to be in my final exam committee. I thank them for reading and correcting my dissertation and spending their precious time attending my thesis final exam.

I would like to thank Professor Joel Roberts for helping me to learn computational algebraic geometry. Also I thank the Institute for Mathematics and its Applications for the chances to participate in their annual program of applications of algebraic geometry and attend a summer school learning those subjects.

Finally, I want to thank my friends, Nai-Chia Chen, Jeonghun Lee, Hsi-Wei Shih, and Yi Zhang for good discussions with me to clarify many basic mathematical concepts and theorems. Also, I thank my friends, Chong Luo and Yi Wang, for their help in using computers for my computations. I want to give special thanks to my parents, Ke-Shee Tsai and Shu-Lin Tsai Chen, and my family. I would never ever have had the chance to carry out my dream studying here without their total support and love.

Abstract

Polynomial systems appear in many different fields of study. Many important problems can be reduced to solving systems of polynomial equations and usually the coefficients involve parameters. This thesis is devoted to finding practical ways to solve such problems from two fields, the studies of central configurations from the Newtonian N-body problem and Maxwell's conjecture about the electric potential created by point charges.

Central configurations play an important role in the study of celestial mechanics. They determine some special solutions of the Newton's laws of motion and lead to explicit expression of the solutions. After some changes of the coordinates, we can describe the central configurations as zeros of a system of polynomials, where the coefficients of each polynomial are polynomials in the masses. Therefore, the problem of counting central configurations becomes counting the positive zeros of parametric polynomial systems.

A problem studied by James C. Maxwell back in the 19th century is about finding an upper bound of the number of nondegenerate equilibrium points of the electric potential created by point charges. In the case of 3 point charges, he conjectured that there are at most 4 such equilibrium points. After given proper coordinates, the problem also becomes to count positive zeros of a parametric polynomial system. In Chapter 1, we will introduce these two problems and derive some parametric polynomial systems for which we will count positive zeros in Chapter 4. Some open questions from these two fields of studies will be given in Chapter 5.

Our methods of counting positive zeros are based on classic tools such as resultants, subresultant sequences, and Hermite quadratic forms. Recently developed tools like Groebner bases make it possible to let computers perform symbolic computations of polynomials and count zeros by applying classic results. A computer algebra system (CAS), for example Mathematica, is the software to do such compu-

tations. In Chapter 2, we present those tools and demonstrate how to count zeros for polynomial systems with real or complex coefficients in a CAS.

When it comes to counting zeros of parametric polynomial systems, we want to count zeros of all the real polynomial systems obtained by substituting real numbers for parameters. For example, when there is one parameter, we may want to know the numbers of positive zeros for real polynomial systems obtained by substituting parameters in an open interval (a, b). When there are two parameters, we may want to count positive zeros for all real polynomial systems obtained by substituting parameters with real pairs in an open region in $\mathbb{R}^2$. Our main contributions in this thesis are finding methods to achieve that goal based on standard computer algebra tools and applying these methods to some enumeration problems of central configurations and some special cases of Maxwell's conjecture. We will outline our methods and develop sufficient tools in Chapter 3.

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Chapter 1

Introduction

A parametric polynomial system is a finite set of equations $f_1 = 0, \cdots, f_m = 0$, where $f_i \in \mathbb{D}[x_1, \ldots, x_n]$ and $\mathbb{D} = \mathbb{R}[a_1, \ldots, a_l]$. That is, the coefficients of each polynomial f_i are real polynomials in $a_1, \ldots, a_l$. We call the x_j's the variables and the a_k's the parameters. Each l-tuple $(a_1, \ldots, a_l) \in \mathbb{R}^l$ gives a polynomial system in $\mathbb{R}[x_1, \ldots, x_n]$. In this chapter, we introduce two different fields of study and show how they are reduced to solving parametric polynomial systems.

1.1 Newtonian N-body problem

1.1.1 Central configurations

The Newtonian N-body problem is the study of the dynamics of N point particles with masses $m_i > 0$ and positions $q_i \in \mathbb{R}^d$, moving according to a second order differential equation, called the Newton's laws of motion:

$$m_j \ddot{q}_j = \sum_{i \neq j} \frac{m_i m_j (q_i - q_j)}{r_{ij}^3}, \quad 1 \leq j \leq N, \tag{1.1}$$

where $r_{ij} = |q_i - q_j|$.

Definition 1.1. *A configuration $q = (q_1, q_2, \cdots, q_N)$ in $\mathbb{R}^{dN} \setminus \triangle$ is called a central configuration if there exists $\lambda > 0$ such that*

$$\lambda(q_j - c) + \sum_{i \neq j} \frac{m_i (q_i - q_j)}{r_{ij}^3} = 0, \quad 1 \leq j \leq N, \tag{1.2}$$

where $c = \frac{1}{M}(m_1 q_1 + \cdots + m_N q_N), M = m_1 + \cdots + m_N,$ *and* $\triangle = \{q_i = q_j, i \neq j\}$.

Central configurations determine some special solutions of (1.1). Given a initial position vector q in $\mathbb{R}^{dN} \backslash \triangle$ satisfying (1.2) and zero initial velocity, the N particles in $\mathbb{R}^d$ accelerate toward c in such a way that the configuration collapses homothetically. When $d = 2$, if we have initial positions satisfying (1.2) and proper initial velocities, each particle will traverse an elliptical orbit around c as in the Kepler problem and the configuration remain similar to the initial configuration throughout the motion, varying only in size. If the initial velocities are just large enough, the orbits will be circular and the particles will rotate uniformly around c like a rigid body [28]. One could introduce a uniformly rotating coordinate system in which such orbits appear fixed. Therefore, we call the corresponding rigid motions determined by central configurations in $\mathbb{R}^2$ relative equilibria.

Finding all the central configurations and their properties is a challenging problem. Smale listed 18 mathematical problems for the 21 century [22] [23]. The 6-th problem is to prove the finiteness of the number of relative equilibria. Since (1.2) is invariant under the action of $\mathbf{O}(d)$ on q by $Aq = (Aq_1, \cdots, Aq_N)$, we have an equivalence relation under this action. Counting the number of central configurations means counting the number of different classes of them. Smale's 6-th problem is to prove that, for any choice of positive masses $m_1, \ldots, m_N$, the number of different classes of q in $\mathbb{R}^{2N} \setminus \triangle$ satisfying (1.2) is finite.

In the 3-body problem, there are 4 relative equilibria. Three of them are collinear configurations studied by Euler and the other is the equilateral triangle found by Lagrange. These are the only possible relative equilibria of 3 bodies. For 4 bodies, the finiteness of relative equilibria has recently been proved by Hampton and Moeckel [20]. Smale's 6-th problem is still open for $N \geq 5$. It is worthwhile to mention here that relative equilibria exist for all choices of the masses of N particles [29].

Even in the 4 bodies case, there are still many interesting open problems. We list two of them to show the lack of knowledge for solutions satisfying the equations (1.2). One is to show that, for any ordering of 4 positive masses, there exists an unique convex central configuration with that order. The existence proof can be found in Xia's paper [30] and the uniqueness is still open. The second is the problem of finding the limiting central configurations of 4 bodies when 3 masses approach zero. It is proved that such central configurations have the body of large mass at the center of a circle which passes through the bodies of zero mass [25]. When the small masses approach zero with the same ratio, it is known that there are three limiting central configurations [7]. However, there are no results for different ratios.

From the point of view of the dimension of the configuration, we know that N bodies form a configuration that spans an affine subspace of dimension at most $(N-1)$. It is proved that for N bodies with positive masses, there is only one $(N-1)$-dimensional central configuration [17]. It is when the N bodies form the regular simplex. However, the $(N-2)$-dimensional central configuration are far from understood. They are called the Dziobek configurations. In this thesis, we will focus on enumeration problems for this kind of central configurations. Also in the problems that we discuss here, we assume some of the masses are zero. We call them restricted N-body problems.

1.1.2 Enumerations of Dziobek configurations

Here we derive three sets of parametric polynomial systems where roots will be counted in Chapter 4.

A restricted 4-body problem

First, we consider a restricted 4 body problem and try to count the number of Dziobek configurations. Let 4 particles with positions $q_1, \cdots, q_4$ have masses $m_1, \cdots, m_4$. Suppose that $m_4 = 0$, then the 3 particles with positive masses also form a central configuration by themselves according to equations (1.2) and they form an equilateral triangle by [17]. Assuming the total mass $M = 1$, without loss of generality, we impose a symmetry condition on the masses by letting $m_1 = m_2 = k, m_4 = 1 - 2k$ for $k \in (0, \frac{1}{2})$. Also we can fix q_1, q_2, q_3 in $\mathbb{R}^2$ such that the length of the sides of the equilateral triangle is 1. These choices imply that the constant λ in equations (1.2) is equal to 1. Then the central configuration equation for the zero mass can be written as $\partial_x \Phi = \partial_y \Phi = 0$, where

$$\Phi = \frac{1}{2}\|q_4 - c\|^2 + \frac{k}{\|q_4 - q_1\|} + \frac{k}{\|q_4 - q_2\|} + \frac{1-2k}{\|q_4 - q_3\|}$$

This means the position of the zero mass forms a central configuration if and only if it is the critical point of Φ. Using the identity,

$$\|q_4 - c\|^2 - \sum_{i=1}^{3} m_i \|q_4 - q_i\|^2 = \|c\|^2 - \sum_{i=1}^{3} m_i \|q_i\|^2,$$

we can change the coordinate of q_4 to $r_i = \|q_4 - q_i\|$ for $i = 1, 2, 3$ and Φ becomes

$$\Phi = \frac{1}{2}(kr_1^2 + kr_2^2 + (1-2k)r_3^2) + \frac{k}{r_1} + \frac{k}{r_2} + \frac{1-2k}{r_3} + C$$

for some constant C depends only on k. With new coordinates, we impose a restriction. That is the restriction of the mutual distances $r_{i,j}$ of 4 points in $\mathbb{R}^2$ given by the Cayley-Menger determinant below [21].

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{1,2}^2 & r_{1,3}^2 & r_{1,4}^2 \\ 1 & r_{2,1}^2 & 0 & r_{2,3}^2 & r_{2,4}^2 \\ 1 & r_{3,1}^2 & r_{3,2}^2 & 0 & r_{3,4}^2 \\ 1 & r_{4,1}^2 & r_{4,2}^2 & r_{4,3}^2 & 0 \end{vmatrix} = 0.$$

Therefore, r_1, r_2, r_3 have to satisfy the equations below.

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & r_1^2 \\ 1 & 1 & 0 & 1 & r_2^2 \\ 1 & 1 & 1 & 0 & r_3^2 \\ 1 & r_1^2 & r_2^2 & r_3^2 & 0 \end{vmatrix} = F = 0.$$

Now we have to find the critical points of Φ restricted to the surface $F = 0$. Using the Lagrange multiplier technique, we find they are zeros of the following equations for a Lagrange multiplier ω.

$$\begin{aligned} \partial_{r_i}\Phi &= \omega \partial_{r_i} F, \ \forall \ i = 1, 2, 3, \\ F &= 0. \end{aligned}$$

Eliminating ω and clearing the denominators, we get polynomial equations. To simplify the problem, we only deal with the case when $r_1 = r_2 = x$ and $r_3 = y$. Then we get two polynomial equations with two variables and one parameter k.

$$\begin{aligned} f_1 &= (1-2k)x^3 - (1-2k)x^5 + (1-2k)x^3y^2 - ky^3 \\ &\quad - 2kx^2y^3 - (1-3k)x^3y^3 + x^5y^3 + 2ky^5 - x^3y^5, \end{aligned} \tag{1.3}$$

$$f_2 = -1 + 2x^2 - x^4 + y^2 + 2x^2y^2 - y^4. \tag{1.4}$$

Here we are trying to count the number of relative equilibria positions of zero mass on a line going through a vertex and the center of an equilateral triangle when the two particles not on that vertex have equal masses. See the lower picture of Figure 5.1. This problem was solved in [27]. In Chapter 4, we give a new proof of it.

Theorem 1.1. *The polynomial system $f_1 = f_2 = 0$ has 2 positive zeros for $0 < k < \alpha$, 4 positive zeros for $\alpha < k < \frac{1}{2}$, and 3 positive zeros when $k = \alpha$, where α is only known approximately $0.288276\ldots$. Therefore, in the restricted 4-body problem with two equal masses k and the third mass $1-2k$ forming a equilateral triangle, there are only $2, 3$, or 4 Dziobek configurations for the 4-th zero mass sitting on the line through the $1-2k$ mass and the center of the triangle.*

A restricted 5-body problem

Using the same technique, we can derive another polynomial system from a similar restricted 5-body problem. The polynomials are given below.

$$\begin{aligned} f_3 &= (1-3k)x^3 - (1-3k)x^5 + (1-3k)x^3y^2 - ky^3 \\ &\quad - 3kx^2y^3 - (1-4k)x^3y^3 + x^5y^3 + 3ky^5 - x^3y^5, \end{aligned} \tag{1.5}$$

$$f_4 = 3 - 6x^2 + 3x^4 - 2y^2 - 6x^2y^2 + 3y^4. \tag{1.6}$$

This time we are trying to count the number of central configuration positions of the zero mass on a line going through a vertex and the center of the tetrahedron when the three particles not on that vertex have equal masses. See the upper picture of Figure 5.1. While theorem 1.1 appears in previous literature, the following theorem is a new result.

Theorem 1.2. *The polynomial system $f_3 = f_4 = 0$ has 2 positive zeros for $0 < k < \alpha$, 4 positive zeros for $\alpha < k < \frac{1}{2}$, and 3 positive zeros when $k = \alpha$, where α is only known approximately $0.246659\ldots$. Therefore, in the restricted 5-body problem with*

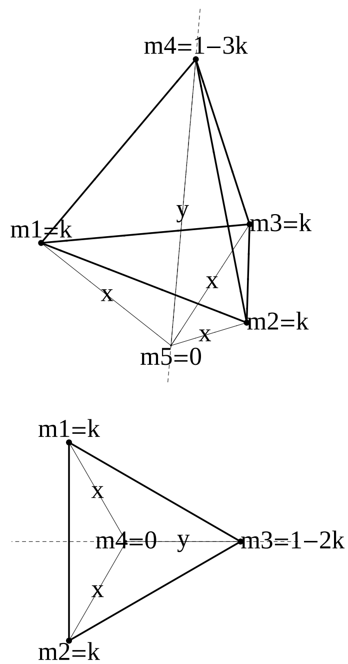

Figure 1.1: Restricted 4 and 5 body problems

three equal masses k and the fourth mass $1-3k$ forming a tetrahedron, there are only $2, 3,$ or 4 Dziobek configurations for the 5-th zero mass sitting on the line through the $1-3k$ mass and the center of the tetrahedron.

A restricted N-body problem

Now we consider a general case of the restricted N-body problem when one mass is zero and the rest are equal and try to count the Dziobek configurations for all N. Let $m_N = 0$. Then the $(N-1)$ bodies with positive equal masses form a regular simplex by [17]. We can again assume that it has unit length in each side. The number of Dziobek configurations of this N-body problem will be the number of the positions q_N of the m_N mass where together with the regular simplex they are a central configuration. We have the following symmetry for the position q_N [19].

Lemma 1.1. *Let p and q be the number of vertices of two complementary subsimplices of the regular simplex formed by the $(N-1)$ positive equal masses. Then the* *zero mass must be on the symmetry line which goes through the centers of a subsimplex with p vertices and its complementary subsimplex with q vertices. Moreover, let x, y be the two possible values of the mutual distances from the body with zero mass to each of the remaining bodies. Then the x, y satisfies $(x^2-y^2)(p+q-\frac{p}{x^3}-\frac{q}{y^3}) = \frac{1}{x^3}-\frac{1}{y^3}$.*

Factoring this equation and canceling the factor $(x-y)$ on both sides, we get the first equation of our polynomial system.

$$f_5 = qx^4 + py^4 + pxy^3 + qx^3y - (p+q)(x^4y^3 + x^3y^4) - x^2 - y^2 - xy = 0. \quad (1.7)$$

Since we use mutual distances $r_{i,j}$ as coordinates, we have again a restriction for N points in the space of dimension $N-2$ given by the Cayley-Menger determinant [21]. Since $m_1, \cdots, m_{N-1}$ forms a regular simplex with side 1, denoting $r_{N,j} = r_{j,N} = r_j$ for $j = 1, \ldots, N-1$, the Cayley-Menger determinant becomes

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & r_1^2 \\ 1 & 1 & 0 & \cdots & 1 & r_2^2 \\ . & . & . & 0 & . & . \\ 1 & 1 & 1 & \cdots & 0 & r_{N-1}^2 \\ 1 & r_1^2 & r_2^2 & \cdots & r_{N-1}^2 & 0 \end{vmatrix} = F(r_1, \cdots, r_{N-1}) = 0.$$

Lemma 1.2. $F = 0 \Leftrightarrow$

$$(N-2)(1 + \sum_{i=1}^{N-1} r_i^4) - 2\sum_{i=1}^{N-1} r_i^2 - 2\sum_{i,j=1}^{N-1} r_i^2 r_j^2 = 0.$$

Proof. By permuting rows and columns, it is easy to see that the polynomial F in $r_1, \cdots, r_{N-1}$ is symmetric. By expanding the determinant in the last row, we find that the cofactors consist of at most one column that is not $0's$ or $1's$. Therefore, we conclude that the polynomial is of the form $f = a + b(r_1^2 + \cdots + r_{N-1}^2) + c(r_1^2 r_2^2 + \cdots + r_{N-2}^2 r_{N-1}^2) + d(r_1^4 + \cdots + r_{N-1}^4)$..

It is well known that the determinant F of the Cayley-Menger matrix equals the square of the generalized volume of the $(N-1)$-dimensional simplex in $\mathbb{R}^{N-1}$ multiplied by a constant. Here the simplex is defined by a regular unit $(N-2)$-simplex in $\mathbb{R}^{N-2}$ and the N^{th} position whose distances to the vertices of the regular simplex are $r_1, \cdots, r_{N-1}$. Let p_1 be one of the vertices of the regular unit $(N-2)$-simplex, p_2

be the center of this simplex, and p_3 be the intersection point of the regular simplex and the line passing through p_1, p_2. Then we have $f(p_1) = f(p_2) = f(p_3) = 0$ since the $(N-1)$-dimensional volume is zero.

At p_1, $(r_1^2, \ldots, r_{N-1}^2) = (0, 1, \ldots, 1)$. So

$$\sum_{i=1}^{N-1} r_i^2 = \sum_{i=1}^{N-1} r_i^4 = N-2, \qquad \sum_{i,j=1}^{N-1} r_i^2 r_j^2 = \frac{(N-2)(N-3)}{2}.$$

At p_2, $(r_1^2, \ldots, r_{N-1}^2) = \frac{N-2}{2(N-1)}(1, \ldots, 1)$. So

$$\sum_{i=1}^{N-1} r_i^2 = \frac{N-2}{2}, \qquad \sum_{i=1}^{N-1} r_i^4 = \frac{(N-2)^2}{4(N-1)}, \qquad \sum_{i,j=1}^{N-1} r_i^2 r_j^2 = \frac{(N-2)^3}{8(N-1)}.$$

At p_3, $r_1^2 = \frac{N-1}{2(N-2)}, r_i^2 = \frac{N-3}{2(N-2)}$ for all $i \geq 2$. So

$$\sum_{i=1}^{N-1} r_i^2 = \frac{N^2-4N+5}{2(N-2)}, \qquad \sum_{i=1}^{N-1} r_i^4 = \frac{N^3-7N^2+19N-17}{4(N-2)^2},$$

and

$$\sum_{i,j=1}^{N-1} r_i^2 r_j^2 = \frac{(N-3)(N^2-4N+7)}{8(N-2)}.$$

Now, we have a linear system

$$\begin{aligned}
a_{11}b + a_{12}c + a_{13}d &= -\,a,\\
a_{21}b + a_{22}c + a_{23}d &= -\,a,\\
a_{31}b + a_{32}c + a_{33}d &= -\,a,
\end{aligned}$$

where $a_{i,j}$ are functions of N as given above. Computing the determinant of this system, we get $\frac{3-N}{16}$. Therefore, it is not zero for all $N > 3$. Let $a = N-2$. By Cramer's rule, we get an unique solution $(b, c, d) = (-2, -2, N-2)$. $\square$

In our situation, we have $r_1 = \cdots = r_p = x, r_{p+1} = \cdots = r_{N-1} = y$, then we get from lemma 1.2 the second equation of our polynomial system.

$$f_6 = (p+q-1) - 2px^2 - 2qy^2 - 2pqx^2y^2 + pqx^4 + pqy^4 = 0. \tag{1.8}$$

The polynomial system $f_5 = f_6 = 0$ has two variables x, y and two parameters

p, q. Note that $p + q + 1 = N$. Therefore, we will count the common positive zeros of it for all parameter values with $p \leq q \in \mathbb{N}$. Each positive zero gives a Dziobek configuration position of the zero mass m_{N+1} on the symmetry line which goes through the centers of a subsimplex with p vertices and its complementary subsimplex with q vertices. And this position is not in the center of the regular simplex since we cancel $(x - y)$ when forming f_5.

Theorem 1.3. *The polynomial system $f_5 = f_6 = 0$ has 3 positive zeros when $p = 1, q > 1$, 4 positive zeros when $p = 2, q = 2, 3$, and 2 positive zeros when $p = 2, q \geq 4$ and $p \geq 3, q \geq p$. Moreover, there are no zeros with $x = y$ for those parameters.*

Using this theorem, we give the exact count of the number of the Dziobek configurations. The case of N= 5 is given in the paper [18]. Also, the following corollary improves the result in [19] where only finiteness is proved.

Corollary 1.1. *The number of Dziobek configurations of the restricted N-body problem with equal masses is $25, 56$ for $N = 5, 6$, $2^{N-1} + N - 2$ for $N = 4$ or $N \geq 7$.*

Proof. We compute the number of the Dziobek configurations for $N = 4$ to 10 in the table below. Since we factored out $(x - y)$ when forming f_5 and the polynomial system $f_5 = f_6 = 0$ has no solution with $x = y$, we add 1 to our summations for the case when the zero mass is at the center of the regular simplex. For $N \geq 7$, we compute that the number equals to $3(N-1)+2(2^{N-2}-1-(N-1))+1 = 2^{N-1}+N-2$. □

$N = p + q + 1$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	Sum+1
4	$3C_1^3 = 9$				10
5	$3C_1^4 = 12$	$4(C_2^4/2) = 12$			25
6	$3C_1^5 = 15$	$4C_2^5 = 40$			56
7	$3C_1^6 = 18$	$2C_2^6 = 30$	$2(C_3^6/2) = 20$		69
8	$3C_1^7 = 21$	$2C_2^7 = 42$	$2C_3^7 = 70$		134
9	$3C_1^8 = 24$	$2C_2^8 = 56$	$2C_3^8 = 112$	$2(C_4^8/2) = 70$	263
10	$3C_1^9 = 27$	$2C_2^9 = 72$	$2C_3^9 = 168$	$2C_4^9 = 252$	520

Table 1.1: Some small N's

1.2 Point charge problems

Given μ positive numbers, $\eta_1, \ldots, \eta_\mu$, ν negative numbers, $\zeta_1, \ldots, \zeta_\nu$, and $\mu+\nu$ points, $p_1, \ldots, p_\mu, n_1, \ldots, n_\nu$, in $\mathbb{R}^3$, where η_i, ζ_j represent the magnitude of electrostatic charges at the points p_i and n_j, respectively, the real analytic function $x \to V(x)$ defined on $\mathbb{R}^3$ except for points $p_1, \ldots, n_\nu$ is called the electric potential created by those $\mu + \nu$ point charges, where $V(x)$ is given by

$$V = \frac{\eta_1}{r_1} + \cdots + \frac{\eta_\mu}{r_\mu} + \frac{\zeta_1}{\rho_1} + \cdots + \frac{\zeta_\mu}{\rho_\mu}, r_i = \|p_i - x\|, \rho_j = \|n_j - x\|, i = 1, \ldots, \mu, j = 1, \ldots, \nu. \tag{1.9}$$

A point of equilibrium is a critical point of V, that is x in $\mathbb{R}^3$ such that each of the first-order partial derivatives of V vanishes. A nondegenerate equilibrium points is a critical point with nonsingular Hessian matrix. In his book [15], J.C. Maxwell studied the number of nondegenerate equilibrium points in section 113 and claimed the following result.

Conjecture 1.1. *(Maxwell's conjecture) The total number of nondegenerate equilibrium points of the electric potential created by l charges of any configuration in $\mathbb{R}^3$ is at most $(l-1)^2$.*

This conjecture is still an open question for $l > 2$. In 2007, Gabrielov, Novikov and Shapiro published a paper [12] showing that when $l = 3$, the number of equilibrium is at most 12 for any configuration in $\mathbb{R}^2$. This is so far the best upper bound proved responding to Maxwell's conjecture when $l = 3$, although the conjectured upper bound is 4. In this thesis, we fix the configuration that the 3 charges form. We consider two cases, an isosceles right triangle and an equilateral triangle. In these special cases, we can prove that the upper bounds are indeed 4 for any charge values of these 3 point charges.

In the same paper [12], it is proved that if the 3 point charges span a 2-dimensional affine subspace L in $\mathbb{R}^3$, then all the critical points must be on L. Since we consider the cases when 3 point charges form special triangles, we can restrict the potential to be a function defined on the plane formed by the 3 points charges. An upper bound for the number of the nondegenerate critical points of the potential restricted on this plane will also be an upper bound for the number of the nondegenerate critical points of the potential in $\mathbb{R}^3$.

We now reduce the problem to a root counting problem of a polynomial system [12]. Without loss of generality, assume three point charges are located at $(0,0),(1,0)$ and (u,v) with charges $t,s,1$, respectively. They create an electric potential restricted in $\mathbb{R}^2$ which equals to

$$V(x_1,x_2)=\frac{t}{\sqrt{x_1^2+x_2^2}}+\frac{s}{\sqrt{(x_1-1)^2+x_2^2}}+\frac{1}{\sqrt{(x_1-u)^2+(x_2-v)^2}}. \tag{1.10}$$

Let

$$\begin{aligned}
h_1&=(x_1^2+x_2^2)^{-\frac{3}{2}},\\
h_2&=((x_1-1)^2+x_2^2)^{-\frac{3}{2}},\\
h_3&=((x_1-u)^2+(x_2-v)^2)^{-\frac{3}{2}},\\
f&=\frac{h_2}{h_1}, g=\frac{h_3}{h_1}.
\end{aligned}$$

The system of equations defining the critical points of V is $\partial_{x_1}V=\partial_{x_2}V=0$. From these, we derive that

$$x_1=\frac{ug+sf}{t+sf+g}, x_2=\frac{vg}{t+sf+g}. \tag{1.11}$$

Using these expressions of x_1 and x_2 in the equations $f=\frac{h_2}{h_1}, g=\frac{h_3}{h_1}$, we get

$$f^{-\frac{2}{3}}=\frac{((u-1)g-t)^2+v^2g^2}{(ug+sf)^2+v^2g^2}, g^{-\frac{2}{3}}=\frac{((u-1)sf+ut)^2+v^2(t+sf)^2}{(ug+sf)^2+v^2g^2}.$$

Now let $f=x^3, g=y^3$. We obtain the following polynomial system with two equations, two variables x,y, and four parameters s,t,u,v. The common positive roots give the critical points of V from equations 1.11.

$$\begin{aligned}
p_1&=t^2x^2-s^2x^6+2tx^2y^3-2utx^2y^3-2usx^3y^3-\\
&\quad u^2y^6-v^2y^6+x^2y^6-2ux^2y^6+u^2x^2y^6+v^2x^2y^6,\\
p_2&=-s^2x^6+u^2t^2y^2+v^2t^2y^2-2ustx^3y^2+2u^2stx^3y^2+2v^2stx^3y^2+\\
&\quad s^2x^6y^2-2us^2x^6y^2+u^2s^2x^6y^2+v^2s^2x^6y^2-2usx^3y^3-u^2y^6-v^2y^6.
\end{aligned} \tag{1.12}$$

When $(u,v)=(0,1)$, the 3 point charges form an isosceles right triangle. After substituting into p_1 and p_2, we get our first polynomial system.

$$f_7 = t^2x^2 - s^2x^6 + 2tx^2y^3 - y^6 + 2x^2y^6, \tag{1.13}$$
$$f_8 = -s^2x^6 + t^2y^2 + 2stx^3y^2 + 2s^2x^6y^2 - y^6. \tag{1.14}$$

When $(u, v) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, the 3 point charges form an equilateral triangle. After substituting (u, v) into p_1 and p_2, we get the second polynomial system.

$$f_9 = t^2x^2 - s^2x^6 + tx^2y^3 - sx^3y^3 - y^6 + x^2y^6, \tag{1.15}$$
$$f_{10} = -s^2x^6 + t^2y^2 + stx^3y^2 + s^2x^6y^2 - sx^3y^3 - y^6. \tag{1.16}$$

We will prove the following results in Chapter 4.

Theorem 1.4. *Both the polynomial systems $f_7 = f_8 = 0$ and $f_9 = f_{10} = 0$ have finitely many zeros and at most 4 positive zeros for all $t, s \in \mathbb{R}$ with $t \neq 0$ and $s \neq 0$. Therefore, Maxwell's conjecture is true in the case of 3 point charges when they form an isosceles right triangle or an equilateral triangle.*

Chapter 2

Standard computer algebra tools

Studying the roots of polynomials, or solving polynomial equations is among the oldest problems in mathematics. Classic root counting methods such as theorem of Hermite or Sturm have been known since the 19th century. While Sturm's theorem counts roots for univariate polynomials, Hermite's theorem can deal with polynomial systems in two or more variables. In 1960's, Bruno Buchberger created Groebner bases which provide us tools to study polynomial systems in an algorithmic or computational approach. With the computer algebra system (CAS) developed around the same time, we can now apply Hermite's theorem to count roots for polynomial systems through computing Groebner bases in a computer.

A CAS is a software program that facilitates symbolic mathematics. In contrast with numerical computations in a floating point system of a computer, a CAS manipulates mathematical expressions in symbolic form. Among many of currently existing CAS, Mathematica is a powerful one that is commonly used by research mathematicians, scientists, and engineers. In this thesis, we use "Mathematica 6.0.0" as our CAS to perform many computations.

Besides Groebner bases, resultants and subresultant sequences also provide us powerful tools to study and manipulate polynomials. We will introduce them here and demonstrate how they can also be applied to our root counting tests. Especially in the cases of a polynomial system with two equations or one univariate polynomials, they provide an alternative computer algebra tools to study polynomials symbolically. There computations involve matrix determinants. Each entry in the matrix involves mathematical symbols. Therefore, a CAS also provides the right environment to study polynomials in this algorithmic approach.

In Mathematica 6.0.0, there are some commands that have already implemented, such as commands to compute a Groebner basis of a polynomial system, the resultant of two polynomials, and number of roots for univariate polynomials. However, we have no commands to compute subresultant sequences or apply the root counting theorems to count roots for a polynomial system. We will introduce in appendix A some of the existing commands that we used and include in appendix B some notebooks, Mathematica files, to show algorithms for running other tests. Those notebooks are works of my adviser, Professor Moeckel, while I contributed to the improvement of the algorithm of forming Hermite's matrix.

2.1 Groebner bases

Let $f_1, \cdots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$, where $\mathbb{F}$ is an infinite field. Consider the *ideal* generated by $f_1, \cdots, f_s$ denoted by $I = \langle f_1, \cdots, f_m \rangle$. Let $f \in \mathbb{F}[x_1, \ldots, x_n]$. Then $f \in I \Leftrightarrow \exists\, a_1, \cdots, a_m \in \mathbb{F}[x_1, \ldots, x_n]$ such that $f = a_1 f_1 + \cdots + a_m f_m$.

On the other hand, consider all the points $(x_1, \ldots, x_n)$ in $\mathbb{F}^n$ that are the common zeros of f_i for all $i = 1, \ldots, m$. We call the set of all these points the *variety* defined by $f_1, \cdots, f_m$ and denote it by $V_{\mathbb{F}} = V_{\mathbb{F}}(f_1, \cdots, f_m)$. The variety is the object we want to understand; we want to count how many points of $V_{\mathbb{F}}$ have positive coordinates. By studying the ideal I, we gain information about $V_{\mathbb{F}}$.

The ideal I gives an equivalence relation in $\mathbb{F}[x_1, \ldots, x_n]$, i.e. $f \sim g \Leftrightarrow f - g \in I$. The quotient space determined by this equivalence relation will be denoted by $A = \mathbb{F}[x_1, \ldots, x_n]/I$. It is an algebra over $\mathbb{F}$ if we define the operations naturally.

Suppose that A is finite dimensional as an $\mathbb{F}$-vector space and consider the classes $[x_1^k]$ for $k = 1, 2, \ldots$. Then there must exist real numbers a_k (not all zero) and some l such that

$$\sum_{j=0}^{l} a_k [x_1^k] = [\sum_{j=0}^{l} a_k x_1^k] = [0].$$

This implies that $\sum_{j=0}^{l} a_k x_1^k \in I$. We can apply the same argument for other $x_j's$. Therefore, we find nonzero polynomials $g_i \in I$ for $i = 1, \ldots, n$ such that g_i only involves x_i for $i = 1, \ldots, n$, respectively. Since nonzero univariate polynomials in $\mathbb{F}$ can only have finitely many zeros in $\mathbb{F}$ and every point in $V_{\mathbb{F}}$ must have x_i coordinate satisfying $g_i = 0$, we conclude that $V_{\mathbb{F}}$ contains only finitely many distinct points. Therefore, the finiteness of dimension of A provides us a sufficient condition for the variety to be finite. It is actually a necessary condition if $\mathbb{F} = \mathbb{C}$, the field of complex numbers. The ideal I and quotient algebra A play central roles of studying the variety $V_{\mathbb{F}}$. Groebner bases serve as useful tools to study I and A. We now give the definition of Groebner bases.

