Integer-Valued Lyapunov Function and its Application to Monotone Cyclic Feedback Systems

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

Master of Science

January 2011

Acknowledgements

I would like to thank my advisors Professors Peter Rejto and Jiaping Wang for their encouragement, advice, guidance, valuable consultations and discussions.

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Chapter 1

Introduction and Preparations

1.1 Two Fundamental Concepts

Given a n-D nonlinear autonomous system

$$\dot{x} = f(x)$$

where $f : \Omega \to \mathbf{R^n}$ is C^1 map, $\Omega \subset \mathbf{R^n}$ is an open set.

When one considers stability of an equilibrium of the above system, Lyapunov function V has played an important role. This paper introduces a special nonlinear autonomous system, the monotone cyclic feedback system and a fundamental tool, integer-valued Lyapunov function. Through the use of the integer-valued Lyapunov function in solving problems of the monotone cyclic feedback system, we will show the power of the integer-valued Lyapunov function in handling some problems. This thesis essentially follows [1].

First we introduce this special system.

We consider system of the form

$$\dot{x}^i = f^i(x^i, x^{i-1}), \quad i = 1, 2, \cdots, n \tag{1.1.1}$$

where n coordinate variables satisfy $x^0 = x^n, x^1 = x^{n+1}$, and so on. That is, all indices of all variables are to be taken mod n.

The nonlinearity $f = (f^1, f^2, \cdots, f^n)$ is defined on a nonempty open set $O \subset \mathbf{R^n}$ with the property that each coordinate projection $O^i \subset \mathbf{R^2}$ of O onto the (x^i, x^{i-1}) plane is convex and that $f^i \in C^1(O^i)$.

Another key assumption is there exist constants $\delta^i \in \{-1, +1\}$, such that

$$\delta^i \frac{\partial f^i(x^i, x^{i-1})}{\partial x^{i-1}} > 0; \text{ for all } (x^i, x^{i-1}) \in O^i \qquad 1 \le i \le n. \tag{1.1.2}$$

The system (1.1.1) satisfying (1.1.2) is called a monotone cyclic feedback system.

In order to study this special system, one applies integer-valued Lyapunov function as a principal tool. This function was first given by Smillie [2] in the study of the behavior of the solution of strongly competitive and strongly cooperatively tridiagonal system of ordinary differential equations.

Now we define the integer-valued Lyapunov function N.

Definition. Let $\mathcal{N}^* = \{y \in \mathbf{R^n} | \ y^i \neq 0, 1 \le i \le n\}$. for each $y \in \mathcal{N}^*$, define

$$N(y) = \text{card}\{i \,|\, \delta^i y^i y^{i-1} < 0\}.$$

Definition. The domain of $N(y)$ can be extended (by continuity) to

$$\mathcal{N} = \{y \in \mathbf{R^n} | \, y^i = 0 \text{ for some } i \text{ implies } \quad \delta^{i+1}\delta^i y^{i+1} y^{i-1} < 0\}.$$

The extended function is still denoted by $N(y)$.

By the definition, $N(y)$ and $\mathcal{N}$ have the following properties:

(1) $N : \mathcal{N} \to \{0, 1, \cdots, n\}$ is a continuous function.

(2) Let $\Delta = \delta^1\delta^2 \cdots \delta^n$. For $y \in \mathcal{N}^*$, $(-1)^{N(y)} = \text{sign} \prod_{i=1}^n \delta^i y^i y^{i-1} = \prod_{i=1}^n \delta^i = \Delta$ and

$$N(y) = \begin{cases} \text{odd} & \text{if } \Delta = -1 \\ \text{even} & \text{if } \Delta = +1. \end{cases}$$

(3) For all $k \in \mathbf{R}$, $k \neq 0, N(ky) = N(y)$.

(4) $\mathcal{N}$ is a open set.

If $y \in \mathbf{R^n} \setminus \mathcal{N}$, then $N(y)$ is undefined.

When $y \notin \mathcal{N}$, $y = (y^1, y^2, \cdots, y^n)$ has at least one zero component, $y^i = 0$. By the definition of $\mathcal{N}$, there are two types of y so that $y \notin \mathcal{N}$:

Type 1. y has no consecutive zero components (mod n) (Equivalently y has only one or more single zero components, i.e., $y^i = 0$ implies $y^{i+1}y^{i-1} \neq 0$) and there exists at least an index k such that for $y^k = 0, \delta^{k+1}\delta^k y^{k+1}y^{k-1} > 0$.

Type 2. y has at least one group of consecutive zero components (mod n) (where one group of consecutive zero components means that for some index i, $y^i(0) \neq 0, y^{i+1}(0) = 0, \cdots, y^{i+p}(0) = 0, y^{i+p+1}(0) \neq 0, \ \ p \geq 2$).

1.2 The Key Proposition

Similar to the continuous case, N is associated with a solution $y(t)$, where $y(t)$ is the derivative $\dot{x}(t)$ of solution $x(t)$ of (1.1.1) with (1.1.2) or the difference $x(t) - \bar{x}(t)$ of two solutions $x(t), \bar{x}(t)$ of (1.1.1) with (1.1.2), and $y(t) = \dot{x}(t)$ or $y(t) = x(t) - \bar{x}(t)$ satisfies a nonautonomous linear system.

For $y(t) = \dot{x}(t)$, by the chain rule, one has

$$\begin{aligned}\dot{y}^i(t) &= \ddot{x}^i(t) = \frac{\partial f^i(x^i(t), x^{i-1}(t))}{\partial x^i}\dot{x}^i(t) + \frac{\partial f^i(x^i(t), x^{i-1}(t))}{\partial x^{i-1}}\dot{x}^{i-1}(t) \\ &= \frac{\partial f^i(x^i(t), x^{i-1}(t))}{\partial x^i}y^i(t) + \frac{\partial f^i(x^i(t), x^{i-1}(t))}{\partial x^{i-1}}y^{i-1}(t)\end{aligned}$$