We start with some terminology. A monomial in $x_1, \ldots, x_n$ is an expression of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with nonnegative integers α_i. A monomial order is a total ordering on $\mathbb{Z}_{\geq 0}^n$ satisfying $\alpha + \gamma > \beta + \gamma$ and $\gamma \geq (0, \ldots, 0)$ if $\alpha > \beta$ and $\gamma \in \mathbb{Z}_{\geq 0}^n$. It gives monomials a total ordering which preserves the order under multiplication and has smallest monomial 1.

For example, the *lexicographic order* (or lex order) $>$ with $x_1 > \cdots > x_n$ is defined by: $x_1^{\alpha_1} \cdots x_n^{\alpha_n} > x_1^{\beta_1} \cdots x_n^{\beta_n}$ if the first non-zero component of the vector $(\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ is positive. Another commonly used order is called the *graded reverse lexicographic order* (or grlex order). The grlex order $>$ with $x_1 > \cdots > x_n$ is defined by: $x_1^{\alpha_1} \cdots x_n^{\alpha_n} > x_1^{\beta_1} \cdots x_n^{\beta_n}$ if $\alpha_1 + \cdots + \alpha_n > \beta_1 + \cdots + \beta_n$ or the last non-zero component of the vector $(\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ is negative.

Given a monomial order, for any $f \in \mathbb{F}[x_1, \ldots, x_n]$, we can define its leading term $LT(f)$, leading monomial $LM(f)$, and the leading coefficient $LC(f)$. $LT(f) = LC(f)LM(f)$. We denote $\langle LT(I) \rangle$ by the ideal generated by $LT(f)$ for all $f \in I$. Note that $\langle LT(I) \rangle$ can be strictly larger than $\langle LT(f_1), \cdots, LT(f_m) \rangle$.

Definition 2.1. *Fix a monomial order. A finite subset $G = \{g_1, \cdots g_l\}$ of I is called a Groebner basis of I in $\mathbb{F}[x_1, \ldots, x_n]$ if $\langle LT(g_1), \cdots, LT(g_l) \rangle = \langle LT(I) \rangle$.*

Here we list some properties that we will use later [8], [11].

Proposition 2.1. *Let $I = \langle f_1, \cdots, f_m \rangle \neq \{0\}$ in $\mathbb{F}[x_1, \ldots, x_n]$ and $V_{\mathbb{F}} = V_{\mathbb{F}}(f_1, \cdots, f_m)$ in $\mathbb{F}^n$. Fixing a monomial order, we have the following properties.*

1. *There exist a Groebner basis $G = \{g_1, \cdots, g_l\}$ with respect to this monomial order. G also generate I. If, moreover, $LC(g_i) = 1$ for all i and no monomial of g_i lies in $\langle LT(G - \{g_i\}) \rangle$, then G is unique. We call it the reduced Groebner basis.*

2. *Let $\mathbb{E}$ be a extension field of $\mathbb{F}$, J be the ideal generated by I in $\mathbb{E}[x_1, \ldots, x_n]$. Then G is also a Groebner basis of J in $\mathbb{E}[x_1, \ldots, x_n]$ with respect to the same monomial order.*

3. *For every $f \in \mathbb{F}[x_1, \ldots, x_n]$, there exists a unique $r \in \mathbb{F}[x_1, \ldots, x_n]$ such that no terms of r are divisible by any of $LT(g_1), \cdots, LT(g_l)$ and $f - r \in I$. r is call the normal form of f. There exists an division algorithm to compute r.*

4. *If, for $i = 1, \ldots, n, \exists\ \beta_i \in\ \mathbb{Z}_{\geq 0}$ such that $x_i^{\beta_i} = LM(g_i)$ for some $g_i \in G$, then $dimA < \infty$. When $\mathbb{F} = \mathbb{C}$, the number of elements in $V_{\mathbb{F}}$ is at most $dimA$. A basis of A is given by $\{[x_1^{\alpha_1} \cdots x_n^{\alpha_i}]\,|x_1^{\alpha_1} \cdots x_n^{\alpha_i} \notin \langle LT(g_1), \ldots, LT(g_l)\rangle\}$.*

A monomial order on $\mathbb{F}[x_1, \ldots, x_n]$ can be described in terms of a $m \times n$ matrix M with entries in $\mathbb{F}$. We write the rows of M as $\mathbf{w_1}, \cdots, \mathbf{w_m}$. Given two monomials with exponent vectors α and β, we have $\alpha > \beta$ if $\mathbf{w_1} \cdot \alpha > \mathbf{w_1} \cdot \beta$. If $\mathbf{w_1} \cdot \alpha = \mathbf{w_1} \cdot \beta$, we compare $\mathbf{w_2} \cdot \alpha$ with $\mathbf{w_2} \cdot \beta$. Repeat the procedure until we find which one is larger. When $\mathbb{F} = \mathbb{Q}$, the set of all rational numbers, and M has only nonnegative entries, $\ker(M) \cap \mathbb{Z}^n = \{0\}$ gives a sufficient condition for M to be a correct matrix to define a monomial order [9]. For example, the lex order is given by the $n \times n$ identity matrix. Also if $m = n$ and $\mathbf{w_i} = (1, \ldots, 1, 0, \ldots, 0)$, where the first $(n-i+1)$ entries are 1 and the rest are 0, such matrix gives the grlex order.

We use an example to show how a Groebner basis works as a powerful tool in studying ideals generated by polynomial systems and the quotient algebra constructed from them. The construction of a Groebner basis can be found in the book [8], where an algorithm is presented. Here we use commands in Mathematica 6.0.0 to compute them. See Appendix A for detail introductions for its commands.

Example 2.1. *Let $p = x^4 - xy - 6x^2 + 11x + y - 6, q = 3x^3 - x^3y - 3y + y^2$, $I = \langle p, q\rangle \subseteq \mathbb{C}[x, y]$, $A = \mathbb{C}[x, y]/I$, and $V_{\mathbb{C}} = \{(x, y) \in \mathbb{C}^2 | p = q = 0\}$. Using the command* $\mathbf{GroebnerBasis}[\{\mathbf{p}, \mathbf{q}\}, \{\mathbf{x}, \mathbf{y}\}]$ *in Mathematica, we obtain a Groebner basis* $\mathbf{gb}$ *in $\mathbb{Q}[x, y]$ with lex order $x > y$, where* $\mathbf{gb} = \{g_1, g_2, g_3\}$ *with*

$$
\begin{aligned}
g_1 &= y^4 - 39y^3 + 359y^2 - 969y + 648, \\
g_2 &= 1729xy - 5187x + 6y^3 - 319y^2 - 531y + 4302, \\
g_3 &= 1729x^4 - 10374x^2 + 13832x + 6y^3 - 319y^2 + 1198y - 6072.
\end{aligned}
$$

By part 2 of proposition 2.1, $\mathbf{gb}$ *is also a Groebner basis of I in $\mathbb{C}[x, y]$. Since $LM(g_1) = y^4$ and $LM(g_3) = x^4$, we know A is finite dimensional by part 4 of proposition 2.1. A basis is given by $\{[x^m y^n]\}$ where $x^m y^n \notin \langle LT(g_1), LT(g_2), LT(g_3)\rangle) = \langle y^4, 1729xy, 1729x^4\rangle = \langle y^4, xy, x^4\rangle$ and it is $\{[1], [x], [x^2], [x^3], [y], [y^2], [y^3]\}$. Therefore, $V_{\mathbb{C}}$ has at most 7 elements, or there are at most 7 complex zeros of f_1 and f_2.*

In fact, since $q = (y - x^3)(y - 3)$*, we can easily solve this system to get totally* 5 *zeros* $(1,1), (2,8), (3,27), (1,3), (-3,3)$.

From now on, we abuse the notation by writing basis as $\{1, x, x^2, x^3, y, y^2, y^3\}$ *and* $f \in A$ *instead of* $[f] \in A$*. We next demonstrate how to compute the normal form of a polynomial* f*, or equivalently, how to express* $f \in A$ *in the basis* $\{1, x, x^2, x^3, y, y^2, y^3\}$*. For example, we consider* $f = xy$*. Using the Mathematica, we apply* $\mathbf{PolynomialReduce}[\mathbf{xy}, \mathbf{gb}, \{\mathbf{x}, \mathbf{y}\}]$ *and get*

$$xy = \frac{1}{1729}g_2 - \frac{4302}{1729} + 3x + \frac{531}{1729}y + \frac{319}{1729}y^2 - \frac{6}{1729}y^3.$$

We can see from g_2 *in* $\mathbf{gb}$ *that this is correct. Since* $g_2 \in I$*, we have* $[xy] = -\frac{4302}{1729}[1] + 3[x] + \frac{531}{1729}[y] + \frac{319}{1729}[y^2] - \frac{6}{1729}[y^3]$*, or* $f = -\frac{4302}{1729} + 3x + \frac{531}{1729}y + \frac{319}{1729}y^2 - \frac{6}{1729}y^3$ *in* A*.*

2.2 Resultants and subresultant sequences

In this section, we will introduce the Sylvester resultants and subresultant sequences. In the first part, we will recall the classic Sylvester resultants, which we call resultants for short. While they come from the study of univariate polynomials, resultants have many applications to the study of two plane curves. Here a plane curve means the variety of a bivariate polynomial in $\mathbb{C}^2$. In the second part, subresultant sequences are introduced. We will follow definitions in Marie-Francoise Roy's paper [14]. Detailed studies of them can be found there.

2.2.1 Resultants and plane curves

Considering univariate polynomials $p, q \in \mathbb{C}[x]$, we form the ideals $I = \langle p \rangle, J = \langle q \rangle$ and the quotient algebras $A = \mathbb{C}[x]/I, B = \mathbb{C}[x]/J$. Let degrees of p and q be m and n. Then A and B are vector spaces over $\mathbb{C}$ of dimensions m and n, respectively. By the Euclidean algorithm, we can express any polynomial in A in the standard basis $\{1, \ldots, x^{m-1}\}$ of A, and similar for B. Therefore, we can view A as a linear subspace of $\mathbb{C}[x]$ consisting of polynomials with degree strictly less than m and B as the subspace of all polynomials with degree strictly less than n.

It is well known that p and q do not have a common complex root if and only if there exist $a, b \in \mathbb{C}[x]$ such that $pb + qa = 1$. In this case, we will claim that $H = pB + qA$, where $H = \mathbb{C}[x]/\langle pq \rangle$, the subspace of all polynomials with degrees less than $m+n$. Since we have $\langle p,q\rangle = \mathbb{C}[x]$, for any polynomial f with degrees less than $m+n$, there exist f_1, f_2 such that $f = pf_1 + qf_2$.

By the Euclidean algorithm, $f_1 = qg_1 + r_1, f_2 = pg_2 + r_2$ for unique remainders r_1, r_2, either 0 or polynomials with degrees less than n, m, respectively. Therefore, $f = p(qg_1 + r_1) + q(pg_2 + r_2) = pq(g_1 + g_2) + pr_1 + qr_2 = pr_1 + qr_2$ in H. Therefore, $H \subseteq pB + qA$. The other direction is trivial. Therefore, we conclude that $H = pB + qA$. Therefore, the standard basis, $\beta_1 = \{1, \ldots, x^{m+n-1}\}$, and $\beta_2 = \{p, \ldots, px^{n-1}, q, \ldots, qx^{m-1}\}$ are all bases of H. The basis transformation matrix is, therefore, invertible.

If we write $p = a_m x^m + \cdots + a_0, q = b_n x^n + \cdots + b_0$, then the basis transformation matrix is the transpose of the following $(m+n) \times (m+n)$ matrix,

$$Sylv\,(p,q) = \begin{pmatrix} a_m & \cdots & . & a_0 & & & \\ & \ddots & & \ddots & & & \\ & & a_m & \cdots & . & a_0 \\ b_n & \cdots & b_0 & & & \\ & \ddots & & \ddots & & \\ & & & b_n & \cdots & b_0 \end{pmatrix}.$$

Definition 2.2. *The resultant of p and q, denoted by $Res(p,q,x)$, is the determinant of the matrix above, called the Sylvester matrix of p and q.*

The definition of resultants applies to two polynomials whose coefficients are in an integral domain $\mathbb{D}$. In this case, $Res(p,q,x) \in \mathbb{D}$. If their coefficients are in the field of complex numbers, the arguments above show that p, q have no common zeros implies $Res(p,q,x) \neq 0$. Conversely, if the linear map from H to itself by sending the standard basis β_1 to the set β_2 (not known as a basis yet) is invertible, then β_2 is a basis, and $H = pB + qA$. So, $pb + qa = 1$ for some a, b and we conclude that p, q do not have a common root. Therefore, we have the following property.

Proposition 2.2. *Let $p, q \in \mathbb{C}[x]$. $Res(p,q,x) = 0 \Leftrightarrow p, q$ have a common factor.*

Another basic property about resultant is given below.

Proposition 2.3. *Let $p, q \in \mathbb{D}[x]$, where $\mathbb{D}$ is an integral domain. Then there exist $f, g \in \mathbb{D}[x]$ such that $Res(p,q,x) = fp + gq$.*

If we multiply the first column of the Sylvester matrix by x^{m+n-1} and add it to the last column, the second column by x^{m+n-2} and add it to the last column, and so on, we will get the following matrix

$$\begin{pmatrix} a_m & \cdots & . & a_0 & & x^{n-1}p \\ & \ddots & & \ddots & & \vdots \\ & & a_m & \cdots & . & p \\ b_n & \cdots & b_0 & & & x^{m-1}q \\ & \ddots & & \ddots & & \vdots \\ & & & b_n & \cdots & q \end{pmatrix}.$$

This matrix has the same determinant as the resultant. Computing the determinant by expanding the last column of the above matrix, we prove proposition 2.3. From this property, we have an important application.

Given two bivariate polynomials p, q in $\mathbb{C}[x, y]$, we view $p, q \in \mathbb{D}[x]$, where $\mathbb{D} = \mathbb{C}[y]$. Then proposition 2.3 tells us there exist $f, g \in \mathbb{D}[x] = \mathbb{C}[x, y]$ such that $Res(p, q, x) = fp + gq$. Here the x in the notation $Res(p, q, x)$ make it clear that we are computing the resultant with respect to the variable x. Moreover, since we know that $Res(p, q, x) \in \mathbb{D} = \mathbb{C}[y]$, it is a univariate polynomial in y.

Corollary 2.1. *Let $p, q \in \mathbb{C}[x, y]$. Then $Res(p, q, x) \in I \cap \mathbb{C}[y]$, where $I = \langle p, q \rangle \subseteq \mathbb{C}[x, y]$.*

In other words, by computing the resultant of p and q in $\mathbb{C}[x, y]$ with respect to x, we eliminate the variable x and get a univariate polynomial $Res(p, q, x)$, denoted by $h(y)$, in the ideal generated by p and q.

Now we turn to study the complex zeros of p and q. We only consider the case that there are only finitely many of them, or the number of points in the variety $V_{\mathbb{C}} = V_{\mathbb{C}}(p, q) \subseteq \mathbb{C}^2$ is finite. Write $V_{\mathbb{C}} = \{z_1 = (x_1, y_1), \ldots, z_k = (x_k, y_k)\}$. We have $p(x_i, y_i) = q(x_i, y_i) = 0$. Since $h = fp + gq$ for some f, g in $\mathbb{C}[x, y]$, we get $h(y) = f(x, y)p(x, y) + g(x, y)q(x, y)$ for any x and y. Substituting $x = x_i, y = y_i$ into the equation, we have $h(y_i) = 0$. for all $i = 1, \ldots, k$. Therefore, the projections of the points of the variety onto the y-axis are all zeros of h.

Moreover, the resultants also "accumulate and preserve the multiplicities" of points in the variety. We first give the definition of the intersection multiplicity of a point in a variety with finite elements and then present another good property about the resultant. It will then be clear what we mean by "accumulate and preserve the multiplicities."

Definition 2.3. *The intersection multiplicity of z_i, denoted by $mul(z_i)$, is equal to $dim(\mathcal{O}_{z_i}/\langle p,q\rangle_{z_i})$, where $\mathcal{O}_{z_i}=\left\{\frac{f}{g}\mid f,g\in\mathbb{C}[x,y],g(z_i)\neq 0\right\}$ and $\langle p,q\rangle_{z_i}=p\mathcal{O}_{z_i}+q\mathcal{O}_{z_i}$. When $mul(z_i)=1$, we say p and q intersect at z_i transversely.*

Write $p=\sum_{i=0}^{m}a_i(y)x^i, q=\sum_{i=0}^{n}b_i(y)x^i$. We have the following proposition whose proof can be found in [4].

Proposition 2.4. *Suppose $a_m(y)$ and $b_n(y)$ do not vanish simultaneously. Then $Res(p,q,x)=c\prod_{j=1}^{s}(y-\alpha_j)^{\beta_j}$, where $\{\alpha_1,\dots\alpha_s\}=\{y_1,\dots,y_k\}$ as sets and $\beta_j=\sum_{z_i\in V|y_i=\alpha_i}mul(z_i)$ and c is a complex number.*

Therefore, that the resultants "accumulate and preserve the multiplicities" is clear since the definition of multiplicities of zeros of a univariate polynomial is clear and they are β_j in the formula of proposition 2.4 in our case. In fact, that formula tells more. It also tells us that any zero of the resultant is actually a projection of some complex zeros of p and q if $a_m(y)$ and $b_n(y)$ do not vanish simultaneously.

We now give an example to explain some definitions and demonstrate some properties mentioned in this section. Mathematica 6.0.0 provides us the command $\mathbf{Resultant}[\mathbf{p},\mathbf{q},\mathbf{var}]$ to compute the resultant of p and q in the variable var.

Example 2.2. *Let p,q be polynomials in the example 2.1. There we computed $\mathbf{gb}=\{g_1,g_2,g_3\}$ and also know that $V_{\mathbb{C}}=\{(1,1),(2,8),(3,27),(1,3),(-3,3)\}$. In this example, we compute the intersection multiplicities of these common zeros and try to verify the formula in proposition 2.4. Let us start with $(1,3)$. First, we perform an linear transformation by letting $x=X+1,y=Y+3$ and get a new system $P=4X^3+X^4-XY, Q=2Y-3XY-3X^2Y-X^3Y+Y^2$. Then we compute $mul(z_0=(0,0))$.*

By definition, $mul(z_0)=dim(\mathcal{O}_{z_0}/P\mathcal{O}_{z_0}+Q\mathcal{O}_{z_0})$. For all $\frac{f}{g}\in P\mathcal{O}_{z_0}+Q\mathcal{O}_{z_0}$, then

$$\begin{aligned}\frac{f}{g}&=(4X^3+X^4-XY)\frac{f_1}{g_1}+(2Y-3XY-3X^2Y-X^3Y+Y^2)\frac{f_2}{g_2}\\&=X^3\frac{f_3}{g_3}+Y\frac{f_4}{g_4}\end{aligned}$$

, where $g_1,\dots,g_4$ are all non-zero at z_0.

Conversely, if $\frac{f}{g} \in X^3\mathcal{O}_{z_0} + Y\mathcal{O}_{z_0}$*, then*

$$\begin{aligned}
\frac{f}{g} &= X^3\frac{f_5}{g_5} + Y\frac{f_6}{g_6} \\
&= X^3(4+X)\frac{f_5}{(4+X)g_5} + Y\frac{f_6}{g_6} \\
&= (P - XY)\frac{f_5}{(4+X)g_5} + Y\frac{f_6}{g_6} \\
&= P\frac{f_5}{(4+X)g_5} + Y(\frac{-Xf_5}{(4+X)g_5} + \frac{f_6}{g_6}) \\
&= P\frac{f_5}{(4+X)g_5} + Y(2 - 3X - 3X^2 - X^3 + Y) \\
&(\frac{-Xf_5}{(2 - 3X - 3X^2 - X^3 + Y)(4+X)g_5} + \frac{f_6}{(2 - 3X - 3X^2 - X^3 + Y)g_6}) \\
&= P\frac{f_7}{g_7} + Q\frac{f_8}{g_8}
\end{aligned}$$

, where $g_5, \ldots, g_8$ *are all non-zero at* z_0*. So* $P\mathcal{O}_{z_0} + Q\mathcal{O}_{z_0} = X^3\mathcal{O}_{z_0} + Y\mathcal{O}_{z_0}$*.*

Let $\frac{f}{g} \in \mathcal{O}_{z_0}$*. Write* $\frac{f}{g} = \frac{k}{h+1}$*, where* $k = \frac{f}{g(z_0)}, h = \frac{g-g(z_0)}{g(z_0)}$*. Note that* $h \in \langle X, Y\rangle$ *since* $g - g(z_0)$ *has no constant term. Writing* $h = Xh_1 + Yh_2$*, we have*

$$\frac{k}{h+1} - k(1 - h + h^2) = \frac{kh^3}{h+1} = X^3\frac{kh_1^3}{h+1} + Y\frac{3kX^2h_1^2h_2 + 3kXYh_1h_2^2 + kY^2h_2^3}{h+1}.$$

It is in $X^3\mathcal{O}_{z_0} + Y\mathcal{O}_{z_0} = P\mathcal{O}_{z_0} + Q\mathcal{O}_{z_0}$*. So,*

$$\begin{aligned}
\left[\frac{f}{g}\right] &= \left[k(1 - h + h^2)\right] \\
&= \left[k - Xkh_1 - Ykh_2 + kX^2h_1^2 + 2kXYh_1h_2 + kY^2h_2^2\right] \\
&= \left[k - kh_1X + kh_1^2X^2\right].
\end{aligned}$$

Therefore, $[1], [X], [X^2]$ *is a basis of* $\mathcal{O}_{z_0}/P\mathcal{O}_{z_0}+Q\mathcal{O}_{z_0}$ *and* $mul(z_0) = dim(\mathcal{O}_{z_0}/P\mathcal{O}_{z_0}+Q\mathcal{O}_{z_0}) = 3$*. Therefore, the intersection multiplicity of* $(1,3)$ *in* $V_{\mathbb{C}}(p,q)$ *is* 3*. Using the similar way, we get* $mul((1,1)) = mul((2,8)) = mul((3,27)) = mul((-3,3)) = 1$*.*

Next, applying the command $\mathbf{Resultant}[\mathbf{p},\mathbf{q},\mathbf{x}]$ *to compute the resultant with respect to the variable* x*, we get* $Resultant(p,q,x) = (y-1)(y-3)^4(y-8)(y-27)$*. Also applying the command* $\mathbf{Resultant}[\mathbf{p},\mathbf{q},\mathbf{y}]$*, we get* $Resultant(p,q,y) = (x+3)(x-1)^4(x-2)(x-3)$*. The formula in proposition 2.4 is verified. Note that the leading coefficients of* p, q *in* x *and* y *are* $1, 3-y$ *and* $1-x, 1$*, respectively.*

2.2.2 Subresultant sequences

Now, we will introduce subresultant sequences. While resultants project points of the variety onto one axis, subresultant sequences can be used as "lifting" the zeros of the resultants to the variety of the system. We will see how to apply such sequences to "lift" zeros in the following chapters.

Let $p, q \in \mathbb{D}[x]$, where $\mathbb{D}$ is a integral domain, and denote their degrees by $deg\,(p)$, and $deg\,(q)$, respectively. Here we assume that $deg\,(p) \leq deg\,(q) + 1$, which is sufficient to cove all the problems that we considered in this thesis. Let $m = n+1$ and $n = deg\,(q)$. We write $p\,(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$ and $q\,(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$. Here $a_m = a_{m-1} = \cdots = a_{deg(p)+1} = 0$. For $j < n$, we first define the j-th Sylvester matrix of p and q to be the $(m+n-2j) \times (m+n-j)$ matrix

$$Sylv_j\,(p,q) = \overbrace{\begin{pmatrix} a_m & \cdots & . & a_0 & & \\ & \ddots & & \ddots & & \\ & & a_m & \cdots & . & a_0 \\ b_n & \cdots & b_0 & & & \\ & \ddots & & \ddots & & \\ & & & b_n & \cdots & b_0 \end{pmatrix}}^{m+n-j}.$$

Note that $Sylv_0\,(p,q) = Sylv\,(p,q)$ only if $deg\,(p) = deg\,(q)+1$. If $deg\,(p) < deg\,(q)+1$, then $deg\,(p) < m$. Therefore, we add $m - deg\,(p)$ zeros, $a_m, a_{m-1}, \cdots, a_{deg(p)+1}$, into the beginning of the coefficient list of p when forming the first n rows of the 0-th Sylvester matrix. However, the first n rows of the Sylvester matrix comes only from the coefficient list of p. Also if, for example, $m - deg\,(p) = 1$, then there are $deg\,(p)+1$ rows from the coefficient list of q in the 0-th Sylvester matrix, while there are only $deg\,(p)$ rows from the coefficient list of q in the Sylvester matrix.

Fix $j < n$. For $k = 0, \ldots, j$, let $Sylv_{j,k}\,(p,q)$ denote the square matrix of dimension $m+n-2j$ obtained by taking the $m+n-2j-1$ first columns and the $(m+n-j-k)$-th column from $Sylv_j\,(p,q)$.

Definition 2.4. *1. The subresultant sequence associated with p and q is given by* $Sres_j\,(p,q,x) = \sum_{k=0}^{j} det\,(Sylv_{j,k}\,(p,q))\, x^k$ *for* $j = 0, \ldots, n-1$, $Sres_n\,(p,q,x) = q$, *and* $Sres_m\,(p,q,x) = p$.

2. The Sylvester-Habicht sequence associated with p and q is $SyHa_j\,(p,q,x) =$

$\delta_{m-j} Sres_j(p,q,x)$, *where* $\delta_k = (-1)^{\frac{k(k-1)}{2}}$.

3. *When* $q = \dfrac{dp}{dx}$, *we denote by* $SyHa_j(p,q,x) = StHa_j(p,x)$ *the Sturm-Habicht sequences of* p.

Definition of subresultant sequences comes from the j-th Sylvester matrix just as the definition of resultant comes from Sylvester matrix. Therefore, we can expect they have similar properties as resultants. We list them here without proofs but they are easy to derive. More properties can be found in papers [14] and [26].

Proposition 2.5. *Let* $p, q \in \mathbb{D}[x]$, *where* $\mathbb{D}$ *is an integral domain.*

1. $Sres_j(p,q,x) \in \mathbb{D}[x]$ *and has degree at most* j.

2. $Sres_0(p,q,x) = ((-1)^n b_n)^\alpha Res(p,q,x)$, *where* $\alpha = 1 + deg(q) - deg(p) \geq 0$.

3. *There exist* $f_j, g_j \in \mathbb{D}[x]$ *such that* $Sres_j(p,q,x) = f_j p + g_j q$. *In particular, if* $\mathbb{D} = \mathbb{C}[y]$, *then* $Sres_j(p,q,x) \in \langle p, q \rangle \subseteq \mathbb{C}[x,y]$.

Now, we compute the subresultant sequence of p, q given in the example 2.2. Since we have no direct command to compute the subresultant sequence from Mathematica 6.0.0, we provide an algorithm in appendix B that computes $SyHa_j(p,q,x)$.

Example 2.3. *We apply the command* $\mathbf{SyHa[p,4,q,3,i]}$, $i = 0, \ldots, 4$ *to get*

$$\begin{aligned}
SyHa_4(p,q,x) =& y - 6 + (11-y)x - 6x^2 + x^4 = p,\\
SyHa_3(p,q,x) =& y^2 - 3y + (3-y)x^3 = q,\\
SyHa_2(p,q,x) =& 54 - 45y + 12y^2 - y^3 - (99 - 66y + 11y^2)x - (-56 + 36y - 6y^2)x^2,\\
SyHa_1(p,q,x) =& 1782 - 1107y - 81y^2 + 159y^3 - 25y^4 -\\
& x(2295 - 2133y + 603y^2 - 31y^3 - 6y^4).\\
SyHa_0(p,q,x) =& (y-1)(y-3)^4(y-8)(y-27) = Res(p,q,x).
\end{aligned}$$

By the command $\mathbf{PolynomialReduce[SyHa_i(p,q,x), gb, \{x,y\}]}$, *we get the remainder* 0 *for all* i, *which verifies the third part of proposition* 2.5.

2.3 Root counting methods

In this section, we will introduce some classic root counting theorems. In particular, we describe Sturm and Sturm-Habicht sequences and Hermite quadratic forms.

2.3.1 Sturm and Sturm-Habicht sequences

Let $p \in \mathbb{D}[x]$ be an univariate polynomial. The *Sturm sequence* of p is the following sequence of polynomials in $\mathbb{K}[x]$. Here $\mathbb{D}$ is an integral domain and $\mathbb{K}$ is its field of fractions.

$$St_0(p,x) = p, St_1(p,x) = \frac{dp}{dx}, St_{i+1}(p,x) = -rem\left(St_{i-1}(p,x), St_i(p,x)\right) for\ i \geq 1,$$

where rem denotes the Euclidean remainder. Let $St(p) = \{St_0(p,x), \ldots, St_n(p,x)\}$, where St_{n-1} is the last remainder that is not identically equal to the zero polynomial. Also recall that the *Sturm-Habicht sequence* of p is defined by $SyHa_j\left(p, \frac{dp}{dx}, x\right) = StHa_j(p,x)$ in $\mathbb{D}[x]$ for $j = 0, \ldots, m$, where $m = deg(p)$. We denote that $StHa(p) = \{StHa_0(p,x), \ldots, StHa_m(p,x)\}$.

When $\mathbb{D} \subseteq \mathbb{R}$ both sequences provide good tools to count the real roots in an open interval. Both root counting methods involve counting the algebraic sign changes of a sequence of real numbers. We first define two rules of counting in a sequence of real numbers obtained from a sequence of polynomials and a number.

Definition 2.5. *Given a sequence of polynomials $F = \{f_0, \ldots, f_k\}$ with $f_j \in \mathbb{R}[x]$ for all j, and $G = \{g_0, \ldots, g_l\} \subseteq F$ by removing zero polynomials in F. $\alpha \in \mathbb{R} \cup \{\pm\infty\}$. Let $G(\alpha) = \{g_0(\alpha), \ldots, g_l(\alpha)\}$ if $\alpha \in \mathbb{R}$, $G(\alpha) = \{LC(g_0), \ldots, LC(g_l)\}$ if $\alpha = \infty$, and $G(\alpha) = \{(-1)^{deg(g_0)}LC(g_0), \ldots, (-1)^{deg(g_l)}LC(g_l)\}$ if $\alpha = -\infty$.*

1. *$V(F(\alpha))$ is the number of sign changes in the sequence of real numbers obtained from $G(\alpha)$ by removing all $0'$s.*

2. *Suppose $G(\alpha)$ has no more than 2 consecutive 0's. $W(F(\alpha))$ is the number of sign changes in $G(\alpha)$ with $W(+,0,+) = W(-,0,-) = 0, W(+,0,0,+) = W(-,0,0,-) = 2, W(+,0,-) = W(-,0,+) = W(+,0,0,-) = W(-,0,0,+) = 1$, where $+,-$ denote positive and negative numbers, respectively.*

Proposition 2.6. *Let $p \in \mathbb{R}[x]$, $a < b$ with $a, b \in \mathbb{R} \cup \{+\infty, -\infty\}$, $p(a) \neq 0, p(b) \neq 0$. Denote $c(p,(a,b))$ the number of distinct real roots of p in the interval (a,b). Then we have $c(p,(a,b)) = V(St(p)(a)) - V(St(p)(b)) = W(StHa(p)(a)) - W(StHa(p)(b))$.*

A proof can be found in [14]. There, it is proved that the new sequence by removing zero polynomials in $StHa(p)$ have no more than 2 consecutive 0 after evaluating at α with $p(\alpha) \neq 0$. Now we give an example to demonstrate these two root counting methods. An algorithm computing Sturm sequence is in appendix B.