For $y(t) = x(t) - \bar{x}(t)$, let

$$u^j(s,t) = sx^j(t) + (1-s)\bar{x}^j(t) \quad j = i, i-1.$$

Then by the fundamental theorem of calculus and the chain rule, one has

$$\begin{aligned}\dot{y}^i(t) &= \dot{x}^i(t) - \dot{\bar{x}}^i(t) = f^i(x^i(t), x^{i-1}(t)) - f^i(\bar{x}^i(t), \bar{x}^{i-1}(t)) \\ &= f^i(u^i(1,t), u^{i-1}(1,t)) - f^i(u^i(0,t), u^{i-1}(0,t)) \\ &= \int_0^1 \frac{\partial f^i(u^i(s,t), u^{i-1}(s,t))}{\partial s}ds \\ &= \int_0^1 \frac{\partial f^i(u^i(s,t), u^{i-1}(s,t))}{\partial u^i}\frac{\partial u^i}{\partial s} + \frac{\partial f^i(u^i(s,t), u^{i-1}(s,t))}{\partial u^{i-1}}\frac{\partial u^{i-1}}{\partial s}ds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{\partial f^i(u^i(s,t),u^{i-1}(s,t))}{\partial u^i}(x^i(t)-\bar{x}^i(t)) \\
&\quad + \frac{\partial f^i(u^i(s,t),u^{i-1}(s,t))}{\partial u^{i-1}}(x^{i-1}(t)-\bar{x}^{i-1}(t))ds \\
&= \int_0^1 \frac{\partial f^i(u^i(s,t),u^{i-1}(s,t))}{\partial u^i}ds(x^i(t)-\bar{x}^i(t)) \\
&\quad + \int_0^1 \frac{\partial f^i(u^i(s,t),u^{i-1}(s,t))}{\partial u^{i-1}}ds(x^{i-1}(t)-\bar{x}^{i-1}(t)) \\
&= \int_0^1 \frac{\partial f^i(u^i(s,t),u^{i-1}(s,t))}{\partial u^i}ds\; y^i(t) \\
&\quad + \int_0^1 \frac{\partial f^i(u^i(s,t),u^{i-1}(s,t))}{\partial u^{i-1}}ds\; y^{i-1}(t)
\end{aligned}
$$

Therefore, for $y = \dot{x}$ and $y = x - \bar{x}$, they satisfy the linear system of the same form :

$$\dot{y}^i(t) = w^{i,i}(t)y^i(t) + w^{i,i-1}(t)y^{i-1}(t), \quad 1 \le i \le n \tag{1.2.3}$$

where for $y(t) = \dot{x}(t)$,

$$w^{i,j}(t) = \frac{\partial f^i(x^i(t),x^{i-1}(t))}{\partial x^j}, \quad j = i, i-1,$$

while for $y(t) = x(t) - \bar{x}(t)$,

$$w^{i,j}(t) = \int_0^1 \frac{\partial f^i(u^i(s,t),u^{i-1}(s,t))}{\partial u^j}ds,$$

where $u^j(s,t) = sx^j(t) + (1-s)\bar{x}^j(t), \quad j = i, i-1$.

Furthermore, since $\delta^i \frac{\partial f^i}{\partial x^{i-1}} > 0$,

$$\delta^i \int_0^1 \frac{\partial f^i}{\partial u^{i-1}}ds = \int_0^1 \delta^i \frac{\partial f^i}{\partial u^{i-1}}ds > 0,$$

one has

$$\delta^i w^{i,i-1}(t) > 0 \qquad \text{and} \qquad w^{i,j} \text{ continuous}, \quad j = i, i-1. \tag{1.2.4}$$

It is clear that (1.2.3) with (1.2.4) is a nonautonomous linear monotone cyclic feedback system.

The integer-valued Lyapunov function N plays an important role through the solution of the linear system (1.2.3) with (1.2.4) in the study of (1.1.1) with (1.1.2).

Along the solution $y = \dot{x}$ or $y = x - \bar{x}$, N has some fine properties shown by the following Proposition 1.1. It is because N has these properties that N becomes a powerful tool to handle the nonlinear system.

Proposition 1.1 (Proposition 1.1 in [1]). Let $y(t)$ be a nontrivial solution of (1.2.3) where (1.2.4) holds. Then

(a) $y(t) \in \mathcal{N}$ except at isolated values of t.

(b) $N(y(t))$ is locally constant where $y(t) \in \mathcal{N}$.

(c) If $y(t_0) \notin \mathcal{N}$, then $N(y(t_0^+)) < N(y(t_0^-))$.

(d) If $y(t) \in \mathcal{N}$, then $(y^i(t), y^{i-1}(t)) \neq (0,0)$ and $(y^i(t), \dot{y}^i(t)) \neq (0,0), 1 \leq i \leq n$.

Proof. The fact that $N(y(t))$ is continuous and takes only finitely many values implies (b). Assume that $y(t) \in \mathcal{N}$. If $y^i(t) = 0$, from the definition of $\mathcal{N}$, $y^{i-1}(t) \neq 0$. By using (1.2.3) and (1.2.4), the latter is equivalent to $\dot{y}^i(t) \neq 0$, provided $y^i(t) = 0$. So we get (d). Thus we need only prove (a) and (c). Suppose that there exists some t_0 such that $y(t_0) \notin \mathcal{N}$. Without loss of generality, assume that $t_0 = 0$. To verify (a) and (c), we will show that $N(y(t))$ is defined in $0 < |t| < \delta$ for some δ and $N(y(0^+)) < N(y(0^-))$.

For each i, by (1.2.3) with (1.2.4), $y^i(t) \not\equiv 0$ since $y(t)$ is nontrivial. Furthermore, for each i, there exist $p_0^i \neq 0$ and a nonnegative integer k, such that

$$y^i(t) = p_0^i t^k + o(t^k) \quad \text{as } t \to 0. \tag{1.2.5}$$

If $y^i(0) \neq 0$, let $p_0^i = y^i(0), k = 0$.

If $y^i(0) = 0$, for the existence of such nonzero p_0^i, see later Lemma 1.4 and $t = 0$ is an isolated zero of $y^i(t)$.

Thus, by (1.2.5) we can give the following

Definition 1.2. Let k and p_0^i be given as in (1.2.5), for $1 \leq i \leq n$, define

$$k(i) \quad = \quad \begin{cases} 0 & \text{if } y^i(0) \neq 0 \\ k & \text{if } y^i(0) = 0, \end{cases}$$

$$
\begin{aligned}
p^i &= \begin{cases} y^i(0) & \text{if } y^i(0) \neq 0 \\ p_0^i & \text{if } y^i(0) = 0, \end{cases} \\
P^i &= \text{sign } p^i.
\end{aligned}
$$

Now we consider the variation of the value of $N(y(t))$ as t increases through zero.

Type 1. Suppose that $y(0) = (y^1(0), y^2(0), \cdots, y^n(0))$ has no consecutive zero components, i.e., it has only one or more single zero components. Here we need only analyze one single zero component, since the analysis of each single zero component is the same.

Assume that $y^j(0) = 0$, then $y^{j-1}(0) \neq 0, y^{j+1}(0) \neq 0$. First, note that $k(j) = 1$ since $\dot{y}^j(0) = w^{j,j}(0)y^j(0)+w^{j,j-1}(0)y^{j-1}(0) = w^{j,j-1}(0)y^{j-1}(0) \neq 0$, by (1.2.3),(1.2.4). Therefore, from (1.2.5), $t = 0$ is an isolated zero of $y^j(t)$.

In order to discuss the sign changes of the functions $\delta^j y^j(t)y^{j-1}(t), \delta^{i+1}y^{j+1}(t)y^j(t)$ as t increases through zero, we will use the derivatives of $\delta^j y^j(t)y^{j-1}(t), \delta^{i+1}y^{j+1}(t)y^j(t)$ and (1.2.3) with (1.2.4). Since $(\delta^{i+1}y^{j+1}(t)y^j(t))\,'|_{t=0}$ can have two signs, there exists two cases:

Case 1. $\delta^{j+1}\delta^j P^{j+1}P^{j-1} = +1$.

$$
\begin{aligned}
(\delta^j y^j(t)y^{j-1}(t))\,'|_{t=0} &= \delta^j \dot{y}^j(0)y^{j-1}(0) = \delta^j w^{j,j-1}(0)(y^{j-1}(0))^2 > 0, \\
(\delta^{j+1}y^{j+1}(t)y^j(t))\,'|_{t=0} &= \delta^{j+1}y^{j+1}(0)\dot{y}^j(0) = \delta^{j+1}y^{j+1}(0)w^{j,j-1}(0)y^{j-1}(0) \\
&= \delta^{j+1}y^{j+1}(0)\delta^j\delta^j w^{j,j-1}(0)y^{j-1}(0) > 0.
\end{aligned}
$$

The single zero component $y^j(0)$ at the jth position contributes a decrease by two in N as t increases through zero.

Case 2. $\delta^{j+1}\delta^j P^{j+1}P^{j-1} = -1$.

$$
(\delta^{j+1}y^{j+1}(t)y^j(t))\,'|_{t=0} = \delta^{j+1}y^{j+1}(0)\delta^j\delta^j w^{j,j-1}(0)y^{j-1}(0) < 0.
$$

Thus, for the single zero component $y^j(0)$, the values of N remain unchanged as t increases through zero.

Note that not all indices j, at which $y^j(0) = 0$, can satisfy $\delta^{j+1}\delta^j P^{j+1}P^{j-1} = -1$ since $y(0) \notin \mathcal{N}$. That is, $y(0) = (y^1(0), y^2(0), \cdots, y^n(0))$ must has some index k such that $\delta^{k+1}\delta^k P^{k+1}P^{k-1} = +1$.

Therefore, the results of the above two cases imply that in Type 1 the change in N is always a negative even integer as t increases through zero, no matter how many single zero components $y(0)$ has.

Type 2. $y(0) = (y^1(0), y^2(0), \cdots, y^n(0))$ has at least one group of consecutive zero components. If $y(0)$ has some single zero components in Type 2, their analysis is similar to that in Type 1. Since the analysis of each group of consecutive zero components is the same, we discuss only a group of consecutive zero components. Without loss of generality let $y^j(0) \neq 0, y^{j+1}(0) = 0, \cdots, y^{j+p}(0) = 0, y^{j+p+1}(0) \neq 0,\ p \geq 2$. For the group of zero components we have the following relations

$$\begin{cases} k(j+r) = r & 1 \leq r \leq p \\ P^{j+r} = \delta^{j+r}P^{j+r-1} & 1 \leq r \leq p \end{cases} \tag{1.2.6}$$

where $p \geq 2$, for the definition of k and P see Definition 1.2.

To show (1.2.6) holds, we need the following

Lemma 1.3. Let $y(t)$ be the solution of

$$\dot{y} = A(t)y + g(t), \quad y(0) = 0 \tag{1.2.7}$$

where $A(t)$ is a continuous $n \times n$ matrix function and $g(t)$ is a continuous n-vector function satisfying

$$g(t) = g_m t^m + o(t^m), \quad t \to 0$$

where $g_m \in \mathbf{R^n}$ and m is a nonnegative integer. Then

$$y(t) = \frac{g_m}{m+1}t^{m+1} + o(t^{m+1}).$$

Proof. Let

$$\dot{y} = A(t)y \tag{1.2.8}$$

be the homogeneous linear system corresponding to the nonhomogeneous linear system in (1.2.7). Assume $\Phi(t)$ is a fundamental matrix for (1.2.8), then from [3], the function

$$\varphi(t) = \Phi(t) \int_0^t \Phi^{-1}(s) g(s) ds \tag{1.2.9}$$

is the solution of the nonhomogeneous linear system satisfying

$$\varphi(0) = 0.$$

That is, $y = \varphi(t)$ is the solution of the initial value problem (1.2.7). By using Taylor's theorem

$$\begin{aligned} \Phi(t) &= \Phi(0) + \frac{\Phi'(0)}{1!} t + o(t), \\ \Phi^{-1}(s) &= \Phi^{-1}(0) + \frac{(\Phi^{-1}(s))'|_{s=0}}{1!} s + o(s). \end{aligned}$$

Substituting them and the expression of $g(t)$ into (1.2.9), one gets

$$\begin{aligned} y(t) &= (\Phi(0) + \frac{\Phi'(0)}{1!} t + o(t)) \int_0^t (\Phi^{-1}(0) + \frac{(\Phi^{-1}(s))'|_{s=0}}{1!} s + o(s)) \\ &\quad (g_m s^m + o(s^m)) ds \\ &= (\Phi(0) + \frac{\Phi'(0)}{1!} t + o(t)) \int_0^t \Phi^{-1}(0) g_m s^m + o(s^m) ds \\ &= (\Phi(0) + \frac{\Phi'(0)}{1!} t + o(t))(\Phi^{-1}(0) \frac{g_m}{m+1} t^{m+1} + o(t^{m+1})) \\ &= \frac{g_m}{m+1} t^{m+1} + o(t^{m+1}). \qquad [] \end{aligned}$$

Now we prove (1.2.6).