Example 2.4. *In example 2.2 of the previous section, we compute a resultant* $f = Resultant(p,q,y) = (x+3)(x-1)^4(x-2)(x-3)$*. We will compute* $St(f)$ *and* $StHa(f)$ *and check that* f *has* 3 *positive roots or* $c(f,(0,\infty)) = 3$ *through Sturm and Sturm-Habicht sequences. Using Mathematica, we apply the command* $\mathbf{SturmSeq[f,x]}$ *to compute Sturm sequence of* f *and* $\mathbf{SyHa[f,7,D[f,x],6,i]}$, $i = 0,\ldots,7$ *to compute Sturm-Habicht sequence of* f*. We have the following results.*

$$
\begin{aligned}
&St_0(f,x) = 18 - 81x + 142x^2 - 117x^3 + 38x^4 + 5x^5 - 6x^6 + x^7,\\
&St_1(f,x) = -81 + 284x - 351x^2 + 152x^3 + 25x^4 - 36x^5 + 7x^6,\\
&St_2(f,x) = -\frac{396}{49} + \frac{1698}{49}x - \frac{2864}{49}x^2 + \frac{2364}{49}x^3 - \frac{948}{49}x^4 + \frac{146}{49}x^5,\\
&St_3(f,x) = \frac{295029}{5329} - \frac{1028804}{5329}x + \frac{1316238}{5329}x^2 - \frac{726180}{5329}x^3 + \frac{143717}{5329}x^4,\\
&St_4(f,x) = -\frac{306950400}{421521961} + \frac{920851200}{421521961}x - \frac{920851200}{421521961}x^2 + \frac{306950400}{421521961}x^3,\\
&St_5(f,x) = 0,\\
&StHa_7(f,x) = 18 - 81x + 142x^2 - 117x^3 + 38x^4 + 5x^5 - 6x^6 + x^7,\\
&StHa_6(f,x) = -81 + 284x - 351x^2 + 152x^3 + 25x^4 - 36x^5 + 7x^6,\\
&StHa_5(f,x) = -396 + 1698x - 2864x^2 + 2364x^3 - 948x^4 + 146x^5,\\
&StHa_4(f,x) = 24084 - 83984x + 107448x^2 - 59280x^3 + 11732x^4,\\
&StHa_3(f,x) = -230400 + 691200x - 691200x^2 + 230400x^3,\\
&StHa_2(f,x) = StHa_1(p,q,x) = StHa_0(p,q,x) = 0.
\end{aligned}
$$

We have,

$$
\begin{aligned}
&V(St(f)(0)) = Number\ of\ sign\ changes\ in\ \{18, -81, -\frac{396}{49}, \frac{295029}{5329}, -\frac{306950400}{421521961}\},\\
&V(St(f)(\infty)) = Number\ of\ sign\ changes\ in\ \{1, 7, \frac{146}{49}, \frac{143717}{5329}, \frac{306950400}{421521961}\},\\
&W(StHa(f)(0)) = Number\ of\ sign\ changes\ in\ \{18, -81, -396, 24084, -230400\},\\
&W(StHa(f)(\infty)) = Number\ of\ sign\ changes\ in\ \{1, 7, 146, 11732, 230400\}.
\end{aligned}
$$

Therefore, we verified that $c(f,(0,\infty)) = V(St(f)(0)) - V(St(f)(\infty)) = 3 - 0 = 3$. $c(f,(0,\infty)) = W(St(f)(0)) - W(St(f)(\infty)) = 3 - 0 = 3$*, too. Note that* $St_j \in \mathbb{Q}[x]$ *and* $StHa_j \in \mathbb{Z}[x]$*. Therefore,* $StHa(p)$ *is in some sense better than* $St(p)$*. We will discuss this in the next chapter.*

Although Sturm and Sturm-Habicht sequences provide us tools to count roots for univariate polynomials, Mathematica 6.0.0's command $\mathbf{CountRoots}[\mathbf{p}, \{\mathbf{x}, \mathbf{a}, \mathbf{b}\}]$ uses a more efficient approach than those sequences and always terminates with correct number of roots of p in $[a, b]$. The paper [10] studies and presents the algorithm of this command. Since many root counting tests in Chapter 4 involve complicated univariate polynomials with large orders and large coefficients, we will use this implemented command as our real root counting tools for univariate polynomials.

Next, we introduce the Hermite quadratic forms and their applications to count zeros of real polynomial systems.

2.3.2 Hermite quadratic forms

Let $f_1, \cdots, f_m \in \mathbb{R}[x_1, \ldots, x_n]$, $I = \langle f_1, \cdots, f_m \rangle$, and $A = \mathbb{R}[x_1, \ldots, x_n]/I$ be the quotient algebra. Write $\mathcal{F} = \{f_1, \cdots, f_m\}$, the set of all polynomials in the polynomial system. Given $f \in A$, we define the linear multiplication map on A by $L(f)(q) := fq$. Let $g \in A$. It can be shown that the function defined by $A \to \mathbb{R} : f \mapsto Trace(L(gf^2))$ is a quadratic form on A. This form is called the *Hermite quadratic form* determined by $\mathcal{F}$ and g.

Given a basis of A, say $\{b_1, \ldots, b_m\}$, then we can form an $m \times m$ matrix representation of the quadratic form, called the Hermite matrix and denoted by $H(\mathcal{F}, g)$. Each entry $H(\mathcal{F}, g)(i, j)$ is given by $Trace(L(gb_ib_j))$. From the basic facts about quadratic forms, we know the rank and the signature are independent of the choice of the basis. We have the following important real root counting theorem. The proof of this theorem can be found in [6].

Proposition 2.7. *Matrix rank of $H(\mathcal{F}, g)$ equals to the number of common complex roots $(x_1, \ldots, x_n)$ of $\mathcal{F}$ such that $g(x_1, \ldots, x_n) \neq 0$. Signature of $H(\mathcal{F}, g)$ equals to the number of common real roots of $\mathcal{F}$ such that $g > 0$ minus the number of common real roots of $\mathcal{F}$ such that $g < 0$.*

Due to the advantages of Groebner bases, we can explicitly find the $m \times m$ real matrix representation of L with respect to this basis $\{b_1, \ldots, b_m\}$. In particular, we can explicitly compute the trace. Therefore, we can explicitly compute the $m \times m$ real matrix $H(\mathcal{F}, g)$. Each entry is the trace of a $m \times m$ real matrix. An algorithm is presented in appendix B. Therefore, we can compute $H(\mathcal{F}, g)$ by Mathematica. We give an example to show how to use proposition 2.7 to count roots of a polynomial system.

Example 2.5. *We use the same polynomial system as in example* 2.1*, again. This system has common zeros* $\{(1,1),(2,8),(3,27),(1,3),(-3,3)\}$*, all in* $\mathbb{R}^2$*. We will use proposition* 2.7 *to verify that there are* 5 *complex common zeros,* 4 *positive common zeros, that is, zeros with both positive coordinates.*

Now we have $\mathcal{F} = \{p,q\}$*. Instead of considering the ideal generated by* p,q *in* $\mathbb{C}[x,y]$*, we let* $I = \langle p,q\rangle$ *in* $\mathbb{R}[x,y]$*.* $A = \mathbb{R}[x,y]/I$*. Using the command* $\mathbf{GroebnerBasis}[\mathcal{F},\{\mathbf{x},\mathbf{y}\},\mathbf{MonomialOrder}\rightarrow\mathbf{DegreeReverseLexicographic}]$*, we compute a Groebner basis* **gb** *of* I *with respect to the grlex order where* $x > y$*.*

A basis of A *is given by* $\{[x^m y^n]\}$ *where* $x^m y^n \notin \langle LT(gb)\rangle) = \langle y^3, xy^2, x^2y, x^4\rangle$ *and it is* $\{1, y, y^2, x, xy, x^2, x^3\}$*. Note that the basis here is different as that we get in example* 2.1*, but the number is the same. Next, we will compute four* 7×7 *Hermite matrices,* $H(\mathcal{F},1)$*,* $H(\mathcal{F},x)$*,* $H(\mathcal{F},y)$ *and* $H(\mathcal{F},xy)$*.*

The multiplication table for the basis $\{1, y, y^2, x, xy, x^2, x^3\}$ *is given by the following.*

	1	y	y^2	x	xy	x^2	x^3
1	1	y	y^2	x	xy	x^2	x^3
y	y	y^2	y^3	xy	xy^2	x^2y	x^3y
y^2	y^2	y^3	y^4	xy^2	xy^3	x^2y^2	x^3y^2
x	x	xy	xy^2	x^2	x^2y	x^3	x^4
xy	xy	xy^2	xy^3	x^2y	x^2y^2	x^3y	x^4y
x^2	x^2	x^2y	x^2y^2	x^3	x^3y	x^4	x^5
x^3	x^3	x^3y	x^3y^2	x^4	x^4y	x^5	x^6

In the case of $H(\mathcal{F},1)$*, we need to compute* 49 *traces,* $Trace(L(b_ib_j)), i,j = 1,\ldots,7$*. However, it is easy to see that in the table above there are only* 19 *different* b_ib_j*. They are* $\{1, y, y^2, y^3, y^4, x, xy, xy^2, xy^3, x^2, x^2y, x^2y^2, x^3, x^3y, x^3y^2, x^4, x^4y, x^5, x^6\}$*. Therefore, we can reduce to only computing* 19 *traces.*

Another observation is that $Trace(L(f)) = \sum_{i=1}^{7} a_i Trace(L(b_i))$*, where* $f = \sum_{i=1}^{7} a_i b_i$ *in* A *and* a_i *are just real numbers. Therefore, we only need to compute* 7 $Trace(L(b_i))$*, where* b_i *are basis elements.*

To compute each $Trace(L(b_i))$*, we need to compute normal forms of* $b_ib_1,\ldots,b_ib_7$ *in* A*, say* $b_ib_j = \sum_{k=1}^{7} a_{ijk} b_k$*, and then* $Trace(L(b_i)) = \sum_{l=1}^{7} a_{ill}$*. However, again we do not need to compute* 49 *normal forms of* b_ib_j*. Instead, we need only to compute* 19 *normal forms. Each normal form is computed by the command*
$\mathbf{PolynomialReduce}[\mathbf{b_ib_j},\mathbf{gb},\{\mathbf{x},\mathbf{y}\},\mathbf{MonomialOrder}\rightarrow$
$\mathbf{DegreeReverseLexicographic}]$

Now we have 19×7 *real numbers* a_{ijk} *after computing* 19 **PolynomialReduce**. *Fixing an order of the* 19 *essentially different* $b_i b_j$, *we then make the following matrices.*

$$ct = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 8 & 5 & 9 & 10 & 11 \\ 3 & 8 & 12 & 9 & 13 & 14 & 15 \\ 4 & 5 & 9 & 6 & 10 & 7 & 16 \\ 5 & 9 & 13 & 10 & 14 & 11 & 17 \\ 6 & 10 & 4 & 7 & 11 & 16 & 18 \\ 7 & 11 & 15 & 16 & 17 & 18 & 19 \end{pmatrix}, List = \begin{pmatrix} a_{ijk} \end{pmatrix}$$

where $ct(i,j)$ *is the order of* $b_i b_j$ *in the* 19*'s different monomials and* $List$ *is the* 19×7 *matrix storing those* a_{ijk}. *Then* $Trace\left(L\left(b_i\right)\right) = \sum_{k=1}^{7} List(ct(i,k),k)$.

Next, all the 19 *different* $Trace\left(L\left(b_i b_j\right)\right)$*'s can therefore be obtained by the formula* $\sum_{k=1}^{7} List(l,k) Trace\left(L\left(b_k\right)\right)$, *where* $l = 1, \ldots, 19$. *Therefore, we form the Hermite matrix* $H\left(\mathcal{F}, 1\right)$ *from them.*

Similarly, we compute $H\left(\mathcal{F}, x\right)$, $H\left(\mathcal{F}, y\right)$ *and* $H\left(\mathcal{F}, xy\right)$. *Applying the command* **MatrixRank** *to compute the matrix rank of* $H\left(\mathcal{F}, x\right) = 5$, *we verify that there are* 5 *complex common zeros. By the command* **Sig**, *we compute the signatures of the four matrices and get* $5, 3, 5, 3$.

Next, let $s, t, u, v, \varepsilon, \delta, \varepsilon', \delta'$ *denote the number of real roots* (x, y) *having the signs* $(+,+), (+,-), (-,+), (-,-), (0,+), (0,-), (+,0), (-,0)$, *respectively. We get*

$$\begin{aligned} 5 &= s + t + u + v + \varepsilon + \delta + \varepsilon' + \delta' \\ 3 &= s + t - u - v + \varepsilon' - \delta' \\ 5 &= s - t + u - v + \varepsilon - \delta \\ 3 &= s - t - u + v \end{aligned}$$

A simple observation tell us that $\varepsilon = \delta = \varepsilon' = \delta' = 0$. *Therefore, we get* $(s, t, u, v) = (4, 0, 1, 0)$. *So, we verify that there is one root with negative* x*-coordinate and positive* y*-coordinate,* $(-3, 3)$, *and* 4 *positive roots of* p *and* q, $\{(1,1), (2,8), (3,27), (1,3)\}$.

Detailed algorithms for computing the Hermite matrix and signatures of real symmetric matrix are presented in appendix B. While it is not hard to compute by hand, **MatrixRank** is implemented in Mathematica to compute the rank of a real matrix.

Chapter 3

Parametric polynomial systems

In this chapter, we develop and present ideas and methods of counting positive zeros of parametric polynomial systems based on the standard computer algebra tools introduced in Chapter 2. Since counting positive zeros of systems $\{f_1, f_2\}, \ldots, \{f_9, f_{10}\}$ is the goal of this thesis, we will especially present methods that work for parametric polynomial systems having properties satisfied by those 5 systems.

3.1 Counting positive zeros

Consider a parametric polynomial system with the same numbers of variables and polynomials and denote it by $\mathcal{F} = \{f_1, \cdots, f_n\}$, where $f_i \in \mathbb{D}[x_1, \ldots, x_n]$ and $\mathbb{D} = \mathbb{Q}[a_1, \ldots, a_l]$. We first define the *specialization* at a point $a \in \mathbb{R}^l$. It is a ring homomorphism $\varphi^a : \mathbb{D}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$ such that $\varphi^a(f)$ is the real polynomial after substituting the parameter with the value a for $f \in \mathbb{D}[x_1, \ldots, x_n]$. We denote $\varphi^a(f) = f^a$ and $\varphi^a(\mathcal{F}) = \{f^a | f \in \mathcal{F}\} = \mathcal{F}^a$.

Assume that $\mathcal{F}^a$ has finitely many complex common zeros for almost all points $a \in \mathbb{R}^l$. Specifically, we assume that there exists a nonzero polynomial $h \in \mathbb{R}[a_1, \ldots, a_l]$ such that $V_{\mathbb{C}}(\mathcal{F}^a)$ have finitely many elements in $\mathbb{C}^n$ for $a \in \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$. We call an open subset $\mathcal{P} \subseteq \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$ the parameter space if it contains all parameters $a \in \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$ where zeros of systems $\mathcal{F}^a$ are meaningful to us. Groebner bases can be applied to find a sufficient condition for $\mathcal{F}^a$ to have finitely many zeros for almost all a and at the same time find such nonzero h when the condition is satisfied. Once h is found and $\mathcal{P}$ is decided, our goal is to count the number of positive zeros of $\mathcal{F}^a$ for every $a \in \mathcal{P}$.

Now we present how to apply Groebner bases in the following steps. First, for each polynomial $f_i \in \mathcal{F}$, we view $f_i \in \mathbb{Q}[a_1, \dots, a_l, x_1, \dots, x_n]$ and form the ideal $I = \langle f_1, \cdots, f_n \rangle \subseteq \mathbb{Q}[a_1, \dots, a_l, x_1, \dots, x_n]$. Note that we have the following property.

Proposition 3.1. *$I^a = \langle f_1^a, \cdots, f_n^a \rangle \subseteq \mathbb{R}[x_1, \dots, x_n]$, for all $a \in \mathbb{R}^l$.*

Proof. If $f \in I^a$, then $f = \varphi_a(h_1 f_1 + \cdots + h_n f_n)$, for some $h_i \in \mathbb{R}[a_1, \dots, a_l, x_1, \dots, x_n]$. So, $f = h_1^a f_1^a + \cdots + h_n^a f_n^a$ by the definition of ring homomorphism. We have $h_i^a \in \mathbb{R}[x_1, \dots, x_n]$. Therefore, $f \in \langle f_1^a, \cdots, f_n^a \rangle$. Conversely, if $f = k_1 f_1^a + \cdots + k_n f_n^a$, where $k_j \in \mathbb{R}[x_1, \dots, x_n]$, then $f = \varphi_a(k_1 f_1 + \cdots + k_n f_n)$ since $k_j = k_j^a$ for all j, a. $\square$

Next, fix a monomial order $<$ such that any monomial involving one of the x_i's is greater than all monomials in $a_1, \dots, a_l$ alone. This monomial order can be given by a $(n+l) \times (n+l)$ matrix $\mathbf{w}$ with two diagonal blocks and zero outside these two blocks. The first $n \times n$ block gives a monomial order matrix for x_i's and the second $l \times l$ block gives a monomial order matrix for a_j's. Monomial orders defined by such matrices are called *block orders.* We denote $<_x$ the monomial order reduced from $<$ on monomials in x_i's. It is actually given by the matrix of the first $n \times n$ block.

Then, by applying the command in Mathematica:

$$\mathbf{GroebnerBasis}[\{\mathbf{f_1}, \cdots, \mathbf{f_n}\}, , \{\mathbf{x_1}, \dots, \mathbf{x_n}, \mathbf{a_1}, \dots, \mathbf{a_l}\}, \mathbf{MonomialOrder} \to \mathbf{w}],$$

we get $G = \{g_1, \dots, g_t\}$, a Groebner basis of $\langle f_1, \cdots, f_n \rangle$ in $\mathbb{Q}[a_1, \dots, a_l, x_1, \dots, x_n]$ with the block order $<$. From [13], we have the important result.

Proposition 3.2. *Let $h_i = LC_{<_x}(g_i) \in \mathbb{Q}[a_1, \dots, a_l]$. G^a is a Groebner basis of $I^a = \langle f_1^a, \cdots, f_n^a \rangle$ in $\mathbb{R}[x_1, \dots, x_n]$ for all $a \in \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$, where $h = h_1 \cdots h_t$.*

Denote $A^a = \mathbb{C}[x_1, \dots, x_n] / \langle f_1^a, \cdots, f_n^a \rangle$ for all $a \in \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$. If, for $i = 1, \dots, n$, there exists $\beta_i \in \mathbb{Z}_{\geq 0}$ such that $x_i^{\beta_i} = LM_{<_x}(g_i)$ for some $g_i \in G$, then, by proposition 2.1, A^a has a basis $\{x_1^{\alpha_1} \cdots x_n^{\alpha_i} \notin \langle LT_{<_x}(g_1^a), \dots, LT_{<_x}(g_t^a) \rangle\}$, and $V_{\mathbb{C}}(f_1^a, \cdots, f_n^a)$ has at most $dim(A^a)$ elements in $\mathbb{C}^n$. Moreover, since the leading coefficients in g_i^a are all nonzero, sets $\{LT_{<_x}(g_1^a), \dots, LT_{<_x}(g_t^a)\}$ are the same and so are the bases of $dim(A^a)$ for all $a \in \mathbb{R}^l \setminus V(h)$. Therefore, $dim(A^a)$ is a constant independent of $a \in \mathbb{R}^l \setminus V(h)$.

In conclusion, existence of $g_i \in G$ such that $x_i^{\beta_i} = LM_{<_x}(g_i)$ for some $\beta_i \in \mathbb{Z}_{\geq 0}$ gives a sufficient condition to guarantee $\mathcal{F}^a$ has finitely many complex zeros for almost all points a in $\mathbb{R}^l$. Under that condition, we have an *universal upper bound* of the number of complex zeros for all real polynomial system $\mathcal{F}^a$.

The set $\mathbb{R}^l \setminus V_{\mathbb{R}}(h)$ is a union of finitely many open connected subsets. Therefore, we can also choose the parameter space $\mathcal{P}$ to be a union of finitely many open connected subsets. Now, we outline a method of counting the number of positive common roots of $\mathcal{F}^a$ for every $a \in \mathcal{P}$.

The first and critical step is to find the *bifurcation set* $\mathcal{B}$ in $\mathcal{P}$. By definition, $\mathcal{B} = \pi(V_{\mathbb{C}}(\mathcal{F}, J) \cap (\mathcal{P} \times \mathbb{R}_+^n))$, where $J = |\dfrac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}|$ is the Jacobian determinant, $\mathbb{R}_+$ is the set of positive real numbers, and π is the canonical projection onto $\mathbb{C}^l$. In the definition, $\mathcal{F}$ and J are viewed as polynomials in $\mathbb{Q}[a_1, \ldots, a_l, x_1, \ldots, x_n]$. By finding $\mathcal{B}$, we mean being able to find a non-zero polynomial f in $\mathbb{Q}[a_1, \ldots, a_l]$ such that $\mathcal{B} \subseteq V_{\mathbb{R}}(f)$. We call f a *bifurcation polynomial*.

We will present methods to find f in the next section. Given a bifurcation polynomial f, $\mathcal{P}$ is then partitioned into $\mathcal{P} \cap V_{\mathbb{R}}(f)$ and $\mathcal{P} \setminus V_{\mathbb{R}}(f)$. The later is again a finite union of open connected subsets of $\mathcal{P}$. We call each subset an open connected component of $\mathcal{P}$. On each component, the number of positive zeros will be a constant as shown in the following proposition.

Proposition 3.3. *Let $\mathcal{N}$ be the open subset of $\mathbb{R}^n$ consisting all points with no coordinates being zero. Assume our system $\mathcal{F}$ satisfies the following three conditions.*

1. *$\mathbb{R}^n \setminus \mathcal{N}$ contains either points which are not zeros for all systems $\{\mathcal{F}^a | a \in \mathcal{P}\}$ or points which are zeros for all systems $\{\mathcal{F}^a, J^a | a \in \mathcal{P}\}$.*
2. *The number of points in the latter case is finite. We denote them by $\{z_1, \ldots, z_t\}$.*
3. *For each i and $a \in \mathcal{P}$, there exist neighborhoods $\mathcal{U}$ containing a and $\mathcal{V}$ containing z_i such that z_i is the only real zeros in $\mathcal{V}$ of systems $\{\mathcal{F}^b | b \in \mathcal{U}\}$.*

Then the number of positive zeros is the same for all systems $\{\mathcal{F}^a | a \in \mathcal{O}\}$, where $\mathcal{O}$ is one of the nonempty open connected component of $\mathcal{P}$.

Proof. We already know that $\mathcal{F}^a$ has at most $d = dim(A^a)$ complex zeros for all $a \in \mathcal{P}$. We can write $\mathcal{O} = \cup_{j=0}^{d} \mathcal{O}_j$, where systems $\{\mathcal{F}^a | a \in \mathcal{O}_j\}$ have j positive zeros. Let $a_0 \in \mathcal{O}$. Assume $\mathcal{F}^{a_0}$ has m positive zeros. Then $m \leq d$ and $\mathcal{O}_m \neq \emptyset$. We will claim that $\mathcal{O}_j$ are open sets for all j. Since $\mathcal{O}$ is connected, we get $\mathcal{O} = \mathcal{O}_m$.

For any $a \in \mathcal{O}_j$, since it is not in $\mathcal{B} = \pi(V_{\mathbb{C}}(\mathcal{F}, J) \cap (\mathcal{P} \times \mathbb{R}_+^n))$, we have $J(a, z) \neq 0$ for all positive zeros z of $\mathcal{F}^a$. Therefore, by the implicit function theorem, we know that z continues in a neighborhood of a. If the neighborhood is sufficient small, all the continuations of positive zeros are still positive zeros. If the boundary of $\mathbb{R}^n_+$ contains all points which are not zeros for all system $\{\mathcal{F}^a | a \in \mathcal{P}\}$, then the number of positive zeros will be the same in a neighborhood of a. Therefore, $\mathcal{O}_j$ is open.

Otherwise, without loss of generality , we can assume that only $(0, \ldots, 0)$ is a zero for all systems $\{\mathcal{F}^a, J^a | a \in \mathcal{P}\}$. By the hypothesis, we can find a neighborhood of a such that all the continuations of positive zeros are away from $(0, \ldots, 0)$ and also that there is no other zeros near $(0, \ldots, 0)$ except for itself. Therefore, the number of positive zeros is again j in a neighborhood of a. So, $\mathcal{O}_j$ is open, too. $\square$

Therefore, if we can find at least one sample point from each open connected components, we can count positive zeros of finitely many real polynomial systems by proposition 2.7 in Chapter 2 to get the generic results.

For $a \in \mathcal{P} \cap V_{\mathbb{R}}(f)$ but not in the bifurcation set, the number of positive zeros of $\mathcal{F}^a$ is the same as that of $\mathcal{F}^b$ for some $b \in \mathcal{P} \setminus V_{\mathbb{R}}(f)$ close enough to a by the same argument in the proof above. The resulting positive root count for $\mathcal{F}^a$ will be among the finitely different numbers from the generic results. As for systems $\mathcal{F}^a$ where $a \in \mathcal{B}$, we will discuss how to count their positive zeros in section 3.3.

3.2 Finding bifurcation sets

In this section, we present two methods of finding a bifurcation polynomial f for system $\mathcal{F}$. We first present some basic facts about intersection multiplicity. Both methods make use of these facts to find such f. In Chapter 2, we defined intersection multiplicities of common zeros of two plane curves with finitely many points of intersection. This definition can easily be extend to more than two variables. Here we state the definition and list some important properties.

Lemma 3.1. *Let $I = \langle f_1, \cdots, f_n \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]$, $A = \mathbb{C}[x_1, \ldots, x_n]/I$ with $dim(A) < \infty$. $\{z_1, \ldots, z_m\} = V_{\mathbb{C}}(f_1, \cdots, f_n) \subseteq \mathbb{C}^n$. The intersection multiplicity of z_i is $mul(z_i) = dim(\mathcal{O}_{z_i}/I\mathcal{O}_{z_i})$, where $\mathcal{O}_{z_i} = \left\{\frac{p}{q} \mid p, q \in \mathbb{C}[x_1, \ldots, x_n], q(z_i) \neq 0\right\}$ and $I\mathcal{O}_{z_i} = f_1\mathcal{O}_{z_i} + \cdots + f_n\mathcal{O}_{z_i}$. We have the following facts.*

1. $mul(z_i) \geq 1$,
2. $dim(A) = \sum_{i=1}^m mul(z_i)$,
3. $J(z_i) = |\dfrac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}(z_i)| = 0 \Leftrightarrow mul(z_i) \geq 2$.

Proof. We prove the first part and one direction of the third part. The proof of second part can be found in [9] and the other direction of part 3 is a result of Nakayama's lemma [3].

For the first part, we have $dim(\mathcal{O}_{z_i}/I\mathcal{O}_{z_i}) = 0 \Leftrightarrow \mathcal{O}_{z_i} = I\mathcal{O}_{z_i} \Leftrightarrow 1 = f_1\frac{p_1}{q_1} + \cdots + f_n\frac{p_n}{q_n}$, for some p_j, q_j with $q_j(z_i)$ all nonzero. If we multiply $q = q_1 \cdots q_n$ on both sides of the equation, we get $q = f_1P_1 + \cdots + f_nP_n, P_j = qp_j/q_j \in \mathbb{C}[x_1, \ldots, x_n]$. Since $z_i \in V_{\mathbb{C}}(f_1, \cdots, f_n)$, the right hand side equals to zero after substituting z_i into it. However, the left hand side is equal to $q_1(z_i) \cdots q_n(z_i) \neq 0$. Therefore, we cannot have $dim(\mathcal{O}_{z_i}/I\mathcal{O}_{z_i}) = 0$, so $mul(z_i) \geq 1$.

Next, we come to the third part. Suppose there exists a complex zero z_i such that $J(z_i) = 0$. Without loss of generality, we may assume that $z_i = 0$. Therefore, we can write $f_j = \sum_{k=1}^{n} a_{jk}x_k + r_j = \triangle_j + r_j$, where a_{jk} are constants and r_j contain terms of order greater than two. Then the Jacobian determinant is $det(a_{jk})$.

Let $M_{z_i} = x_1\mathcal{O}_{z_i} + \cdots + x_n\mathcal{O}_{z_i}$ and $M_{z_i}^2$ be the ideal in $\mathcal{O}_{z_i}$ generated by all terms of the form x_ux_v. Since $det(a_{jk}) = 0$, $\triangle_j$'s are not linearly independent, say $\triangle_n$ is a linear combination of the others. Then there exist $b_1, \ldots, b_{n-1}$ such that $\triangle_n = \sum_{j=1}^{n-1} b_j\triangle_j$. Therefore, $(I\mathcal{O}_{z_i} + M_{z_i}^2)/M_{z_i}^2 = Span(\triangle_1, \ldots, \triangle_{n-1})$. So, $dim((I\mathcal{O}_{z_i} + M_{z_i}^2)/M_{z_i}^2) \leq n - 1$.

Also, by the inclusion $I\mathcal{O}_{z_i} \subseteq M_{z_i} \subseteq \mathcal{O}_{z_i}$, we get $dim(\mathcal{O}_{z_i}/M_{z_i}) + dim(M_{z_i}/I\mathcal{O}_{z_i}) = dim(\mathcal{O}_{z_i}/I\mathcal{O}_{z_i})$. Similarly, by the inclusion $M_{z_i}^2 \subseteq M_{z_i} \subseteq \mathcal{O}_{z_i}$, we have $dim(\mathcal{O}_{z_i}/M_{z_i}) + dim(M_{z_i}/M_{z_i}^2) = dim(\mathcal{O}_{z_i}/M_{z_i}^2)$. It is also easy to see that $\mathcal{O}_{z_i}/M_{z_i} = Span(1)$, and $\mathcal{O}_{z_i}/M_{z_i}^2 = Span(1, x_1, \ldots, x_n)$. Note that $\{1, x_1, \ldots, x_n\}$ are linearly independent. Therefore, we have $dim(\mathcal{O}_{z_i}/M_{z_i}) = 1, dim(\mathcal{O}_{z_i}/M_{z_i}^2) = n + 1$. Then the two equations above becomes $1 + dim(M_{z_i}/I\mathcal{O}_{z_i}) = dim(\mathcal{O}_{z_i}/I\mathcal{O}_{z_i}), 1 + dim(M_{z_i}/M_{z_i}^2) = n + 1$.

If we have $mul(z_i) = 1$, then $dim(M_{z_i}/I\mathcal{O}_{z_i}) = 0$ and $M_{z_i} = I\mathcal{O}_{z_i}$. Therefore, $(I\mathcal{O}_{z_i} + M_{z_i}^2)/M_{z_i}^2 = (M_{z_i} + M_{z_i}^2)/M_{z_i}^2 = M_{z_i}/M_{z_i}^2$. But, the dimension of the most right is n, while the dimension of the most left is strictly smaller than n. We come to a contradiction. □

3.2.1 Hermite matrix method

Recall that our system $\mathcal{F} = \{f_1, \cdots, f_n\}$ has the property that there exist $g_i \in G$ such that $x_i^{\beta_i} = LM_{<_x}(g_i)$ for some $\beta_i \in \mathbb{Z}_{\geq 0}$, where G is a Groebner basis of $I = \langle f_1, \cdots, f_n \rangle \subseteq \mathbb{Q}[a_1, \ldots, a_l, x_1, \ldots, x_n]$. There are at most d complex zeros for all systems $\{\mathcal{F}^a | a \in \mathcal{P}\}$, where $d = dim(A^a), A^a = \mathbb{C}[x_1, \ldots, x_n]/I^a$ for all $a \in \mathcal{P}$.

We now view $f_i \in \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K} = \{\frac{p}{q} \mid p, q \in \mathbb{Q}[a_1, \ldots, a_l], q \neq 0\}$, the field of fraction of $\mathbb{Q}[a_1, \ldots, a_l]$. Forming the ideal $J = \langle f_1, \cdots, f_n \rangle \subseteq \mathbb{K}[x_1, \ldots, x_n]$, we have the following property.

Lemma 3.2. *Viewing $G \subseteq \mathbb{K}[x_1, \ldots, x_n]$, we have that G is a Groebner basis of J with respect to the monomial order $<_x$.*

Proof. We consider two cases. Suppose first that there exists a $g \in G$ such that $g \in \mathbb{Q}[a_1, \ldots, a_l]$. Since $g \in I$, there exist $h_1, \ldots, h_n \in \mathbb{Q}[a_1, \ldots, a_l, x_1, \ldots, x_n]$ such that $g = \sum_{j=1}^n h_j f_j$. Therefore, $1 = \sum_{j=1}^n \frac{h_j}{g} f_j$. So, $1 \in J$ and $J = \mathbb{K}[x_1, \ldots, x_n]$. We have $\langle LT_{<_x}(J) \rangle = \mathbb{K}[x_1, \ldots, x_n] = \langle LT_{<_x}(g) \rangle$. Therefore, by definition, G is a Groebner basis of J.

Next, suppose none of elements in G belong to $\mathbb{Q}[a_1, \ldots, a_l]$. For any $f \in J$, there exist $H_1, \ldots, H_n \in \mathbb{K}[x_1, \ldots, x_n]$ such that $f = \sum_{k=1}^n H_i f_i$. Let $H \in \mathbb{Q}[a_1, \ldots, a_l]$ be the least common factor of all denominator in the coefficients of H_i's. We have $Hf = \sum_{k=1}^n HH_i f_i \in I$. Since G is a Groebner basis of of I, $\langle LT_<(I) \rangle = \langle LT_<(g_1), \ldots, LT_<(g_t) \rangle$. Therefore, $LT_<(Hf) \in \langle LT_<(g_1), \ldots, LT_<(g_t) \rangle$ and also $LM_<(Hf) \in \langle LM_<(g_1), \ldots, LM_<(g_t) \rangle$. Since $LM_<(Hf)$ contains only one monomial, we have that it is divided by $LM_<(g_f)$ for some $g_f \in G$.