Lemma 1.4. In Type 2, if $y(0)$ has a group of consecutive zero components $y^{j+1}(0) = 0, \cdots, y^{j+p}(0) = 0$, but $y^j(0) \neq 0, y^{j+p+1}(0) \neq 0$, then one has the following results

$$\begin{cases} k(j+r) = r & 1 \leq r \leq p \\ P^{j+r} = \delta^{j+r} P^{j+r-1} & 1 \leq r \leq p \end{cases} \tag{1.2.6}$$

where $p \geq 2$, and the definitions of k and P are given by Definition 1.2.

Proof. We use induction to prove (1.2.6).

For $r = 1$

$$\begin{aligned}\dot{y}^{j+1}(t) &= w^{j+1,j+1}(t)y^{j+1}(t) + w^{j+1,j}(t)y^j(t),\\ \dot{y}^{j+1}(0) &= w^{j+1,j+1}(0)y^{j+1}(0) + w^{j+1,j}(0)y^j(0) = w^{j+1,j}(0)y^j(0) \neq 0.\end{aligned}$$

From $y^j(0) \neq 0, p^j = y^j(0)$ and $y^{j+1}(0) = 0$, by Taylor expansion,

$$\begin{aligned}y^{j+1}(t) &= y^{j+1}(0) + \frac{\dot{y}^{j+1}(0)}{1!}t + o(t) = \frac{\dot{y}^{j+1}(0)}{1!}t + o(t)\\ &= w^{j+1,j}(0)y^j(0)t + o(t) = w^{j+1,j}(0)p^j\, t + o(t).\end{aligned}$$

Since the coefficient $w^{j+1,j}(0)p^j \neq 0$, by (1.2.5) and Definition 1.2, $p^{j+1} = w^{j+1,j}(0)p^j$.

Therefore, for $r = 1,\ k(j+1) = 1$ and

$$P^{j+1} = \text{sign } p^{j+1} = \text{sign } \delta^{j+1}\delta^{j+1}w^{j+1,j}(0)p^j = \text{sign } \delta^{j+1}p^j = \delta^{j+1}P^j.$$

The assertions hold for $r = 1$.

Now suppose that the assertions hold for r and we prove that the results hold for $r+1, (r+1 \leq p)$. That is, from

$$\begin{cases} k(j+r) = r \\ P^{j+r} = \delta^{j+r}P^{j+r-1} \end{cases} \tag{1.2.10}$$

we must show that the results

$$\begin{cases} k(j+r+1) = r+1 \\ P^{j+r+1} = \delta^{j+r+1}P^{j+r} \end{cases}$$

hold.

The condition (1.2.10) of the induction means that there exists $p^{j+r} \neq 0$ so that

$$y^{j+r}(t) = p^{j+r}t^r + o(t^r). \tag{1.2.11}$$

By (1.2.3) (1.2.4) and by $r+1 \leq p$

$$\dot{y}^{j+r+1}(t) = w^{j+r+1,j+r+1}(t)y^{j+r+1}(t) + w^{j+r+1,j+r}(t)y^{j+r}(t), \quad y^{j+r+1}(0) = 0.$$

If one considers $y^{j+r+1}(t)$ and $w^{j+r+1,j+r}(t)y^{j+r}(t)$ as $y(t)$ and $g(t)$ in Lemma 1.3, respectively, for $y^{j+r+1}(0) = 0, y^{j+r+1}(t)$ also satisfies the initial condition $y(0) = 0$ in (1.2.7).

Now, we expand $w^{j+r+1,j+r}(t)y^{j+r}(t)$.

Since $w^{j+r+1,j+r}(t)$ is continuous, substitute (1.2.11) into $w^{j+r+1,j+r}(t)y^{j+r}(t)$, one gets

$$\begin{aligned} w^{j+r+1,j+r}(t)y^{j+r}(t) &= w^{j+r+1,j+r}(t)(p^{j+r}t^r + o(t^r)) \\ &= w^{j+r+1,j+r}(t)p^{j+r}t^r + o(t^r), \quad \text{as} \ \ t \to 0. \end{aligned}$$

The fact that $\frac{w^{j+r+1,j+r}(t)p^{j+r}t^r}{w^{j+r+1,j+r}(0)p^{j+r}t^r} \to 1$ as $t \to 0$ leads us to obtain

$$\begin{aligned} &\frac{w^{j+r+1,j+r}(t)p^{j+r}t^r - w^{j+r+1,j+r}(0)p^{j+r}t^r}{w^{j+r+1,j+r}(0)p^{j+r}t^r} \to 0 \quad \text{and} \\ &w^{j+r+1,j+r}(t)p^{j+r}t^r = w^{j+r+1,j+r}(0)p^{j+r}t^r + o(t^r) \quad \text{as} \ \ t \to 0. \end{aligned}$$

Hence the expansion of $w^{j+r+1,j+r}(t)y^{j+r}(t)$ is

$$w^{j+r+1,j+r}(t)y^{j+r}(t) = w^{j+r+1,j+r}(0)p^{j+r}t^r + o(t^r), \quad \text{as} \ \ t \to 0.$$

By using Lemma 1.3

$$y^{j+r+1}(t) = \frac{w^{j+r+1,j+r}(0)p^{j+r}}{r+1}t^{r+1} + o(t^{r+1}).$$

Since $\frac{w^{j+r+1,j+r}(0)p^{j+r}}{r+1} \neq 0$, by Definition 1.2

$$p^{j+r+1} = \frac{w^{j+r+1,j+r}(0)p^{j+r}}{r+1} \quad \text{and} \quad k(j+r+1) = r+1.$$

Moreover,

$$\text{sign } p^{j+r+1} = \text{sign } \frac{\delta^{j+r+1}\delta^{j+r+1}w^{j+r+1,j+r}(0)p^{j+r}}{r+1} = \delta^{j+r+1}\text{sign } p^{j+r}.$$

Therefore,

$$P^{j+r+1} = \delta^{j+r+1}P^{j+r}.$$

By induction, (1.2.6) holds for all r, $1 \leq r \leq p$. []

From Lemma 1.4 one has $t=0$ is isolated zero of $y^i(t), \ \ i=j+1, j+2, \cdots, j+p$.

Now consider the change of the values of $N(y(t))$ as t increases through zero.

For $q=1,2,\cdots,p, \ \ 0<|t|$ and $|t|$ small, by using the formulas (1.2.5) and (1.2.6), we get

$$\begin{aligned} & \operatorname{sign}(\delta^{j+q} y^{j+q} y^{j+q-1}) = \operatorname{sign}(\delta^{j+q} p^{j+q} t^q p^{j+q-1} t^{q-1}) \\ = & \operatorname{sign}(\delta^{j+q} p^{j+q} p^{j+q-1} t^{2q-1}) = \delta^{j+q} P^{j+q} P^{j+q-1} \operatorname{sign} t^{2q-1} \\ = & (\delta^{j+q} P^{j+q-1})^2 \operatorname{sign} t^{2q-1} = \operatorname{sign} t^{2q-1}. \end{aligned}$$

Moreover, for $t \neq 0$ and $|t|$ small

$$\begin{aligned} & \operatorname{sign}(\delta^{j+p+1} y^{j+p+1} y^{j+p}) = \operatorname{sign}(\delta^{j+p+1} y^{j+p+1} p^{j+p} t^p) \\ = & \delta^{j+p+1} P^{j+p+1} P^{j+p} \operatorname{sign} t^p. \end{aligned}$$

Thus, as t increases through zero ($|t|$ small), the change in N contributed by the group of consecutive zero components is

$$\triangle N = \begin{cases} -p & p \text{ even} \\ -(p+1) & p \text{ odd and } \delta^{j+p+1} P^{j+p+1} P^{j+p} > 0 \\ -(p-1) & p \text{ odd and } \delta^{j+p+1} P^{j+p+1} P^{j+p} < 0. \end{cases} \qquad (1.2.12)$$

In Type 2, for each single zero component N decreases by two or keeps no change as t increases through zero. For each group of consecutive zero components N decreases by an even integer shown by (1.2.12). Hence for Type 2 the change in N is also a negative even integer.

In summary, for both types the value of N always decreases by a positive even integer as t increases through zero. Thus we get (c).

Observe that for each $y^i(0) \neq 0$, there is a $\delta^i > 0$, such that $y^i(t) \neq 0$, $t \in (-\delta^i, \delta^i)$. For each $y^j(0)=0$, by the results of the two types, $t=0$ is an isolated zero of $y^j(t)$. So there exists a $\delta_0^j > 0$, such that $y^j(t) \neq 0$, $0<|t|<\delta_0^j$.

Therefore, there exists a $\delta > 0$, such that $y^k(t) \neq 0, \ 0<|t|<\delta, \ k=1,2,\ldots,n$. That is, $y(t) \in \mathcal{N}^* \subset \mathcal{N}$, $t \in (-\delta, 0) \bigcup (0, \delta)$ and $t=0$ is an isolated point of $y(t)$. So (a) is proved. $\square$

Corollary 1.5. Let $y(t)$ be a nontrivial solution of (1.2.3) with (1.2.4). Then there exist only finite points $t_1, t_2, \cdots, t_p$, such that $y(t_j) \notin \mathcal{N}$. Moreover, as t increases through t_j, $1 \leq j \leq p$, $N(y(t))$ decreases by a positive even integer.

Corollary 1.6. Let $y(t)$ be a nontrivial solution of (1.2.3) with (1.2.4) which exists for $t \in \mathbf{R}$. If $y(t)$ is periodic with period τ, then for all $t \in \mathbf{R}$, $y(t) \in \mathcal{N}$ and $N(y(t)) = k$ is constant.

Proof. If this result is not true, then there exists some t_0 such that $y(t_0) \notin \mathcal{N}$, and from Proposition 1.1(a) there are two values t_1, t_2 satisfying $t_1 < t_0 < t_2$, such that $y(t_1), y(t_2) \in \mathcal{N}$. By using Proposition 1.1(c), one has $N(y(t_1)) > N(y(t_2))$. Because there exists integer n such that $t_2' = t_2 - n\tau < t_1$, thus from the periodicity of $y(t)$, one has the following contradiction

$$N(y(t_2)) = N(y(t_2')) \geq N(y(t_1)) > N(y(t_2)).$$

It is easy to see that $N(y(t)) = k$ is constant. $\square$

Chapter 2

Application of Integer-Valued Lyapunov Function

2.1 Poincaré-Bendixson Theorem

The continuous Lyapunov function V is applied to determine the stability of an equilibrium of a nonlinear system, while in [1] the integer-valued Lyapunov function N is used to prove Poincaré-Bendixson theorem holds for monotone cyclic feedback system in $\mathbf{R^n}$. So we need some the following preliminary work.

We consider the following nonlinear autonomous system

$$\dot{x} = f(x)$$

where $x \in \mathbf{R^2}$, f is a C^1 function. Let $x(t)$ be a solution through $x(0) = x_0, \gamma(x_0)$ or γ be an orbit corresponding to $x(t)$ and $\omega(x_0)$ or $\omega(\gamma)$ be a ω-limit set of $\gamma(x_0)$.

The following theorem gives a criterion for the detection of periodic orbit in the plane.

Theorem (Poincaré-Bendixson). Let $\omega(\gamma)$ be the ω -limit set of an orbit γ in $\mathbf{R^2}$. If $\omega(\gamma)$ is nonempty, bounded and does not contain an equilibrium, then $\omega(\gamma)$ is a periodic orbit.

In general, there is no Poincaré-Bendixson theorem in $\mathbf{R^n}$ (n ≥ 3). However, for some special system in $\mathbf{R^n}$, such as the monotone cyclic feedback system, Poincaré-Bendixson theorem still holds. The main tool for the proof of the theorem is just integer-valued Lyapunov function.