Write $LM_<(g_f) = a_1^{\alpha_1} \cdots a_l^{\alpha_l} x_1^{\beta_1} \cdots x_n^{\beta_n}$ and $LM_<(Hf) = a_1^{\alpha'_1} \cdots a_l^{\alpha'_l} x_1^{\beta'_1} \cdots x_n^{\beta'_n}$. We have $\alpha_i \leq \alpha'_i$ and $\beta_j \leq \beta'_j$ for all i, j. We will claim that $LM_{<_x}(g_f) = x_1^{\beta_1} \cdots x_n^{\beta_n}$, and $LM_{<_x}(f) = x_1^{\beta'_1} \cdots x_n^{\beta'_n}$. That is, $LM_{<_x}(g_f)$ divide $LM_{<_x}(f)$. Since f is an arbitrary element in J, we have $\langle LT_{<_x}(J) \rangle = \langle LT_{<_x}(g_1), \ldots, LT_{<_x}(g_t) \rangle$. Therefore, by definition, G is a Groebner basis of J.

Now, we prove our claims. Suppose $LM_{<_x}(g_f) = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, where $(\beta_1, \ldots, \beta_n) <_x (\gamma_1, \ldots, \gamma_n)$. Since $g_f \in G \subseteq \mathbb{Q}[a_1, \ldots, a_l, x_1, \ldots, x_n]$, there exist $\delta_1, \ldots, \delta_l$ such that $a_1^{\delta_1} \cdots a_l^{\delta_l} x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ is a monomial of g_f. Since the monomial order $<$ is a block order where x_i's dominate over a_j's, we get $LM_<(g_f) = a_1^{\delta_1} \cdots a_l^{\delta_l} x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, a contradiction.

Finally, suppose $LM_{<_x}(f) = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, where $(\beta'_1, \ldots, \beta'_n) <_x (\epsilon_1, \ldots, \epsilon_n)$. Since $Hf \in \mathbb{Q}[a_1, \ldots, a_l, x_1, \ldots, x_n]$ and $H \in \mathbb{Q}[a_1, \ldots, a_l]$, Hf contains two monomials $a_1^{\alpha'_1} \cdots a_l^{\alpha'_l} x_1^{\beta'_1} \cdots x_n^{\beta'_n}$ and $a_1^{\eta_1} \cdots a_l^{\eta_l} x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ for some η_j's. The latter is greater than the former with respect to $<$ since again x_i's dominate in $<$. Therefore, $LM_<(Hf) \neq a_1^{\alpha'_1} \cdots a_l^{\alpha'_l} x_1^{\beta'_1} \cdots x_n^{\beta'_n}$, which is a contradiction. $\square$

We consider the algebra $A = \mathbb{K}[x_1, \ldots, x_n] / J$ over $\mathbb{K}$. Since there exist $g_i \in G$ such that $x_i^{\beta_i} = LM_{<_x}(g_i)$ for some $\beta_i \in \mathbb{Z}_{\geq 0}$, we can find a finite basis for A and there are d elements in it. It is the set $\{x_1^{\alpha_1} \cdots x_n^{\alpha_i} \notin \langle LT_{<_x}(g_1), \ldots, LT_{<_x}(g_t)\rangle\}$ denoted by $\mathcal{B} = \{b_1, \ldots, b_d\}$.

We also have the Hermite quadratic form $A \to \mathbb{K} : f \mapsto Trace\,(L\,(gf^2))$ and the Hermite matrices $H\,(\mathcal{F}, g)$, where $g \in \mathbb{Q}\,[x_1, \ldots, x_n]$ and L is the linear multiplication map on A. Each entry $H\,(\mathcal{F}, g)\,(i, j)$ is given by $Trace\,(L\,(gb_ib_j))$. Let $h = LC_{<_x}(g_1) \cdots LC_{<_x}(g_t) \in \mathbb{Q}\,[a_1, \ldots, a_l]$ as before. We have the specialization property for Hermite matrices.

Lemma 3.3. *$\varphi^a(H\,(\mathcal{F}, g)) = H\,(\mathcal{F}^a, g^a)$ for all $a \in \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$.*

Here applying φ^a on a matrix with entries in $\mathbb{K}$ means the specialization of both denominator and numerator of each entries. Why this makes sense is explained in the proof. By this lemma, if we want to compute $H\,(\mathcal{F}^a, g^a)$ for all $a \in \mathcal{P}$, we just have to compute $H\,(\mathcal{F}, g)$ and then substitute a into entries of it to get the real symmetric Hermite matrix $H\,(\mathcal{F}^a, g^a)$.

The algorithm to compute $H\,(\mathcal{F}, g)$ is similar to that in the appendix B. There is one major difference. In the command **PolynomialReduce**, the input variables should be only x_i's, the monomial order is $<_x$ and we need to assign the coefficient domain to be **RationalFunctions**. See an example of the usage of this function in appendix A. Our input **gb** should be G that is indeed a Groebner basis if J in $\mathbb{K}\,[x_1, \ldots, x_n]$. Therefore, command **PolynomialReduce** correctly computes the normal forms in A.

Proof. Fix $a \in \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$. Since we only consider $g \in \mathbb{Q}\,[x_1, \ldots, x_n]$ and $g^a = g$, the role of g is not important when we deal with specializations. Therefore, we can assume $g = 1$ without loss of generality. By the discussion in example 2.5, it suffices to prove that, for each b_k, $\varphi_a(Trace\,(L\,(b_k))) = Trace\,(L^a\,(b_k))$, where $L^a\,(b_k)$ is the multiplication map by b_k on the algebra $A^a = \mathbb{R}\,[x_1, \ldots, x_n]\,/I^a$.

Since $\mathcal{B}$ is a basis for both A and A^a, we can express these two multiplication maps as matrices in the basis $\mathcal{B}$ and also denote them by $L\,(b_k)$ and $L^a\,(b_k)$. Note that $L\,(b_k)\,(i, j)$ are in $\mathbb{K}$, while $L^a\,(b_k)\,(i, j)$ are in $\mathbb{R}$. We need to prove that $\varphi^a(L\,(b_k)\,(i, j)) = L^a\,(b_k)\,(i, j)$ for all i, j. Also note that b_k are just monomials in $x_1, \ldots, x_n$ and, therefore, $\varphi^a(b_k) = (b_k)$.

For each b_k, $b_kb_j = \sum_{i=1}^m L\,(b_k)\,(i, j)b_i$ in A. Equivalently, we have $b_kb_j = \sum_{i=1}^t h_ig_i + \sum_{i=1}^m L\,(b_k)\,(i, j)b_i$ for some $h_i \in \mathbb{K}\,[x_1, \ldots, x_n]$. Therefore, $b_kb_j = \varphi^a(b_kb_j) = \sum_{i=1}^t h_i^a g_i^a + \sum_{i=1}^m \varphi^a(L\,(b_k)\,(i, j))b_i$. Note here, by the division algorithm

[8], each $L\left(b_k\right)(i,j)$ and coefficients of h_i can only have denominators of the form ch^α for some $\alpha \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{C}$. Therefore, specializations of them at $a \in \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$ make sense. Since $a \in \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$, G^a is a Groebner basis for I^a. Therefore, by the uniqueness of remainder of $b_k b_j$, we have $\varphi^a(L\left(b_k\right)(i,j)) = L^a\left(b_k\right)(i,j)$. □

Proposition 3.4. *$det(H\left(\mathcal{F},1\right)) \in \mathbb{K}$ is not the zero element if and only if the set $\pi(V_{\mathbb{C}}(\mathcal{F},J) \cap (\mathcal{P} \times \mathbb{C}^n))$ is strictly contained in $\mathcal{P}$. Moreover, if $det(H\left(\mathcal{F},1\right))$ is not the zero and has numerator $f \in \mathbb{Q}\left[a_1,\ldots,a_l\right]$, then $\pi(V_{\mathbb{C}}(\mathcal{F},J) \cap (\mathcal{P} \times \mathbb{C}^n)) = V_{\mathbb{R}}(f) \cap \mathcal{P}$, and, therefore, $\mathcal{B} \subseteq V_{\mathbb{R}}(f)$.*

Proof. $det(H\left(\mathcal{F},1\right)) \equiv 0 \Leftrightarrow det(H\left(\mathcal{F}^a,1\right)) = 0,\ \forall a \in \mathcal{P} \Leftrightarrow$ rank of $H\left(\mathcal{F}^a,1\right) < d,\ \forall a \in \mathcal{P} \Leftrightarrow$ the number of distinct complex zeros is strictly smaller than d for all systems $\{\mathcal{F}^a | a \in \mathcal{P}\} \Leftrightarrow$ there exist complex zeros $z(a)$ with $mul(z(a)) \geq 2$ for all systems $\{\mathcal{F}^a | a \in \mathcal{P}\} \Leftrightarrow \pi(V_{\mathbb{C}}(\mathcal{F},J) \cap (\mathcal{P} \times \mathbb{C}^n)) = \mathcal{P}$.

If $det(H\left(\mathcal{F},1\right))$ has numerator f not identically equal to zero, we have similarly $\pi(V_{\mathbb{C}}(\mathcal{F},J) \cap (\mathcal{P} \times \mathbb{C}^n)) = V_{\mathbb{R}}(f) \cap \mathcal{P}$. Since $\pi(V_{\mathbb{C}}(\mathcal{F},J) \cap (\mathcal{P} \times \mathbb{R}^n_+)) \subseteq \pi(V_{\mathbb{C}}(\mathbb{F},J) \cap (\mathcal{P} \times \mathbb{C}^n))$, we get $\mathcal{B} \subseteq V_{\mathbb{R}}(f)$. □

Here we use Groebner bases to compute $H\left(\mathcal{F},1\right)$ and find a bifurcation polynomial of the system $\mathcal{F}$. An example is given in Chapter 4 in solving system $\{f_5, f_6\}$. We need $det(H\left(\mathcal{F},1\right))$ is not identically equal to zero. When there exist common zeros for all systems $\{\mathcal{F}^a, J^a | a \in \mathcal{P}\}$, we get $\pi(V_{\mathbb{C}}(\mathcal{F},J) \cap (\mathcal{P} \times \mathbb{C}^n)) = \mathcal{P}$ and $det(H\left(\mathcal{F},1\right)) \equiv 0$. Therefore, we can not find a nonzero bifurcation polynomial directly by proposition 3.4.

Assume those common zeros are $\{z_1,\ldots,z_t\}$ and none of them are positive zeros. Our strategy is to find another parametric polynomial system $\mathcal{G}$ with m variables $\{x_1,\ldots,x_n,x_{n+1},\ldots,x_m\}$ and m polynomial equations in $\mathbb{Q}\left[a_1,\ldots,a_l\right]\left[x_1,\ldots,x_m\right]$ such that $V_{\mathbb{C}}(\mathcal{G}^a)$ is in one to one correspondence with $V_{\mathbb{C}}(\mathcal{F}^a) \setminus \{z_1,\ldots,z_t\}$ by the canonical projection from $\mathbb{C}^n \times \mathbb{C}^{m-n}$ onto $\mathbb{C}^n$ for all $a \in \mathcal{P}$.

If $H\left(\mathcal{G},1\right)$ is not identically equal to zero, then $\pi(V_{\mathbb{C}}(\mathcal{G},J) \cap (\mathcal{P} \times \mathbb{C}^m)) = V_{\mathbb{R}}(g) \cap \mathcal{P}$ for some nonzero polynomial g. Counting positive zeros for $\mathcal{F}$ is then equivalent to counting real zeros of $\mathcal{G}$ with positive first n coordinates. Then, by defining the bifurcation set as $\pi(V_{\mathbb{C}}(\mathcal{G},J) \cap (\mathcal{P} \times \mathbb{R}^n_+ \times \mathbb{C}^{m-n}))$, we get a bifurcation polynomial g for the system $\mathcal{G}$. We will demonstrate how to apply this strategy when solving system $\{f_3, f_4\}$ in Chapter 4.

The strategy above increases the number of variables and equations and hence the complexity of computing Groebner bases in order to compute $H\left(\mathcal{G},1\right)$. Therefore, this strategy may not be practical in some situations. Alternatively, when there are only two variables and equations, resultant method provides another strategy to find a bifurcation polynomial when there are common zeros for all systems $\{\mathcal{F}^a, J^a | a \in \mathcal{P}\}$. This will not require more complicated Groebner basis computations.

3.2.2 Resultant method

Let $\mathcal{F} = \{p, q\}$, $p, q \in \mathbb{Q}[a_1, \ldots, a_l][x, y]$. As before, assume there exists h such that $\mathcal{F}^a$ has finitely many complex zeros for all $a \in \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$. $\mathcal{P} \subseteq \mathbb{R}^l \setminus V_{\mathbb{R}}(h)$ is the parameter space. And, $\mathcal{B} = \pi(V_{\mathbb{C}}(p, q, J) \cap (\mathcal{P} \times \mathbb{R}^2_+))$ is the bifurcation set, where $J = |\dfrac{\partial(p,q)}{\partial(x,y)}|$.

Writing $p = \sum_{i=0}^m \alpha_i(y)x^i, q = \sum_{i=0}^n \beta_i(y)x^i$, where $\alpha_i(y), \beta_i(y) \in \mathbb{Q}[a_1, \ldots, a_l][y]$, we compute the resultant $Res(p, q, x)$. It is in $\mathbb{Q}[a_1, \ldots, a_l][y]$. After restricting $m \leq n+1$, we compute subresultant sequences $Sres_j(p, q, x) \in \mathbb{Q}[a_1, \ldots, a_l][y][x] = \mathbb{Q}[a_1, \ldots, a_l][x, y]$. They have *good specialization* properties.

Proposition 3.5. *For all $a \in \mathbb{R}^l$, $\varphi^a(Res(p, q, x)) = Res(p^a, q^a, x) \in \mathbb{R}[y]$ and $\varphi^a(Sres_j(p, q, x)) = Sres_j(p^a, q^a, x) \in \mathbb{R}[y][x] = \mathbb{R}[x, y]$ for all j.*

These properties are easy to get just by observing that the definition of resultants and subresultant sequences come from computing the determinants of Sylvester matrix and j-th Sylvester matrices, whose entries are in the domain $\mathbb{Q}[a_1, \ldots, a_l][y]$. Also, we have similar results for $Res(p, q, y)$ and $Sres_j(p, q, y)$ when writing $p = \sum_{i=0}^s \gamma_i(x)y^i, q = \sum_{i=0}^t \delta_i(x)y^i$ with $s \leq t+1$, where $\gamma_i(x), \delta_i(x) \in \mathbb{Q}[a_1, \ldots, a_l][x]$.

Proposition 3.6. *If, for all $a \in \mathcal{P}$, $\alpha_m^a(y), \beta_n^a(y)$ do not vanish simultaneously and the same for $\gamma_s^a(x), \delta_t^a(x)$, then $\mathcal{B} \subseteq V_{\mathbb{R}}(r_x, r_y) \cap \mathcal{P}$, where $r_x = Res(h_x, \partial_x h_x, x), r_y = Res(h_y, \partial_y h_y, y)$, and $h_y = Res(p, q, x), h_x = Res(p, q, y)$.*

Proof. Let $a_0 \in \mathcal{B}$. Then, there exists a positive zero (x_0, y_0) such that $J^{a_0}(x_0, y_0) = 0$. By lemma 3.1, we have $mul(x_0, y_0) \geq 2$ in the system $\mathcal{F}^{a_0}$. By proposition 2.4, we know that $h_y^{a_0} = Res(p^{a_0}, q^{a_0}, x)$ has y_0 as a zero with multiplicity ≥ 2 and $h_x^{a_0} = Res(p^{a_0}, q^{a_0}, y)$ has x_0 as a zero with multiplicity ≥ 2. Therefore, $h_y^{a_0}, (\partial_y h_y)^{a_0}$ has common zero y_0 and $h_x^{a_0}, (\partial_x h_x)^{a_0}$ has common zero x_0. So, by proposition 2.2 and 3.5, $r_y^{a_0} = \varphi^{a_0}(Res(h_y, \partial_y h_y, y)) = Res(h_y^{a_0}, (\partial_y h_y)^{a_0}, y) = 0$ and $r_x^{a_0} = \varphi^{a_0}(Res(h_x, \partial_x h_x, x)) = Res(h_x^{a_0}, (\partial_x h_x)^{a_0}, x) = 0$. □

Note that since $p, q \in \mathbb{Q}[a_1, \ldots, a_l][x, y]$, we have $r_x, r_y \in \mathbb{Q}[a_1, \ldots, a_l]$. If either r_x or r_y is not identically equal to zero, then we pick a nonzero common divisor f of r_x and r_y. If $\mathcal{B} \cap V_{\mathbb{R}}(\frac{r_x}{f}, \frac{r_y}{f}) = \emptyset$, then $\mathcal{B} \subseteq V_{\mathbb{R}}(f)$ and the nonzero f is a bifurcation polynomial for $\mathcal{F}$.

However, if there exist finite common zeros for all systems $\{\mathcal{F}^a, J^a | a \in \mathcal{P}\}$, then both r_x and r_y are identically equal to zero and $V_{\mathbb{R}}(r_x, r_y) = \mathbb{R}^l$, which gives no information about finding bifurcation polynomials.

Without loss of generality, we assume that $(0, 0)$ is a common zero for all systems $\{\mathcal{F}^a, J^a | a \in \mathcal{P}\}$. Moreover, we assume that the intersection multiplicity at $(0, 0)$ is a constant $m \geq 2$ for all $a \in \mathcal{P}$. Then, we have that y^m divides $Res(p, q, x)$ and x^m divides $Res(p, q, y)$. Redefining $h_y = Res(p, q, x)/y^m$ and $h_x = Res(p, q, y)/x^m$, we also have $\mathcal{B} \subseteq V_{\mathbb{R}}(r_x, r_y) \cap \mathcal{P}$, where r_x and r_y are defined as in proposition 3.6. Now if both r_x and r_y are not identically equal to zero, this means no more common zeros for all systems $\{\mathcal{F}^a, J^a | a \in \mathcal{P}\}$ other than $(0, 0)$ and we can try the procedure above to find a nonzero bifurcation polynomial.

Systems $\{f_1, f_2\}$, $\{f_7, f_8\}$, and $\{f_9, f_{10}\}$ will use the resultant method to find bifurcation polynomials. In systems $\{f_7, f_8\}$ and $\{f_9, f_{10}\}$, it is not hard to see that $(0, 0)$ is a common zero of multiplicity 4 as we did in example 2.2. In system $\{f_1, f_2\}$, we find both $(1, 0)$ and $(-1, 0)$ are common zeros of multiplicity 2. We will show in Chapter 4 how we *remove* $(\pm 1, 0)$ from $Res(p, q, x)$ and $Res(p, q, y)$ to get the *correct* h_x and h_y in our process to find nonzero bifurcation polynomials.

3.3 On the bifurcation sets

In the previous sections, we learned how to count positive zeros generically for parametric polynomial systems. Namely, we find bifurcation set $\mathcal{B}$ and count the number of positive zeros for all systems $\{\mathcal{F}^a | a \in \mathcal{P} \setminus \mathcal{B}\}$. In this section, we will present methods to count positive zeros for systems $\{\mathcal{F}^a | a \in \mathcal{B}\}$.

When there are only two variables and two equations, we will use resultants and subresultant sequences together with generic results to achieve our goal. The generic results are necessary in this method when there is more than one parameter. If there is only one parameter in the system, then $\mathcal{B}$ consists of only finitely many points in the parameter space. In this case, we can make use of both resultants and subresultant sequences and Hermite matrices to count positive zeros for $\{\mathcal{F}^a | a \in \mathcal{B}\}$ without first knowing the generic results. The latter method works for systems with more than two variables and equations.

3.3.1 General cases

We use same notations as in section 3.2.2 where we consider parametric systems $\mathcal{F}$ with two variables x, y, two equations p, q, the parameter space $\mathcal{P}$, the bifurcation set $\mathcal{B}$, and a bifurcation polynomial f. We have $\mathcal{B} \subseteq (V_{\mathbb{R}}(f) \cap \mathcal{P}) \subseteq \mathcal{P} \subseteq \mathbb{R}^l, l \geq 1$.

As in proposition 3.6, we have $h_y = Res(p, q, x), r_y = Res(h_y, \partial_y h_y, y), h_x = Res(p, q, y), r_x = Res(h_x, \partial_x h_x, x)$. We denote $SyHa_1(p, q, x) = h_{1y}x + h_{0y}$ and define $r_{0y} = Res(h_y, h_{0y}, y), r_{1y} = Res(h_y, h_{1y}, y)$. Similarly, we define r_{0x}, r_{1x} from $StHa_1(p, q, y) = h_{1x}y + h_{0x}$. Note that $h_y, h_{0y}, h_{1y} \in \mathbb{Q}[a_1, \ldots, a_l][y], h_x, h_{0x}, h_{1x} \in \mathbb{Q}[a_1, \ldots, a_l][x]$, and $r_y, r_{0y}, r_{1y}, r_x, r_{0x}, r_{1x} \in \mathbb{Q}[a_1, \ldots, a_l]$.

Fix j and $a_0 \in \mathcal{P}$. Denote $b_{0j} = \pi_j(a_0) \subseteq \mathbb{R}^{l-1}$, where π_j is the canonical projection onto the $l-1$ coordinates other than the jth one and write $a_0 = (a_{0j}, b_{0j}) \in \mathcal{P}$. The jth coordinate of a_0 is a_{0j} and the other $j-1$ coordinates are b_{0j}.

We assume that the leading coefficients of $h_y^{a_0} \in \mathbb{R}[y]$ are nonzero for all $a_0 \in \mathcal{P}$. For any j, we can view $h_y \in \mathbb{Q}[a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_l][a_j, y]$ and write $a_0 = (a_{0j}, b_{0j})$. This assumption means that $V_{\mathbb{R}}(h_y^{b_{0j}}) \subseteq \mathbb{R}^2$ does not contain the line $a_j = a_{0j}$ and that y does not approach $\pm\infty$ as a_j approaches a_{0j} on $V_{\mathbb{R}}(h_y^{b_{0j}})$. Similarly, we assume that the leading coefficients of $h_x^{a_0} \in \mathbb{R}[x]$ are nonzero for all $a_0 \in \mathcal{P}$, too.

Let $a_0 \in \mathcal{P}$. We also assume that the leading coefficient of $f^{b_{0j}} \in \mathbb{R}[a_j]$ is nonzero when viewing $a_0 = (a_{0j}, b_{0j})$ for any j. If $a_0 \in \mathcal{B}$, this assumption means that, for any j, there exists an open interval $\mathcal{I}_j$ containing a_{0j} such that (a, b_{0j}) are not in $\mathcal{B}$ for all $a \in \mathcal{I}_j \setminus \{a_{0j}\}$. Therefore, the numbers of positive zeros are constants on both side of a_{0j} in $\mathcal{I}_j$. We denote them by m_j and n_j. The only possible values for them are given in the generic results. Our goal here is to express the number of positive zeros, denoted by ω, of the system $\mathcal{F}^{a_{0j}, b_{0j}}$ in terms of some m_j and n_j.

Since $a_0 = (a_{0j}, b_{0j}) \in \mathcal{B}$, we know from definition that there exists a positive zero (x_0, y_0) of $\mathcal{F}^{a_0}$ such that $J^{a_0}(x_0, y_0) = 0$. Therefore, we have $h_y^{a_0}$ has y_0 as a zero with multiplicity ≥ 2. So, $r_y^{a_0} = 0$ or $r_y^{b_{0j}}$ has a_{0j} as a zero for all j. Similarly, $r_x^{b_{0j}}$ has a_{0j} as a zero for all j. Our methods in the following two propositions are to consider the multiplicity of the zero a_{0j} of $r_y^{b_{0j}}$ or $r_x^{b_{0j}}$.

Proposition 3.7. *Let $a_0 \in \mathcal{B}$. If there exists j such that $r_y^{b_{0j}}$ has a_{0j} as a zero with multiplicity* 1 *when viewing $a_0 = (a_{0j}, b_{0j})$ and that $r_{0y}^{a_0} \neq 0, r_{1y}^{a_0} \neq 0$, then*

$|m_j - n_j| = 2$ *and* $\omega = \frac{m_j+n_j}{2}$. *Similarly, if there exists* j *such that* $r_x^{b_{0j}}$ *has* a_{0j} *as a zero with multiplicity* 1 *and* $r_{0x}^{a_0} \neq 0, r_{1x}^{a_0} \neq 0$, *then we have the same conclusion.*

Proof. Suppose there is another positive zero (x_1, y_1) with multiplicity ≥ 2. If $y_1 \neq y_0$, then there exist two zeros of $h_y^{a_0}$ with multiplicity ≥ 2. Therefore, $r_y^{b_{0j}}$ has a_{0j} as a zero with multiplicity ≥ 2, a contradiction. If $y_1 = y_0$ but $x_1 \neq x_0$, then the linear polynomial $h_{1y}^{a_0}(y_0)x + h_{0y}^{a_0}(y_0)$ has two zeros by the proposition 2.5. So, $h_{1y}^{a_0}(y_0) = h_{0y}^{a_0}(y_0) = 0$ and we get $r_{0y}^{a_0} = r_{1y}^{a_0} = 0$, a contradiction. In conclusion, the positive zero (x_0, y_0) with multiplicity ≥ 2 is unique.

Next, since $r_y^{b_{0j}}$ has a_{0j} as a zero with multiplicity 1, $h_y^{b_{0j}}$ and $\partial_y(h_y^{b_{0j}})$ intersect at (a_{0j}, y_0) transversely. Therefore, we have

$$\begin{vmatrix} \partial_y(h_y^{b_{0j}}) & \partial_{a_j}(h_y^{b_{0j}}) \\ \partial_{yy}(h_y^{b_{0j}}) & \partial_{a_j y}(h_y^{b_{0j}}) \end{vmatrix} (a_{0j}, y_0) = \begin{vmatrix} 0 & \partial_{a_j}(h_y^{b_{0j}}) \\ \partial_{yy}(h_y^{b_{0j}}) & \partial_{a_j y}(h_y^{b_{0j}}) \end{vmatrix} (a_{0j}, y_0) \neq 0$$

Therefore, $\partial_{yy}(h_y^{b_{0j}})(a_{0j}, y_0) \neq 0, \partial_{a_j}(h_y^{b_{0j}})(a_{0j}, y_0) \neq 0$. From these, we conclude that $h_y^{b_{0j}}$ experiences a saddle node bifurcation [16] at $(a_j, y) = (a_{0j}, y_0)$. The geometry of the curve $h_y^{b_{0j}} = 0$ nearby (a_{0j}, y_0) is illustrated in graph 4.3, where a_{0j} is s_0 and b_{0j} is t_0 in that graph. Therefore, the positive y_0 splits into $y_1(a)$ and $y_2(a)$ continuously in one side of a neighborhood of a_{0j} and disappear on the other side.

Since $r_{0y}^{b_{0j}}(a_{0j}) = r_{0y}^{a_0} \neq 0, r_{1y}^{b_{0j}}(a_{0j}) = r_{1y}^{a_0} \neq 0$, $r_{0y}^{b_{0j}}$ and $r_{1y}^{b_{0j}}$ are nonzero near a_{0j}, too. Therefore, $h_{0y}^{b_{0j}}(a), h_y^{b_{0j}}(a) \in \mathbb{R}[y]$ have no common zeros for a near a_{0j}, and $h_{1y}^{b_{0j}}(a), h_y^{b_{0j}}(a) \in \mathbb{R}[y]$ have no common zeros for a near a_{0j}. We can extend uniquely $y_1(a), y_2(a)$ to zeros $(x_1(a), y_1(a)), (x_2(a), y_2(a))$ of the system $\mathcal{F}^{a,b_{0j}}$ continuously in one side of a neighborhood of a_{0j} by defining $x_1(a) = \frac{h_{0y}^{b_{0j}}(a,y_1(a))}{h_{1y}^{b_{0j}}(a,y_1(a))}, x_2(a) = \frac{h_{0y}^{b_{0j}}(a,y_2(a))}{h_{1y}^{b_{0j}}(a,y_2(a))}$. On the other side, there are no zeros near by (x_0, y_0), since y_0 disappears on that side.

Finally, since we have ω positive zeros of $\mathcal{F}^{a_{0j},b_{0j}}$, there are exactly $\omega - 1$ positive zeros has multiplicities 1 and they all continue uniquely in a neighborhood of a_{0j} with y coordinates away from y_0 since $r_{1y}^{a_0} \neq 0$. Together with the behavior of (x_0, y_0) as a_j varies near a_{0j} as we discussed above, we conclude that either m_j or n_j is $\omega - 1$ and the other is $\omega + 1$ since x, y do not approach $\pm\infty$ as a_j approaches a_{0j}. $\square$

In the cases when both $r_x^{b_{0j}}$ and $r_y^{b_{0j}}$ have a_{0j} as a zero with multiplicity ≥ 2 for all j, we have the following results.

Proposition 3.8. *Let $a_0 \in \mathcal{B}$. If there exists j such that $r_y^{b_{0j}}$ has a_{0j} as a zero with multiplicity 2 when viewing $a_0 = (a_{0j}, b_{0j})$ and that $r_{0y}^{a_0} \neq 0, r_{1y}^{a_0} \neq 0, r_{a_j y}^{a_0} \neq 0$, where $r_{a_j y} = Res(h_y, \partial_{a_j} h_y, y)$, then $|m_j - n_j| = 0, 2$ or 4 and $\omega = \frac{m_j+n_j}{2}$. Similarly, if there exists j such that $r_x^{b_{0j}}$ has a_{0j} as a zero with multiplicity 2 and $r_{0x}^{a_0} \neq 0, r_{1x}^{a_0} \neq 0, r_{a_j x}^{a_0} \neq 0$, where $r_{a_j x} = Res(h_x, \partial_{a_j} h_x, x)$, then we have the same conclusion.*

Proof. We already have (x_0, y_0) as a positive zero of $\mathcal{F}^{a_0}$ such that $J^{a_0}(x_0, y_0) = 0$. Since $r_{a_j y}^{a_0} \neq 0$, we have that $(\partial_{a_j} h_y)^{b_{0j}}(a_{0j}, y_0) = (\partial_{a_j} h_y)^{a_0}(y_0) \neq 0$. Therefore, in the variety $V_{\mathbb{R}}(h_y^{b_{0j}}) \subseteq \mathbb{R}^2$, a is a real analytic function of y nearby (a_{0j}, y_0) by the implicit function theorem. Therefore, either y_0 is uniquely continued in a neighborhood of a_{0j} or $h_y^{b_{0j}}$ behaves as a saddle node bifurcation in graph 4.3 near (a_{0j}, y_0). Since $r_{0y}^{a_0} \neq 0$ and $r_{1y}^{a_0} \neq 0$, we extend positive zeros behaviors nearby a_{0j} as in the previous proof. Suppose there are no other zeros with multiplicity ≥ 2. Then the first case leads to $|m - n| = 0, \omega = \frac{m+n}{2}$, while the second case leads to $|m - n| = 2, \omega = \frac{m+n}{2}$.

Suppose there is another positive zero (x_1, y_1) of $\mathcal{F}^{a_0}$ such that $J^{a_0}(x_1, y_1) = 0$. If $y_1 \neq y_0$, then $h_y^{b_{0j}}$ and $\partial_y(h_y^{b_{0j}})$ intersect at (a_{0j}, y_0) and (a_{0j}, y_1) transversely, since $r_y^{b_{0j}}$ has a_{0j} as a zero with multiplicity 2. In this case, two points (a_{0j}, y_0) and (a_{0j}, y_1) experience saddle node bifurcations. Again, since $r_{0y}^{a_0} \neq 0$ and $r_{1y}^{a_0} \neq 0$, this behavior extends to that of zeros of systems $\mathcal{F}^{a, b_{0j}}$ for all a near a_{0j}. Therefore, $|m - n| = 0$ if the opening of the two saddle nodes are in the different direction, $|m - n| = 4$, otherwise, and $\omega = \frac{m+n}{2}$ for both situations. Next, the case of $y_1 = y_0$ and $x_1 \neq x_0$ contradict to the fact that $r_{1y}^{a_0} \neq 0$ again. Finally, we can not have more than two zeros such that $J^{a_0} = 0$ by the fact that $r_y^{b_{0j}}$ has a_{0j} as a zero with multiplicity 2. $\square$

For system $\{f_5, f_6\}$, proposition 3.7 is enough for our needs, while system $\{f_7, f_8\}$ requires both proposition 3.7 and 3.8. However, for system $\{f_9, f_{10}\}$, it may happen that, for some points $a_0 \in \mathcal{B}$, both $r_x^{b_{0j}}$ and $r_y^{b_{0j}}$ have a zero $a_{0j}$with multiplicity ≥ 3 for all j or $r_{a_j y}^{a_0} = r_{a_j x}^{a_0} = 0$ when $r_x^{b_{0j}}$ or $r_x^{b_{0j}}$ have a_{0j} as a zero with multiplicity 2 for some j. In both situations, we cannot apply proposition 3.8. We will show how we still can count the number of positive zeros for $\{f_9^{a_0}, f_{10}^{a_0}\}$ in Chapter 4. Also we will see how the bifurcation polynomial f helps us to verify the hypothesis in both propositions 3.7 and 3.8 when we solve these concrete problems.

3.3.2 One parameter

Resultants and subresultant sequences method

We now consider $l = 1$. Just denoting a_1 by a, we have $p, q \in \mathbb{Q}[a][x, y]$. Without loss of generality, we assume that the parameter space $\mathcal{P}$ is a open interval (b, c) and $a_0 \in (b, c)$ is the only point in the bifurcation set $\mathcal{B}$ and is a zero of a bifurcation polynomial $f \in \mathbb{Q}[a]$.