The proof of Poincaré-Bendixson theorem in $\mathbf{R^2}$ involves usually two steps:

1. If the ω -limit set $\omega(x_0)$ satisfies the conditions in the above theorem, then $\omega(x_0)$ must contain a periodic orbit $\gamma(p_0)$, where $p_0 \in \omega(x_0)$, i.e., $\omega(x_0) \supset \gamma(p_0)$.

2. If $\omega(x_0)$ contains a periodic orbit, then it is equal to the periodic orbit, i.e., $\omega(x_0) = \gamma(p_0)$.

Using some results in [1], one can get the same theorem, for the special system in $\mathbf{R^n}$, as that in $\mathbf{R^2}$ and correspondingly the proof of this theorem is also related to the two steps :

1. $\omega(x_0)$ which has no equilibrium contains definitely a periodic orbit. This result can be obtained from Lemma 3.6 in [1].

2. If $\omega(x_0)$ contains a periodic orbit $\gamma(p_0)$, it is equal to $\gamma(p_0)$. This result is given by Lemma 3.10 in [1].

Now we give the statement of the two Lemmas. (We will add some conditions for the following two lemmas and Lemma 2.2: Let $x(t)$, through $x(0) = x_0$, whose forward orbit is bounded, with closure $\overline{\gamma^+(x_0)} \subset O$.)

Lemma (Lemma 3.6 in [1]). If $y_0 \in \omega(x_0)$, then $\omega(y_0)$ contains at least one equilibrium or periodic orbit. The same is true of $\alpha(y_0)$. Thus $\omega(x_0)$ contains at least one equilibrium or periodic orbit.

Lemma (Lemma 3.10 in [1]). Let $p_0 \in \omega(x_0)$ and suppose the solution $p(t)$ through p_0 is either a nonconstant periodic solution or else an equilibrium point. Assume, moreover, that

$$\Delta \det(-D(f(p_0))) < 0$$

if p_0 is an equilibrium. Then in fact

$$\omega(x_0) = \gamma(p_0).$$

Having the two Lemmas we can get the Poincaré-Bendixson theorem for monotone cyclic feedback system in $\mathbf{R^n}$.

From Lemma 3.6 in [1], under the required conditions, the ω-limit set $\omega(x_0)$ in $\mathbf{R^n}$ contains an equilibrium or periodic orbit. Thus, $\omega(x_0)$ which has no equilibrium must contain a periodic orbit, say $\gamma(p_0)$. Then from Lemma 3.10 in [1], the ω-limit set $\omega(x_0)$ is just the periodic orbit. That is, $\omega(x_0) = \gamma(p_0)$.

In addition, there is a stronger conclusion in [1]. In fact, the Poincaré-Bendixson theorem in $\mathbf{R^2}$ has the generalized form with the ω-limit set containing equilibrium points.

Theorem (Theorem 1.3, p.55 in [4], Theorem 1.1.19 in [5]). Let γ^+ be a forward orbit in a closed bounded subset K of $\mathbf{R^2}$ and suppose K has only a finite number of equilibria. Then one of the following is satisfied:

(i) $\omega(\gamma^+)$ is an equilibrium;

(ii) $\omega(\gamma^+)$ is a periodic orbit;

(iii) $\omega(\gamma^+)$ contains a finite number of equilibria and a set of orbits γ_i with $\alpha(\gamma_i)$ and $\omega(\gamma_i)$ consisting of an equilibrium for each orbit γ_i.

In [1], the Poincaré-Bendixson theorem is just given in this generalized form.

Theorem (Main Theorem in [1]). Let $x(t)$ be the solution of monotone cyclic feedback system (1.1.1), (1.1.2) through $x(0) = x_0$, and suppose the forward orbit $\gamma^+(x_0)$ is bounded, with closure $\overline{\gamma^+(x_0)} \subset O$. Then the ω-limit set $\omega(x_0)$ is one of the following:

(i)an equilibrium,

(ii)a nonconstant periodic orbit, or

(iii)a set $E \cup H$, where E is a set of equilibria and H is the set of orbits which connect the equilibria in E.

That is, for each orbit $\gamma \in H$, $\alpha(\gamma) = \{e_i\}$, $\omega(\gamma) = \{e_j\}$, $e_i, e_j \in E$.

2.2 Proofs of Proposition 3.1 and Lemma 3.5 in [1]

In [1], the proof of lemma 3.6 requires Lemma 3.5 while the proof of Lemma 3.10 needs Proposition 3.1 (through Lemma 3.9).

So we use the two proofs of Proposition 3.1 and Lemma 3.5 in [1] as our examples to show what role the integer-valued Lyapunov function plays.

Proposition 2.1 (Proposition 3.1 in [1]). Let $p(t)$ be a periodic solution of (1.1.1) with (1.1.2) of least period $\tau > 0$. Then for each i the maps

$$t \to (p^i(t), \dot{p}^i(t)) \;\text{ and }\; t \to (p^i(t), p^{i-1}(t)) \tag{2.2.1}$$

are one-to-one on $[0,\tau)$ and have nonzero derivative throughout $[0,\tau)$.

Moreover, there is an integer k_0 such that for each $\theta \in (0, \tau)$

$$N(p(t+\theta) - p(t)) = N(\dot{p}(t)) = k_0 \tag{2.2.2}$$

holds for all t.

Proof. First we prove the two maps are one-to-one on $[0,\tau)$. That is, we must show that for any $t' \neq t$, $\;t', t \in [0, \tau)$,

$$(p^i(t'), \dot{p}^i(t')) \neq (p^i(t), \dot{p}^i(t)) \;\text{ and }\; (p^i(t'), p^{i-1}(t')) \neq (p^i(t), p^{i-1}(t)) \quad \text{ or}$$

$$(p^i(t') - p^i(t), \dot{p}^i(t') - \dot{p}^i(t)) \neq (0, 0) \;\text{ and }\; (p^i(t') - p^i(t), p^{i-1}(t') - p^{i-1}(t)) \neq (0, 0)$$

Assume without loss of generality that $t' > t$ and let $t' - t = \theta$ be fixed, then $\theta \in (0, \tau)$. Note that $p(t') = p(t + \theta)$ and $p(t)$ are two different solutions of (1.1.1) with (1.1.2), then their difference $z(t) = p(t+\theta) - p(t)$ is a nontrivial solution of the linear monotone cyclic feedback system (1.2.3), (1.2.4) with coefficients

$$w^{i,j} = \int_0^1 \frac{\partial f^i(u^i(s,t), u^{i-1}(s,t))}{\partial u^j} ds$$

where

$$u^j(s,t) = sp^j(t+\theta) + (1-s)p^j(t) \qquad j = i, i-1.$$

Because $p(t+\theta), p(t)$ are periodic solutions of (1.1.1) with (1.1.2) of least period $\tau > 0$, $z(t) = p(t+\theta) - p(t)$ is also periodic. By Corollary 1.6, $z(t) \in \mathcal{N}$, for all $t \in \mathbf{R}$ and $N(z(t)) \equiv k$ is constant in t (with k possibly depending on θ). Then from Proposition 1.1(d), for all $t \in \mathbf{R}$

$$(z^i(t), \dot{z}^i(t)) \neq (0,0) \ \text{ and } \ (z^i(t), z^{i-1}(t)) \neq (0,0). \tag{2.2.3}$$

Since $\theta \in (0, \tau)$ can be arbitrary, from (2.2.3), one has

$$(p^i(t') - p^i(t), \dot{p}^i(t') - \dot{p}^i(t)) \neq (0,0) \ \text{ and } \ (p^i(t') - p^i(t), p^{i-1}(t') - p^{i-1}(t)) \neq (0,0)$$

for any $t' \neq t$, $t', t \in [0, \tau)$. That is, the two maps are one-to-one on $[0, \tau)$.

Now we prove that the two maps have nonzero derivative, i.e.,

$$(\dot{p}^i(t), \ddot{p}^i(t)) \neq (0,0) \quad \text{and} \quad (\dot{p}^i(t), \dot{p}^{i-1}(t)) \neq (0,0).$$

Let $q(t) = \dot{p}(t)$, then $q(t)$ satisfies the linear monotone cyclic feedback system

$$\dot{q} = Df(p(t))q.$$

Moreover, since $p(t)$ is periodic, $q(t) = \dot{p}(t)$ is also periodic. Therefore by Corollary 1.6, $q(t) \in \mathcal{N}$, for all $t \in \mathbf{R}$ and $N(q(t)) \equiv k_0$ is constant. Also by Proposition 1.1(d), one has for all $t \in \mathbf{R}$

$$(q^i(t), \dot{q}^i(t)) \neq (0,0) \quad \text{and} \quad (q^i(t), q^{i-1}(t)) \neq (0,0) \qquad \text{or}$$

$$(\dot{p}^i(t), \ddot{p}^i(t)) \neq (0,0) \quad \text{and} \quad (\dot{p}^i(t), \dot{p}^{i-1}(t)) \neq (0,0).$$

Hence the two maps have nonzero derivative.

Finally we prove (2.2.2). For $\theta \in (0, \tau)$,

$$\lim_{\theta \to 0} \frac{p(t+\theta) - p(t)}{\theta} = \dot{p}(t).$$

When θ is sufficiently small, $\frac{p(t+\theta)-p(t)}{\theta}$ is near $\dot{p}(t)$, so $\theta^{-1}(p(t+\theta) - p(t)) \in \mathcal{N}$, and

$$N(p(t+\theta) - p(t)) = N(\theta^{-1}(p(t+\theta) - p(t)) = N(\dot{p}(t)) = k_0.$$

In order to prove that for all values of $\theta \in (0, \tau)$ (2.2.2) holds one needs only to observe that for each fixed t, $p(t+\theta) - p(t) \in \mathcal{N}$, $\theta \in (0, \tau)$. Then from the local constancy of N on $\mathcal{N}$, one can get that (2.2.2) holds for all $\theta \in (0, \tau)$. []

Lemma 2.2 (Lemma 3.5 in [1]). If the orbit through $y_0 \in \omega(x_0)$ is neither an equilibrium nor a periodic solution, then for each i the maps

$$t \to (y^i(t), \dot{y}^i(t)) \quad \text{and} \quad t \to (y^i(t), y^{i-1}(t))$$

are one-to- one with nonzero derivative for all $t \in \mathbf{R}$.

Proof. Let $y(t)$ be the solution of (1.1.1) with (1.1.2) through $y_0 \in \omega(x_0)$. Now we prove the two maps

$$t \to (y^i(t), \dot{y}^i(t)) \quad \text{and} \quad t \to (y^i(t), y^{i-1}(t))$$

are one-to- one from $\mathbf{R}$ to $\mathbf{R^2}$. Like the proof of Proposition 2.1, we need only to show that for any $\theta \in \mathbf{R}$, $\theta \neq 0$, the following relations

$$(y^i(t+\theta), \dot{y}^i(t+\theta)) \neq (y^i(t), \dot{y}^i(t)) \text{ and } (y^i(t+\theta), y^{i-1}(t+\theta)) \neq (y^i(t), y^{i-1}(t))$$

or

$$(y^i(t+\theta) - y^i(t), \dot{y}^i(t+\theta) - \dot{y}^i(t)) \neq (0,0)$$
$$(y^i(t+\theta) - y^i(t), y^{i-1}(t+\theta) - y^{i-1}(t)) \neq (0,0)$$

hold. Thus, we consider the difference of two different solutions of (1.1.1) with (1.1.2) $z(t) = y(t+\theta) - y(t)$.

If we can prove that for all $t \in \mathbf{R}$, $z(t) \in \mathcal{N}$, or equivalently $N(z(t)) = k$ is constant, then from Proposition 1.1(d), we can obtain above results.