Here we use resultants and subresultant sequences to count positive zeros of the system $\{p^{a_0}, q^{a_0}\}$. An example of applying this method is given in Chapter 4 when we study system $\{f_1, f_2\}$. Unfortunately, this method only applies for systems satisfying that there exists one coordinate, say y, such that the y coordinate of real zeros of $\{p^{a_0}, q^{a_0}\}$ are all positive.

Writing $p = \sum_{i=0}^{m} \alpha_i(y)x^i, q = \sum_{i=0}^{n} \beta_i(y)x^i$, where $\alpha_i(y), \beta_i(y) \in \mathbb{Q}[a][y]$, we assume $m \leq n + 1$ and $Res(\alpha_m, \beta_n, y)(a) \neq 0$ for all $a \in (b, c)$. Similarly, writing $p = \sum_{i=0}^{s} \gamma_i(x)y^i, q = \sum_{i=0}^{t} \delta_i(x)y^i$, where $\gamma_i(x), \delta_i(x) \in \mathbb{Q}[a][x]$, we assume $s \leq t+1$ and $Res(\gamma_s, \delta_t, x)(a) \neq 0$ for all $a \in (b, c)$.

By the assumptions, we know that, for any fixed $a \in (b, c)$, any zero of $h_x^a = Res(p^a, q^a, y)$ extends to a complex zero of p^a, q^a and any zero of $h_y^a = Res(p^a, q^a, x)$ extends to a complex zero of p^a, q^a. If we have $r_{1x}^a = Res(h_x^a, h_{1x}^a, x) \neq 0, r_{1y}^a = Res(h_y^a, h_{1y}^a, y) \neq 0$, where h_{1x}, h_{1y} as before come from $SyHa_1(p, q, x) = h_{1y}x + h_{0y}, SyHa_1(p, q, y) = h_{1x}y + h_{0x}$, then each extension is unique. Since $h_x^a, h_{1x}^a, h_{0x}^a \in \mathbb{R}[x], h_y^a, h_{1y}^a, h_{0y}^a \in \mathbb{R}[y]$, any real zero of h_x^a extends uniquely to a real zero of p^a, q^a and any real zero of h_y^a extends uniquely to a real zero of p^a, q^a. Therefore, the number of distinct real zeros of h_x^a is equal to the number of distinct real zeros of h_y^a and is equal to the number of distinct real zeros of p^a, q^a. If all the real zeros of h_y^a are positive, then the number of positive zero of p^a, q^a is equal to the number of positive zeros of h_x^a.

Our goal is to count positive zeros of p^{a_0}, q^{a_0}, where $a_0 \in (b, c)$ is a zero of the bifurcation polynomial f. The only work left now is to count positive zeros of $h_x^{a_0}$ and $h_y^{a_0}$. We give an example to explain how we count real zeros of one univariate polynomial with one parameter where the parameter value is only known implicitly as a zero of a real polynomial.

Example 3.1. *Let $g = ax^4 - x^2 - x + a^2 + 1$. We will count the number of positive roots of g^{a_0}, where a_0 is a zero of $f = 256a^8 + 768a^6 - 128a^5 + 768a^4 - 400a^3 +$*

$272a^2 - 299a + 20$ *in* $(\frac{1}{2}, 1)$. *Compute the Sturm-Habicht sequence of* g*, we get*

$$\begin{aligned}
StHa_4\,(g) &= g = ax^4 - x^2 - x + (a^2+1),\\
StHa_3\,(g) &= g' = 4ax^3 - 2x - 1,\\
StHa_2\,(g) &= 8a^2x^2 + 12a^2x + (-16a^2 - 16a^4),\\
StHa_1\,(g) &= -4a^2(8a^3 + 17a - 2)x + 4a^2(1 + 12a + 12a^3),\\
StHa_0\,(g) &= Res(g, \partial_x g, x) = a^2 f.
\end{aligned}$$

Numerically, $a_0 = 0.627\ldots$ *Choosing* $\alpha = 0.6$ *and* $\beta = 0.7$*, we use proposition* 2.6 *to verify that* f *has only one root in* $[\alpha, \beta]$*, and that the leading coefficients and constant terms* $-4a^2(8a^3 + 17a - 2), 4a^2(1 + 12a + 12a^3), 8a^2, -16a^2 - 16a^4, 4a, -1, a, a^2 + 1$ *of* $StHa_j, j > 0$ *have no real roots in* $[\alpha, \beta]$*.*

When $a = \alpha$*, the signs of the Sturm-Habicht sequence at* $x = -\infty, 0, \infty$ *are* $+ - + + -, + - - + -, + + + - -$*, respectively, the number of sign changes are* $3, 3, 1$*, respectively. Because of the leading coefficients and constant terms have no real roots in* $[\alpha, \beta]$*, when* $a = a_0, \beta$*, we have the same first* 4 *signs in the Sturm-Habicht sequence at* $x = -\infty, 0, \infty$ *as when* $a = \alpha$*. Because of* a_0 *is a zero of* f *in* $[\alpha, \beta]$*, the last sign is* 0 *at* $x = -\infty, 0, \infty$ *when* $a = a_0$*. At* $a = \beta$*, it is easy to see the last sign is* $+$*. We list these results in the following* 3 *tables.*

$a = \alpha$	$StHa_4$	$StHa_3$	$StHa_2$	$StHa_1$	$StHa_0$	*Sign changes*
$-\infty$	$+$	$-$	$+$	$+$	$-$	3
0	$+$	$-$	$-$	$+$	$-$	3
∞	$+$	$+$	$+$	$-$	$-$	1

$a = a_0$	$StHa_4$	$StHa_3$	$StHa_2$	$StHa_1$	$StHa_0$	*Sign changes*
$-\infty$	$+$	$-$	$+$	$+$	0	2
0	$+$	$-$	$-$	$+$	0	2
∞	$+$	$+$	$+$	$-$	0	1

$a = \beta$	$StHa_4$	$StHa_3$	$StHa_2$	$StHa_1$	$StHa_0$	*Sign changes*
$-\infty$	$+$	$-$	$+$	$+$	$+$	2
0	$+$	$-$	$-$	$+$	$+$	2
∞	$+$	$+$	$+$	$-$	$+$	2

In conclusion, by the root counting method in proposition 2.6*, the number of* *positive roots of g^{a_0} is 1, and there is no negative roots of g^{a_0}. At the same time, we know the number of positive roots of g^{α} is 2, there is no negative roots of g^{α}, and there are no real roots of g^{β}.*

Hermite matrices method

In the last part of this chapter, we return back to tools of Groebner basis and Hermite matrices. We consider a system $\mathcal{F} = \{f_1, \cdots, f_n\}$, where $f_i \in \mathbb{Q}[a][x_1, \ldots, x_n]$, with the parameter space (b, c) and a unique bifurcation point $a_0 \in (a, b)$ known as a zero of the bifurcation polynomial f. As in proposition 3.4, f may be obtained from the numerator of $det(H(\mathcal{F}, 1))$. By proposition 2.7, signature of $H(\mathcal{F}^{a_0}, 1)$ is the number of real zeros of the system $\mathcal{F}^{a_0}$. Since a_0 is only known implicitly as a real zero of $f \in \mathbb{Q}[a]$, we can not compute its signature by substituting a_0 into $H(\mathcal{F}, 1)(a)$.

However, we can still make use of parameter values near a_0 and compute signatures by principle minors to compute the signature of $H(\mathcal{F}^{a_0}, 1)$. If we want to know the number of positive zeros, we need to compute three more signatures of matrices $H(\mathcal{F}^{a_0}, x), H(\mathcal{F}^{a_0}, y)$ and $H(\mathcal{F}^{a_0}, xy)$ in the case of two variables x, y as in the example 2.5. Therefore, it suffices for us to demonstrate how to compute the signature of $H(\mathcal{F}^{a_0}, 1)$.

Our method is based on Jacobi's method of counting the signature of a real symmetric matrix [25]. Let M be a real symmetric matrix with rank r and denote D_i by the determinant of the matrix from the first i columns and the first i rows of M. Suppose $D_r \neq 0$ and that there are no more than two consecutive zeros in the sequence $1, D_1, \ldots, D_r$. Then the signature of M is given by $1 - 2W(1, D_1, \ldots, D_r)$, where W is the sign variation as defined in definition 2.5.

Suppose that the dimension of $H(\mathcal{F}, 1)$ is d and $det(H(\mathcal{F}, 1)) = D_d(a)$ is not identically equal to zero. The idea of computing the signature of $H(\mathcal{F}^{a_0}, 1)$ is again first choose α, β near a_0 such that $D_d(a)$ has only one zero a_0 in $[\alpha, \beta]$ and, for $i < d$, D_i has no zeros in $[\alpha, \beta]$ if it is not identically equal to zero. Then, the signs of $D_i^{a_0}$ can be decided by D_i^{α} or D_i^{β}. Since $D_d^{a_0} = 0$, rank of $H(\mathcal{F}^{a_0}, 1)$ will be $d-1$ if $D_{d-1}^{a_0} \neq 0$. Therefore, by Jacobi's method the signature of $H(\mathcal{F}^{a_0}, 1)$ is equal to $1 - 2W(1, D_1^{a_0}, \ldots, D_{d-1}^{a_0}) = 1 - 2W(1, D_1^{\alpha}, \ldots, D_{d-1}^{\alpha}) = 1 - 2W(1, D_1^{\beta}, \ldots, D_{d-1}^{\beta})$ if there is no more than two consecutive identically zeros in the sequence $1, D_1(a), \ldots, D_{d-1}(a)$. An example will be given in Chapter 4 when we study system $\{f_3, f_4\}$.

Chapter 4

Solving concrete problems

In this chapter, we will use the methods developed in Chapter 3 to count positive zeros of five parametric polynomial systems. In each of these five systems, we will first prove the finiteness of complex zeros for almost every parameter, find an nonzero bifurcation polynomial f, obtain the generic results, and count positive zeros of systems where parameters are on the bifurcation set.

Three systems come from the studies of central configurations. One is from a restricted 4-body problem. Another is from a restricted 5-body problem. The other is from a restricted N-body problem. We will study these three problems in the first section and prove theorems 1.1, 1.2, and 1.3. The other two systems are from the study of Maxwell's conjecture of 3 point charges. Fixing two configurations of the 3 point charges, we get two systems. We will count positive zeros of them in the second section and prove theorem 1.4.

4.1 Central configurations problems

4.1.1 A restricted 4-body problem

Finiteness of complex zeros

Let $\mathcal{F} = \{f_1, f_2\}$, where $f_1, f_2 \in \mathbb{Q}[k][x,y]$ are given in the equations 1.3 and 1.4. Applying $\mathbf{GroebnerBasis}[\mathcal{F}, \{\mathbf{x},\mathbf{y},\mathbf{k}\}, \mathbf{MonomialOrder} \to \mathbf{w}]$, where $\mathbf{w}$ is the block ordering $<$ given by $\mathbf{w} = \{\{\mathbf{1},\mathbf{1},\mathbf{0}\},\{\mathbf{1},\mathbf{0},\mathbf{0}\},\{\mathbf{0},\mathbf{0},\mathbf{1}\}\}$, we get a set $G = \{g_1, \ldots, g_4\}$, where $LC_{<_{x,y}}(g_i) = 1$, for all i, and $LM_{<_{x,y}}(g_1) = x^4, LM_{<_{x,y}}(g_4) = y^9$. Therefore, G^k is a Groebner basis of $I^k = \langle f_1^k, f_2^k \rangle$ for all $k \in \mathbb{R}$ and $A^k = \mathbb{C}[x,y]/I^k$

is of finite dimensional and has a basis $\{x^m y^n \notin \langle LT_{<_{x,y}}(g_1^k), \ldots, LT_{<_{x,y}}(g_4^k)\rangle\} = \{x^m y^n \notin \langle x^4, xy^7, x^3y^5, y^9\rangle\} = \{1, y, \ldots, y^8, x, xy, \ldots, xy^6, x^2, \ldots, x^2y^6, x^3, \ldots, x^3y^4\}$. So, the system $\mathcal{F}^k$ has at most 28 complex zeros for all $k \in \mathbb{R}$. Since k and $1-2k$ represent masses of the bodies, we define the parameter space $\mathcal{P} = (0, \frac{1}{2})$ so that all bodies have positive masses. In conclusion, $\mathcal{F}^k$ has at most 28 complex zeros for all $k \in \mathcal{P}$.

Bifurcation polynomial

Computing resultants, we get $Res(f_1, f_2, x) = 288y^4h_y$ and $Res(f_1, f_2, y) = 288(x^2-1)^2h_x$, where $h_y^k(0) \neq 0, h_x^k(\pm 1) \neq 0$ for all $k \in \mathcal{P}$. Note that we can show $\mathcal{F}^k$ has $(\pm 1, 0)$ as zeros with $mul(1,0) = mul(-1,0) = 2$ for all $k \in \mathcal{P}$ as we did in example 2.2. Computing $r_x = Res(h_x, \partial_x h_x, x)$ and $r_y = Res(h_y, \partial_y h_y, y)$, we find an nonzero polynomial f dividing both r_x and r_y. Computing $Res(\frac{r_x}{f}, \frac{r_y}{f}, k) \neq 0$, we conclude that f is a bifurcation polynomial. Numerically solving $f = 0$, we get 3 zeros in $\mathcal{P}$. They are $k_1 = 0.288276\ldots, k_2 = 0.30516\ldots, k_3 = 0.329314\ldots$.

Generic results

Fix $k_0 \in \mathcal{P}$. Since $h_y^{k_0}(0) \neq 0, h_x^{k_0}(\pm 1) \neq 0$ and h_y, h_x are continuous in k, x, y, there exist a neighborhood $\mathcal{U} \subseteq \mathcal{P}$ of k_0 and neighborhoods $\mathcal{V}_1$ containing $(1,0)$, $\mathcal{V}_2$ containing $(-1,0)$ such that $h_y^k(y) \neq 0, h_x^k(x) \neq 0$ for all $k \in \mathcal{U}, (x,y) \in \mathcal{V}_1$ or $\mathcal{V}_2$, respectively. Therefore, $(1,0)$ and $(-1,0)$ are the only real zeros of systems $\{\mathcal{F}^k | k \in \mathcal{U}\}$ in $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. For example, suppose there exist $k_0 \in (0, \frac{1}{2})$ and $(x_0, y_0) \in \mathcal{V}_1$ such that $f_1^{k_0}(x_0, y_0) = f_2^{k_0}(x_0, y_0)$ and $x_0 \neq 1$ or $y_0 \neq 0$. If $y_0 \neq 0$, then $0 = Res(f_1^{k_0}, f_2^{k_0}, x)(y_0) = 288y_0^4h_y(y_0)$. But, the right hand side is not zero, which leads to a contradiction. Therefore, by proposition 3.3, the numbers of positive zeros are constant in each interval $(0, k_1), (k_1, k_2), (k_2, k_3), (k_3, \frac{1}{2})$.

Let sample points be $\alpha_1 = 0.2, \alpha_2 = 0.3, \alpha_3 = 0.32, \alpha_4 = 0.33$. Using the Hermite's root counting method, proposition 2.7, we get that the number of positive zeros of systems $\mathcal{F}^{\alpha_1}, \mathcal{F}^{\alpha_2}, \mathcal{F}^{\alpha_3}, \mathcal{F}^{\alpha_4}$ are $2, 4, 4, 4$, respectively. Therefore, the number of positive zeros is generically 2 or 4. Here, the two bifurcation points k_2 and k_3 are in fact parameters where the number of negative zeros changes. Therefore, the number of positive zeros does not change at those points.

On the bifurcation set

Computing Sylvester-Habicht sequences, we get $SyHa_1(f_1, f_2, x) = -48y^2h_{1y}x + 48ky^5h_{0y}$ and $SyHa_1(f_1, f_2, y) = -48(x^2-1)^2h_{1x}y + 48(2k-1)x^3(x^2-1)h_{0x}$. We denote $r_{1y} = Res(h_y, h_{1y}, y), r_{0y} = Res(h_y, h_{0y}, y)$, and $r_{1x} = Res(h_x, h_{1x}, x), r_{0x} = Res(h_x, h_{0x}, x)$. From $Res(r_{1y}, f, k) \neq 0, Res(r_{1x}, f, k) \neq 0$, and $Res(f, \frac{df}{dk}, k) \neq 0$, we know that, for all $k \in \mathcal{B}$, $r_{1y}^k \neq 0, r_{1x}^k \neq 0$ and r_x, r_y have k as a zero of multiplicity 1. In particular, $r_{1y}^{k_i} \neq 0, r_{1x}^{k_i} \neq 0$ and r_x, r_y have k_i as a zero of multiplicity 1 for $i = 1, 2, 3$. Therefore, by proposition 3.7, we know the number of positive zeros for systems $\mathcal{F}^{k_i}, i = 1, 2, 3$, are $3, 4, 4$, respectively, from the generic results.

In particular at k_1

We can verify our result of 3 positive zeros for system $\mathcal{F}^{k_1}$ by directly applying methods in section 3.3.2. Let $\alpha = 0.288276, \beta = 0.288277$ be the points around k_1. Computing $SyHa_j(h_x, \partial_x h_x, x)$ and $SyHa_j(h_y, \partial_y h_y, y)$, we check that there is only one zero k_1 in $[\alpha, \beta]$ for $SyHa_0(h_x, \partial_x h_x, x)$ and $SyHa_0(h_y, \partial_y h_y, y)$ and there is no zero in $[\alpha, \beta]$ for other leading coefficients and constant terms of $SyHa_j(h_x, \partial_x h_x, x)$ and $SyHa_j(h_y, \partial_y h_y, y)$.

As in example 3.1, we count the sign variations of the leading coefficients and constant terms of $SyHa_j(h_x^\alpha, \partial_x h_x^\alpha, x)$ to conclude that h_x^α has $14 - 12 = 2$ negative and $12 - 10 = 2$ positive zeros, $h_x^{k_1}$ has $14 - 12 = 2$ negative and $12 - 9 = 3$ positive zeros, and h_x^β has $15 - 13 = 2$ negative and $13 - 9 = 4$ positive zeros. Similarly, from the sign variations of the leading coefficients and constant terms of $SyHa_j(h_y^\alpha, \partial_y h_y^\alpha, y)$, we get h_y^α has $14-14 = 0$ negative and $14-10 = 4$ positive zeros, $h_y^{k_1}$ has $14 - 14 = 0$ negative and $14 - 9 = 5$ positive zeros, and h_y^β has $15 - 15 = 0$ negative and $15 - 9 = 6$ positive zeros.

As the argument in section 3.3.2, we conclude there are 2 negative and 2 positive zeros for the system $\mathcal{F}^\alpha$, 2 negative and 3 positive zeros for the system $\mathcal{F}^{k_1}$, and 2 negative and 4 positive zeros for the system $\mathcal{F}^\beta$.

4.1.2 A restricted 5-body problem

Finiteness of complex zeros

Let $\mathcal{F} = \{f_3, f_4\}$, where $f_3, f_4 \in \mathbb{Q}[k][x, y]$ are given in equations 1.5 and 1.6. This system has much in common with the system $\{f_1, f_2\}$. They both have at most 28 complex zeros and two of them are $(\pm 1, 0)$ each with multiplicity 2 for all parameters. Now the parameter space is $\mathcal{P} = (0, \frac{1}{3})$.

Bifurcation polynomial

It is easy to see that $det(H(\mathcal{F}), 1)$ is identically equal to zero due to the existence of common zeros $(\pm 1, 0)$ with multiplicities 2. We rewrite $f_3 = (3k-1)(x^2-1)x^3 + y^2(x^3 - 3kx^3 - ky - 3kx^2y - x^3y + 4kx^3y + x^5y + 3ky^3 - x^3y^3)$, $f_4 = 3(x^2-1)^2 + y^2(-2 - 6x^2 + 3y^2)$. Letting $z = \frac{x^2-1}{y^2}$, dividing f_3 and f_4 by y^2, and substituting z into f_3, f_4, we get another system $\mathcal{G} = \{g_1, g_2, g_3\}$, where $g_1 = (3k-1)zx^3 + x^3 - 3kx^3 - ky - 3kx^2y - x^3y + 4kx^3y + x^5y + 3ky^3 - x^3y^3$, $g_2 = 3(x^2-1)z - 2 - 6x^2 + 3y^2$, $g_3 = x^2 - 1 - y^2z$.

It is easy to see that $\mathcal{G}$ has no zeros with any of the coordinates being zero and that the positive zeros of $\mathcal{F}$ except for $(\pm 1, 0)$ is in one to one correspondence with real zeros of $\mathcal{G}$ with the first two coordinates being positive. Therefore, we turn to study the system $\mathcal{G}$ and count real zeros of $\mathcal{G}^k$ for all $k \in \mathcal{P} = (0, \frac{1}{3})$.

Applying $\mathbf{GroebnerBasis}[\mathcal{G}, \{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{k}\}, \mathbf{MonomialOrder} \rightarrow \mathbf{w}]$, where $\mathbf{w}$ is the block ordering $<$ given by $\mathbf{w} = \{\{\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{0}\}, \{\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}\}, \{\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}\}, \{\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}\}\}$, we get a set G consisting 11 polynomials in $\mathbb{Q}[k][x, y, z]$. The leading coefficients of them with respect to $<_{x,y,z}$ are $1, 3, 48, 16, 3, 81(3k-1)^2, 27(3k-1), 27(3k-1), 108k, 48, 108$ and 3 leading monomials are x^4, y^5, z^5 . Therefore, G^k is a Groebner basis of $I^k = \langle g_1^k, g_2^k, g_3^k \rangle$ and that $A^k = \mathbb{C}[x, y, z]/I^k$ is of finite dimensional for all $k \in \mathcal{P}$. We pick an ordered basis $\mathcal{S} = \{z^2, z^3, z^4, 1, y, z, y^2, y^3, y^4, x^3, xz, xz^2, xz^3, xy, xyz, xyz^2, xy^2, x^2, x^2y, x^2y^2, x, yz, yz^2, yz^3\}$ for A^k. Note that $dim(A^k) = 24$, which is what we expect from removing zeros $(\pm 1, 0)$ from the original system $\mathcal{F}$.

With the basis $\mathcal{S}$ of A^k, we compute $det(H(\mathcal{G}, 1)) = \frac{ck^{10}f}{(3k-1)^{16}}$, where c is a constant, and get an nonzero bifurcation polynomial f. Numerically solving $f = 0$, we get 3 zeros in $(0, \frac{1}{3})$. They are $k_1 = 0.235517\ldots, k_2 = 0.246659\ldots, k_3 = 0.248606\ldots$.

On the bifurcation set

We now demonstrate how to count real zeros with positive first two coordinates for the system $\mathcal{G}^{k_2}$. Let $\alpha = 0.24665, \beta = 0.24666$ be points around k_2. We check that the determinants D_i of the matrix from the first i rows and columns of $H(\mathcal{G}, 1)$ has no zeros in $[\alpha, \beta]$ for $i < 24$ and D_{24} has k_2 as the only zero in $[\alpha, \beta]$. Similarly, we check this property for principal minors of $H(\mathcal{G}, x), H(\mathcal{G}, y), H(\mathcal{G}, xy)$, too.

Now, from the sign variations of $1, D_1^\alpha, \ldots, D_{24}^\alpha$, we get the signatures of $H(\mathcal{G}^\alpha, 1)$,

$H(\mathcal{G}^{k_2},1), H(\mathcal{G}^{\beta},1)$ are $6,7,8$, respectively. In the same way, we get the signatures of $H(\mathcal{G}^{\alpha},x), H(\mathcal{G}^{k_2},x), H(\mathcal{G}^{\beta},x), H(\mathcal{G}^{\alpha},y), H(\mathcal{G}^{k_2},y), H(\mathcal{G}^{\beta},y), H(\mathcal{G}^{\alpha},xy), H(\mathcal{G}^{k_2},xy)$, $H(\mathcal{G}^{\beta},xy)$ are $2,3,4,2,3,4,-2,-1,0$, respectively.

From those information, we can count the number of positive zeros as we did in example 2.5 and conclude that the numbers of real zero with $x>0, y>0$ of the systems $\mathcal{G}^{\alpha}, \mathcal{G}^{k_0}, \mathcal{G}^{\beta}$ are $2,3,4$, respectively. Therefore, the number of positive zeros of systems $\mathcal{F}^{k_2}$ is 3.

Generic results

Since $\mathcal{G}$ has no zeros with any of the coordinates being zero, we know that the number of real zero with $x>0, y>0$ of the systems $\{\mathcal{G}^k | k \in (k_1,k_2)\}$ is 2 and the number of real zero with $x>0, y>0$ of the systems $\{\mathcal{G}^k | k \in (k_2,k_3)\}$ is 4.

As for points k_1, k_3 and intervals $(0,k_1), (k_3,\frac{1}{3})$, we use the same way to get that the number of real zero with $x>0, y>0$ of the systems $\{\mathcal{G}^k | k \in (0,k_1]\}$ is 2 and that of the systems $\{\mathcal{G}^k | k \in [k_3,\frac{1}{3})\}$ is 4. In conclusion, the number of positive zeros of systems $\{\mathcal{F}^k | k \in (0,k_2)\}$ is 2 and the number of positive zeros of systems $\{\mathcal{F}^k | k \in (k_2,\frac{1}{3})\}$ is 4.

4.1.3 A restricted N-body problem

Let $\mathcal{F} = \{f_5, f_6\}$, where $f_5, f_6 \in \mathbb{Q}[p,q][x,y]$ are given in equations 1.7 and 1.8. Our goal here is to count positive zeros of systems $\mathcal{F}^{p,q}$, where $p,q \in \mathbb{N}, q \geq p > 1$, or $q > p = 1$. Here, although p,q are positive integers, we will treat them as real parameters initially.

Finiteness of complex zeros

Applying $\mathbf{GroebnerBasis}[\mathcal{F}, \{\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q}\}, \mathbf{MonomialOrder} \to \mathbf{w}]$, where $\mathbf{w}$ is the block ordering $<$ given by $\mathbf{w} = \{\{\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0}\},\{\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}\},\{\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1}\},\{\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0}\}\}$, we get a set G consisting of 10 polynomials in $\mathbb{Q}[p,q][x,y]$. The leading coefficients of them with respect to $<_{x,y}$ are $pq, pq(p+q), p+q, q^2, 2pq(p+q)^2, 2pq(p+q), 2(p+q)^2, 2q^2(p+q), 2pq(p+q)^2, pq(p+q)$ and 2 leading monomials are x^4, y^9. Therefore, $G^{p,q}$ is a Groebner basis of $I^{p,q} = \langle f_5^{p,q}, f_6^{p,q} \rangle$ and that $A^{p,q} = \mathbb{C}[x,y]/I^{p,q}$ is of finite dimensional for all $p>0, q>0$. As before, we can choose an basis for $A^{p,q}$ and find $dim(A^{p,q}) = 26$.

Bifurcation polynomial

Using the basis obtained above to compute the Hermite matrix $H(\mathcal{F},1)$, we find that $det(H(\mathcal{F},1)) = \frac{549755813888(p-1)^2(q-1)^2(p+q-1)^2 f}{p^{38}q^{38}(p+q)^{21}}$, where f is a nonzero polynomial. We consider two cases, $q \geq p > 1$ and $q \geq p = 1$. In the first case, if we define our parameter space $\mathcal{P} = \{(p,q) \in \mathbb{R}^2 | p > 1, q > 1\}$, then f is our bifurcation polynomial. The second case involves only one parameter. We will treat it separately.

When $p \geq 9, q \geq p$ and $f \neq 0$

Computing $Res(f, \partial_q f, q)$, we get a polynomial in p of degree 3330. Using the Mathematica command **CountRoots** [10], we find that it has no real zero for $p \geq 9$. Also we find that $f(9,q) = 0$ has only one zero which is greater than or equal to 9 and that $f(q,q) = 0$ has no real zero for $q \geq 9$. Therefore, we know that in the region of $q \geq p, p \geq 9$, there are only two open connected sets separated by the curve $f = 0$. Since the real zero of $f(9,q) = 0$ for $q \geq 9$ is approximately 32.7, we pick two points $(9,30)$ and $(9,40)$ as sample points from the two open connected regions. Since there is no zero with one of the coordinates being zero for systems $\{\mathcal{F}^{p,q} | p > 1, q > 1\}$, the numbers of positive zeros in those two components are constants. Using Hermite root counting method to count zeros for systems $\mathcal{F}^{9,30}$ and $\mathcal{F}^{9,40}$, we find that there are 2 positive zeros of $\mathcal{F}^{p,q}$ when $q \geq p, p \geq 9$.

When $p \geq 9, q \geq p$ **and** $f = 0$

We first compute the following Sylvester-Habicht sequences and resultants.

$$\begin{aligned}
&Res\,(f_5, f_6, x) = -(p+q)^2 h_y,\\
&Res\,(f_5, f_6, y) = -(p+q)^2 h_x,\\
&SyHa_1(f_5, f_6, x) = p^2 q(p+q) y h_{1y} x + p^2 q(p+q) h_{0y},\\
&SyHa_1(f_5, f_6, y) = p q^2 (p+q) x h_{1x} y + p q^2 (p+q) h_{0x},\\
&r_y = Res(h_y, \partial_y h_y, y) = c C_1 a^2 f,\\
&r_x = Res(h_x, \partial_x h_x, x) = c C_2 b^2 f,\\
&r_{1y} = Res\,(h_y, h_{1y}, y) = C_3 a^2,\\
&r_{0y} = Res\,(h_y, h_{0y}, y) = C_4 a^2,\\
&r_{1x} = Res\,(h_x, h_{1,x}, x) = C_5 b^2,\\
&r_{0x} = Res\,(h_x, h_{0x}, x) = C_6 b^2, where
\end{aligned}$$

$$\begin{aligned}
C_1 =& p^{89} q^{55} (p-1)^8 (q-1)^{14} (1+4p+7p^2)^2 (p+q-1)^6 (p+q)^{38},\\
C_2 =& p^{55} q^{89} (p-1)^{14} (q-1)^8 (1+4q+7q^2)^2 (p+q-1)^6 (p+q)^{38},\\
C_3 =& -p^{62} q^{52} (p-1)^6 (q-1)^{10} (1+4p+7p^2)^2 (p+q)^{30},\\
C_4 =& -p^{64} q^{54} (p-1)^8 (q-1)^{12} (1+4p+7p^2)^2 (p+q-1)^4 (p+q)^{30},\\
C_5 =& -p^{52} q^{62} (p-1)^{10} (q-1)^6 (1+4q+7q^2)^2 (p+q)^{30},\\
C_6 =& -p^{54} q^{64} (p-1)^{12} (q-1)^8 (1+4q+7q^2)^2 (p+q-1)^4 (p+q)^{30}.
\end{aligned}$$

Here $h_x^{p,q}, h_{1x}^{p,q}, h_{0x}^{p,q} \in \mathbb{R}[x]$ and have degrees $26, 17, 19$, respectively, for all $p, q > 0$. $h_y^{p,q}, h_{1y}^{p,q}, h_{0y}^{p,q} \in \mathbb{R}[y]$ and have degrees $26, 17, 19$, respectively, for all $p, q > 0$. $f, a, b \in \mathbb{Q}[p, q]$ and have total degrees $68, 69, 69$ as polynomials in p, q. Note that $C_1, \cdots, C_6 \neq 0$ for $p, q > 1$. Also, $c = -4503599627370496$.

We next claim that f, a and b have no common positive zeros. Compute two resultants, $Res(a, f, q)$ and $Res(b, f, q)$, we get 2 polynomials in p of degrees 3412 and 3398, respectively. Using Mathematica command **CountRoots** to separate their positive zeros, we find that the union of the open intervals $(0, \frac{5}{4}), (\frac{4}{3}, \frac{29}{30}), (\frac{3}{2}, 3), (4, \frac{407}{100})$, $(15, \frac{153}{10}), (223, 1288)$ contains all positive zeros of $Res(a, f, q)$ and no positive zeros of $Res(b, f, q)$. Therefore, a, b and f has no common positive zeros.

Let $p_0 \geq 9, q_0 \geq p$ and $f^{p_0, q_0} = 0$. We assume that $b^{p_0, q_0} \neq 0$. Therefore, we get that $r_x^{p_0}$ has q_0 as a zero with multiplicity 1 since $Res(f, \partial_q f, q) \neq 0$ and

$r_{0x}^{p_0,q_0} \neq 0, r_{1x}^{p_0,q_0} \neq 0$ since $b^{p_0,q_0} \neq 0$. Therefore, by proposition 3.7, we conclude that the numbers of positive zeros are 2 for all systems $\{\mathcal{F}^{p,q} | p \geq 9, q \geq p, f^{p,q} = 0\}$.

When $p = 2, \ldots, 8, q \geq p$ **and** $p, q \in \mathbb{N}$

For $p_0 = 2$, we find that $f(2, q) = 0$ has 2 positive zeros greater than 2. They are approximately 3.31 and 6.69. So, we count the positive zeros at $(p, q) = (2, 3), (2, 5)$ and $(2, 7)$ to find that there are $4, 2, 2$ positive zeros, respectively. Therefore, there are 4 positive zeros for the systems $\mathcal{F}^{2,2}$ and $\mathcal{F}^{2,3}$ and there are 2 positive zeros for all systems $\{\mathcal{F}^{2,q} | q \in \mathbb{N}, q \geq 4\}$.

For $p_0 = 3, \ldots, 8$, we find that $f(p_0, q) = 0$ all have one zero which is greater than or equal to p_0 and not a integer. We find two sample points in each case and the number of positive zeros are all 2 at those points. Therefore, we conclude that the systems have 2 positive zeros for all $p = 3, \ldots, 8, q \geq p$ and $p, q \in \mathbb{N}$.

When $p = 1, q > p$

Recall that $det(H(\mathcal{F}, 1)) = \frac{549755813888(p-1)^2(q-1)^2(p+q-1)^2 f}{p^{38}q^{38}(p+q)^{21}}$. Therefore, in the case of $p = 1$, we get that $det(H(\mathcal{F}, 1))$ is identically equal to zero. This is because we have common zeros $(0, \pm 1)$ of multiplicities 2 for all system $\{\mathcal{F}^{1,q} | q > 0\}$. Therefore, we need to *get rid of* common zeros $(0, \pm 1)$. As we did in the system $\{f_3, f_4\}$, we consider the system $\mathcal{G} = \{g_1, g_2, g_3\}$, where $g_1 = (x + y)(y^2 - 1)y - (1 - qx^2 - qxy + x^2y^3 + qx^2y^3 + xy^4 + qxy^4)$, $g_2 = qz(y^2 - 1) - 2 + qx^2 - 2qy^2$, $g_3 = y^2 - 1 - zx^2$. There is no zero with one of the coordinates being zero for all system $\mathcal{G}^q, q > 1$. We find a bifurcation polynomial of $\mathcal{G}$ and find that there is no real zero for $q > 1$. The number of real zeros of $\mathcal{G}^2$ with positive x, y coordinates is 2. Therefore, for all $q > 1$, the number of real zeros of $\mathcal{G}^q$ with positive x, y coordinates is 3. The number of positive zeros of $\mathcal{F}^{1,q}$ is 3 for all $q > 1$.

No zeros with $x = y$ **for all** $p, q \in \mathbb{N}$

Finally, to finish the proof of 1.3, we need to show that the system $f_5 = f_6 = 0$ has no common zeros with $x = y$ for all $p, q \in \mathbb{N}$. Let $x = y$ in the polynomials f_5 and f_6. We get the two polynomials in y, $-y^2(3 - 2py^2 - 2qy^2 + 2py^5 + 2qy^5)$ and $-1 + p + q - 2py^2 - 2qy^2$. Computing the resultant of $3 - 2py^2 - 2qy^2 + 2py^5 + 2qy^5$ and $-1 + p + q - 2py^2 - 2qy^2$ with respect to y, we get a polynomial $k(p, q) = -4(p + q)^2(a_0(p) + a_1(p)q + a_2(p)q^2 + a_3(p)q^3 + a_4(p)q^4 + 7q^5)$, where $a_0 =$

$1 - 5p + 10p^2 + 118p^3 - 59p^4 + 7p^5, a_1 = -5 + 20p + 354p^2 - 236p^3 + 35p^4, a_2 = 10 + 354p - 354p^2 + 70p^3, a_3 = 118 - 236p + 70p^2, a_4 = -59 + 35p$.

We find that $a_0, \ldots, a_4$ have no real zeros for $p \geq 6$ and they are all positive at $p = 6$. Therefore, $a_0, \ldots, a_4 > 0$ for $p \geq 6$. So, $k(p,q) < 0$ for $p \geq 6, q > 0$. Also, for $p = 1, 2, 3, 4, 5$, $k(p,q) = 0$ has no positive integer zero. In conclusion, $k(p,q) \neq 0$ for all $p, q \in \mathbb{N}$. This means that system $\mathcal{F}^{p,q}$ has no common zeros with $x = y$ for all $p, q \in \mathbb{N}$.

4.2 Maxwell's conjecture of 3 point charges

In this section, we consider systems $\{f_7, f_8\}$ and $\{f_9, f_{10}\}$, where $f_7, f_8, f_9, f_{10} \in \mathbb{Q}[s,t][x,y]$ are given by equations 1.13, 1.14, 1.15, and 1.16. Here the ratio of charge values of 3 point charges is $1 : s : t$. Therefore, we define the parameter space $\mathcal{P} = \{(s,t) \in \mathbb{R}^2 | s \neq 0, t \neq 0\}$ for both systems.

Finiteness of complex zeros

Again, applying $\mathbf{GroebnerBasis}[\{\mathbf{f_7}, \mathbf{f_8}\}, \{\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t}\}, \mathbf{MonomialOrder} \to \mathbf{w}]$ and $\mathbf{GroebnerBasis}[\{\mathbf{f_9}, \mathbf{f_{10}}\}, \{\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t}\}, \mathbf{MonomialOrder} \to \mathbf{w}]$, where $\mathbf{w}$ is the block ordering $<$ given by $\mathbf{w} = \{\{\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}\}, \{\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}\}, \{\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}\}, \{\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}\}\}$, we get sets G_1 and G_2 both consisting 4 polynomials in $\mathbb{Q}[s,t][x,y]$. The leading coefficients of polynomials in G_1 with respect to $<_{x,y}$ are $2, 2s^2, 4s^4, 8$ and 2 leading monomials are x^{10}, y^{12}. The leading coefficients of polynomials in G_2 with respect to $<_{x,y}$ are $1, s^2, s^4, 1$ and 2 leading monomials are x^{10}, y^{12}, too. Therefore, $G_1^{s,t}$ is a Groebner basis of $I_1^{s,t} = \langle f_7^{s,t}, f_8^{s,t} \rangle$ and $A_1^{s,t} = \mathbb{C}[x,y]/I_1^{s,t}$ is of finite dimensional for all $s \neq 0$, and we find $dim(A_1^{s,t}) = 56$. Similarly, $G_2^{s,t}$ is a Groebner basis of $I_2^{s,t} = \langle f_9^{s,t}, f_{10}^{s,t} \rangle$ and $A_2^{s,t} = \mathbb{C}[x,y]/I_2^{s,t}$ is of finite dimensional for all $s \neq 0$, and we find $dim(A_2^{s,t}) = 56$, too. Therefore, systems $\{f_7^{s,t}, f_8^{s,t}\}$ and $\{f_9^{s,t}, f_{10}^{s,t}\}$ have at most 56 complex zeros for all $(s,t) \in \mathcal{P}$.

Bifurcation polynomial

First, we compute some resultants from equations f_7, f_8 and f_9, f_{10}.

$$\begin{aligned}
&Res\,(f_7, f_8, x) = s^8y^4(t^2+2ty^3+2t^6)^3h_y,\\
&Res\,(f_7, f_8, y) = x^4(t^2+2tsx^3+2s^2x^6)^3h_x,\\
&r_y = Res(h_y, \partial_y h_y, y) = c_1t^{238}s^{44}a^2f,\\
&r_x = Res(h_x, \partial_x h_x, x) = c_2t^{238}s^{236}b^2f,\\
&Res\,(f_9, f_{10}, x) = s^8y^4(t^2+ty^3+y^6)^3k_y,\\
&Res\,(f_9, f_{10}, y) = x^4(t^2+tsx^3+s^2x^6)^3k_x,\\
&v_y = Res(k_y, \partial_y k_y, y) = c_3t^{226}s^{44}(t^2-1)^{30}c^2g,\\
&v_x = Res(k_x, \partial_x k_x, x) = c_4t^{226}s^{236}(t^2-s^2)^{30}d^2g.
\end{aligned}$$

Here $h_x^{s,t}, k_x^{s,t} \in \mathbb{R}[x]$ and have degree 34 for all $(s,t) \in \mathcal{P}$. $h_y^{s,t}, k_y^{s,t} \in \mathbb{R}[y]$ and also have degree 34 for all $(s,t) \in \mathcal{P}$. $h_x(0) = h_y(0) = k_x(0) = k_y(0) = t^2$. So, $h_x^{s,t}, h_y^{s,t}, k_x^{s,t}, k_y^{s,t}$ are all non-zero at 0 for all $(s,t) \in \mathcal{P}$. $f, g, a, b, c, d \in \mathbb{Z}[s^2, t^2]$ and have total degrees $24, 24, 26, 26, 14, 14$ as polynomials in s^2 and t^2. $c_1, \ldots, c_4$ are non-zero integers. Note that $t^2+2ty^3+2y^6, t^2+ty^3+y^6$ and $t^2+2tsx^3+2s^2x^6, t^2+tsx^3+s^2x^6$ have no real zeros.

Proposition 4.1. *f, g are bifurcation polynomials for $\{f_7, f_8\}, \{f_9, f_{10}\}$, respectively.*

Proof. We know from above that $Res\,(f_7, f_8, x)\,, Res\,(f_7, f_8, y)$ have positive zeros in h_y, h_x, respectively, and $Res\,(f_9, f_{10}, x)\,, Res\,(f_9, f_{10}, y)$ have positive zeros in k_y, k_x, respectively. According to section 3.2.2, it suffices for us to show that $\mathcal{B}_1 \cap V_{\mathbb{R}}(\frac{r_x}{f}, \frac{r_y}{f})$ and $\mathcal{B}_2 \cap V_{\mathbb{R}}(\frac{v_x}{g}, \frac{v_y}{g})$ are empty sets, where $\mathcal{B}_1, \mathcal{B}_2$ are the bifurcation sets for systems $\{f_7, f_8\}, \{f_9, f_{10}\}$, respectively. We only demonstrate the case of $\mathcal{B}_1 \cap V_{\mathbb{R}}(\frac{r_x}{f}, \frac{r_y}{f}) = \emptyset$ here. The other case can be shown in the same way.

Since $s \neq 0, t \neq 0$ in $\mathcal{B}_1$, we need to show that $\mathcal{B}_1 \cap V_{\mathbb{R}}(a, b) = \emptyset$. Let $(s_0, t_0) \in \mathcal{B}_1 \cap V_{\mathbb{R}}(a, b)$. Note first that, by a technique of separating positive zeros as we did in the study of system $\{f_5, f_6\}$, we can show that a, b and f have no common positive zeros. Open intervals separating positive zeros for $Res(f, a, t^2)$ and $Res(a, b, t^2)$ are listed in table 4.7. Therefore, $f^{s_0,t_0} \neq 0$.

Denote $J = |\dfrac{\partial(f_7, f_8)}{\partial(x, y)}|$, the Jacobian determinate of f_7, f_8. We compute some resultants.

$$Res(f_7, f_8, s) = x^{12}(t^2 + 2ty^3 + 2y^6)A_1,$$
$$Res(J, f_8, s) = x^{20}A_2,$$
$$Res(A_1, A_2, x) = y^{16}(t^2 + 2ty^3 + 2y^6)^2 B_1^4 B_2^2 B_3^2.$$

Both A_1 and A_2 have degree 4 in x. B_1, B_2, B_3 have degree $2, 24, 34$, respectively, in y. Next, we compute

$$Res(h_y, B_1, y) = d_1 s^4 C_1,$$
$$Res(h_y, B_2, y) = d_2 t^{180} C_2 f,$$
$$Res(h_y, B_3, y) = d_3 t^{216} C_3 C_4.$$

Here C_1, C_2, C_3, C_4 have degrees $2, 24, 34, 34$ in s^2 and d_1, d_2, d_3 are integers. Again, we use the same technique to separate positive roots of $Res(a, b, s^2)$ from those of $Res(a, C_1, s^2), Res(a, C_2, s^2), Res(a, C_3, s^2), Res(a, C_4, s^2)$, respectively. See table 4.7 for details.

Since $a^{s_0,t_0} = b^{s_0,t_0} = 0$, C_1, C_2, C_3, C_4 are all nonzero. Since $(s_0, t_0) \in \mathcal{B}_1$, there exist $x_0 > 0, y_0 > 0$ such that $f_7^{s_0,t_0}(x_0, y_0) = f_8^{s_0,t_0}(x_0, y_0) = 0$ and also the Jacobian polynomial $J^{s_0,t_0}(x_0, y_0) = 0$. Then we will have that $Res(A_1, A_2, x)$ and h_y have (y_0, t_0) as a common root and at least one of $Res(h_y, B_1, y), Res(h_y, B_2, y)$, or $Res(h_y, B_3, y)$ is zero, which is a contradiction. $\square$

4.2.1 Generic results

Finding sample points in each open connected component

Here we will focus on the study of the curve $f(t, s) = 0$ and try to find at least one sample points from each open connected components determined by f. Similarly, for the system $f_9 = f_{10} = 0$, we study the curve $g(t, s) = 0$ and try to find sample points. Note that f and g are polynomials in t^2 and s^2. So the zero sets will be symmetric with respect to the t-axis and the s-axis. Also, for both curves, it is easy to see that they do not pass through the two coordinate axes. Therefore, it suffices to find sample points from the first quadrant of the $s - t$ plane.

Mathematica 6.0 provides the command, **SemialgebraicComponentInstance**, to give at least one point in each open connected component. We get 64 points for $f < 0$ and 407 points for $f > 0$. This command is based on the cylindrical algebraic decomposition [1] [5]. However, from the numerical plot evidence of $f = 0$ and $g = 0$, figure 4.1 and 4.2, it seems that there are 5 connected components in each quadrant in both cases. Here, we use different approach than just applying the Mathematica command. We actually reduced the number of sample points in each quadrant to 5.

Proposition 4.2. *There are at most* 5 *open connected components determined by* $f = 0$ *in the first quadrant. Simple points are* $(\frac{1}{5}, \frac{2}{25}), (\frac{1}{4}, 1), (3, 13), (3, 1), (1, 1)$. *Similarly, there are at most* 5 *open connected components determined by* $g = 0$ *in the first quadrant. Simple points are* $(\frac{1}{5}, \frac{1}{5}), (\frac{4}{5}, 1), (\frac{6}{5}, \frac{29}{5}), (2, 1), (5, 1)$.

Proof. Here we focus on the case of $f = 0$. The other case can be shown in the same way. We first compute $Res(f, \partial_s f, s)$ to get a integral polynomial $p(t)$ of degree 2136. Using Mathematica command **CountRoots** to isolate its positive roots [10], we pick 34 points, denoted by $t_1, \ldots, t_{34}$, where $0 < t_1 < \cdots < t_{34}$, such that each interval (t_i, t_{i+1}) contains an unique positive root for all $i = 1, \ldots, 33$ and that there are no positive roots outside the interval (t_1, t_{34}). We denote the 33 positive roots of $p(t)$ as $r_1, \ldots, r_{33}$ and also denote 0 as r_0 and ∞ as r_{34}.

Then for each t_i, we isolate positive roots of $f(t_i, s) = 0$ and pick $s_{i,j}$, where $j = 1, \ldots, n_i$ and $0 < s_{i,1} < \cdots < s_{i,n_i}$, such that each interval $(s_{i,j}, s_{i,j+1})$ contains an unique positive root of $f(t_i, s)$ and that there are no positive roots outside the interval $(s_{i,1}, s_{i,n_i})$. So, there are $(n_i - 1)$ positive roots of $f(t_i, s)$. If $f(t_i, s)$ has no positive roots, then $n_i = 1$ and we let $s_{i,1} = 30$ in the case of f and let $s_{i,1} = 10$ in the case of g.

In each region $(r_{i-1}, r_i) \times (0, \infty)$ for $i = 1, \ldots, 34$, the curve of $f = 0$ is non-singular with $\partial_s f \neq 0$. Therefore, the curve is smooth and does not have vertical tangent line. For all $t \in (r_{i-1}, r_i)$, the numbers of positive roots of $f(s) = 0$ are the same. Since $t_i \in (r_{i-1}, r_i)$, the number of positive roots is $(n_i - 1)$. Therefore, there are n_i open connected subsets of $(r_{i-1}, r_i) \times (0, \infty)$ determined by the curve of $f = 0$. Moreover, each pair $(t_i, s_{i,j})$ for all $j = 1, \ldots, n_i$ gives one sample point of those open connected subsets. They are listed in table 4.1.

Finally, we connect 96 samples points $(t_i, s_{i,j})$ by line segments avoiding the bifurcation curve to reduce the number of sample points to 5. We get five sample points in the first quadrant, $(\frac{1}{5}, \frac{2}{25}), (\frac{1}{4}, 1), (3, 13), (3, 1), (1, 1)$. Table 4.2 gives the details about other sample points connecting to them. Similarly, table 4.3 and 4.4 show sample points for $g = 0$. $\square$

The idea in our proof above is also based on and similar to that of the cylindrical algebraic decomposition [1] [5]. However, the numerical plots of $f = 0$ and $g = 0$ help us to select sample points $(t_i, s_{i,j})$ to make it easy to connect them by line segments. Good choices of sample points and the reference of the numerical plots together give us ideas about which points may be connected to which points. This makes it not such hard work to reduce the 96 sample points $(t_i, s_{i,j})$ to only 5 points in the case of $f = 0$ and the 54 sample points $(t_i, s_{i,j})$ to only 5 points in the case of $g = 0$.

Generically 2 or 4 positive zeros

By proposition 4.2, there are 5 sample points in the first quadrant. By symmetry, we have 20 sample points in the $s - t$ plane for both the system $\{f_7, f_8\}$ and $\{f_9, f_{10}\}$. Using Hermite's root counting method to count the positive zeros for 40 real polynomial systems, we get that the number of positive zeros are either 2 or 4. Table 4.5 and 4.6 give the detail results for each system.

For both systems $\{f_7, f_8\}$ and $\{f_9, f_{10}\}$, it is easy to see that they both have $(0, 0)$ as a real zero of intersection multiplicity 4 for all $(s, t) \in \mathcal{P}$. Also, for both systems, the only real zero with $x = 0$ or $y = 0$ is the origin. Therefore, by proposition 3.3, in order to claim that the number of positive roots is a constant when parameters are in one of the open connected component determined by $f = 0$ or $g = 0$, we need to show that, for each $(s_0, t_0) \in \mathcal{P}$ there exist neighborhoods $\mathcal{U}$ containing (s_0, t_0) and $\mathcal{V}$ containing $(0, 0)$ such that $(0, 0)$ is the only real zeros in $\mathcal{V}$ of systems $\{f_7^{s,t}, f_8^{s,t} | (s, t) \in \mathcal{U}\}$ or $\{f_9^{s,t}, f_{10}^{s,t} | (s, t) \in \mathcal{U}\}$. As we did in studying the system $\{f_1, f_2\}$, these are obtained by the facts that $h_x^{s,t}, h_y^{s,t}, k_x^{s,t}, k_y^{s,t}$ are all nonzero at 0 for all $(s, t) \in \mathcal{P}$. Therefore, we get the generic results for both systems. That is, there are 2 or 4 positive zeros for systems $\{f_7^{s,t}, f_8^{s,t} | (s, t) \in \mathcal{P}, f^{s,t} \neq 0\}$ and $\{f_9^{s,t}, f_{10}^{s,t} | (s, t) \in \mathcal{P}, g^{s,t} \neq 0\}$.

4.2.2 On the bifurcation set

Recall that we have $r_y = c_1 t^{238} s^{44} a^2 f, r_x = c_2 t^{238} s^{236} b^2 f, v_y = c_3 t^{226} s^{44} (t^2 - 1)^{30} c^2 g$, and $v_x = c_4 t^{226} s^{236} (t^2 - s^2)^{30} d^2 g$. We compute Sylvester-Habicht sequences and more resultants in the following.

$$
\begin{aligned}
SyHa_1(f_8, f_7, x) &= -2tsy^8(t^2+2ty^3+2y^6)^2h_{1y}x - s^{10}y^2(t^2+2ty^3+2y^6)^2h_{0y},\\
SyHa_1(f_7, f_8, y) &= 2tx^8(t^2+2tsx^3+2s^2x^6)^2h_{1x}y + x^2(t^2+2tsx^3+2s^2x^6)^2h_{0x},\\
r_{1y} &= Res\,(h_y, h_{1y}, y) = c_5t^{104}s^{40}a^2,\\
r_{0y} &= Res\,(h_y, h_{0y}, y) = c_6t^{208}s^{64}a^2,\\
r_{1x} &= Res\,(h_x, h_{1,x}, x) = c_7t^{104}s^{156}b^2,\\
r_{0x} &= Res\,(h_x, h_{0x}, x) = c_8t^{208}s^{196}b^2,\\
SyHa_1(f_{10}, f_9, x) &= -s^{11}y^5(k^2+ky^3+y^6)^2k_{1y}x + s^{10}y^2(k^2+ky^3+y^6)^2k_{0y},\\
SyHa_1(f_9, f_{10}, y) &= -x^5(k^2+ksx^3+s^2x^6)^2k_{1x}y + x^2(k^2+ksx^3+s^2x^6)^2k_{0x},\\
v_{1y} &= Res\,(k_y, k_{1y}, y) = c_9t^{156}s^{40}(t^2-1)^{30}c^2,\\
v_{0y} &= Res\,(k_y, k_{0y}, y) = c_{10}t^{196}s^{64}(t^2-1)^{30}c^2,\\
v_{1x} &= Res\,(k_x, k_{1,x}, x) = c_{11}t^{156}s^{156}(t^2-s^2)^{30}d^2,\\
v_{0x} &= Res\,(k_x, k_{0x}, x) = c_{12}t^{196}s^{196}(t^2-s^2)^{30}d^2,
\end{aligned}
$$

where, $c_5, \ldots, c_{12}$ are nonzero integers.

Non-singular points on $f = 0$ or $g = 0$

Proposition 4.3. *There are 3 positive zeros for systems $\{f_7^{s,t}, f_8^{s,t}\}$ for all (s,t) that are in the bifurcation set and non-singular points on $f = 0$. Similarly, there are 3 positive zeros for systems $\{f_7^{s,t}, f_8^{s,t}\}$ for all (s,t) that is in the bifurcation set and a non-singular point on $g = 0$.*

Proof. Here we only demonstrate the proof for the case of the system $\{f_7, f_8\}$. The other case can be shown in the same way. Let (s_0, t_0) be in the bifurcation set and a non-singular point on $f = 0$. We will claim that either $r_x^{t_0}$ has s_0 as a zero with multiplicity 1, $r_{0x}^{s_0,t_0} \neq 0, r_{1x}^{s_0,t_0} \neq 0$, or $r_y^{t_0}$ has s_0 as a zero with multiplicity 1, $r_{0y}^{s_0,t_0} \neq 0, r_{1y}^{s_0,t_0} \neq 0$, or $r_x^{s_0}$ has t_0 as a zero with multiplicity 1, $r_{0x}^{s_0,t_0} \neq 0, r_{1x}^{s_0,t_0} \neq 0$, or $r_y^{s_0}$ has t_0 as a zero with multiplicity 1, $r_{0y}^{s_0,t_0} \neq 0, r_{1y}^{s_0,t_0} \neq 0$. If so, by proposition 3.7 and the generic results, we get there are exactly $\frac{2+4}{2} = 3$ positive zeros for systems $\{f_7^{s_0,t_0}, f_8^{s_0,t_0}\}$.

Since (s_0, t_0) is non-singular on $f = 0$, we get either $\partial_s f(s_0, t_0) \neq 0$ or $\partial_t f(t_0, s_0) \neq 0$. Since $f(s_0, t_0) = 0$ and f, a, b have no common real zeros, we have either $a(s_0, t_0) \neq 0$ or $b(s_0, t_0) \neq 0$. If $\partial_s f(s_0, t_0) \neq 0$ and $a(s_0, t_0) \neq 0$, then we get $r_y^{t_0}$ has s_0 as a zero with multiplicity 1, $r_{0y}^{s_0,t_0} \neq 0, r_{1y}^{s_0,t_0} \neq 0$. The remaining three situations are in correspondence to three other situations in the last paragraph. $\square$

Singular points on $f = 0$

Proposition 4.4. *Let (s,t) be in the bifurcation set and a singular point on $f = 0$. There are $2, 3$ or 4 positive zeros for systems $\{f_7^{s,t}, f_8^{s,t}\}$.*

Proof. Let (s_0, t_0) be in the bifurcation set and a singular point on $f = 0$. We will claim that either $r_x^{t_0}$ has s_0 as a zero with multiplicity 2, $r_{0x}^{s_0,t_0} \neq 0, r_{1x}^{s_0,t_0} \neq 0, r_{sx}^{s_0,t_0} \neq 0$, or $r_y^{t_0}$ has s_0 as a zero with multiplicity 2, $r_{0y}^{s_0,t_0} \neq 0, r_{1y}^{s_0,t_0} \neq 0, r_{sy}^{s_0,t_0} \neq 0$, where $r_{sx} = Res(h_x, \partial_s h_x, x)$ and $r_{sy} = Res(h_y, \partial_s h_y, y)$. If so, by proposition 3.8 and the generic results, we get there are $\frac{2+2}{2} = 2, \frac{2+4}{2} = 3$, or $\frac{4+4}{2} = 4$ positive zeros for systems $\{f_7^{s_0,t_0}, f_8^{s_0,t_0}\}$. We first compute $r_{sy} = c_{13} t^{248} s^{34} a^2 w_1, r_{sx} = c_{14} t^{248} s^{210} b^2 w_2$, where c_{13}, c_{14} are nonzero integers and $w_1, w_2 \in \mathbb{Z}[s,t]$.

Computing $Res(f, \partial_s f, s)$ and $Res(f, \partial_{ss} f, s)$, we get polynomials in t of order 2136 and 2104. They are actually polynomials in t^2 of order 1068 and 1052. Separating their positive roots by finding union of the open intervals in table 4.7 such that it contains all of one's positive root and excludes all positive roots of the other one, we know that $f, \partial_s f, \partial_{ss} f$ have no common real zeros. Since $f(s_0, t_0) = \partial_s f(s_0, t_0) = 0$, we get $\partial_{ss} f(t_0, s_0) \neq 0$. Again, since $f(s_0, t_0) = 0$, we have either $a(s_0, t_0) \neq 0$ or $b(s_0, t_0) \neq 0$.

If $a(s_0, t_0) \neq 0$, then we get $r_y^{t_0}$ has s_0 as a zero with multiplicity 2, $r_{0y}^{s_0,t_0} \neq 0, r_{1y}^{s_0,t_0} \neq 0$. What left is to show $r_{sy}^{s_0,t_0} \neq 0$. But, $r_{sy} = c_{13} t^{248} s^{34} a^2 w_1$. Therefore, it suffices to show $w_1(s_0, t_0) \neq 0$. On the other hand, if $b(s_0, t_0) \neq 0$, then we get $r_x^{t_0}$ has s_0 as a zero with multiplicity 2, $r_{0x}^{s_0,t_0} \neq 0, r_{1x}^{s_0,t_0} \neq 0$. What left is to show that $r_{sx}^{s_0,t_0} \neq 0$. Since $r_{sx} = c_{13} t^{248} s^{210} b^2 w_2$, it suffices to show that $w_2(s_0, t_0) \neq 0$.

Now, we prove that $w_1(s_0, t_0) \neq 0$ and $w_2(s_0, t_0) \neq 0$. For w_1, since $t_0 \neq 0$ and $\partial_t f$ has $12t$ as a factor, we compute $Res(\frac{\partial_t f}{12t}, w_1, t^2)(s^2)$ and $Res(\frac{\partial_s f}{12t}, w_1, t^2)(s^2)$ and separate their positive roots in s^2. So, $w_1(s_0, t_0) \neq 0$ since $\partial_t f(s_0, t_0) = \partial_s f(s_0, t_0) = 0$. By similar computations for w_2, we can separate positive zeros and conclude that $w_2(s_0, t_0) \neq 0$. The open sets separating positive zeros are listed in table 4.7. $\square$

Singular points on $g = 0$

Now we consider (t_0, s_0) such that it is singular on $g = 0$ and in the bifurcation set of $\{f_9, f_{10}\}$. It can also be shown that $g, (t^2 - 1)c$, and $(t^2 - s^2)d$ have no common real roots. See table 4.8 for details. We follow the same approach as we did in the proof of proposition 4.4. The roles of r_{sy}, r_{sx} in that proof are replaced by v_{sy}, v_{sx}, where $v_{sy} = Res(k_y, \partial_s k_y, y) = c_{15}t^{236}s^{34}(t^2-1)^{30}c^2w_3, v_{sx} = Res(k_x, \partial_s k_x, x) = c_{16}t^{236}s^{210}(t^2-s^2)^{30}d^2w_4$. The roles of w_1, w_2 in that proof are replaced by w_3, w_4. In order to get the results as in proposition 4.4, we need to show that

$$\begin{aligned} Part(1) &: \partial_{ss}g(t_0, s_0) \neq 0, \\ Part(2) &: w_3(t_0, s_0) \neq 0, \\ Part(3) &: w_4(t_0, s_0) \neq 0. \end{aligned}$$

However, we find that the each inequality fails for some singular points (t_0, s_0) on $g_{34} = 0$. For $Part(1)$, we find that t_0 is a zero of t^2-1 and s_0 is a zero of $p_1 = 256 - 576s^2 + 27s^4$ when $\partial_{ss}g(t_0, s_0) = 0$. We say these parameters are in $Group(1)$.

$$Group(1) : t^2 - 1 = 0, p_1 = 0$$

Examples of singular points in $Group(1)$ are shown in figure 4.2 as points with label 1. We show how to find these t_0 and s_0 and count the positive zeros of the system $\{f_9^{t_0,s_0}, f_{10}^{t_0,s_0}\}$ directly in the following proposition.

Proposition 4.5. *Let (t_0, s_0) be a singular points on $g = 0$ and satisfy $\partial_{ss}g_{34}(t_0, s_0) = 0$. Then (t_0, s_0) is in the $Group(1)$. For such (t_0, s_0), the system $\{f_9^{t_0,s_0}, f_{10}^{t_0,s_0}\}$ has 2 positive zeros.*

Proof. When we compute $Res(g, \partial_s g, s)$ and $Res(g, \partial_{ss}g, s)$, we find that they have common factor $(t^2-1)^8$. After dividing by this factor, we can separate their positive zeros. See table 4.8 for detail. Therefore, we conclude that $t = \pm 1$ are the only real common zeros. Substituting $t = \pm 1$ into $g = 0$, we have $g(\pm 1, s) = (1024 - 828s^2 + 27s^4)p_1^3p_2^2$. Therefore, $g(\pm 1, s)$ have s_0 as zeros of multiplicity 3 when s_0 are zeros of p_1. They are approximate ± 0.67 and ± 4.56.

We now apply Hermite's theorem and only demonstrate the positive root counting here for the case when $(t, s) = (1, 0.67\ldots)$. We consider the system of three equations in three variables x, y, s. The first two equations are $f_9(x, y, 1, s) = 0$ and $f_{10}(x, y, 1, s) = 0$ and third one is $p_1 = 256 - 576s^2 + 27s^4 = 0$. Since $0.67\ldots$ is the only real zero of the p_1 between $(0, 2)$, our goal now is to find the common zeros of the three equations in the region $(0, +\infty) \times (0, +\infty) \times (0, 2)$.

Let $I = \langle f_9(x, y, 1, s), f_{10}(x, y, 1, s), p_1 \rangle \subset \mathbb{R}[x, y, s]$ be the ideal generated by the three polynomials and $A = \mathbb{R}[x,y,s]/I$. Computing a Groebner basis of I with respect to the grlex order where $x > y > s$, we get 13 polynomials with leading monomials $s^4, x^2y^6, x^3y^3s^3, x^6y^2s, x^6y^5, x^9y^2, y^{10}s^2, x^{10}s^2, x^8y^4, x^{12}, y^{12}s, y^{13}, xy^{12}$. Therefore, there are 224 monomials that are not divided by them. Therefore, the dimension of A is 224.

Next, we compute 8 Hermite matrices $H(I,1), H(I,x), H(I,y), H(I,p)$, $H(I,xy), H(I,xp), H(I,yp)$, and $H(I,xyp)$ where $p = 1-(s-1)^2$. Their signatures are $36, 0, 0, -18, 0, 0, 0, 0$. Let $a_1, \cdots, a_8$ denote the number of common real zeros (x,y,s) having the signs $(+,+,+), \cdots, (-,-,-)$, respectively. We get

$$\begin{aligned}
36 &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + 4\\
0 &= a_1 + a_2 + a_3 + a_4 - a_5 - a_6 - a_7 - a_8\\
0 &= a_1 + a_2 - a_3 - a_4 + a_5 + a_6 - a_7 - a_8\\
-18 &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + 1 - 3\\
0 &= a_1 + a_2 - a_3 - a_4 - a_5 - a_6 + a_7 + a_8\\
0 &= a_1 - a_2 + a_3 - a_4 - a_5 + a_6 - a_7 + a_8\\
0 &= a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8\\
0 &= a_1 - a_2 - a_3 + a_4 - a_5 + a_6 + a_7 - a_8
\end{aligned}$$

In the first equation, 4 is the number of zeros $(0,0,\pm 0.67\ldots)$ and $(0,0,\pm 4.56\ldots)$. In the forth equation, 1 is the number of zero $(0,0,0.67\ldots)$ and 3 is the number of zeros $(0,0,-0.67\ldots)$ and $(0,0,\pm 4.56\ldots)$. Solving this linear system, we get $(a_1, \cdots, a_8) = (2,6,2,6,2,6,2,6)$. In particular, $a_1 = 2$. So, there are 2 positive common zeros for the system $\{f_9^{t,s}, f_{10}^{t,s}\}$ when $(t,s) = (1, 0.67\ldots)$. $\square$

Among the parameters which are singular points on $g = 0$ and not in $Group(1)$, we find that each inequality in $Part(2)$ and $Part(3)$ may fail only in the following groups of parameters presented as zeros of polynomials as listed below.

$$\begin{aligned}
&Group(2) : t^2 - 1 = 0, p_2 = 0 \text{ or } 1 - 46t^2 + t^4 = 0, p_3 = 0.\\
&Group(3) : 256 - 576t^2 + 27t^4 = 0, p_1 = 0.
\end{aligned}$$

Here $p_2 = -1836660096 + \cdots + 452984832s^{16}$ and $p_3 = 27 - 576s^2 + 256s^4$. Examples of singular points in $Group(2), Group(3)$ are shown in figure 4.2 as points with label 2, 3, respectively. We show how to find such t_0 and s_0 and count the positive zeros of the system $\{f_9^{t_0,s_0}, f_{10}^{t_0,s_0}\}$ directly in the following proposition.