Since $y_0 \in \omega(x_0)$, there exists a sequence $\{t_m\}, t_m \to \infty$ such that $x(t_m) \to y_0 = y(0)$. By the continuity of flow [6], one has

$$x(t + t_m + \theta) \to y(t+\theta) \text{ and } x(t+t_m) \to y(t), \quad \text{as } m \to \infty.$$

Set the sequence of the difference of solutions

$$\begin{aligned} z_1(t) &= x(t+t_1+\theta) - x(t+t_1) \\ z_2(t) &= x(t+t_2+\theta) - x(t+t_2) \\ &\vdots \\ z_m(t) &= x(t+t_m+\theta) - x(t+t_m) \\ &\vdots \end{aligned}$$

Then $z_m(t) \to z(t)$ as $m \to \infty$ and $z_m(t)$ satisfies the monotone cyclic feedback system (1.2.3), (1.2.4) with coefficients

$$w_m^{i,j}(t) = \int_0^1 \frac{\partial f^i(u_m^i(s,t), u_m^{i-1}(s,t))}{\partial u_m^j} ds$$

where

$$u_m^j(s,t) = sx^j(t+t_m+\theta) + (1-s)x^j(t+t_m), \qquad j = i, i-1,$$

and $z(t)$ satisfies limiting system (1.2.3), (1.2.4) with coefficients

$$w^{i,j}(t) = \int_0^1 \frac{\partial f^i(u^i(s,t), u^{i-1}(s,t))}{\partial u^j} ds$$

where

$$u^j(s,t) = sy^j(t+\theta) + (1-s)y^j(t), \qquad j = i, i-1.$$

Since $z_m(t) = x(t+t_m+\theta) - x(t+t_m)$ is the difference of solutions of (1.1.1) with (1.1.2), from Corollary 1.5, for each $t \in \mathbf{R}$, there exists a $M_t > 0$, such that if $m > M_t$, then $z_m(t) \in \mathcal{N}$ and $N(z_m(t))$ is a constant which does not depend on m and t. Thus, there is a constant k such that

$$\lim_{m\to\infty} N(z_m(t)) = k$$

where k is independent of t.

Furthermore, since $z(t) = y(t+\theta) - y(t)$ is the difference of two solutions of (1.1.1) with (1.1.2) , from Proposition 1.1(a), there must exists some t such that $z(t) \in \mathcal{N}$. Therefore, from $z_m(t) \to z(t)$ one has $N(z(t)) = k$ whenever $z(t) \in \mathcal{N}$.

In fact, one can get stronger results: for all $t \in \mathbf{R}$, $z(t) \in \mathcal{N}$. Suppose that the assertion is not true, that is, there exists some t_0 such that $z(t_0) \notin \mathcal{N}$. Then from Proposition 1.1(a) there are two values t_1, t_2 which satisfy $t_1 < t_0 < t_2$, such that $z(t_1), z(t_2) \in \mathcal{N}$. By using Proposition 1.1(c) one has $N(z(t_2)) < N(z(t_1))$. But this contradicts above fact: $N(z(t)) = k$ whenever $z(t) \in \mathcal{N}$.

Because for all $t \in \mathbf{R}$, $z(t) \in \mathcal{N}$ and $z(t)$ is the nontrivial solution of (1.2.3) with (1.2.4), by Proposition 1.1(d) one has

$$(z^i(t), \dot{z}^i(t)) \neq (0,0) \text{ and } (z^i(t), z^{i-1}(t)) \neq (0,0).$$

That is, for all $t, \theta \in \mathbf{R}, \theta \neq 0$

$$\begin{aligned} &(y^i(t+\theta) - y^i(t), \dot{y}^i(t+\theta) - \dot{y}^i(t)) \neq (0,0) \qquad \text{and} \\ &(y^i(t+\theta) - y^i(t), y^{i-1}(t+\theta) - y^{i-1}(t)) \neq (0,0) \end{aligned}$$

Hence the two maps

$$t \to (y^i(t), \dot{y}^i(t)) \text{ and } t \to (y^i(t), y^{i-1}(t))$$

are one-to- one for all $t \in \mathbf{R}$.

To prove

$$(\dot{y}^i(t), \ddot{y}^i(t)) \neq (0,0) \text{ and } (\dot{y}^i(t), \dot{y}^{i-1}(t)) \neq (0,0),$$

similar to the proof of Lemma 6, we consider the derivative of $y(t)$.

Let $q(t) = \dot{y}(t)$. Similar to the above case of the difference of two solutions, if for all $t \in \mathbf{R}$, $q(t) \in \mathcal{N}$ or for all $t \in \mathbf{R}$, $N(q(t)) = k$ is a constant, one can get the two maps have nonzero derivative.

When $x(t + t_m) \to y(t)$, its derivative $\dot{x}(t + t_m) \to \dot{y}(t)$ since $f^i(x^i, x^{i-1})$ is continuous, $1 \leq i \leq n$. Set $q_m(t) = \dot{x}(t + t_m)$, then $q_m(t) \to \dot{y}(t) = q(t)$ as $m \to \infty$, and $q_m(t)$ satisfies monotone cyclic feedback system (1.2.3),(1.2.4) with coefficients

$$w_m^{i,j}(t) = \frac{\partial f^i(x^i(t+t_m), x^{i-1}(t+t_m))}{\partial x^j}, \quad j = i, i-1,$$

and $q(t)$ satisfies (1.2.3),(1.2.4) with coefficients

$$w^{i,j}(t) = \frac{\partial f^i(y^i(t), y^{i-1}(t))}{\partial y^j}, \quad j = i, i-1.$$

Note that $q_m(t) = \dot{x}(t + t_m)$ is the derivative of solution of (1.1.1) with (1.1.2) and the preceding analysis about the difference $z_m(t)$ is applicable to this case of $q_m(t)$. Hence, there exists k such that

$$\lim_{m\to\infty} N(q_m(t)) = k$$

where k is independent of t.

Since $q(t) = \dot{y}(t)$ is the derivative of solution of (1.1.1), (1.1.2) from Proposition 1.1(a) there definitely exists some t, such that $q(t) \in \mathcal{N}$. Thus, as $q_m(t) \to q(t)$, one has

$$N(q(t)) = \lim_{m\to\infty} N(q_m(t)) = k,$$

provided $q(t) \in \mathcal{N}$. In fact, based on Proposition 1.1, for all $t \in \mathbf{R}$, $q(t) \in \mathcal{N}$ and $N(q(t)) = k$.

Because $q(t)$ is the nontrivial solution of linear system (1.2.3) with (1.2.4), by Proposition 1.1(d)

$$(q^i(t), \dot{q}^i(t)) \neq (0,0) \ \text{ and } \ (q^i(t), q^{i-1}(t)) \neq (0,0).$$

Therefore

$$(\dot{y}^i(t), \ddot{y}^i(t)) \neq (0,0) \ \text{ and } \ (\dot{y}^i(t), \dot{y}^{i-1}(t)) \neq (0,0),$$

that is, the two maps have nonzero derivative. []

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