Proposition 4.6. *The only parameters that are singular points on $g = 0$ not in $Group(1)$ and satisfy $w_3 = 0, w_4 = 0$ are in the $Group(2), Gruop(3)$, respectively. If (t_0, s_0) is in the bifurcation set and is such a parameter, then there are at most 4 positive zeros for the system $\{f_9^{t_0,s_0}, f_{10}^{t_0,s_0}\}$.*

Proof. First, we find (t_0, s_0) such that $w_3(t_0, s_0) = 0$. Computing $Res(\frac{\partial_t g}{12t}, w_3, t^2)$ and $Res(\frac{\partial_s g}{12t}, w_3, t^2)$, we find that they have common factors p_1, p_2, and p_3. We separate the positive roots of two quotients of these resultants divided by the common factors. See table 4.8 for details. So, the s_0's such that $w_3(t_0, s_0) = 0$ can only be zeros of these polynomials p_1, p_2 or p_3.

On the other hand, we compute $Res(\frac{\partial_t g}{12t}, w_3, s^2)$ and $Res(\frac{\partial_s g}{12t}, w_3, s^2)$ and find their common factors $t^2 - 1$ and $1 - 46t^2 + t^4$. We can separate the positive roots of two quotients again. Therefore, the t_0's such that $w_3(t_0, s_0) = 0$ can only be zeros of polynomials $t^2 - 1$ and $1 - 46t^2 + t^4$. Next, we match these s_0's and t_0's.

We make use of the fact that $g(t_0, s_0) = 0$. We have that $g(\pm 1, s) = (1024 - 828s^2 + 27s^4)p_1^3 p_2^2$. Computing $Res(g, 1 - 46t^2 + t^4, t)$, we find that p_3 is a factor. Note that p_1, p_2 and p_3 have different real roots. From these, we conclude that when $t_0 = \pm 1$, s_0 is a zero of p_1 or p_2. When t_0 is a zero of $1 - 46t^2 + t^4$, s_0 is a zero of p_3. Parameters with $t_0 = \pm 1$ and s_0 as a zero of p_1 are in $Group(1)$. Therefore, $Group(2)$ consists of parameters satisfying $t_0 = \pm 1$ and s_0 is a zero of p_2 or t_0 is a zero of $1 - 46t^2 + t^4$ and s_0 is a zero of p_3.

Similarly, to find (t_0, s_0) such that $w_4(t_0, s_0) = 0$, we compute $Res(\frac{\partial_t g}{12t}, w_4, t^2)$ and $Res(\frac{\partial_s g}{12t}, w_4, t^2)$, we find that they have common factors p_1. Again, we separate the positive roots of two quotients of these resultants divided by the common factors. See table 4.8 for details. So, the s_0's such that $w_4(t_0, s_0) = 0$ can only be zeros of p_1.

On the other hand, when computing $Res(\frac{\partial_t g}{12t}, w_4, s^2)$ and $Res(\frac{\partial_s g}{12t}, w_4, s^2)$, we find that they have common factors $t^2 - 1$ and $256 - 576t^2 + 27t^4$. We separate the positive roots of two quotients and conclude that the t_0's such that $w_4(t_0, s_0) = 0$ can only be zeros of polynomials $t^2 - 1$ or $256 - 576t^2 + 27t^4$.

Finally, computing $Res(g, 256 - 576t^2 + 27t^4, t)$, we find p_1 divides it. Therefore, we conclude that only when t_0 is a zero of $256 - 576t^2 + 27t^4$ or ± 1 and s_0 is a zero of $p_1 = 256 - 576s^2 + 27s^4$ may (t_0, s_0) be a zero of w_4. Again, parameters with $t_0 = \pm 1$ and s_0 as a zero of p_1 are in $Group(1)$.

Next, we prove that, when fixing t_0's from $Group(2)$ or $Group(3)$, there exists a neighborhood of the corresponding s_0 such that $f_9^{t_0,s_0} = f_{10}^{t_0,s_0} = 0$ has 2 positive zeros in this neighborhood except at the point (t_0, s_0). We demonstrate only one (t_0, s_0). Cases of other points can be handled in the same way. We consider the positive zero $0.14\ldots$ of $1 - 46t^2 + t^4$ and the positive zero $0.21\ldots$ of p_1 .

Fixing $t_0 = 0.14\ldots$, we find a neighborhood of $s_0 = 0.21\ldots$ by the followings. First, we consider the line segments connecting $(\frac{15}{100}, \frac{22}{100})$ to $(\frac{14}{100}, \frac{22}{100})$ to $(\frac{14}{100}, \frac{21}{100})$ to $(\frac{15}{100}, \frac{21}{100})$, and find they do not intersect with $g = 0$. Next, counting the positive zeros of $f_9^{\frac{14}{100},\frac{21}{100}} = f_{10}^{\frac{14}{100},\frac{21}{100}} = 0$, we get there are 2. Finally, we argue that, in the interval $(\frac{21}{100}, \frac{22}{100})$, $g(0.14\ldots, s)$ has $0.21\ldots$ as the only real root by showing that $Res(g, 1 - 46t^2 + t^4, t)$ has only one real root in this interval. In conclusion, $f_9^{t_0,s} = f_{10}^{t_0,s} = 0$ has 2 positive zeros for all $s \in (s_0 - \epsilon, s_0 + \epsilon) \setminus \{s_0\}$, for some $\epsilon > 0$.

Finally, we show there are at most 4 positive zeros for the system $\{f_9^{t_0,s_0}, f_{10}^{t_0,s_0}\}$, where (t_0, s_0) is a singular point of $g = 0$ in the bifurcation set and not in $Group(1)$ but in $Group(2)$ or $Gruop(3)$. Projecting real zeros on to the x or y axis, we know that $h_x^{t_0,s}$ or $h_y^{t_0,s}$ has 2 positive zeros for all $s \in (s_0 - \epsilon, s_0 + \epsilon) \setminus \{s_0\}$ since either $(t^2 - 1)c \neq 0$ or $(t^2 - s^2)d \neq 0$.

Suppose $(t^2-1)c \neq 0$ and $h_y^{t_0,s}$ has 2 positive zeros for all $s \in (s_0-\epsilon, s_0+\epsilon)\setminus\{s_0\}$. There must exist at least one positive zero of $\{f_9^{t_0,s_0}, f_{10}^{t_0,s_0}\}$ such that its multiplicity is greater than 1 since (t_0, s_0) is in the bifurcation set. Suppose there are 5 or more positive zeros for the system $\{f_9^{t_0,s_0}, f_{10}^{t_0,s_0}\}$. Since (t_0, s_0) is not in $Group(1)$, $\partial_{ss}g(t_0, s_0) \neq 0$ and $r_y^{t_0}$ has s_0 as a zero of multiplicity 2. As arguments in proposition 3.8, we know there are at most two zeros of the system with intersection multiplicity greater or equal to 2. Therefore, among those 5 or more positive zeros of the systems, projections of 3 or more of them will be positive zeros of $h_y^{t_0,s_0}$ that can be extend continuously as zeros of $h_y^{t_0,s}$ for s nearby s_0. But, this contradicts to the fact that there is only 2 positive zeros of $h_y^{t_0,s}$ for all s near s_0. $\square$

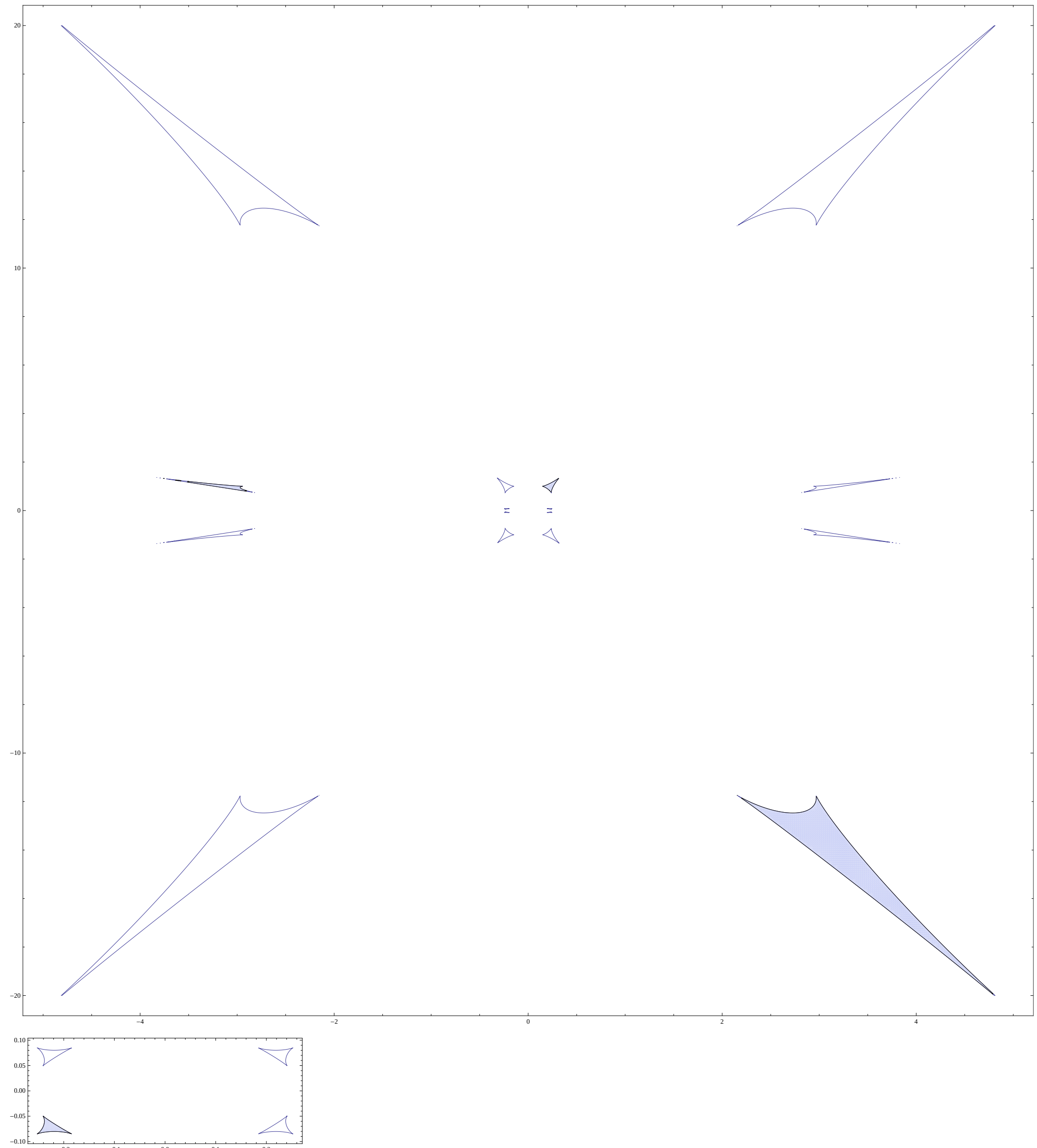

Figure 4.1: Plots of $f = 0$. System $\{f_7, f_8\}$ has 4 positive roots when parameters are on the shaded regions and 2 positive roots on the other regions. The left lower graph is the blow up around $(0, 0)$.

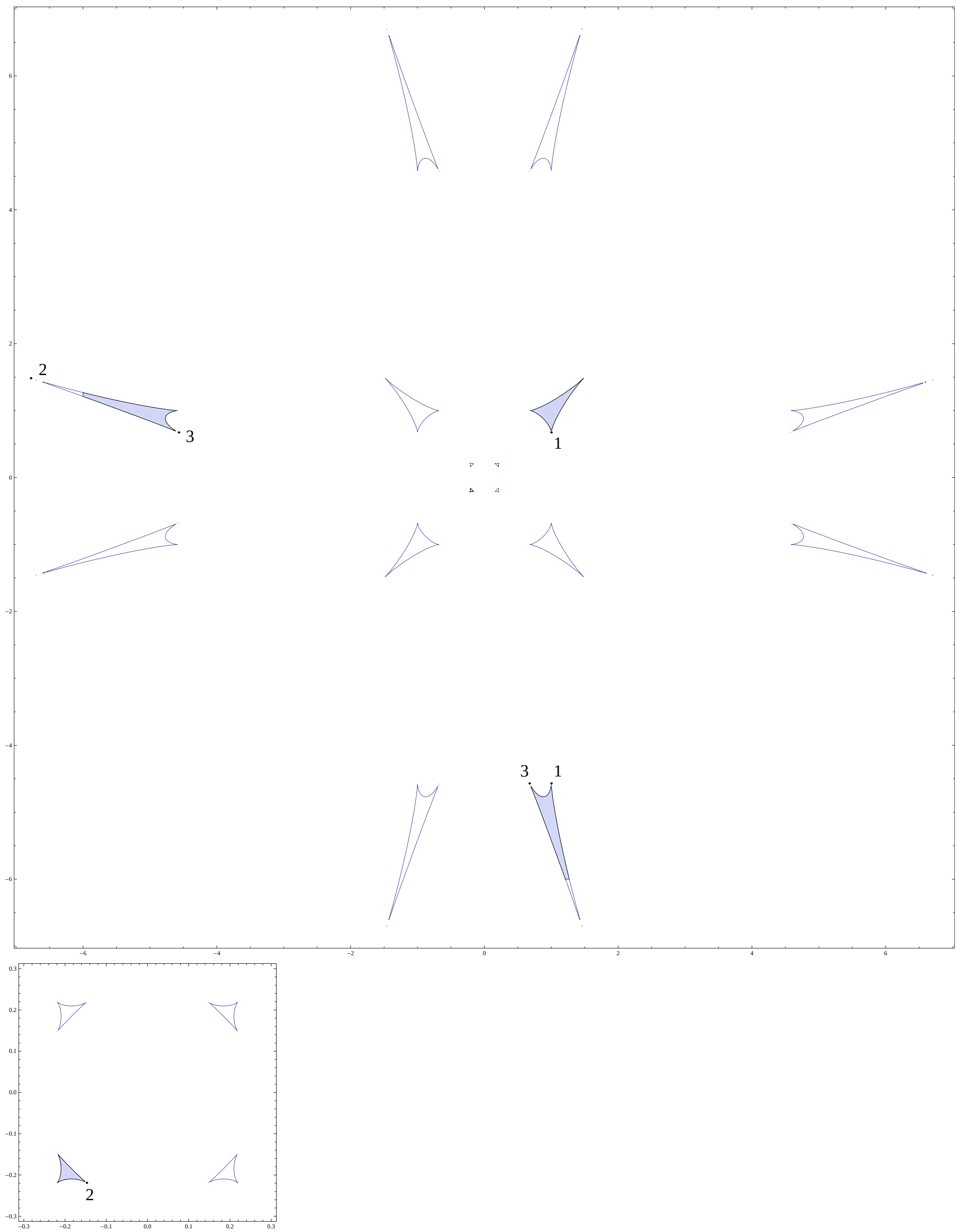

Figure 4.2: Plots of $g = 0$. System $\{f_9, f_{10}\}$ has 4 positive roots when parameters are on the shaded regions and 2 positive roots on the other regions. The left lower graph is the blow up around $(0, 0)$.

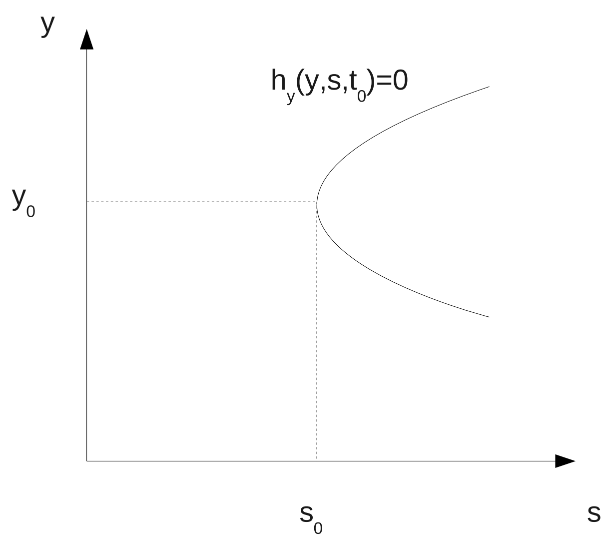

Figure 4.3: Saddle-Note bifurcation.

$t_1 = \frac{4}{100}$	$s_{1,1} = 30$
$t_2 = \frac{6}{100}$	$s_{2,1} = 30$
$t_3 = \frac{1}{10}$	$s_{3,1} = 30$
$t_4 = \frac{12}{100}$	$s_{4,1} = 30$
$t_5 = \frac{14}{100}$	$s_{5,1} = 30$
$t_6 = \frac{144}{1000}$	$s_{6,1} = \frac{5}{10}, s_{6,2} = 1, s_{6,3} = 30$
$t_7 = \frac{16}{100}$	$s_{7,1} = \frac{5}{10}, s_{7,2} = 1, s_{7,3} = 30$
$t_8 = \frac{2}{10}$	$s_{8,1} = \frac{2}{100}, s_{8,2} = \frac{8}{100}, s_{8,3} = \frac{5}{10}, s_{8,4} = 1, s_{8,5} = 30$
$t_9 = \frac{2384}{10000}$	$s_{9,1} = \frac{2}{100}, s_{9,2} = \frac{52}{1000}, s_{9,3} = \frac{6}{100}, s_{9,4} = \frac{8}{100}, s_{9,5} = \frac{5}{10}$, $s_{9,6} = \frac{725}{1000}, s_{9,7} = \frac{76}{100}, s_{9,8} = 1, s_{9,9} = 30$
$t_{10} = \frac{24}{100}$	$s_{10,1} = \frac{2}{100}, s_{10,2} = \frac{512}{10000}, s_{10,3} = \frac{6}{100}$, $s_{10,4} = \frac{8}{100}, s_{10,5} = \frac{5}{10}, s_{10,6} = 1, s_{10,7} = 30$
$t_{11} = \frac{25}{100}$	$s_{11,1} = \frac{2}{100}, s_{11,2} = \frac{84}{1000}, s_{11,3} = \frac{5}{10}, s_{11,4} = 1, s_{11,5} = 30$
$t_{12} = \frac{3}{10}$	$s_{12,1} = \frac{5}{10}, s_{12,2} = \frac{128}{100}, s_{12,3} = 30$
$t_{13} = \frac{36}{100}$	$s_{13,1} = 30$
$t_{14} = \frac{4}{10}$	$s_{14,1} = 30$
$t_{15} = \frac{5}{10}$	$s_{15,1} = 30$
$t_{16} = \frac{572}{1000}$	$s_{16,1} = 30$
$t_{17} = \frac{6}{10}$	$s_{17,1} = 30$
$t_{18} = \frac{72}{100}$	$s_{18,1} = 30$
$t_{19} = \frac{8}{10}$	$s_{18,1} = 30$
$t_{20} = \frac{9}{10}$	$s_{20,1} = 30$
$t_{21} = 1$	$s_{21,1} = 30$
$t_{22} = 2$	$s_{22,1} = 30$
$t_{23} = \frac{25}{10}$	$s_{23,1} = \frac{5}{10}, s_{23,2} = \frac{125}{10}, s_{23,3} = 30$
$t_{24} = \frac{292}{100}$	$s_{24,1} = \frac{5}{10}, s_{24,2} = \frac{82}{100}, s_{24,3} = 10, s_{24,4} = 13, s_{24,5} = 30$
$t_{25} = \frac{296}{100}$	$s_{25,1} = \frac{5}{10}, s_{25,2} = \frac{9}{10}, s_{25,3} = \frac{95}{100}$, $s_{25,4} = 1, s_{25,5} = 10, s_{25,6} = 13, s_{25,7} = 30$
$t_{26} = \frac{2968}{1000}$	$s_{26,1} = \frac{5}{10}, s_{26,2} = \frac{9}{100}, s_{26,3} = \frac{95}{100}, s_{26,4} = 1, s_{26,5} = 10$, $s_{26,6} = \frac{1176}{100}, s_{26,7} = \frac{118}{10}, s_{26,8} = 13, s_{26,9} = 30$
$t_{27} = 3$	$s_{27,1} = \frac{5}{10}, s_{27,2} = 1, s_{27,3} = 10, s_{27,4} = 13, s_{27,5} = 30$
$t_{28} = \frac{308}{100}$	$s_{28,1} = \frac{5}{10}, s_{28,2} = 1, s_{28,3} = 10, s_{28,4} = 13, s_{28,5} = 30$
$t_{29} = \frac{36}{10}$	$s_{29,1} = \frac{5}{10}, s_{29,2} = \frac{124}{100}, s_{29,3} = 10, s_{29,4} = 16, s_{29,5} = 30$
$t_{30} = 4$	$s_{30,1} = 10, s_{30,2} = 17, s_{30,3} = 30$
$t_{31} = 10$	$s_{31,1} = 30$
$t_{32} = 20$	$s_{32,1} = 30$
$t_{33} = \frac{356}{10}$	$s_{33,1} = 30$
$t_{34} = 40$	$s_{34,1} = 30$

Table 4.1: 96 sample points for $f = 0$

5 sample points	Other sample points connected by line segments
$(t_8, s_{8,2}) = (\frac{1}{5}, \frac{2}{25})$	$(t_9, s_{9,2}), (t_9, s_{9,4}), (t_{10}, s_{10,2}), (t_{10}, s_{10,4}), (t_{11}, s_{11,2})$
$(t_{11}, s_{11,4}) = (\frac{1}{4}, 1)$	$(t_6, s_{6,2}), (t_7, s_{7,2}), (t_8, s_{8,4}), (t_9, s_{9,6}),$ $(t_9, s_{9,8}), (t_{10}, s_{10,6}), (t_{12}, s_{12,2})$
$(t_{27}, s_{27,4}) = (3, 13)$	$(t_{23}, s_{23,2}), (t_{24}, s_{24,4}), (t_{25}, s_{25,6}), (t_{26}, s_{26,8})$ $(t_{26}, s_{26,6}), (t_{28}, s_{28,4}), (t_{29}, s_{29,4}), (t_{30}, s_{30,2})$
$(t_{27}, s_{27,2}) = (3, 1)$	$(t_{24}, s_{24,2}), (t_{25}, s_{25,2}), (t_{25}, s_{25,4}), (t_{26}, s_{26,2}),$ $(t_{26}, s_{26,4}), (t_{28}, s_{28,2}), (t_{29}, s_{29,2})$
$(1, 1)$	All other sample points

Table 4.2: 5 sample points for $f = 0$

$t_1 = \frac{2}{100}$	$s_{1,1} = 10$
$t_2 = \frac{5}{100}$	$s_{2,1} = 10$
$t_3 = \frac{1}{100}$	$s_{3,1} = 10$
$t_4 = \frac{2}{100}$	$s_{4,1} = 10$
$t_5 = \frac{32}{1000}$	$s_{5,1} = 10$
$t_6 = \frac{5}{100}$	$s_{6,1} = 10$
$t_7 = \frac{1}{10}$	$s_{7,1} = 10$
$t_8 = \frac{16}{100}$	$s_{8,1} = \frac{1}{10}, s_{8,2} = \frac{21}{100}, s_{8,3} = 10$
$t_9 = \frac{2}{10}$	$s_{9,1} = \frac{1}{10}, s_{9,2} = \frac{21}{100}, s_{9,3} = 10$
$t_{10} = \frac{21}{100}$	$s_{10,1} = \frac{1}{10}, s_{10,2} = \frac{16}{100}, s_{10,3} = \frac{18}{100}, s_{10,4} = \frac{21}{100}, s_{10,5} = 10$
$t_{11} = \frac{4}{10}$	$s_{11,1} = 10$
$t_{12} = \frac{45}{100}$	$s_{12,1} = 10$
$t_{13} = \frac{5}{10}$	$s_{13,1} = 10$
$t_{14} = \frac{6}{10}$	$s_{14,1} = 10$
$t_{15} = \frac{8}{10}$	$s_{15,1} = \frac{1}{10}, s_{15,2} = 1, s_{15,3} = 3, s_{15,4} = \frac{48}{10}, s_{15,5} = 10$
$t_{16} = \frac{12}{10}$	$s_{16,1} = \frac{1}{10}, s_{16,2} = \frac{12}{10}, s_{16,3} = 3, s_{16,4} = \frac{58}{10}, s_{16,5} = 10$
$t_{17} = \frac{15}{10}$	$s_{17,1} = 10$
$t_{18} = 2$	$s_{18,1} = 10$
$t_{19} = \frac{225}{100}$	$s_{19,1} = 10$
$t_{20} = 3$	$s_{20,1} = 10$
$t_{21} = \frac{475}{100}$	$s_{21,1} = \frac{1}{10}, s_{21,2} = \frac{8}{10}, s_{21,3} = \frac{9}{10}, s_{21,4} = 1, s_{21,5} = 10$
$t_{22} = 5$	$s_{22,1} = \frac{1}{10}, s_{22,2} = 1, s_{23,3} = 10$
$t_{23} = 6$	$s_{23,1} = \frac{1}{10}, s_{23,2} = \frac{125}{100}, s_{23,3} = 10$
$t_{24} = 10$	$s_{24,1} = 10$
$t_{25} = 20$	$s_{25,1} = 10$
$t_{26} = 31$	$s_{26,1} = 10$
$t_{27} = 50$	$s_{27,1} = 10$
$t_{28} = 100$	$s_{28,1} = 10$
$t_{29} = 200$	$s_{29,1} = 10$
$t_{30} = 300$	$s_{30,1} = 10$

Table 4.3: 54 sample points for $g = 0$

5 sample points	Other sample points connected by line segments
$(\frac{1}{5}, \frac{1}{5})$	$(t_8, s_{8,2}), (t_9, s_{9,2}), (t_{10}, s_{10,2}), (t_{10}, s_{10,4})$
$(t_{15}, s_{15,2}) = (\frac{4}{5}, 1)$	$(t_{16}, s_{16,2})$
$(t_{16}, s_{16,4}) = (\frac{6}{5}, \frac{29}{5})$	$(t_{15}, s_{15,4})$
$(t_{22}, s_{22,2}) = (5, 1)$	$(t_{21}, s_{21,2}), (t_{21}, s_{21,4}), (t_{23}, s_{23,3})$
$(2, 1)$	All other sample points

Table 4.4: 5 sample points for $g = 0$

20 sample points	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
$(\frac{1}{5}, \frac{2}{25})$	2	4	2	2
$(-\frac{1}{5}, \frac{2}{25})$	2	2	4	2
$(\frac{1}{5}, -\frac{2}{25})$	2	2	2	4
$(-\frac{1}{5}, -\frac{2}{25})$	4	2	2	2
$(\frac{1}{4}, 1)$	4	2	2	2
$(-\frac{1}{4}, 1)$	2	2	2	4
$(\frac{1}{4}, -1)$	2	2	4	2
$(-\frac{1}{4}, -1)$	2	4	2	2
$(3, 13)$	2	2	4	2
$(-3, 13)$	2	4	2	2
$(3, -13)$	4	2	2	2
$(-3, -13)$	2	2	2	4
$(3, 1)$	2	2	2	4
$(-3, 1)$	4	2	2	2
$(3, -1)$	2	4	2	2
$(-3, -1)$	2	2	4	2
$(1, 1)$	2	2	2	2
$(-1, 1)$	2	2	2	2
$(1, -1)$	2	2	2	2
$(-1, -1)$	2	2	2	2

Table 4.5: Number of real zeros at 20 sample points for system $\{f_7, f_8\}$

20 sample points	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
$(\frac{1}{5},\frac{1}{5})$	2	4	2	2
$(-\frac{1}{5},\frac{1}{5})$	2	2	4	2
$(\frac{1}{5},-\frac{1}{5})$	2	2	2	4
$(-\frac{1}{5},-\frac{1}{5})$	4	2	2	2
$(\frac{4}{5},1)$	4	2	2	2
$(-\frac{4}{5},1)$	2	2	2	4
$(\frac{4}{5},-1)$	2	2	4	2
$(-\frac{4}{5},-1)$	2	4	2	2
$(\frac{6}{5},\frac{29}{5})$	2	2	4	2
$(-\frac{6}{5},\frac{29}{5})$	2	4	2	2
$(\frac{6}{5},-\frac{29}{5})$	4	2	2	2
$(-\frac{6}{5},-\frac{29}{5})$	2	2	2	4
$(5,1)$	2	2	2	4
$(-5,1)$	4	2	2	2
$(5,-1)$	2	4	2	2
$(-5,-1)$	2	2	4	2
$(2,1)$	2	2	2	2
$(-2,1)$	2	2	2	2
$(2,-1)$	2	2	2	2
$(-2,-1)$	2	2	2	2

Table 4.6: Number of real zeros at 20 sample points for system $\{f_9, f_{10}\}$

Polynomials in t^2 or s^2	Open sets to separate their positive roots
$Res(f, \partial_s f, s)$ $Res(f, \partial_{ss} f, s)$	$(\frac{1}{36}, \frac{1}{35}), (\frac{1}{18}, \frac{347}{6144}), (\frac{56706}{1000000}, \frac{56707}{1000000}), (\frac{8}{141}, \frac{5}{88}), (\frac{585}{10000}, \frac{1}{17}),$ $(\frac{1}{12}, \frac{1}{11}), (6, 7), (\frac{26}{3}, \frac{35}{4}), (\frac{8795}{1000}, \frac{8796}{1000}), (\frac{8817}{1000}, \frac{8818}{1000}), (\frac{53}{6}, 9), (16, 17)$
$Res(\frac{\partial_t f}{12t}, w_1, t^2)$ $Res(\frac{\partial_s f}{12s}, w_1, t^2)$	$(\frac{1}{4432175}, \frac{1}{323}), (\frac{1}{217}, \frac{1}{183}), (\frac{1}{151}, \frac{1}{150}), (\frac{7205}{1000000}, \frac{7206}{1000000}), (\frac{1}{97}, \frac{1}{22}),$ $(\frac{35}{100}, \frac{36}{100}), (\frac{53}{100}, \frac{535}{1000}), (\frac{77}{100}, \frac{78}{100}), (\frac{945}{1000}, \frac{946}{1000}), (\frac{108}{100}, \frac{109}{100}),$ $(\frac{7}{6}, \frac{6}{5}), (\frac{18768}{10000}, \frac{18769}{10000}), (3, 4), (\frac{1387}{10}, \frac{1388}{10}), (330, 335)$
$Res(\frac{\partial_t f}{12t}, w_2, t^2)$ $Res(\frac{\partial_s f}{12t}, w_2, t^2)$	$(\frac{1}{352}, \frac{1}{153}), (\frac{493}{896}, \frac{247}{448}), (\frac{15}{26}, \frac{3}{5}), (\frac{5}{6}, \frac{16}{17}), (\frac{151}{152}, 1),$ $(\frac{27}{26}, \frac{5}{4}), (112, 113), (126, \frac{1265}{10}), (134, \frac{6363}{46}), (\frac{277}{2}, 156)$
$Res(a, f, t^2)$ $Res(a, b, t^2)$	$(0, \frac{33}{10000}), (\frac{247}{39424}, \frac{65712}{10000000}), (\frac{1}{151}, \frac{1}{22}), (\frac{8}{15}, \frac{62}{100}), (\frac{2}{3}, \frac{9}{10}),$ $(\frac{101}{100}, \frac{108}{100}), (\frac{11}{10}, \frac{9}{8}), (\frac{16}{10}, \frac{15}{8}), (124, 141), (153, 161), (270, 287)$
$Res(a, b, s^2)$ $Res(a, C_1, s^2)$	$(0, \frac{1}{101}), (\frac{1}{41}, \frac{1}{40}), (\frac{1}{34}, \frac{1}{26}), (\frac{524}{10000}, \frac{5245}{100000}), (\frac{1}{18}, \frac{1}{9}), (\frac{1135}{10000}, \frac{1}{7}),$ $(\frac{26}{100}, \frac{3}{10}), (\frac{373}{1000}, \frac{374}{1000}), (\frac{45}{100}, \frac{46}{100}), (\frac{1}{2}, \frac{551}{1000}), (\frac{5}{9}, 3), (9, 14)$
$Res(a, b, s^2)$ $Res(a, C_2, s^2)$	$(0, \frac{1}{101}), (\frac{1}{41}, \frac{1}{40}), (\frac{1}{34}, \frac{1}{29}), (\frac{346}{10000}, \frac{1}{28}), (\frac{1}{27}, \frac{375}{10000}),$ $(\frac{1}{20}, \frac{1}{19}), (\frac{8}{141}, \frac{614}{10000}), (\frac{63}{1000}, \frac{1}{15}), (\frac{1}{11}, \frac{2}{21}), (\frac{1}{10}, \frac{2}{19}),$ $(\frac{1}{9}, 3), (\frac{93}{10}, \frac{94}{10}), (\frac{19}{2}, 10), (\frac{135}{10}, 14)$
$Res(a, b, s^2)$ $Res(a, C_3, s^2)$	$(0, \frac{1}{101}), (\frac{1}{46}, \frac{31795}{1194858}), (\frac{1}{35}, \frac{1}{29}), (\frac{346}{10000}, \frac{1}{28}), (\frac{1}{27}, \frac{1}{26}),$ $(\frac{1}{20}, \frac{1}{19}), (\frac{5}{88}, \frac{1}{6}), (\frac{1}{5}, \frac{1}{3}), (\frac{11}{32}, \frac{2}{5}), (\frac{3}{7}, \frac{4}{7}),$ $(\frac{2}{3}, \frac{5}{6}), (\frac{20}{22}, \frac{21}{17}), (\frac{5}{2}, 3), (9, \frac{138}{10})$
$Res(a, b, s^2)$ $Res(a, C_4, s^2)$	$(0, \frac{1}{138}), (\frac{1}{121}, \frac{1}{101}), (\frac{245}{10000}, \frac{246}{10000}), (\frac{1}{36}, \frac{1}{29}), (\frac{348}{10000}, \frac{175}{5014}),$ $(\frac{1}{27}, \frac{1}{26}), (\frac{1}{20}, \frac{1}{19}), (\frac{2}{35}, \frac{1}{7}), (\frac{1}{4}, \frac{4}{7}), (\frac{2}{3}, 2),$ $(\frac{5}{2}, 3), (9, 10), (\frac{1375}{100}, \frac{111}{8})$

Table 4.7: Open sets for separation in the case of system $\{f_7, f_8\}$

Polynomials in t^2 or s^2	Open sets to separate their positive roots
$\frac{Res(g,\partial_s g,s)}{(t^2-1)^8}$ $\frac{Res(g,\partial_{ss} g,s)}{(t^2-1)^8}$	$(\frac{20}{455},\frac{1}{22}),(\frac{4}{7},\frac{7}{4}),(22,\frac{455}{20})$
$\frac{Res(\frac{\partial_t g}{12t},w_3,t^2)}{p_1^{12}p_2^6p_3^2}$ $\frac{Res(\frac{\partial_s g}{12s},w_3,t^2)}{p_1^{12}p_2^6p_3^2}$	$(0,\frac{1}{30}),(\frac{1}{29},\frac{1}{20}),(\frac{75}{1000},\frac{1}{4}),(\frac{1}{2},\frac{7}{10})$ $(\frac{115}{100},\frac{117}{100}),(\frac{5}{4},2),(25,26),(31,220)$
$\frac{Res(\frac{\partial_t g}{12t},w_3,s^2)}{(t^2-1)^{138}(t^4-46t^2+1)^2}$ $\frac{Res(\frac{\partial_s g}{12s},w_3,s^2)}{(t^2-1)^{138}(t^4-46t^2+1)^2}$	$(\frac{1}{403},\frac{1}{153}),(\frac{1}{53},\frac{1}{52}),(\frac{2}{51},30),(52,53),(153,403)$
$\frac{Res(\frac{\partial_t g}{12t},w_4,t^2)}{p_1^4}$ $\frac{Res(\frac{\partial_s g}{12s},w_4,t^2)}{p_1^4}$	$(0,\frac{1}{52}),(\frac{1}{51},\frac{1}{36}),(\frac{6}{133},\frac{1}{16}),(\frac{1}{13},\frac{4}{19}),$ $(1,\frac{51}{50}),(\frac{3}{2},12),(\frac{165}{10},17),(22,194)$
$\frac{Res(\frac{\partial_t g}{12t},w_4,s^2)}{(t^2-1)^4(27t^4-576t^2+256)^2}$ $\frac{Res(\frac{\partial_s g}{12s},w_4,s^2)}{(t^2-1)^4(27t^4-576t^2+256)^2}$	$(0,\frac{1}{212}),(\frac{1}{130},\frac{1}{40}),(\frac{1}{24},\frac{1}{23}),(\frac{2}{45},\frac{1}{12}),$ $(\frac{2}{5},\frac{1}{2}),(\frac{7}{12},\frac{12}{13}),(\frac{12}{10},\frac{215}{10}),(37,38)$
$Res(c,g,t^2)$ $Res(c,d,t^2)$	$(\frac{1}{31},\frac{1}{10}),(\frac{1}{2},3),(671,5246)$

Table 4.8: Open sets for separation in the case of system $\{f_9, f_{10}\}$

Chapter 5

Open questions and future studies

One restricted version of the study of central configurations in $\mathbb{R}^2$ is to consider one large positive mass, assuming 1, and n small masses, $\mu_i \varepsilon$, when $\varepsilon \to 0$. Results of counting the number of such limiting central configurations when μ_i's are equal are given in [7], [2] for $n = 3, 4$. Such central configurations have the body of large mass at the center of a circle which passes through the bodies of small masses [25]. Imaging those small bodies as satellites, we have that those n satellites are co-orbital. In the followings, we call such problems *n satellites problems*.

About n satellites problems, the newest result is in the paper [2](2009) that discussed about four identical satellites. So far, there are no results about different μ_i's even for $n = 3$. When $n = 3$, we found a parametric polynomial system with 4 equations, 4 variables, and 2 parameters and reduced the problem to finding real zeros of this system in an open region of $\mathbb{R}^4$. According to our numerical experiments by testing some parameter values, it seems there are $3, 4, 5, 6$ or 7 real zeros in such region. Therefore, we conjectured that there are $3, 4, 5, 6$ or 7 central configurations. However, we still have no rigorous proof.

The challenge now is that the Groebner basis computation can not be carried out due to the complexity of the system and the limited computer memories. Therefore, one idea to solve this problem is to reduce the problem to another easier parametric polynomial system. Another idea is to improve algorithms of Groebner basis computations to manipulate the current system. The other is just to learn other tools than Groebner bases from computational algebraic geometry. we expect to try those ideas and learn more tools to solve this problem. In the first section, we will show how to derive our current parametric polynomial system.

About the problem of Maxwell's conjecture, we have made some progress on cases of general right triangles. In the cases of fixed configurations of the 3 point charges, the problems are reduced to counting positive zeros of a parametric polynomial system with 2 equations, 2 variables, and 2 parameters. When considering the cases of general right triangles, we have a parametric polynomial system with one more parameter. So far, we can find the bifurcation set in $\mathbb{R}^3$. It is contained in a smooth surface defined by the zero set of a bifurcation polynomial in 3 variables.

The challenge for this problem is that such polynomial is too complicated to perform resultant computations which are necessary in our method to study the bifurcation set. Therefore, either finding other algorithms to compute resultants or finding other methods than resultants to study bifurcation sets will be helpful in overcome the difficulties. In the second section, we will focus on how to compute the bifurcation polynomial.

5.1 3 satellites

It is a classic result that when all bodies lie on the same plane, the equations 1.2 are equivalent to the Dziobek equations $\sum_k'' m_k \triangle_{i,j,k} (\frac{1}{r_{ik}^3} - \frac{1}{r_{jk}^3}) = 0$, where m_k is the mass of the kth body, r_{ik} is the distance between the ith and kth bodies, $\triangle_{i,j,k}$ is the signed area of the i, j, and k bodies, and the summation runs over all $k \neq i, j$. If we denote the positions of the three bodies by $(x_i, y_i), (x_j, y_j)$, and (x_k, y_k), then

$$\triangle_{i,j,k} = det \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}.$$

In the case of the 3 satellites problem, we assume, without lose of generality, that four bodies have positions $q_1 = (x_1, y_1), q_2 = (x_2, y_2), q_3 = (-1, 0), q_4 = (0, 0)$ and their masses are $m_1 = \mu_1\varepsilon, m_2 = \mu_2\varepsilon, m_3 = \mu_3\varepsilon$ and $m_4 = 1$. When $\varepsilon \to 0$, the limiting positions of q_1 and q_2 will also be on the unit circle. We use the coordinate (r_1, r_2, r_3, r_4) instead of (x_1, y_1, x_2, y_2), where r_3, r_4 are the y-coordinates of the intersection points between q_3 and q_1, q_2, respectively, and r_1, r_2 are distances between q_3 and those intersection points. It is easy to see that (x_1, y_1) and (x_2, y_2) can be parametrized by $(\frac{1-r_3^2}{1+r_3^2}, \frac{2r_3}{1+r_3^2})$ and $(\frac{1-r_4^2}{1+r_4^2}, \frac{2r_4}{1+r_4^2})$. The following figure gives the picture of our setting.

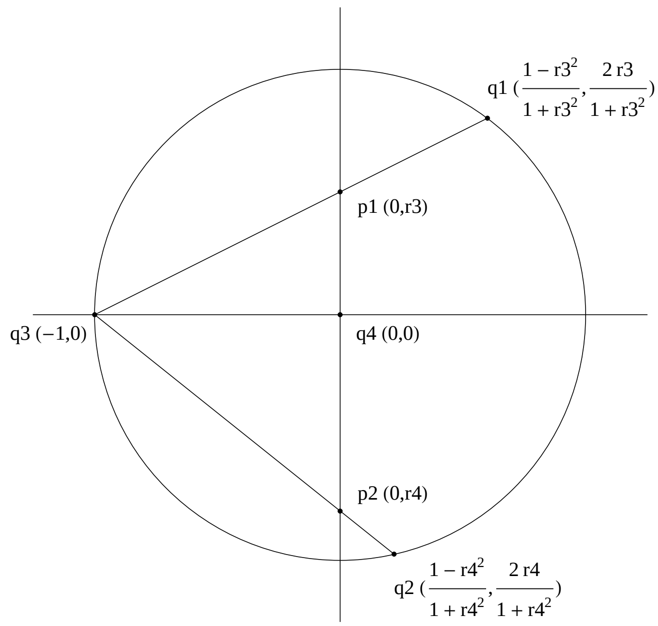

Figure 5.1: 3 satellites problem

Assuming $t = \frac{\mu_1}{\mu_3}, s = \frac{\mu_2}{\mu_3}$, using $q_1 = (\frac{1-r_3^2}{1+r_3^2}, \frac{2r_3}{1+r_3^2}), q_2 = (\frac{1-r_4^2}{1+r_4^2}, \frac{2r_4}{1+r_4^2}), q_3 = (-1, 0)$, and $q_4 = (0, 0)$ in the Dziobek equations, and letting $\varepsilon \to 0$, we obtained the following parametric polynomial systems with 4 variables, 4 equations, and 2 parameters.

$$f_1 = r_1^2 - r_3^2 - 1, \tag{5.1}$$

$$f_2 = r_2^2 - r_4^2 - 1, \tag{5.2}$$

$$f_3 = -8tr_2^2r_3 + tr_1^3r_2^2r_3 - 8sr_1^2r_4 + sr_1^2r_2^3r_4, \tag{5.3}$$

$$\begin{aligned} f_4 = & tr_1^3r_2^3 - 8tr_3^3 + tr_1^3r_2^3r_3r_4 + 24tr_3^2r_4 - 8r_1^2r_3^2r_4 + r_1^2r_2^3r_3^2r_4 - 8tr_3^4r_4 - 24tr_3r_4^2 \\ & 16r_1^2r_3r_4^2 - 2r_1^2r_2^3r_3r_4^2 + 24tr_3^3r_4^2 + 8tr_4^3 - 8r_1^2r_4^3 + r_1^2r_2^3r_4^3 - 24r_3^2r_4^3 + 8tr_3r_4^3. \end{aligned} \tag{5.4}$$

In this system, the meaningful zeros are such that $r_1 > 0, r_2 > 0, r_3, r_4 \in \mathbb{R}$. In order to count the essential different positions, we always consider $r_3 > r_4$. According to our experiments by trying some (s, t) pairs, it seems there are generically $3, 5$, or 7 solutions with $r_1 > 0, r_2 > 0$ and $r_3 - r_4 > 0$. Therefore, we concluded the following conjecture.

Conjecture 5.1. *The* 3 *satellites problem has* $3, 4, 5, 6$ *or* 7 *central configurations.*

5.2 General right triangles

In chapter 1.2, we derived the parametric polynomial system 1.12 for Maxwell's conjecture of 3 point charges. The cases of general right triangles correspond to $(u, v) = (1, c)$ for $c > 0$. Substituting such (u, v) into equations 1.12, we get the following parametric polynomial system with 2 equations, 2 variables, and 3 parameters s, t, c. The goal now is to prove that there are at most 4 positive zeros for all systems when $s \neq 0, t \neq 0$, and $c > 0$.

$$f_{11}x^2 + cf_{12}y^2 = 0, f_{11} = y^6 - s^2x^4 + 2ty^3 + t^2, f_{12} = -y^4 + x^2y^4, \tag{5.5}$$

$$f_{21}x^2 + cf_{22}y^2 = 0, f_{21} = -x^4s^2 + x^4y^2s^2, f_{22} = -y^4 + s^2x^6 + 2stx^3 + t^2. \tag{5.6}$$

Like the two systems $\{f_7, f_8\}$ and $\{f_9, f_{10}\}$, the above system has $(0, 0)$ as a zero of multiplicity 4 for all $s \neq 0, t \neq 0$, and $c > 0$. In Chapter 4.2, we use the resultant method from Chapter 3.2.2 to obtain bifurcation polynomials. Here we will use the Hermite matrix method in Chapter 3.2.1 to obtain bifurcation polynomial after changing the system into another parametric polynomial system where its real zeros are in one to one correspondence to nonzero real zeros of the system above.

Proposition 5.1. *For all $s \neq 0, t \neq 0$, and $c > 0$, the nonzero real solutions of $f_{11}x^2 + cf_{12}y^2 = f_{21}x^2 + cf_{22}y^2 = 0$ are one to one correspondent to real zeros of the system $f_{11} + zf_{12} = f_{21} + zf_{22} = zx^2 - cy^2 = 0$.*

Proof. Fix $s_0, t_0 \neq 0, c_0 > 0$. If (x_0, y_0) is a real zero with $x_0, y_0 \neq 0$, we get $f_{11}^{s_0,t_0,c_0}(x_0, y_0)x_0^2 + c_0f_{12}^{s_0,t_0,c_0}(x_0, y_0)y_0^2 = f_{21}^{s_0,t_0,c_0}(x_0, y_0)x_0^2 + c_0f_{22}^{s_0,t_0,c_0}(x_0, y_0)y_0^2 = 0$. Defining $z_0 = \frac{c_0y_0^2}{x_0}$, we have (x_0, y_0, z_0) is a real zero of $f_{11}^{s_0,t_0,c_0} + zf_{12}^{s_0,t_0,c_0} = f_{21}^{s_0,t_0,c_0} + zf_{22}^{s_0,t_0,c_0} = zx^2 - c_0y^2 = 0$.

On the other hand, suppose $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ is a zero of $f_{11}^{s_0,t_0,c_0} + zf_{12}^{s_0,t_0,c_0} = f_{21}^{s_0,t_0,c_0} + zf_{22}^{s_0,t_0,c_0} = zx^2 - c_0y^2 = 0$. We will claim that $\alpha \neq 0, \beta \neq 0$ and (α, β) is a zero of $f_{11}^{s_0,t_0,c_0}x^2 + c_0f_{12}^{s_0,t_0,c_0}y^2 = f_{21}^{s_0,t_0,c_0}x^2 + c_0f_{22}^{s_0,t_0,c_0}y^2 = 0$.

If $\alpha = \beta = 0$, then $f_{11}^{s_0,t_0,c_0}(\alpha, \beta) + \gamma f_{12}^{s_0,t_0,c_0}(\alpha, \beta) = t_0^2 = 0$, a contradiction. If $\alpha = 0, \beta \neq 0$, $\gamma 0 - \beta^2c_0 = 0$. So, $\beta = 0$, a contradiction. Finally, if $\alpha \neq 0, \beta = 0$, then $\gamma = 0$ and $f_{21}^{s_0,t_0,c_0}(\alpha, \beta) = 0$. But, $f_{21}^{s_0,t_0,c_0}(\alpha, \beta) = -\alpha^4s_0^2 \neq 0$, a contradiction. Finally, it is easy to see that (α, β) is a zero of $f_{11}^{s_0,t_0,c_0}x^2 + c_0f_{12}^{s_0,t_0,c_0}y^2 = f_{21}^{s_0,t_0,c_0}x^2 + c_0f_{22}^{s_0,t_0,c_0}y^2 = 0$, since $\gamma = \frac{c_0\beta^2}{\alpha^2}$. $\square$

Next, we will find the bifurcation polynomial of the system $p_1 = p_2 = p_3 = 0$, where $p_1 = f_{11} + zf_{12}, p_2 = f_{21} + zf_{22}, p_3 = zx^2 - cy^2 \in \mathbb{Q}[s,t,c][x,y,z]$. Let $\mathcal{F} = \{p_1, p_2, p_3\}$. Applying $\mathbf{GroebnerBasis}[\mathcal{F}, \{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{s}, \mathbf{t}, \mathbf{c}\}, \mathbf{MonomialOrder} \to \mathbf{w}]$, where $\mathbf{w}$ is the block ordering $<$ given by $\mathbf{w} = \{\{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}\}, \{\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}\}, \{\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}\}, \{\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{1}\}, \{\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{0}\}, \{\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{0}\}\}$, we get a set $G = \{g_1, \dots, g_{28}\}$, where $LC_{<_{x,y,z}}(g_i) \neq 0$, for all i and $s,t \neq 0, c > 0$. Also we have $LM_{<_{x,y,z}}(g_8) = y^6, LM_{<_{x,y,z}}(g_{15}) = x^6$, and $LM_{<_{x,y,z}}(g_{16}) = z^7$.

Therefore, $G^{s,t,c}$ is a Groebner basis of $I^{s,t,c} = \langle p_1^{s,t,c}, p_2^{s,t,c}, p_3^{s,t,c} \rangle$ for all $s,t \neq 0, c > 0$ and $A^{s,t,c} = \mathbb{C}[x,y,z]/I^{s,t,c}$ is of finite dimensional and has a basis $\{x^m y^n z^l \notin \langle LT_{<_{x,y,z}}(g_1^{s,t,c}), \dots, LT_{<_{x,y,z}}(g_{28}^{s,t,c}) \rangle\}$. There are 52 of them. So, the system $\mathcal{F}^{s,t,c}$ has at most 52 complex zeros for all $s \neq 0, t \neq 0, c > 0$.

Using such G, we compute the Hermite matrix H_1 and find that $H = (1+c)^5 s^4 H_1$ has entries all polynomials in s, t, c. Therefore, we will compute $det(H)$ and obtain $det(H_1)$ from $(1+c)^{-260} s^{-208} det(H)$. Directly applying the Mathematica command $\mathbf{det}[\mathbf{H}]$ gave us no results after one week. Therefore, we use a property of polynomials to help us compute $det(H)$.

Proposition 5.2. *$det(H) = (1+c)^{190} c^{92} s^{230} t^{76} h^2 g$, where $h, g \in \mathbb{Z}[s,t,c]$ and $h > 0$ for $s,t \neq 0, c > 0$. Therefore, $det(H_1) = \frac{s^{22} t^{76} c^{92} h^2}{(1+c)^{70}} g$. So, g the bifurcation polynomial.*

Proof. Computing $det(H)(s_0, t_0, c)$ for some random choices of (s_0, t_0), we guessed the form of $det(H)$ is $(1+c)^{190} c^{92} \sum_{k=0}^{58} a_k(s,t) c^k$. Now, let $\{c_1, \dots, c_{30}, c_{31}, \dots, c_{59}\} = \{1, 2, \dots, 30, -2, -3, \dots, -30\}$. We compute $b_k = \frac{det(H)(c_k)}{(1+c_k)^{190} c_k^{92}} \in \mathbb{Z}[s,t]$ for $k = 1, \dots, 59$. We have the following matrix identity $\mathbf{MA} = \mathbf{B}$, where $\mathbf{A} = (a_{58}, \dots, a_0)^{\perp}$, $\mathbf{B} = (b_1, \dots, b_{59})^{\perp}$, and

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 30^{58} & 30^{57} & \cdots & 30^2 & 30 & 1 \\ (-2)^{58} & (-2)^{57} & \cdots & (-2)^2 & -2 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ (-30)^{58} & (-30)^{57} & \cdots & (-30)^2 & -30 & 1 \end{pmatrix}.$$

Therefore, we obtained $\mathbf{A}$ by computing $\mathbf{M^{-1}B}$. Factoring $\sum_{k=0}^{58} a_k(s,t) c^k$, we get $\sum_{k=0}^{58} a_k(s,t) c^k = s^{230} t^{76} h^2 g$. So, $det(H) = (1+c)^{190} c^{92} s^{230} t^{76} h^2 g$.

□

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Appendix A

This appendix will introduce syntax and usage of some commands that are implemented in Mathematica 6.0.0 and used in this thesis.

1. **GroebnerBasis[polylist, varlist, options]** : This command computes a semi reduced Groebner basis of **polylist** in the variables **varlist**. Suppose $G = \{g_1, \cdots g_l\}$ is the unique Groebner basis of I, a semi reduced Groebner basis is the set $\{h_1, \cdots h_l\}$, where $h_i = c_i g_i$ for some invertible elements of the field of coefficient domain. **options** includes the assignments of coefficient domain, monomial order, etc. By default, the coefficient domain is the field generated by numeric coefficients presented in the input, and the monomial order is the lex order with $x_1 > \cdots > x_n$ where $\{x_1, \cdots, x_n\}$ is the ordered sequence input of **varlist**. We demonstrate some examples in the followings.

$In[1] := \mathbf{gb_1 = GroebnerBasis[\{x^2 + 3y, y^2 - 2x^3\}, \{x, y\}]}$

This computes a Groebner basis of $I = \langle x^2 + 3y, y^2 - 2x^3 \rangle$ in $\mathbb{Q}[x, y]$ with the lex order where $x > y$. Here, "$In[1] :=$" is the Mathematica prompt.

$In[2] := \mathbf{gb_2 = GroebnerBasis[\{x^2 + ay, y^2 - bx^3\}, \{x, y\}, MonomialOrder \to DegreeReverseLexicographic, CoefficientDomain \to RationalFunctions]}$

This computes a Groebner basis of $J = \langle x^2 + ay, y^2 - bx^3 \rangle$ in $\mathbb{Q}(a, b)[x, y]$ with the grlex order where $x > y$. We can also assign the monomial order by a matrix.

$In[3] := \mathbf{w = \{\{1, 1, 0, 0\}, \{1, 0, 0, 0\}, \{0, 0, 1, 1\}, \{0, 0, 1, 0\}\}};$

$In[4] := \mathbf{GroebnerBasis[\{x^2 + ay, y^2 - bx^3\}, \{x, y, a, b\}, MonomialOrder \to w]}$

This first defines a 4×4 matrix whose first row is $\{1, 1, 0, 0\}$ and last row is $\{0, 0, 1, 0\}$ and then computes a Groebner basis of $L = \langle x^2 + ay, y^2 - bx^3 \rangle$ in $\mathbb{Q}[x, y, a, b]$ with the monomial order defined by $\mathbf{w}$. This monomial order is an example of block order mentioned in Chapter 3.1 with $\{x, y\} > \{a, b\}$ and the order in $\{x, y\}$ is the grlex order with $x > y$ and the order in $\{a, b\}$ is the grlex order with $a > b$.

2. $\mathbf{PolynomialReduce}[\mathbf{f}, \mathbf{gb}, \mathbf{varlist}, \mathbf{options}]$: Referring to the third part of proposition 2.1 in Chapter 2, this command computes the unique r (the normal form of f) such that $\mathbf{f} - r$ is in the ideal generated by the Groebner basis $\mathbf{gb}$ and no terms of r is divisible by any of $LT(g_1), \cdots, LT(g_l)$ where $g_i \in \mathbf{gb}$. Its algorithm is call the division algorithm described in [8]. The output of this command is a list of two entries: the first is a list of quotients and the second is the remainder r. **options** and **varlist** are the same as we introduce in the **GroebnerBasis** command. Following the examples above, we have Groebner bases $\mathbf{gb_1} = \{g_1, g_2, g_3\}$ of $I = \langle x^2 + 3y, y^2 - 2x^3 \rangle$ in $\mathbb{Q}[x, y]$ with lex order where $x > y$ and $\mathbf{gb_2} = \{g_4, g_5, g_6\}$ of $J = \langle x^2 + ay, y^2 - bx^3 \rangle$ in $\mathbb{Q}(a, b)[x, y]$ with grlex order where $x > y$. We have the followings. Lines after "$Out[n]$:=" in Mathematica is the output after executing the input "$In[n]$:=."

$In[5] := \mathbf{PolynomialReduce}[\mathbf{x^4}, \mathbf{gb_1}, \{\mathbf{x}, \mathbf{y}\}]$

$Out[5] := \{\{-\frac{\mathbf{1}}{\mathbf{12}}, -\frac{\mathbf{x}}{\mathbf{2}} + \frac{\mathbf{y}}{\mathbf{12}}, \mathbf{x^2}\}, \mathbf{9y^2}\}$

This means that

$$x^4 = -\frac{1}{12}g_1 + (-\frac{x}{2} + \frac{y}{12})g_2 + x^2 g_3 + 9y^2,$$

and $9y^2$ is the unique normal form of x^4 module I in $\mathbb{Q}[x, y]$.

$In[6] := \mathbf{PolynomialReduce}[\mathbf{xy^3}, \mathbf{gb_2}, \{\mathbf{x}, \mathbf{y}\}, \mathbf{MonomialOrder} \to$
$\mathbf{DegreeReverseLexicographic}, \mathbf{CoefficientDomain} \to \mathbf{RationalFunctions}]$

$Out[6] := \{\{\frac{\mathbf{y^2}}{\mathbf{ab}}, \mathbf{0}, -\mathbf{a^2b} + \frac{\mathbf{y}}{\mathbf{ab}}\}, -\mathbf{a^5b^3y^2}\}$

Here, we have

$$xy^3 = \frac{y^2}{ab}g_4 + (-a^2b + \frac{y}{ab})g_6 - a^5b^3y^2,$$

and $-a^5b^3y^2$ is the unique normal form of xy^3 module J in $\mathbb{Q}(a, b)[x, y]$. We can examine the two equations above by just substituting the polynomials below in the Groebner bases into them.

$$\begin{aligned} g_1 &= 108y^2 + y^3, \\ g_2 &= 6xy + y^2, \\ g_3 &= x^2 + 3y, \\ g_4 &= abxy + y^2, \\ g_5 &= x^2 + ay, \\ g_6 &= -a^3b^2y^2 - y^3. \end{aligned}$$

Appendix B

This appendix will present algorithms of some commands that we use and are not implemented in Mathematica 6.0.0. We show them in programming language of Mathematica.

1. **SyHa[p, a, q, b, z]** : This computes $SyHa_z\,(p, q, x)$. $b = deg(q), a = b + 1$.

```
SyHa[p_, a_, q_, b_, z_, var : x_] := Module[{A = a, sgn},
GetCoefficientList[p_, d_, var : x_] :=
Table[Coefficient[p, var^z], {z, d, 1, −1}] ∽ Join ∽ {p/.var → 0};
Sylv[p_, a_, q_, b_, z_, var : x_] :=
Table[CoefficientList[var^j p, a + b − z − 1, var], {j, b − z − 1, 0, −1}] ∽ Join
∽ Table[CoefficientList[var^j q, a + b − z − 1, var], {j, a − z − 1, 0, −1}];
PolynomialDet[A_, var : x_] := Module[{m, n, At, At1}, m = Length[A];
At = Transpose[A]; n = Length[At]; At1 = Take[At, m − 1];
Table[Det[At1 ∽ Join ∽ {At[[m + j]]}], {j, 0, n − m}].
Table[var^(n−m−j), {j, 0, n − m}]];
PolynomialSubresultant[p_, a_, q_, b_, z_, var : x_] :=
PolynomialDet[Sylv[p, a, q, b, z], var];
If[z > A, Return[0]]; If[z == A, Return[p]]; If[z == A − 1, Return[q]];
sgn = (−1)^((a−z)(a−z−1)/2); sgn ∗ PolynomialSubresultant[p, a, q, b, z, var]]
```

2. **SturmSeq[p, x]** : This command computes Sturm sequences of p in x.

```
SturmSeq[p_, x_] := Module[{f = p, g = D[p, x]//Simplify, h, s}, s := {f, g};
While[Exponent[g, x] > 0, h = −PolynomialRemainder[f, g, x];
s = s ∽ Join ∽ {h}; f = g; g = h; ]; Return[s]]
```

3. **HermiteMatrix[g, gb, mb, cla]** : This command computes the Hermite matrix determined by $\mathcal{F} = \{f_1, \cdots, f_m\}$ and g. Here **gb** is a Groebner basis **gb** of $\langle f_1, \cdots, f_m \rangle$ in the grlex order with $x > y$, $\mathbf{mb} = \{\mathbf{b_1}, \ldots, \mathbf{b_m}\}$ is a basis of $A = \mathbb{R}[x,y] / \langle f_1, \cdots, f_m \rangle$, and **cla** is the set of different $b_i b_j$'s. In the algorithm, **list** and **ct** produce the matrices *List* and *ct*, respectively, as in example 2.5, and **mt** produces $\{\mathbf{Trace}(\mathbf{L}(\mathbf{b_1})), \ldots, \mathbf{Trace}(\mathbf{L}(\mathbf{b_m}))\}$.

```
HermiteMatrix[g_, gb_, mb_, cla_] := Module[{h, i, j, k, l, m, tr, list, ct, mt},
h := Table[0, {i, 1, Length[mb]}, {j, 1, Length[mb]}];
GetCoefficient[p_, {a, b}_] := Module[{q}, q = Expand[p];
If[{a, b} == {0, 0}, q/.{x → 0, y → 0}, If[a == 0, Coefficient[q, y^b]/.x → 0,
If[b == 0, Coefficient[q, y^a]/.y → 0, Coefficient[q, x^a y^b]]]]]];
PolyReduceVector[p_, gb_, mb_] := Module[{r},
r = PolynomialReduce[p, gb, {x, y},
MonomialOrder → DegreeReverseLexicographic][[2]];
Map[GetCoefficient[r, ♯]&, mb]];
list[g_, gb_, mb_, cla_] := Module[{i, j, k, l}, l = Table[0, {i, 1, Length[cla]}];
For[k = 1, k ≤ Length[cla], k + +,
l[[k]] = PolyReduceVector[gx^(mb[[First[cla[[k]]][[1]]]]+mb[[First[cla[[k]]][[2]]]])[[1]] *
y^(mb[[First[cla[[k]]][[1]]]]+mb[[First[cla[[k]]][[2]]]])[[2]], gb, mb]; l];
list = list[g, gb, mb, cla];
ct[mb_, cla_] := Module[{M, i, j, k, l},
M := Table[0, {i, 1, Length[mb]}, {j, 1, Length[mb]}];
For[k = 1, k ≤ Length[cla], k + +, For[l = 1, l ≤ Length[cla[[k]]], l + +,
M[[cla[[k, l, 1]], cla[[k, l, 2]]]] = k; ]; ]; M];
ct = ct[mb, cla];
mt[mb_] := Module[{M, i, j, n}, M := Table[0, {i, 1, Length[mb]}];
For[j = 1, j ≤ Length[mb], j + +,
M[[j]] = Plus@@Table[list[[ct[[j]][[n]]]][[n]],
{n, 1, Length[mb]}]; ]; M];
mt = mt[mb];
For[k = 1, k ≤ Length[cla], k + +,
tr = Plus@@Table[list[[k]][[m]]mt[[m]], {m, 1, Length[mb]}];
For[l = 1, l ≤ Length[cla[[k]]], l + +, h[[cla[[k, l, 1]], cla[[k, l, 2]]]] = tr; ]; ]; h]
```

4. **Sig[A]** : This computes the signature of a real symmetric matrix A.

```
Sig[A_] := Module[{A1, i, diagsigns, p, n},
ClearCol[A_, i_] := Module[{B = A, d, ri, j}, ri = A[[i]]; d = ri[[i]];
For[j = i + 1, j ≤ Length[A], j + +, B[[j]] = B[[j]] − B[[j, i]] ∗ ri/d; ]; B];
ClearRow[A_, i_] := Transpose[ClearCol[Transpose[A], i]];
ClearCR[A_, i_] := ClearRow[ClearCol[A, i], i];
SwapRow[A_, i_, j_] := Module[{B = A, }, B[[i]] = A[[i]]; B[[j]] = A[[i]]; B];
SwapCol[A_, i_, j_] := Transpose[SwapRow[Transpose[A], i, j]];
SwapCR[A_, i_, j_] := SwapRow[SwapCol[A, i, j], i, j];
RowSumDiff[A_, i_, j_] := Module[{B = A, v, w},
v = A[[i]] + A[[j]]; w = −A[[i]] + A[[j]]; B[[i]] = v; B[[j]] = w; B];
ColSumDiff[A_, i_, j_] := Transpose[RowSumDiff[Transpose[A], i, j]];
SumDiffCR[A_, i_, j_] := RowSumDiff[ColSumDiff[A, i, j], i, j];
SymmetricReduce[A_] := Module[{B = A, n = Length[A], i, j, k},
For[i = 1, i ≤ n, i + +,
If[B[[i, j]] == 0, For[j = i + 1, j ≤ n, j + +,
If[B[[i, j]] ≠ 0, B = SwapCR[B, i, j]; Break[]; ]; ]; ];
If[B[[i, j]] == 0, For[j = i + 1, j ≤ n, j + +,
If[B[[i, j]] ≠ 0, B = SumDiffCR[B, i, j]; Break[]; ]; ]; ];
If[B[[i, j]] ≠ 0, B = ClearCR[B, i]; ]; ]; B];
A1 = SymmetricReduce[A];
diagsigns = Table[Sign[A1[[i, i]]], i, 1, Length[A1]]; Plus@@diagsigns]
```